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$\mathbb{Q}$-FACTORIAL LAURENT RINGS

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Abstract. Dolgachev proves that the ring naturally associated to a generic Laurent polynomial in d variables, $d \geq 4$, is factorial [4, 5] (for any field k). We prove a sufficient condition for the ring associated to a very general complex Laurent polynomial in $d = 3$ variables to be $\mathbb{Q}$-factorial.

1. Introduction

In [4] and Dolgachev [5] proves that the ring A_F naturally associated to generic Laurent polynomial F in d variables, $d \geq 4$, with coefficients in any field k, is factorial. The basic ingredient in Dolgachev's proof is Grothendieck's Lefschetz-type theorem ([6], Prop. 3.12) which, among other things, shows that under suitable conditions, the natural restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$, where X is a scheme and Y is subvariety corresponding to an ideal sheaf in $\mathcal{O}_X$, is an isomorphism. This result can be applied only when $d \geq 4$.

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In this paper we consider the case $d = 3$, assuming that $k = \mathbb{C}$, and prove a sufficient condition for the ring A_F to be $\mathbb{Q}$-factorial (Theorem 3.1). The proof of this fact follows the lines of Dolgachev's proof, with Grothendieck's result replaced by a Noether-Lefschetz theorem for hypersurfaces in toric 3-folds (Theorem 2.5) that we proved in [2].

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2. Preliminaries

We follow the notation in [1] and [2]. Let M be a d-dimensional lattice, $N = \mathrm{Hom}(M, \mathbb{Z})$ and $\mathbf{T}_N = N \otimes \mathbb{C}^*$ the associated algebraic torus. Let $\Sigma \subset N_{\mathbb{R}}$ be a complete simplicial fan, and denote by X_Σ the corresponding complete toric variety. The torus $\mathbf{T}_N$ naturally acts on X_Σ; $\mathbf{T}_\tau \subset X_\Sigma$ denotes the orbit of a subset of X_Σ corresponding to a face τ of Σ under this action; the open dense orbit is denoted by $\mathbf{T}_0$.

Definition 2.1. [1, Def. 4.13] *A hypersurface X in X_Σ is* nondegenerate *if $X \cap \mathbf{T}_\tau$ is a smooth 1-codimensional subvariety of $\mathbf{T}_\tau$ for all faces τ in Σ.*

X_Σ has only abelian quotient singularities, and is therefore an orbifold.

Proposition 2.2. [1, Prop. 3.5, 4.15] *Let L be a ample line bundle on X_Σ. The hypersurface $X \subset X_\Sigma$ given by the zero locus of a generic section of L is nondegenerate. Moreover, X is an orbifold.*

Since X is an orbifold, its complex cohomology has a pure Hodge structure [9]. This is an essential point in the proof of our Theorem 2.5.

Definition 2.3 (The Cox Ring [3])**.** *Consider a variable z_i for each 1-dimensional cone ς_i, $i = 1, \ldots, n$ in Σ, and let $S(\Sigma)$ be the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$.*

The Cox ring has a natural gradation given by its class group $Cl(\Sigma)$ of X_Σ.

Let L be an ample line bundle on X_Σ, and let $f \in H^0(X_\Sigma, L) \simeq S(\Sigma)_\beta$, where $\beta = \deg(L)$.

Definition 2.4. *The* Jacobian ring *of f is the quotient $R(f) = S(\Sigma)/J(f)$, where $J(f)$ is the ideal in $S(\Sigma)$ generated by the derivatives of f.*

The Jacobian ring $R(f)$ inherits a natural gradation from $S(\Sigma)$.

The next theorem was proved in [2], and will be key to proving our result about Laurent rings. We assume $d = 3$. We recall that the Picard number is defined as the rank of the class group.

Theorem 2.5. [2] *Let X_Σ a complete simplicial toric variety, and $X \subset X_\Sigma$ a very general hypersurface cut by a section f of an ample line bundle L such that the multiplication morphism*

$$R(f)_\beta \otimes R(f)_{\beta-\beta_0} \to R(f)_{2\beta-\beta_0} \tag{1}$$

is surjective (here $\beta = \deg(L)$ and $\beta_0 = -\deg(K_{X_\Sigma})$, where K_{X_Σ} is the canonical sheaf of X_Σ). Then X has the same Picard number as X_Σ.

Recall that a property is very general if it holds in the complement of countably many proper subvarieties.

If X is a quartic surface in $\mathbb{P}^3$, or more generally a $K3$ surface defined by a section of the anticanonical divisor in a simplicial toric variety, then the above map is surjective [2]. It is a classical result that the map is not surjective if X is a cubic in $\mathbb{P}^3$.

3. $\mathbb{Q}$-factorial Laurent rings

The ring $\mathbb{C}[M]$ may be identified with the ring of regular functions on the torus $\mathbf{T}_N \simeq \mathbf{T}_0 \subset X_\Sigma$. An element $F \in \mathbb{C}[M]$ is called a *Laurent polynomial*; F may be regarded as a section of the ample line bundle L, and it defines a hypersurface X_F in X_Σ.

Let $\Delta \subset M \otimes_{\mathbb{Z}} \mathbb{R}$ be the polytope uniquely determined by the fan Σ and L (see [8], Lemma 2.14). To each Laurent polynomial F on can associate a polytope Δ_F, called the *Newton polytope* of F. This is most easily described by choosing an isomorphism $M \simeq \mathbb{Z}^d$, writing

$$F = \sum_{i_1,\dots,i_d \in \mathbb{Z}^d} a_{i_1,\dots,i_d}\, t_1^{i_1} \cdots t_d^{i_d}$$

and defining

$$\operatorname{supp}(F) = \{i_1, \dots, i_d \in \mathbb{Z}^d \,|a_{i_1,\dots,i_d} \neq 0\}.$$

Δ_F is then defined to be the convex hull of $\operatorname{supp}(F)$ and $\Gamma(\Delta)$ the set of all Laurent polynomials such that $\Delta_F \subset \Delta$. $\Gamma(\Delta)$ is a finite dimensional vector space over $\mathbb{C}$.

By results given in [7] (see also [8], Chapter 2) a Laurent polynomial F extends to a meromorphic function on X_Σ, which is a section of an ample line bundle L_F. Thus, F

may be regarded as an element in $S(\Sigma)_\beta$, where $\beta = \deg(L_F)$. Denote by A_F the ring $\mathbb{C}[M]/(F)$.

Theorem 3.1. *Let $d = 3$, and let F be a very general Laurent polynomial in $\Gamma(\Delta)$; set $\beta = \deg(L_F)$ and $\beta_0 = -\deg(K_{X_\Sigma})$. If the multiplication morphism*

$$R(F)_\beta \otimes R(F)_{\beta-\beta_0} \to R(F)_{2\beta-\beta_0} \tag{2}$$

is surjective, the ring A_F is $\mathbb{Q}$-factorial.

The proof that A_F is $\mathbb{Q}$-factorial follows closely the proof of Theorem 1.1 in [4]. The basic idea is to formulate the problem in a geometric way:

Proof. Let $X_F \subset X_\Sigma$ be the hypersurface cut by F (as a section of L_F). By Proposition 2.2 the hypersurface X_F is nondegenerate, and is an orbifold.

Note that the ring A_F may be identified with the ring of regular functions on the affine part $U_F = X_F \cap \mathbf{T}_0$ of X_F. Since the Picard group of $\mathbf{T}_0$ is trivial, every Cartier divisor in X_Σ is linearly equivalent to a divisor supported in $X_\Sigma - \mathbf{T}_0$. By Theorem 2.5, X_F has the same Picard number as X_Σ, i.e., $\rho(X_F) = \rho(X_\Sigma)$. Then any Cartier divisor in X_F is linearly equivalent modulo torsion to a divisor supported in $X_F - U_F$, so that $\operatorname{Pic}(U_F) \otimes \mathbb{Q} = 0$. Since U_F is normal (actually smooth), then $Cl(U_F) \otimes \mathbb{Q} = 0$. As $U_F \simeq \operatorname{Spec}(A_F)$, we have $Cl(A_F) \otimes \mathbb{Q} = 0$. □

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