

THE GYSIN SEQUENCE FOR QUANTUM LENS SPACES

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ABSTRACT. We define quantum lens spaces as ‘direct sums of line bundles’ and exhibit them as ‘total spaces’ of certain principal bundles over quantum projective spaces. For each of these quantum lens spaces we construct an analogue of the classical Gysin sequence in K-theory. We use the sequence to compute the K-theory of the quantum lens spaces, in particular to give explicit geometric representatives of their K-theory classes. These representatives are interpreted as ‘line bundles’ over quantum lens spaces and generically define ‘torsion classes’. We work out explicit examples of these classes.

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1. INTRODUCTION

This paper is devoted to the study of the noncommutative topology of quantum lens spaces via their K-theory. We construct an exact sequence – a noncommutative analogue of the classical Gysin sequence – which relates the K-theory of quantum lens spaces to the K-theory of quantum projective spaces. Our construction enables us not only to compute the K-theory of the quantum lens spaces but also to obtain geometric representatives of the K-theory classes, generically torsion ones, in terms of ‘line bundles’.

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Noncommutative (or quantum) lens spaces have been the subject of increasing interest of late. They first appeared in [23] in the context of what we would now call ‘theta-deformed’ topology; they later surfaced in [15] in the guise of graph C^* -algebras, with certain more recent special cases (*cf.* [3, 12]). The particular case of the quantum three-dimensional real projective space was studied in [25] and [20].

Lens spaces arise in classical geometry as quotients of odd-dimensional spheres by an action of a finite cyclic group. In parallel with this, quantum lens spaces are usually introduced in terms of fixed point algebras for suitable actions of finite cyclic groups on function algebras over odd dimensional quantum spheres. Indeed, the key result of [15] is the realization of the C^* -algebra of continuous functions on a quantum lens space as the Cuntz-Krieger algebra of a directed graph. From this, and quite importantly, one deduces the K-theory of the algebra as the kernel and cokernel of a certain ‘incidence matrix’ associated to the graph. This computation of the K-theory is, one has to say, very direct, although somewhat implicit and obtained via some rather ugly isomorphisms.

In the present paper we task ourselves with finding a more elegant intuitive and geometric approach to the K-theory of quantum lens spaces. To this end, our starting point is the ‘algebraic’ approach to the K-theory of quantum projective spaces presented in [10]. Center stage there is taken by (polynomial) bimodules $\mathcal{L}_N$ of sections of noncommutative ‘line bundles’ over the projective space; line bundles which determine the K-theory of the C^* -algebra $C(\mathbb{CP}_q^n)$. Out of this algebraic approach to K-theory there come several important advantages that we list in the remainder of this introduction, by way of summarizing some of the main results of the present paper.

Given a pair of positive integers n, r , the coordinate algebra $\mathcal{A}(L_q^{(n,r)})$ of the *quantum lens space* of dimension $2n + 1$ (and index r) is defined to be

$$(4.1) \quad \mathcal{A}(L_q^{(n,r)}) := \bigoplus_{N \in \mathbb{Z}} \mathcal{L}_{rN}.$$

Then $L_q^{(n,r)}$ is the ‘total space’ of a principal bundle over the quantum projective space $\mathbb{CP}_q^n$ with structure group $\tilde{U}(1) := U(1)/\mathbb{Z}_r$. This parallels the $U(1)$ principal bundle over $\mathbb{CP}_q^n$ having total space the quantum sphere S_q^{2n+1} , the latter being obtained for $r = 1$ in the previous decomposition:

$$(4.3) \quad \mathcal{A}(S_q^{2n+1}) := \bigoplus_{N \in \mathbb{Z}} \mathcal{L}_N.$$

One is then able to show *a posteriori* that the algebra $\mathcal{A}(L_q^{(n,r)})$ is made of all elements of $\mathcal{A}(S_q^{2n+1})$ which are invariant under a certain action of the cyclic group $\mathbb{Z}_r$. With these principal bundles there comes a way to ‘pull-back’ line bundles from $\mathbb{CP}_q^n$ to $L_q^{(n,r)}$:

$$\begin{array}{ccc} \tilde{\mathcal{L}}_N & \xleftarrow{j_*} & \mathcal{L}_N \\ \vdots & & \vdots \\ \mathcal{A}(L_q^{(n,r)}) & \xleftarrow{j} & \mathcal{A}(\mathbb{CP}_q^n). \end{array}$$

That is to say, the algebra inclusion $j : \mathcal{A}(\mathbb{CP}_q^n) \rightarrow \mathcal{A}(L_q^{(n,r)})$ also induces a map

$$(4.6) \quad j_* : K_0(C(\mathbb{CP}_q^n)) \rightarrow K_0(C(L_q^{(n,r)})).$$

The marked difference between a line bundle $\mathcal{L}_N$ over $\mathbb{CP}_q^n$ versus its pull-back $\tilde{\mathcal{L}}_N$ to $L_q^{(n,r)}$ is that, while each $\mathcal{L}_N$ is not free when $N \neq 0$, this need not be the case for $\tilde{\mathcal{L}}_N$:

the pull-back $\tilde{\mathcal{L}}_{-r}$ of $\mathcal{L}_{-r}$ is tautologically free, that is to say it is trivial in the group $K_0(C(L_q^{(n,r)}))$. It follows that $(\tilde{\mathcal{L}}_{-N})^{\otimes r} \simeq \tilde{\mathcal{L}}_{-rN}$ also has trivial class for any $N \in \mathbb{Z}$ and thus such line bundles $\tilde{\mathcal{L}}_{-N}$ define *torsion classes*; they generate the group $K_0(C(L_q^{(n,r)}))$.

In addition, there is a multiplicative structure on the group $K_0(C(\mathbb{CP}_q^n))$:

Proposition. 3.4. *It holds that*

$$K_0(C(\mathbb{CP}_q^n)) \simeq \mathbb{Z}[[\mathcal{L}_{-1}]]/(1 - [\mathcal{L}_{-1}])^{n+1} \simeq \mathbb{Z}[u]/u^{n+1}$$

where $u = \chi([\mathcal{L}_{-1}]) := 1 - [\mathcal{L}_{-1}]$ is the Euler class of the line bundle $\mathcal{L}_{-1}$.

Out of this one is led to a map

$$(4.7) \quad \alpha : K_0(C(\mathbb{CP}_q^n)) \rightarrow K_0(C(\mathbb{CP}_q^n))$$

where α is now multiplication by the Euler class $\chi(\mathcal{L}_{-r}) := 1 - [\mathcal{L}_{-r}]$ of $\mathcal{L}_{-r}$. Central for us is the assembly of this map with the pull-back map (4.6) into an exact sequence

$$0 \rightarrow K_1(C(L_q^{(n,r)})) \xrightarrow{\text{Ind}} K_0(C(\mathbb{CP}_q^n)) \xrightarrow{\alpha} K_0(C(\mathbb{CP}_q^n)) \xrightarrow{j_*} K_0(C(L_q^{(n,r)})) \xrightarrow{\text{Ind}} 0,$$

the *Gysin sequence* for our quantum lens space $L_q^{(n,r)}$, with the maps Ind being suitable index maps explicitly described below. Having arrived at this sequence, one could easily be content simply by admiring its sheer elegance. It has, however, some very practical and doubtlessly important applications, which we present in the final sections.

Notably, there is the computation of the K-theory of the quantum lens spaces $L_q^{(n,r)}$. Owing to Prop. 3.4, the map α can be given as an $(n+1) \times (n+1)$ matrix with respect to the $\mathbb{Z}$ -module basis $\{1, u, \dots, u^n\}$ of $K^0(C(\mathbb{CP}_q^n)) \simeq \mathbb{Z}^{n+1}$. This leads to the identifications

$$(6.1) \quad K_1(C(L_q^{(n,r)})) \simeq \ker(\alpha), \quad K_0(C(L_q^{(n,r)})) \simeq \text{coker}(\alpha).$$

We stress that our construction is structurally different from the one in [15], the only point of contact being that the K-theory is obtained out of a matrix. First, our matrix is different from the incidence matrix of [15]. Second, and more importantly, the structure of the map α and Prop. 3.4 allow us to give *geometric* generators of both the groups $K_1(C(L_q^{(n,r)}))$ and $K_0(C(L_q^{(n,r)}))$, for the latter in particular as (combinations) of pulled-back line bundles from $\mathbb{CP}_q^n$ to $L_q^{(n,r)}$. All of this is described in full detail in §6.

Some of the dual constructions pertinent to the K-homology of the quantum lens spaces $L_q^{(n,r)}$, stemming from a sequence dual to the previous one, will be reported elsewhere.

Notation. By a $*$ -algebra we mean a complex associative unital involutive algebra. An unadorned tensor product is meant to be over $\mathbb{C}$. As it is customary, a noncommutative C^* -algebra A is thought of as being the algebra of continuous functions on an underlying ‘quantum’ topological space, and we use the notation $K_\bullet(A)$ for the K-theory of this C^* -algebra, together with $K^\bullet(A)$ for its K-homology.

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2. THE CLASSICAL GYSIN SEQUENCE

In this section we simply follow Karoubi's book [16]. Let $\mathbb{CP}^n$ denote the complex projective space of $\mathbb{C}^{n+1}$ and let V be a complex vector bundle over $\mathbb{CP}^n$ equipped with a Hermitian fibre metric. We write $B(V)$ for the 'ball bundle' of V , the bundle over $\mathbb{CP}^n$ whose fibre $B(V)_x$ at the point $x \in \mathbb{CP}^n$ is the closed unit ball of the fibre V_x of V . Similarly we write $S(V)$ for the 'sphere bundle' of V , whose fibre $S(V)_x$ at $x \in \mathbb{CP}^n$ is the unit sphere of the fibre V_x . Then $B(V) - S(V)$ denotes the open ball bundle.

Since $S(V)$ is closed in $B(V)$, the short exact sequence of topological spaces

$$(2.1) \quad \emptyset \rightarrow S(V) \rightarrow B(V) \rightarrow B(V) - S(V) \rightarrow \emptyset$$

gives rise to the following six term exact sequence in topological K-theory [16, IV.1.13]:

$$(2.2) \quad \begin{array}{ccccc} K^0(B(V) - S(V)) & \longrightarrow & K^0(B(V)) & \longrightarrow & K^0(S(V)) \\ \delta_{10} \uparrow & & & & \downarrow \delta_{01} \\ K^1(S(V)) & \longleftarrow & K^1(B(V)) & \longleftarrow & K^1(B(V) - S(V)). \end{array}$$

The horizontal arrows are the ones induced by the maps in the sequence (2.1), whereas the vertical arrows are the usual 'connecting homomorphisms' [16, II.3.21].

Observing that the total space of $B(V)$ is homotopic to $\mathbb{CP}^n$ (via the inclusion of the latter into $B(V)$ determined by the zero section of V), we have isomorphisms of K-groups

$$K^*(B(V)) \simeq K^*(\mathbb{CP}^n).$$

Moreover, the total space of the fibre bundle $B(V) - S(V)$ is homeomorphic to the total space of V . This, followed by the Thom isomorphism combined with Bott periodicity thus gives rise to the isomorphisms of K-groups

$$K^*(B(V) - S(V)) \simeq K^*(V) \simeq K^*(\mathbb{CP}^n).$$

Now let L be the tautological line bundle over $\mathbb{CP}^n$, whose total space is $\mathbb{C}^{n+1}$ and whose fibre L_x at $x \in \mathbb{CP}^n$ is the one-dimensional complex vector subspace of $\mathbb{C}^{n+1}$ which defines that point. Via the usual associated bundle construction, the bundle L may be identified with the quotient of $S^{2n+1} \times \mathbb{C}$ by the equivalence relation

$$(x, t) \sim (\lambda x, \lambda^{-1}t), \quad \lambda \in S^1 \subseteq \mathbb{C}.$$

Similarly, its r -th tensor power $L^{\otimes r}$ may be identified with the quotient of $S^{2n+1} \times \mathbb{C}$ by the equivalence relation $(x, t) \sim (\lambda x, \lambda^{-r}t)$. Moreover, $L^{\otimes r}$ can be given the fibre metric defined by $\varphi((x, t'), (x, t)) = t'\bar{t}$. It follows that the sphere bundle $S(L^{\otimes r})$ can be identified with the 'lens space' $L^{(n,r)} := S^{2n+1}/\mathbb{Z}_r$ (where the cyclic group $\mathbb{Z}_r$ of order r acts upon the sphere S^{2n+1} via the r -th roots of unity) by the map $(x, t) \mapsto \sqrt[r]{t} \cdot x$.

Taking $V = L^{\otimes r}$ in the above construction in (2.1), one find just as in [16, IV.1.14] that the exact sequence (2.2) in this case reads

$$(2.3) \quad \begin{array}{ccccc} K^0(\mathbb{CP}^n) & \xrightarrow{\alpha} & K^0(\mathbb{CP}^n) & \xrightarrow{\pi^*} & K^0(L^{(n,r)}) \\ \delta_{10} \uparrow & & & & \downarrow \delta_{01} \\ K^1(L^{(n,r)}) & \longleftarrow & K^1(\mathbb{CP}^n) & \longleftarrow & K^1(\mathbb{CP}^n). \end{array}$$

The ‘pull-back’ homomorphism π^* is induced by the bundle projection $\pi : L^{(n,r)} \rightarrow \mathbb{CP}^n$. The homomorphism α is multiplication by the *Euler class*

$$\chi(L^{\otimes r}) := 1 - [L^{\otimes r}]$$

of the vector bundle $L^{\otimes r}$. In fact, it is known (*cf.* [16, Cor. IV.2.8]) that one has $K^1(\mathbb{CP}^n) = 0$. The sequence (2.3) thus reduces to the exact sequence

$$(2.4) \quad 0 \longrightarrow K^1(L^{(n,r)}) \xrightarrow{\delta_{10}} K^0(\mathbb{CP}^n) \xrightarrow{\alpha} K^0(\mathbb{CP}^n) \xrightarrow{\pi^*} K^0(L^{(n,r)}) \xrightarrow{\delta_{01}} 0,$$

the K-theoretic *Gysin sequence* for the lens space $L^{(n,r)}$.

3. QUANTUM PROJECTIVE SPACES

We first describe the class of noncommutative projective spaces that we need. We recall both the algebras of coordinate and continuous functions on quantum projective space, together with the noncommutative ‘line bundles’ which represent the K-theory.

3.1. Functions on quantum projective spaces. In the following, without loss of generality, the real deformation parameter is restricted to the interval $0 < q < 1$. We recall from [29] that the coordinate algebra of the unit quantum sphere S_q^{2n+1} is the $*$ -algebra $\mathcal{A}(S_q^{2n+1})$ generated by $2n+2$ elements $\{z_i, z_i^*\}_{i=0,\dots,n}$ subject to the relations:

$$(3.1) \quad \begin{aligned} z_i z_j &= q^{-1} z_j z_i & 0 \leq i < j \leq n, \\ z_i^* z_j &= q z_j z_i^* & i \neq j, \\ [z_n^*, z_n] &= 0, \quad [z_i^*, z_i] = (1 - q^2) \sum_{j=i+1}^n z_j z_j^* & i = 0, \dots, n-1, \\ 1 &= z_0 z_0^* + z_1 z_1^* + \dots + z_n z_n^*. \end{aligned}$$

The notation of [29] is obtained by setting $q = e^{h/2}$, while the relationship with the generators x_i used in [13] is given by $x_i = z_{n+1-i}^*$ together with the replacement $q \rightarrow q^{-1}$.

We write $\mathcal{A}(\mathbb{CP}_q^n)$ for the $*$ -subalgebra of $\mathcal{A}(S_q^{2n+1})$ generated by the elements $p_{ij} := z_i^* z_j$ for $i, j = 0, 1, \dots, n$, which we think of as the coordinate algebra of the quantum projective space $\mathbb{CP}_q^n$. It is easy to see that the algebra $\mathcal{A}(\mathbb{CP}_q^n)$ is made of the invariant elements for the action of $U(1)$ on the algebra $\mathcal{A}(S_q^{2n+1})$ given by

$$(3.2) \quad (z_0, z_1, \dots, z_n) \mapsto (\lambda z_0, \lambda z_1, \dots, \lambda z_n), \quad \lambda \in U(1).$$

From the relations of $\mathcal{A}(S_q^{2n+1})$ one gets relations for $\mathcal{A}(\mathbb{CP}_q^n)$:

$$(3.3) \quad \begin{aligned} p_{ij} p_{kl} &= q^{\text{sign}(k-i) + \text{sign}(j-l)} p_{kl} p_{ij} & \text{if } i \neq l \text{ and } j \neq k, \\ p_{ij} p_{jk} &= q^{\text{sign}(j-i) + \text{sign}(j-k) + 1} p_{jk} p_{ij} - (1 - q^2) \sum_{l>j} p_{il} p_{lk} & \text{if } i \neq k, \\ p_{ij} p_{ji} &= q^{2\text{sign}(j-i)} p_{ji} p_{ij} + (1 - q^2) \left(\sum_{l>i} q^{2\text{sign}(j-i)} p_{jl} p_{li} - \sum_{l>j} p_{il} p_{li} \right) & \text{if } i \neq j, \end{aligned}$$

with $\text{sign}(0) := 0$. The elements p_{ij} are the matrix entries of a projection $P = (p_{ij})$, that is to say it obeys $P^2 = P = P^*$, or rather that $\sum_{j=0}^n p_{ij} p_{jk} = p_{ik}$ and $p_{ij}^* = p_{ji}$. This projection has q -trace equal to one:

$$(3.4) \quad \text{Tr}_q(P) := \sum_{i=0}^n q^{2i} p_{ii} = 1.$$

To the best of our knowledge, the algebra $\mathcal{A}(\mathbb{CP}_q^n)$ first appeared in [30].

The C^* -algebra $C(S_q^{2n+1})$ of continuous functions on the quantum sphere S_q^{2n+1} is the completion of $\mathcal{A}(S_q^{2n+1})$ in the universal C^* -norm. The C^* -algebra $C(\mathbb{CP}_q^n)$ of continuous

functions on the quantum projective space is the completion of $\mathcal{A}(\mathbb{CP}_q^n)$ in the universal C^* -norm. By definition, the $*$ -algebra inclusion $\mathcal{A}(\mathbb{CP}_q^n) \hookrightarrow \mathcal{A}(S_q^{2n+1})$ extends to an inclusion of C^* -algebras $C(\mathbb{CP}_q^n) \hookrightarrow C(S_q^{2n+1})$.

There is a marked difference between S_q^{2n+1} and $\mathbb{CP}_q^n$ which is reflected in their (K_0, K_1) -groups: for the odd-dimensional spheres S_q^{2n+1} these are equal to $(\mathbb{Z}, \mathbb{Z})$ regardless of the dimension, while for $\mathbb{CP}_q^n$ they are equal to $(\mathbb{Z}^{n+1}, 0)$. A set of generators for the K-theory and K-homology of the sphere algebras $C(S_q^{2n+1})$ can be found in [13].

That $K_0(C(\mathbb{CP}_q^n)) \simeq \mathbb{Z}^{n+1}$ can be proved by viewing the C^* -algebra $C(\mathbb{CP}_q^n)$ as the Cuntz–Krieger algebra of a graph [14]. The group K_0 is the cokernel of the incidence matrix canonically associated to the graph (while K_1 is the kernel of the matrix). The dual result for K-homology is obtained using the same techniques: the group K^0 is now the kernel of the transposed matrix [7] and this leads to $K^0(C(\mathbb{CP}_q^n)) \simeq \mathbb{Z}^{n+1}$ (and K^1 is the cokernel of the transposed matrix).

Generators of the homology group $K^0(C(\mathbb{CP}_q^n))$ were given explicitly in [10] as (classes of) even Fredholm modules

$$(3.5) \quad \mu_k = (\mathcal{A}(\mathbb{CP}_q^n), \mathcal{H}_{(k)}, \pi^{(k)}, \gamma_{(k)}, F_{(k)}), \quad \text{for } 0 \leq k \leq n.$$

Generators of the K-theory $K_0(C(\mathbb{CP}_q^n))$ were also given in [10] (*cf.* also [11]) as projections whose entries are polynomial functions, that is to say its entries are in the coordinate algebra $\mathcal{A}(\mathbb{CP}_q^n)$ rather than the C^* -algebra $C(\mathbb{CP}_q^n)$.

Before we recall these generators explicitly for later use, we need to pause for some notation. The q -analogue of an integer $n \in \mathbb{Z}$ is given by

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}};$$

it is defined for $q \neq 1$ and is equal to n in the limit $q \rightarrow 1$. For any $n \geq 0$, one defines the factorial of the q -number $[n]$ by setting $[0]! := 1$ and then $[n]! := [n][n-1] \cdots [1]$. The q -multinomial coefficients are in turn defined by

$$[j_0, \dots, j_n]! := \frac{[j_0 + \dots + j_n]!}{[j_0]! \cdots [j_n]!}.$$

For $N \in \mathbb{Z}$, let $\Psi_N := (\psi_{j_0, \dots, j_n}^N)$ be the vector-valued function on S_q^{2n+1} with components

$$(3.6) \quad \psi_{j_0, \dots, j_n}^N := \begin{cases} [j_0, \dots, j_n]!^{\frac{1}{2}} q^{-\frac{1}{2} \sum_{r < s} j_r j_s} (z_0^{j_0})^* \cdots (z_n^{j_n})^* & \text{for } N \geq 0, \\ [j_0, \dots, j_n]!^{\frac{1}{2}} q^{\frac{1}{2} \sum_{r < s} j_r j_s + \sum_{r=0}^n r j_r} z_0^{j_0} \cdots z_n^{j_n} & \text{for } N \leq 0, \end{cases}$$

with $j_0 + \dots + j_n = |N|$. Then $\Psi_N^* \Psi_N = 1$ and $P_N := \Psi_N \Psi_N^*$ is a projection in a matrix algebra of a certain size:

$$(3.7) \quad P_N \in M_{d_N}(\mathcal{A}(\mathbb{CP}_q^n)), \quad d_N := \binom{|N| + n}{n},$$

(this was proven in [8], generalizing the special case where $n = 2$ in [9]). By construction, the entries of the matrix P_N are $U(1)$ -invariant and so they are indeed elements of the algebra $\mathcal{A}(\mathbb{CP}_q^n)$. In particular we see that $P_1 = P$ is the ‘defining’ projection of the algebra $\mathcal{A}(\mathbb{CP}_q^n)$ given before with relations in (3.3).

We let $[P_N]$ denote the class in $K_0(C(\mathbb{CP}_q^n))$ of the projection P_N and let $[\mu_k]$ denote the classes in $K^0(C(\mathbb{CP}_q^n))$ of the Fredholm modules (3.5). The following result was proved in [10] (cf. Props. 4 and 5 there).

Proposition 3.1. *For all $N \in \mathbb{N}$ and for all $0 \leq k \leq n$ it holds that*

$$\langle [\mu_k], [P_{-N}] \rangle := \text{Tr}_{\mathcal{H}_k}(\gamma_{(k)}(\pi^{(k)}(\text{Tr } P_{-N})) = \binom{N}{k},$$

with $\binom{N}{k} := 0$ when $k > N$. Moreover, the elements $[\mu_0], \dots, [\mu_n]$ are generators of $K^0(C(\mathbb{CP}_q^n))$, and the elements $[P_0], \dots, [P_{-n}]$ are generators of $K_0(C(\mathbb{CP}_q^n))$.

Indeed, the matrix of couplings $M \in M_{n+1}(\mathbb{Z})$ with $M_{ij} := \langle [\mu_i], [P_{-j}] \rangle = \binom{j}{i}$, for $i, j = 0, 1, \dots, n$, has inverse with integer entries $(M^{-1})_{ij} = (-1)^{i+j} \binom{j}{i}$. Thus the aforementioned elements are a basis of $\mathbb{Z}^{n+1}$ as a $\mathbb{Z}$ -module, which is equivalent to saying that they generate $\mathbb{Z}^{n+1}$ as an Abelian group.

3.2. Line bundles. It is well known that the algebra inclusion $\mathcal{A}(\mathbb{CP}_q^n) \hookrightarrow \mathcal{A}(S_q^{2n+1})$ is a quantum principal bundle with structure group $U(1)$. To each projection P_N there corresponds a line bundle associated to this principal bundle, as we now describe.

The column vector Ψ_N has d_N entries, all of which are elements of $\mathcal{A}(S_q^{2n+1})$. We consider the collection

$$(3.8) \quad \mathcal{L}_N := \left\{ \varphi_N := v \cdot \Psi_N = \sum_{j_0 + \dots + j_n = N} v_{j_0, \dots, j_n} \psi_{j_0, \dots, j_n}^N \right\},$$

where $v = (v_{j_0, \dots, j_n}) \in (\mathcal{A}(\mathbb{CP}_q^n))^{d_N}$. Each $\mathcal{L}_N$ consists of elements of $\mathcal{A}(S_q^{2n+1})$ which transform under the $U(1)$ action given in (3.2), as $\varphi_N \mapsto \varphi_N \lambda^{-N}$. In particular we have $\mathcal{L}_0 = \mathcal{A}(\mathbb{CP}_q^n)$. By their very definition each $\mathcal{L}_N$ is an $\mathcal{A}(\mathbb{CP}_q^n)$ -bimodule – the bimodule of equivariant maps for the irreducible representation of $U(1)$ with weight N . It also holds that

$$(3.9) \quad \mathcal{L}_N \otimes_{\mathcal{A}(\mathbb{CP}_q^n)} \mathcal{L}_{N'} \simeq \mathcal{L}_{N+N'}$$

(cf. [11, Lem. 7.5] and also [18, Prop. 3.1]) and so, in particular,

$$(3.10) \quad (\mathcal{L}_N)^{\otimes_{\mathcal{A}(\mathbb{CP}_q^n)} M} \simeq \mathcal{L}_{MN}.$$

An argument as in [10, Prop. 3.3] yields isomorphisms $\mathcal{L}_N \simeq (\mathcal{A}(\mathbb{CP}_q^n))^{d_N} P_N$ as left $\mathcal{A}(\mathbb{CP}_q^n)$ -modules and $\mathcal{L}_N \simeq P_{-N} (\mathcal{A}(\mathbb{CP}_q^n))^{d_N}$ as right $\mathcal{A}(\mathbb{CP}_q^n)$ -modules. Clearly then, we have to make a choice: we always use the left $\mathcal{A}(\mathbb{CP}_q^n)$ -module identification and denote the class of the projection P_N by $[\mathcal{L}_N]$ as an element of the group $K_0(C(\mathbb{CP}_q^n))$.

For each $N \in \mathbb{Z}$ the module $\mathcal{L}_N$ describes a line bundle, in the sense that its ‘rank’ (as computed by pairing with $[\mu_0]$) is equal to 1. It is completely characterized by its ‘first Chern number’ (as computed by pairing with the class $[\mu_1]$). Indeed, using an argument similar to that of the proof of Prop. 3.1 one shows the following.

Proposition 3.2. *For all $N \in \mathbb{Z}$ it holds that*

$$\langle [\mu_0], [\mathcal{L}_N] \rangle = 1 \quad \text{and} \quad \langle [\mu_1], [\mathcal{L}_N] \rangle = -N.$$

From the above discussion, the line bundle $\mathcal{L}_{-1}$ emerges as a central character: from Prop. 3.1 its only non-vanishing charges are $\langle [\mu_0], [\mathcal{L}_{-1}] \rangle = 1$ and $\langle [\mu_1], [\mathcal{L}_{-1}] \rangle = 1$. The bundle $\mathcal{L}_{-1}$ is the *tautological line bundle* for the quantum projective space $\mathbb{CP}_q^n$.

Now consider the element in $K_0(C(\mathbb{CP}_q^n))$ given by

$$(3.11) \quad u := 1 - [\mathcal{L}_{-1}],$$

of which we can take powers using the identification (3.10). For $j \geq 0$, as elements in K-theory, one has then

$$(3.12) \quad u^j = (1 - [\mathcal{L}_{-1}])^j \simeq \sum_{N=0}^j (-1)^N \binom{j}{N} [\mathcal{L}_{-N}].$$

Proposition 3.3. *For $0 \leq j \leq n$ and for $0 \leq k \leq n$, it holds that*

$$(3.13) \quad \langle [\mu_k], u^j \rangle = \begin{cases} 0 & \text{for } j \neq k \\ (-1)^j & \text{for } j = k \end{cases},$$

while for all $0 \leq k \leq n$ it holds that

$$(3.14) \quad \langle [\mu_k], u^{n+1} \rangle = 0.$$

Proof. Denoting as before by $[\mathcal{L}_{-N}]$ the class of the projection P_{-N} and setting $\binom{N}{k} := 0$ when $k > N$, we compute using Prop. 3.1 that

$$\langle [\mu_k], u^j \rangle = \sum_{N=0}^j (-1)^N \binom{j}{N} \langle [\mu_k], [\mathcal{L}_{-N}] \rangle = \sum_{N=k}^j (-1)^N \binom{j}{N} \binom{N}{k}.$$

If $k > j$ this vanishes again due to $\binom{N}{k} := 0$ for $k > N$. On the other hand, if $k \leq j$, it is

$$\langle [\mu_k], u^j \rangle = \frac{j!}{k!} \sum_{N=k}^j \frac{(-1)^N}{(j-N)!(N-k)!}$$

and an act of direct computation yields (3.13). Similarly, one computes that

$$\langle [\mu_k], u^{n+1} \rangle = \frac{(n+1)!}{k!} \sum_{N=k}^{n+1} \frac{(-1)^N}{(n+1-N)!(N-k)!} = 0,$$

thus completing the proof. $\square$

The element $u = \chi([\mathcal{L}_{-1}]) := 1 - [\mathcal{L}_{-1}]$ shall be named the *Euler class* of the line bundle $\mathcal{L}_{-1}$, in analogy with the classical case (cf. [16, IV.1.13]). Since for $0 \leq k \leq n$ the elements $[\mu_k]$ are generators of $K^0(C(\mathbb{CP}_q^n))$, the fact that $\langle [\mu_k], u^{n+1} \rangle = 0$ for $0 \leq k \leq n$ amounts to saying that $u^{n+1} = 0$ in $K_0(C(\mathbb{CP}_q^n))$. On the other hand, since the elements $[\mathcal{L}_{-k}]$ for $0 \leq k \leq n$ are generators of $K_0(C(\mathbb{CP}_q^n))$, the results in (3.13) says that the elements $[\mu_k]$ and $(-u)^j$ for $0 \leq k, j \leq n$ form dual bases. These two facts lead to the following analogue of the classical result (cf. [16, Cor. IV.2.11]).

Proposition 3.4. *It holds that*

$$K_0(C(\mathbb{CP}_q^n)) \simeq \mathbb{Z}[\mathcal{L}_{-1}]/(1 - [\mathcal{L}_{-1}])^{n+1} \simeq \mathbb{Z}[u]/u^{n+1}$$

where $u = \chi([\mathcal{L}_{-1}]) := 1 - [\mathcal{L}_{-1}]$ is the Euler class of the line bundle $\mathcal{L}_{-1}$.

4. QUANTUM LENS SPACES

Next we come to describe the algebras of functions on quantum lens spaces and non-commutative ‘line bundles’ thereon. Indeed, we define quantum lens spaces as ‘direct sums of line bundles’ and algebras of their functions in terms of corresponding ‘modules of sections’. *A posteriori* these algebra of functions are seen as subalgebras of functions on odd-dimensional quantum spheres which are invariant for the action of a cyclic group.

In such a manner there are natural principal and associated fibrations. This gives rise to a natural family of representatives of classes in the K-theory of quantum lens spaces.

4.1. Functions on quantum lens spaces. Fix an integer $r \geq 2$ and define

$$(4.1) \quad \mathcal{A}(\mathbb{L}_q^{(n,r)}) := \bigoplus_{N \in \mathbb{Z}} \mathcal{L}_{rN}.$$

Then proving the following is straightforward.

Proposition 4.1. *The vector space $\mathcal{A}(\mathbb{L}_q^{(n,r)})$ is a $*$ -algebra made of all elements of $\mathcal{A}(\mathbb{S}_q^{2n+1})$ which are invariant under the action $\alpha_r : \mathbb{Z}_r \rightarrow \text{Aut}(\mathcal{A}(\mathbb{S}_q^{2n+1}))$ of the cyclic group $\mathbb{Z}_r$ generated by the map*

$$(4.2) \quad (z_0, z_1, \dots, z_n) \mapsto (e^{2\pi i/r} z_0, e^{2\pi i/r} z_1, \dots, e^{2\pi i/r} z_n).$$

We think of the algebra $\mathcal{A}(\mathbb{L}_q^{(n,r)})$ as the coordinate algebra of an underlying quantum space $\mathbb{L}_q^{(n,r)}$, which is named the *quantum lens space* of dimension $2n+1$ (and index r); it is a deformation of the classical lens space $L^{(n,r)} = \mathbb{S}^{2n+1}/\mathbb{Z}_r$ of the same dimension. The C^* -algebra $C(\mathbb{L}_q^{(n,r)})$ of continuous functions on the quantum lens space, the universal C^* -completion of $\mathcal{A}(\mathbb{L}_q^{(n,r)})$, is part of the general family of lens spaces defined in [15].

Of course, the value $r = 1$ is also possible but this does not yield anything new. Indeed, in that case one has $\mathbb{L}_q^{(n,1)} = \mathbb{S}_q^{2n+1}$ and the above expression (4.1) is nothing other than the well known vector space decomposition

$$(4.3) \quad \mathcal{A}(\mathbb{S}_q^{2n+1}) = \bigoplus_{N \in \mathbb{Z}} \mathcal{L}_N.$$

Clearly $\mathcal{A}(\mathbb{CP}_q^n)$ is a subalgebra of $\mathcal{A}(\mathbb{L}_q^{(n,r)})$. In parallel with the $U(1)$ quantum principal bundle $\mathcal{A}(\mathbb{CP}_q^n) \hookrightarrow \mathcal{A}(\mathbb{S}_q^{2n+1})$ there is indeed more structure. We state this in two propositions whose direct proofs, given for the sake of completeness, are deferred to App. C.

Proposition 4.2. *The algebra inclusion $\mathcal{A}(\mathbb{CP}_q^n) \hookrightarrow \mathcal{A}(\mathbb{L}_q^{(n,r)})$ is a quantum principal bundle with structure group $\tilde{U}(1) := U(1)/\mathbb{Z}_r \simeq U(1)$. In particular, one finds that*

$$\mathcal{A}(\mathbb{CP}_q^n) = \mathcal{A}(\mathbb{L}_q^{(n,r)})^{\tilde{U}(1)},$$

in analogy with the identification $\mathcal{A}(\mathbb{CP}_q^n) = \mathcal{A}(\mathbb{S}_q^{2n+1})^{U(1)}$ as defined before.

The $U(1)$ and $\tilde{U}(1)$ principal bundles over the quantum projective space $\mathbb{CP}_q^n$ as in Prop. 4.2 are related by a $\mathbb{Z}_r$ principal bundle structure over the quantum lens space $\mathbb{L}_q^{(n,r)}$. Indeed, it is also not difficult to verify the following.

Proposition 4.3. *The algebra inclusion $\mathcal{A}(\mathbb{L}_q^{(n,r)}) \hookrightarrow \mathcal{A}(\mathbb{S}_q^{2n+1})$ is a quantum principal bundle with structure group $\mathbb{Z}_r$.*

Remark 4.4. Note that, with $\tilde{\mathbb{Z}}$ denoting the Pontryagin dual of the quotient group $\tilde{U}(1)$ (so that there is an injection $\mathbb{Z} \hookrightarrow \tilde{\mathbb{Z}}$ given by multiplication by r), the decomposition (4.1) may be written, in parallel with the decomposition (4.3), as

$$(4.4) \quad \mathcal{A}(\mathbb{L}_q^{(n,r)}) = \bigoplus_{N \in \tilde{\mathbb{Z}}} \mathcal{L}_N.$$

4.2. Pulling back line bundles. Having the principal bundle $j : \mathcal{A}(\mathbb{CP}_q^n) \hookrightarrow \mathcal{A}(\mathbb{L}_q^{(n,r)})$, we proceed now to ‘pull-back’ the associated line bundles from $\mathbb{CP}_q^n$ to $\mathbb{L}_q^{(n,r)}$. We are led to the following natural definition.

Definition 4.5. For each $\mathcal{A}(\mathbb{CP}_q^n)$ -bimodule $\mathcal{L}_N$ as in (3.8) (a line bundle over $\mathbb{CP}_q^n$), its ‘pull-back’ to $\mathbb{L}_q^{(n,r)}$ is the $\mathcal{A}(\mathbb{L}_q^{(n,r)})$ -bimodule

$$(4.5) \quad j_*(\mathcal{L}_N) := \left\{ \tilde{\varphi}_N = v \cdot \Psi_N = \sum_{j_0+\dots+j_n=N} v_{j_0,\dots,j_n} \psi_{j_0,\dots,j_n}^N \right\},$$

for $v = (v_{j_0,\dots,j_n}) \in (\mathcal{A}(\mathbb{L}_q^{(n,r)}))^{d_N}$. We shall often use the shorthand $j_*(\mathcal{L}_N) := \tilde{\mathcal{L}}_N$.

By embedding the cyclic group $\mathbb{Z}_r$ into $U(1)$ via the r -th roots of unity, each $\tilde{\mathcal{L}}_N$ is made of elements of $\mathcal{A}(\mathbb{S}_q^{2n+1})$ which transform as $\tilde{\varphi}_N \mapsto \tilde{\varphi}_N e^{-2\pi i N/r}$ under the $U(1)$ -action of Prop. 4.1. By its very definition, $\tilde{\mathcal{L}}_N$ is an $\mathcal{A}(\mathbb{L}_q^{(n,r)})$ -bimodule. Once again, arguments like those of [10, Prop. 3.3] for the $\tilde{\mathcal{L}}_N$ yield the following.

Proposition 4.6. There are left $\mathcal{A}(\mathbb{L}_q^{(n,r)})$ -module isomorphisms

$$\tilde{\mathcal{L}}_N \simeq (\mathcal{A}(\mathbb{L}_q^{(n,r)}))^{d_N} P_N$$

and right $\mathcal{A}(\mathbb{L}_q^{(n,r)})$ -module isomorphisms

$$\tilde{\mathcal{L}}_N \simeq P_{-N}(\mathcal{A}(\mathbb{L}_q^{(n,r)}))^{d_N}.$$

We stress that the projections P_N here are those constructed before, around (3.6) and (3.7), taken now as elements of the group $K_0(C(\mathbb{L}_q^{(n,r)}))$. Just as for the modules $\mathcal{L}_N$, we need to make a choice of representatives: we use the left $\mathcal{A}(\mathbb{L}_q^{(n,r)})$ -module identification and denote by $[\tilde{\mathcal{L}}_N]$ the class of the projection P_N as an element in $K_0(C(\mathbb{L}_q^{(n,r)}))$.

The pull-back of line bundles in Defn. 4.5 induces a map

$$(4.6) \quad j_* : K_0(C(\mathbb{CP}_q^n)) \rightarrow K_0(C(\mathbb{L}_q^{(n,r)})).$$

The marked difference between the module $\mathcal{L}_N$ versus its pull-back $\tilde{\mathcal{L}}_N$ is that, while each $\mathcal{L}_N$ is not free when $N \neq 0$ (as a consequence of Prop. 3.2), this need not be the case for $\tilde{\mathcal{L}}_N$, that is the projection P_N could be trivial (i.e. equivalent to 1) in $K_0(C(\mathbb{L}_q^{(n,r)}))$.

Indeed, the pull-back $\tilde{\mathcal{L}}_{-r}$ of the line bundle $\mathcal{L}_{-r}$ from the projective space $\mathbb{CP}_q^n$ to the lens space $\mathbb{L}_q^{(n,r)}$ is free: recall that the corresponding projection is $P_{-r} := \Psi_{-r} \Psi_{-r}^*$ and here the vector-valued function Ψ_{-r} has entries in the algebra $\mathcal{A}(\mathbb{L}_q^{(n,r)})$ itself. Thus the condition $\Psi_{-r}^* \Psi_{-r} = 1$ implies that the projector P_{-r} is equivalent to 1, that is to say, the class of the module $\tilde{\mathcal{L}}_{-r}$ is trivial in the group $K_0(C(\mathbb{L}_q^{(n,r)}))$. It follows that $(\tilde{\mathcal{L}}_{-N})^{\otimes r} \simeq \tilde{\mathcal{L}}_{-rN}$ also has trivial class for any $N \in \mathbb{Z}$, the tensor product being taken over $\mathcal{A}(\mathbb{L}_q^{(n,r)})$. Such pulled-back line bundles $\tilde{\mathcal{L}}_{-N}$ thus define *torsion classes* and, as we shall see later on, they generate the group $K_0(C(\mathbb{L}_q^{(n,r)}))$.

At this point it is pertinent to introduce a second crucial ingredient to the discussion, in the form of a natural map

$$(4.7) \quad \alpha : K_0(C(\mathbb{CP}_q^n)) \rightarrow K_0(C(\mathbb{CP}_q^n)),$$

where α is multiplication by the Euler class $\chi(\mathcal{L}_{-r}) := 1 - [\mathcal{L}_{-r}]$ of the line bundle $\mathcal{L}_{-r}$. The central idea of our paper is to combine this map with the pull-back map (4.6) into

a sequence for our quantum lens spaces that parallels the classical Gysin sequence (2.4). Such is the topic of the next section.

5. THE GYSIN SEQUENCE FOR QUANTUM LENS SPACES

In this section we arrive at a Gysin sequence for quantum lens spaces by invoking some general properties of K-theory associated to circle actions on C^* -algebras. We refer to App. B for a review of the various constructions in bivariant K-theory that we need, as well as for the notation that we use in this section.

5.1. Construction of the sequence. Let us lighten our notation by writing

$$A := C(L_q^{(n,r)}), \quad F := C(\mathbb{CP}_q^n).$$

By the universality property of these C^* -algebras, the action of $\tilde{U}(1)$ on $\mathcal{A}(L_q^{(n,r)})$ extends uniquely to a strongly continuous circle action $\sigma : \tilde{U}(1) \rightarrow \text{Aut}(A)$ on A . From Prop. 4.2 the C^* -algebra F sits inside A as the fixed point subalgebra, namely

$$F = \{a \in A : \sigma_t(a) = a \text{ for all } t \in \tilde{U}(1)\}.$$

Following [6], in this situation one has a faithful conditional expectation

$$\tau : A \rightarrow F, \quad \tau(a) := \int_0^{2\pi} \sigma_t(a) dt,$$

from which one obtains an F -valued inner product on A by defining

$$\langle \cdot, \cdot \rangle_F : A \times A \rightarrow F, \quad \langle a, b \rangle_F := \tau(a^*b).$$

The properties of the conditional expectation imply that this equips A with the structure of a right pre-Hilbert F -module and we write X for the Hilbert module obtained by completing A in the resulting topology. The infinitesimal generator of the circle action $\sigma : \tilde{U}(1) \rightarrow \text{Aut}(A)$ on A determines an unbounded self-adjoint regular operator

$$\mathfrak{D} : \mathfrak{Dom}(\mathfrak{D}) \rightarrow X$$

on the Hilbert module X . It is not difficult to check as in [18, Prop. 3.1] that the action of $\tilde{U}(1)$ on A has full spectral subspaces (*cf.* Defn. B.4) and so according to [6, Prop. 2.9] the pair $(X, \mathfrak{D})$ determines a class in the bivariant K-theory $KK_1(A, F)$. The internal Kasparov product with the class $[(X, \mathfrak{D})]$ thus furnishes us with a pair of maps

$$\text{Ind}_{\mathfrak{D}} : K_*(A) \rightarrow K_{*+1}(F), \quad \text{Ind}_{\mathfrak{D}}(-) := - \hat{\otimes}_A [(X, \mathfrak{D})].$$

Remark 5.1. We give more detail of this construction in App. B, where we also present a brief review of Kasparov's bivariant KK-theory and the internal Kasparov product

$$(5.1) \quad - \hat{\otimes}_A - : KK_i(A, B) \times KK_j(B, C) \rightarrow KK_{i+j}(A, C)$$

for separable C^* -algebras A, B, C (*cf.* [17, 2] for full details).

In our new notation, the multiplication in (4.7) by the Euler class $\chi(\mathcal{L}_{-r}) := 1 - [\mathcal{L}_{-r}]$ of the line bundle $\mathcal{L}_{-r}$ yields a map

$$\alpha : K_0(F) \rightarrow K_0(F),$$

whereas the map in (4.6) induced by the inclusion map $j : F \hookrightarrow A$ gives

$$j_* : K_0(F) \rightarrow K_0(A).$$

Assembling all of these together yields a sequence

$$(5.2) \quad 0 \rightarrow K_1(A) \xrightarrow{\text{Ind}_{\mathfrak{D}}} K_0(F) \xrightarrow{\alpha} K_0(F) \xrightarrow{j_*} K_0(A) \xrightarrow{\text{Ind}_{\mathfrak{D}}} 0,$$

where we already use the fact that in the last term of the sequence we have $K_1(F) = 0$. We claim that it is an *exact* sequence: this is proved in the next section. The sequence (5.2) will be called the *Gysin sequence* for the quantum lens space $L_q^{(n,r)}$.

Remark 5.2. Let us stress that it is not invidious to put zeros at the beginning and end of the sequence (5.2). At this point we are saying nothing about exactness of the sequence, so we are not yet saying that the maps $\text{Ind}_{\mathfrak{D}}$ and j_* are respectively injective and surjective: for the time being this is merely a claim.

5.2. Exactness of the sequence. The *mapping cone* of the pair (F, A) is the C^* -algebra

$$M(F, A) := \{f \in C([0, 1], A) \mid f(0) = 0, f(1) \in F\}.$$

With $S(A) := C_0((0, 1)) \otimes A$ being the *suspension* of the C^* -algebra A , it is clear that

$$(5.3) \quad 0 \rightarrow S(A) \xrightarrow{i} M(F, A) \xrightarrow{\text{ev}} F \rightarrow 0,$$

where $i(f \otimes a)(t) := f(t)a$ and $\text{ev}(f) := f(1)$, is an exact sequence of C^* -algebras.

The six term exact sequence in K-theory corresponding to (5.3) is of the form

$$(5.4) \quad \begin{array}{ccccc} K_1(A) & \longrightarrow & K_0(M(F, A)) & \longrightarrow & K_0(F) \\ \uparrow & & & & \downarrow \\ K_1(F) & \longleftarrow & K_1(M(F, A)) & \longleftarrow & K_0(A), \end{array}$$

having used the isomorphism $K_0(S(A)) = K_1(A)$ and Bott periodicity in the top left and bottom right corners. Using the vanishing of $K_1(F)$, the sequence (5.4) degenerates to

$$(5.5) \quad 0 \rightarrow K_1(A) \xrightarrow{i_*} K_0(M(F, A)) \xrightarrow{\text{ev}_*} K_0(F) \xrightarrow{j_*} K_0(A) \rightarrow K_1(M(F, A)) \rightarrow 0.$$

As before, the inclusion $i : S(A) \rightarrow M(F, A)$ induces the map $i_* : K_1(A) \rightarrow K_0(M(F, A))$ whereas, up to Bott periodicity, the map

$$j_* : K_0(F) \rightarrow K_0(A) \cong K_1(S(A))$$

is essentially induced by the algebra inclusion $j : F \rightarrow A$ (cf. also [5, Lem. 3.1]). Finally, the map $\text{ev}_* : K_0(M(F, A)) \rightarrow K_0(F)$ is best given by means of a very practical realization of $K_0(M(F, A))$ in terms of partial isometries [26, 5]. Let us write $V_m(F, A)$ for the collection of partial isometries $v \in M_m(A)$ such that the associated projections v^*v and vv^* belong to $M_m(F)$. Then defining

$$V(F, A) := \bigcup_m V_m(F, A)$$

one finds that a suitable addition can be defined in $V(F, A)$ and, modulo certain equivalence relations, the quotient space of $V(F, A)$ by the equivalence relations and $K_0(M(F, A))$ are isomorphic as groups (see App. B for more details),

$$(5.6) \quad : V(F, A)/\sim \rightarrow K_0(M(F, A)).$$

Then, the map ev_* in (5.5) can be given by

$$\text{ev}_* : K_0(M(F, A)) \rightarrow K_0(F), \quad \text{ev}_*([v]) := [v^*v] - [vv^*],$$

for $v \in M_m(A)$ a partial isometry representing a class in $K_0(M(F, A))$ under the isomorphism in (5.6) (cf. [26, Lem. 2.3]).

It is shown in [5, §4] that the pair $(X, \mathfrak{D})$ defining the class in $KK_1(A, F)$ used in §5.1 above has a canonical lift to a pair $(\hat{X}, \hat{\mathfrak{D}})$ representing a class in $KK_0(M(F, A), F)$. The internal Kasparov product of $K_0(M(F, A))$ with this class yields a map

$$\text{Ind}_{\hat{\mathfrak{D}}} : K_0(M(F, A)) \rightarrow K_0(F).$$

The output of this map is explicitly represented by the index of a certain operator:

$$(5.7) \quad \text{Ind}_{\hat{\mathfrak{D}}}([v]) = \text{Ker}(PvP)|_{v^*v(P\mathcal{E})} - \text{Ker}(Pv^*P)|_{vv^*(P\mathcal{E})}.$$

Here $\mathcal{E} = X^m$ denotes the domain of the projection $v^*v \in M_m(F)$ and P is the spectral projection for the self-adjoint operator $\mathfrak{D}$ associated to the non-negative real axis. We also give more details of this construction in App. B.

Now recall that A is a Cuntz-Krieger algebra associated to a graph which is connected, row-finite and has neither sources nor sinks [15]. It follows [5, Lem. 6.7] that $K_1(M(F, A)) = 0$ and that the map (5.7) is an isomorphism [5, Prop. 6.8]. Thus,

$$(5.8) \quad K_0(M(F, A)) \simeq K_0(F) \simeq \mathbb{Z}^{n+1}.$$

As a consequence, there is a very easy description of the partial isometries which generate $K_0(M(F, A))$. Recall from the discussion at the end of §4 that, upon pulling-back to A , one finds that for all $N \in \mathbb{Z}$ the projections P_{rN} become equivalent to the identity. This is equivalent to say that for any $M \in \mathbb{Z}$ the projections P_{rN} and $P_{r(N+M)}$ are equivalent for all $N \in \mathbb{Z}$. Indeed, one can explicitly exhibit partial isometries relating these projectors. Taking the particular case $M = 1$, these partial isometries are the elements $v_N \in M_{(d_{r(N+1)}, d_{rN})}(A)$, with the integers $d_{(\cdot)}$ as in (3.7), given by

$$(5.9) \quad v_N = \Psi_{r(N+1)} \Psi_{rN}^\dagger, \quad N = 0, -1, \dots, -n;$$

clearly $v_N^*v_N = P_{rN}$ and $v_N v_N^* = P_{r(N+1)}$ for $N = 0, -1, \dots, -n$. With our conventions, the entries of v_N are elements of A which are homogeneous of degree $-r$ for the action of $U(1)$.

Proposition 5.3. *The partial isometries (5.9) form a basis of $K_0(M(F, A))$.*

Proof. From (5.8) we just need $n + 1$ independent generators. Now, since the map (5.7) is an isomorphism, the partial isometries v_N are independent (and thus a basis for $K_0(M(F, A))$) if and only if the classes $\text{Ind}_{\hat{\mathfrak{D}}}([v_N])$ are so. Since $Pv_N P$ is essentially a ‘left degree shift’ operator on the elements of non-negative homogeneous degree in $PX^{d_{rN}}$ it has no cokernel. Its kernel thus determines the index:

$$\text{Ind}_{\hat{\mathfrak{D}}}([v_N]) = \left[P_{rN} X_0^{d_{rN}} \right] = [P_{rN}].$$

Now, it follows from Prop. 3.1 that the matrix of pairings $\{\langle [\mu_k], [P_{-rN}] \rangle = \binom{rN}{k}\}$ is invertible, thus proving that the elements P_{rN} for $N = 0, -1, \dots, -n$ are independent. We note that these projections do not form a basis for $K_0(F)$ since the matrix of pairings, while invertible over $\mathbb{Q}$, is not so over $\mathbb{Z}$, that is it does not belong to $GL(n + 1, \mathbb{Z})$. $\square$

We are ready to state and prove¹ our central theorem; it directly implies exactness of the Gysin sequence (5.2).

Theorem 5.4. *There is a diagram*

$$(5.10) \quad \begin{array}{ccccccccc} 0 & \longrightarrow & K_1(A) & \xrightarrow{i_*} & K_0(M(F, A)) & \xrightarrow{\text{ev}_*} & K_0(F) & \xrightarrow{j_*} & K_0(A) & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow \text{Ind}_{\widehat{\mathfrak{D}}} & & \downarrow B_F & & \downarrow B_A & & \\ 0 & \longrightarrow & K_1(A) & \xrightarrow{\text{Ind}_{\widehat{\mathfrak{D}}}} & K_0(F) & \xrightarrow{\alpha} & K_0(F) & \xrightarrow{j_*} & K_0(A) & \longrightarrow & 0 \end{array}$$

in which every square commutes and each vertical arrow is an isomorphism of groups.

Proof. That the first square commutes is precisely [5, Thm. 5.1]. The maps B_F and B_A are defined to be multiplication by the class $-[P_{-r}]$ in $K_0(F)$ and its image in $K_0(A)$. For the second square we compute that for each $N = 0, -1, \dots, -n$, we have

$$\alpha(\text{Ind}_{\widehat{\mathfrak{D}}}([v_N])) = (1 - [P_{-r}])[P_{rN}] = -[P_{-r}](P_{rN} - [P_{r(N+1)}]) = B_F(\text{ev}_*([v_N])).$$

Commutativity of the third square is of course obvious. $\square$

6. THE K-THEORY OF QUANTUM LENS SPACES

We put to work the Gysin sequence (5.2) by using it to compute the K-theory of our quantum lens spaces. We stress that the strength of our construction is not only the matter of computing the K-theory groups: this could be done by means of graph algebras as in [14]. We shall also obtain explicit generators as classes of ‘line bundles’, generically torsion ones. This is illustrated by working out some explicit examples.

Since the map j_* in (5.2) is surjective, the group $K_0(C(L_q^{(n,r)}))$ can be obtained by ‘pulling back’ classes from $K_0(C(\mathbb{CP}_q^n))$. Now, as shown in Prop. 3.4,

$$K_0(C(\mathbb{CP}_q^n)) \cong \mathbb{Z}[u]/u^{n+1}$$

with $u := 1 - [\mathcal{L}_{-1}]$ the Euler class of the line bundle $\mathcal{L}_{-1}$. Moreover, the Euler class $\chi(\mathcal{L}_{-r})$ of the line bundle $\mathcal{L}_{-r}$ is just

$$\chi(\mathcal{L}_{-r}) = 1 - [\mathcal{L}_{-r}] = 1 - [\mathcal{L}_{-1}]^r = 1 - (1 - u)^r.$$

As a consequence, the map α in (5.2) can be given as an $(n+1) \times (n+1)$ matrix A with respect to the $\mathbb{Z}$ -module basis $\{1, u, \dots, u^n\}$ of $K^0(C(\mathbb{CP}_q^n)) \simeq \mathbb{Z}^{n+1}$. This leads to

$$(6.1) \quad K_1(C(L_q^{(n,r)})) \simeq \ker(\alpha) = \ker(A), \quad K_0(C(L_q^{(n,r)})) \simeq \text{coker}(\alpha) = \text{coker}(A),$$

as $\mathbb{Z}$ -module identifications via the surjective ‘pull-back’ map j_* .

Simple algebra allows one to compute explicitly the matrix A of the map α with respect to the $\mathbb{Z}$ -module basis $\{1, u, \dots, u^n\}$. Using the condition $u^{n+1} = 0$ one has

$$\chi(\mathcal{L}_{-r}) = 1 - (1 - u)^r = \sum_{j=1}^{\min(r,n)} (-1)^{j+1} \binom{r}{j} u^j.$$

¹We thank Adam Rennie for explaining to us the explicit forms of the various maps in the exact sequence (5.5), a conversation from which the proof of our theorem followed very naturally.

Thus A is an $(n+1) \times (n+1)$ strictly lower triangular matrix with entries on the j -th sub-diagonal equal to $(-1)^{j+1} \binom{r}{j}$ for $j \leq \min(r, n)$ and zero otherwise:

$$(6.2) \quad A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ r & 0 & 0 & \dots & 0 \\ -\binom{r}{2} & r & 0 & \dots & 0 \\ \binom{r}{3} & -\binom{r}{2} & r & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & r & 0 \end{pmatrix}.$$

The following is then immediate.

Proposition 6.1. *The $(n+1) \times (n+1)$ matrix A has rank n , whence*

$$K_1(C(L_q^{(n,r)})) \simeq \mathbb{Z}.$$

On the other hand, the structure of the cokernel of the matrix A depends on the divisibility properties of the integer r . Since $\text{coker}(A) \simeq \mathbb{Z}^{n+1}/\text{Im}(A)$ and $\text{Im}(A)$ being generated by the columns of A , the vanishing of these columns yields conditions on the generators making them torsion classes in general. Indeed, upon pulling back to the lens space, the vanishing of the j -th column is just the condition that the pulled back line bundles satisfy $\tilde{\mathcal{L}}_{-(r+j)} = \tilde{\mathcal{L}}_{-j}$; thus this vanishing contains geometric information.

However, to quickly determine $\text{coker}(A)$ (although not directly its generators) one can make use of the *Smith normal form* for matrices over a principal ideal domain, such as $\mathbb{Z}$. The basic facts of the theory are recalled in App. A, whilst we mention here that in general there exist invertible matrices P and Q having integer entries which transform A to a diagonal matrix

$$(6.3) \quad \text{Sm}(A) := PAQ = \text{diag}(\alpha_1, \dots, \alpha_n, 0),$$

with integer entries $\alpha_i \geq 1$, ordered in such a way that $\alpha_i \mid \alpha_{i+1}$ for $1 \leq i \leq n$. These integers are algorithmically and explicitly given by $\alpha_1 = d_1(A)$, and $\alpha_i = d_i(A)/d_{i-1}(A)$ for each $2 \leq i \leq n$, where $d_i(A)$ is the greatest common divisor of the non-zero determinants of the minors of order i of the matrix A . The above leads directly to the following.

Proposition 6.2. *It holds that*

$$\text{coker}(A) \simeq \text{coker}(\text{Sm}(A)) = \mathbb{Z} \oplus \mathbb{Z}/\alpha_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/\alpha_n\mathbb{Z}.$$

As a consequence,

$$K_0(C(L_q^{(n,r)})) \simeq \mathbb{Z} \oplus \mathbb{Z}_{\alpha_1} \oplus \dots \oplus \mathbb{Z}_{\alpha_n},$$

with the convention that $\mathbb{Z}_1 = \mathbb{Z}/1\mathbb{Z}$ is the trivial group.

As already mentioned, the merit of our construction is not in the computation of the K-theory groups – these are found for instance by using graph algebras as in [14]. Owing to the explicit diagonalization as in (6.3) and to Prop. 3.4, we also obtain explicit generators as integral combinations of powers of the pull-back to the lens space $L_q^{(n,r)}$ of the generator $u := 1 - [\mathcal{L}_{-1}]$. We illustrate this by computing $K_0(C(L_q^{(n,r)}))$ in some examples.

Example 6.3. If $r = 2$ one computes $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 1$ and $\alpha_n = 2^n$. Hence for $L_q^{(n,2)} = S_q^{2n+1}/\mathbb{Z}_2 = \mathbb{R}P_q^{2n+1}$, the quantum real projective space, we get

$$K_0(C(\mathbb{R}P_q^{2n+1})) = \mathbb{Z} \oplus \mathbb{Z}_{2^n},$$

in agreement with [14, §4.2] (with a shift $n \rightarrow n+1$ from there to here). Moreover, we can construct explicitly the generator of the torsion part of the K-theory group. We claim this is given by $1 - [\tilde{\mathcal{L}}_{-1}]$. First of all, owing to $\tilde{\mathcal{L}}_{-2} \simeq \tilde{\mathcal{L}}_0$ one has

$$(1 - [\tilde{\mathcal{L}}_{-1}])^2 = 2(1 - [\tilde{\mathcal{L}}_{-1}]),$$

and iterating:

$$(1 - [\tilde{\mathcal{L}}_{-1}])^k = 2^{k-1}(1 - [\tilde{\mathcal{L}}_{-1}]).$$

Thus, in a sense one can switch from multiplicative to additive notation. Furthermore, from Prop. (3.4) we know that $u^{n+1} = 0$, with $u = 1 - [\mathcal{L}_{-1}]$. When pulled back to the lens space, owing to $\tilde{\mathcal{L}}_{2N} \simeq \tilde{\mathcal{L}}_0$ and $\tilde{\mathcal{L}}_{2N+1} \simeq \tilde{\mathcal{L}}_{-1}$, this implies that

$$0 = (1 - [\tilde{\mathcal{L}}_{-1}])^{n+1} = 2^n(1 - [\tilde{\mathcal{L}}_{-1}]).$$

This amounts to saying that the generator $1 - [\tilde{\mathcal{L}}_{-1}]$ is cyclic with the correct order 2^n .

Example 6.4. For $n = 1$ there is only one $\alpha_1 = r$. Then in this case one has

$$K_0(C(L_q^{(1,r)})) = \mathbb{Z} \oplus \mathbb{Z}_r.$$

From its very definition $[\tilde{\mathcal{L}}_{-r}] = 1$, thus $\tilde{\mathcal{L}}_{-1}$ generates the torsion part. Alternatively, from $u^2 = 0$ it follows that $\mathcal{L}_{-j} = -(j-1) + j\mathcal{L}_{-1}$ for all $j > 0$; upon lifting to $L_q^{(1,r)}$, for $j = r$ this yields $r(1 - [\tilde{\mathcal{L}}_{-1}]) = 0$, that is $1 - [\tilde{\mathcal{L}}_{-1}]$ is cyclic of order r .

Example 6.5. For $n = 2$ there are two cases, according to whether r is even or odd. For the α 's in Prop. 6.2 one finds:

$$(\alpha_1, \alpha_2) = \begin{cases} (r/2, 2r) & \text{if } r \text{ even} \\ (r, r) & \text{if } r \text{ odd} \end{cases}.$$

As a consequence one has that

$$K_0(C(L_q^{(2,r)})) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{2}} \oplus \mathbb{Z}_{2r} & \text{if } r \text{ even} \\ \mathbb{Z} \oplus \mathbb{Z}_r \oplus \mathbb{Z}_r & \text{if } r \text{ odd} \end{cases}.$$

This is in agreement with [15, Prop. 2.3] (once again with a shift $n \rightarrow n+1$). In particular, for $r = 2$ we get back the case of Example 6.3. In order to identify generators in the two cases, we start from $[\tilde{\mathcal{L}}_{-r}] = 1$. Direct computations from the conditions $[\tilde{\mathcal{L}}_{-(r+j)}] = [\tilde{\mathcal{L}}_{-j}]$ for $j = 0, \dots, r-1$ lead to

$$(6.4) \quad \frac{1}{2}r(r-1)\tilde{u}^2 - r\tilde{u} = 0 \quad \text{and} \quad r\tilde{u}^2 = 0,$$

where $\tilde{u} = 1 - [\tilde{\mathcal{L}}_{-1}]$. Indeed these are just the lifts to the lens space $L_q^{(2,r)}$ of the non-vanishing columns of the corresponding matrix A in (6.2).

When $r = 2k$ is even, we have conditions coming from $(\tilde{\mathcal{L}}_{-2})^k \simeq \tilde{\mathcal{L}}_0$. In fact, due to $[\tilde{\mathcal{L}}_{-2k}] = 1$, one has $(1 - [\tilde{\mathcal{L}}_{-k}])^2 = 2(1 - [\tilde{\mathcal{L}}_{-k}])$, leading to

$$0 = (1 - [\tilde{\mathcal{L}}_{-k}])^3 = 4(1 - [\tilde{\mathcal{L}}_{-k}]) = 4k\tilde{u} - 2k(k-1)\tilde{u}^2.$$

Together with the conditions (6.4) this yields

$$\frac{1}{2}r(\tilde{u}^2 + 2\tilde{u}) = 0 \quad \text{and} \quad 2r\tilde{u} = 0,$$

that is $\tilde{u}^2 + 2\tilde{u}$ is of order $r/2$ while $\tilde{u}$ is of order $2r$ (again, for $r = 2$ this is consistent with the result of Example 6.3, the first ‘generator’ collapsing to the condition $\tilde{u}^2 + 2\tilde{u} = 0$).

When $r = 2k + 1$ is odd, the conditions (6.4) just say that $\tilde{u}$ and $\tilde{u}^2$ are cyclic of order r :

$$r\tilde{u} = 0 \quad \text{and} \quad r\tilde{u}^2 = 0.$$

Example 6.6. When $n = 3$ the selection of generators for the torsion groups is more involved but still ‘doable’. We compute explicitly in App. D the cokernel of the matrix A in (6.2) and list here the K-theory groups as well as the generators obtained by lifting to the lens space the cokernel of A via the surjective map j_* . As before we denote $\tilde{u} = 1 - [\tilde{\mathcal{L}}_{-1}]$. There are now four possibilities. For the α ’s in Prop. 6.2 one finds:

	$6 \mid r$	$2 \mid r, 3 \nmid r$	$2 \nmid r, 3 \mid r$	$2 \nmid r, 3 \nmid r$
α_1	$r/6$	$r/2$	$r/3$	r
α_2	$r/2$	$r/2$	r	r
α_3	$12r$	$4r$	$3r$	r

As a consequence:

Case $r \equiv 0 \pmod{6}$:

$$K_0(C(L_q^{(3,r)})) = \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{6}} \oplus \mathbb{Z}_{\frac{r}{2}} \oplus \mathbb{Z}_{12r}$$

with generators

$$\tilde{u}^3 + 12\tilde{u}, \quad \tilde{u}^2 + 6\tilde{u}, \quad \tilde{u},$$

of order $r/6$, $r/2$ and $12r$, respectively. For the particular case $r = 6$, the first torsion part is absent, one has $\tilde{u}^3 + 12\tilde{u} = 0$, and

$$K_0(C(L_q^{(3,6)})) = \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{72}.$$

Case $r \equiv 2, 4 \pmod{6}$:

$$K_0(C(L_q^{(3,r)})) = \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{2}} \oplus \mathbb{Z}_{\frac{r}{2}} \oplus \mathbb{Z}_{4r}$$

with generators

$$\tilde{u}^3 + 2\tilde{u}^2, \quad \tilde{u}^2 + 2\tilde{u}, \quad \tilde{u},$$

of order $r/2$, $r/2$ and $4r$, respectively. The particular case $r = 2$ goes back to Example 6.3 with the first and second torsion parts absent and the condition $\tilde{u}^2 + 2\tilde{u} = 0$ as in there.

Case $r \equiv 3 \pmod{6}$:

$$K_0(C(L_q^{(3,r)})) = \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{3}} \oplus \mathbb{Z}_r \oplus \mathbb{Z}_{3r}$$

with generators

$$\tilde{u}^3 + 3\tilde{u}, \quad \tilde{u}^2, \quad \tilde{u},$$

of order $r/3$, r and $3r$, respectively. For the particular case $r = 3$ the first torsion part is absent, one has $\tilde{u}^3 + 3\tilde{u} = 0$, and

$$K_0(C(L_q^{(3,3)})) = \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9.$$

Case $r \equiv 1, 5 \pmod{6}$:

$$K_0(C(L_q^{(3,r)})) = \mathbb{Z} \oplus \mathbb{Z}_r \oplus \mathbb{Z}_r \oplus \mathbb{Z}_r$$

with the three generators of order r given by

$$\tilde{u}^3, \quad \tilde{u}^2, \quad \tilde{u}.$$

To further illustrate the construction, we mention the next case of the dimension n , for which we list the K-theory groups.

Example 6.7. When $n = 4$ there are 8 possibilities. For the α 's in Prop. 6.2 one finds:

	$24 \mid r$	$12 \mid r; 8 \nmid r$	$8 \mid r; 6 \nmid r$	$6 \mid r; 4 \nmid r$	$4 \mid r; 3, 8 \nmid r$	
α_1	$r/24$	$r/12$	$r/8$	$r/6$	$r/4$	
α_2	$r/6$	$r/12$	$r/4$	$r/6$	$r/4$	
α_3	$6r$	$12r$	$4r$	$4r$	$2r$	
α_4	$24r$	$12r$	$8r$	$12r$	$8r$	

....		$3 \mid r; 2 \nmid r$	$2 \mid r; 3, 4 \nmid r$	$2 \nmid r; 3 \nmid r$	
		$r/3$	$r/2$	r	
		$r/3$	$r/2$	r	
		r	$r/2$	r	
		$9r$	$8r$	r	

As a consequence,

$$K_0(C(L_q^{(4,r)})) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{24}} \oplus \mathbb{Z}_{\frac{r}{6}} \oplus \mathbb{Z}_{6r} \oplus \mathbb{Z}_{24r} & r \equiv 0 \pmod{24} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{12}} \oplus \mathbb{Z}_{\frac{r}{12}} \oplus \mathbb{Z}_{12r} \oplus \mathbb{Z}_{12r} & r \equiv 12 \pmod{24} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{8}} \oplus \mathbb{Z}_{\frac{r}{4}} \oplus \mathbb{Z}_{4r} \oplus \mathbb{Z}_{8r} & r \equiv 8, 16 \pmod{24} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{6}} \oplus \mathbb{Z}_{\frac{r}{6}} \oplus \mathbb{Z}_{4r} \oplus \mathbb{Z}_{12r} & r \equiv 6 \pmod{12} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{4}} \oplus \mathbb{Z}_{\frac{r}{4}} \oplus \mathbb{Z}_{2r} \oplus \mathbb{Z}_{8r} & r \equiv 4, 20 \pmod{24} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{3}} \oplus \mathbb{Z}_{\frac{r}{3}} \oplus \mathbb{Z}_r \oplus \mathbb{Z}_{9r} & r \equiv 3, 9 \pmod{12} \\ \mathbb{Z} \oplus \mathbb{Z}_{\frac{r}{2}} \oplus \mathbb{Z}_{\frac{r}{2}} \oplus \mathbb{Z}_{\frac{r}{2}} \oplus \mathbb{Z}_{8r} & r \equiv 2 \pmod{12} \\ \mathbb{Z} \oplus \mathbb{Z}_r \oplus \mathbb{Z}_r \oplus \mathbb{Z}_r \oplus \mathbb{Z}_r & r \equiv 1, 5 \pmod{6} \end{cases}.$$

We leave to the diligent reader the determination of the corresponding generators.

APPENDIX A. THE SMITH NORMAL FORM

The Smith normal form of a matrix over a principal ideal domain is a simple tool for computing its kernel and cokernel. The important result relevant for the present paper is the following [28] (cf. [22, Thm. 26.2 and Thm. 27.1]).

Theorem A.1. *Let A be a $n \times n$ matrix over a principal ideal domain R . Then there exist invertible matrices P and Q over R such that*

$$(A.1) \quad \text{Sm}(A) := PAQ$$

is a diagonal matrix:

$$(A.2) \quad \text{Sm}(A) = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & 0 \\ \vdots & & & \alpha_k & \vdots \\ & & & 0 & \\ & & & & \ddots \\ 0 & & \cdots & & 0 \end{pmatrix}.$$

The non-zero entries $\alpha_i \in R$, which are ordered in such a way that $\alpha_i \mid \alpha_{i+1}$, are given by

$$(A.3) \quad \alpha_1 = d_1(A), \quad \alpha_i = \frac{d_i(A)}{d_{i-1}(A)},$$

for each $2 \leq i \leq k$; here $d_i(A)$ is the greatest common divisor of the nonzero determinants of the minors of order i of the matrix A .

The matrix $\text{Sm}(A)$ is called a *Smith normal form* of A . It is then clear that

$$(A.4) \quad \begin{aligned} \ker(A) &\simeq \ker(\text{Sm}(A)) = R^{n-k}, \\ \text{coker}(A) &\simeq \text{coker}(\text{Sm}(A)) = R^{n-k} \oplus R/\alpha_1 R \oplus \cdots \oplus R/\alpha_k R. \end{aligned}$$

For $R = \mathbb{Z}$ this is equivalent to the classification of finitely generated Abelian groups.

APPENDIX B. BIVARIANT K-THEORY AND INDEX MAPS

Here we briefly recall some of the constructions involved in Kasparov's bivariant K-theory [17]. Throughout we only consider separable C^* -algebras with trivial grading.

For the general background of unbounded operators on Hilbert modules we refer to [19]. Given a (countably generated) right Hilbert B -module $X \rightleftharpoons B$ over a C^* -algebra B , the right B -Hermitian structure on X is denoted $\langle \cdot | \cdot \rangle_B$; we write $\text{End}_B(X)$ for the C^* -algebra of adjointable operators on X . We denote an unbounded linear operator $\mathfrak{D}$ on X with dense domain $\mathfrak{Dom}(\mathfrak{D}) \subseteq X$ simply by $\mathfrak{D} : \mathfrak{Dom}(\mathfrak{D}) \rightarrow X$.

B.1. Kasparov's bivariant K-theory. Given C^* -algebras A and B , a Hilbert A - B -bimodule, denoted $A \rightarrow X \rightleftharpoons B$, is a (possibly $\mathbb{Z}_2$ -graded) right Hilbert B -module X equipped with a $*$ -homomorphism $\phi : A \rightarrow \text{End}_B(X)$. Let $\mathfrak{D} : \mathfrak{Dom}(\mathfrak{D}) \rightarrow X$ be an unbounded self-adjoint regular operator. With these one makes the following definition.

Definition B.1. *The datum $(X, \phi, \mathfrak{D})$ is an odd unbounded Kasparov A - B -bimodule if:*

- (1) *the operators $\phi(a)(1 + \mathfrak{D}^2)^{-1/2}$ are compact endomorphisms for all $a \in A$;*
- (2) *the $*$ -subalgebra*

$$\mathcal{A} := \{a \in A : [\mathfrak{D}, \phi(a)] \in \text{End}_B(X)\} \subseteq A$$

is dense in A .

An even unbounded Kasparov A - B -bimodule is a datum $(X, \phi, \mathfrak{D})$ as above, together with a $\mathbb{Z}_2$ -grading $\Gamma \in \text{End}_B(X)$ which makes the action of A even and the operator $\mathfrak{D}$ odd, that is to say $\phi(a)\Gamma = \Gamma\phi(a)$ for all $a \in A$ and $\mathfrak{D}\Gamma + \Gamma\mathfrak{D} = 0$.

These unbounded bimodules are representatives of the elements of Kasparov's KK-groups. Associated to a given operator $\mathfrak{D}$ on X as before is its *bounded transform*

$$\mathfrak{b}(\mathfrak{D}) := \mathfrak{D}(1 + \mathfrak{D}^2)^{-1/2},$$

which determines $\mathfrak{D}$ uniquely [19]. The triple $(X, \phi, \mathfrak{b}(\mathfrak{D}))$ is a *bounded Kasparov bimodule*: these are defined to be triples (X, ϕ, S) with $S \in \text{End}_B(X)$ such that the operators

$$\phi(a)(S^2 - 1), \quad [S, \phi(a)], \quad \phi(a)(S - S^*)$$

are all compact endomorphisms of X for all $a \in A$. It is shown in [1] that, for every unbounded Kasparov bimodule $(X, \phi, \mathfrak{D})$, its bounded transform $(X, \phi, \mathfrak{b}(\mathfrak{D}))$ is a bounded Kasparov bimodule; conversely every bounded Kasparov bimodule arises in this way as the bounded transform of some unbounded representative.

Given the C^* -algebras A and B , on the set of all bounded Kasparov A - B -bimodules (X, ϕ, S) there is an equivalence relation generated by

- (1) unitary equivalence: this says that $(X, \phi, S) \sim (UX, U\phi U^*, USU^*)$;
- (2) homology: this says that $(X_1, \phi_1, S_1) \sim (X_2, \phi_2, S_2)$ if $S_1\phi_1(a) - S_2\phi_2(a)$ is a compact endomorphism for all $a \in A$.

Definition B.2. *The set of all pairs (X, ϕ, S) of odd bounded Kasparov bimodules modulo equivalence is the bivariant K -theory $KK_1(A, B)$. The set of all even Kasparov bimodules (X, ϕ, S) modulo equivalence is the bivariant K -theory $KK_0(A, B)$.*

An important feature of the KK-groups $KK_*(A, B)$ is that they admit an internal product $KK_i(A, B) \times KK_j(B, C) \rightarrow KK_{i+j}(A, C)$, for $i, j = 0, 1$, with $i + j$ modulo two [17]. As is customary, we do not say anything about the rather complicated construction of this product, since in the situation of the present paper we are rescued by the following very general result, which says that the product with odd K -theory $K_1(A) = KK_1(\mathbb{C}, A)$ is represented by the index of a suitable operator [24, App. A], [5, Thm. 2.3].

Theorem B.3. *Let $(X, \mathfrak{D})$ be an odd unbounded Kasparov A - B -bimodule. Assume $\mathfrak{D}$ has a spectral gap around zero. Then the Kasparov product of $K_1(A)$ with the class of $(X, \mathfrak{D})$ in $KK_1(A, B)$ (resulting in an element in $KK_0(\mathbb{C}, B) = K_0(B)$) is represented by an index map. The latter is explicitly given by*

$$\text{Ind}_{\mathfrak{D}}([u]) := [\text{Ker } PuP] - [\text{Coker } PuP] \in K_0(B),$$

where P denotes the non-negative spectral projection for the self-adjoint operator $\mathfrak{D}$.

B.2. Bivariant K-theory for circle actions. Let A be a C^* -algebra equipped with a strongly continuous circle action

$$(B.1) \quad \sigma : U(1) \rightarrow \text{Aut}(A).$$

It is convenient to think of this action as a 2π -periodic one-parameter group of automorphisms. In this way, one defines the *fixed point* C^* -algebra $F \subseteq A$ to be the C^* -subalgebra

$$F = \{a \in A : \sigma_t(a) = a \text{ for all } t \in \mathbb{R}\}.$$

Since $U(1)$ is compact, following [6] there is a faithful conditional expectation on A defined by

$$\tau : A \rightarrow F, \quad \tau(a) := \int_0^{2\pi} \sigma_t(a) dt.$$

In turn, using this one obtains an F -valued inner product on A by setting

$$\langle \cdot, \cdot \rangle_F : A \times A \rightarrow F, \quad \langle a, b \rangle_F := \tau(a^*b).$$

The properties of the conditional expectation imply that this equips A with the structure of a right pre-Hilbert F -module. Let $X \rightrightarrows F$ be the right Hilbert module completion of A in the topology determined by the norm $\|a\|_X^2 := \|\langle a, a \rangle_F\|$. For each $k \in \mathbb{Z}$ we denote the corresponding eigenspace of the action σ on A by

$$A_k := \{a \in A : \sigma_t(a) = e^{ikt}a \text{ for all } t \in \mathbb{R}\}.$$

In particular $A_0 = F$. This leads to the following definition [6, Defn. 2.2], which is in some sense a noncommutative analogue of having a free circle action.

Definition B.4. *The circle action (B.1) is said to have full spectral subspaces if it holds that $\overline{A_k^* A_k} = F$ for all $k \in \mathbb{Z}$.*

Next we construct an unbounded operator on the Hilbert module X . The operator

$$(B.2) \quad \mathfrak{D} : X_{\mathfrak{D}} \rightarrow X, \quad \mathfrak{D} \left(\sum_{k \in \mathbb{Z}} x_k \right) := \sum_{k \in \mathbb{Z}} k x_k$$

on the dense domain $X_{\mathfrak{D}} \subset X$,

$$(B.3) \quad X_{\mathfrak{D}} := \left\{ x = \sum_{k \in \mathbb{Z}} x_k \in X \mid x_k \in A_k, \left\| \sum_{k \in \mathbb{Z}} k^2 \langle x_k, x_k \rangle \right\| < \infty \right\},$$

defines a self-adjoint and regular operator on X (cf. [24, Prop. 4.6] and [6, Prop. 2.7]). Indeed, the operator $\mathfrak{D}$ is nothing other than the infinitesimal generator of the circle action $\sigma : U(1) \rightarrow \text{Aut}(A)$. Of utmost importance for the present paper is the following result, which is a slight weakening of [6, Prop. 2.9].

Theorem B.5. *Let $\sigma : U(1) \rightarrow \text{Aut}(A)$ be a circle action with full spectral subspaces. Then $(X, \mathfrak{D})$ defined in (B.2) and (B.3) is an odd unbounded Kasparov module whose bounded transform determines a class in the bivariant K-theory $KK_1(A, F)$.*

Then the Kasparov product of the class of $(X, \mathfrak{D})$ with $K_*(A) = KK_*(\mathbb{C}, A)$ immediately equips us with a pair of maps

$$\text{Ind}_{\mathfrak{D}} : K_*(A) \rightarrow K_{*+1}(F), \quad \text{Ind}_{\mathfrak{D}}(-) := - \widehat{\otimes}_A [(X, \mathfrak{D})].$$

For the case of interest in the present paper one has $K_1(F) = 0$, thus one of these maps is just the zero map. On the other hand, by its definition the operator $\mathfrak{D}$ has a spectral gap around zero, whence the map $\text{Ind}_{\mathfrak{D}} : K_1(A) \rightarrow K_0(F)$ is given explicitly as in Thm. B.3.

B.3. K-theory of mapping cones. Let A be a C^* -algebra equipped with a circle action having full spectral subspaces and let F denote its fixed point C^* -subalgebra. Recall that the *mapping cone* of the pair (F, A) is the C^* -algebra

$$M(F, A) := \{f \in C([0, 1], A) \mid f(0) = 0, f(1) \in F\}.$$

The group $K_0(M(F, A))$ has a particularly elegant description in terms of partial isometries. Indeed, let us write $V_m(F, A)$ for the set of partial isometries $v \in M_m(A)$ such that the associated projections v^*v and vv^* belong to $M_m(F)$. Using the inclusion $V_m(F, A) \hookrightarrow V_{m+1}(F, A)$ given by setting $v \mapsto v \oplus 0$, one defines

$$V(F, A) := \bigcup_m V_m(F, A)$$

and then generates an equivalence relation $\sim$ on $V(F, A)$ by declaring that:

- (1) $v \sim v \oplus p$ for all $v \in V(F, A)$ and $p \in M_m(F)$;
- (2) if $v(t)$, $t \in [0, 1]$, is a continuous path in $V(F, A)$ then $v(0) \sim v(1)$.

This construction results in the following identification [26, Lem. 2.5].

Lemma B.6. *There is a well defined bijection between $V(F, A)/\sim$ and $K_0(M(F, A))$.*

Indeed, in [26, Lem. 2.5], it is also shown that if v and w are partial isometries with the same image in $K_0(M(F, A))$ one can arrange them to have the same initial projection, i.e. $v^*v = w^*w$, without changing their class in $V(F, A)/\sim$. Having done so, an addition is defined [5, Lem. 3.3] in $V(F, A)/\sim$ by $[v \oplus w^*] = [v] + [w^*] = [v] - [w] = [vw^*]$ so that $V(F, A)/\sim$ and $K_0(M(F, A))$ are isomorphic as Abelian groups.

The class of $(X, \mathfrak{D})$ in $KK_1(A, F)$, as defined previously, has a canonical lift to the group $KK_0(M(F, A), F)$. Let P be the spectral projection for $\mathfrak{D}$ corresponding to the non-negative real axis. Following [5, §4], we write $T_{\pm} := \pm \partial_t \otimes 1 + 1 \otimes \mathfrak{D}$ for the unbounded operators with domains

$$\begin{aligned} \mathfrak{Dom}(T_{\pm}) &:= \left\{ f \in C_c^{\infty}([0, \infty)) \otimes X_{\mathfrak{D}} \mid f = \sum_{i=1}^n f_i \otimes x_i, x_i \in X_{\mathfrak{D}}, \right\} \\ \text{and} \quad P(f(0)) &= 0 \text{ (+ case), } (1 - P)(f(0)) = 0 \text{ (- case),} \end{aligned}$$

where smoothness at the boundary of $[0, \infty)$ is defined by taking one-sided limits. With $\mathcal{E} := L^2([0, \infty)) \otimes X$, one finds that

$$\widehat{\mathfrak{D}} : \mathfrak{Dom}(T_+) \oplus \mathfrak{Dom}(T_-) \rightarrow \mathcal{E} \oplus \mathcal{E}, \quad \widehat{\mathfrak{D}} := \begin{pmatrix} 0 & T_- \\ T_+ & 0 \end{pmatrix},$$

is a densely defined unbounded symmetric linear operator. By modifying the domains slightly, one obtains a $\mathbb{Z}_2$ -graded Hilbert $M(F, A)$ - F -bimodule $\widehat{X}$ equipped with an odd unbounded linear operator $\widehat{\mathfrak{D}} : \mathfrak{Dom}(\widehat{\mathfrak{D}}) \rightarrow \widehat{X}$ which is self-adjoint and regular [5, Prop. 4.13]. The following is then the content of [5, Prop. 4.14].

Theorem B.7. *The pair $(\widehat{X}, \widehat{\mathfrak{D}})$ is an even unbounded Kasparov bimodule whose bounded transform determines a class in the bivariant K-theory $KK_0(M(F, A), F)$.*

Thus one obtains the following representation of the pairing of the K-theory $K_0(M(F, A))$ with the class of the pair $(\widehat{X}, \widehat{\mathfrak{D}})$ [5, Thm. 5.1], which generalizes the case of Thm. B.3.

Theorem B.8. *Writing $\mathcal{E} := X^m$, the internal Kasparov product of $K_0(M(F, A))$ with the class of $(\widehat{X}, \widehat{\mathcal{D}})$ in the K -theory $KK_0(M(F, A), F)$ is represented by*

$$\text{Ind}_{\widehat{\mathcal{D}}}([v]) := \text{Ker}(PvP)|_{v^*vP\mathcal{E}} - \text{Ker}(Pv^*P)|_{vv^*P\mathcal{E}},$$

*the result being an element of $KK_0(\mathbb{C}, F) = K_0(F)$. Here $v \in M_m(A)$ is a partial isometry representing a class in $K_0(M(F, A))$ and considered as a map $v : v^*vP\mathcal{E} \rightarrow vv^*P\mathcal{E}$.*

APPENDIX C. PRINCIPAL BUNDLE STRUCTURES

A *noncommutative principal bundle* is a triple $(\mathcal{A}, H, \mathcal{F})$, where $\mathcal{A}$ is the $*$ -algebra of functions on ‘the total space’, H is the Hopf $*$ -algebra of functions on the ‘structure group’, with $\mathcal{A}$ being a right (say) H -comodule $*$ -algebra, that is there is a right coaction

$$\Delta_R : \mathcal{A} \rightarrow \mathcal{A} \otimes H.$$

The functions on ‘the base space’ are given by the $*$ -subalgebra of coinvariant elements:

$$\mathcal{F} := \{a \in \mathcal{A} \mid \Delta_R(a) = a \otimes 1\}.$$

Conditions to be satisfied are imposed via a suitable sequence. Let $\Omega_{un}^1(\mathcal{B})$ denote the bimodule of universal differential forms over a unital algebra $\mathcal{B}$ and ϵ_H be the counit of the Hopf algebra H . Principality of the bundle is expressed by requiring the sequence

$$(C.1) \quad 0 \rightarrow \mathcal{A}(\Omega_{un}^1(\mathcal{F}))\mathcal{A} \rightarrow \Omega_{un}^1(\mathcal{A}) \xrightarrow{\text{ver}} \mathcal{A} \otimes \ker \epsilon_H \rightarrow 0$$

to be exact. Here the first map is inclusion while the second one, $\text{ver}(a \otimes b) := (a \otimes 1)\Delta_R(b)$, generates ‘vertical one-forms’. When H is cosemisimple and has an invertible antipode, exactness of the sequence (C.1) is equivalent to the statement that the canonical map

$$(C.2) \quad \chi : \mathcal{A} \otimes_{\mathcal{F}} \mathcal{A} \rightarrow \mathcal{A} \otimes H, \quad \chi(a \otimes b) := (a \otimes 1)\Delta_R(b),$$

is an isomorphism (sometimes this is also known as the statement that the triple $(\mathcal{A}, H, \mathcal{F})$ is a *Hopf-Galois extension*). Furthermore, things are easier for a cosemisimple Hopf algebra H with bijective antipode, since then the map (C.2) is injective whenever it is surjective and thus it is enough to check surjectivity [27, Thm. I].

C.1. Proof of Prop. 4.3. The algebra $\mathcal{A}(\mathbb{Z}_r)$ of functions on the cyclic group $\mathbb{Z}_r$ is the $*$ -algebra generated by a single element ζ modulo the relation $\zeta^r = 1$. The Hopf structures are given by coproduct $\Delta(\zeta) = \zeta \otimes \zeta$, counit $\epsilon(\zeta) = 1$ and antipode $S(\zeta) = \zeta^*$.

Thus the algebra $\mathcal{A}(L_q^{(n,r)})$ can be also obtained as the algebra of coinvariant elements with respect to the coaction of $\mathcal{A}(\mathbb{Z}_r)$ on $\mathcal{A}(S_q^{2n+1})$ defined on generators by

$$\Delta_R(z_i) = z_i \otimes \zeta, \quad \Delta_R(z_i^*) = z_i^* \otimes \zeta^*$$

and extended as algebra map to the whole of $\mathcal{A}(S_q^{2n+1})$.

The Hopf algebra $\mathcal{A}(\mathbb{Z}_r)$ is certainly cosemisimple and so, to show that the datum $(\mathcal{A}(S_q^{2n+1}), \mathcal{A}(\mathbb{Z}_r), \mathcal{A}(L_q^{(n,r)}))$ is a principal bundle, it is enough to establish surjectivity of the map χ defined as in (C.2). For this we use a strategy borrowed from [21].

Writing $\mathcal{A} = \mathcal{A}(S_q^{2n+1})$, $H = \mathcal{A}(\mathbb{Z}_r)$ and $\mathcal{F} = \mathcal{A}(L_q^{(n,r)})$, a generic element in $\mathcal{A} \otimes H$ is a sum of elements of the form $f \otimes \zeta^{*N}$ with $N = 0, \dots, r-1$ and $f \in \mathcal{A}(S_q^{2n+1})$. By left

$\mathcal{A}(S_q^{2n+1})$ -linearity of χ , it is enough to exhibit a pre-image for elements of the form $1 \otimes \zeta^{*N}$, since if $\gamma \in \mathcal{A} \otimes_{\mathcal{F}} \mathcal{A}$ is such that $\chi(\gamma) = 1 \otimes \zeta^{*N}$, then $\chi(f\gamma) = f(1 \otimes \zeta^{*N}) = f \otimes \zeta^{*N}$.

Let $\psi_{j_0, \dots, j_n}^N$ be the vector-valued functions given in (3.6), and define the element

$$(C.3) \quad \gamma := \sum_j \psi_j^{N*} \otimes \psi_j^N,$$

which is clearly in $\mathcal{A} \otimes_{\mathcal{F}} \mathcal{A}$. Denote $\beta_j^N := [j_0, \dots, j_n]! q^{-\sum_{r < s} j_r j_s}$, to lighten notation. Upon applying χ one obtains

$$\begin{aligned} \chi(\gamma) &= \sum_j \chi(\psi_j^{N*} \otimes \psi_j^N) = \sum_{j_0 + \dots + j_n = N} \beta_j^N (z_n^{j_n} \dots z_0^{j_0} \otimes 1) \Delta_R((z_0^{j_0})^* \dots (z_n^{j_n})^*) \\ &= \sum_j \beta_j^N (z_n^{j_n} \dots z_0^{j_0} \otimes 1) \cdot ((z_0^{j_0})^* \dots (z_n^{j_n})^* \otimes \xi^{*N}) \\ &= \sum_j \beta_j^N (z_n^{j_n} \dots z_0^{j_0} (z_0^{j_0})^* \dots (z_n^{j_n})^* \otimes \xi^{*N}) \\ &= \left(\sum_j \beta_j^N z_n^{j_n} \dots z_0^{j_0} (z_0^{j_0})^* \dots (z_n^{j_n})^* \right) \otimes \xi^{*N} \\ &= 1 \otimes \xi^{*N}, \end{aligned}$$

which is all one needs for proving surjectivity of the map χ .

C.2. Proof of Prop. 4.2. Let $\mathcal{A}(U(1) = \mathbb{C}[\xi, \xi^*]/\langle \xi^* \xi - 1 \rangle$ denote the coordinate algebra of the group $U(1)$. With $\tilde{U}(1) := U(1)/\mathbb{Z}_r$, the corresponding coordinate algebra,

$$\mathcal{A}(\tilde{U}(1)) := \mathcal{A}(U(1)/\mathbb{Z}_r) = \mathcal{A}(U(1))^{\mathbb{Z}_r},$$

is the Hopf $*$ -subalgebra of $\mathcal{A}(U(1))$ generated by the powers ξ^r and ξ^{*r} .

Denote $\mathcal{A}' = \mathcal{A}(L_q^{(n,r)})$, $H' = \mathcal{A}(\tilde{U}(1))$ and $\mathcal{F}' = \mathcal{A}(\mathbb{CP}_q^n)$. As before, the datum $(\mathcal{A}', H', \mathcal{F}')$ is seen as a quantum principal bundle via surjectivity of the canonical map

$$\chi' : \mathcal{A}' \otimes_{\mathcal{F}'} \mathcal{A}' \rightarrow \mathcal{A}' \otimes H',$$

surjectivity proved again by exhibiting a pre-image for elements of the kind $1 \otimes \xi^{*rN}$.

For this, we observe that the vectors $\psi_{j_0, \dots, j_n}^N$ in 3.6 have entries in $\mathcal{A}' = \mathcal{A}(L_q^{(n,r)})$ precisely when N is a multiple of r . Then, in parallel with (C.3), the element of $\mathcal{A}' \otimes_{\mathcal{F}'} \mathcal{A}'$,

$$(C.4) \quad \gamma' := \sum_j \psi_j^{rN*} \otimes \psi_j^{rN},$$

is mapped to $1 \otimes \zeta^{*rN}$ by the canonical map χ' ; enough for the surjectivity of the latter.

APPENDIX D. COMPUTING COKERNELS

We compute explicitly the cokernel of the matrix A in (6.2) when $n = 3$. This is a bit involved and, depending on the divisibility properties of the integer r , requires considering different cases for r . Now $\text{coker}(A) \simeq \mathbb{Z}^4 / \text{Im}(A)$; since $\text{Im}(A)$ is generated by the columns of A , the vanishing of these columns yields conditions on the generators of $\text{coker}(A)$.

$r = 6k$. The columns of the matrix A yield the constraints

$$\begin{cases} 6ku - 3k(6k-1)u^2 + k(6k-1)(6k-2)u^3 = 0, \\ 6ku^2 - 3k(6k-1)u^3 = 0, \\ 6ku^3 = 0. \end{cases}$$

Substituting the third equation into the first and second ones yields the relations

$$\begin{cases} 6ku + 3k(1 - 6k)u^2 + 2ku^3 = 0, \\ 3k(2u^2 \pm u^3) = 0 \quad \Rightarrow \quad 12ku^2 = 0, \\ 6ku^3 = 0. \end{cases}$$

By multiplying the first equation by two and using the second, one gets

$$k(12u + u^3) = 0,$$

that is $12u + u^3$ has order $k = r/6$. On the other and, the first equation can be multiplied by three, after which the use of the third equation yields $18ku + 9k(1 - 6k)u^2 = 0$. Now, modulo $12ku^2$, one has $9k(1 - 6k)u^2 \equiv 3ku^2$, which transforms the previous equation into

$$3k(6u + u^2) = 0,$$

that is $(6u + u^2)$ has order $3k = r/2$. Finally, multiplying the first equation by 6 and using $6ku^3 = 0$ or the second equation by 4 and using $12ku^2 = 0$, it follows that

$$72ku = 0,$$

i.e. u has order $12r$.

$r = 6k + 2$ and $r = 6k - 2$. For the first case, the columns of A yield the constraints

$$\begin{cases} 2(3k + 1)u - (3k + 1)(6k + 1)u^2 + 2k(3k + 1)(6k + 1)u^3 = 0, \\ 2(3k + 1)u^2 - (3k + 1)(6k + 1)u^3 = 0, \\ 2(3k + 1)u^3 = 0. \end{cases}$$

These can be rewritten as

$$\begin{cases} 2(3k + 1)u - (3k + 1)(6k + 1)u^2 = 0, \\ 2(3k + 1)u^2 \mp (3k + 1)u^3 = 0 \quad \Rightarrow \quad 4(3k + 1)u^2 = 0, \\ 2(3k + 1)u^3 = 0 \end{cases}$$

and from the second equation one immediately gets

$$(3k + 1)(2u^2 + u^3) = 0,$$

which says that $(2u^2 + u^3)$ has order $3k + 1 = r/2$. On the other hand, modulo $4(3k + 1)u^2$ one has that $(3k + 1)(6k + 1)u^2 \equiv (-1)^k(3k + 1)u^2$ which transforms the first equation to

$$(3k + 1)(2u + (-1)^{k+1}u^2) = 0,$$

that is $2u + (-1)^{k+1}u^2$ has order $3k + 1 = r/2$. Moreover, using again $4(3k + 1)u^2 = 0$ this also yields

$$4(3k + 1)2u = 0,$$

that is u has order $8(3k + 1) = 4r$.

Analogous computations and results holds for $r = 6k - 2$.

$r = 6k + 3$. The columns of the matrix A yield the constraints

$$\begin{cases} 3(2k+1)u - 3(3k+1)(2k+1)u^2 + (2k+1)(3k+1)(6k+1)u^3 = 0, \\ 3(2k+1)u^2 - 3(3k+1)(2k+1)u^3 = 0, \\ 3(2k+1)u^3 = 0. \end{cases}$$

These can be rewritten as

$$\begin{cases} 3(2k+1)u + (2k+1)(3k+1)(6k+1)u^3 = 0, \\ 3(2k+1)u^2 = 0, \\ 3(2k+1)u^3 = 0, \end{cases}$$

that is both u^3 and u^2 have order $3(2k+1) = r$. Moreover, using twice the last equation in the first one leads to

$$\begin{aligned} 0 &= (6k+3)u - 2(2k+1)(3k+1)u^3 = (2k+1)(3u - (6k+2)u^3) \\ &= (2k+1)(3u + u^3), \end{aligned}$$

which says that $3u + u^3$ has order $2k+1 = r/3$. Finally

$$9(2k+1)u = 3(2k+1)u^3 = 0,$$

hence u has order $9(2k+1) = 3r$.

$r = 6k + 1$ and $r = 6k - 1$. The columns of the matrix A yield the constraints

$$\begin{cases} (6k+1)u - 3k(6k+1)u^2 + k(6k+1)(6k-1)u^3 = 0, \\ (6k+1)u^2 - 3k(6k+1)u^3 = 0, \\ (6k+1)u^3 = 0. \end{cases}$$

These just tell us that u, u^2, u^3 all have order $6k+1 = r$.

Analogous computations and results hold for $r = 6k - 1$.

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