Γ-CONVERGENCE AND HOMOGENISATION FOR FREE DISCONTINUITY FUNCTIONALS WITH LINEAR GROWTH IN THE SPACE OF FUNCTIONS WITH BOUNDED DEFORMATION

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Abstract. We study the Γ-convergence of sequences of free discontinuity functionals with linear growth defined in the space BD of functions with bounded deformation. We prove a compactness result with respect to Γ-convergence and outline the main properties of the Γ-limits, which lead to an integral representation result. The corresponding integrands are obtained by taking limits of suitable minimisation problems on small cubes. These results are then used to study the deterministic and stochastic homogenisation problem for a large class of free discontinuity functionals defined in BD.

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1. Introduction

Given a bounded open set $U \subset \mathbb{R}^d$, the space $\mathrm{BD}(U)$ of *functions of bounded deformation* is defined as the space of vector fields $u \in L^1(U;\mathbb{R}^d)$ such that the symmetric part $Eu := \frac{1}{2}(Du + Du^T)$ of the distributional gradient Du is a bounded Radon measure with values in the space $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ of $(d \times d)$ symmetric matrices. It was introduced in [46, 48, 49] to provide a variational framework for the study of elasto-plasticity in the small strain regime and has since been object of a large number of contributions, where the authors investigated trace and fine properties [2,5,6,23,32,43,44], approximation by means of more regular functions [7,18,19,21,24], and rigidity estimates [20, 22, 35, 40, 41].

In addition to its applications to elasto-plasticity, this space is particularly useful in the mathematical modelling of fracture mechanics in the variational formulation of Francfort and Marigo introduced in [39]. For this application, it is convenient to consider the decomposition of the measure Eu introduced in [2] and given by

$$Eu = E^a u + E^c u + E^j u, \tag{1.1}$$

where $E^a u$ is the absolutely continuous part of Eu with respect to the Lebesgue measure $\mathcal{L}^d$, whose density is denoted by $\mathcal{E}u$, $E^j u$ is the jump part of Eu, defined as the restriction of Eu to the jump set J_u of u, and the Cantor part $E^c u$ is the restriction of the singular part of the measure Eu to the complement of J_u in U. It is known that $E^j u := ([u] \odot \nu_u)\mathcal{H}^{d-1} \llcorner J_u$, where $[u] := u^+ - u^-$ is the difference of the unilateral traces u^+ and u^- of u on J_u, ν_u is the unit normal to J_u, $\odot$ denotes the symmetrised tensor product, $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, and $\llcorner$ denotes the restriction of a Borel measure to a Borel set. Moreover, it is known that $E^c u$ is singular with respect to the Lebesgue measure and vanishes on all Borel sets with finite $\mathcal{H}^{d-1}$-measure.

In some *cohesive* fracture models, the approach of [39] naturally leads to the minimisation of energies of the form (see, for instance, [10, Sections 4.2 and 5.2])

$$F(u,U) := \int_U f(x,\mathcal{E}u)\,\mathrm{d}x + \int_U f^\infty\Big(x, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u\cap U} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{1.2}$$

where the *bulk* energy density $f\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ is a Borel function with

$$c_1|A| - c_2 \le f(x,A) \le c_3|A| + c_4 \quad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \tag{1.3}$$

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for some constants $0 < c_1 \leq c_3$, $c_2 \geq 0$, and $c_4 \geq 0$, $f^\infty\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ denotes the *recession function* of f with respect to A, defined by

$$f^\infty(x,A) := \limsup_{t\to+\infty} \frac{f(x,tA)}{t} \quad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym},$$

and the *surface* energy density $g\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ is a Borel function satisfying

$$c_1|\zeta \odot \nu| \leq g(x,\zeta,\nu) \leq c_3|\zeta \odot \nu| \quad \text{for every } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d, \text{ and } \nu \in \mathbb{S}^{d-1}, \tag{1.4}$$

where $\mathbb{S}^{d-1} := \{\nu \in \mathbb{R}^d : |\nu| = 1\}$.

In this paper we study the Γ-limits of sequences of the form (1.2) with respect to the topology induced by $L^1_{\rm loc}$. The first result in this direction is Theorem 4.1, which states that for every sequence $\{F_n\}_{n\in\mathbb{N}}$, with $c_1,\dots,c_4$ independent of n, there exists a subsequence, not relabelled, such that for every bounded open set $U \subset \mathbb{R}^d$ the sequence $\{F_n(\cdot,U)\}_{n\in\mathbb{N}}$ Γ-converges to a functional $F(\cdot,U)$ defined on $\mathrm{BD}(U)$.

This limit functional F satisfies suitable regularity properties (see Definition 3.4), the most important being that for every bounded open set $U \subset \mathbb{R}^d$ and $u \in \mathrm{BD}(U)$ the set function defined for every Borel set $B \subset U$ by

$$B \mapsto F(u,B) := \inf\{F(u,V) : V \text{ open, } B \subset V \subset U\}$$

is a bounded Radon measure. Let $F^a(u,\cdot)$ and $F^s(u,\cdot)$ be the absolutely continuous and singular parts of the measure $F(u,\cdot)$ with respect to $\mathcal{L}^d$, respectively. In analogy to (1.1), we decompose the limit functional F as

$$F(u,B) = F^a(u,B) + F^c(u,B) + F^j(u,B),$$

where $F^j(u,B) := F^s(u,B\cap J_u)$ and $F^c(u,B) := F^s(u,B\setminus J_u)$. It is possible to check (see Remark 5.2) that $F^c(u,\cdot)$ is the absolutely continuous part of $F(u,\cdot)$ with respect to E^cu and that $F^j(u,\cdot)$ is the absolutely continuous part with respect to E^ju.

By recent results of Caroccia, Focardi, and Van Goethem [17] (see also [37]) it is possible to represent F^a and F^j by means of integral functionals in the form

$$F^a(u,B) = \int_B f(x,\mathcal{E}u)\,\mathrm{d}x, \tag{1.5}$$

$$F^j(u,B) = \int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{1.6}$$

where f and g are Borel functions satisfying (1.3) and (1.4). As customary in integral representation results for free discontinuity functionals (see, for instance, [9, 12]), the functions f and g are defined by means of limits of auxiliary minimisation problems on small cubes. More precisely, for every bounded open set $W \subset \mathbb{R}^d$ with Lipschitz boundary, and $w \in \mathrm{BD}(W)$ we set

$$\mathfrak{m}^F(w,W) := \inf\{F(u,W) : u \in \mathrm{BD}(W) \text{ and } u = w \text{ on } \partial W\}.$$

The integrands f and g that appear in (1.5) and (1.6) are given by

$$f(x,A) := \limsup_{\rho\to0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d}, \tag{1.7}$$

$$g(x,\zeta,\nu) := \limsup_{\rho\to0^+} \frac{\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}},$$

where ℓ_A is the linear function defined by $\ell_A(y) = Ay$, $Q(x,\rho)$ is the cube with centre x, side-length $\rho > 0$, and sides parallel to the axes, $u_{x,\zeta,\nu}$ is the pure jump function equal to ζ on $\{y \in \mathbb{R}^d : (y-x)\cdot\nu \geq 0\}$ and to 0 on $\{y \in \mathbb{R}^d : (y-x)\cdot\nu < 0\}$, while $Q_\nu(x,\rho)$ is a cube with centre x, side-length $\rho > 0$, and two faces orthogonal to ν.

Under suitable additional hypotheses (see Definition 6.4) on the sequence $\{F_n\}_{n\in\mathbb{N}}$, and assuming that (1.7) holds with f independent of x and lim sup replaced by lim, we show in Theorem

7.1 that the complete limit functional F, including its Cantor part F^c, can be represented in the form

$$F(u,B)=\int_B f(\mathcal{E}u)\,\mathrm{d}x+\int_B f^\infty\Big(\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\Big)\,\mathrm{d}|E^cu|+\int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{1.8}$$

for every $u\in \mathrm{BD}(U)$, with $U\subset\mathbb{R}^d$ bounded open set, and every Borel set $B\subset U$. Thanks to (1.5) and (1.6) the crucial point is to show that

$$F^c(u,B)=\int_B f^\infty\Big(\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\Big)\,\mathrm{d}|E^cu|.$$

In Caroccia, Focardi, and Van Goethem [17], this is obtained by exploiting the uniform continuity of F with respect to horizontal translations, which is one of their key assumptions. However, this hypothesis is not natural for our approach to stochastic homogenisation, so we are forced to use a different technique. First of all, we use the characterisation of

$$\frac{\mathrm{d}F^c(u,\cdot)}{\mathrm{d}|E^cu|}(x)$$

obtained in [17, Lemma 5.3], which holds even if $\{F_n\}_{n\in\mathbb{N}}$ is not uniformly continuous, and then, using the property introduced in Definition 6.4, we adapt to the BD setting some arguments used in [26, 30] for the BV case.

Combining the compactness result and the integral representation (1.8) we are able to obtain in Theorem 8.3 a characterisation of the integrands of the Γ-limits by means of limits of minimum values on small cubes. More precisely, assuming that F_n satisfy the property described in Definition 6.4 uniformly with respect to n and that for some functions $\hat f$ and $\hat g$ we have

$$\hat f(A)=\lim_{\rho\to0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A,Q(x,\rho))}{\rho^d}=\lim_{\rho\to0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A,Q(x,\rho))}{\rho^d}, \tag{1.9}$$

$$\hat g(x,\zeta,\nu)=\limsup_{\rho\to0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}}=\limsup_{\rho\to0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu},Q_\nu(x,\rho))}{\rho^{d-1}} \tag{1.10}$$

for every $x\in\mathbb{R}^d$, $A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\zeta\in\mathbb{R}^d$, and $\nu\in\mathbb{S}^{d-1}$, then for every bounded open set $U\subset\mathbb{R}^d$ we prove that the sequence $\{F_n(\cdot,U)\}_{n\in\mathbb{N}}$ Γ-converges to the functional $F(\cdot,U)$ in (1.8) with $f=\hat f$ and $g=\hat g$.

We conclude the paper by applying these results to a general class of integral functionals to deduce homogenisation results. More precisely, in Theorems 9.7 and 10.6 we will consider functionals F_n given by

$$F_n(u,U):=\int_U f\Big(\frac{x}{\varepsilon_n},\mathcal{E}u\Big)\,\mathrm{d}x+\int_U f^\infty\Big(\frac{x}{\varepsilon_n},\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\Big)\,\mathrm{d}|E^cu|+\int_{J_u\cap U}\varepsilon_n g\Big(\frac{x}{\varepsilon_n},\frac{1}{\varepsilon_n}[u],\nu_u\Big)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{1.11}$$

where f and g satisfy (1.3), (1.4), and some conditions related to Definition 6.4 (see Definition 6.1), while $\{\varepsilon_n\}_{n\in\mathbb{N}}\subset(0,1)$ is a sequence converging to 0 as $n\to+\infty$. We remark that, in general the functionals $\{F_n\}_{n\in\mathbb{N}}$ do not coincide with those normally considered in the homogenisation of free discontinuity problems, unless one assumes that $[0,+\infty)\ni t\mapsto g(x,t\zeta,\nu)$ is positively homogeneous of degree one, in which case $F_n(u,U)$ reads as

$$\int_U f\Big(\frac{x}{\varepsilon_n},\mathcal{E}u\Big)\,\mathrm{d}x+\int_U f^\infty\Big(\frac{x}{\varepsilon_n},\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\Big)\,\mathrm{d}|E^cu|+\int_{J_u\cap U} g\Big(\frac{x}{\varepsilon_n},[u],\nu_u\Big)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{1.12}$$

which is the standard functional considered in the homogenisation problem for free discontinuity functionals (see [13–16]).

So far, in the BD setting our technique allows us to prove the Γ-convergence of (1.12) only when g is positively 1-homogeneous in the variable ζ, while in the BV setting (see [16, 26, 30]) the Γ-convergence of the functionals corresponding to (1.12) has been proved assuming only that the function g is sufficiently close to a 1-homogeneous function for $|\zeta|$ small enough. Unfortunately, this approach heavily relies on sophisticated truncation arguments that we are not able to extend to BD.

The particular choice of the scaling for the surface term in (1.11) will allow us to circumvent the use of vertical truncations and will allow us to obtain suitable change of variable formulas used in the proof, which avoids any truncation argument.

A similar problem for bulk energies of the form

$$\Phi_n(u,U) := \begin{cases} \displaystyle\int_U f\Big(\frac{x}{\varepsilon_n}, \mathcal{E}u\Big)\,\mathrm{d}x & \text{if } Eu << \mathcal{L}^d, \\ +\infty & \text{otherwise,} \end{cases}$$

was studied in [17], when f is 1-periodic with respect to x and satisfies (1.3), and in [4] in the stochastically periodic case.

To obtain the Γ-limit of $\{F_n\}_{n\in\mathbb{N}}$ using (1.9) and (1.10) is convenient to rewrite these formulas using the change of variables $y = \frac{x}{\varepsilon_n}$. Setting $v(y) := \frac{1}{\varepsilon_n}u(\varepsilon_n y)$, one then checks that

$$\begin{aligned} F_n(u, Q(x,\rho)) = \varepsilon_n^d \Big(\int_{Q(\frac{x}{\varepsilon_n}, \frac{\rho}{\varepsilon_n})} f(y, \mathcal{E}v)\,\mathrm{d}y + \int_{Q(\frac{x}{\varepsilon_n}, \frac{\rho}{\varepsilon_n})} f^\infty\Big(y, \frac{\mathrm{d}E^c v}{\mathrm{d}|E^c v|}\Big)\,\mathrm{d}|E^c v| \\ + \int_{J_v} g(y, [v], \nu_u)\,\mathrm{d}\mathcal{H}^{d-1}\Big), \end{aligned}$$

where we have exploited the scaling chosen for the surface term in (1.11). Setting $r_n := \frac{\rho}{\varepsilon_n}$, we have

$$\frac{1}{\rho^d}\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho)) = \frac{1}{r_n^d}\mathfrak{m}^F(\ell_A, Q(r_n\tfrac{x}{\rho}, r_n)),$$

where F is given by (1.2). Hence, if there exists the limit

$$f_{\mathrm{lim}}(A) := \lim_{n\to+\infty} \frac{1}{r_n^d}\mathfrak{m}^F(\ell_A, Q(x, r_n)) \tag{1.13}$$

and is independent of x, condition (1.9) is satisfied by $\hat{f}(x,A) = f_{\mathrm{lim}}(A)$.

To deal with (1.10) we consider F^∞ the functional obtained by replacing f and g by f^∞ and g^∞ in (1.2), where

$$g^\infty(x,\zeta,\nu) := \limsup_{t\to+\infty} \frac{g(x, t\zeta, \nu)}{t}$$

(see Definition 6.1). Performing a change of variables similar to the one that led to (1.13), we obtain that if the limit

$$g_{\mathrm{lim}}(\zeta,\nu) := \lim_{n\to+\infty} \frac{1}{r_n^{d-1}}\mathfrak{m}^{F^\infty}(u_{r_n x,\zeta,\nu}, Q(x, r_n)) \tag{1.14}$$

exists for every $\zeta \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$ and is independent of x, then condition (1.10) holds with $\hat{g}(x,\zeta,\nu) = g_{\mathrm{lim}}(\zeta,\nu)$.

As (1.9) and (1.10) are sufficient conditions for the Γ-convergence, we obtain the following result (see Theorem 9.7): suppose that the limits (1.13) and (1.14) exist for every $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and are independent of x, then for every bounded open set $U \subset \mathbb{R}^d$ the sequence $\{F_n(\cdot, U)\}_{n\in\mathbb{N}}$ Γ-converges with respect to the L^1_{loc}-convergence to the functional $F_{\mathrm{lim}}(\cdot, U)$ defined by

$$F_{\mathrm{lim}}(u,U) := \int_U f_{\mathrm{lim}}(\mathcal{E}u)\,\mathrm{d}x + \int_U f^\infty_{\mathrm{lim}}\Big(\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u} g_{\mathrm{lim}}([u], \nu_u)\,\mathrm{d}\mathcal{H}^{d-1}.$$

We also prove a related result in the probabilistic setting. Under the standard assumptions of stochastic homogenisation (see Section 10), we will assume that the integrands f and g appearing in (1.11) are random functions, and we will prove that the limit (1.13) exists almost surely thanks to the Subadditive Ergodic Theorem of Akcoglu and Krengel [1], and show that the almost sure existence of the limit in (1.14) can be obtained by using the same theorem in dimension $d-1$, arguing as in [15, 16]. This will allow us to prove in Theorem 10.6 the almost sure Γ-convergence of $\{F_n\}_{n\in\mathbb{N}}$ to F_{lim}. Of course, the stochastic result immediately implies a related result in the deterministic periodic case (see Corollary 10.7).

2. Notation

In this section we present the notation used throughout the paper.

(a) The space $\mathbb{R}^d$ is endowed with the usual scalar product, denoted by $\cdot$, while the Euclidean norm is denoted by $|\ |$. The unit sphere of $\mathbb{R}^d$ is denoted by $\mathbb{S}^{d-1} := \{\nu \in \mathbb{R}^d : |\nu| = 1\}$. We also set $\mathbb{S}^{d-1}_{\pm} := \{\nu \in \mathbb{S}^{d-1} : \pm\nu_{i(\nu)} > 0\}$, where $i(\nu) \in \{1, ..., d\}$ is the largest index such that $\nu_{i(\nu)} \neq 0$.

(b) We identify the vector space $\mathbb{R}^{d\times d}$ with the space of $d \times d$ matrices. The subspace of $\mathbb{R}^{d\times d}$ of $d\times d$ symmetric matrices (resp. antisymmetric) is denoted by $\mathbb{R}^{d\times d}_{\rm sym}$ (resp. $\mathbb{R}^{d\times d}_{\rm skew}$). If $A \in \mathbb{R}^{d\times d}$ its symmetric part is denoted by $A^{\rm sym} := \frac{1}{2}\big(A + A^T\big)$. For $A \in \mathbb{R}^{d\times d}$ and $x \in \mathbb{R}^d$, the vector $Ax \in \mathbb{R}^d$ is given by the usual matrix by vector multiplication. Given a matrix $A = (A_{ij}) \in \mathbb{R}^{d\times d}$, its Frobenius norm is given by

$$|A| := \Big(\sum_{i,j=1}^{d} |A_{ij}|^2\Big)^{1/2}.$$

The identity matrix is denoted by I.

(c) The i-th vector of the canonical basis of $\mathbb{R}^d$ is denoted by e_i. Given $x \in \mathbb{R}^d$ and $\rho > 0$, we consider the cube $Q(x, \rho) := \big\{y \in \mathbb{R}^d : |(y - x) \cdot e_i| < \rho/2 \text{ for every } i \in \{1, ..., d\}\big\}$.

(d) $SO(d)$ is the space of $d \times d$ orthonormal matrices R with $\det(R) = 1$. We fix once and for all a map $\mathbb{S}^{d-1} \ni \nu \mapsto R_\nu \in SO(d)$ satisfying the following properties: $R_\nu e_d = \nu$ and $R_\nu(Q(0, \rho)) = R_{-\nu}(Q(0, \rho))$ for every $\nu \in \mathbb{S}^{d-1}$, $R_{e_d} = I$, and the restrictions of $\nu \mapsto R_\nu$ to $\mathbb{S}^{d-1}_{\pm}$ are continuous (for an example of such map see, for instance, [15, Example A.1]).

(e) Given $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $\rho > 0$, we consider the open cube defined by

$$Q_\nu(x, \rho) := x + R_\nu\big(Q(0, \rho)\big).$$

(f) For every $A \in \mathbb{R}^{d\times d}$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ let $\ell_A \colon \mathbb{R}^d \to \mathbb{R}^d$ and $u_{x,\zeta,\nu} \colon \mathbb{R}^d \to \mathbb{R}^d$ be the functions defined for every $y \in \mathbb{R}^d$ by

$$\ell_A(y) := Ay \quad \text{and} \quad u_{x,\zeta,\nu}(y) := \begin{cases} \zeta & \text{if } (y - x) \cdot \nu > 0, \\ 0 & \text{if } (y - x) \cdot \nu \leq 0. \end{cases}$$

(g) Given an open set $U \subset \mathbb{R}^d$, the collection of all open subsets (resp. Borel) of U is denoted by $\mathcal{U}(U)$ (resp. $\mathcal{B}(U)$). If $V \in \mathcal{U}(\mathbb{R}^d)$, we write $V \subset\subset U$ if the closure of V is contained in U. The collection of all these open sets is denoted by $\mathcal{U}_c(U)$.

(h) Given $U \in \mathcal{U}(\mathbb{R}^d)$ and a finite dimensional real normed vector space X, the space of all bounded Radon measures with values in X is denoted by $\mathcal{M}_b(U; X)$. The indication of X is omitted if $X = \mathbb{R}$. Given a positive measure $\lambda \in \mathcal{M}_b(U)$, and a measure $\mu \in \mathcal{M}_b(U; X)$ with $\mu << \lambda$, the function $\mathrm{d}\mu/\mathrm{d}\lambda$ denotes the Radon-Nikodým derivative of μ with respect to λ, defined as the density of μ with respect to λ. The total variation measure of $\mu \in \mathcal{M}_b(U; X)$ with respect to the norm $|\ |$ on X, is denoted by $|\mu|$. Given a positive measure $\lambda \in \mathcal{M}_b(U)$ and an integrable function $f \colon U \to X$, we denote by $f\lambda$ the X-valued measure defined for every $B \in \mathcal{B}(U)$ by

$$f\lambda(B) := \int_B f \,\mathrm{d}\lambda.$$

Given a Borel measure μ on U and $E \in \mathcal{B}(U)$, the Borel measure $\mu \llcorner E$ is defined by $\mu \llcorner E(B) := \mu(B \cap E)$ for every $B \in \mathcal{B}(U)$.

(i) Given a function $u \in L^1_{\rm loc}(\mathbb{R}^d; \mathbb{R}^d)$, the *jump set* J_u of u is the set of all points $x \in U$ with the following property: there exists a triple $(u^+(x), u^-(x), \nu_u(x)) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$, with $u^+(x) \neq u^-(x)$, such that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^d} \int_{B_\rho(x)\cap H^\pm(x)} |u(y) - u^\pm(x)| \,\mathrm{d}y = 0,$$

where $H^{\pm}(x) := \{y \in \mathbb{R}^d : \pm(y-x)\cdot\nu_u(x) > 0\}$. The triple $(u^+(x), u^-(x), \nu_u(x))$ is uniquely determined, up to changing the sign of $\nu(x)$ and swapping $u^+(x)$ and $u^-(x)$. Having fixed ν_u, we set $[u] := u^+ - u^-$ on J_u. It can be shown that J_u is a Borel set and that there is a choice of $u^{\pm}$ and ν_u that makes them Borel measurable on J_u (see [3, Proposition 3.69]). Moreover, Del Nin has recently showed (see [34]) that J_u is always countably $(\mathcal{H}^{d-1}, d-1)$-rectifiable in the sense of [38, 3.2.14].

(j) Given $U \in \mathcal{U}(\mathbb{R}^d)$, a function $u\colon U \to \mathbb{R}^d$ is said to be of *bounded deformation* if $u \in L^1(U;\mathbb{R}^d)$ and its *distributional symmetric gradient* $Eu := \frac{1}{2}(Du + Du^T)$ belongs to $\mathcal{M}_b(U;\mathbb{R}^{d\times d}_{\mathrm{sym}})$. The collection of all functions of bounded deformation on U is denoted by $\mathrm{BD}(U)$, while $\mathrm{BD}_{\mathrm{loc}}(U)$ is the collection of all $\mathcal{L}^d$-measurable functions $u\colon U \to \mathbb{R}^d$ such that $u|_V \in \mathrm{BD}(V)$ for every $V \in \mathcal{U}_c(U)$. We refer the reader to [49] for an introduction to this space and to [2] for the fine properties of its functions.

(k) Given $U \in \mathcal{U}(\mathbb{R}^d)$ and $u \in \mathrm{BD}(U)$, the measure Eu can be decomposed as

$$Eu = E^a u + E^c u + E^j u = \mathcal{E}u\mathcal{L}^d + E^c u + [u]\odot\nu_u\mathcal{H}^{d-1} \llcorner J_u, \tag{2.1}$$

where:

(i) the *absolutely continuous part* $E^a u$ is the absolutely continuous part of Eu with respect to $\mathcal{L}^d$, whose density is denoted by $\mathcal{E}u \in L^1(U;\mathbb{R}^{d\times d}_{\mathrm{sym}})$,

(ii) the *Cantor part* $E^c u$ is defined as $E^c u := E^s u \llcorner (\Omega\setminus J_u)$, where $E^s u$ is the singular part of Eu with respect to $\mathcal{L}^d$; it is known that $E^c u$ vanishes on all $B \in \mathcal{B}(U)$ which are σ-finite with respect to $\mathcal{H}^{d-1}$,

(iii) the *jump part* $E^j u$ is defined by $E^j u := Eu \llcorner J_u = E^s u \llcorner J_u$; it is known that $E^j u = [u]\odot\nu_u\mathcal{H}^{d-1}\llcorner J_u$, where $\odot$ denotes the symmetric tensor product defined by $(a\odot b)_{ij} = \frac{1}{2}(a_i b_j + a_j b_i)$ for every $a, b \in \mathbb{R}^d$.

(l) A function $u\colon \mathbb{R}^d \to \mathbb{R}^d$ is said to be a *rigid motion* if there exist $A \in \mathbb{R}^{d\times d}_{\mathrm{skew}}$ and $b \in \mathbb{R}^d$ such that $u(x) = Ax + b$ for every $x \in \mathbb{R}^d$. The collection of all rigid motions is denoted by $\mathcal{R}$.

3. THE INTEGRANDS AND THE FUNCTIONALS

In this section we introduce the collections of integrands and of functionals that are going to be our main objects of study.

We fix non-negative constants $c_1, ..., c_5 \in [0,+\infty)$, with $0 < c_1 \le c_3$, and a constant $\alpha \in (0,1)$. We also fix a continuous function $\sigma\colon [0,+\infty) \to [0,+\infty)$ with the property that $\sigma(0) = 0$ and

$$\sigma^\infty := \limsup_{t\to+\infty}\frac{\sigma(t)}{t} < +\infty. \tag{3.1}$$

Definition 3.1. Let $\mathcal{F}$ be the collection of all integrands $f\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ satisfying the following conditions:

(f1) f is Borel measurable on $\mathbb{R}^d \times \mathbb{R}^{d\times d}_{\mathrm{sym}}$;

(f2) for every $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we have

$$c_1|A| - c_2 \le f(x,A) \le c_3|A| + c_4;$$

(f3) for every $x \in \mathbb{R}^d$ and $A_1, A_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we have

$$|f(x,A_1) - f(x,A_2)| \le c_5|A_1 - A_2|.$$

Let $\mathcal{G}$ be the collection of all integrands $g\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ satisfying the following conditions:

(g1) g is Borel measurable on $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}$;

(g2) for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$g(x,\zeta,\nu) = g(x,-\zeta,-\nu);$$

(g3) for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$c_1|\zeta \odot \nu| \le g(x,\zeta,\nu) \le c_3|\zeta \odot \nu|;$$

(g4) for every $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$|g(x,\zeta_1,\nu) - g(x,\zeta_2,\nu)| \le \sigma(|\zeta_1 - \zeta_2|).$$

To every function $f\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ we associate its *recession function* f^∞ (with respect to the variable A), defined by

$$f^\infty(x,A) := \limsup_{t\to+\infty} \frac{f(x,tA)}{t} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}. \tag{3.2}$$

Remark 3.2. Given $f\colon \mathbb{R}^d \times \mathbb{R}^{d\times d}_{\rm sym}$ satisfying (f2) and (f3), one can check that for every $x \in \mathbb{R}^d$, $A, A_1, A_2 \in \mathbb{R}^{d\times d}_{\rm sym}$ the function f^∞ satisfies

$$\begin{gathered} c_1|A| \le f^\infty(x,A) \le c_3|A|, \qquad (3.3)\\ |f^\infty(x,A_1) - f^\infty(x,A_2)| \le c_5|A_1 - A_2|. \end{gathered}$$

We now introduce a class of integral functionals associated with $\mathcal{F}$ and $\mathcal{G}$.

Definition 3.3. Given $f \in \mathcal{F}$ and $g \in \mathcal{G}$, for every open set $U \subset \mathbb{R}^d$ and $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ we set

$$F^{f,g}(u,U) := \int_U f(x,\mathcal{E}u)\,{\rm d}x + \int_U f^\infty\Big(x, \frac{{\rm d}E^cu}{{\rm d}|E^cu|}\Big)\,{\rm d}|E^cu| + \int_{J_u\cap U} g(x,[u],\nu_u)\,{\rm d}\mathcal{H}^{d-1} \tag{3.4}$$

if $u|_U \in {\rm BD}_{\rm loc}(U)$, while $F^{f,g}(u,U) = +\infty$ otherwise. The definition is then extended to every Borel set $B \in \mathcal{B}(\mathbb{R}^d)$ by setting

$$F^{f,g}(u,B) := \inf\{F^{f,g}(u,U) : U \subset \mathbb{R}^d \text{ open, with } U \supset B\}. \tag{3.5}$$

Note that, if $U \subset \mathbb{R}^d$ is a bounded open set, $B \in \mathcal{B}(U)$, $u|_U \in {\rm BD}_{\rm loc}(U)$, and $|Eu|(U) < +\infty$, then

$$F^{f,g}(u,B) = \int_B f(x,\mathcal{E}u)\,{\rm d}x + \int_B f^\infty\Big(x, \frac{{\rm d}E^cu}{{\rm d}|E^cu|}\Big)\,{\rm d}|E^cu| + \int_{J_u\cap B} g(x,[u],\nu_u)\,{\rm d}\mathcal{H}^{d-1}. \tag{3.6}$$

Indeed, in this case the upper bounds in (f2), (3.3), and (g3) imply

$$\int_U f(x,\mathcal{E}u)\,{\rm d}x + \int_U f^\infty\Big(x, \frac{{\rm d}E^cu}{{\rm d}|E^cu|}\Big)\,{\rm d}|E^cu| + \int_{J_u\cap U} g(x,[u],\nu_u)\,{\rm d}\mathcal{H}^{d-1} < +\infty,$$

so that (3.6) follows from (3.4) and (3.5).

We now introduce a collection of abstract functionals related to the integral functionals considered above.

Definition 3.4. Let $\mathfrak{F}$ be the collection of functionals $F\colon L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ with the following properties:

(a) *measure property*: for every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ the set function $F(u,\cdot)$ is a Borel measure that for every $B \in \mathcal{B}(\mathbb{R}^d)$ satisfies

$$F(u,B) := \inf\{F(u,U) : U \subset \mathbb{R}^d \text{ open, } U \supset B\}; \tag{3.7}$$

(b) *locality on open sets:* for every bounded open set $U \subset \mathbb{R}^d$ and $u, v \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ with $u = v$ $\mathcal{L}^d$-a.e on U we have $F(u,U) = F(v,U)$;

(c) *upper and lower bounds:* if $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ and $U \subset \mathbb{R}^d$ is a bounded open set, then

$$c_1|Eu|(U) - c_2\mathcal{L}^d(U) \le F(u,U) \le c_3|Eu|(U) + c_4\mathcal{L}^d(U) \quad \text{if } u|_U \in {\rm BD}(U),$$

while $F(u,U) = +\infty$ if $u|_U \notin {\rm BD}(U)$;

(d) *invariance under rigid motions:* for every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, $w \in \mathcal{R}$, and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$F(u+w,B) = F(u,B).$$

(e) *bulk continuity estimate:* for every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$F(u+\ell_A, B) \leq F(u,B) + c_5|A|\mathcal{L}^d(B);$$

(f) *surface continuity estimate:* for every $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $B \in \mathcal{B}(\mathbb{R}^d)$ we have

$$F(u+u_{x,\zeta,\nu}, B) \leq F(u,B) + \sigma(|\zeta|)\mathcal{H}^{d-1}(B\cap \Pi^\nu_x),$$

where $\Pi^\nu_x := \{y \in \mathbb{R}^d : (y-x)\cdot\nu = 0\}$ is the hyperplane orthogonal to ν containing x.

We denote by $\mathfrak{F}_{\rm sc}$ the collection of functionals $F \in \mathfrak{F}$ such that for every open $U \subset \mathbb{R}^d$ the functional $F(\cdot,U)$ is $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$-lower semicontinuous.

Remark 3.5. Combining properties (a) and (b) we obtain that the functionals in $\mathfrak{F}$ satisfy the following locality property: if $B \in \mathcal{B}(\mathbb{R}^d)$, $u, v \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, and $u = v$ $\mathcal{L}^d$-a.e. on a neighbourhood of B, then $F(u,B) = F(v,B)$. Easy examples show that, in general, this equality does not hold if we have only $u = v$ $\mathcal{L}^d$-a.e. on B.

For this reason, if $U \subset \mathbb{R}^d$ is an open set and $u \in L^1_{\rm loc}(U;\mathbb{R}^d)$, for every $B \in \mathcal{B}(U)$ we can define $F(u,B) := F(v,B)$, where $v \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ is any function such that $v = u$ $\mathcal{L}^d$-a.e. on U. The locality property described above implies that the value of $F(u,B)$ does not depend on the chosen extension v of u.

Remark 3.6. From (3.7) and (c) we deduce the following inequalities for Borel sets. If $U \subset \mathbb{R}^d$ is a bounded open set and $u \in {\rm BD}(U)$, then for every $B \in \mathcal{B}(U)$ we have

$$c_1|Eu|(B) - c_2\mathcal{L}^d(B) \leq F(u,B) \leq c_3|Eu|(B) + c_4\mathcal{L}^d(B).$$

Remark 3.7. Thanks to property (a) a functional $F \in \mathfrak{F}$ belongs to $\mathfrak{F}_{\rm sc}$ if and only if for every open bounded set $U \subset \mathbb{R}^d$ the functional $F(\cdot,U)$ is $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$-lower semicontinuous.

We now show that the functional introduced in Definition 3.3 belongs to $\mathfrak{F}$.

Proposition 3.8. *Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Then the functional $F^{f,g}$ introduced in Definition 3.3 belongs to $\mathfrak{F}$.*

Proof. By (3.5) $F^{f,g}$ satisfies (3.7), while the rest of property (a) is obvious when $u \in {\rm BD}(\mathbb{R}^d)$. The general case $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ can be obtained by a straightforward argument; it can also be treated using the De Giorgi-Letta criterion for measures (see Lemma 4.4 below). Property (b) is obvious. Property (c) follows from (f2), (3.3), and (g3). To see that property (d) is satisfied by $F^{f,g}$ is enough to observe that for every $w \in \mathcal{R}$ we have $Ew = 0$.

To see that $F^{f,g}$ satisfies property (e) we observe that by (f3) for every $B \in \mathcal{B}(\mathbb{R}^d)$ and $A \in \mathbb{R}^{d\times d}_{\rm sym}$ we have

$$|F^{f,g}(u+\ell_A, B) - F^{f,g}(u,B)| \leq c_5|A|\mathcal{L}^d(B).$$

A similar argument shows that $F^{f,g}$ satisfies property (f). □

4. A COMPACTNESS RESULT FOR $\mathfrak{F}$

In this section we state and prove a compactness result for sequences of functionals in $\mathfrak{F}$. The proof is based on the classical localisation method for Γ-convergence (see [11, Section 3.3] or [25, Chapter 18]).

Theorem 4.1. *Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathfrak{F}$. Then there exist a functional $F \in \mathfrak{F}_{\rm sc}$ and a subsequence, not relabelled, such that for every bounded open set $U \subset \mathbb{R}^d$ the sequence $\{F_n(\cdot,U)\}_{n\in\mathbb{N}}$ Γ-converges to $F(\cdot,U)$ in the topology of $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$.*

Proof. For every open set $U \subset \mathbb{R}^d$ we consider the functionals defined on $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ by

$$F'(\cdot,U) = \Gamma\text{-}\liminf_{n\to+\infty} F_n(\cdot,U) \quad \text{and} \quad F''(\cdot,U) = \Gamma\text{-}\limsup_{n\to+\infty} F_n(\cdot,U), \tag{4.1}$$

$$F'_-(\cdot,U) = \sup_{U'\in\mathcal{U}_c(U)} F'(\cdot,U') \quad \text{and} \quad F''_-(\cdot,U) = \sup_{U'\in\mathcal{U}_c(U)} F''(\cdot,U'), \tag{4.2}$$

where the Γ-liminf and Γ-limsup are computed with respect to the $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$-convergence and $\mathcal{U}_c(U)$ is the collection of open sets defined in (g) of Section 2.

It is immediate to check that for every $u\in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ the set functions $F_n(u,\cdot)$, $F'(u,\cdot)$, $F''(u,\cdot)$, $F'_-(u,\cdot)$, and $F''_-(u,\cdot)$ are increasing. Thus, we can apply the compactness theorem for sequences of increasing set functionals [25, Theorem 16.9] to obtain that there exist a subsequence of $\{F_n\}_{n\in\mathbb{N}}$, not relabelled, and a functional $F\colon L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)\times\mathcal{U}(\mathbb{R}^d)$ such that

$$F(u,U)=F'_-(u,U)=F''_-(u,U)\quad\text{for every } u\in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)\text{ and } U\in\mathcal{U}(\mathbb{R}^d). \tag{4.3}$$

For every $u\in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ and $B\in\mathcal{B}(\mathbb{R}^d)$ we set

$$F(u,B):=\inf\{F(u,U):U\in\mathcal{U}(\mathbb{R}^d)\text{ with } U\supset B\}. \tag{4.4}$$

We want to prove that $F\in\mathfrak{F}$. The approximation property (3.7) is obvious, while the measure property (a) will be proved in Lemma 4.4 below. The locality property (b) and the lower semicontinuity of F follow from general results about Γ-limits of local functionals (see [25, Proposition 16.15 and Remark 16.3]).

We note that it is enough to prove properties (d)-(f) for $U\in\mathcal{U}_c(\mathbb{R}^d)$. Indeed, by the measure property (a) they can be extended to $U\in\mathcal{U}(\mathbb{R}^d)$ using the inner regularity of Borel measures. The further extension to Borel sets follows immediately from (4.4).

Let us fix $u\in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ and $U\in\mathcal{U}_c(\mathbb{R}^d)$. We begin by proving the upper and lower bounds in (c). We consider first the case $u\in{\rm BD}(U)$. Under this hypothesis, the inequality

$$F''(u,U)\le c_3|Eu|(U)+c_4\mathcal{L}^d(U) \tag{4.5}$$

is a consequence of the corresponding inequality for F_n and of the fact that the Γ-limsup is less than or equal to the pointwise limsup. The upper bound in (c) for F follows immediately from (4.2), (4.3), and (4.5). To prove the lower bound for F'', we observe that our assumption $u\in{\rm BD}(U)$ implies that $F''(u,U)<+\infty$. Given $U'\in\mathcal{U}_c(U)$, we fix a sequence $\{u_n\}_{n\in\mathbb{N}}\subset L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ converging to u in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ such that $\limsup_n F_n(u_n,U')=F''(u,U')<+\infty$. Since for every $n\in\mathbb{N}$ the functional F_n satisfies (c) we have $u_n\in{\rm BD}(U')$ for $n\in\mathbb{N}$ large enough and

$$c_1|Eu_n|(U')-c_2\mathcal{L}^d(U')\le F_n(u_n,U').$$

From the lower semicontinuity of $v\mapsto|Ev|(U')$ with respect to the L^1-convergence and from the previous inequality we obtain

$$c_1|Eu|(U')-c_2\mathcal{L}^d(U')\le F''(u,U').$$

Using again (4.2) and (4.3), we deduce the corresponding lower bound for $F(u,U)$.

Assume now that $u\notin{\rm BD}(U)$. We want to prove that $F(u,U)=+\infty$. Assume by contradiction that $F(u,U)<+\infty$. Given $U'\in\mathcal{U}_c(U)$, we consider a sequence $\{u_n\}_{n\in\mathbb{N}}\subset L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ converging to u in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ such that $\limsup_n F_n(u_n,U')=F''(u,U')\le F(u,U)<+\infty$. Arguing as before, we obtain that $u_n\in{\rm BD}(U')$ for $n\in\mathbb{N}$ large enough and

$$\limsup_{n\to+\infty}|Eu_n|(U')\le\frac{1}{c_1}\limsup_{n\to+\infty}F_n(u_n,U')+\frac{c_2}{c_1}\mathcal{L}^d(U')\le\frac{F(u,U)}{c_1}+\frac{c_2}{c_1}\mathcal{L}^d(U),$$

which implies $u\in{\rm BD}(U')$ and $|Eu|(U')\le\frac{1}{c_1}F(u,U)+\frac{c_2}{c_1}\mathcal{L}^d(U)$. As this holds for every $U'\in\mathcal{U}_c(U)$, we obtain $u\in{\rm BD}(U)$, contradicting our hypothesis. This concludes the proof of the implication $u\notin{\rm BD}(U)\implies F(u,U)=+\infty$.

Properties (d)-(f) for F follow immediately from the same properties for F_n.

In Lemma 4.5 we will prove that for every $u\in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ and $U\in\mathcal{U}_c(\mathbb{R}^d)$ we have

$$F(u,U)=F'(u,U)=F''(u,U), \tag{4.6}$$

hence for every $U\in\mathcal{U}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,U)\}_{n\in\mathbb{N}}$ Γ-converges to $F(\cdot,U)$ in the topology of $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$. □

In the rest of this section we assume that F' and F'' are defined by (4.1) and that F satisfies (4.3). Our aim is to prove equality (4.6) and the measure property (a) of Definition 3.4 for this functional. The proof of the measure property (a) is based on the De Giorgi-Letta criterion for measures, which is based on the subadditivity of the function $U \mapsto F(u, U)$. For this reason, we begin by proving that F'' satisfies the *nested* subadditivity condition.

Lemma 4.2. *Let $u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ and $U', U, V \in \mathcal{U}_c(\mathbb{R}^d)$, with $U' \subset\subset U$. Then*

$$F''(u, U' \cup V) \leq F''(u, U) + F''(u, V). \tag{4.7}$$

Proof. Without loss of generality, we may assume that $F''(u, U) + F''(u, V) < +\infty$. Since $F'' \geq F$ and F satisfies property (c), this implies that $u \in {\rm BD}(U \cup V)$. Recalling the definition of F'' (see (4.1)), we can find two sequences $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, converging to u in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ and such that

$$F''(u, U) = \limsup_{n\to+\infty} F_n(u_n, U) \quad \text{and} \quad F''(u, V) = \limsup_{n\to+\infty} F_n(v_n, V). \tag{4.8}$$

Thanks to property (c) for F_n and the finiteness of $F''(u, U)$ and $F''(u, V)$ we can assume that $\{u_n\}_{n\in\mathbb{N}} \subset {\rm BD}(U)$, $\{v_n\}_{n\in\mathbb{N}} \subset {\rm BD}(V)$, and that

$$M := c_3 \sup_{n\in\mathbb{N}} \Big(|Eu_n|(U) + |Ev_n|(V)\Big) + c_4\mathcal{L}^d(U \cup V) < +\infty. \tag{4.9}$$

Let us fix $m \in \mathbb{N}$ and set $\eta := {\rm dist}(U', \partial U) > 0$. For $j \in \{0, ..., m\}$ we also consider the sets $U_j := \{x \in U : {\rm dist}(x, \partial U) > \eta - \frac{j\eta}{m}\}$. Clearly, we have $U' \subset U_0 \subset\subset \cdots \subset\subset U_{m-1} \subset U_m = U$.

For every $j \in \{1, ..., m\}$ we consider a function $\varphi_j \in C^\infty_c(\mathbb{R}^d; [0, 1])$ satisfying $\varphi_j = 1$ on a neighbourhood of $\overline{U}_{j-1}$, $\varphi_j = 0$ on a neighbourhood of $\mathbb{R}^d \setminus U_j$, and

$$|\nabla\varphi_j| \leq \frac{2m}{\eta} \quad \text{on } \mathbb{R}^d. \tag{4.10}$$

We set

$$w^j_n := \varphi_j u_n + (1 - \varphi_j)v_n \quad \text{and} \quad S_j := (U_j \setminus \overline{U}_{j-1}) \cap V, \tag{4.11}$$

and observe that $w^j_n \in {\rm BD}(U' \cup V)$ and

$$Ew^j_n = \varphi_j Eu_n + (1 - \varphi_j)Ev_n + (u_n - v_n) \odot \nabla\varphi_j\, \mathcal{L}^d \tag{4.12}$$

as Radon measures on $U' \cup V$. Moreover, one checks immediately that for every choice of $j_n \in \{1, ..., m\}$ the sequence $\{w^{j_n}_n\}_{n\in\mathbb{N}}$ converges to u in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ as $n \to +\infty$.

Let us fix $j \in \{1, ..., m\}$ and $n \in \mathbb{N}$. We observe that by the measure property (a), by the properties of φ_j, and by the locality property of Remark 3.5 we have

$$\begin{aligned} F_n(w^j_n, U' \cup V) &\leq F_n(w^j_n, \overline{U}_{j-1}) + F_n(w^j_n, S_j) + F_n(w^j_n, V \setminus U_j) \\ = F_n(u_n, \overline{U}_{j-1}) + F_n(w^j_n, S_j) + F_n(v_n, V \setminus U_j) &\leq F_n(u_n, U) + F_n(w^j_n, S_j) + F_n(v_n, V). \end{aligned} \tag{4.13}$$

We now estimate the term involving the strip S_j in (4.13).

$$F_n(w^j_n, S_j) \leq c_3|Ew^j_n|(S_j) + c_4\mathcal{L}^d(S_j),$$

and by (4.10) and (4.12) we get

$$|Ew^j_n|(S_j) \leq |Eu_n|(S_j) + |Ev_n|(S_j) + \frac{2m}{\eta}\int_{S_j} |u_n - v_n|\, {\rm d}x.$$

From these two inequalities, we deduce that for every $n \in \mathbb{N}$ we can find $j_n \in \{1, .., m\}$ such that, setting $w_n := w^{j_n}_n$, we have

$$\begin{aligned} F_n(w_n, S_{j_n}) &\leq \frac{1}{m}\sum_{j=1}^m \Big(c_3|Eu_n|(S_j) + c_3|Ev_n|(S_j) + \frac{2c_3m}{\eta}\int_{S_j} |u_n - v_n|\, {\rm d}x + c_4\mathcal{L}^d(S_j)\Big) \\ &\leq \frac{1}{m}\big(c_3|Eu_n|(U) + c_3|Ev_n|(V) + c_4\mathcal{L}^d(U \cup V)\big) + \frac{2c_3}{\eta}\int_{U\cap V} |u_n - v_n|\, {\rm d}x \end{aligned}$$

$$\leq \frac{M}{m} + \frac{2c_3}{\eta} \int_{U\cap V} |u_n - v_n| \,\mathrm{d}x, \tag{4.14}$$

where in the second inequality we used (4.9).

Finally, combining (4.13) and (4.14) we deduce that

$$F_n(w_n, U' \cup V) \leq F_n(u_n, U) + F_n(v_n, V) + \frac{M}{m} + \frac{2c_3}{\eta} \int_{U\cap V} |u_n - v_n| \,\mathrm{d}x \tag{4.15}$$

for every $n \in \mathbb{N}$. Recalling that $\{u_n - v_n\}_{n\in\mathbb{N}}$ converges to 0 strongly in $L^1(U \cap V; \mathbb{R}^d)$, that $\{w_n\}_{n\in\mathbb{N}}$ converges to u in $L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$, we may let $n \to +\infty$ in (4.15) and from (4.8) we deduce that

$$F''(u, U' \cup V) \leq \limsup_{n\to+\infty} F_n(w_n, U' \cup V) \leq F''(u, U) + F''(u, V) + \frac{M}{m}.$$

Letting $m \to +\infty$ in this inequality, we obtain (4.7). □

We are now ready to prove the subadditivity of $F(u, \cdot)$ on $\mathcal{U}(\mathbb{R}^d)$.

Lemma 4.3. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ and $U, V \in \mathcal{U}(\mathbb{R}^d)$. Then*

$$F(u, U \cup V) \leq F(u, U) + F(u, V).$$

Proof. The result is a consequence of Lemma 4.2 and of classical arguments concerning increasing set functions (see, for instance, [25, Lemma 18.4]). □

We now prove that F satisfies the measure property (a).

Lemma 4.4. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$. Then $F(u, \cdot)$ is a Borel measure.*

Proof. It is enough to use the De Giorgi-Letta criterion for measures (see [31, Théorème 5.1] or [25, Theorem 14.21]), which ensures that the claim is proved, provided that we show that the set function $U \mapsto F(u, U)$ is subadditive, superadditive, and inner regular on $\mathcal{U}(\mathbb{R}^d)$. Subadditivity follows from Lemma 4.3, superadditivity is a consequence of classic results of Γ-convergence (see [25, Proposition 16.12]), and inner regularity follows from (4.3), recalling (4.2). □

Finally, we conclude the section by showing that Γ-convergence takes place for every $U \in \mathcal{U}_c(\mathbb{R}^d)$.

Lemma 4.5. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{R}^d)$ and $U \in \mathcal{U}_c(\mathbb{R}^d)$. Then*

$$F(u, U) = F'(u, U) = F''(u, U).$$

Proof. Since the inequalities $F(u, U) \leq F'(u, U) \leq F''(u, U)$ are obvious, we only have to show that $F''(u, U) \leq F(u, U)$, assuming that $F(u, U) < +\infty$. By property (c) for the functional F, this implies that $u \in \mathrm{BD}(U)$.

Let $\varepsilon > 0$ and consider a compact set $K \subset U$ such that

$$c_3|Eu|(U \setminus K) + c_4\mathcal{L}^d(U \setminus K) < \varepsilon.$$

Consider now $U' \in \mathcal{U}_c(\mathbb{R}^d)$ with $K \subset U' \subset\subset U$. Using Lemma 4.2 with $V = U \setminus K$, from the previous inequality and (4.5) we obtain

$$F''(u, U) \leq F''(u, U') + F''(u, U \setminus K) < F(u, U) + \varepsilon.$$

The conclusion then follows from the arbitrariness of ε. □

5. A partial integral representation for abstract functionals

In this section we use the results proved in [17] to provide a partial integral representation of every functional $F \in \mathfrak{F}_{\mathrm{sc}}$. More precisely, we show that the "absolutely continuous" part and the "jump" part of every such F can be written as integrals associated to suitable integrands $f \in \mathcal{F}$ and $g \in \mathcal{G}$.

We begin by introducing a decomposition of functionals in $\mathfrak{F}$ that reflects the usual decomposition of the measure Eu described in (2.1).

Definition 5.1. Let $F \in \mathfrak{F}$, $U \in \mathcal{U}_c(\mathbb{R}^d)$, and $u \in \mathrm{BD}(U)$. By (a) and (c) of Definition 3.4 $F(u,\cdot)$ is a bounded Radon measure on U. Its absolutely continuous and the singular part with respect to $\mathcal{L}^d$, defined on every $B \in \mathcal{B}(U)$, are denoted by $F^a(u,B)$ and $F^s(u,B)$, respectively. We also introduce the bounded Radon measures $F^c(u,\cdot)$ and $F^j(u,\cdot)$ defined by

$$F^c(u,B) := F^s(u,B\setminus J_u) \text{ and } F^j(u,B) := F(u,B\cap J_u)$$

for every $B \in \mathcal{B}(U)$. Note that we have

$$F(u,B) = F^a(u,B) + F^c(u,B) + F^j(u,B) \tag{5.1}$$

for every $B \in \mathcal{B}(U)$.

Remark 5.2. Let $F \in \mathfrak{F}$, $U \in \mathcal{U}_c(\mathbb{R}^d)$, and $u \in \mathrm{BD}(U)$. It follows directly from the bounds in (c) of Definition 3.4 that $F^c(u,\cdot)$ and $F^j(u,\cdot)$, considered as measures defined on $\mathcal{B}(U)$, are absolutely continuous with respect to $|E^cu|$ and $\mathcal{H}^{d-1} \llcorner J_u$, respectively.

In the rest of the paper, we will often work with functions defined by using auxiliary minimisation problems. For this reason, given a functional $F \in \mathfrak{F}$, a set $U \in \mathcal{U}_c(\mathbb{R}^d)$ with Lipschitz boundary ∂U, and a function $v \in \mathrm{BD}(U)$, we set

$$\mathfrak{m}^F(v,U) := \inf\{F(u,U) : u \in \mathrm{BD}(U) \text{ and } u = v \text{ on } \partial U\},$$

where the equality $u = v$ is understood in the sense of traces. As v is a competitor for the minimisation problem $\mathfrak{m}^F(v,U)$, by the upper bounds in Definition 3.4 it follows that

$$\mathfrak{m}^F(v,U) \le F(v,U) \le c_3|Ev|(U) + c_4\mathcal{L}^d(U) < +\infty.$$

Given $F \in \mathfrak{F}$, for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, and $\nu \in \mathbb{S}^{d-1}$ we set

$$f(x,A) := \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d}, \tag{5.2}$$

$$g(x,\zeta,\nu) := \limsup_{\rho\to 0^+} \frac{\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}}, \tag{5.3}$$

where $Q(x,\rho)$ and $Q_\nu(x,\rho)$ are the cubes defined in (c) and (e) of Section 2, while the functions ℓ_A and $u_{x,\zeta,\nu}$ are those introduced in (f) of the same section.

We now state the partial integral representation result, whose proof is given in [17, Lemma 5.1], following the lines of [9] and of [37].

Lemma 5.3. *Let $F \in \mathfrak{F}_{\mathrm{sc}}$, $U \in \mathcal{U}_c(\mathbb{R}^d)$, and $u \in \mathrm{BD}(U)$. Let f and g be the two functions defined by* (5.2) *and* (5.3)*. Then $f \in \mathcal{F}$, $g \in \mathcal{G}$, and*

$$F^a(u,B) = \int_B f(x,\mathcal{E}u)\,\mathrm{d}x, \tag{5.4}$$

$$F^j(u,B) = \int_{J_u\cap B} g(x,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{5.5}$$

for every set $B \in \mathcal{B}(U)$.

Proof. The proof of equalities (5.4) and (5.5) can be found in [17, Lemma 5.1]. The inclusions $f \in \mathcal{F}$ and $g \in \mathcal{G}$ are proved in Lemmas 5.6 and 5.8 below. ☐

Remark 5.4. We remark that properties (e) and (f) of Definition 3.4 are not used to obtain (5.4) and (5.5). Properties (e) and (f) are used only to prove that f and g satisfy (f3) and (g4).

To prove that f is Borel measurable we will need the following lemma, which will be used also in Section 9 for different purposes.

Lemma 5.5. *Let $F \in \mathfrak{F}$, $U', U \in \mathcal{U}_c(\mathbb{R}^d)$, with Lipschitz boundary and $U' \subset\subset U$, and let $w \in \mathrm{BD}(U)$. Then*

$$\mathfrak{m}^F(w,U) \le \mathfrak{m}^F(w,U') + c_3|Ew|(U\setminus U') + c_4\mathcal{L}^d(U\setminus U'). \tag{5.6}$$

Proof. Let $\eta > 0$ and $u \in \mathrm{BD}(U')$ with $u = w$ on $\partial U'$ be such that

$$F(u, U') \le \mathfrak{m}^F(w, U') + \eta. \tag{5.7}$$

We can then extend u to U by setting $u = w$ on $U \setminus U'$, so that $u = w$ on ∂U, which implies

$$\mathfrak{m}^F(w, U) \le F(u, U).$$

Using properties (a) and (c) of Definition 3.4 we get

$$F(u, U) = F(u, U') + F(u, U \setminus U') \le F(u, U') + c_3|Eu|(U \setminus \overline{U}') + c_3|Eu|(\partial U') + c_4\mathcal{L}^d(U \setminus U'). \tag{5.8}$$

Since $u = w$ on the open set $U \setminus \overline{U}'$, we have $Eu = Ew$ as measures defined on $\mathcal{B}(U \setminus \overline{U}')$, hence $|Eu|(U \setminus \overline{U}') = |Ew|(U \setminus \overline{U}')$. Moreover, since $|Eu| \llcorner \partial U' = |(u^+ - u^-) \odot \nu_{\partial U'}|\mathcal{H}^{d-1} \llcorner (J_u \cap \partial U')$ and $|Ew| \llcorner \partial U' = |(w^+ - w^-) \odot \nu_{\partial U'}|\mathcal{H}^{d-1} \llcorner (J_w \cap \partial U')$, the equalities $u^+ = w^+$ and $u^- = w^-$ on $\partial U'$ imply that $|Eu|(\partial U') = |Ew|(\partial U')$. Combining this equality with (5.7)-(5.8) we get

$$\mathfrak{m}^F(w, U) \le \mathfrak{m}^F(w, U') + c_3|Ew|(U \setminus \overline{U}') + c_3|Ew|(\partial U') + c_4\mathcal{L}^d(U \setminus U') + \eta,$$

which, by arbitrariness of η, implies (5.6). □

We are ready to prove that $f \in \mathcal{F}$.

Lemma 5.6. *Let $F \in \mathfrak{F}$. Then the function f defined by* (5.2) *belongs to $\mathcal{F}$.*

Proof. The proof of property (f1) can be obtained following the lines of the proof of [29, Lemma 5.3], replacing the Lemma 4.11 of that paper by our Lemma 5.5.

Let us fix $x \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. We want to show that

$$f(x, A) \le c_3|A| + c_4. \tag{5.9}$$

To this aim, we note that for every $\rho > 0$ from the upper bound in (c) of Definition 3.4 and the fact that ℓ_A is a competitor for the minimisation problem $\mathfrak{m}^F(\ell_A, Q(x, \rho))$ it follows that

$$\mathfrak{m}^F(\ell_A, Q(x, \rho)) \le F(\ell_A, Q(x, \rho)) \le (c_3|A| + c_4)\rho^d.$$

We now divide by ρ^d, take the limsup as $\rho \to 0^+$, and use (5.2) to obtain (5.9).

To conclude the proof of (f2) we show that

$$c_1|A| - c_2 \le f(x, A). \tag{5.10}$$

To prove this, let us fix $\rho > 0$ and observe that the lower bound in (c) of Definition 3.4 we have

$$c_1|Eu|(Q(x, \rho)) - c_2\rho^d \le F(u, Q(x, \rho))$$

for every $u \in \mathrm{BD}(Q(x, \rho))$, so that

$$c_1 \min\{|Eu|(Q(x, \rho)) : u \in \mathrm{BD}(Q(x, \rho)) \text{ with } u = \ell_A \text{ on } \partial Q(x, \rho)\} - c_2\rho^d \le \mathfrak{m}^F(\ell_A, Q(x, \rho)).$$

By Jensen's inequality the function ℓ_A minimises the problem on the left-hand side of the previous inequality. Hence,

$$c_1|A|\rho^d - c_2\rho^d \le \mathfrak{m}^F(\ell_A, Q(x, \rho)).$$

Dividing by ρ^d, taking the limsup as $\rho \to 0^+$, and recalling (5.2), we obtain (5.10).

We now prove (f3). To this aim, let us fix $x \in \mathbb{R}^d$, $A_1, A_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\rho > 0$, and $\eta > 0$. Consider a function $u_2 \in \mathrm{BD}(Q(x, \rho))$ with $u_2 = \ell_{A_2}$ on $\partial Q(x, \rho)$ and such that

$$F(u_2, Q(x, \rho)) \le \mathfrak{m}^F(\ell_{A_2}, Q(x, \rho)) + \eta\rho^d.$$

Then, we set $u_1 := u_2 - \ell_{A_2} + \ell_{A_1}$ and observe that $u_1 = \ell_{A_1}$ on $\partial Q(x, \rho)$. Using property (e) of Definition 3.4 and the previous inequality, it follows that

$$\begin{aligned} \mathfrak{m}^F(\ell_{A_1}, Q(x, \rho)) &\le F(u_1, Q(x, \rho)) \le F(u_2, Q(x, \rho)) + c_5|A_1 - A_2|\rho^d \\ &\le \mathfrak{m}^F(\ell_{A_2}, Q(x, \rho)) + c_5|A_1 - A_2|\rho^d + \eta\rho^d. \end{aligned} \tag{5.11}$$

Dividing by ρ^d, taking first the limsup as $\rho \to 0^+$ and then as $\eta \to 0^+$, by (5.2) we obtain

$$f(x, A_1) \le f(x, A_2) + c_5|A_1 - A_2|.$$

Exchanging the roles of A_1 and A_2, we obtain (f3). □

To prove that g is Borel measurable we will need the following lemma, which will be used also in Section 9 for different purposes.

Lemma 5.7. *Let* $F \in \mathfrak{F}$. *Then there exists a continuous function* $\omega\colon [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, *increasing with respect to each variable and with* $\omega(0,0) = 0$, *such that for every* $x_1, x_2 \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu_1, \nu_2 \in \mathbb{S}^{d-1}$, *and* $0 < \rho_1 < \rho_2$, *the inclusion* $Q_{\nu_1}(x_1,\rho_1) \subset\subset Q_{\nu_2}(x_2,\rho_2)$ *implies*

$$\begin{aligned}\mathfrak{m}^F(u_{x_2,\zeta,\nu_2}, Q_{\nu_2}(x_2,\rho_2)) \le \mathfrak{m}^F(u_{x_1,\zeta,\nu_1}, Q_{\nu_1}(x_1,\rho_1)) + c_3|\zeta|(\rho_2^{d-1} - \rho_1^{d-1})\\ + c_4(\rho_2^d - \rho_1^d) + c_3|\zeta|\omega(\tfrac{|x_2-x_1|}{\rho_1}, |\nu_1 - \nu_2|)\rho_1^{d-1}.\end{aligned}$$

Proof. The proof is obtained by the same argument used in the proof of [29, Lemma 5.9], replacing the use of the distributional gradient D by that of the symmetric distributional gradient E. □

We are now in position to prove that $g \in \mathcal{G}$.

Lemma 5.8. *Let* $F \in \mathfrak{F}$. *Then the function* g *defined by* (5.3) *belongs to* $\mathcal{G}$.

Proof. The proof of property (g1) can be obtained arguing as in [29, Lemma 5.12], replacing Lemma 5.9 of that paper by our Lemma 5.7.

Property (g2) follows immediately from (d) of Definition 3.4 and (5.3) once we observe that $u_{x,-\zeta,-\nu} = u_{x,\zeta,\nu} - \zeta$ and that $Q_\nu(x,\rho) = Q_{-\nu}(x,\rho)$ by (d) and (e) of Section 2.

We prove property (g3). Let us fix $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$. As for every $\rho > 0$ the function $u_{x,\zeta,\nu}$ is a competitor for the minimisation problem $\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho))$, by the upper bound in (c) of Definition 3.4 and (k) of Section 2 we obtain that

$$\mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) \le F(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) \le c_3|\zeta \odot \nu|\rho^{d-1} + c_4\rho^d.$$

Dividing by ρ^{d-1}, letting $\rho \to 0^+$, and recalling (5.3) we obtain the upper bound in (g3).

To obtain the lower bound in (g3) we argue as follows. For every $\rho > 0$ let $u \in \mathrm{BD}(Q_\nu(x,\rho))$ be a function with $u = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$ and observe that by the lower bound in (c) of Definition 3.4 we have

$$c_1|Eu|(Q_\nu(x,\rho)) - c_2\rho^d \le F(u, Q_\nu(x,\rho)),$$

so that

$$c_1 \inf\{|Eu|(Q_\nu(x,\rho)) : u = u_{x,\zeta,\nu} \text{ on } \partial Q_\nu(x,\rho)\} - c_2\rho^d \le \mathfrak{m}^F(u_{x,\zeta,\nu}, Q_\nu(x,\rho)). \tag{5.12}$$

Consider the function $\widetilde{g}\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ defined for every $\zeta \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$ by

$$\widetilde{g}(\zeta,\nu) := \limsup_{\rho\to 0^+} \frac{\inf\{|Eu|(Q_\nu(x,\rho)) : u = u_{x,\zeta,\nu} \text{ on } \partial Q_\nu(x,\rho)\}}{\rho^{d-1}}, \tag{5.13}$$

where x is an arbitrary point of $\mathbb{R}^d$. By Lemma 5.3 applied to the functional $\widetilde{F}(u,U) := |Eu|(U)$, we have

$$\widetilde{F}^j(u,U) = \int_{J_u\cap U} \widetilde{g}([u],\nu_u)\, \mathrm{d}\mathcal{H}^{d-1} \tag{5.14}$$

for every $U \in \mathcal{U}_c(\mathbb{R}^d)$ and $u \in \mathrm{BD}(U)$. In particular, for $U = Q_\nu(x,1)$ and $u = u_{x,\zeta,\nu}$ by (k) of Section 2 equality (5.14) leads to $|\zeta \odot \nu| = \widetilde{g}(\zeta,\nu)$. Dividing (5.12) by ρ^{d-1}, letting $\rho \to 0^+$, and using (5.3) and (5.13), we obtain the lower bound in (g3).

We conclude by showing that g satisfies property (g4). Let $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, $\rho > 0$, and $\eta > 0$. Consider a function $u_2 \in \mathrm{BD}(Q_\nu(x,\rho))$ with $u_2 = u_{x,\zeta_2,\nu}$ on $\partial Q_\nu(x,\rho)$ and such that

$$F(u_2, Q_\nu(x,\rho)) \le \mathfrak{m}^F(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + \eta\rho^{d-1}.$$

Then, we set $u_1 := u_2 - u_{x,\zeta_2,\nu} + u_{x,\zeta_1,\nu}$ and observe that $u_1 = u_{x,\zeta_1,\nu}$ on $\partial Q_\nu(x,\rho)$. Using property (f) of Definition 3.4 and the previous inequality, it follows that

$$\begin{aligned}\mathfrak{m}^F(u_{x,\zeta_1,\nu}, Q_\nu(x,\rho)) \le F(u_{x,\zeta_1,\nu}, Q_\nu(x,\rho)) \le F(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + \sigma(|\zeta_1 - \zeta_2|)\rho^{d-1}\\ \le \mathfrak{m}^F(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + \sigma(|\zeta_1 - \zeta_2|)\rho^{d-1} + \eta\rho^{d-1}.\end{aligned} \tag{5.15}$$

Dividing this inequality by ρ^{d-1}, taking first the limsup for $\rho \to 0^+$ and then the limit for $\eta \to 0^+$, by (5.3) we obtain

$$g(x,\zeta_1,\nu) \leq g(x,\zeta_2,\nu) + \sigma(|\zeta_1 - \zeta_2|).$$

Exchanging the roles of ζ_1 and ζ_2, we obtain (g4). □

6. A SMALLER COLLECTION OF FUNCTIONALS

In this section we introduce two subcollections of $\mathcal{F}$ and $\mathcal{G}$, denoted by $\mathcal{F}^\alpha$ and $\mathcal{G}^\infty$, by prescribing suitable conditions on the integrands, related to their behaviour when $|A| \to +\infty$ and $|\zeta| \to +\infty$ (see Remarks 6.2 and 6.3 below). We then define a subcollection of $\mathfrak{F}$, denoted by $\mathfrak{F}^{\alpha,\infty}$, which is stable by Γ-convergence (see Proposition 6.6) and includes the functionals $F^{f,g}$ with $f \in \mathcal{F}^\alpha$ and $g \in \mathcal{G}^\infty$. We shall see in Section 7 that all lower semicontinuos functionals $F \in \mathfrak{F}^{\alpha,\infty}$ admit a complete integral representation, including their Cantor part F^c, provided they satisfy an additional condition (see (7.1) below).

In the rest of the paper we fix three constants $0 < \alpha < 1$ and $c_6, c_7 \geq 0$. We are ready to introduce the collections $\mathcal{F}^\alpha$ and $\mathcal{G}^\infty$.

Definition 6.1. Let $\mathcal{F}^\alpha$ be the collection of all functions $f \in \mathcal{F}$ such that for every $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, and $s,t > 0$ we have

$$\Big|\frac{f(x,sA)}{s} - \frac{f(x,tA)}{t}\Big| \leq \frac{c_6}{s} f(x,sA)^{1-\alpha} + \frac{c_6}{s} + \frac{c_6}{t} f(x,tA)^{1-\alpha} + \frac{c_6}{t}. \tag{f4}$$

Let $\mathcal{G}^\infty$ be the collection of all functions $g \in \mathcal{G}$ such that for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $s,t > 0$ we have

$$\Big|\frac{g(x,s\zeta,\nu)}{s} - \frac{g(x,t\zeta,\nu)}{t}\Big| \leq c_7\Big(\frac{g(x,s\zeta,\nu)}{s} + \frac{g(x,t\zeta,\nu)}{t}\Big)\Big(\frac{1}{s} + \frac{1}{t}\Big). \tag{g5}$$

Remark 6.2. Property (f4) can be interpreted as a condition specifying the rate at which $f(x,tA)/t$ approaches its recession function $f^\infty(x,A)$ as $t \to +\infty$. Indeed (see [16, Remark 3.4]), the upper bound in (f2) and (f4), are equivalent to the two conditions

$$f^\infty(x,A) = \lim_{t\to+\infty} \frac{f(x,tA)}{t} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}, \tag{6.1}$$

$$\Big|\frac{f(x,tA)}{t} - f^\infty(x,A)\Big| \leq \frac{c_6}{t} + \frac{c_6}{t} f(x,tA)^{1-\alpha} \qquad \text{for every } x \in \mathbb{R}^d,\ t > 0, \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}. \tag{6.2}$$

Observe that by the growth conditions in (f2), inequality (6.2) implies that

$$\Big|\frac{f(x,tA)}{t} - f^\infty(x,A)\Big| \leq \frac{C_A}{t^\alpha} \qquad \text{for every } x \in \mathbb{R}^d,\ t \geq 1, \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}, \tag{6.3}$$

where $C_A := c_6 + c_4^{1-\alpha} + c_6 c_3^{1-\alpha}|A|^{1-\alpha}$. Conditions similar to (6.2) and to (6.3) have already been considered in the literature (see, for instance, [9, Property (H4)] and [16, 26, 30, 45]).

Remark 6.3. If $g \in \mathcal{G}^\infty$, then for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ there exists the limit

$$g^\infty(x,\zeta,\nu) := \lim_{t\to+\infty} \frac{g(x,t\zeta,\nu)}{t}. \tag{6.4}$$

Indeed, (g3) and (g5) imply the Cauchy condition for the function $t \mapsto g(x,t\zeta,\nu)/t$. Passing to the limit in (g3) we obtain

$$c_1|\zeta \odot \nu| \leq g^\infty(x,\zeta,\nu) \leq c_3|\zeta \odot \nu| \tag{6.5}$$

for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, while passing to the limit in (g4) and using (3.1) we get

$$|g^\infty(x,\zeta_1,\nu) - g^\infty(x,\zeta_2,\nu)| \leq \sigma^\infty|\zeta_1 - \zeta_2| \tag{g4$'$}$$

for every $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$, where σ^∞ is the non-negative constant introduced in (3.1).

Letting $s\to+\infty$ in (g5), we also obtain

$$\Big|\frac{g(x,t\zeta,\nu)}{t}-g^\infty(x,\zeta,\nu)\Big|\le c_7\Big(\frac{g(x,t\zeta,\nu)}{t}+g^\infty(x,\zeta,\nu)\Big)\frac{1}{t} \tag{6.6}$$

for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^d$, $\nu\in\mathbb{S}^{d-1}$, and $t>0$. Thanks to the bounds in (g3) and (6.5), this inequality implies that

$$\Big|\frac{g(x,t\zeta,\nu)}{t}-g^\infty(x,\zeta,\nu)\Big|\le \frac{2c_3c_7|\zeta\odot\nu|}{t}, \tag{6.7}$$

while the inequality

$$\Big|\frac{g(x,t\zeta,\nu)}{t}-g^\infty(x,\zeta,\nu)\Big|\le \frac{2c_1c_7|\zeta\odot\nu|}{t}$$

implies (6.6).

Conversely, assume that g is a function in $\mathcal{G}$ such that the limit in (6.4) exists and the inequality

$$g^\infty(x,\zeta,\nu)\le\frac{g(x,t\zeta,\nu)}{t} \tag{6.8}$$

is satisfied for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^d$, $\nu\in\mathbb{S}^{d-1}$, and $t>0$. If g satisfies also (6.6), then $g\in\mathcal{G}^\infty$. Indeed, by (6.6) and (6.8) we have

$$\Big|\frac{g(x,t\zeta,\nu)}{t}-g^\infty(x,\zeta,\nu)\Big|\le c_7\Big(\frac{g(x,t\zeta,\nu)}{t}+\frac{g(x,s\zeta,\nu)}{s}\Big)\frac{1}{t} \tag{6.9}$$

for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^d$, $\nu\in\mathbb{S}^{d-1}$, and $s,t>0$. Exchanging the roles of s and t, we obtain

$$\Big|\frac{g(x,s\zeta,\nu)}{s}-g^\infty(x,\zeta,\nu)\Big|\le c_7\Big(\frac{g(x,s\zeta,\nu)}{s}+\frac{g(x,t\zeta,\nu)}{t}\Big)\frac{1}{s}. \tag{6.10}$$

Using the triangle inequality, from (6.9) and (6.10) we obtain (g5).

Note that the existence of the limit in (6.4) and inequality (6.8) is always satisfied when $t\mapsto g(x,t\zeta,\nu)$ is concave.

We now introduce a collection of functionals closely related to conditions (f4) and (g5).

Definition 6.4. Let $\mathfrak{F}^{\alpha,\infty}$ be the collection of all functionals $F\in\mathfrak{F}$ such that for every $U\in\mathcal{U}_c(\mathbb{R}^d)$, $u\in BD(U)$, and $s,t>0$ we have

$$\begin{aligned}\Big|\frac{F(su,U)}{s}-\frac{F(tu,U)}{t}\Big|&\le\frac{c_6}{s}\mathcal{L}^d(U)^\alpha F(su,U)^{1-\alpha}+\frac{c_6}{s}\mathcal{L}^d(U)\\&+\frac{c_6}{t}\mathcal{L}^d(U)^\alpha F(tu,U)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(U)+c_7\Big(\frac{F(su,U)}{s}+\frac{F(tu,U)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).\end{aligned} \tag{g}$$

The collection of functionals $F\in\mathfrak{F}^{\alpha,\infty}$ such that for every $U\in\mathcal{U}(\mathbb{R}^d)$ the functional $F(\cdot,U)$ is $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$-lower semicontinuous is denoted by $\mathfrak{F}^{\alpha,\infty}_{\rm sc}$.

We now show that integral functionals of the form $F^{f,g}$ belong to $\mathfrak{F}^{\alpha,\infty}$ whenever $f\in\mathcal{F}^\alpha$ and $g\in\mathcal{G}^\infty$.

Proposition 6.5. *Let $f\in\mathcal{F}^\alpha$ and $g\in\mathcal{G}^\infty$. Then the functional $F^{f,g}$ defined by* (3.3) *belongs to $\mathfrak{F}^{\alpha,\infty}$.*

Proof. Thanks to Proposition 3.8, we only need to prove that $F^{f,g}$ satisfies property (g). To this aim, let us fix $U\in\mathcal{U}_c(\mathbb{R}^d)$, $u\in BD(U)$, and $s,t>0$. Using (f4) and Hölder's inequality we see that

$$\begin{aligned}\Big|\int_U\frac{f(x,s\mathcal{E}u)}{s}\,dx-\int_U\frac{f(x,t\mathcal{E}u)}{t}\,dx\Big|&\le\int_U\Big(\frac{c_6}{s}f(x,s\mathcal{E}u)^{1-\alpha}+\frac{c_6}{s}+\frac{c_6}{t}f(x,t\mathcal{E}u)^{1-\alpha}+\frac{c_6}{t}\Big)dx\\&\le\frac{c_6}{s}\mathcal{L}^d(U)^\alpha F(su,U)^{1-\alpha}+\frac{c_6}{s}\mathcal{L}^d(U)+\frac{c_6}{t}\mathcal{L}^d(U)^\alpha F(tu,U)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(U).\end{aligned} \tag{6.11}$$

Since f^∞ is positively 1-homogeneous, we immediately see that

$$\frac{1}{s}\int_U f^\infty\Big(x,s\frac{dE^cu}{d|E^cu|}\Big)\,d|E^cu|=\frac{1}{t}\int_U f^\infty\Big(x,t\frac{dE^cu}{d|E^cu|}\Big)\,d|E^cu|. \tag{6.12}$$

Condition (g5) implies that

$$\begin{aligned}
&\Big|\int_{J_u\cap U}\Big(\frac{g(x,s[u],\nu_u)}{s}-\frac{g(x,t[u],\nu_u)}{t}\Big)\mathrm{d}\mathcal{H}^{d-1}\Big|\\
&\le c_7\Big(\int_{J_u\cap U}\frac{g(x,s[u],\nu_u)}{s}\mathcal{H}^{d-1}+\int_{J_u\cap U}\frac{g(x,t[u],\nu_u)}{t}\mathcal{H}^{d-1}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big)\\
&\le c_7\Big(\frac{F(su,U)}{s}+\frac{F(tu,U)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).
\end{aligned}\tag{6.13}$$

Finally, combining (6.11)-(6.13) we obtain that $F^{f,g}$ satisfies property (g). □

We now prove that the closure of the class $\mathfrak{F}^{\alpha,\infty}$ with respect to Γ-convergence is $\mathfrak{F}^{\alpha,\infty}_{\mathrm{sc}}$.

Proposition 6.6. *Let $\{F_n\}_{n\in\mathbb{N}}\subset\mathfrak{F}^{\alpha,\infty}$. Assume that for every $U\in\mathcal{U}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,U)\}_n$ Γ-converges to $F(\cdot,U)$ with respect to the topology of $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. Then $F\in\mathfrak{F}^{\alpha,\infty}_{\mathrm{sc}}$.*

Proof. Thanks to Theorem 4.1 we have $F\in\mathfrak{F}_{\mathrm{sc}}$. Hence, to conclude we only have to show that F satisfies property (g). The argument we use is a variant of the one presented in [26, Proposition 6.11] and [30, Theorem 4.10].

Let us fix $U\in\mathcal{U}_c(\mathbb{R}^d)$, $u\in\mathrm{BD}(U)$, and $s,t>0$. To prove that (g) holds it is enough to show

$$\begin{aligned}
&\frac{F(su,U)}{s}-\frac{c_6}{s}\mathcal{L}^d(U)^\alpha F(su,U)^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(U)-c_7\frac{F(su,U)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\
&\le\frac{F(tu,U)}{t}+\frac{c_6}{t}\mathcal{L}^d(U)^\alpha F(tu,U)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(U)+c_7\frac{F(tu,U)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big).
\end{aligned}\tag{6.14}$$

Indeed, exchanging the role of s and t we obtain (g). We note that, if the left-hand side of (6.14) is less than or equal to zero, then the inequality is trivial because the right-hand side is non-negative. Hence, we may assume that the left-hand side is positive.

Let $\{u_n\}_{n\in\mathbb{N}}\subset\mathrm{BD}(U)$ be a sequence converging to u in $L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$ and such that $F_n(tu_n,U)$ converges to $F(tu,U)$ as $n\to+\infty$. It follows from property (g) applied to F_n that for every $n\in\mathbb{N}$ we have

$$\begin{aligned}
&\frac{F_n(su_n,U)}{s}-\frac{c_6}{s}\mathcal{L}^d(U)^\alpha F_n(su_n,U)^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(U)-c_7\frac{F_n(su_n,U)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\
&\le\frac{F_n(tu_n,U)}{t}+\frac{c_6}{t}\mathcal{L}^d(U)^\alpha F_n(tu_n,U)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(U)+c_7\frac{F_n(tu_n,U)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big).
\end{aligned}\tag{6.15}$$

Thanks to our choice of $\{u_n\}_{n\in\mathbb{N}}$, we have

$$\begin{aligned}
&\lim_{n\to+\infty}\Big(\frac{F_n(tu_n,U)}{t}+\frac{c_6}{t}\mathcal{L}^d(U)^\alpha F_n(tu_n,U)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(U)+c_7\frac{F_n(tu_n,U)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big)\Big)\\
&\quad=\frac{F(tu,U)}{t}+\frac{c_6}{t}\mathcal{L}^d(U)^\alpha F(tu,U)^{1-\alpha}+\frac{c_6}{t}\mathcal{L}^d(U)+c_7\frac{F(tu,U)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big).
\end{aligned}$$

Hence, by (6.15) to conclude we only need to show that

$$\begin{aligned}
&\frac{F(su,U)}{s}-\frac{c_6}{s}\mathcal{L}^d(U)^\alpha F(su,U)^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(U)-c_7\frac{F(su,U)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\
&\le\liminf_{n\to+\infty}\Big(\frac{F_n(su_n,U)}{s}-\frac{c_6}{s}\mathcal{L}^d(U)^\alpha F_n(su_n,U)^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(U)-c_7\frac{F_n(su_n,U)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\Big).
\end{aligned}\tag{6.16}$$

We consider the function $\Phi\colon[0,+\infty)\to\mathbb{R}$ defined for every $z\in[0,+\infty)$ by

$$\Phi(z):=\frac{z}{s}-\frac{c_6}{s}\mathcal{L}^d(U)^\alpha z^{1-\alpha}-\frac{c_6}{s}\mathcal{L}^d(U)-c_7\frac{z}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

We observe that $\Phi(F(su,U))$ coincides with the left-hand side of (6.14) and (6.16), while the right-hand side of (6.16) coincides with $\liminf_n\Phi(F_n(su_n,U))$. If $1-c_7(1/s+1/t)\le 0$ then $\Phi(z)\le 0$ for every $z\in[0,+\infty)$, so that (6.14) is satisfied. If $1-c_7(1/s+1/t)>0$, then we set $z_0:=\big(c_6(1-\alpha)\mathcal{L}^d(U)^\alpha\big)^{1/\alpha}\big(1-c_7\big(1/s+1/t\big)\big)^{-1/\alpha}$. By direct computation of Φ' we see that Φ is strictly decreasing in $[0,z_0]$ and strictly increasing in $[z_0,+\infty)$. Since $\Phi(0)\le 0$ we deduce that $\Phi(z)\le 0$ for $z\in[0,z_0]$. This implies that if $\Phi(z)>0$ then $z>z_0$. Since $\Phi(F(su,U))$ coincides with the left-hand side of (6.16), which we assumed to be positive, we have that $F(su,U) > z_0$. Moreover, by the Γ-liminf inequality we have $F(su,U) \leq \liminf_n F_n(su_n,U)$, so that, using that Φ is increasing on $[z_0,+\infty)$, we deduce that

$$\Phi(F(su,U)) \leq \liminf_{n\to+\infty} \Phi(F_n(su_n,U)).$$

This proves (6.16) and concludes the proof of the proposition. □

We now present an inequality that allows to estimate the difference between the minimum values of some minimisation problems with Dirichlet boundary conditions involving functionals in $\mathfrak{F}^{\alpha,\infty}$. The aim of this lemma is twofold. On the one hand, it is useful to establish that the functions f and g defined by (5.2) and (5.3) satisfy (f4) and (g5). On the other hand, this lemma will be crucial in the proof of the integral representation of the Cantor part of functionals in $\mathfrak{F}^{\alpha,\infty}_{\rm sc}$. The proof of this lemma closely resembles that of Proposition 6.6. It follows the lines of the proofs of [26, Lemmas 6.12 and 6.13], removing any truncation argument, which is not available in our context.

Lemma 6.7. *Let $F \in \mathfrak{F}^{\alpha,\infty}$, let $U \in \mathcal{U}_c(\mathbb{R}^d)$ with Lipschitz boundary, and let $w \in \mathrm{BD}(U)$. Then*

$$\Big|\frac{\mathfrak{m}^F(sw,U)}{s} - \frac{\mathfrak{m}^F(tw,U)}{t}\Big| \leq \frac{c_6}{s}\mathcal{L}^d(U)^\alpha \mathfrak{m}^F(sw,U)^{1-\alpha} + \frac{c_6}{s}\mathcal{L}^d(U)$$
$$\frac{c_6}{t}\mathcal{L}^d(U)^\alpha \mathfrak{m}^F(tw,U)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(U) + c_7\Big(\frac{\mathfrak{m}^F(sw,U)}{s} + \frac{\mathfrak{m}^F(tw,U)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

Proof. It is enough to prove that

$$\begin{aligned}&\frac{\mathfrak{m}^F(sw,U)}{s} - \frac{c_6}{s}\mathcal{L}^d(U)^\alpha \mathfrak{m}^F(sw,U)^{1-\alpha} - \frac{c_6}{s}\mathcal{L}^d(U) - c_7\frac{\mathfrak{m}^F(sw,U)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\ &\leq \frac{\mathfrak{m}^F(tw,U)}{t} + \frac{c_6}{t}\mathcal{L}^d(U)^\alpha \mathfrak{m}^F(tw,U)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}(U) + c_7\frac{\mathfrak{m}^F(tw,U)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big)\end{aligned} \tag{6.17}$$

whenever the left-hand side is positive.

Let $\eta > 0$ and consider a function $u \in \mathrm{BD}(U)$ with $u = w$ on ∂U and such that

$$F(tu,U) \leq \mathfrak{m}(tw,U) + \eta.$$

Since F satisfies property (g) of Definition 6.4, from this inequality we deduce that

$$\begin{aligned}&\frac{F(su,U)}{s} - \frac{c_6}{s}\mathcal{L}^d(U)^\alpha F(su,U)^{1-\alpha} - \frac{c_6}{s}\mathcal{L}^d(U) - c_7\frac{F(su,U)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\ &\leq \frac{F(tu,U)}{t} + \frac{c_6}{t}\mathcal{L}^d(U)^\alpha F(tu,U)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(U) + c_7\frac{F(su,U)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\ &\leq \frac{\mathfrak{m}^F(tw,U)}{t} + \frac{\eta}{t} + \frac{c_6}{t}\mathcal{L}^d(U)^\alpha \mathfrak{m}^F(tw,U)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(U)^\alpha \eta^{1-\alpha}\\ &\quad + \frac{c_6}{t}\mathcal{L}^d(U) + c_7\frac{\mathfrak{m}^F(tw,U)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big) + c_7\frac{\eta}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big).\end{aligned}$$

Exploiting the fact that $\mathfrak{m}^F(sw,U) \leq F(su,U)$ and using the same argument employed at the end of the proof of Proposition 6.6 we conclude that

$$\begin{aligned}&\frac{\mathfrak{m}^F(sw,U)}{s} - \frac{c_6}{s}\mathcal{L}^d(U)^\alpha \mathfrak{m}^F(sw,U)^{1-\alpha} - \frac{c_6}{s}\mathcal{L}^d(U) - c_7\frac{\mathfrak{m}^F(sw,U)}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big)\\ &\leq \frac{\mathfrak{m}^F(tw,U)}{t} + \frac{\eta}{t} + \frac{c_6}{t}\mathcal{L}^d(U)^\alpha \mathfrak{m}^F(tw,U)^{1-\alpha} + \frac{c_6}{t}\mathcal{L}^d(U)^\alpha \eta^{1-\alpha}\\ &\quad + \frac{c_6}{t}\mathcal{L}^d(U) + c_7\frac{\mathfrak{m}^F(tw,U)}{t}\Big(\frac{1}{s}+\frac{1}{t}\Big) + c_7\frac{\eta}{s}\Big(\frac{1}{s}+\frac{1}{t}\Big).\end{aligned}$$

As η is arbitrary, we obtain (6.17). □

We conclude the section by proving a further property of the functionals $F \in \mathfrak{F}^{\alpha,\infty}$.

Lemma 6.8. *Let $F \in \mathfrak{F}^{\alpha,\infty}$. Then the functions f and g defined by* (5.2) *and* (5.3) *belong to $\mathcal{F}^\alpha$ and $\mathcal{G}^\infty$, respectively.*

Proof. Thanks to Lemmas 5.6 and 5.8, to conclude it is enough to show that f and g satisfy properties (f4) and (g5).

We first prove that f satisfies (f4). Let us fix $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, and $s,t > 0$. Thanks to Lemma 6.7 applied with $U = Q(x,\rho)$ and $w = \ell_A$, we have that

$$\frac{\mathfrak{m}^F(s\ell_A, Q(x,\rho))}{s} \leq \frac{\mathfrak{m}^F(t\ell_A, Q(x,\rho))}{t} + \frac{c_6}{s}\rho^{\alpha d}\mathfrak{m}^F(s\ell_A, Q(x,\rho))^{1-\alpha} + \frac{c_6}{s}\rho^d$$
$$\frac{c_6}{t}\rho^{\alpha d}\mathfrak{m}^F(t\ell_A, Q(x,\rho))^{1-\alpha} + \frac{c_6}{t}\rho^d + c_7\Big(\frac{\mathfrak{m}^F(s\ell_A, Q(x,\rho))}{s} + \frac{\mathfrak{m}^F(t\ell_A, Q(x,\rho))}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

Dividing this inequality by ρ^d, letting $\rho \to 0^+$, and recalling (5.2), we obtain

$$\frac{f(x,sA)}{s} \leq \frac{f(x,tA)}{t} + \frac{c_6}{s}f(x,sA)^{1-\alpha} + \frac{c_6}{s} + \frac{c_6}{t}f(x,tA)^{1-\alpha}$$
$$+\frac{c_6}{t} + c_7\Big(\frac{f(x,sA)}{s} + \frac{f(x,tA)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

Exchanging the roles of s and t we obtain

$$\frac{f(x,tA)}{t} \leq \frac{f(x,sA)}{s} + \frac{c_6}{s}f(x,sA)^{1-\alpha} + \frac{c_6}{s} + \frac{c_6}{t}f(x,tA)^{1-\alpha}$$
$$+\frac{c_6}{t} + c_7\Big(\frac{f(x,sA)}{s} + \frac{f(x,tA)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

As these inequalities hold for every s and t, we may let $s \to +\infty$ and obtain for every $t > 0$ that

$$\big|f^\infty(x,A) - \frac{f(x,tA)}{t}\big| \leq \frac{c_6}{t}f(x,tA)^{1-\alpha} + \frac{c_6}{t} + \frac{c_7}{t}\Big(f^\infty(x,A) + \frac{f(x,tA)}{t}\Big).$$

Let us fix $\lambda > 0$ and apply this inequality with A replaced by λA to get

$$\big|\lambda f^\infty(x,A) - \frac{f(x,t\lambda A)}{t}\big| \leq \frac{c_6}{t}f(x,t\lambda A)^{1-\alpha} + \frac{c_6}{t} + \frac{c_7}{t}\Big(\lambda f^\infty(x,A) + \frac{f(x,t\lambda A)}{t}\Big)$$

for every $t > 0$, hence

$$\big|f^\infty(x,A) - \frac{f(x,t\lambda A)}{\lambda t}\big| \leq \frac{c_6}{\lambda t}f(x,t\lambda A)^{1-\alpha} + \frac{c_6}{\lambda t} + \frac{c_7}{t}\Big(f^\infty(x,A) + \frac{f(x,t\lambda A)}{\lambda t}\Big)$$

for every $t > 0$ and $\lambda > 0$. Setting $\tau = t\lambda$

$$\big|f^\infty(x,A) - \frac{f(x,\tau A)}{\tau}\big| \leq \frac{c_6}{\tau}f(x,\tau A)^{1-\alpha} + \frac{c_6}{\tau} + \frac{c_7}{t}\Big(f^\infty(x,A) + \frac{f(x,\tau A)}{\tau}\Big)$$

for every $t > 0$ and $\tau > 0$, so that, letting $t \to +\infty$ we obtain

$$\big|f^\infty(x,A) - \frac{f(x,\tau A)}{\tau}\big| \leq \frac{c_6}{\tau}f(x,\tau A)^{1-\alpha} + \frac{c_6}{\tau} \tag{6.18}$$

for every $\tau > 0$. Recalling that f satisfies (f2), we may use Remark 6.2 to deduce from (6.18) that f satisfies (f4), concluding the proof of the inclusion $f \in \mathcal{F}^\alpha$.

We now prove that g satisfies property (g5). Let $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$. Lemma 6.7 applied with $U = Q_\nu(x,\rho)$ and $w = u_{x,\zeta,\nu}$ implies that for every $s,t > 0$ we have

$$\frac{\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} \leq \frac{\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))}{t} + \frac{c_6}{s}\rho^{\alpha d}\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha} + \frac{c_6}{s}\rho^d$$
$$\frac{c_6}{t}\rho^{\alpha d}\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))^{1-\alpha} + \frac{c_6}{t}\rho^d$$
$$+c_7\Big(\frac{\mathfrak{m}^F(su_{x,\zeta,\nu}, Q_\nu(x,\rho))}{s} + \frac{\mathfrak{m}^F(tu_{x,\zeta,\nu}, Q_\nu(x,\rho))}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

Dividing this inequality by ρ^{d-1}, letting $\rho \to 0^+$, and recalling (5.3) we deduce

$$\frac{g(x,s\zeta,\nu)}{s} \leq \frac{g(x,t\zeta,\nu)}{t} + c_7\Big(\frac{g(x,s\zeta,\nu)}{s} + \frac{g(x,t\zeta,\nu)}{t}\Big)\Big(\frac{1}{s}+\frac{1}{t}\Big).$$

Exchanging the roles of s and t we obtain (g5). $\square$

7. Full integral representation

In this section we prove a complete integral representation result for functionals F in $\mathfrak{F}_{\rm sc}^{\alpha,\infty}$, including the representation of the Cantor part F^c, assuming that the function f, defined in (5.2), does not depend on x.

A full integral representation result for functionals $F \in \mathfrak{F}_{\rm sc}$ has recently been proved in [17], under the additional assumption of uniform continuity of F with respect to translations of the independent variables. This condition was originally considered in [9, Lemma 3.11] for the corresponding integral representation problem in BV.

In the case of periodic homogenisation one can easily cheque that the Γ-limit functional $F_{\rm hom}$ (see Theorem 9.7 below) is invariant under translations (see, for instance, [17, Theorem 6.14]). However, in general, this invariance (nor the continuity with respect to translations) cannot be checked directly in the case of non-periodic homogenisation. For this reason, we will prove Theorem 7.1 below under the sole additional assumption (7.1), which is much weaker than invariance under translations (see Example 7.4). We shall see in Section 9 that condition (7.1) is satisfied almost surely under the standard hypotheses of stochastic homogenisation.

The following theorem is the main result of this section.

Theorem 7.1. *Let $F \in \mathfrak{F}_{\rm sc}^{\alpha,\infty}$ and let g be the function defined by* (5.3)*. Assume that there exist a function $f\colon \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ and a Borel set $N \subset \mathbb{R}^d$, σ-finite with respect to $\mathcal{H}^{d-1}$, such that*

$$f(A) = \lim_{\rho\to 0^+} \frac{\mathfrak{m}^F(\ell_A, Q(x,\rho))}{\rho^d} \quad \textit{for every } x \in \mathbb{R}^d \setminus N \textit{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}. \tag{7.1}$$

Then for every $U \in \mathcal{U}_c(\mathbb{R}^d)$ and $u \in {\rm BD}(U)$ we have

$$F(u,B) = \int_B f(\mathcal{E}u)\,{\rm d}x + \int_B f^\infty\Big(\frac{{\rm d}E^c u}{{\rm d}|E^c u|}\Big)\,{\rm d}|E^c u| + \int_{J_u\cap B} g(x,[u],\nu_u)\,{\rm d}\mathcal{H}^{d-1}$$

for every $B \in \mathcal{B}(U)$.

We first state a useful lemma, obtained by Caroccia, Focardi, and Van Goethem in [17, Lemma 5.3] (see [9, Lemma 3.7] for an analogous result in the BV-setting), which characterises the Radon-Nikodým derivative ${\rm d}F^c(u,\cdot)/{\rm d}|E^c u|$ by means of suitable minimum values of minimisation problems on small parallelograms. To state this result, we set some further notation.

We recall that De Philippis and Rindler proved in [32, Theorem 1.18] (see also the survey [33] and the book [47]) the following remarkable theorem: given an open set $U \in \mathcal{U}(\mathbb{R}^d)$ and $u \in {\rm BD}(U)$ there exist two Borel maps $a, b\colon U \to \mathbb{R}^d$ such that

$$\frac{{\rm d}E^c u}{{\rm d}|E^c u|} = a \odot b \quad \text{ and } \quad |a\odot b| = 1 \quad |E^c u|\text{-a.e. in } U. \tag{7.2}$$

For every $\lambda > 0$ and every pair $(a,b) \in \mathbb{R}^d \times \mathbb{R}^d$ with $a \neq \pm b$ we set

$$P_\lambda^{a,b} := \big\{z \in U : |z\cdot b| < \tfrac{\lambda}{2},\ |z\cdot a| < \tfrac12,\ |z\cdot\theta_i| < \tfrac12 \text{ for } i \in \{1,...,n-2\}\big\},$$

where $\{\theta_i\}_{i=1}^{n-2} \subset \mathbb{S}^{d-1}$ is such that $\{a, b, \theta_1, ..., \theta_{n-2}\}$ is a basis of $\mathbb{R}^d$, while if $a = \pm b$ we set

$$P_\lambda^{a,b} := \big\{z \in U : |z\cdot b| < \tfrac{\lambda}{2},\ |z\cdot\theta_i| < \tfrac12 \text{ for } i \in \{1,...,n-1\}\big\},$$

where $\{\theta_i\}_{i=1}^{n-1} \subset \mathbb{S}^{d-1}$ is such that $\{b, \theta_1, ..., \theta_{n-1}\}$ is a basis of $\mathbb{R}^d$. The choice of the vectors $\{\theta_i\}_i$ in the previous definitions is irrelevant for the arguments that follow. Given $x \in \mathbb{R}^d$ and $\rho > 0$, we also set $P_\lambda^{a,b}(x,\rho) := x + \rho P_\lambda^{a,b}$. Moreover, for every point $x \in U$ such that (7.2) holds, we set $P_\lambda^x(x,\rho) := P_\lambda^{a(x),b(x)}(x,\rho)$.

The following result characterises ${\rm d}F^c(u,\cdot)/{\rm d}|E^c u|$ in terms of the double limit of infima of problems related to parallelograms of the form P_λ^x.

Lemma 7.2 ([17, Lemma 5.3])**.** *Let $F \in \mathfrak{F}_{\rm sc}$, $U \in \mathcal{U}(\mathbb{R}^d)$, and $u \in {\rm BD}(U)$. Then there exists $C(u) \in \mathcal{B}(U)$, with $|E^c u|(U \setminus C(u)) = 0$, such that for every $x \in C(u)$ equality* (7.2) *holds and* *there exist a positive sequence* $\{\lambda_j\}_{j\in\mathbb{N}}$*, converging to* 0 *as* $j\to+\infty$*, and for every* $j\in\mathbb{N}$ *a positive sequence* $\{\rho_{i,j}\}_{i\in\mathbb{N}}$*, converging to* 0 *as* $i\to+\infty$*, such that, setting*

$$A=A(x):=a(x)\odot b(x)\quad\textit{and}\quad s_{i,j}:=\frac{|Eu|(P^x_{\lambda_j}(x,\rho_{i,j}))}{\mathcal{L}^d(P^x_j(x,\rho_{i,j}))},\tag{7.3}$$

we have

$$\textit{for every } j\in\mathbb{N} \textit{ the sequence } \{s_{i,j}\}_{i\in\mathbb{N}} \textit{ tends to } +\infty \textit{ as } i\to+\infty,\tag{7.4}$$

$$\frac{\mathrm{d}F^c(u,\cdot)}{\mathrm{d}|E^cu|}(x)=\lim_{j\to+\infty}\limsup_{i\to+\infty}\frac{\mathfrak{m}^F(s_{i,j}\ell_A,P^x_{\lambda_j}(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_{\lambda_j}(x,\rho_{i,j}))}.\tag{7.5}$$

Proof. This result is proved in [17, Lemma 5.3] under slightly different hypotheses on the functional F. In particular, they assume a condition, which they denote by (H4), that prescribes a joint continuity of the functional F with respect to translations both of the dependent and of the independent variables. Not all functionals in $\mathfrak{F}^{\alpha,\infty}$ satisfy this property. However, examining the proof of [17, Lemma 5.3] one can see that the use of property (H4) can be replaced by the property (e) of Definition 3.4. This invariance property also allows us to consider the boundary conditions appearing in (7.5) instead of those used in [17, (5.6)], which differ from ours by a rigid motion. ☐

The following result shows that, under suitable assumptions on F, in definition (5.2) we may replace the cube $Q(x,\rho)$ by parallelograms of the form $P^{a,b}_\lambda(x,\rho)$.

Lemma 7.3. *Let* $F\in\mathfrak{F}$*,* $A\in\mathbb{R}^{d\times d}_{\rm sym}$*,* $a,b\in\mathbb{R}^d$*, and* $\lambda>0$*. Assume that there exists* $\mu\geq 0$ *such that*

$$\mathfrak{m}^F(\ell_A,Q(y,\rho))\leq\mu\rho^d$$

for every $y\in\mathbb{R}^d$ *and* $\rho>0$*. Then*

$$\mathfrak{m}^F(\ell_A,P^{a,b}_\lambda(y,\rho))\leq\mu\mathcal{L}^d(P^{a,b}_\lambda(y,\rho))$$

for every $y\in\mathbb{R}^d$ *and* $\rho>0$*. If, in addition, for some* $x\in\mathbb{R}^d$ *we have*

$$\lim_{\rho\to0^+}\frac{\mathfrak{m}^F(\ell_A,Q(x,\rho))}{\rho^d}=\mu,$$

then

$$\lim_{\rho\to0^+}\frac{\mathfrak{m}^F(\ell_A,P^{a,b}_\lambda(x,\rho))}{\mathcal{L}^d(P^{a,b}_\lambda(x,\rho))}=\mu.$$

Proof. The proof can be obtained arguing as in [30, Lemma 5.3], observing that in the last part of that lemma limsup can be replaced by lim. We also remark that, in contrast with [30, Lemma 5.3], we deal with minimisation problems without constraints on the oscillation of the competitors, which simplifies the proof. ☐

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. By (7.1) we have

$$f(A)=\limsup_{\rho\to0^+}\frac{\mathfrak{m}^F(\ell_A,Q(x,\rho))}{\rho^d}\qquad\text{for }\mathcal{L}^d\text{-a.e. }x\in\mathbb{R}^d\text{ and every }A\in\mathbb{R}^{d\times d}_{\rm sym}.\tag{7.6}$$

Let us fix $U\in\mathcal{U}_c(\mathbb{R}^d)$ and $u\in BD(U)$. Thanks to (5.1), (7.6), and Proposition 5.3, it is enough to show that

$$F^c(u,B)=\int_B f^\infty\Big(\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\Big)\mathrm{d}|E^cu|$$

for every $B\in\mathcal{B}(U)$. To prove this we will show that

$$\frac{\mathrm{d}F^c(u,\cdot)}{\mathrm{d}|E^cu|}(x)=f^\infty\Big(\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}(x)\Big)\tag{7.7}$$

for $|E^c u|$-a.e. $x \in U$. Let $C(u) \subset U$ be the set of Lemma 7.2 and let us fix $x \in C(u) \setminus N$, where N is as in the statement.

Thanks to Lemma 7.2 there exist a positive sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ converging to 0 as $j \to +\infty$, and for every $j \in \mathbb{N}$ a positive sequence $\{\rho_{i,j}\}_{i\in\mathbb{N}}$, converging to 0 as $i \to +\infty$, such that (7.4) and (7.5) hold. For simplicity of notation, we set $P^x_j(x,\rho_{i,j}) := P^x_{\lambda_j}(x,\rho_{i,j})$. To obtain (7.7) is enough to show that

$$f^\infty\Big(\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(x)\Big) = \lim_{j\to+\infty}\limsup_{i\to+\infty}\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}, \tag{7.8}$$

where $A = A(x)$ and $s_{i,j}$ are defined by (7.3).

By (5.2), (5.4), and (7.1) we have

$$\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}) \le F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j})) = f(s_{i,j}A)\mathcal{L}^d(P^x_j(x,\rho_{i,j})),$$

so that by (7.4) we have

$$\limsup_{i\to+\infty}\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} \le \limsup_{i\to+\infty}\frac{f(s_{i,j}A)}{s_{i,j}} = f^\infty(A), \tag{7.9}$$

for every $j \in \mathbb{N}$.

Then, we observe that for every $y \in \mathbb{R}^d$, $\rho > 0$, and $t > 0$ by (5.2), (5.4), and (7.1) we have

$$\mathfrak{m}^F(t\ell_A, Q(y,\rho)) \le F(t\ell_A, Q(y,\rho)) = f(tA)\rho^d \quad \text{and} \quad \lim_{\rho\to0^+}\frac{\mathfrak{m}^F(t\ell_A, Q(x,\rho))}{\rho^d} = f(tA).$$

Hence, we can use Lemma 7.3 to deduce that

$$\lim_{\rho\to0^+}\frac{\mathfrak{m}^F(t\ell_A, P^x_\lambda(x,\rho))}{\mathcal{L}^d(P^x_\lambda(x,\rho))} = f(tA) \tag{7.10}$$

for every $\lambda, t > 0$. Recalling (6.1), from the previous equality it follows that

$$\lim_{t\to+\infty}\lim_{\rho\to0^+}\frac{\mathfrak{m}^F(t\ell_A, P^x_\lambda(x,\rho))}{t\mathcal{L}^d(P^x_\lambda(x,\rho))} = f^\infty(A). \tag{7.11}$$

Then, we apply Lemma 6.7 with $w = \ell_A$ and $U = P^x_j(x,\rho_{i,j})$ to get for every $t > 0$

$$\begin{aligned}\Big|\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} - \frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big| &\le \frac{c_6}{s_{i,j}}\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)^{1-\alpha}\\ &\quad+\frac{c_6}{s_{i,j}} + \frac{c_6}{t}\Big(\frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)^{1-\alpha} + \frac{c_6}{t}\\ &\quad+c_7\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A, P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))} + \frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)\Big(\frac{1}{s_{i,j}} + \frac{1}{t}\Big).\end{aligned} \tag{7.12}$$

We observe that for every $\tau > 0$ it follows from the upper bound in (c) of Definition 3.4 that

$$\mathfrak{m}^F(\tau\ell_A, P^x_j(x,\rho_{i,j})) \le F(\tau\ell_A,, P^x_j(x,\rho_{i,j})) \le (c_3\tau|A| + c_4)\mathcal{L}^d(P^x_j(x,\rho_{i,j})). \tag{7.13}$$

Let us fix $\varepsilon > 0$. Recalling (6.1), we may find $t > 0$ such that

$$\Big|\frac{f(tA)}{t} - f^\infty(A)\Big| < \varepsilon \quad \text{and} \quad \frac{c_6}{t} + \frac{c_6}{t^\alpha}(c_3|A| + \frac{c_4}{t})^{1-\alpha} + c_7\big(2c_3|A| + \frac{c_4}{t}\big)\frac{1}{t} < \varepsilon. \tag{7.14}$$

Thanks to (7.10), from the first inequality in (7.14) we deduce that for every $j \in \mathbb{N}$ we have

$$f^\infty(A) - \varepsilon < \frac{f(tA)}{t} = \lim_{i\to+\infty}\frac{\mathfrak{m}^F(t\ell_A, P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}.$$

Combining this inequality with (7.12), we get

$$\begin{aligned}f^\infty(A)-\varepsilon &\le \lim_{j\to+\infty}\lim_{i\to+\infty}\frac{\mathfrak{m}^F(t\ell_A,P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\le \limsup_{j\to+\infty}\limsup_{i\to+\infty}\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A,P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\\ &+\frac{c_6}{s_{i,j}}\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A,P^x_j(x,\rho_{i,j}))}{\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)^{1-\alpha}+\frac{c_6}{s_{i,j}}+\frac{c_6}{t}\Big(\frac{\mathfrak{m}^F(t\ell_A,P^x_j(x,\rho_{i,j}))}{\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)^{1-\alpha}+\frac{c_6}{t}\\ &+c_7\Big(\frac{\mathfrak{m}^F(s_{i,j}\ell_A,P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}+\frac{\mathfrak{m}^F(t\ell_A,P^x_j(x,\rho_{i,j}))}{t\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}\Big)\Big(\frac{1}{s_{i,j}}+\frac{1}{t}\Big)\Big).\end{aligned}\tag{7.15}$$

Using (7.13), we see that the right-hand side of the previous chain of inequalities can be bounded from above by

$$\begin{aligned}\limsup_{j\to+\infty}\limsup_{i\to+\infty}\Big(&\frac{\mathfrak{m}^F(s_{i,j}\ell_A,P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}+\frac{c_6}{s^\alpha_{i,j}}\Big(c_3|A|+\frac{c_4}{s_{i,j}}\Big)^{1-\alpha}+\frac{c_6}{s_{i,j}}+\frac{c_6}{t^\alpha}(c_3|A|+\frac{c_4}{t})^{1-\alpha}+\frac{c_6}{t}\\ &+c_7\Big(2c_3|A|+\frac{c_4}{s_{i,j}}+\frac{c_4}{t}\Big)\Big(\frac{1}{s_{i,j}}+\frac{1}{t}\Big)\Big).\end{aligned}\tag{7.16}$$

Recalling (7.9) and (7.4), from (7.14)-(7.16) we infer

$$f^\infty(A)-\varepsilon<\lim_{j\to+\infty}\limsup_{i\to+\infty}\frac{\mathfrak{m}^F(s_{i,j}\ell_A,P^x_j(x,\rho_{i,j}))}{s_{i,j}\mathcal{L}^d(P^x_j(x,\rho_{i,j}))}+\varepsilon\le f^\infty(A)+\varepsilon.$$

As $\varepsilon>0$ is arbitrary, we obtain (7.8). Since $x\in C(u)\backslash N$, $|E^cu|(U\backslash C(u))=0$, and $|E^cu|(N)=0$ (see part (ii) of (k) of Section 2), this concludes the proof. ☐

It is easy to produce examples of functionals $F\in\mathfrak{F}^{\alpha,\infty}_{\rm sc}$ which are not invariant under translation of the independent variable, but satisfy (7.1).

Example 7.4. Assume that $0<c_1<c_3$ and let f and g be the functions defined for every $A\in\mathbb{R}^{d\times d}_{\rm sym}$, $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^d$, and $\nu\in\mathbb{S}^{d-1}$ by

$$f(A):=c_3|A|\quad\text{and}\quad g(x,\zeta,\nu):=\begin{cases}c_3|\zeta\odot\nu| & \text{if } x\in\mathbb{R}^d\setminus\Pi^{e_d},\\ c_1|\zeta\odot\nu| & \text{if } x\in\Pi^{e_d},\end{cases}$$

where $e_d=(0,\dots,1)$ and $\Pi^{e_d}:=\{y\in\mathbb{R}^d: y\cdot e_d=0\}$. Clearly $f\in\mathcal{F}$ and $g\in\mathcal{G}$, and since they are positively homogeneous of degree one we also have $f\in\mathcal{F}^\alpha$ and $g\in\mathcal{G}^\infty$. Hence, the functional $F:=F^{f,g}$ belongs to $\mathfrak{F}^{\alpha,\infty}$ by Proposition 6.5. Let $\psi\colon\mathbb{R}^d\to\mathbb{R}$ be the function defined by

$$\psi(x):=\begin{cases}c_3 & \text{if } x\in\mathbb{R}^d\setminus\Pi^{e_d},\\ c_1 & \text{if } x\in\Pi^{e_d}.\end{cases}$$

We observe that for every $U\in\mathcal{U}_c(\mathbb{R}^d)$ and $u\in{\rm BD}(U)$ we have

$$F(u,U)=\int_U\psi\,{\rm d}|Eu|.$$

Since ψ is lower semicontinuous in $\mathbb{R}^d$, this equality shows that $F(\cdot,U)$ is $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$-lower semicontinuous, hence $F\in\mathfrak{F}^{\alpha,\infty}_{\rm sc}$.

8. CHARACTERISATION OF Γ-CONVERGENCE USING MINIMA ON SMALL CUBES

In this section we determine the integrands of the Γ-limit of a sequence $\{F_n\}_{n\in\mathbb{N}}\subset\mathfrak{F}^{\alpha,\infty}$ by means of limits of the minimum values of suitable minimisation problems for F_n on small cubes.

The next lemma shows that the Γ-convergence of a sequence $\{F_n\}_{n\in\mathbb{N}}\subset\mathfrak{F}$ to a limit functional F allows us to compare $\mathfrak{m}^F$ with $\{\mathfrak{m}^{F_n}\}_{n\in\mathbb{N}}$.

Lemma 8.1. *Let* $\{F_n\}_{n\in\mathbb{N}} \subset \mathfrak{F}$. *Assume that there exists* $F \in \mathfrak{F}_{\rm sc}$ *such that for every* $U \in \mathcal{U}_c(\mathbb{R}^d)$ *the sequence* $\{F_n(\cdot,U)\}_{n\in\mathbb{N}}$ Γ*-converges to* $F(\cdot,U)$ *with respect to the topology of* $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$. *Let* $U,W \in \mathcal{U}_c(\mathbb{R}^d)$ *with Lipschitz boundary and* $W \subset\subset U$, *and let* $w \in {\rm BD}(U)$. *Then*

$$\mathfrak{m}^F(w,U) \le \liminf_{n\to+\infty} \mathfrak{m}^{F_n}(w,W) + c_3|Ew|(U\setminus W) + c_4\mathcal{L}^d(U\setminus W), \tag{8.1}$$

$$\limsup_{n\to+\infty} \mathfrak{m}^{F_n}(w,U) \le \mathfrak{m}^F(w,U). \tag{8.2}$$

Proof. We consider a subsequence, not relabelled, such that the liminf in the right-hand side of (8.1) is actually a limit. To prove (8.1) we fix $\delta > 0$ and for every $n \in \mathbb{N}$ we select $z_n \in {\rm BD}(W)$ such that $u_n = w$ on ∂W and

$$F_n(z_n,W) < \mathfrak{m}^{F_n}(w,W) + \delta. \tag{8.3}$$

We observe that by the upper bound in (c) of Definition 3.4, we have

$$F_n(z_n,W) < \mathfrak{m}^{F_n}(w,W) + \eta \le F_n(w,W) + \delta \le c_3|Ew|(W) + c_4\mathcal{L}^d(W) + \delta,$$

so that by the lower bound in (c) we also get that $|Ez_n|(W)$ is bounded uniformly with respect to n. We extend z_n to U by setting $z_n = w$ on $U\setminus W$ and observe that $|Ez_n|(U)$ is uniformly bounded in n as well. Hence, there exist a subsequence, not relabelled, and a function $z \in {\rm BD}(U)$ such that $z_n \to z$ in $L^1(U;\mathbb{R}^d)$ and $z = w$ on $U\setminus W$, so that

$$\mathfrak{m}^F(w,U) \le F(z,U) \le \liminf_{n\to+\infty} F_n(z_n,U), \tag{8.4}$$

where the last inequality is due to Γ-convergence. Since the inner trace of z_n on ∂W equals the inner trace of w, we have $|Ez_n|(\partial W) = |Ew|(\partial W)$. Hence, the upper bound in (c) of Definition 3.4 gives

$$F_n(z_n,U\setminus W) \le c_3|Ez_n|(U\setminus W) + c_4\mathcal{L}^d(U\setminus W) = c_3|Ew|(U\setminus W) + c_4\mathcal{L}^d(U\setminus W),$$

where the last equality is due to the fact that $z_n = w$ in the open set $U\setminus\overline{W}$ and $|Ez_n|(\partial W) = |Ew|(\partial W)$. Combining this inequality with (8.3) and (8.4), and letting $\delta\to 0^+$ we obtain (8.1).

To prove (8.2), let us fix $\delta > 0$ and let $u \in {\rm BD}(U)$ be such that $u = w$ on ∂U and $F(u,U) < \mathfrak{m}^F(w,U) + \delta$. By Γ-convergence there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset {\rm BD}(U)$ converging to u strongly in $L^1(U;\mathbb{R}^d)$ and such that

$$\lim_{n\to+\infty} F_n(u_n,U) = F(u,U) < \mathfrak{m}^F(w,U) + \delta. \tag{8.5}$$

We now fix a compact set $K \subset U$ such that

$$c_3|Eu|(U\setminus K) + c_4\mathcal{L}^d(U\setminus K) < \delta. \tag{8.6}$$

We also consider two additional open sets U'' and U' with the property that $K \subset U'' \subset\subset U' \subset\subset U$. We now argue as in the part of the proof of Lemma 4.2 that starts from (4.11), with $V = U\setminus U''$, v_n replaced by u, and for every $n \in \mathbb{N}$ we construct a function $z_n \in {\rm BD}(U)$ such that $z_n = u$ in a neighbourhood of ∂U and

$$F_n(z_n,U) \le \ F_n(u_n,U) + F_n(u,U\setminus U'') + \frac{M}{m} + \frac{2c_3}{\eta}\|u_n - u\|_{L^1(U;\mathbb{R}^d)},$$

where $M \ge 0$ is the constant introduced in (4.9) and $\eta := {\rm dist}(U,\partial U')$. Since $z_n = u = w$ on ∂U, we have $\mathfrak{m}^{F_n}(w,U) \le F_n(z_n,U)$. Hence, using the upper bounds in property (c) of Definition 3.4 for F_n, (8.6), and the inclusion $K \subset U''$, from the previous displayed formula we obtain

$$\mathfrak{m}^{F_n}(w,U) \le F_n(u_n,U) + \delta + \frac{M}{m} + \frac{2c_3}{\eta}\|u_n - u\|_{L^1(U;\mathbb{R}^d)}.$$

We now pass to the limsup as $n\to+\infty$ and, recalling that $u_n \to u$ in $L^1(U;\mathbb{R}^d)$ as $n\to+\infty$, from (8.5) we get

$$\limsup_{n\to+\infty} \mathfrak{m}^{F_n}(w,U) \le \mathfrak{m}^F(w,U) + 2\delta + \frac{M}{m}.$$

Taking the limit as $m\to+\infty$ and $\delta\to 0^+$ we obtain (8.2). □

The next result shows that for a functional $F \in \mathfrak{F}_{\rm sc}$ arising as Γ-limit of a sequence of functionals in $\mathfrak{F}$ the integrands f and g of the bulk and surface parts can be obtained using limits of the minimum values of the minimisation problems $\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))$ and $\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))$ for the functionals F_n.

Lemma 8.2. *Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathfrak{F}$. Assume that there exists $F \in \mathfrak{F}_{\rm sc}$ such that for every $U \in \mathcal{U}_c(\mathbb{R}^d)$ the sequence $\{F_n(\cdot,U)\}_{n\in\mathbb{N}}$ Γ-converges to $F(\cdot,U)$ with respect to the topology of $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, and let f and g be the functions associated to F by* (5.2) *and* (5.3). *Then for every $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ we have*

$$f(x,A) = \limsup_{\rho\to 0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d} = \limsup_{\rho\to 0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d},$$

$$g(x,\zeta,\nu) = \limsup_{\rho\to 0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} = \limsup_{\rho\to 0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}}.$$

Proof. The proof can be obtained arguing exactly as in [30, Lemma 3.3], replacing D with E, ∇ with $\mathcal{E}$, and Propositions 3.1 and 3.2 of that paper with our Lemma 8.1. □

We conclude this section by proving a sufficient condition for the Γ-convergence of a sequence F_n, based on limits of minimum values of minimisation problems for F_n on small cubes. Note that we require that the limits corresponding to the volume integrand do not depend on x.

Theorem 8.3. *Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathfrak{F}^{\alpha,\infty}$. Assume that there exist $\hat f\colon \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ and $\hat g\colon \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that*

$$\hat f(A) = \lim_{\rho\to 0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d} = \lim_{\rho\to 0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d}, \tag{8.7}$$

$$\hat g(x,\zeta,\nu) = \limsup_{\rho\to 0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} = \limsup_{\rho\to 0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}}, \tag{8.8}$$

for every $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$. Then $\hat f \in \mathcal{F}^\alpha$, $\hat g \in \mathcal{G}^\infty$, and for every $U \in \mathcal{U}_c(\mathbb{R}^d)$ we have that $\{F_n(\cdot,U)\}_{n\in\mathbb{N}}$ Γ-converges to $F^{\hat f,\hat g}(\cdot,U)$ with respect to the topology of $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, where $F^{\hat f,\hat g}$ is the functional introduced in Definition 3.3.

Proof. By Theorem 4.1 there exists a subsequence, not relabelled, and a functional $F \in \mathfrak{F}_{\rm sc}$ such that $\{F_n(\cdot,U)\}_{n\in\mathbb{N}}$ Γ-converges ot $F(\cdot,U)$ for every $U \in \mathcal{U}_c(\mathbb{R}^d)$. Let f and g be the functions defined by (5.2) and (5.3) corresponding to F. Using Lemmas 8.1 and 8.2, together with (8.7) and (8.8), we see that $f(x,A) = \hat f(A)$ and $g(x,\zeta,\nu) = \hat g(x,\zeta,\nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}_{\rm sym}$, and $\nu \in \mathbb{S}^{d-1}$. Moreover, thanks to Lemma 8.1 we obtain that

$$\hat f(A) = \lim_{\rho\to 0^+}\limsup_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho))}{\rho^d} \le \liminf_{\rho\to 0^+}\frac{\mathfrak{m}^{F}(\ell_A, Q(x,\rho))}{\rho^d},$$

$$\hat f(A) = \lim_{\rho\to 0^+}\liminf_{n\to+\infty}\frac{\mathfrak{m}^{F_n}(\ell_A, Q(x,\rho-\rho^2))}{(\rho-\rho^2)^d} \ge \limsup_{\rho\to 0^+}\frac{\mathfrak{m}^{F}(\ell_A, Q(x,\rho))}{\rho^d},$$

hence

$$\hat f(A) = \lim_{\rho\to 0^+}\frac{\mathfrak{m}^{F}(\ell_A, Q(x,\rho))}{\rho^d} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}.$$

Thus, the functional F satisfies the hypotheses of Theorem 7.1, so that $F = F^{\hat f,\hat g}$.

Moreover, by (8.7) and (8.8) the functions $\hat f$ and $\hat g$, and hence the functional $F^{\hat f,\hat g}$, do not depend on the subsequence chosen at the beginning of the proof, so that by the Urysohn property of Γ-convergence (see [25, Proposition 8.3]) for every $U \in \mathcal{U}_c(\mathbb{R}^d)$ the whole sequence $\{F_n(\cdot,U)\}_{n\in\mathbb{N}}$ Γ-converges to $F^{\hat f,\hat g}(\cdot,U)$. □

9. FUNCTIONALS OBTAINED BY RESCALING

In this section we use the theory developed in the previous sections to deal with the problem of Γ-convergence of oscillating functionals obtained by rescaling of a single functional. In particular, we prove a general theorem which provides sufficient conditions for the Γ-convergence of these functionals (see Theorem 9.7).

We now introduce the notation we will use in the rest of the work. Throughout this section we keep fixed $f \in \mathcal{F}^\alpha$ and $g \in \mathcal{G}^\infty$ and set

$$F := F^{f,g} \in \mathfrak{F}^{\alpha,\infty},$$

where $F^{f,g}$ is the functional introduced in Definition 3.3. We also assume that the modulus of continuity introduced in (3.1) satisfies

$$\sigma(\tau) = \sigma_1 \tau \quad \text{for every } \tau \geq 0, \tag{9.1}$$

for some constant $\sigma_1 > 0$. With this hypothesis, condition (g4) of Definition 3.1 reads

$$|g(x,\zeta_1,\nu) - g(x,\zeta_2,\nu)| \leq \sigma_1 |\zeta_1 - \zeta_2|,$$

while the surface continuity estimate (f) of Definition 3.4 becomes

$$F(u + u_{x,\zeta,\nu}, B) \leq F(u,B) + \sigma_1 |\zeta| \mathcal{H}^{d-1}(\Pi_x^\nu \cap B).$$

Definition 9.1. For every $\varepsilon > 0$ we consider the integrands $f_\varepsilon \in \mathcal{F}^\alpha$ and $g_\varepsilon \in \mathcal{G}^\infty$ defined by

$$f_\varepsilon(x,A) := f\big(\tfrac{x}{\varepsilon}, A\big) \qquad \text{for } x \in \mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \tag{9.2}$$

$$g_\varepsilon(x,\zeta,\nu) := \varepsilon g\big(\tfrac{x}{\varepsilon}, \tfrac{\zeta}{\varepsilon}, \nu\big) \qquad \text{for } x \in \mathbb{R}^d, \zeta \in \mathbb{R}^d, \text{ and } \nu \in \mathbb{S}^{d-1}, \tag{9.3}$$

and we set $F_\varepsilon := F^{f_\varepsilon, g_\varepsilon} \in \mathfrak{F}^{\alpha,\infty}$.

Remark 9.2. If for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$ the function $t \mapsto g(x, t\zeta, \nu)$ is positively homogeneous of degree one, that is to say

$$g(x,t\zeta,\nu) = t g(x,\zeta,\nu) \quad \text{for every } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d,\ \nu \in \mathbb{S}^{d-1}, \text{ and } t \geq 0, \tag{9.4}$$

then

$$g_\varepsilon(x,\zeta,\nu) = g\big(\tfrac{x}{\varepsilon}, \zeta, \nu\big) \quad \text{for every } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d,\ \nu \in \mathbb{S}^{d-1}, \text{ and } \varepsilon > 0.$$

In this case, for $U \in \mathcal{U}_c(\mathbb{R}^d)$ and $u \in \mathrm{BD}(U)$ the functionals $F_\varepsilon(u,U)$ of Definition 9.1 become

$$\int_U f\Big(\frac{x}{\varepsilon}, \mathcal{E}u\Big)\,\mathrm{d}x + \int_U f^\infty\Big(\frac{x}{\varepsilon}, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| + \int_{J_u \cap U} g\Big(\frac{x}{\varepsilon}, [u], \nu_u\Big)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{9.5}$$

which are the functionals commonly considered in homogenisation of free discontinuity problems.

Our choice in the definition of g_ε given by (9.3) is justified by the fact that the corresponding functional F_ε defined by $F^{f_\varepsilon,g_\varepsilon}$ satisfies good change of variables formulas (see Lemmas 9.4 and 9.6 below) even when g does not satisfy (9.4). This will allow us to prove a very general Γ-convergence result for $F^{f_\varepsilon,g_\varepsilon}$ (see Theorem 9.7), which implies the Γ-convergence of the functionals in (9.5) when g satisfies the additional condition (9.4). Unfortunately, in the BD case we are not able to extend the truncation arguments that were crucial to study the analogue of (9.5) in BV (see [16, 26, 30]).

In the rest of the section we need the following technical result about a change of variables formula involving the Cantor part of a BD function.

Lemma 9.3. *Let $\varepsilon, \rho \in (0,1)$, $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, $u \in \mathrm{BD}(Q_\nu(x,\rho))$, and $v \in \mathrm{BD}(Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon}))$. Assume that $v(z) = \frac{1}{\varepsilon} u(\varepsilon z)$ for every $z \in Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$. Then*

$$\int_{Q_\nu(x,\rho)} f_\varepsilon^\infty\Big(y, \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| = \varepsilon^d \int_{Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})} f^\infty\Big(z, \frac{\mathrm{d}E^c v}{\mathrm{d}|E^c v|}\Big)\,\mathrm{d}|E^c v|. \tag{9.6}$$

Proof. Let $\psi \in C_c^\infty(Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon});\mathbb{R}^{d\times d}_{\rm sym})$ and let $\psi_\varepsilon := \psi\big(\frac{\cdot}{\varepsilon}\big) \in C_c^\infty(Q_\nu(x,\rho);\mathbb{R}^{d\times d}_{\rm sym})$. By change of variables and integration by parts we have

$$\begin{aligned}
-\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \psi \,\mathrm{d}Ev &= \int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} v \operatorname{div}\psi \,\mathrm{d}z = \int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \tfrac{1}{\varepsilon}u(\varepsilon z)\operatorname{div}\psi\,\mathrm{d}z \\
&= \frac{1}{\varepsilon^d}\int_{Q_\nu(x,\rho)} \frac{u(y)}{\varepsilon}\operatorname{div}\psi\big(\frac{y}{\varepsilon}\big)\,\mathrm{d}y = \frac{1}{\varepsilon^d}\int_{Q_\nu(x,\rho)} u\operatorname{div}\psi_\varepsilon\,\mathrm{d}y \\
&= -\frac{1}{\varepsilon^d}\int_{Q_\nu(x,\rho)} \psi_\varepsilon\,\mathrm{d}Eu = -\frac{1}{\varepsilon^d}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \psi\,\mathrm{d}\Big(\big(\frac{\cdot}{\varepsilon}\big)_{\#}Eu\Big),
\end{aligned}$$

where $\big(\frac{\cdot}{\varepsilon}\big)_{\#}Eu$ denotes the push-forward of Eu via the function $y \mapsto \frac{y}{\varepsilon}$ (see, for instance, [8, Theorem 3.6.1]). This implies that

$$Ev = \frac{1}{\varepsilon^d}\Big(\frac{\cdot}{\varepsilon}\Big)_{\#}Eu \quad \text{as Borel measures on } Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})$$

and, passing to their singular parts,

$$E^s v = \frac{1}{\varepsilon^d}\Big(\frac{\cdot}{\varepsilon}\Big)_{\#}E^s u \quad \text{as Borel measures on } Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}),$$

where $E^s v$ and $E^s u$ denote the singular part of Ev and Eu with respect to the Lebesgue measure. Restricting the previous equality to $Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})\setminus J_v = Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})\setminus(\frac{1}{\varepsilon}J_u)$, we deduce that

$$E^c v = \frac{1}{\varepsilon^d}\Big(\frac{\cdot}{\varepsilon}\Big)_{\#}E^c u \quad \text{as Borel measures on } Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}).$$

This implies that

$$\frac{\mathrm{d}E^c v}{\mathrm{d}|E^c v|}\big(\frac{y}{\varepsilon}\big) = \frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}(y) \quad \text{for } |E^c u|\text{-a.e. } y \in Q_\nu(x,\rho).$$

Hence, by the integration formula for the push-forward of measures (see, for instance, [8, Theorem 3.6.1]) we get (9.6). □

We state two preliminary results that allow us to rewrite the minimisation problems on small cubes for the functionals F_ε by means of minimisation problems on large cubes for F and F^{f^∞,g^∞}, where f^∞ and g^∞ are the functions defined by (6.1) and (6.4), respectively.

Lemma 9.4. *Let $\varepsilon,\rho \in (0,1)$, $x \in \mathbb{R}^d$, and $A \in \mathbb{R}^{d\times d}_{\rm sym}$. Then*

$$\mathfrak{m}^{F_\varepsilon}(\ell_A, Q(x,\rho)) = \varepsilon^d \mathfrak{m}^{F^{f,g}}(\ell_A, Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})).$$

Proof. Let $u \in \mathrm{BD}(Q(x,\rho))$, with $u = \ell_A$ on $\partial Q(x,\rho)$, and let $v(z) := \frac{1}{\varepsilon}u(\varepsilon z)$ for $z \in Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$. It is easy to see that $v = \ell_A$ on $\partial Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$. Thanks to (9.2) and (9.3), a change of variables and Lemma 9.3 show that

$$\begin{aligned}
\int_{Q(x,\rho)} f_\varepsilon(y,\mathcal{E}u)\,\mathrm{d}y &= \varepsilon^d\int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f(z,\mathcal{E}v)\,\mathrm{d}z, \\
\int_{Q(x,\rho)} f_\varepsilon^\infty\Big(y,\frac{\mathrm{d}E^c u}{\mathrm{d}|E^c u|}\Big)\,\mathrm{d}|E^c u| &= \varepsilon^d\int_{Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} f^\infty\Big(z,\frac{\mathrm{d}E^c v}{\mathrm{d}|E^c v|}\Big)\,\mathrm{d}|E^c v|, \\
\int_{J_u\cap Q(x,\rho)} g_\varepsilon(y,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1} &= \varepsilon^d\int_{J_v\cap Q(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} g(z,[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

Thus, we infer

$$\mathfrak{m}^{F_\varepsilon}(\ell_A, Q(x,\rho)) = \varepsilon^d\mathfrak{m}^{F}(\ell_A, Q(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})),$$

concluding the proof. □

Definition 9.5. We set

$$F^\infty := F^{f^\infty,g^\infty} \in \mathfrak{F}^{\alpha,\infty},$$

where f^∞ and g^∞ are the functions defined by (6.1) and (6.4).

Lemma 9.6. *Let $\varepsilon\in(0,1/(2c_6))$, $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^d$, $\nu\in\mathbb{S}^{d-1}$, and $\rho\in(0,1)$. Then*

$$\big|\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho))-\varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}))\big|\le C\rho^{d-1+\alpha}+C\varepsilon\rho^{d-1},\tag{9.7}$$

where $C=C_\zeta>0$ is a constant depending only on $|\zeta|$ and on the structural constants α,c_3, c_6, and c_7, but is independent of ε, x, ν, and ρ.

Proof. Let $u\in\mathrm{BD}(Q_\nu(x,\rho))$ with $u=u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$ and let $v(z):=u(\varepsilon z)$ for $z\in Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$, so that $v=u_{\frac{x}{\varepsilon},\zeta,\nu}$ on $\partial Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$. Recalling (9.2) and (9.3), a change of variables and Lemma 9.3 imply that

$$\int_{Q_\nu(x,\rho)}f_\varepsilon(y,\mathcal{E}u)\,\mathrm{d}y=\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}\varepsilon f\Big(z,\frac{1}{\varepsilon}\mathcal{E}v\Big)\,\mathrm{d}z,\tag{9.8}$$

$$\int_{Q_\nu(x,\rho)}f_\varepsilon^\infty\Big(y,\frac{\mathrm{d}E^cu}{\mathrm{d}|E^cu|}\Big)\,\mathrm{d}|E^cu|=\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}f^\infty\Big(z,\frac{\mathrm{d}E^cv}{\mathrm{d}|E^cv|}\Big)\,\mathrm{d}|E^cv|,\tag{9.9}$$

$$\int_{J_u\cap Q_\nu(x,\rho)}g_\varepsilon(y,[u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}=\varepsilon^{d-1}\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}\varepsilon g\Big(z,\frac{1}{\varepsilon}[v],\nu_v\Big)\,\mathrm{d}\mathcal{H}^{d-1}.\tag{9.10}$$

We can now exploit (6.2) of Remark 6.2 to obtain for $\mathcal{L}^d$-a.e. $z\in Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$

$$|\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)-f^\infty(z,\mathcal{E}v)|\le c_6\varepsilon+c_6\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)^{1-\alpha},\tag{9.11}$$

while using (6.7) of Remark 6.3 and inequalities (g3) and (6.5), for $\mathcal{H}^{d-1}$-a.e. $z\in J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$ we get

$$|\varepsilon g(z,\tfrac{1}{\varepsilon}[v],\nu_v)-g^\infty(z,[v],\nu_v)|\le 2c_3c_7\varepsilon|[v]\odot\nu_v|\le C_1\varepsilon^2 g(z,\tfrac{1}{\varepsilon}[v],\nu_v),\tag{9.12}$$

$$|\varepsilon g(z,\tfrac{1}{\varepsilon}[v],\nu_v)-g^\infty(z,[v],\nu_v)|\le 2c_3c_7\varepsilon|[v]\odot\nu_v|\le C_1\varepsilon g^\infty(z,[v],\nu_v),\tag{9.13}$$

where we have set $C_1:=(2c_3c_7)/c_1$. From (9.11) we then deduce for $\mathcal{L}^d$-a.e. $z\in Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$ that

$$f^\infty(z,\mathcal{E}v)\le\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)+c_6\varepsilon+c_6\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)^{1-\alpha},\tag{9.14}$$

$$\begin{aligned}f^\infty(z,\mathcal{E}v)&\ge\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)-c_6\varepsilon-c_6\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)^{1-\alpha}\\&\ge\tfrac{\varepsilon}{2}f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)-c_6\varepsilon-c_6\varepsilon(2c_6(1-\alpha))^{\frac{1-\alpha}{\alpha}},\end{aligned}\tag{9.15}$$

where we have used the inequality $\tau^{1-\alpha}\le\frac{1}{2c_6}\tau+(2c_6(1-\alpha))^{\frac{1-\alpha}{\alpha}}$ for every $\tau\ge0$ and $\varepsilon\in(0,1/(2c_6))$, while from (9.12) and (9.13) we obtain for $\mathcal{H}^{d-1}$-a.e. $z\in J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$

$$g^\infty(z,[v],\nu_v)\le\varepsilon g(z,\tfrac{1}{\varepsilon}[v],\nu_v)+C_1\varepsilon^2 g(z,\tfrac{1}{\varepsilon}[v],\nu_v),\tag{9.16}$$

$$g^\infty(z,[v],\nu_v)\ge\varepsilon g(z,\tfrac{1}{\varepsilon}[v],\nu_v)-C_1\varepsilon g^\infty(z,[v],\nu_v).\tag{9.17}$$

Note that (9.15) implies that

$$\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)\le 2f^\infty(z,\mathcal{E}v)+C_2\varepsilon,$$

where $C_2:=2c_6+2c_6(2c_6(1-\alpha))^{\frac{1-\alpha}{\alpha}}$, so that from (9.11) we infer

$$\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)\le f^\infty(z,\mathcal{E}v)+c_6\varepsilon+c_6\varepsilon^\alpha(2f^\infty(z,\mathcal{E}v)+C_2\varepsilon)^{1-\alpha}.\tag{9.18}$$

Integrating (9.14) on $Q_\nu\big(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}\big)$, by Hölder's inequality we get

$$\begin{aligned}\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}f^\infty(z,\mathcal{E}v)\,\mathrm{d}z&\le\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)\,\mathrm{d}z+c_6\rho^d\\&\quad+c_6\Big(\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)\,\mathrm{d}z\Big)^{1-\alpha}\rho^{\alpha d},\end{aligned}\tag{9.19}$$

while from (9.18) we get

$$\begin{aligned}\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}\varepsilon f(z,\tfrac{1}{\varepsilon}\mathcal{E}v)\,\mathrm{d}z&\le\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}f^\infty(z,\mathcal{E}v)\,\mathrm{d}z+c_6\rho^d\\&\quad+c_6\Big(2\varepsilon^{d-1}\int_{Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})}f^\infty(z,\mathcal{E}v)\,\mathrm{d}z+C_2\rho^d\Big)^{1-\alpha}\rho^{\alpha d}.\end{aligned}\tag{9.20}$$

Integrating (9.16) and (9.17) on $J_v \cap Q_\nu(\frac{x}{\varepsilon}, \frac{\rho}{\varepsilon})$ we obtain

$$\begin{aligned}\varepsilon^{d-1}\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} g^\infty(z,[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1} \le \varepsilon^{d-1}&\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \varepsilon g(z,\tfrac{1}{\varepsilon}[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1}\\ +\varepsilon^d C_1&\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \varepsilon g(z,\tfrac{1}{\varepsilon}[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1},\end{aligned}\tag{9.21}$$

and

$$\begin{aligned}\varepsilon^{d-1}\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} g^\infty(z,[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1} \ge \varepsilon^{d-1}&\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} \varepsilon g(z,\tfrac{1}{\varepsilon}[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1}\\ -\varepsilon^d C_1&\int_{J_v\cap Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})} g^\infty(z,[v],\nu_v)\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}\tag{9.22}$$

Recalling (9.8)-(9.10) and combining (9.19) and (9.21), we obtain

$$\begin{aligned}\varepsilon^{d-1}F^\infty(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) &\le F_\varepsilon(u,Q_\nu(x,\rho)) + c_6\rho^d\\ &+c_6F_\varepsilon(u,Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d} + C_1\varepsilon F_\varepsilon(u,Q_\nu(x,\rho)),\end{aligned}$$

while from (9.20) and (9.22) we get

$$\begin{aligned}F_\varepsilon(u,Q_\nu(x,\rho)) &\le \varepsilon^{d-1}F^\infty(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + c_6\rho^d\\ +c_6\Big(2\varepsilon^{d-1}F^\infty(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}))+C_2\rho^d\Big)^{1-\alpha}&\rho^{\alpha d} + C_1\varepsilon^dF^\infty(v,Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})).\end{aligned}$$

Since $u = u_{x,\zeta,\nu}$ on $\partial Q_\nu(x,\rho)$ if and only $v = u_{\frac{x}{\varepsilon},\zeta,\nu}$ on $\partial Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon})$, the last two inequalities imply

$$\begin{aligned}\varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) &\le \mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) + c_6\rho^d\\ +c_6\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}&\rho^{\alpha d} + C_1\varepsilon\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)),\end{aligned}\tag{9.23}$$

and

$$\begin{aligned}\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) &\le \varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + c_6\rho^d\\ +c_6\Big(2\varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon}))+C_2\rho^d\Big)^{1-\alpha}&\rho^{\alpha d} + C_1\varepsilon^d\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})).\end{aligned}\tag{9.24}$$

To conclude, as $u_{x,\zeta,\nu}$ and $u_{\frac{x}{\varepsilon},\zeta,\nu}$ are competitors for the problems $\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho))$ and $m^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\frac{x}{\varepsilon},\frac{\rho}{\varepsilon}))$, respectively, using the upper bounds in (g3) and (6.5) we obtain

$$\begin{aligned}\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) &\le c_3|\zeta\odot\nu|\rho^{d-1},\\ \mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) &\le c_3|\zeta\odot\nu|\frac{\rho^{d-1}}{\varepsilon^{d-1}}.\end{aligned}$$

Finally, combining these two inequalities with (9.23) and (9.24) we get

$$\begin{aligned}\varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) &\le \mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) + c_6\rho^d\\ &+c_6c_3|\zeta\odot\nu|\rho^{d-1+\alpha} + C_1c_3|\zeta\odot\nu|\varepsilon\rho^{d-1},\end{aligned}$$

and

$$\begin{aligned}\mathfrak{m}^{F_\varepsilon}(u_{x,\zeta,\nu},Q_\nu(x,\rho)) &\le \varepsilon^{d-1}\mathfrak{m}^{F^\infty}(u_{\frac{x}{\varepsilon},\zeta,\nu},Q_\nu(\tfrac{x}{\varepsilon},\tfrac{\rho}{\varepsilon})) + c_6\rho^d\\ &+c_6\big(2c_3|\zeta\odot\nu|+C_2\big)^{1-\alpha}\rho^{d-1+\alpha} + C_1c_3|\zeta\odot\nu|\varepsilon\rho^{d-1},\end{aligned}$$

which imply (9.7) for $C := \max\{c_6, c_6c_3|\zeta|, C_1c_3|\zeta|, c_6\big(2c_3|\zeta|+C_2\big)^{1-\alpha}\}$. □

The following theorem constitutes the main result of this section. We shall see in the next section that its hypotheses are satisfied under the standard hypotheses of periodic or stochastic homogenisation.

Theorem 9.7. *Let $f \in \mathcal{F}^\alpha$, $g \in \mathcal{G}^\infty$, and let $F_\varepsilon := F^{f_\varepsilon, g_\varepsilon}$ be the functionals introduced in Definition 9.1. Assume that there exist functions $f_{\rm lim}\colon \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ and $g_{\rm lim}\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that*

$$f_{\rm lim}(A) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f,g}}(\ell_A, Q(rx,r))}{r^d} \quad \text{for all } x\in\mathbb{R}^d \text{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}, \tag{9.25}$$

$$g_{\rm lim}(\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \quad \text{for all } x\in\mathbb{R}^d,\ \zeta\in\mathbb{R}^d, \text{ and } \nu\in\mathbb{S}^{d-1}. \tag{9.26}$$

Then $f_{\rm lim} \in \mathcal{F}^\alpha$, $g_{\rm lim} \in \mathcal{G}^\infty$, $g_{\rm lim}$ satisfies (9.4), and the following property holds: for every positive sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to 0 as $n\to+\infty$ and for every $U\in\mathcal{U}_c(\mathbb{R}^d)$ the sequence $\{F_{\varepsilon_n}(\cdot,U)\}_{n\in\mathbb{N}}$ Γ-converges to $F^{f_{\rm lim},g_{\rm lim}}(\cdot,U)$ with respect to the topology of $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$. Moreover, we have the equality

$$g_{\rm lim}(\zeta,\nu) = f^\infty_{\rm lim}(\zeta\odot\nu) \quad \text{for all } \zeta\in\mathbb{R}^d \text{ and } \nu\in\mathbb{S}^{d-1}, \tag{9.27}$$

which implies that for every $U\in\mathcal{U}(\mathbb{R}^d)$, $u\in {\rm BD}(U)$, and $B\in\mathcal{B}(U)$ we have

$$F^{f_{\rm lim},g_{\rm lim}}(u,B) = \int_B f_{\rm lim}(\mathcal{E}u)\,{\rm d}x + \int_B f^\infty_{\rm lim}\Big(\frac{{\rm d}E^s u}{{\rm d}|E^s u|}\Big)\,{\rm d}|E^s u|, \tag{9.28}$$

where $E^s u$ denotes the singular part of Eu with respect to the Lebesgue measure.

Proof. Let us fix a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ converging to 0 as $n\to+\infty$. Since $\{F_{\varepsilon_n}\}_{n\in\mathbb{N}} \subset \mathfrak{F}^{\alpha,\infty}$, to prove the Γ-convergence part of the statement it is enough to check that the hypotheses of Theorem 8.3 are satisfied. Using Lemma 9.4 we see that for every $\rho\in(0,1)$, $n\in\mathbb{N}$, $x\in\mathbb{R}^d$, and $A\in\mathbb{R}^{d\times d}_{\rm sym}$ we have

$$\frac{1}{\rho^d}\mathfrak{m}^{F_{\varepsilon_n}}(\ell_A, Q(x,\rho)) = \frac{\varepsilon_n^d}{\rho^d}\mathfrak{m}^F(\ell_A, Q(\tfrac{x}{\varepsilon_n}, \tfrac{\rho}{\varepsilon_n})),$$

so that, setting $r_n := \rho/\varepsilon_n$, we may rewrite the previous equality as

$$\frac{1}{\rho^d}\mathfrak{m}^{F_{\varepsilon_n}}(\ell_A, Q(x,\rho)) = \frac{1}{r_n^d}\mathfrak{m}^F(\ell_A, Q(r_n\tfrac{x}{\rho}, r_n)).$$

By (9.25) applied with x replaced by x/ρ, we then obtain for every $A\in\mathbb{R}^{d\times d}_{\rm sym}$ and $\rho\in(0,1)$ that

$$\lim_{n\to+\infty}\frac{1}{\rho^d}\mathfrak{m}^{F_{\varepsilon_n}}(\ell_A, Q(x,\rho)) = f_{\rm lim}(A),$$

which proves (8.7) for every $x\in\mathbb{R}^d$ and $A\in\mathbb{R}^{d\times d}_{\rm sym}$.

Let us fix $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^d$, and $\nu\in\mathbb{S}^{d-1}$. Using Lemma 9.6 we see that for every $\rho\in(0,1)$ and n large enough we have

$$\begin{aligned}\frac{\varepsilon_n^{d-1}}{\rho^{d-1}}\mathfrak{m}^{F^{f^\infty,g}}(u_{\frac{x}{\varepsilon_n},\zeta,\nu}, Q_\nu(\tfrac{x}{\varepsilon_n},\tfrac{\rho}{\varepsilon_n})) - C(\rho^\alpha+\varepsilon_n) &\le \frac{1}{\rho^{d-1}}\mathfrak{m}^{F_{\varepsilon_n}}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))\\ &\le \frac{\varepsilon_n^{d-1}}{\rho^{d-1}}\mathfrak{m}^{F^{f^\infty,g}}(u_{\frac{x}{\varepsilon_n},\zeta,\nu}, Q_\nu(\tfrac{x}{\varepsilon_n},\tfrac{\rho}{\varepsilon_n})) + C(\rho^\alpha+\varepsilon_n),\end{aligned}$$

so that, setting again $r_n := \rho/\varepsilon_n$, by (9.26) we may pass to the limit first as $n\to+\infty$ and then as $\rho\to0^+$ to obtain

$$\lim_{\rho\to0^+}\lim_{n\to+\infty}\frac{1}{\rho^{d-1}}\mathfrak{m}^{F_{\varepsilon_n}}(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) = g_{\rm lim}(\zeta,\nu),$$

proving (8.8) for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^d$, and $\nu\in\mathbb{S}^{d-1}$.

The fact that $f_{\rm lim}\in\mathcal{F}^\alpha$ and $g_{\rm lim}\in\mathcal{G}^\infty$ follows again by Theorem 8.3, while property (9.4) for $g_{\rm lim}$ follows from the observation that, being f^∞ and g^∞ both positively homogeneous of degree one (in the variables A and ζ, respectively), so is the function $t\mapsto \mathfrak{m}^{F^\infty}(u_{x,t\zeta,\nu}, Q_\nu(x,\rho))$. By (9.26) this leads to the positive homogeneity of $g_{\rm lim}$.

Finally, since for every $U\in\mathcal{U}_c(\mathbb{R}^d)$ the functional $F^{f_{\rm lim},g_{\rm lim}}(\cdot,U)$ is $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ lower semicontinuous and the functions $f_{\rm lim}$ and $g_{\rm lim}$ are independent of the variable x, equality (9.27) follows from Lemma 9.10 below. Equality (9.28) follows immediately from Definition 3.3. □

For the application to stochastic homogenisation it is useful to obtain the conclusion of the previous theorem assuming only that the limits in (9.25) and (9.26) hold on countable dense collections of A, ζ, and ν. This is made possible by the following two lemmas.

Lemma 9.8. *Let $f \in \mathcal{F}$, $g \in \mathcal{G}$, and $\mathbb{D} \subset \mathbb{R}^{d\times d}_{\rm sym}$ be a dense subset. Assume that there exists a function $f_{\rm lim}\colon \mathbb{D} \to [0,+\infty)$ such that*

$$f_{\rm lim}(A) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f,g}}(\ell_A, Q(rx,r))}{r^d} \quad \textit{for every } x \in \mathbb{R}^d \textit{ and } A \in \mathbb{D}. \tag{9.29}$$

Then there exists a unique continuous extension $f_{\rm lim}\colon \mathbb{R}^{d\times d}_{\rm sym} \to [0,+\infty)$ and this extension is Lipschitz continuous with Lipschitz constant c_5 and satisfies

$$f_{\rm lim}(A) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f,g}}(\ell_A, Q(rx,r))}{r^d} \quad \textit{for every } x \in \mathbb{R}^d \textit{ and } A \in \mathbb{R}^{d\times d}_{\rm sym}. \tag{9.30}$$

Proof. Arguing exactly as in the proof of (5.11), we obtain

$$|\mathfrak{m}^{F^{f,g}}(\ell_{A_1}, Q(rx,r)) - \mathfrak{m}^{F^{f,g}}(\ell_{A_2}, Q(rx,r))| \le c_5|A_2 - A_1| r^d \quad \text{for every } A_1, A_2 \in \mathbb{R}^{d\times d}_{\rm sym}.$$

Combining the previous inequality with (9.29), we obtain that there exists a unique continuous extension of $f_{\rm lim}$ and that this extension is c_5-Lipschitz continuous and satisfies (9.30). □

Lemma 9.9. *Let $f \in \mathcal{F}$, $g \in \mathcal{G}$, and $\mathbb{D}_1 \subset \mathbb{R}^d$ and $\mathbb{D}_2 \subset \mathbb{S}^{d-1}$ be dense subsets. Assume that there exists a function $g_{\rm lim}\colon \mathbb{D}_1 \times \mathbb{D}_2 \to [0,+\infty)$ such that*

$$g_{\rm lim}(\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \quad \textit{for all } x \in \mathbb{R}^d,\ \zeta \in \mathbb{D}_1,\ \textit{and } \nu \in \mathbb{D}_2. \tag{9.31}$$

Then there exists a unique extension $g_{\rm lim}\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that

$$g_{\rm lim}(\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \quad \textit{for all } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d,\ \textit{and } \nu \in \mathbb{S}^{d-1}. \tag{9.32}$$

Proof. We begin by proving that there exists an extension $g_{\rm lim}\colon \mathbb{R}^d \times \mathbb{D}_2 \to [0,+\infty)$ such that

$$g_{\rm lim}(\zeta,\nu) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \quad \text{for all } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^d, \text{ and } \nu \in \mathbb{D}_2. \tag{9.33}$$

To this aim, we observe that by (g4′) and (9.1) we have

$$|\mathfrak{m}^{F^\infty}(u_{x,\zeta_1,\nu}, Q_\nu(rx,r)) - \mathfrak{m}^{F^\infty}(u_{x,\zeta_2,\nu}, Q_\nu(rx,r))| \le \sigma_1|\zeta_1 - \zeta_2| r^{d-1}$$

for every $x \in \mathbb{R}^d$, $\zeta_1, \zeta_2 \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$. This can be proved arguing as in the proof of inequality (5.15) in Lemma 5.8. Combining this inequality with (9.31), we obtain that the limit in (9.33) exists for every $\zeta \in \mathbb{R}^d$ and $\nu \in \mathbb{D}_2$.

We now prove that there exists and extension of $g_{\rm lim}$ satisfying (9.32). We begin by observing that, given $0 < \eta < 1$, by the continuity condition on the map $\nu \mapsto R_\nu$ introduced in (d) of Section 2 there exists $0 < \delta_\eta < \eta$ such that

$$Q_{\nu_1}(0,1) \subset\subset Q_{\nu_2}(0, 1+\eta) \quad \text{for every } \nu_1, \nu_2 \in \mathbb{S}^{d-1}_\pm \text{ with } |\nu_1 - \nu_2| < \delta_\eta. \tag{9.34}$$

Let us fix $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, $r > 0$, and $\eta > 0$. We observe that (9.34) gives

$$Q_{\nu_1}(rx, r) \subset\subset Q_{\nu_2}(rx, (1+\eta)r) \quad \text{for every } \nu_1, \nu_2 \in \mathbb{S}^{d-1}_\pm \text{ with } |\nu_1 - \nu_2| < \delta_\eta.$$

Hence, we can apply Lemma 5.7 with $x_1 = x_2 = rx$, $\rho_1 = r$, $\rho_2 = (1+\eta)r$, and $\nu_1, \nu_2 \in \mathbb{S}^{d-1}_\pm$ with $|\nu_1 - \nu_2| < \delta_\eta$. We observe that F^∞ satisfies the upper estimate in (c) of Definition 3.4 with $c_4 = 0$ because of (3.3) and Proposition 3.8, so that we can omit the term containing c_4 in Lemma 5.7. Using the inequality $(1+\eta)^{d-1} - 1 \le 2^{d-1}\eta$, for every $\nu_1, \nu_2 \in \mathbb{S}^{d-1}_\pm$ with $|\nu_1 - \nu_2| < \delta_\eta < \eta$ we then obtain

$$\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu_2}, Q_{\nu_2}(rx, (1+\eta)r)) \le \mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu_1}, Q_{\nu_1}(rx,r)) + c_3|\zeta|(2^{d-1}\eta + \omega(0,\eta)) r^{d-1}.$$

We set $x_\eta := \frac{x}{1+\eta}$ and $r_\eta := (1+\eta)r$. Dividing the previous inequality by r^{d-1}, we obtain

$$\frac{\mathfrak{m}^{F^\infty}(u_{r_\eta x_\eta,\zeta,\nu_2}, Q_{\nu_2}(r_\eta x_\eta, r_\eta))}{r_\eta^{d-1}} \leq \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu_1}, Q_{\nu_1}(rx,r))}{r^{d-1}} + c_3|\zeta|(2^{d-1}\eta + \omega(0,\eta)). \tag{9.35}$$

Exchanging the roles of ν_1 and ν_2 we also get

$$\frac{\mathfrak{m}^{F^\infty}(u_{r_\eta x_\eta,\zeta,\nu_1}, Q_{\nu_1}(r_\eta x_\eta, r_\eta))}{r_\eta^{d-1}} \leq \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu_2}, Q_{\nu_2}(rx,r))}{r^{d-1}} + c_3|\zeta|(2^{d-1}\eta + \omega(0,\eta)). \tag{9.36}$$

Moreover, applying (9.35) with x replaced by $x^\eta := (1+\eta)x$ we obtain

$$\frac{\mathfrak{m}^{F^\infty}(u_{r_\eta x,\zeta,\nu_2}, Q_{\nu_2}(r_\eta x, r_\eta))}{r_\eta^{d-1}} \leq \frac{\mathfrak{m}^{F^\infty}(u_{rx^\eta,\zeta,\nu_1}, Q_{\nu_1}(rx^\eta,r))}{r^{d-1}} + c_3|\zeta|(2^{d-1}\eta + \omega(0,\eta)). \tag{9.37}$$

Let us fix $\nu \in \mathbb{S}^{d-1}_\pm$. For every $\eta > 0$ we can find $\nu_\eta \in \mathbb{S}^{d-1}_\pm \cap \mathbb{D}_2$ with $|\nu_\eta - \nu| < \delta_\eta$, where $0 < \delta < \eta$ is the constant given by (9.34). Using (9.36) with $\nu_1 = \nu_\eta$ and $\nu_2 = \nu$, by (9.33), which clearly holds with r replaced by r_η, we obtain

$$g_{\lim}(\zeta,\nu_\eta) - c_3|\zeta|(2^{d-1}\eta + \omega(0,\eta)) \leq \liminf_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}}, \tag{9.38}$$

while using (9.37) with $\nu_1 = \nu_\eta$ and $\nu_2 = \nu$ by (9.33) we get

$$\limsup_{n\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \leq g_{\lim}(\zeta,\nu_\eta) + c_3|\zeta|(2^{d-1}\eta + \omega(0,\eta)), \tag{9.39}$$

where we have also used that by (9.33) the limit appearing the definition of $g_{\lim}(\zeta,\nu_\eta)$ is the same for x and x^η. Combining (9.38) and (9.39), we obtain

$$\limsup_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} - \liminf_{r\to+\infty} \frac{\mathfrak{m}^{F^\infty}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \leq 2c_3|\zeta|(2^{d-1}\eta + \omega(0,\eta)).$$

Letting $\eta \to 0^+$ we conclude the proof of the lemma. ☐

We conclude this section by showing that, if $F^{f,g}(\cdot,U)$ is $L^1(U;\mathbb{R}^d)$ lower semicontinuous for some bounded open set $U \subset \mathbb{R}^d$, then $f^\infty(\zeta\odot\nu) = g(\zeta,\nu)$, provided that f and g are independent of x and g is positively homogeneous of degree one in ζ. Although variants of this result are well-known (see, for instance, [17, Step 3 of Theorem 6.14]), we present here a complete proof in order to conclude the proof of Theorem 9.7.

Lemma 9.10. *Let* $f\colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \to [0,+\infty)$ *and* $g\colon \mathbb{R}^d\times\mathbb{S}^{d-1} \to [0,+\infty)$ *be Borel functions satisfying* (f2), (g2), (g3), *and* (9.4). *Assume that there exists a non-empty bounded open set* $U \subset \mathbb{R}^d$ *such that the functional* $F^{f,g}(\cdot,U)$ *is* $L^1(U;\mathbb{R}^d)$*-lower semicontinuous. Then we have*

$$g(\zeta,\nu) = f^\infty(\zeta\odot\nu)$$

for every $\zeta \in \mathbb{R}^d$ *and* $\nu \in \mathbb{S}^{d-1}$.

Proof. Let us fix $\zeta \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $x \in \mathbb{R}^d$ such that

$$0 < \mathcal{H}^{d-1}(\Pi^\nu_x \cap U), \tag{9.40}$$

$$0 \text{ is a Lebesgue point of the function } \mathbb{R} \ni t \mapsto \mathcal{H}^{d-1}(\Pi^\nu_{x+t\nu} \cap U), \tag{9.41}$$

where we recall that for $\nu_0 \in \mathbb{S}^{d-1}$ and $x_0 \in \mathbb{R}^d$ the symbol $\Pi^{\nu_0}_{x_0}$ denotes the hyperplane passing through x_0 and orthogonal to ν_0, given by $\{y \in \mathbb{R}^d : (y-x_0)\cdot\nu_0 = 0\}$. Thanks to the Fubini Theorem and the Lebesgue Differentiation Theorem the set of points $x \in \mathbb{R}^d$ for which the two previous conditions are satisfied has positive $\mathcal{L}^d$ measure.

We now show that

$$g(\zeta,\nu) \leq f^\infty(\zeta\odot\nu). \tag{9.42}$$

To prove this, we approximate the function $u_{x,\zeta,\nu}$ by the sequence of piecewise affine functions $\{u_n\}_{n\in\mathbb{N}}\subset \mathrm{BD}(U)$ defined for every $y\in U$ by

$$u_n(y):=\begin{cases}0 & \text{if } (y-x)\cdot\nu\le -\frac{1}{2n},\\ n((y-x)\cdot\nu+\frac{1}{2n})\zeta & \text{if } -\frac{1}{2n}\le (y-x)\cdot\nu\le\frac{1}{2n},\\ \zeta & \text{if } (y-x)\cdot\nu\ge\frac{1}{2n}.\end{cases}$$

Clearly $\{u_n\}_{n\in\mathbb{N}}$ converges to $u_{x,\zeta,\nu}$ strongly in $L^1(U;\mathbb{R}^d)$ as $n\to+\infty$.

Setting $H_n:=\{y\in U:-\frac{1}{2n}\le (y-x)\cdot\nu\le\frac{1}{2n}\}$, one checks immediately that,

$$\mathcal{E}u_n=n\zeta\odot\nu\chi_{H_n}\qquad \mathcal{L}^d\text{-a.e. in } U, \tag{9.43}$$

where χ_{H_n} is the characteristic function of H_n. Moreover, the Fubini Theorem implies that

$$\mathcal{L}^d(H_n)=\int_{-\frac{1}{2n}}^{\frac{1}{2n}}\mathcal{H}^{d-1}(\Pi^\nu_{x+t\nu}\cap U)\,\mathrm{d}t,$$

which together with (9.41) this gives

$$\lim_{n\to+\infty} n\mathcal{L}^d(H_n)=\mathcal{H}^{d-1}(\Pi^\nu_x\cap U). \tag{9.44}$$

In light of the lower semicontinuity of $F^{f,g}(\cdot,U)$, a direct computation shows that

$$f(0)\mathcal{L}^d(U)+g(\zeta,\nu)\mathcal{H}^{d-1}(\Pi^\nu_x\cap U)=F^{f,g}(u_{x,\zeta,\nu},U)\le\liminf_{n\to+\infty}F^{f,g}(u_n,U)$$
$$=\liminf_{n\to+\infty}\Big(\int_{H_n}f(\mathcal{E}u_n)\,\mathrm{d}x+f(0)\mathcal{L}^d(U\setminus H_n)\Big).$$

Since $\mathcal{L}^d(H_n)$ converges to 0 as $n\to+\infty$, the previous inequality gives

$$g(\zeta,\nu)\mathcal{H}^{d-1}(\Pi^\nu_x\cap U)\le\liminf_{n\to+\infty}\int_{H_n}f(\mathcal{E}u_n)\,\mathrm{d}x. \tag{9.45}$$

Using (9.43) we see that

$$\int_{H_n}f(\mathcal{E}u_n)\,\mathrm{d}x=f(n\zeta\odot\nu)\mathcal{L}^d(H_n). \tag{9.46}$$

Combining (9.44)-(9.46), recalling the definition of f^∞ given by (3.2), we have

$$g(\zeta,\nu)\mathcal{H}^{d-1}(\Pi^\nu_x\cap U)\le\liminf_{n\to+\infty}\frac{f(n\zeta\odot\nu)}{n}(n\mathcal{L}^d(H_n))\le f^\infty(\zeta\odot\nu)\mathcal{H}^{d-1}(\Pi^\nu_x\cap U),$$

which by (9.40) implies (9.42).

We now prove

$$f^\infty(\zeta\odot\nu)\le g(\zeta,\nu). \tag{9.47}$$

To this aim, we set $T>\mathrm{diam}(U)$ and let $x\in U$. By the Fubini Theorem we have

$$\mathcal{L}^d(U)=\int_{-T}^{T}\mathcal{H}^{d-1}(\Pi^\nu_{x+t\nu}\cap U)\,\mathrm{d}t.$$

By the well-know approximation properties of Lebesgue integrals by means of Riemann sums (see [42], [36, Page 63], or [27, Lemma 4.12]) we can select $x\in U$ in such a way that

$$\mathcal{L}^d(U)=\lim_{n\to+\infty}\frac{T}{n}\sum_{i=-n+1}^{n}\mathcal{H}^{d-1}\big(\Pi^\nu_{x+\frac{i-1}{n}T\nu}\cap U\big). \tag{9.48}$$

We set $A:=\zeta\otimes\nu$ and fix $t>0$. To prove (9.47), we construct a sequence of pure jump functions approximating ℓ_{tA} in $L^1(U;\mathbb{R}^d)$. Given $n\in\mathbb{N}$, let $\sigma_n\colon(-T,T)\to\mathbb{R}$ be defined by

$$\sigma_n(s):=\sum_{i=-n+1}^{n}\tfrac{iT}{n}\chi_{(\frac{i-1}{n}T,\frac{i}{n}T)}(s)\qquad\text{for every } s\in(-T,T).$$

For every $n \in \mathbb{N}$, let $u_n \in \mathrm{BD}(U)$ be the function defined by

$$u_n(y) := t\zeta\sigma_n((y-x)\cdot\nu) \quad \text{for every } y \in U.$$

It is easy to see that $\{u_n\}_{n\in\mathbb{N}}$ converges to ℓ_{tA} strongly in $L^1(U;\mathbb{R}^d)$ as $n \to +\infty$. Thus, by the L^1-lower semicontinuity of $F^{f,g}(\cdot,U)$ we obtain

$$f(tA^{\mathrm{sym}})\mathcal{L}^d(U) = \int_U f(tA^{\mathrm{sym}})\,\mathrm{d}x = F^{f,g}(\ell_{tA},U) \le \liminf_{n\to+\infty} F^{f,g}(u_n,U). \tag{9.49}$$

On the other hand, using the homogeneity of g given by (9.4) we immediately check that

$$\begin{aligned} F^{f,g}(u_n,U) &= f(0)\mathcal{L}^d(U) + \sum_{i=-n+1}^{n} g\Big(\frac{t\zeta}{n}T,\nu\Big)\mathcal{H}^{d-1}(\Pi^\nu_{x+\frac{i-1}{n}T\nu}\cap U)\\ &= f(0)\mathcal{L}^d(U) + tg(\zeta,\nu)\frac{T}{n}\sum_{i=-n+1}^{n}\mathcal{H}^{d-1}(\Pi^\nu_{x+\frac{i-1}{n}T\nu}\cap U). \end{aligned}$$

Taking the liminf as $n \to +\infty$ and using (9.48) and (9.49) we get

$$f(tA^{\mathrm{sym}})\mathcal{L}^d(U) \le f(0)\mathcal{L}^d(U) + tg(\zeta,\nu)\mathcal{L}^d(U).$$

Hence, we get $\frac{1}{t}f(tA^{\mathrm{sym}}) \le \frac{1}{t}f(0) + g(\zeta,\nu)$, and letting $t \to +\infty$ we obtain (9.47). □

10. Stochastic homogenisation

In this section we use the results of the previous section to study the problem of stochastic homogenisation of free discontinuity functionals defined on the space BD. In the following we still assume that the modulus of continuity introduced in (3.1) satisfies (9.1). We begin by introducing the probabilistic setting we will use to deal with this problem.

Throughout this section $(\Omega,\mathcal{T},P)$ is a probability space endowed with a group $(\tau_z)_{z\in\mathbb{Z}^d}$ of P-preserving transformations on $(\Omega,\mathcal{T},P)$, i.e., a family $(\tau_z)_{z\in\mathbb{Z}^d}$ of $\mathcal{T}$-measurable bijective maps $\tau_z\colon\Omega\to\Omega$ such that

(a) $\tau_{z_1}\circ\tau_{z_2} = \tau_{z_1+z_2}$ for every $z_1,z_2\in\mathbb{Z}^d$;
(b) $P(\tau_z^{-1}(E)) = P(E)$ for every $E\in\mathcal{T}$ and $z\in\mathbb{Z}^d$.

Note that from (a) and the bijectivity it follows that for $z=0$ the map τ_0 is the identity on Ω. We say that a group $(\tau_z)_{z\in\mathbb{Z}^d}$ of P-preserving transformations is ergodic if every set $E\in\mathcal{T}$ with $\tau_z(E)=E$ for every $z\in\mathbb{Z}^d$ has probability either 0 or 1. In analogy with [26,30], we introduce the following two classes of stochastic integrands.

Definition 10.1. $\mathcal{F}^\alpha_{\mathrm{stoc}}$ denotes the set of all $\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^d\times\mathbb{R}^{d\times d}_{\mathrm{sym}})$-measurable functions $f\colon\Omega\times\mathbb{R}^d\times\mathbb{R}^{d\times d}_{\mathrm{sym}}\to[0,+\infty)$ such that for every $\omega\in\Omega$ the function $f(\omega):=f(\omega,\cdot,\cdot)$ belongs to $\mathcal{F}^\alpha$ and the following stochastic periodicity condition holds:

$$f(\omega,x+z,A) = f(\tau_z(\omega),x,A)$$

for every $\omega\in\Omega$, $z\in\mathbb{Z}^d$, $x\in\mathbb{R}^d$, and $A\in\mathbb{R}^{d\times d}_{\mathrm{sym}}$. $\mathcal{G}^\infty_{\mathrm{stoc}}$ denotes the set of all $\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1})$-measurable functions $g\colon\Omega\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}\to[0,+\infty)$ such that for every $\omega\in\Omega$ the function $g(\omega):=g(\omega,\cdot,\cdot,\cdot)$ belongs to $\mathcal{G}^\infty$, and the following stochastic periodicity condition holds:

$$g(\omega,x+z,\zeta,\nu) = g(\tau_z(\omega),x,\zeta,\nu)$$

for every $\omega\in\Omega$, $z\in\mathbb{Z}^d$, $x,\zeta\in\mathbb{R}^d$, and $\nu\in\mathbb{S}^{d-1}$.

We recall the definition of subadditive process. Let $\mathcal{R}$ the collection of half-closed rectangles defined by

$$\mathcal{R} := \{R\subset\mathbb{R}^d\colon R=[a_1,b_1)\times\ldots\times[a_d,b_d) \text{ with } a_i<b_i \text{ for } i\in\{1,\ldots,d\}\}.$$

Given $R\in\mathcal{R}$ its interior is denoted by R°. We also introduce the completion of $(\Omega,\mathcal{T},P)$, denoted by $(\Omega,\widehat{\mathcal{T}},\widehat{P})$. It is immediate to see that $(\tau_z)_z$ is a group of P-preserving transformation also on $(\Omega,\widehat{\mathcal{T}},\widehat{P})$.

Definition 10.2. A function $\mu\colon \Omega \times \mathcal{R} \to \mathbb{R}$ is said to be a covariant subadditive process with respect to $(\tau_z)_{z\in\mathbb{Z}^d}$ if the following properties are satisfied

(a) for every $R \in \mathcal{R}$ the function $\mu(\cdot, R)$ is $\widehat{\mathcal{T}}$-measurable;
(b) for every $\omega \in \Omega$, $R \in \mathcal{R}$, and $z \in \mathbb{Z}^d$ we have $\mu(\omega, R+z) = \mu(\tau_z(\omega), R)$;
(c) given $R \in \mathcal{R}$ and a finite partition $(R_i)_{i=1}^n \subset \mathcal{R}$ of R, we have

$$\mu(\omega, R) \leq \sum_{i=1}^n \mu(\omega, R_i)$$

for every $\omega \in \Omega$;
(d) there exists $C > 0$ such that $0 \leq \mu(\omega, R) \leq C\mathcal{L}^d(R)$ for every $\omega \in \Omega$ and $R \in \mathcal{R}$.

In the following we will make substantial use of the Subadditive Ergodic Theorem of Akcoglu and Krengel [1, Theorem 2.7]. In particular, we will use the version of this theorem stated in [28, Proposition 1].

Theorem 10.3. *Let μ be a subadditive process with respect to the group $(\tau_z)_{z\in\mathbb{Z}^d}$. Then there exist $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, and $\varphi\colon \Omega \to [0, +\infty)$ such that*

$$\lim_{r\to+\infty} \frac{\mu(\omega, \widetilde{Q}(rx, r))}{r^d} = \varphi(\omega)$$

for every $x \in \mathbb{R}^d$ and every $\omega \in \Omega'$, where $\widetilde{Q}(rx, r) := [rx_1 - \frac{r}{2}, rx_1 + \frac{r}{2}) \times \cdots \times [rx_d - \frac{r}{2}, rx_d + \frac{r}{2})$.

If the group $(\tau_z)_{z\in\mathbb{Z}^d}$ is also ergodic, then φ is constant P-a.e.

The following is the main result concerning the bulk part.

Proposition 10.4. *Let $f \in \mathcal{F}^\alpha_{\rm stoc}$ and $g \in \mathcal{G}^\infty_{\rm stoc}$. Then there exist $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, and a function $f_{\rm lim}\colon \Omega \times \mathbb{R}^{d\times d}_{\rm sym} \to [0, +\infty)$, with $f_{\rm lim}(\omega, \cdot)$ continuous for every $\omega \in \Omega'$ and $f_{\rm lim}(\cdot, A)$ $\mathcal{T}$-measurable for every $A \in \mathbb{R}^{d\times d}_{\rm sym}$, such that*

$$f_{\rm lim}(\omega, A) = \lim_{r\to+\infty} \frac{\mathfrak{m}^{E^{f(\omega),g(\omega)}}(\ell_A, Q(rx, r))}{r^d} \tag{10.1}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^d$, and $A \in \mathbb{R}^{d\times d}_{\rm sym}$. If, in addition, $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Ω' appropriately we have that $f_{\rm lim}$ is independent of ω.

Proof. Let us fix $A \in \mathbb{R}^{d\times d}_{\rm sym}$ and consider the function $\Phi_A\colon \Omega \times \mathcal{R} \to [0, +\infty)$ defined by

$$\Phi_A(\omega, R) := \mathfrak{m}^{E^{f(\omega),g(\omega)}}(\ell_A, R^\circ).$$

This function defines a covariant subadditive process. Indeed, it is easy to check that properties (b), (c), and (d) of Definition 10.2 hold (see [16, Theorem 9.1] for the details), while the proof of the measurability property (a) can be obtained by slightly modifying the Appendix of [16], replacing Du and ∇u by Eu and $\mathcal{E}u$.

Let $\mathbb{D}$ be a countable dense subset of $\mathbb{R}^{d\times d}_{\rm sym}$. We may then apply Theorem 10.3 with $\mu = \Phi_A$ for every $A \in \mathbb{D}$ to obtain a set $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, and a function $\phi_A\colon \Omega \to [0, +\infty)$ such that

$$\phi_A(\omega) = \lim_{r\to+\infty} \frac{\Phi_A(\omega, \widetilde{Q}(rx, r))}{r^d} = \lim_{r\to+\infty} \frac{\mathfrak{m}^{F^{f(\omega),g(\omega)}}(\ell_A, Q(rx, r))}{r^d}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^d$, and $A \in \mathbb{D}$. By Lemma 9.8 for every $\omega \in \Omega'$ there exists a unique continuous function $f_{\rm lim}(\omega, \cdot)\colon \mathbb{R}^{d\times d}_{\rm sym} \to [0, +\infty)$ such that $f_{\rm lim}(\omega, A) = \phi_A(\omega)$ for every $A \in \mathbb{R}^{d\times d}_{\rm sym}$. Moreover, $A \mapsto f_{\rm lim}(\omega, A)$ is c_5-Lipschitz continuous and satisfies (10.1).

Finally, the statement concerning the ergodic hypothesis follows from the last part of Theorem 10.3. □

The following proposition collects the main results concerning the surface term.

Proposition 10.5. *Let $f \in \mathcal{F}^\alpha_{\rm stoc}$ and $g \in \mathcal{G}^\infty_{\rm stoc}$. Then there exist $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, and a function $g_{\rm lim} \colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0, +\infty)$ such that*

$$\zeta \mapsto g_{\rm lim}(\omega, \zeta, \nu) \text{ is Lipschitz continuous for every } \omega \in \Omega' \text{ and } \nu \in \mathbb{S}^{d-1},$$
$$\mathbb{S}^{d-1}_\pm \ni \nu \mapsto g_{\rm lim}(\omega, \zeta, \nu) \text{ are continuous for every } \omega \in \Omega' \text{ and } \zeta \in \mathbb{R}^d,$$
$$\omega \mapsto g_{\rm lim}(\omega, \zeta, \nu) \text{ is } \mathcal{T}\text{-measurable for every } \zeta \in \mathbb{R}^d \text{ and } \nu \in \mathbb{S}^{d-1},$$

and

$$g_{\rm lim}(\omega, \zeta, \nu) = \lim_{r \to +\infty} \frac{\mathfrak{m}^{F^\infty(\omega)}(u_{rx,\zeta,\nu}, Q_\nu(rx, r))}{r^{d-1}} \tag{10.2}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^d$, and $\nu \in \mathbb{S}^{d-1}$, where F^∞ is the functional introduced in Definition 9.5. If, in addition, $(\tau_z)_{z \in \mathbb{Z}^d}$ is ergodic, by choosing Ω' appropriately we have that $g_{\rm lim}$ is independent of ω.

Proof. It is enough to repeat the arguments of [16, Proposition 9.4 and 9.5], replacing Step 2 of their Proposition 9.4 and 9.5 by our Lemma 9.9. ☐

Collecting the results of Propositions 10.4 and 10.5 we obtain the following almost sure convergence result.

Theorem 10.6. *Let $f \in \mathcal{F}^\alpha_{\rm stoc}$, $g \in \mathcal{G}^\infty_{\rm stoc}$ and let $F_\varepsilon(\omega) := F^{f_\varepsilon(\omega), g_\varepsilon(\omega)}$ be the functionals introduced in Definition 9.1. Then there exist $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, and functions $f_{\rm lim} \in \mathcal{F}^\alpha_{\rm stoc}$ and $g_{\rm lim} \in \mathcal{G}^\infty_{\rm stoc}$, with $g_{\rm lim}(\omega)$ satisfying* (9.4) *for every $\omega \in \Omega'$, and with the following property: for every sequence $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset (0, 1)$ converging to 0 as $n \to +\infty$, $U \in \mathcal{U}_c(\mathbb{R}^d)$, and $\omega \in \Omega'$ we have that $\{F_{\varepsilon_n}(\omega)(\cdot, U)\}_{n \in \mathbb{N}}$ Γ-converges, with respect to the topology of $L^1_{\rm loc}(\mathbb{R}^d; \mathbb{R}^d)$, to $F^{f_{\rm lim}(\omega), g_{\rm lim}(\omega)}(\cdot, U)$ as $n \to +\infty$. Moreover, we have the equality*

$$g_{\rm lim}(\omega, \zeta, \nu) = f^\infty_{\rm lim}(\omega, \zeta \odot \nu) \quad \text{for every } \zeta \in \mathbb{R}^d \text{ and } \nu \in \mathbb{S}^{d-1}. \tag{10.3}$$

If, in addition, $(\tau_z)_{z \in \mathbb{Z}^d}$ is ergodic, by choosing Ω' appropriately we have that $f_{\rm lim}$ and $g_{\rm lim}$ are independent of ω.

Proof. Thanks to Propositions 10.4 and 10.5, there exist $\Omega' \in \mathcal{T}$, with $P(\Omega') = 1$, $f_{\rm lim} \colon \Omega \times \mathbb{R}^{d \times d}_{\rm sym} \to [0, +\infty)$, and $g_{\rm lim} \colon \Omega \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0, +\infty)$ such that (10.1) and (10.2) hold. Thus, by Theorem 9.7 we obtain the two inclusions $f_{\rm lim} \in \mathcal{F}^\alpha_{\rm stoc}$ and $g_{\rm lim} \in \mathcal{G}^\infty_{\rm stoc}$, together with the fact that $g_{\rm lim}(\omega)$ satisfies (9.4) for every $\omega \in \Omega'$, as well as the Γ-convergence part of the statement and equality (10.3).

The last part of the statement follows from the final parts of the statements of Propositions 10.4 and 10.5. ☐

As a particular case of the previous theorem we deduce the following result in the periodic deterministic setting.

Corollary 10.7. *Let $f \in \mathcal{F}^\alpha$, $g \in \mathcal{G}^\infty$, and let $F_\varepsilon := F^{f_\varepsilon, g_\varepsilon}$ be the functional introduced in Definition 9.1. Assume that both f and g are 1-periodic in the x variable. Then there exist functions $f_{\rm lim} \in \mathcal{F}^\alpha$ and $g_{\rm lim} \in \mathcal{G}^\infty$, with $f_{\rm lim}$ satisfying* (9.25)*, and with $g_{\rm lim}$ satisfying* (9.4) *and* (9.26)*, and consequently the following property holds: for every sequence $\{\varepsilon_n\}_{n \in \mathbb{N}} \subset (0, 1)$ converging to 0 as $n \to +\infty$ and for every $U \in \mathcal{U}_c(\mathbb{R}^d)$ the sequence $\{F_{\varepsilon_n}(\cdot, U)\}_{n \in \mathbb{N}}$ Γ-converges to $F^{f_{\rm lim}, g_{\rm lim}}(\cdot, U)$ with respect to the topology of $L^1_{\rm loc}(\mathbb{R}^d; \mathbb{R}^d)$. Moreover, we have the equality*

$$g_{\rm lim}(\zeta, \nu) = f^\infty_{\rm lim}(\zeta \odot \nu) \quad \text{for every } \zeta \in \mathbb{R}^d \text{ and } \nu \in \mathbb{S}^{d-1}.$$

Proof. It is enough to apply Theorem 10.6 in the case where Ω consists of a single point. ☐

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