

A CANONICAL FRAME FOR NONHOLONOMIC RANK TWO DISTRIBUTIONS OF MAXIMAL CLASS

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ABSTRACT. In 1910 E. Cartan constructed the canonical frame and found the most symmetric case for maximally nonholonomic rank 2 distributions in \mathbb{R}^5 . We solve the analogous problems for rank 2 distributions in \mathbb{R}^n for arbitrary $n > 5$. Our method is a kind of symplectification of the problem and it is completely different from the Cartan method of equivalence.

1. INTRODUCTION

A rank l vector distribution D on an n -dimensional manifold M or an (l, n) -distribution (where $l < n$) is a subbundle of the tangent bundle TM with l -dimensional fibers. The group of germs of diffeomorphisms of M acts naturally on the set of germs of (l, n) -distributions and defines the equivalence relation there. The question is *when two germs of distributions are equivalent?* Distributions are naturally associated with Pfaffian systems and with control systems linear in the control. So the problem of equivalence of distributions can be reformulated as the problem of equivalence of the corresponding Pfaffian systems and the state-feedback equivalence of the corresponding control systems. The obvious (but very rough in the most cases) discrete invariant of a distribution D at q is so-called *the small growth vectors* at q . It is the tuple $\{\dim D^j(q)\}_{j \in \mathbb{N}}$, where D^j is the j -th power of the distribution D , i.e., $D^j = D^{j-1} + [D, D^{j-1}]$, $D^1 = D$. A simple estimation shows that at least $l(n-l) - n$ functions of n variables are required to describe generic germs of (l, n) -distribution, up to the equivalence (see [8] and [10] for precise statements). There are only three cases, where $l(n-l) - n$ is not positive: $l = 1$ (line distributions), $l = n - 1$, and $(l, n) = (2, 4)$. Moreover, it is well known that in these cases generic germs of distributions are equivalent. For $l = 1$ it is just the classical theorem about the rectification of vector fields without stationary points, for $l = n - 1$ all generic germs are equivalent to Darboux's model, while for $(l, n) = (2, 4)$ they are equivalent to Engel's model (see, for example, [3]). In all other cases generic (l, n) -distributions have functional invariants.

In the present paper we restrict ourselves to the case of rank 2 distributions. The model examples of such distributions come from so-called underdetermined ODE's of the type

$$z^{(r)}(x) = F(x, y(x), \dots, y^{(s)}(x), z(x), \dots, z^{(r-1)}(x)), \quad r + s = n - 2,$$

for two functions $y(x)$ and $z(x)$. Setting $p_i = y^{(i)}$, $0 \leq i \leq s$, and $q_j = z^{(j)}$, $0 \leq j \leq r - 1$, with each such equation one can associate the rank 2 distribution in \mathbb{R}^n with coordinates $(x, p_0, \dots, p_s, q_0, \dots, q_{r-1})$ given by the intersection of the annihilators of the following $n - 2$ one-forms:

$$\begin{aligned} dp_i - p_{i+1}dx, \quad 0 \leq i \leq s - 1, \quad dq_j - q_{j+1}dx, \quad 0 \leq j \leq r - 2, \\ dq_{r-1} - F(x, p_0, \dots, p_s, q_0, \dots, q_{r-1})dx. \end{aligned}$$

For $n = 3$ and 4 all generic germs of rank 2 distribution are equivalent to the distribution, associated with the underdetermined ODE $z'(x) = y(x)$ (Darboux and Engel models respectively). The case $n = 5$ (the smallest dimension, when functional parameters appear) was treated by E.

Cartan in [4] with his reduction-prolongation procedure. First, for any $(2, 5)$ -distribution with the small growth vector $(2, 3, 5)$ he constructed the canonical coframe in some 14-dimensional manifold, which implied that the group of symmetries of such distributions is at most 14-dimensional. Second, he showed that any $(2, 5)$ -distribution with 14-dimensional group of symmetries is locally equivalent to the distribution, associated with the underdetermined ODE $z'(x) = (y''(x))^2$, and its group of symmetries is isomorphic to the real split form of the exceptional Lie group G_2 . Historically it was the first natural appearance of this group.

After the work of Cartan the open question was **to construct the canonical frame and to find the most symmetric cases for $(2, n)$ -distributions with $n > 5$** . The Cartan equivalence method was systematized and generalized by N. Tanaka and T. Morimoto (see [6, 5]). Their theory is heavily based on the notion of so-called *symbol algebra* of the distribution at a point, which is a special *graded nilpotent Lie algebra*, naturally associated with the distribution at a point: the symbol algebras have to be isomorphic at different points and all constructions strongly depend on the type of the symbol. Note that already in the case of $(2, 6)$ -distributions with maximal possible small growth vector $(2, 3, 5, 6)$ three different symbol algebras are possible, while for $n = 9$ the set of all possible symbol algebras depends on continuous parameters, which implies in particular that generic distributions do not have a constant symbol.

In the present paper we give an answer to the question, underlined in the previous paragraph, for rank 2 distributions from some generic class. Our constructions are based on a completely different, variational approach, developed in [2] and [9]. Roughly speaking, we make a kind of symplectification of the problem by lifting the distribution to the cotangent bundle T^*M of the manifold M .

2. THE CLASS OF RANK 2 DISTRIBUTION

Assume that $\dim D^2(q) = 3$ and $\dim D^3(q) > 3$ for any $q \in M$. Denote by $(D^l)^\perp \subset T^*M$ the annihilator of the l th power D^l , namely

$$(D^l)^\perp = \{(q, p) \in T^*M : p \cdot v = 0 \forall v \in D^l(q)\}.$$

First, we distinguish a characteristic 1-foliation on the codimension 3 submanifold $(D^2)^\perp \setminus (D^3)^\perp$ of T^*M . For this let $\pi: T^*M \mapsto M$ be the canonical projection. For any $\lambda \in T^*M$, $\lambda = (p, q)$, $q \in M$, $p \in T_q^*M$, let $\mathfrak{s}(\lambda)(\cdot) = p(\pi_* \cdot)$ be the canonical Liouville form and $\sigma = d\mathfrak{s}$ be the standard symplectic structure on T^*M . Since the submanifold $(D^2)^\perp$ has odd codimension in T^*M , the kernels of the restriction $\sigma|_{(D^2)^\perp}$ of σ on $(D^2)^\perp$ are not trivial. Moreover for the points of $(D^2)^\perp \setminus (D^3)^\perp$ these kernels are one-dimensional. They form the *characteristic line distribution* in $(D^2)^\perp \setminus (D^3)^\perp$, which will be denoted by \mathcal{C} . The line distribution \mathcal{C} defines a *characteristic 1-foliation* of $(D^2)^\perp \setminus (D^3)^\perp$. The leaves of this foliation are called the *characteristic curves*. In Control Theory these characteristic curves are also called *regular abnormal extremals of D* .

In the sequel given two submanifold S_1 and S_2 of the tangent bundle of some manifold W such that $S_i(w) = S_i \cap T_w W$, $i = 1, 2$, are linear subspaces of $T_w W$ (not necessary of the same dimensions for different w) we will denote by $[S_1, S_2]$ the subset $\{[S_1, S_2](w)\}_{w \in W}$ of TW such that

$$[S_1, S_2](w) = \text{span}\{[Z_1, Z_2](w) : Z_i \text{ are vector fields tangent to } S_i, i = 1, 2\}.$$

It is easy to show that with such definition $S_i \subset [S_1, S_2]$, $i = 1, 2$.

Now, following [9], let

$$(2.1) \quad \mathcal{J}(\lambda) = (T_\lambda(T_{\pi(\lambda)}^*M) + \ker \sigma|_{D^\perp}(\lambda)) \cap T_\lambda(D^2)^\perp = \{v \in T_\lambda(D^2)^\perp : \pi_* v \in D(\pi(\lambda))\}.$$

(here $T_\lambda(T_{\pi(\lambda)}^*M)$ is tangent to the fiber $T_{\pi(\lambda)}^*M$ at the point λ). Note that $\dim \mathcal{J}(\lambda) = n - 1$. Actually, \mathcal{J} is the pull-back of the distribution D on $(D^2)^\perp \setminus (D^3)^\perp$ by the canonical projection π . Another important property of the distribution \mathcal{J} , which follows from its definition, is $\sigma|_{\mathcal{J}} = 0$. Define a sequence of subspaces $\mathcal{J}^{(i)}(\lambda)$, $\lambda \in (D^2)^\perp \setminus (D^3)^\perp$, by the following recursive formulas:

$$(2.2) \quad \mathcal{J}^{(i)} = [\mathcal{C}, \mathcal{J}^{(i-1)}] \quad \mathcal{J}^{(0)} = \mathcal{J}.$$

By [9] (Proposition 3.1 and formula (3.9) there),

$$(2.3) \quad \dim \mathcal{J}^{(1)}(\lambda) - \dim \mathcal{J}(\lambda) = 1$$

Actually,

$$(2.4) \quad \mathcal{J}^{(1)}(\lambda) = \{v \in T_\lambda(D^2)^\perp : \pi_* v \in D^2(\pi(\lambda))\}.$$

Relation (2.3) together with the definition (2.2) implies that

$$(2.5) \quad \dim \mathcal{J}^{(i)}(\lambda) - \dim \mathcal{J}^{(i-1)}(\lambda) \leq 1, \quad i \in \mathbb{N}.$$

Lemma 2.1. *The following inequality holds*

$$(2.6) \quad \dim \mathcal{J}^{(i)}(\lambda) \leq 2n - 4.$$

Proof. By definition the line distribution \mathcal{C} forms the characteristic of the corank 1 distribution on $(D^2)^\perp$, given by the Pfaffian equation $\mathfrak{s}|_{(D^2)^\perp} = 0$. Since by construction $\mathcal{J} \subset \{\mathfrak{s}|_{(D^2)^\perp} = 0\}$, one has

$$(2.7) \quad \mathcal{J}^{(i)} \subset \{\mathfrak{s}|_{(D^2)^\perp} = 0\} \quad i \in \mathbb{N}.$$

Our lemma follows from the fact that the distribution $\{\mathfrak{s}|_{(D^2)^\perp} = 0\}$ has rank $2n - 4$. \square

Further for any point $q \in M$ denote by $(D^l)^\perp(q) = (D^l)^\perp \cap T_q^*M$. Let us define the following two integer-valued functions:

$$\nu(\lambda) = \min\{i \in \mathbb{N} : \mathcal{J}^{(i+1)}(\lambda) = \mathcal{J}^{(i)}(\lambda)\}, \quad m(q) = \max\{\nu(\lambda) : \lambda \in (D^2)^\perp(q) \setminus (D^3)^\perp(q)\}.$$

Definition 1. *The number $m(q)$ is called the class of distribution D at the point q .*

By (2.3), (2.5), and the previous lemma $1 \leq m(q) \leq n - 3$. It is easy to show that the integer-valued functions $\nu(\cdot)$ and $m(\cdot)$ are lower semicontinuous. Hence they are locally constant on the open and dense subset of $(D^2)^\perp \setminus (D^3)^\perp$ and M correspondingly. We say that the point $q \in M$ is *the regular point* of the distribution D , if the function $m(\cdot)$ is constant in some neighborhood U of q , i.e. the distribution D has constant class on U . Obviously, all points, where the distribution D has maximal class $n - 3$, are regular. Moreover,

Proposition 2.1. *Germes of $(2, n)$ -distributions of the maximal class $n - 3$ are generic.*

This Proposition follows directly from Proposition 3.4 of [9]. In the present paper we treat the germes of $(2, n)$ distributions of the maximal class $n - 3$. In the cases $n = 5$ and $n = 6$ a rank 2 distribution has maximal class if and only if it has maximal possible small growth vector, namely, $(2, 3, 5)$ in the case $n = 5$ and $(2, 3, 5, 6)$ in the case $n = 6$ (see Propositions 3.5 and 3.6 of [9] respectively).

3. THE CANONICAL PROJECTIVE STRUCTURE ON CHARACTERISTIC CURVES.

From now on D is a $(2, n)$ -distribution of maximal constant class $m = n - 3$. Let

$$\mathcal{R}_D = \{\lambda \in (D^2)^\perp \setminus (D^3)^\perp : \nu(\lambda) = n - 3\}, \quad \mathcal{R}_D(q) = \mathcal{R}_D \cap T_q^*M.$$

Note that the set $\mathcal{R}_D(q)$ is a nonempty open set in Zariski topology on the linear space $(D^2)^\perp(q)$ (see again [9, Proposition 3.4]). The crucial observation is that *any segment of a characteristic curve γ of D , belonging to \mathcal{R}_D , can be endowed with a canonical projective structure* (for more detailed description than below see [1],[2], and [9]). By a projective structure on a curve we mean a set of parameterizations such that the transition function from one such parameterization to another is a Möbius transformation. To construct this canonical projective structure on γ first we associate with γ a special curve in a Grassmannian $G_m(W)$ of m -dimensional subspaces of a $2m$ -dimensional linear space W , the Jacobi curve, in the following way: Let O_γ be a neighborhood of γ in $(D^2)^\perp$ such that the factor $N = O_\gamma / (\text{the characteristic one-foliation})$ is a well-defined smooth manifold. Its dimension is equal to $2(n - 2)$. Let $\phi: O_\gamma \rightarrow N$ be the canonical projection on the factor. Define the mapping $J_\gamma: \gamma \mapsto G_{n-2}(T_\gamma N)$ by $J_\gamma(\lambda) = \phi_*(\mathcal{J}(\lambda))$ for all $\lambda \in \gamma$, where $\mathcal{J}(\lambda)$ is as above. Actually, the symplectic form σ of T^*M induces naturally the symplectic form $\bar{\sigma}$ on $T_\gamma N$ and $J_\gamma(\lambda)$ for all $\lambda \in \gamma$ are Lagrangian subspace of $T_\gamma N$. Besides, if e is the Euler field (i.e., the infinitesimal generator of homotheties on the fibers of T^*M), then the vector $\bar{e} = \phi_*e(\lambda)$ is the same for any $\lambda \in \gamma$ and lies in $J_\gamma(\lambda)$. Therefore the curve $\lambda \mapsto \tilde{J}_\gamma(\lambda) = J_\gamma(\lambda) / \{\mathbb{R}\bar{e}\}$, $\lambda \in \gamma$, is a curve in $G_m(W)$, where $W = \{v \in T_\gamma N : \sigma(v, \bar{e}) = 0\} / \{\mathbb{R}\bar{e}\}$. The curve \tilde{J}_γ is called *the Jacobi curve of γ* .

Second, we construct the canonical projective structure on \tilde{J}_γ (and therefore on γ itself), using the notion of the generalized cross-ratio of 4 points in $G_m(W)$. Namely, let $\{\Lambda_i\}_{i=1}^4$ be any 4 points of $G_m(W)$. For simplicity suppose that $\Lambda_i \cap \Lambda_j = 0$ for $i \neq j$. Assume that in some coordinates $W \cong \mathbb{R}^m \times \mathbb{R}^m$ and $\Lambda_i = \{(x, S_i x) : x \in \mathbb{R}^m\}$ for some $m \times m$ -matrix S_i . Then the conjugacy class of the following matrix

$$(S_1 - S_4)^{-1}(S_4 - S_3)(S_3 - S_2)^{-1}(S_2 - S_1)$$

does not depend on the choice of the coordinates in W . This conjugacy class is called *the cross-ratio of the tuple $\{\Lambda_i\}_{i=1}^4$* and it is denoted by $[\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4]$.

Now take some parametrization $\varphi: \gamma \mapsto \mathbb{R}$ of γ and let $\Lambda_\varphi(t) = \tilde{J}_\gamma(\varphi^{-1}(t))$. Assume that in some coordinates on W we have $\Lambda_\varphi(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$. The following fact follows from [9, Proposition 2.1] : For all parameters t_1 the functions $t \rightarrow \det(S_t - S_{t_1})$ have zero of the the same order $k = m^2$ at $t = t_1$. Consider the following function

$$(3.1) \quad \mathcal{G}_{\Lambda_\varphi}(t_1, t_2, t_3, t_4) = \ln \left(\det[\Lambda_\varphi(t_1), \Lambda_\varphi(t_2), \Lambda_\varphi(t_3), \Lambda_\varphi(t_4)] ([t_1, t_1, t_2, t_3])^{-k} \right),$$

where $[t_1, t_2, t_3, t_4] = \frac{(t_1 - t_2)(t_2 - t_3)}{(t_3 - t_4)(t_1 - t_4)}$ is the usual cross-ratio of 4 numbers $\{t_i\}_{i=1}^4$. Then, by above, it is not hard to see that $\mathcal{G}_{\Lambda_\varphi}(t_1, t_2, t_3, t_4)$ is smooth at diagonal points (t, t, t, t) and the Taylor expansions up to the order two of it at these points have the form

$$(3.2) \quad \mathcal{G}_{\Lambda_\varphi}(t_0, t_1, t_2, t_3) = \rho_{\Lambda_\varphi}(t)(\xi_1 - \xi_3)(\xi_2 - \xi_4) + \dots, \quad \xi_i = t_i - t.$$

Now let $\psi: \mathbb{R} \mapsto \mathbb{R}$ be a smooth monotonic function. Then by (3.1)

$$(3.3) \quad \mathcal{G}_{\Lambda_\varphi}(t_1, t_2, t_3, t_4) = \mathcal{G}_{\Lambda_{\psi \circ \varphi}}(\psi(t_1), \psi(t_2), \psi(t_3), \psi(t_4)) + k \ln \left(\frac{[\psi(t_1), \psi(t_2), \psi(t_3), \psi(t_4)]}{[t_1, t_2, t_3, t_4]} \right),$$

By direct computation it can be shown that the function $(t_0, t_1, t_2, t_3) \mapsto \ln \left(\frac{[\psi(t_1), \psi(t_2), \psi(t_3), \psi(t_4)]}{[t_1, t_2, t_3, t_4]} \right)$ has the following Taylor expansion up to the order two at the point (t, t, t, t) :

$$(3.4) \quad \ln \left(\frac{[\psi(t_1), \psi(t_2), \psi(t_3), \psi(t_4)]}{[t_1, t_2, t_3, t_4]} \right) = \frac{1}{3} \mathbb{S}\psi(t)(\xi_1 - \xi_3)(\xi_2 - \xi_4) + \dots, \quad \xi_i = t_i - t,$$

where $\mathbb{S}\psi$ is Schwarz derivative of ψ , $\mathbb{S}\psi = \frac{1}{2} \frac{\psi^{(3)}}{\psi'} - \frac{3}{4} \left(\frac{\psi''}{\psi'} \right)^2$. Combining (3.2) and (3.4) we get the following reparameterization rule for ρ_{Λ_φ} :

$$(3.5) \quad \rho_{\Lambda_\varphi}(t) = \rho_{\Lambda_{\psi \circ \varphi}}(\psi(t))(\psi'(t))^2 + \frac{k}{3} \mathbb{S}\psi(t).$$

From the last formula and the fact that $\mathbb{S}\psi \equiv 0$ if and only if the function ψ is Möbius it follows that *the set of all parametrizations φ of γ such that $\rho_{\Lambda_\varphi}(t) \equiv 0$ defines the canonical projective structure on γ .*

4. THE MAIN THEOREM

Now we are ready to describe the manifold, on which the canonical frame for $(2, n)$ -distribution of maximal class, $n > 5$, can be constructed. Given $\lambda \in \mathcal{R}_D$ denote by \mathfrak{P}_λ the set of all projective parameterizations $\varphi: \gamma \mapsto \mathbb{R}$ on the characteristic curve γ , passing through λ , such that $\varphi(\lambda) = 0$. Denote

$$\Sigma_D = \{(\lambda, \varphi) : \lambda \in \mathcal{R}_D, \varphi \in \mathfrak{P}_\lambda\}.$$

Actually, Σ_D is a principal bundle over \mathcal{R}_D with the structural group of all Möbius transformations, preserving 0 and $\dim \Sigma_D = 2n - 1$.

Theorem. *For any $(2, n)$ -distribution, $n > 5$, of maximal class there exist two canonical frames on the corresponding $(2n - 1)$ -dimensional manifold Σ_D , obtained one from another by a reflection. The group of symmetries of such distribution is at most $(2n - 1)$ -dimensional. Any $(2, n)$ -distribution of maximal class with $(2n - 1)$ -dimensional group of symmetries is locally equivalent to the distribution, associated with the underdetermined ODE $z'(x) = (y^{(n-3)}(x))^2$. The algebra of infinitesimal symmetries of this distribution is isomorphic to a semidirect sum of $\mathfrak{gl}(2, \mathbb{R})$ and $(2n - 5)$ -dimensional Heisenberg algebra \mathfrak{n}_{2n-5} .*

Sketch of the proof. Define the following two fiber-preserving flows on Σ_D :

$$F_{1,s}(\lambda, \varphi) = (\lambda, e^{2s}\varphi), \quad F_{2,s}(\lambda, \varphi) = \left(\lambda, \frac{\varphi}{s\varphi + 1} \right), \quad \lambda \in \mathcal{R}_D, \varphi \in \mathfrak{P}_\lambda.$$

Further, let δ_s be the flow of homotheties on the fibers of T^*M : $\delta_s(p, q) = (e^s p, q)$, where $q \in M$, $p \in T_q^*M$ (actually the Euler field e generates this flow). The following flow

$$F_{0,s}(\lambda, \varphi) = (\delta_s(\lambda), \varphi \circ \delta_s^{-1})$$

is well-defined on Σ_D (here we use that δ_s preserves the characteristic 1-foliation). For any $0 \leq i \leq 2$ let g_i be the vector field on Σ_D , generating the flow $F_{i,s}$. Besides, the characteristic 1-foliation on $(D^2)^\perp$ can be lifted to the *parameterized* 1-foliation on Σ_D , which gives one more canonical vector field on Σ_D . Indeed, let $u = (\lambda, \varphi) \in \Sigma_D$ and γ be the characteristic curve, passing through λ (so, φ maps γ to \mathbb{R}). Then the the mapping

$$\Gamma_u(t) = (\varphi^{-1}(t), \varphi(\cdot) - t)$$

defines the parametrized curve on Σ_D , the lift of γ to Σ_D , and $\Gamma_u(0) = u$. The additional canonical vector field h on Σ_D is defined by $h(u) = \frac{d}{dt}\Gamma_u(t)|_{t=0}$. It can be shown easily that

$$(4.1) \quad [g_1, g_2] = 2g_2, \quad [g_1, h] = -2h, \quad [g_2, h] = g_1, \quad [g_0, h] = 0, \quad [g_0, g_i] = 0$$

Therefore the linear span (over \mathbb{R}) of the vector fields g_0, g_1, g_2 , and h is endowed with a structure of the Lie algebra isomorphic to $\mathfrak{gl}(2, \mathbb{R})$.

Now we will construct one more canonical, up to the sign, vector field on Σ_D . For this let

$$(4.2) \quad \mathcal{J}_{(i)}(\lambda) = \{v \in T_\lambda((D^2)^\perp) : \sigma(v, w) = 0 \forall w \in \mathcal{J}^{(i)}\}, \quad V_i(\lambda) = \{\lambda \in \mathcal{J}_{(i)} : \pi_*(v) = 0\}.$$

Since $\mathcal{J}^{(i)} \subseteq \mathcal{J}^{(i+1)}$, we have $\mathcal{J}_{(i+1)} \subseteq \mathcal{J}_{(i)}$. If $\lambda \in \mathcal{R}$, then $\dim \mathcal{J}^{(i)}(\lambda) = n - 1 + i$, which implies that $\dim \mathcal{J}_{(i)}(\lambda) = n - 1 - i$. Besides, it is easy to show that $\mathcal{J}_{(i)} = V_i \oplus \mathcal{C}$. Therefore $\dim V_i(\lambda) = n - 2 - i$. In particular, $\dim V_{n-4}(\lambda) = 2$. Also the Euler field $e \in V_{(n-4)}$. Fix a point $\lambda \in \mathcal{R}$. Let γ be the characteristic curve, passing through λ , and let φ be a parametrization on γ such that $\varphi(\lambda) = 0$. As before, let $m = n - 3$. Then there exist a vector $\varepsilon_\varphi(\lambda) \in V_{n-4}(\lambda)$ such that if E and H are two vector fields, satisfying $E \in V_{n-4}$, $H \in \mathcal{C}$, $E(\lambda) = \varepsilon_\varphi(\lambda)$, and $H(\lambda) = \frac{d}{dt}\varphi^{-1}(t)|_{t=0}$, then

$$|\sigma((\text{ad}H)^m E(\lambda), (\text{ad}H)^{m-1} E(\lambda))| = 1.$$

Such vector is defined up to the transformations $\varepsilon_\varphi(\lambda) \rightarrow \pm \varepsilon_\varphi(\lambda) + \mu e(\lambda)$.

Further, denote by $\Pi: \Sigma \mapsto \mathcal{R}$ the canonical projection. Let ε_1 be a vector field on Σ such that

$$(4.3) \quad \forall u = (\lambda, \varphi) \in \Sigma \quad \Pi_* \varepsilon_1(u) = \pm \varepsilon_\varphi(\lambda) \text{ mod } \{\mathbb{R}e(\lambda)\}.$$

Such fields ε_1 are defined modulo span $\{g_0, g_1, g_2\}$ and the sign. How to choose among them the canonical field, up to the sign? Fix some vector field ε_1 , satisfying (4.3). Denote by

$$\varepsilon_i = (\text{ad } h)^{i-1} \varepsilon_1, \quad 2 \leq i \leq 2m, \quad \eta = [\varepsilon_1, \varepsilon_{2m}].$$

First the tuple $(h, \{g_i\}_{i=0}^2, \{\varepsilon_i\}_{i=1}^{2m}, \eta)$ is a frame on Σ_D . Let

$$\mathcal{L}_j = \text{span} \{h, \{g_i\}_{i=0}^2, \{\varepsilon_i\}_{i=1}^j\}, \quad 0 \leq j \leq 2m.$$

Then one can show that

$$[\varepsilon_1, \varepsilon_2] = \kappa_1 \varepsilon_2 \text{ mod } \mathcal{L}_1$$

and, in the case $n > 5$,

$$[\varepsilon_1, \varepsilon_4] = \kappa_2 \varepsilon_3 + \kappa_3 \varepsilon_4 \text{ mod } \mathcal{L}_2.$$

It turns out that among all fields ε_1 , satisfying (4.3), there exists the unique, up to the sign, field $\tilde{\varepsilon}_1$ such that the functions κ_i , $1 \leq i \leq 3$, are identically zero. Then two frames $(h, \{g_i\}_{i=0}^2, \{\tilde{\varepsilon}_i\}_{i=1}^{2m}, \eta)$ and $(h, \{g_i\}_{i=0}^2, \{-\tilde{\varepsilon}_i\}_{i=1}^{2m}, \eta)$ are canonically defined. This immediately implies that the groups of symmetries is at most $(2n - 1)$ -dimensional.

If a $(2, n)$ -distribution of maximal class has a $(2n - 1)$ -dimensional group of symmetries, then all structural functions of its canonical frames have to be constant. It can be shown that the only nonzero commutative relations of each of these frames in addition to the mentioned above are

$$(4.4) \quad \begin{aligned} [\tilde{\varepsilon}_i, \tilde{\varepsilon}_{2m-i+1}] &= (-1)^{i+1} \eta, \quad [g_1, \tilde{\varepsilon}_i] = (2m - 2i + 1) \tilde{\varepsilon}_i, \quad [g_2, \tilde{\varepsilon}_i] = (i - 1)(2m + 1 - i) \tilde{\varepsilon}_{i-1}, \\ [g_0, \tilde{\varepsilon}_i] &= -\tilde{\varepsilon}_i, \quad [g_1, \eta] = 2m\eta, \quad [g_0, \eta] = -2\eta, \end{aligned}$$

which implies the uniqueness of such distribution, up to the equivalence. Besides, from these relations it follows that the algebra of infinitesimal symmetries of such distribution is isomorphic to the semi-direct sum of $\mathfrak{gl}(2, \mathbb{R})$ ($\sim \text{span}_{\mathbb{R}} \{g_0, g_1, g_2, h\}$) and the Heisenberg group \mathfrak{n}_{2m+1} (\sim

$\text{span}_{\mathbb{R}}\{\tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_{2m}, \eta\}$). Finally, it is easy to show that for $(2, n)$ -distribution, associated with $z'(x) = y^{(n-3)}(x)$, the canonical frames satisfy the previous commutative relations. \square

5. DISCUSSION

5.1. Distributions of non-maximal rank. From [9, Remark 3.4] it follows that a rank 2 distribution D has the smallest possible class 1 at a point q iff $\dim D^3(q) = 4$. Suppose that D satisfies $\dim D^3(q) = 4$ on some open set M^o . It is easy to show that the distribution D^2 has a one-dimensional characteristic distribution C . Then (locally) we can consider the quotient M' of the manifold M^o by the corresponding one-dimensional foliation together with a new rank 2 distribution D' obtained by the factorization of D^2 .

In fact, D can be uniquely reconstructed from D' . Let $P(D')$ be a submanifold in $P(TM')$ consisting of all lines lying in D' . Similarly to the canonical contact system on $P(TM')$, we can define lifts of integral curves of D' to $P(D')$ and a canonical rank 2 distribution on $P(D')$ generated by tangent vectors to these lifts. It can be proved that this contact system on $P(D')$ is locally equivalent to D .

Iterating this procedure, we end up either at a non-holonomic rank 2 distribution on a three-dimensional manifold or at a distribution \tilde{D} , satisfying $\dim \tilde{D}^3 = 5$. In the former case the original distribution D is locally equivalent to the Goursat distribution and has an infinite-dimensional symmetry algebra. In other words, the case of non-Goursat distributions of constant class 1 can be reduced to the case of distributions of class greater than 1.

This leaves the following question open: *Do there exist completely nonholonomic rank 2 distributions of constant class $2 \leq m \leq n - 4$?* We know only that the answer is negative for $m = 2$ ($n > 5$), which means that any such example, if it exists, should live on at least 7-dimensional manifold.

5.2. Connection with Tanaka theory. After the symplectification procedure described above, the results of this paper can be interpreted in terms of Tanaka–Morimoto theory of structures on filtered manifolds [5, 6]. The original distribution D (even of maximal class) has, in general, a non-constant symbol, which makes this theory very difficult to apply to the filtered manifold defined by the distribution D itself. However, given rank 2 distribution D of maximal class there is a natural rank 2 distribution on the manifold $P(\mathcal{R}_D)$ obtained from \mathcal{R}_D via the factorization by the trajectories of the Euler vector field (or, in other words, by the projectivization of the fibers of \mathcal{R}_D). It is generated by the projection of the sum $V_{n-4} \oplus C$ w.r.t. this factorization. It is possible to show that this distribution has already a fixed symbol isomorphic to the Lie algebra generated by the vector fields $\{h, \tilde{\varepsilon}_1, \dots, \tilde{\varepsilon}_{2m}, \eta\}$ from the proof of the main theorem (see equation (4.4)).

Moreover, there is a natural decomposition of this distribution into the sum of two line distributions equal to the projections of V_{n-4} and C . This decomposition can be interpreted as a G -structure on a filtered manifold in terms of Tanaka theory and is called a *pseudo-product* structure [7]. The prolongation of this structure (in terms of filtered manifolds) is of finite type and is isomorphic to the maximal symmetry algebra from the main theorem.

We shall dwell into the details of this approach in the forthcoming paper.

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