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Leptonic Decay Constants of Charm and Beauty Mesons in QCD: An Update [1]

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Abstract

An update is given of the determination of leptonic decay constants of charm and beauty mesons in the framework of relativistic Hilbert moments and Laplace transform QCD sum rules.

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In this talk I present an update of relativistic QCD sum rule estimates of the leptonic decay constants

$$\langle 0|A_\mu|P(k)\rangle = i\,\sqrt{2}\,f_P\,k_\mu, \qquad (1)$$

$(P = D, D_s, B, B_s)$, so that in this convention $f_\pi = 93.2$ MeV. The current theoretical status of this problem is rather confusing, due to the existence of a plethora of predictions not always in mutual agreement, even if obtained in the same framework. The major difference between the various determinations to be found in the literature [1, 2, 3, 4, 5] is due to the particular choice of input parameters in the sum rules. Among these, the values of the heavy quark mass and the asymptotic freedom threshold impact the most on the prediction for f_P. To a lesser extent, sum rule windows or stability criteria have also an influence on the results. Hence, it is very important to keep this in mind when comparing different determinations of the same quantity. It is then not very illuminating to present a table showing all the various existing predictions. Instead, I shall present a single prediction for each f_P, from Hilbert moments and, separately, from Laplace transform QCD sum rules, using the criterion that the experimental value of the meson mass should be reproduced by the sum rules, when implemented by our current best knowledge of the input parameters. Considerable progress has now been made in improving the accuracy and reliability of the determinations of quark masses [6],[7], and vacuum condensates [8] entering the QCD sum rules. The uncertainties in these parameters reflect in the uncertainty on f_P, for a given type of sum rule. The two types of sum rules do not give exactly the same answer for a given f_P, although the results are not all that different. A comparison between the two separate determinations can provide a feeling for the systematic uncertainties involved in this approach.

In order to estimate f_P one considers the two-point function

$$\psi_5(q^2) = i \int d^4x\, e^{iqx}\langle 0|T\left(\partial^\mu\, A_\mu(x)\,\partial^\nu\, A_\nu^\dagger(0)\right)|0\rangle\,, \qquad (2)$$

where $\partial^\mu\, A_\mu(x) = (m_Q + m_q)\, :\, \bar{q}(x) i\gamma_5\, Q(x)$: with $q(Q)$ being the light (heavy) quark field and $m_q(m_Q)$ its corresponding QCD (current) mass. The function $\psi_5(Q^2), Q^2 = -q^2$, satisfies a dispersion relation

$$\psi_5(Q^2) = \frac{1}{\pi} \int ds \, \frac{\mathrm{Im}\psi_5(s)}{s+Q^2} + \mathrm{subtractions}, \tag{3}$$

defined up to two subtractions, which can be disposed of by taking at least two derivatives in (3). In this fashion one obtains the Hilbert power moments, which at $Q^2 = 0$ become

$$\varphi^n(0) = \frac{(-)^{n+1}}{(n+1)!} \left(\frac{d}{dQ^2}\right)^{n+1} \psi_5(Q^2)|_{Q^2=0} = \frac{1}{\pi} \int_0^\infty \frac{ds}{s^{n+2}} \, \mathrm{Im}\psi_5(s) \; . \tag{4}$$

The point $Q^2 = 0$ is appropriate for heavy-light quark currents, to the extent that φ^n can be computed in perturbative QCD, adding non-perturbative power corrections which fall off by powers of the heavy quark mass. These corrections are parametrized by vacuum expectation values of the quark and gluon fields in the QCD Lagrangian, and are organized according to their dimension. For instance, in the limit $m_q \to 0$, well justified for $D_{u,d}$ and $B_{u,d}$ mesons, the perturbative contribution to $\varphi^n(0)$ to order $\mathcal{O}(\alpha_s)$ is given by [9]

$$\varphi^{(n)}(0)|_{PT} = \frac{3}{8\pi^2} \left(\frac{1}{m_Q^2}\right)^{n-1} B(n,3)[1 + a_n^0 \; \alpha_s], \tag{5}$$

where $B(x,y)$ is the beta function, $\alpha_s \equiv \alpha_s(m_Q^2)$, and a_n^0 are the rational numbers

$$\frac{3\pi}{4} \, a_n^0 - \frac{\pi^2}{6} = 1 - \frac{2}{n+1} - \frac{6}{n+2} + \sum_{r=1}^{n+2} \left[\frac{1}{r^2} + \left(\frac{3}{2} - \frac{1}{n} \right. \right.$$

$$\left. \left. - \frac{1}{(n+1)} - \frac{1}{(n+2)} + \frac{3}{(n+3)} \right) \frac{1}{r} \right] \; . \tag{6}$$

The non-perturbative part, always in the limit $m_q \to 0$, becomes [9]

$$\begin{aligned} \varphi^{(n)}(0)|_{NP} \;\; &= \;\; \frac{-m_Q < \bar{q}q >}{m_Q^{2n+2}} \Bigg[1 - \frac{< \alpha_s G^2 >}{12\pi m_Q < \bar{q}q >} - \frac{1}{4}(n+2)(n+1)\frac{M_0^2}{m_Q^2} \\ &- \;\; \frac{4}{81}(n+2)(n^2+10n+9)\pi \; \alpha_s \; \rho \; \frac{\langle \bar{q}q \rangle}{m_Q^3} \Bigg], \end{aligned} \tag{7}$$

where ρ is a measure of the deviation from the vacuum saturation approximation of the four-quark condensate ($\rho|_{VS} = 1$). In the case of the D_s and B_s mesons, where the approximation $m_q = 0$ should not be made, the full expressions given in [9] must be used for $\varphi^{(n)}(0)$. Finally, the hadronic spectral function appearing on the r.h.s. of (4) is parametrized by the ground state pseudoscalar meson pole plus a continuum starting at some threshold s_0. This continuum is expected to be well approximated by the QCD spectral function, computed in perturbation theory, provided s_0 is high enough, i.e.

$$\frac{1}{\pi}\mathrm{Im}\psi_5(s)|_{HAD} = 2f_P^2 M_P^4\ \delta(s - m_P^2) + \theta(s - s_0)\frac{1}{\pi}\ \mathrm{Im}\psi_5(s)|_{PT}\ . \tag{8}$$

By taking the ratio of any two consecutive moments one obtains an expression for M_P^2 as a function of s_0, the latter being *a - priori* unknown. The calculation will be meaningful provided M_P does not depend strongly on s_0, i.e. there should be a relatively wide range of values of s_0 leading to a value of M_P with a reasonably small spread. This is certainly the case for D and D_s, where one obtains using the first two moments ($n = 1, 2$) [2]

$$s_0 = 2M_D^2 - 3M_D^2 \qquad , \qquad M_D = 1.85 \pm 0.15 GeV\ , \tag{9}$$

$$s_0 = 2M_{D_s}^2 - 3M_{D_s}^2 \qquad , \qquad M_{D_s} = 1.9 \pm 0.1 GeV\ , \tag{10}$$

to be compared with the experimental values: $M_D|_{EXP} = 1.87$ GeV, and $M_{D_s}|_{EXP} = 1.97$ GeV. With increasing heavy quark mass, the perturbative contribution increases in importance relative to the non-perturbative part. Therefore, the stability region in s_0 becomes narrower. For instance, for Q=b one finds [2] that with $s_0 \simeq (1.1 - 1.3)M_B^2$, the predicted mass is $M_B = 5.2 \pm 0.2$ GeV ($M_B|_{EXP} = 5.27$ GeV). An additional *confidence criterion* often invoked is that of the hierachy of the non-perturbative power corrections. An inspection of Eq.(7) shows that this hierarchy is not respected in the case of Q=c, as the dimension d=5 term can easily become bigger than the d=4 term. In addition, the d=5 contribution could become bigger than the perturbative contribution. In fact, there are no real solutions for f_D and f_{D_s}, unless $M_0^2 \leq 0.5 GeV^2$, and $n \leq 2$, as noticed in [2]. These problems are absent for Q = b, where the perturbative term dominates the sum rule. In this case there are real solutions for all values of n, and M_0^2 could be as high as 1 GeV^2. In summary, there are certain advantages and shortcomings of the Hilbert moments, which should be kept in mind when comparing results in this framework with those from Laplace transform sum rules.

The criterion I shall adopt here is to fix s_0 in such a way as to reproduce the experimental value of the pseudoscalar meson mass, for a given set of input parameters. The latter are chosen as follows: $m_c = 1.35 \pm 0.05$ GeV, m_b $=4.72 \pm 0.05$ GeV, $m_s \simeq 190$ MeV, $\Lambda = 200 \pm 100$ MeV, $\rho = 3 \pm 1$, $< \alpha_s G^2 >= 0.038 - 0.11\ GeV^4$, $< \bar{q}q > |_c = -0.010\ GeV^3$, $< \bar{q}q > |_b = -0.014\ GeV^3$, and $M_0^2 = 0.5 - 1.0\ GeV^2$, except for Q=c where $M_0^2 = 0.5\ GeV^2$, as mentioned above. Concerning f_{D_s} and f_{B_s}, previous analyses [2, 4] have been made keeping $m_s \neq 0$ in the perturbative part, but not in the non - perturbative expression of the two-point function (2). This can be improved by keeping $m_s \neq 0$ everywhere (details are given in [10]). An additional improvement, in the case of f_{B_s} is possible thanks to the recent measurement of the B_s mass [11]: $M_{B_s}|_{EXP} \simeq 5.37$ GeV.

The results from these Hilbert moment QCD sum rules are shown in Tables 1 and 2, for the two extreme values of the corresponding heavy quark masses. The minimum and maximum values of the leptonic decay constants are obtained by varying the input parameters within the limits given above. Of particular importance are the results for the ratios f_{D_s}/f_D, and f_{B_s}/f_B, which are basically independent of the heavy quark mass, and quite stable against changes in the rest of the input parameters.

Table 1: Hilbert moment QCD sum rule results for Q=c

	$m_c = 1.3$ GeV	$m_c = 1.4$ GeV
f_D/f_π	1.29 – 1.79	1.06 – 1.51
$s_0\|_D$ (GeV^2)	4.5 – 8.0	4.0 – 6.0
f_{D_s}/f_π	1.59 – 2.06	1.33 – 1.74
$s_0\|_{D_s}$ (GeV^2)	5.0 – 7.5	4.5 – 7.0
f_{D_s}/f_D	1.23 – 1.15	1.25 – 1.15

It is possible to choose a different kernel in the dispersion relation (3) and obtain other types of QCD sum rules, e.g. the Laplace sum rules

Table 2: Hilbert moment QCD sum rule results for Q = b

	$m_b = 4.67$ GeV	$m_b = 4.77$ GeV
f_B/f_π	1.19 – 1.40	1.02 – 1.18
$s_0\|_B$ (GeV^2)	32.5 – 34.0	32.0 – 33.0
f_{B_s}/f_π	1.48 – 1.68	1.26 – 1.44
$s_0\|_{B_s}$ (GeV^2)	34.0 – 35.0	33.0 – 34.0
f_{B_s}/f_B	1.24 – 1.20	1.24 – 1.22

$$2f_P^2 M_P^4 exp(-M_P^2/M^2) = \int_{m_Q^2}^{so} ds\ exp(-s/M^2)\frac{1}{\pi}\mathrm{Im}\psi_5(s)|_{QCD}$$

$$+m_Q^2 exp(-m_Q^2/M^2)\Bigg[< (\alpha_s/12\pi)G^2 - m_Q\bar{q}q > -\frac{1}{4}\frac{m_Q}{M^2}\left(1-\frac{m_Q^2}{2M^2}\right)2M_0^2 < \bar{q}q >$$

$$-\frac{1}{6M^2}\left(2-\frac{m_Q^2}{2M^2}-\frac{m_Q^4}{6M^4}\right)(16/9)\pi\alpha_s\rho < \bar{q}q >^2\Bigg]\ . \qquad (11)$$

In (11) $m_s = 0$ is understood (for details of the case $m_s \neq 0$ see [10]). Notice that the sign of the gluon condensate is positive, contrary to that in [1] which is incorrect. Taking the first derivative with respect to M^2 in (11) gives an independent sum rule which can be used together with (11) in order to get an expression for the meson mass M_P, independent of f_P. This procedure, which essentially fixes s_0, is quite important, i.e. a determination of f_P will be reliable provided that M_P comes out right. This point has not been fully appreciated in some of the existing analyses.

It has often been claimed that Laplace transform QCD sum rules are superior to the Hilbert moments for the determination of f_P. I do not quite agree with this claim. In fact, in spite of the exponential kernel in the dispersion relation, results are rather sensitive to s_0. For instance, in the case of Q=c, a 20% change in s_0, around the value which gives the correct meson mass, induces typically a 10 % change in that mass as well as in f_D. For Q=b, the situation is somewhat more unstable, i.e. a given relative variation in s_0 is accompanied by roughly the same relative variation of M_B and f_B.

On the other hand, for a fixed value of s_0, the predicted M_D and M_B change by 15 % and 25 %, respectively, inside the sum rule windows in the Laplace parameter M^2. In spite of all this, it is true that the final uncertainty in f_P, due to the uncertainties in the input parameters, is smaller with the Laplace transform QCD sum rules than with the Hilbert moments. This can be appreciated from Tables 3 and 4, where I present the results obtained with the Laplace sum rules.However, the two types of sum rules exhibit different sensitivities to changes in the input parameters, in addition to having different advantages and shortcomings. For this reason they should be viewed as complementary methods within the general framework of QCD sum rules.

Table 3: Laplace transform QCD sum rule results for Q = c

	$m_c = 1.3$ GeV	$m_c = 1.4$ GeV
f_D/f_π	1.30 – 1.46	1.15 – 1.29
$s_0\|_D$ (GeV^2)	5.5 – 5.5	5.0 – 5.0
f_{D_s}/f_π	1.56 – 1.75	1.45 – 1.64
$s_0\|_{D_s}$ (GeV^2)	6.0 – 6.0	5.5 – 5.5
f_{D_s}/f_D	1.20 – 1.20	1.26 – 1.27

Table 4: Laplace transform QCD sum rule results for Q = b

	$m_b = 4.67$ GeV	$m_b = 4.77$ GeV
f_B/f_π	1.32 – 1.40	1.14 – 1.18
$s_0\|_B$ (GeV^2)	35.0 – 35.5	34.5 – 35.0
f_{B_s}/f_π	1.58 – 1.68	1.38 – 1.44
$s_0\|_{B_s}$ (GeV^2)	35.5 – 36.0	35.0 – 35.5
f_{B_s}/f_B	1.20 – 1.20	1.21 – 1.22

Given the fact that results from the Laplace sum rules have less of a spread than those from the Hilbert moments, one may be tempted to consider the former as the best determination of the leptonic decay constants. However, the two methods are not indepenedent, as the various vacuum condensates enter **both** sum rules, albeit with different weight factors, and different signs in the case of d=5 and d=6. Short of performing a correlation analysis, I feel one should not discard the results from the Hilbert moments (nor perform any average from the two methods), but rather read the absolute minimum and maximum values from both sum rules in conjunction. In any case, the predictions for the ratios f_{D_s}/f_D, and f_{B_s}/f_B turn out to be far less dependent on the values of the heavy quark masses, and the particular sum rule, leading to the accurate and stable predictions

$$f_{D_s}/f_D = 1.21 \pm 0.06, \tag{12}$$

$$f_{B_s}/f_B = 1.22 \pm 0.02. \tag{13}$$

These results are in nice agreement with the expectation that the ratio between (12) and (13) should be close to unity (see e.g. [12]).

A comparison of the results listed in Tables 1 - 4 with predictions from lattice QCD (see e.g. [13]) shows reasonable agreement for f_D, and f_{D_s}, but not quite for f_B, although the ratio f_{B_s}/f_B does compare well. In the framework of fully relativistic QCD sum rules, it is simply not possible to obtain values of f_B bigger than what is shown in Tables 1 - 4, if one uses the current best values of the input parameters. There is a recent claim to the contrary by Narison [14], but I have been unable to reproduce his results, which I believe to be incorrect.

It is only when one considers the infinite quark mass limit, and after resumming the large logarithms [15], that one can approach lattice QCD predictions. The price to pay, though, is a two - loop correction at the 100 % level in the expression for f_B^2.

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