Non-Abelian duality from vortex moduli: a dual model of color-confinement

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Abstract

It is argued that the dual transformation of non-Abelian monopoles occurring in a system with gauge symmetry breaking $G \longrightarrow H$ is to be defined by setting the low-energy H system in Higgs phase, so that the dual system is in confinement phase. The transformation law of the monopoles follows from that of monopole-vortex mixed configurations in the system (with a large hierarchy of energy scales, $v_1 \gg v_2$)

$$ G \xrightarrow{v_1} H \xrightarrow{v_2} \emptyset, $$

under an unbroken, exact color-flavor diagonal symmetry $H_{C+F} \sim \tilde{H}$. The transformation property among the regular monopoles characterized by $\pi_2(G/H)$, follows from that among the non-Abelian *vortices* with flux quantized according to $\pi_1(H)$, via the isomorphism $\pi_1(G) \sim \pi_1(H)/\pi_2(G/H)$. Our idea is tested against the concrete models – softly-broken $\mathcal{N} = 2$ supersymmetric $SU(N)$, $SO(N)$ and $USp(2N)$ theories, with appropriate number of flavors. The results obtained in the semiclassical regime (at $v_1 \gg v_2 \gg \Lambda$) of these models are consistent with those inferred from the fully quantum-mechanical low-energy effective action of the systems (at $v_1, v_2 \sim \Lambda$).

1 Introduction and discussion

A system in which the gauge symmetry is spontaneously broken

$$G \stackrel{\langle \phi_1 \rangle \neq 0}{\longrightarrow} H \tag{1.1}$$

where H is some non-Abelian subgroup of G, possesses a set of regular magnetic monopole solutions in the semi-classical approximation, which are natural generalizations of the 't Hooft-Polyakov monopoles [1] found in the system $G = SO(3)$, $H = U(1)$. A straightforward generalization of the Dirac's quantization condition leads to the GNOW (Goddard-Nuyts-Olive-E.Weinberg) conjecture, *i.e.*, that they form a multiplet of the group $\tilde{H}$, dual of H. The group $\tilde{H}$ is generated by the dual root vectors

$$\alpha^* = \frac{\alpha}{\alpha \cdot \alpha}, \tag{1.2}$$

where α are the non-vanishing roots of H [2]-[4]. There are however well-known difficulties in such an interpretation. The first concerns the topological obstruction discussed in [5]: in the presence of the classical monopole background, it is not possible to define a globally well-defined set of generators isomorphic to H. As a consequence, no "colored dyons" exist. In the simplest example of a system with the symmetry breaking,

$$SU(3) \stackrel{\langle \phi_1 \rangle \neq 0}{\longrightarrow} SU(2) \times U(1), \tag{1.3}$$

this means that no monopoles exist which carry the quantum number, *e.g.*,

$$(\underline{2}, 1^*) \tag{1.4}$$

where the asterisk indicates the dual, magnetic $U(1)$ charge.

The second can be regarded as the infinitesimal version of the same difficulty: certain bosonic zero-modes around the monopole solution, corresponding to the H gauge transformations, are non-normalizable (behaving as $r^{-1/2}$ asymptotically). Thus the standard procedure of semiclassical quantization leading to the H multiplet of the monopoles does not work. Some progress on the check of GNO duality along this orthodox approach has been reported nevertheless in [6] for $\mathcal{N} = 4$ supersymmetric gauge theories, which however requires the consideration of particular multi-monopole systems neutral with respect to the non-Abelian group (more precisely, non-Abelian part of) H.

Both of these difficulties concern the transformation properties of the monopoles under the subgroup H, while the truly relevant question is how they transform under the dual group, $\tilde{H}$. As field transformation groups, H and $\tilde{H}$ are relatively non-local; the latter should look like a non-local transformation group in the original, electric description.

Another related question concerns the multiplicity of the monopoles; take again the case of the system with breaking pattern Eq. (1.3). One might argue that there is only one monopole, as all the degenerate solutions are related by the unbroken *gauge* group $H = SU(2)$.[1] Or one might say that there are two monopoles as, according to the semiclassical GNO classification, they are

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[1] This interpretation however encounters the difficulties mentioned above. Also there are cases in which degenerate monopoles occur, which are not simply related by the group H, see below.

supposed to belong to a doublet of the dual $SU(2)$ group. Or, perhaps, one should conclude that there are infinitely many, continuously related solutions, as the two solutions obtained by embedding the 't Hooft solutions in $(1,3)$ and $(2,3)$ subspaces, are clearly part of the continuous set of (moduli of) solutions. In short, what is the multiplicity (#) of the monopoles:

$$\# = 1, \quad 2, \quad \text{or} \quad \infty \; ? \tag{1.5}$$

Clearly the very concept of the *dual gauge group* or *dual gauge transformation* must be better understood. In attempting to gain such an improved insight on the nature of these objects, we are naturally led to several general considerations.

The first is the fact when H and $\tilde{H}$ groups are non-Abelian the dynamics of the system should enter the problem in an essential way. It should not be surprising if the understanding of the concept of non-Abelian duality required a full quantum mechanical treatment of the system.

For instance, the non-Abelian H interactions can become strongly-coupled at low energies and can break itself dynamically. This indeed occurs in pure $\mathcal{N} = 2$ super Yang-Mills theories (*i.e.*, theories without quark hypermultiplets), where the exact quantum mechanical result is known in terms of the Seiberg-Witten curves [8]. Consider for instance, a pure $\mathcal{N} = 2$, $SU(N + 1)$ gauge theory. Even though partial breaking, *e.g.*, $SU(N + 1) \rightarrow SU(N) \times U(1)$ looks perfectly possible semi-classically, in an appropriate region of classical degenerate vacua, no such vacua exist quantum mechanically. In *all* vacua the light monopoles are Abelian, the effective, magnetic gauge group being $U(1)^N$.

Generally speaking, the concept of a dual group multiplet is well-defined only when $\tilde{H}$ interactions are weak (or, at worst, conformal). This however means that one must study the original, electric theory in the regime of strong coupling, which would usually make the task of finding out what happens in the system at low energies exceedingly difficult. Fortunately, in $\mathcal{N} = 2$ supersymmetric gauge theories, the exact Seiberg-Witten curves describe the fully quantum mechanical consequences of the strong-interaction dynamics in terms of weakly-coupled dual magnetic variables. And this is how we know that the non-Abelian monopoles do exist in fully quantum theories [9]: in the so-called r-vacua of softly broken $\mathcal{N} = 2$ SQCD, the light monopoles interact as a point-like particle in a fundamental multiplet $\underline{r}$ of the effective, dual $SU(r)$ gauge group. In the system of the type Eq. (1.3) with appropriate number of quark multiplets ($N_f \geq 4$), we know that light magnetic monopoles carrying the non-Abelian quantum number

$$(\underline{2}^*, 1^*) \tag{1.6}$$

under the dual $SU(2) \times U(1)$ appear in the low-energy effective action (*cfr.* Eq. (1.4)). The distinction between H and $\tilde{H}$ is crucial here.

In general $\mathcal{N} = 2$ SQCD with N_f flavors, light non-Abelian monopoles with $SU(r)$ dual gauge group appear for $r \leq \frac{N_f}{2}$ only. Such a limit clearly reflects the dynamical properties of the soliton monopoles under renormalization group: the effective low-energy gauge group must be either infrared free or conformal invariant, in order for the monopoles to emerge as recognizable low-energy degrees of freedom [10]-[14].

A closely related point concerns the phase of the system. Even if there is an ample evidence for the non-Abelian monopoles, as explained above, we might still wish to understand them in terms of something more familiar, such as semiclassical 't Hooft-Polyakov solitons. An analogous question can be (and should be) asked about the Seiberg's "dual quarks" in $\mathcal{N} = 1$ SQCD

[15]. Actually, the latter can be interpreted as the GNOW monopoles becoming light due to the dynamics, at least in $SU(N)$ theories [16]. For $SO(N)$ or in $USp(2N)$ theories the relation between Seiberg duals and GNOW monopoles are less clear [16]. For instructive discussions on the relation between Seiberg duals and semiclassical monopoles in a class of $\mathcal{N} = 1\ SO(N)$ models with matter fields in vector and spinor representations, see Strassler [17].

Dynamics of the system is thus a crucial ingredient: if the dual group were in Higgs phase, the multiplet structure among the monopoles would get lost, generally. Therefore one must study the dual ($\tilde{H}$) system in confinement phase.[2] *But then, according to the standard electromagnetic duality argument, one must analyze the electric system in Higgs phase.* The monopoles will appear confined by the confining strings which are nothing but the vortices in the H system in Higgs phase.

We are thus led to study the system with a hierarchical symmetry breaking,

$$G \xrightarrow{\langle\phi_1\rangle\neq 0} H \xrightarrow{\langle\phi_2\rangle\neq 0} \emptyset, \tag{1.7}$$

where

$$|\langle\phi_1\rangle| \gg |\langle\phi_2\rangle|, \tag{1.8}$$

instead of the original system Eq. (1.1). The smaller VEV breaks H completely. Also, in order for the degeneracy among the monopoles not to be broken by the breaking at the scale $|\langle\phi_2\rangle|$, we assume that some global color-flavor diagonal group

$$H_{C+F} \subset H_{color} \otimes G_F \tag{1.9}$$

remains unbroken.

It is hardly possible to emphasize the importance of the role of the massless flavors too much. This manifests in several different aspects.

(i) In order that H must be non-asymptotically free, there must be sufficient number of massless flavors: otherwise, H interactions would become strong at low energies and H group can break itself dynamically;

(ii) The physics of the r vacua [10, 12] indeed shows that the non-Abelian dual group $SU(r)$ appear only for $r \leq \frac{N_f}{2}$. This limit can be understood from the renormalization group: in order for a non-trivial r vacuum to exist, there must be at least $2r$ massless flavors in the fundamental theory;

(iii) Non-Abelian vortices [18, 19], which as we shall see are closely related to the concept of non-Abelian monopoles, require a flavor group. The non-Abelian flux moduli arise as a result of an exact, unbroken color-flavor diagonal symmetry of the system, broken by individual soliton vortex.

The idea that the dual group transformations among the monopoles at the end of the vortices follow from those among the vortices (monopole-vortex flux matching, etc.), has been discussed in several occasions, in particular in [20]. The main aim of the present paper is to enforce this

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[2] The non-Abelian monopoles in the Coulomb phase suffer from the difficulties already discussed.

argument, by showing that the degenerate monopoles do indeed transform as a definite multiplet under a group transformation, which is non-local in the original, electric variables, and involves flavor non-trivially, even though this is not too obvious in the usual semiclassical treatment. The flavor dependence enters through the infrared regulator. The resulting, exact transformation group is *defined* to be the dual group of the monopoles.

2 $SU(N+1)$ model with hierarchical symmetry breaking

Our aim is to show that all the difficulties about the non-Abelian monopole moduli discussed in the Introduction are eliminated by reducing the problem to that of the vortex moduli, related to the former by the topology and symmetry argument.

2.1 $U(N)$ model with Fayet-Iliopoulos term

The model frequently considered in the recent literature in the discussion of various solitons [22]-[28], is a $U(N)$ theory with gauge fields W_μ, an adjoint (complex) scalar ϕ, and $N_f = N$ scalar fields in the fundamental representation of $SU(N)$, with the Lagrangian,

$$
\begin{aligned}
\mathcal{L} &= \text{Tr}\left[-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} - \frac{2}{g^2} \mathcal{D}_\mu \phi^\dagger \mathcal{D}^\mu \phi - \mathcal{D}_\mu H \mathcal{D}^\mu H^\dagger - \lambda \left(c\, \mathbf{1}_N - H\, H^\dagger \right)^2 \right] \\
&+ \text{Tr}\left[(H^\dagger \phi - M\, H^\dagger)(\phi\, H - H\, M) \right]
\end{aligned}
\tag{2.1}
$$

where $F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\nu + i\,[W_\mu, W_\nu]$ and $\mathcal{D}_\mu H = (\partial_\mu + i\, W_\mu)\, H$, and H represents the fields in the fundamental representation of $SU(N)$, written in a color-flavor $N \times N$ matrix form, $(H)^i_\alpha \equiv q^i_\alpha$, and M is a $N \times N$ mass matrix. Here, g is the $U(N)_\text{G}$ gauge coupling, λ is a scalar coupling. For

$$
\lambda = \frac{g^2}{4}
\tag{2.2}
$$

the system is BPS saturated. For such a choice, the model can be regarded as a truncation

$$
(H)^i_\alpha \equiv q^i_\alpha, \qquad \tilde{q}^\alpha_i \equiv 0
\tag{2.3}
$$

of the bosonic sector of an $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theory. In the supersymmetric context the parameter c is the Fayet-Iliopoulos parameter. In the following we set $c > 0$ so that the system be in Higgs phase, and so as to allow stable vortex configurations. For generic, unequal quark masses,

$$
M = diag\,(m_1, m_2, \ldots, m_N),
\tag{2.4}
$$

the adjoint scalar VEV takes the form,

$$
\langle \phi \rangle = M = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & m_N \end{pmatrix},
\tag{2.5}
$$

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which breaks the gauge group to $U(1)^N$. In the equal mass case,

$$M = diag\,(m, m, \ldots, m), \tag{2.6}$$

the adjoint and squark fields have the vacuum expectation value (VEV)

$$\langle \phi \rangle = m\,\mathbf{1}_N, \qquad \langle H \rangle = \sqrt{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.7}$$

The squark VEV breaks the gauge symmetry completely, while leaving an unbroken $SU(N)_{C+F}$ color-flavor diagonal symmetry (remember that the flavor group acts on H from the right while the $U(N)_{\rm G}$ gauge symmetry acts on H from the left). The BPS vortex equations are

$$(\mathcal{D}_1 + i\mathcal{D}_2)\,H = 0, \quad F_{12} + \frac{g^2}{2}\left(c\,\mathbf{1}_N - H\,H^\dagger\right) = 0. \tag{2.8}$$

The matter equation can be solved by use of the $N \times N$ moduli matrix $H_0(z)$ whose components are holomorphic functions of the complex coordinate $z = x^1 + ix^2$, [26, 30, 28]

$$H = S^{-1}(z, \bar{z})\,H_0(z), \quad W_1 + i\,W_2 = -2\,i\,S^{-1}(z, \bar{z})\,\bar{\partial}_z S(z, \bar{z}). \tag{2.9}$$

The gauge field equations then take the simple form ("master equation") [26, 30, 28]

$$\partial_z\left(\Omega^{-1}\partial_{\bar{z}}\,\Omega\right) = \frac{g^2}{4}\left(c\,\mathbf{1}_N - \Omega^{-1}\,H_0\,H_0^\dagger\right). \tag{2.10}$$

The moduli matrix and S are defined up to a redefinition,

$$H_0(z) \to V(z)\,H_0(z), \qquad S(z, \bar{z}) \to V(z)\,S(z, \bar{z}), \tag{2.11}$$

where $V(z)$ is any non-singular $N \times N$ matrix which is holomorphic in z.

2.2   The Model

Actually the model we are interested here is not exactly this model, but is a model which contains it as a low-energy approximation. We take as our model the standard $\mathcal{N} = 2$ SQCD with N_f quark hypermultiplets, with a larger gauge symmetry, *e.g.*, $SU(N+1)$, which is broken at a much larger mass scale as

$$SU(N+1) \xrightarrow{v_1 \neq 0} \frac{SU(N) \times U(1)}{\mathbb{Z}_N}. \tag{2.12}$$

The unbroken gauge symmetry is completely broken at a lower mass scale, as in Eq. (2.7).

Clearly one can attempt a similar embedding of the model Eq. (2.1) in a larger gauge group broken at some higher mass scale, in the context of a non-supersymmetric model, even though in such a case the potential must be judiciously chosen and the dynamical stability of the scenario would have to be carefully monitored. Here we choose to study the softly broken $\mathcal{N} = 2$ SQCD for concreteness, and above all because the dynamical properties of this model are well understood:

this will provide us with a non-trivial check of our results. Another motivation is purely of convenience: it gives a definite potential with desired properties.[3]

The underlying theory is thus

$$\mathcal{L} = \frac{1}{8\pi} \text{Im } S_{cl} \left[\int d^4\theta\, \Phi^\dagger e^V \Phi + \int d^2\theta\, \frac{1}{2} WW \right] + \mathcal{L}^{(\text{quarks})} + \int d^2\theta\, \mu\, \text{Tr}\, \Phi^2 + h.c.; \qquad (2.13)$$

$$\mathcal{L}^{(\text{quarks})} = \sum_i \left[\int d^4\theta\, \{ Q_i^\dagger e^V Q_i + \tilde{Q}_i e^{-V} \tilde{Q}_i^\dagger \} + \int d^2\theta\, \{ \sqrt{2} \tilde{Q}_i \Phi Q^i + m_i\, \tilde{Q}_i Q^i \} + h.c. \right] \qquad (2.14)$$

where m is the bare mass of the quarks and we have defined the complex coupling constant

$$S_{cl} \equiv \frac{\theta_0}{\pi} + \frac{8\pi i}{g_0^2}. \qquad (2.15)$$

We also added the parameter μ, the mass of the adjoint chiral multiplet, which breaks the supersymmetry softly to $\mathcal{N} = 1$. The bosonic sector of this model is described, after elimination of the auxiliary fields, by

$$\mathcal{L} = \frac{1}{4g^2} F_{\mu\nu}^2 + \frac{1}{g^2} |\mathcal{D}_\mu \Phi|^2 + |\mathcal{D}_\mu Q|^2 + \left| \mathcal{D}_\mu \tilde{\bar{Q}} \right|^2 - V_1 - V_2, \qquad (2.16)$$

where

$$V_1 = \frac{1}{8} \sum_A \left(t_{ij}^A \left[\frac{1}{g^2} (-2) [\Phi^\dagger, \Phi]_{ji} + Q_j^\dagger Q_i - \tilde{Q}_j \tilde{Q}_i^\dagger \right] \right)^2; \qquad (2.17)$$

$$\begin{aligned} V_2 &= g^2 |\mu\, \Phi^A + \sqrt{2}\, \tilde{Q}\, t^A Q|^2 + \tilde{Q}\, [m + \sqrt{2}\Phi]\, [m + \sqrt{2}\Phi]^\dagger\, \tilde{Q}^\dagger \\ &+ Q^\dagger\, [m + \sqrt{2}\Phi]^\dagger\, [m + \sqrt{2}\Phi]\, Q. \end{aligned} \qquad (2.18)$$

In the construction of the approximate monopole and vortex solutions we shall consider only the VEVs and fluctuations around them which satisfy

$$[\Phi^\dagger, \Phi] = 0, \qquad Q_i = \tilde{Q}_i^\dagger, \qquad (2.19)$$

and hence the D-term potential V_1 can be set identically to zero throughout.

In order to keep the hierarchy of the gauge symmetry breaking scales, Eq. (1.8), we choose the masses such that

$$m_1 = \ldots = m_{N_f} = m, \qquad (2.20)$$

$$m \gg \mu \gg \Lambda. \qquad (2.21)$$

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[3] Recent developments [33, 34] allow us actually to consider systems of this sort within a much wider class of $\mathcal{N} = 1$ supersymmetric models, whose infrared properties are very much under control. We stick ourselves to the standard $\mathcal{N} = 2$ SQCD, however, for concreteness.

Although the theory described by the above Lagrangian has many degenerate vacua, we are interested in the vacuum where (see [12] for the detail)

$$\langle \Phi \rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \dots & m & 0 \\ 0 & \dots & 0 & -Nm \end{pmatrix}; \tag{2.22}$$

$$Q = \tilde{Q}^{\dagger} = \begin{pmatrix} d & 0 & 0 & 0 & \dots \\ 0 & \ddots & 0 & \vdots & \dots \\ 0 & 0 & d & 0 & \dots \\ 0 & \dots & 0 & 0 & \dots \end{pmatrix}, \qquad d = \sqrt{(N+1)\,\mu\,m}. \tag{2.23}$$

This is a particular case of the so-called r vacuum, with $r = N$. Although such a vacuum certainly exists classically, the existence of the quantum $r = N$ vacuum in this theory requires $N_f \geq 2\,N$, which we shall assume.[4]

To start with, ignore the smaller squark VEV, Eq. (2.23). As $\pi_2(G/H) \sim \pi_1(H) = \pi_1(U(1)) = \mathbb{Z}$, the symmetry breaking Eq. (2.22) gives rise to regular magnetic monopoles with mass of order of $O(\frac{v_1}{g})$, whose continuous transformation property is our main concern here. The semiclassical formulas for their mass and fluxes are well known [4, 35] and will not be repeated here.

2.3 Low-energy approximation

At scales much lower than $v_1 = m$ but still neglecting the smaller squark VEV $v_2 = d = \sqrt{(N+1)\,\mu\,m} \ll v_1$, the theory reduces to an $SU(N) \times U(1)$ gauge theory with N_f light quarks $q_i, \tilde{q}^i$ (the first N components of the original quark multiplets $Q_i, \tilde{Q}^i$). By integrating out the massive fields, the effective Lagrangian valid between the two mass scales has the form,

$$\begin{aligned} \mathcal{L} &= \frac{1}{4g_N^2}(F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2}(F_{\mu\nu}^0)^2 + \frac{1}{g_N^2}|\mathcal{D}_\mu\phi^a|^2 + \frac{1}{g_1^2}|\mathcal{D}_\mu\phi^0|^2 + |\mathcal{D}_\mu q|^2 + |\mathcal{D}_\mu\tilde{q}|^2 \\ &\quad - g_1^2\left| -\mu\,m\sqrt{N(N+1)} + \frac{\tilde{q}\,q}{\sqrt{N(N+1)}} \right|^2 - g_N^2|\sqrt{2}\,\tilde{q}\,t^a q|^2 + \dots \end{aligned} \tag{2.24}$$

where $a = 1, 2, \dots N^2 - 1$ labels the $SU(N)$ generators, t^a; the index 0 refers to the $U(1)$ generator $t^0 = \frac{1}{\sqrt{2N(N+1)}}\,diag(1, \dots, 1, -N)$. We have taken into account the fact that the $SU(N)$ and $U(1)$ coupling constants (g_N and g_1) get renormalized differently towards the infrared.

The adjoint scalars are fixed to its VEV, Eq. (2.22), with small fluctuations around it,

$$\Phi = \langle\Phi\rangle(1 + \langle\Phi\rangle^{-1}\,\tilde{\Phi}), \qquad |\tilde{\Phi}| \ll m. \tag{2.25}$$

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[4] This might appear to be a rather tight condition as the original theory loses asymptotic freedom for $N_f \geq 2\,N + 2$. This is not so. An analogous discussion can be made by considering the breaking $SU(N) \to SU(r) \times U(1)^{N-r}$. In this case the condition for the quantum non-Abelian vacuum is $2\,N > N_f \geq 2\,r$, which is a much looser condition. Also, although the corresponding $U(N)$ theory Eq. (2.1) with such a number of flavor has semilocal strings [36, 28, 24], these moduli are not directly related to the derivation of the dual gauge symmetry, which is our interest in this paper. We shall come back to these questions elsewhere.

In the consideration of the vortices of the low-energy theory, they will be in fact replaced by the constant VEV. The presence of the small terms Eq. (2.25), however, makes the low-energy vortices not strictly BPS (and this will be important in the consideration of their stability below).[5]

The quark fields are replaced, consistently with Eq. (2.19), as

$$\tilde{q} \equiv q^\dagger, \qquad q \to \frac{1}{\sqrt{2}}\, q, \tag{2.26}$$

where the second replacement brings back the kinetic term to the standard form.

We further replace the singlet coupling constant and the $U(1)$ gauge field as

$$e \equiv \frac{g_1}{\sqrt{2N(N+1)}}; \qquad \tilde{A}_\mu \equiv \frac{A_\mu}{\sqrt{2N(N+1)}}, \qquad \tilde{\phi}^0 \equiv \frac{\phi^0}{\sqrt{2N(N+1)}}. \tag{2.27}$$

The net effect is

$$\mathcal{L} = \frac{1}{4g_N^2}(F_{\mu\nu}^a)^2 + \frac{1}{4e^2}(\tilde{F}_{\mu\nu})^2 + |\mathcal{D}_\mu q|^2 - \frac{e^2}{2}\,|\,q^\dagger\,q - c\,\mathbf{1}\,|^2 - \frac{1}{2}\,g_N^2\,|\,q^\dagger\,t^a q\,|^2, \tag{2.28}$$

$$c = N(N+1)\sqrt{2\,\mu\,m}. \tag{2.29}$$

Neglecting the small terms left implicit, this is identical to the $U(N)$ model Eq. (2.1), except for the fact that $e \neq g_N$ here. The transformation property of the vortices can be determined from the moduli matrix, as was done in [21]. Indeed, the system possesses BPS saturated vortices described by the linearized equations

$$(\mathcal{D}_1 + i\mathcal{D}_2)\, q = 0, \tag{2.30}$$

$$F_{12}^{(0)} + \frac{e^2}{2}\left(c\,\mathbf{1}_N - q\,q^\dagger\right) = 0; \qquad F_{12}^{(a)} + \frac{g_N^2}{2}\,q_i^\dagger\,t^a\,q_i = 0. \tag{2.31}$$

The matter equation can be solved exactly as in [26, 30, 28] $(z = x^1 + ix^2)$ by setting

$$q = S^{-1}(z, \bar{z})\, H_0(z), \quad A_1 + i A_2 = -2\,i\,S^{-1}(z, \bar{z})\,\bar{\partial}_z S(z, \bar{z}), \tag{2.32}$$

where S is an $N \times N$ invertible matrix over whole of the z plane, and H_0 is the moduli matrix, holomorphic in z.

The gauge field equations take a slightly more complicated form than in the $U(N)$ model Eq. (2.1):

$$\partial_z\left(\Omega^{-1}\partial_{\bar{z}}\,\Omega\right) = -\frac{g_N^2}{2}\,\mathrm{Tr}\,(\,t^a\,\Omega^{-1}\,q\,q^\dagger)\,t^a - \frac{e^2}{4N}\,\mathrm{Tr}\,(\Omega^{-1}q\,q^\dagger - \mathbf{1}), \qquad \Omega = S\,S^\dagger. \tag{2.33}$$

The last equation reduces to the master equation Eq. (2.10) in the $U(N)$ limit, $g_N = e$.

The advantage of the moduli matrix formalism is that all the moduli parameters appear in the holomorphic, moduli matrix $H_0(z)$. Especially, the transformation property of the vortices under the color-flavor diagonal group can be studied by studying the behavior of the moduli matrix.

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[5]In the terminology used in Davis et al. [37] in the discussion of the Abelian vortices in supersymmetric models, our model corresponds to an F model while the models of [22, 24, 30] correspond to a D model. In the approximation of replacing Φ with a constant, the two models are equivalent: they are related by an $SU_R(2)$ transformation [39, 40].

3 Topological stability, vortex-monopole complex and confinement

The fact that there must be a continuous set of monopoles, which transform under the color-flavor G_{C+F} group, follows from the following exact homotopy sequence

$$\cdots \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \xrightarrow{f} \pi_1(G) \to \cdots , \tag{3.1}$$

applied to our systems with a hierarchical symmetry breaking, Eq. (1.7), with an exact unbroken symmetry, Eq. (1.9). $\pi_2(G) = \emptyset$ for any Lie group, and $\pi_1(G)$ depends on the group considered. Eq. (3.1) was earlier used to obtain the relation between the regular, soliton monopoles (represented by $\pi_2(G/H)$) and the singular Dirac monopoles, present if $\pi_1(G)$ is non-trivial. The isomorphism

$$\pi_1(G) \sim \pi_1(H)/\pi_2(G/H) \tag{3.2}$$

implied by Eq. (3.1) shows that among the magnetic monopole configurations $A_i^a(x)$ classified according to $\pi_1(H)$ [43], the regular monopoles correspond to the kernel of the map $f : \pi_1(H) \to \pi_1(G)$ [7].

When the homotopy sequence Eq. (3.1) is applied to a system with hierarchical breaking, in which H is completely broken at low energies,

$$G \xrightarrow{v_1} H \xrightarrow{v_2} \emptyset,$$

it allows an interesting re-interpretation. $\pi_1(H)$ classifies the quantized flux of the vortices in the low-energy H theory in Higgs phase. Vice versa, the high-energy theory (in which the small VEV is negligible) has 't Hooft-Polyakov monopoles quantized according to $\pi_2(G/H)$. However, there is something of a puzzle: when the small VEV's are taken into account, which break the "unbroken" gauge group completely, these monopoles must disappear somehow. A related puzzle is that the low-energy vortices with $\pi_1(H)$ flux, would have to disappear in a theory where $\pi_1(G)$ is trivial.

What happens is that the massive monopoles are confined by the vortices and disappear from the spectrum; on the other hand, the vortices of the low-energy theory end at the heavy monopoles once the latter are taken into account, having mass large but not infinite (Fig. 2). The low-energy vortices become unstable also through heavy monopole pair productions which break the vortices in the middle (albeit with small, tunneling rates [41]), which is really the same thing. Note that, even if the effect of such string breaking is neglected, a monopole-vortex-antimonopole configuration is not topologically stable anyway: its energy would become smaller if the string becomes shorter (so such a composite, generally, *will* get shorter and shorter and eventually disappear).

In the case $G = SU(N+1)$, $H = \frac{SU(N) \times U(1)}{\mathbb{Z}_N}$ we have a trivial $\pi_1(G)$, so

$$\pi_2\left(\frac{SU(N+1)}{U(N)}\right) = \pi_2(\mathbf{C}P^N) \sim \pi_1(U(N)) = \mathbb{Z} : \tag{3.3}$$

each non-trivial element of $\pi_1(U(N))$ is associated with a non-trivial element of $\pi_2(\frac{SU(N+1)}{U(N)})$. Each vortex confines a regular monopole. The monopole transformation properties follow from those of the vortices, as will be more concretely studied in the next section.
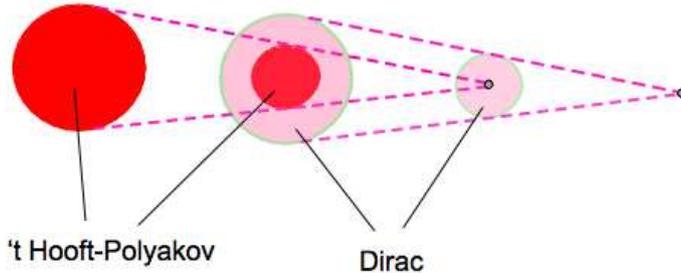
Figure 1: A pictorial representation of the exact homotopy sequence, Eq. (3.1), with the leftmost figure corresponding to $\pi_2(G/H)$.

In theories with a non-trivial $\pi_1(G)$ such as $SO(N)$, the application of these ideas is slightly subtle: these points will be discussed in Section 5.

In all cases, as long as the group H is completely broken at low energies and because $\pi_2(G) = \emptyset$ always, none of the vortices (if $\pi_1(G) = \emptyset$) and monopoles are truly stable, as static configurations. They can be only approximately so, in an effective theory valid in respective regions ($v_1 \simeq \infty$ or $v_2 \simeq 0$).

However, this does not mean that, for instance, a monopole-vortex-antimonopole composite configuration cannot be *dynamically* stabilized, or that they are not relevant as a physical configuration. A rotation can stabilize easily such a configuration dynamically, except that it will have a small non-vanishing probability for decay through a monopole-pair production, if such a decay is allowed kinematically.

After all, we believe that the real-world mesons are quark-string-antiquark bound states of this sort, the endpoints rotating almost with a speed of light! An excited meson can and indeed do decay through quark pair productions into two lighter mesons (or sometimes to a baryon-antibaryon pair, if allowed kinematically and by quantum numbers). Only the lightest mesons are truly stable. The same occurs with our monopole-vortex-antimonopole configurations. The lightest such systems, after the rotation modes are appropriately quantized, are truly stable bound states of solitons, even though they might not be stable as static, semiclassical configurations.

Our model is thus a reasonably faithful (dual) model of the quark confinement in QCD.

A related point, more specific to the supersymmetric models we consider here as a concrete testing ground, is the fact that monopoles in the high-energy theory and vortices in the low-energy theory, are both BPS saturated. It is crucial in our argument that they are both BPS only approximately; *they are almost BPS but not exactly.*[6] They are unstable in the full theory. But the fact that there exists a limit (of a large ratio of the mass scales, $\frac{v_1}{v_2} \to \infty$) in which these solitons become exactly BPS and stable, means that the magnetic flux through the surface of a small sphere surrounding the monopole and the vortex magnetic flux through a plane perpendicular to the vortex axis, must match exactly. These questions (the flux matching) have been discussed extensively already in [20].

Our argument, applied to the simplest case, $G = SO(3)$, and $H = U(1)$, is precisely the one

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[6] The importance of almost BPS soliton configurations have also been emphasized by Strassler [17].
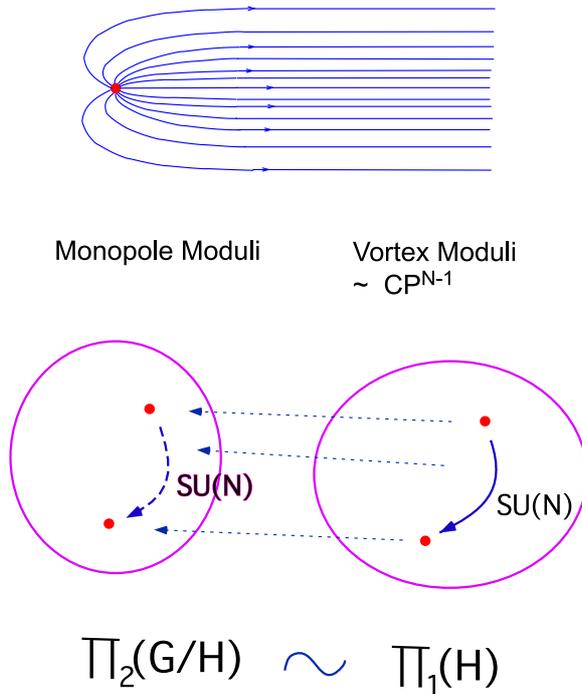
Figure 2: The non-trivial vortex moduli implies a corresponding moduli of monopoles.

adopted by 't Hooft in his pioneering paper [1] to argue that there must be a regular monopole of charge two (with respect to the Dirac's minimum unit): as the vortex of winding number $k = 2$ must be trivial in the full theory ($\pi_1(SO(3)) = \mathbb{Z}_2$), such a vortex must end at a regular monopole. What is new here, as compared to the case discussed by 't Hooft [1] is that now the unbroken group H is non-Abelian and that the low-energy vortices carry continuous, non-Abelian flux moduli. The monopoles appearing as the endpoints of such vortices must carry the same continuous moduli (Fig. 2). This is how we define the non-Abelian monopoles and their dual transformations, as discussed below more explicitly.

4   Dual gauge transformation among the monopoles

The concepts such as the low-energy BPS vortices or the high-energy BPS monopole solutions are thus only approximate: their explicit forms are valid only in the lowest-order approximation, in the respective kinematical regions. Nevertheless, there is a property of the system which is exact and does not depend on any approximation: the full system has an exact, global $SU(N)_{C+F}$ symmetry, which is neither broken by the interactions nor by both sets of VEVs, v_1 and v_2. This symmetry is broken by individual soliton vortex, endowing the latter with non-Abelian orientational moduli, analogous to the translational zero-modes of a kink. Note that the vortex breaks the color-flavor symmetry as

$$SU(N)_{C+F} \to SU(N-1) \times U(1), \tag{4.1}$$

leading to the moduli space of the minimum vortices which is

$$\mathcal{M} \simeq \mathbf{C}P^{N-1} = \frac{SU(N)}{SU(N-1) \times U(1)}. \tag{4.2}$$

11

The fact that this moduli coincides with the moduli of the quantum states of an N-state quantum mechanical system, is a first hint that the monopoles appearing at the endpoint of a vortex, transform as a fundamental multiplet $\underline{N}$ of a group $SU(N)$.

The moduli space of the vortices is described by the moduli matrix (we consider here the vortices of minimal winding, $k = 1$)

$$H_0(z) \simeq \begin{pmatrix} 1 & 0 & 0 & -a_1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -a_{N-1} \\ 0 & \dots & 0 & z \end{pmatrix}, \tag{4.3}$$

where the constants a_i, $i = 1, 2, \ldots, N - 1$ are the coordinates of $\mathbf{C}P^{N-1}$. Under $SU(N)_{C+F}$ transformation, the squark fields transform as

$$q \to U^{-1} q U, \tag{4.4}$$

but as the moduli matrix is defined *modulo* holomorphic redefinition Eq. (2.11), it is sufficient to consider

$$H_0(z) \to H_0(z) U. \tag{4.5}$$

Now, for an infinitesimal $SU(N)$ transformation acting on a matrix of the form Eq. (4.3), U can be taken in the form,

$$U = \mathbf{1} + X, \qquad X = \begin{pmatrix} \mathbf{0} & \vec{\xi} \\ -(\vec{\xi})^\dagger & 0 \end{pmatrix}, \tag{4.6}$$

where $\vec{\xi}$ is a small $N - 1$ component constant vector. Computing $H_0 X$ and making a V transformation from the left to bring back H_0 to the original form, we find

$$\delta a_i = -\xi_i - a_i (\vec{\xi})^\dagger \cdot \vec{a}, \tag{4.7}$$

which shows that a_i's indeed transform as the inhomogeneous coordinates of $\mathbf{C}P^{N-1}$. In other words, the vortex represented by the moduli matrix Eq. (4.3) transforms as a fundamental multiplet of $SU(N)$.[7]

As an illustration consider the simplest case of $SU(2)$ theory. In this case the moduli matrix is simply [42]

$$H_0^{(1,0)} \simeq \begin{pmatrix} z - z_0 & 0 \\ -b_0 & 1 \end{pmatrix}; \qquad H_0^{(0,1)} \simeq \begin{pmatrix} 1 & -a_0 \\ 0 & z - z_0 \end{pmatrix}. \tag{4.8}$$

with the transition function between the two patches:

$$b_0 = \frac{1}{a_0}. \tag{4.9}$$

---

[7] Note that, if a $\underline{N}$ vector $\vec{c}$ transforms as $\vec{c} \to (\mathbf{1} + X) \vec{c}$, the inhomogeneous coordinates $a_i = c_i/c_N$ transform as in Eq. (4.7).

The points on this $\mathbf{C}P^1$ represent all possible $k = 1$ vortices. Note that points on the space of a quantum mechanical two-state system,

$$|\Psi\rangle = a_1|\psi_1\rangle + a_2|\psi_2\rangle, \qquad (a_1, a_2) \sim \lambda(a_1, a_2), \quad \lambda \in \mathbf{C}, \tag{4.10}$$

can be put in one-to-one correspondence with the inhomogeneous coordinate of a $\mathbf{C}P^1$,

$$a_0 = \frac{a_1}{a_2}, \qquad b_0 = \frac{a_2}{a_1}. \tag{4.11}$$

In order to make this correspondence manifest, note that the minimal vortex Eq. (4.8) transforms under the $SU(2)_{C+F}$ transformation, as

$$H_0 \rightarrow V H_0 U^\dagger, \qquad U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \tag{4.12}$$

where the factor $U^\dagger$ from the right represents a flavor transformation, V is a holomorphic matrix which brings H_0 to the original triangular form [21]. The action of this transformation on the moduli parameter, for instance, a_0, can be found to be

$$a_0 \rightarrow \frac{\alpha\, a_0 + \beta}{\alpha^* - \beta^*\, a_0}. \tag{4.13}$$

But this is precisely the way a doublet state Eq. (4.10) transforms under $SU(2)$,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{4.14}$$

The fact that the vortices (seen as solitons of the low-energy approximation) transform as in the $\underline{N}$ representation of $SU(N)_{C+F}$, implies that there exist a set of monopoles which transform accordingly, as $\underline{N}$. The existence of such a set follows from the exact $SU(N)_{C+F}$ symmetry of the theory, broken by the individual monopole-vortex configuration. This answers questions such as Eq. (1.5) unambiguously.

Note that in our derivation of continuous transformations of the monopoles, the explicit, semiclassical form of the latter is not utilized.

A subtle point is that in the high-energy approximation, and to lowest order of such an approximation, the semiclassical monopoles are just certain non-trivial field configurations involving $\phi(x)$ and $A_i(x)$ fields, and therefore apparently transform under the color part of $SU(N)_{C+F}$ only. When the full monopole-vortex configuration $\phi(x), A_i(x), q(x)$ (Fig. 2) are considered, however, only the combined color-flavor diagonal transformations keep the energy of the configuration invariant. In other words, the monopole transformations must be regarded as part of more complicated transformations involving flavor, when higher order effects in $O(\frac{v_1}{v_2})$ are taken into account.[8]

And this means that the transformations are among physically distinct states, as the vortex moduli describe obviously physically distinct vortices [19].

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[8] Another independent effect due to the massless flavors is that of Jackiw-Rebbi [38]: due to the normalizable zero-modes of the fermions, the semi-classical monopole is converted to some irreducible multiplet of monopoles in the *flavor* group $SU(N_f)$. The "clouds" of the fermion fields surrounding the monopole have an extension of $O(\frac{1}{v_1})$, which is much smaller than the distance scales associated with the infrared effects discussed here and should be regarded as distinct effects.
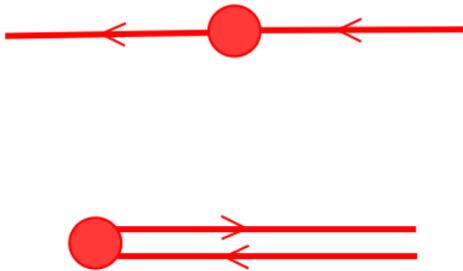
Figure 3: Monopoles in $U(N)$ systems with abelianization are confined by two Abelian vortices.

4.1 $SU(N)$ gauge symmetry breaking and Abelian monopole-vortex systems

Recently there has been considerable amount of research activity [18],[22]-[30], on systems closely related to ours. As the terminology used and concepts involved are often similar but physically distinct, a confusion might possibly arise.

As should be clear from what we said so far, it is crucial that the color-flavor diagonal symmetry $SU(N)$ remains exactly conserved, for the emergence of non-Abelian dual gauge group. Consider, instead, the cases in which the gauge $U(N)$ (or $SU(N) \times U(1)$) symmetry is broken to Abelian subgroup $U(1)^N$, either by small quark mass differences (*cfr.* Eq. (2.5) and Eq. (2.7)) or dynamically, as in the $\mathcal{N} = 2$ models with $N_f < 2N$ [23, 24]. From the breaking of various $SU(2)$ subgroups to $U(1)$ there appear light 't Hooft-Polyakov monopoles of mass $O(\frac{\Delta m}{g})$ (in the case of an explicit breaking) or $O(\Lambda)$ (in the case of dynamical breaking). As the $U(1)^N$ gauge group is further broken by the squark VEVs, the system develops ANO vortices. The light magnetic monopoles, carrying magnetic charges of two different $U(1)$ factors, look confined by the two vortices (Fig. 3). These cases have been discussed extensively, within the context of $U(N)$ model of Subsection 2.1, in [18],[22]-[30]. In Hanany et al. [22, 23] and Shifman et al. [24, 25], furthermore, the dynamics of the fluctuation of the orientational modes along the vortex, described as a two-dimensional $\mathbf{C}P^{N-1}$ model, is studied. It is shown that the kinks of the two-dimensional sigma model precisely correspond to these light monopoles, to be expected in the underlying $4D$ gauge theory. In particular, it was noted that there is an elegant matching between the dynamics of two-dimensional sigma model (describing the dynamics of the vortex orientation modes in the Higgs phase of the $4D$ theory) and the dynamics of the $4D$ gauge theory in the Coulomb phase [29, 22, 23, 24].

Note that this is also a reasonably close (dual) model of *what would occur* in QCD if the color $SU(3)$ symmetry were to dynamically break itself to $U(1)^2$, *i.e.*, with generators $Q^1 = diag\,(1, -1, 0)$, $Q^2 = diag\,(0, 1, -1)$, respectively. Confinement would be described in this case by the condensation of magnetic monopoles carrying the Abelian charges Q^1, or Q^2, and the resulting ANO vortices will be of two types, 1 and 2 carrying the related fluxes. The quark q_1 will be confined by the vortex 1, the quark q_2 by the composite of the vortices $\bar{1}$ and 2 (just as the light monopoles discussed above – Fig. 3) and the quark q_3 by the vortex $\bar{2}$.

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4.2 Non-Abelian duality requires an exact flavor symmetry

In the $\mathcal{N} = 2$ supersymmetric QCD, the presence of massless flavor and the exact color-flavor diagonal symmetry is fundamental for the emergence of the dual (non-Abelian) gauge transformations. It is well known in fact that the continuous non-Abelian vortex flux moduli - hence the non-Abelian vortex - disappear as soon as non-zero mass differences $m_i - m_j$ are introduced.[9] Also in order for the $SU(N)_{C+F}$ color-flavor symmetry not to be destroyed by the gauge dynamics itself, it is necessary to have the number of flavors such that $N_f > 2N$. These points have been emphasized already in the first paper on the subject [19].

It is illuminating that the same phenomenon can be seen in the fully quantum behavior of the theory of Section 2.2, in another regime,

$$\mu, m_i \sim \Lambda \tag{4.15}$$

(*cfr.* Eq. (2.21)). Indeed, this model was analyzed thoroughly in this regime in [12]. The so-called r vacua with the low-energy effective $SU(r) \times U(1)^{N+1-r}$ gauge symmetry emerges in the equal mass limit $m_i \to m$ in which the global symmetry group $SU(N_f) \times U(1)$ of the underlying theory become exact. When the bare quark masses are almost equal but distinct, the theory possesses a group of $\binom{N_f}{r}$ nearby vacua, each of which is an Abelian $U(1)^N$ theory, with N massless Abelian magnetic monopole pairs. The jump from the $U(1)^N$ to $SU(r) \times U(1)^{N+1-r}$ theory in the exact $SU(N_f)$ limit might appear a discontinuous change of physics, but is not so. What happens is that the range of validity of Abelian description in each Abelian vacuum, neglecting the light monopoles and gauge bosons (including massless particles of the neighboring vacua, and other light particles which fill up a larger gauge multiplet in the limit the vacua coalesce), gradually tends to zero as the vacua collide. The non-Abelian, enhanced gauge symmetry of course only emerges in the strictly degenerate limit, in which the underlying theory has an exact $SU(N_f)$ global symmetry.

5 $\quad SO(2N+1) \to SU(r) \times U(1)^{N-r-1} \to \emptyset$

Let us now test our ideas about duality transformations against another class of theories,

$$SO(2N+1) \xrightarrow{\langle \phi_1 \rangle \neq 0} SU(r) \times U(1)^{N-r+1} \xrightarrow{\langle \phi_2 \rangle \neq 0} \emptyset. \tag{5.1}$$

One of the reasons why this case is interesting is that the semiclassical monopoles arising from the symmetry breaking $SO(2N+1) \xrightarrow{\langle \phi_1 \rangle \neq 0} U(N)$ appear to belong to the second-rank symmetric tensor representation of $SU(N)$ [6, 35]. Another, related reason is the fact that since $\pi_1(G) = \pi_1(SO(2N+1)) = \mathbb{Z}_2$, the homotopy map Eq. (3.1) is less trivial in this case. Thirdly, according to the detailed analysis of the softly-broken $\mathcal{N} = 2$ theories with $SO(N)$ gauge group [13] the quantum mechanical behavior of the monopoles is different for $r = N$ and for $r < N$. Non-Abelian monopoles belonging to the fundamental representation of the dual $SU(r)$ group appears only for $r < N_f/2$, and because of the requirement of asymptotic freedom of the original theory ($N_f < 2N - 1$), this is possible only for $r < N$. It is very encouraging that such a difference in

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[9] Such an alignment of the vacuum with the bare mass parameters is characteristic of supersymmetric theories, familiar also in the $\mathcal{N} = 1$ SQCD [31]. In real QCD we do not expect such a strict alignment.

the behavior of non-Abelian monopoles indeed follows, as we shall see, from the way we define the dual group though the transformation properties of mixed monopole-vortex configurations and homotopy map.

5.1   Maximal SU factor; $SO(5) \to U(2) \to \emptyset$

Let us first consider the case the $SU(N)$ factor has the maximum rank,

$$SO(2N + 1) \stackrel{\langle \phi_1 \rangle \neq 0}{\longrightarrow} U(N).$$

To be concrete, let us consider the case of an $SO(5)$ theory, where a scalar VEV of the form

$$\langle \Phi \rangle = \begin{pmatrix} 0 & i\,v & 0 & 0 & 0 \\ -i\,v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\,v & 0 \\ 0 & 0 & -i\,v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.2}$$

breaking the gauge group as $SO(5) \to H = SU(2) \times U(1)/\mathbb{Z}_2 = U(2)$. We assume that at lower energies some other scalar VEVs break H completely, leaving however a color-flavor diagonal $SU(2)$ group unbroken. This model arises semiclassically in softly broken $\mathcal{N} = 2$ supersymmetric $SO(5)$ gauge theory with large, equal bare quark masses, m, and with a small adjoint scalar mass μ, with scalar VEVs given by $v = m/\sqrt{2}$ in Eq. (5.2) and

$$Q = \tilde{Q}^\dagger = \sqrt{\frac{\mu m}{2}} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ i & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & i & 0 & \cdots \\ 0 & 0 & 0 & \cdots \end{pmatrix}. \tag{5.3}$$

(See Appendix A, also the Section 2 of [13], for more details).

The $SO(4) \sim SU(2) \times SU(2)$ subgroup living on the upper-left corner is broken to $SU(2) \times U(1)$, giving rise to a single 't Hooft-Polyakov monopole. On the other hand, by embedding the 't Hooft-Polyakov monopole in the two $SO(3)$ subgroups (in the (125) and (345) subspaces), one finds two more monopoles. All three of them are degenerate. Actually, E. Weinberg [44] has found a continuous set of degenerate monopole solutions interpolating these, and noted that the transformations among them are not simply related to the unbroken $SU(2)$ group.[10]

From the point of view of stability argument, Eq. (3.1), this case is very similar to the case considered by 't Hooft, as $\pi_1(SO(5)) = \mathbb{Z}_2$: a singular $\mathbb{Z}_2$ Dirac monopole can be introduced in the theory. The minimal vortex of the low-energy theory is truly stable in this case, as a minimal non-trivial element of $\pi_1(H)$ represents also a non-trivial element of $\pi_1(G)$. This can be seen as follows. A minimum element of $\pi_1(H) = \pi_1(U(2)) \sim \mathbb{Z}$ corresponds to simultaneous rotations of angle π in the (12) and (34) planes (which is a half circle of $U(1)$), which brings the origin to the $\mathbb{Z}_2$ element of $SU(2)$, $diag\,(-1, -1, -1, -1, 1)$, followed by an $SU(2)$ transformation back to the origin, an angle $-\pi$ rotation in the (12) plane and an angle π rotation around (34)

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[10] This and similar cases are sometimes referred to as "accidentally degenerate case" in the literature.

plane. The net effect is a 2π rotation in the (34) plane, which is indeed a non-trivial element of $\pi_1(SO(5)) = \mathbb{Z}_2$. Such a vortex would confine the singular Dirac monopole, if introduced into the theory (See Fig. 1).

On the other hand, there are classes of vortices which appear to be stable in the low-energy approximation, but are not so in the full theory. In fact non-minimal $k = 2$ elements of $\pi_1(H) = \pi_1(SU(2) \times U(1)/\mathbb{Z}_2) \sim \mathbb{Z}$ are actually trivial in the full theory. This means that the $k = 2$ vortices must end at a regular monopole. Vice versa, as $\pi_2(SO(5)) = \emptyset$, the regular 't Hooft Polyakov monopoles of high-energy theory must be confined by these non-minimal vortices and disappear from the spectrum.

The transformation property of $k = 2$ vortices has been studied recently in [46, 45], and in particular, in [21]. It turns out that the moduli space of the $k = 2$ vortices is a $\mathbf{C}P^2$ with a conic singularity. It was shown that the generic $k = 2$ vortices transform under the $SU(2)_{C+F}$ group as a *triplet*. At a particular point of the moduli - an orbifold singularity - the vortex is Abelian: it is a *singlet* of $SU(2)_{C+F}$. [11]

As the full theory has an exact, unbroken $SU(2)_{C+F}$ symmetry, it follows from the homotopy-group argument of Section 3 that *the monopoles in the high-energy $SO(5) \to U(2)$ theory have components transforming as a triplet and a singlet of $SU(2)_{C+F}$.*

Note that it is not easy to see this result – and is somewhat misleading to attempt to do so – based solely on the semi-classical construction of the monopoles or on the zero-mode analysis around such solutions, where the unbroken color-flavor symmetry is not appropriately taken into account. Generically, the "unbroken" color $SU(2)$ group suffers from the topological obstruction [5] (or perturbatively, from the pathology of non-normalizable gauge zero-modes [5, 6]), as we noted already.

Nevertheless, there are indications that the findings by E. Weinberg [44] are consistent with the properties of the $k = 2$ vortices. In the standard way to embed $SU(2)$ subgroups through the Cartan decomposition (we follow here the notation of [44]),

$$t_1(\nu) = \frac{1}{(2\,\nu^2)^{-1/2}} (E_\nu + E_{-\nu}); \quad t_2(\nu) = \frac{-i}{(2\,\nu^2)^{-1/2}} (E_\nu - E_{-\nu}); \quad t_3 = (\nu^2)^{-1}\,\nu_j\,T_j, \quad (5.4)$$

where ν denotes the non-vanishing root vectors of $SO(5)$ (Fig. 4), the unbroken $SU(2)$ group is generated by γ. The monopole associated with the root vector β and the (equivalent) one given by μ naturally form a doublet of the "unbroken" $SU(2)$, while the monopole with the α charges is a singlet. The continuous set of monopoles interpolating among these monopoles found by Weinberg are analogous to the continuous set of vortices we found, which form the points of the $\mathbf{C}P^2$, which transform as a triplet. (See the Fig. 5 taken from [21]).

An even more concrete hint of consistency comes from the structure of the moduli space of the monopoles. The moduli metric found in [44] is

$$ds^2 = M\,d\mathbf{x}^2 + \frac{16\pi^2}{M} d\chi^2 + k \left[\frac{d\,b^2}{b} + b\,(d\alpha^2 + \sin^2\alpha\,d\beta^2 + (d\,\gamma + \cos\alpha\,d\,\beta)^2) \right]. \quad (5.5)$$

By performing a simple change of coordinate, $B \equiv 2\sqrt{b}$, it becomes evident that the moduli

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[11] In another complex codimension-one subspace, they appear to transform as a *doublet*. However quantum states of *any* triplet of $SU(2)$ contains such an orbit. The state of maximum S_z, $|1,1\rangle$, transforms under $SU(2)$ as an $SO(3)$ vector, staying on a subspace $S^2 \sim CP^1 \subset CP^2$.
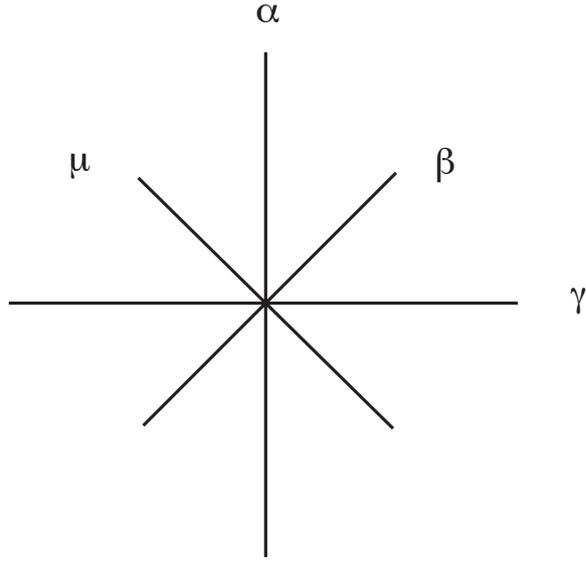
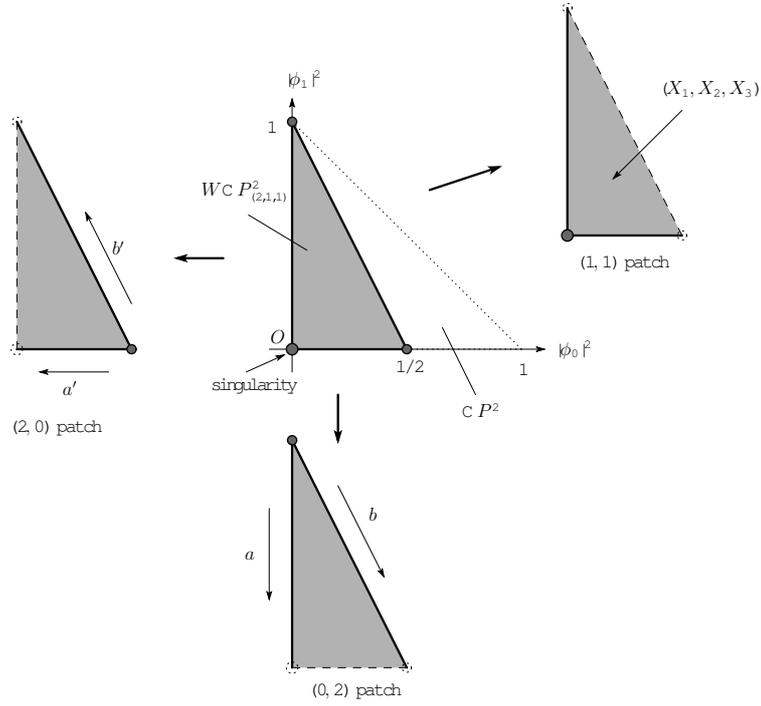Figure 4: Non-zero root vectors of $SO(5)$

Figure 5: Moduli space of $k = 2$ vortices of $U(2)$ theory. See [21] for more details.

space has the structure

$$\mathbf{C}^2/\mathbb{Z}_2, \tag{5.6}$$

apart from the irrelevant factor $\mathbf{R}^3$ (the position of the monopole) and S^1 ($U(1)$ phase).[12] Eq. (5.6) coincides with the moduli space of the $k = 2$ co-axial *vortices*, seen in the central $(1,1)$ patch [21].

These considerations strengthen our conclusion that the continuous set of monopoles found in [44] belongs to a singlet and a triplet representations of the dual $SU(2)$ group. Although the detailed properties of the moduli spaces for monopoles and vortices are different[13], this could be related to the fact that one should ultimately consider a smooth monopole-vortex mixed configurations in the full theory, not each of them separately. Also, related to this point, there remains the fact that the dual group which is exact and under which monopoles transform, is *not* the original $SU(2)$ subgroup but involves the flavor group essentially.

Note that our conclusion is based on the exact symmetry, and should be reliable. However, the degeneracy among all the vortices (or the monopoles) lying in the entire moduli space $\mathbf{C}P^2/\mathbb{Z}_2$ found in the BPS limits, is an artifact of the lowest-order approximation. Only the degeneracy among the vortices (or among the monopoles) belonging to the same multiplet is expected to survive quantum mechanically. $\underline{1}$ and $\underline{3}$ vortex tensions (monopole masses) will split. Which of the multiplets ($\underline{1}$ or $\underline{3}$) will remain stable, after quantum corrections are taken into account, is a question just lying beyond the power of semiclassical considerations.

In the context of asymptotically-free $\mathcal{N} = 2$ supersymmetric models, there are no indications that the triplet monopoles of $SO(5) \to U(2)$ theory survive quantum mechanically. This result can be actually understood by a simple renormalization-group argument:

- In a $SO(2N + 1)$ theories with $\mathcal{N} = 2, 1$ supersymmetries, the condition for the original theory to be asymptotic-free (N_f less than $2N-1$, $\frac{3(2N-2)}{2}$, respectively)[14] is not compatible with the low-energy $SU(N)$ theory being non-asymptotic-free ($N_f \geq 2N$ and $N_f \geq 3N$, respectively.)

The problem would not arise if the rank of the unbroken $SU(r)$ were smaller. That such a "sign-flip" of the beta function is a necessary condition for the emergence of low-energy non-Abelian monopoles has been pointed out some time ago by one of the authors [48], even though the validity of such an argument for non-supersymmetric theories is perhaps not obvious.

If the condition of asymptotic freedom of the ultraviolet theory is dropped, then there are no such constraints, and it makes sense to consider symmetry breaking patterns such as $SO(2N + 1) \to U(N)$. Our conclusion that the monopoles of $SO(5) \to U(2)$ system transform as a triplet or a singlet would apply under such conditions. Analogously, we expect the monopoles in the system $SO(2N + 1) \to U(N)$ to transform as a second-rank symmetric or antisymmetric representation.

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[12] The monopole modulus due to the unbroken $U(1) \subset U(2)$ is not present in the full system, where the gauge group is completely broken.

[13] The first is known to be hyper-Kähler and the second Kähler – indeed $\mathbf{C}P^2/\mathbb{Z}_2$ does not admit hyper-Kähler structure.

[14]The counting is made for the appropriate supersymmetry multiplets, N_f hypermultiplets for $\mathcal{N} = 2$; N_f chiral multiplets for $\mathcal{N} = 1$ supersymmetric $SO(N)$ theory.

5.2 $SO(2N+1) \to SU(r) \times U(1)^{N-r-1} \to \emptyset \ (r < N)$

Consider now the cases in which the unbroken $SU(r)$ factor has a smaller rank, $SO(2N+1) \to SU(r) \times U(1)^{N-r+1} \to \emptyset$, where $r < N$. For concreteness, let us discuss the case of an $SO(7)$ theory,

$$SO(7) \xrightarrow{\langle \phi_1 \rangle \neq 0} U(2) \times U(1) \xrightarrow{\langle \phi_2 \rangle \neq 0} \emptyset. \tag{5.7}$$

As we are interested in a concrete dynamical realization of this, we consider the softly broken $\mathcal{N} = 2$ theory, with $N_f = 4$ quark hypermultiplets. Such a number of flavors ensures both the original $SO(7)$ theory being asymptotically free and the $SU(2)$ subgroup being non-asymptotically free. The low-energy gauge group $U(2) \times U(1)$ is completely broken by the squark VEV's similar to Eq. (5.3). The large VEV $\langle \phi_1 \rangle$ has the form:

$$\langle \phi_1 \rangle = \begin{pmatrix} 0 & iv_0 & 0 & 0 & 0 & 0 & 0 \\ -iv_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & iv_0 & 0 & 0 & 0 \\ 0 & 0 & -iv_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & iv_1 & 0 \\ 0 & 0 & 0 & 0 & -iv_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad v_1 \neq v_0. \tag{5.8}$$

The "unbroken" $U(2)$ lies in $SO(4)_{1234} \sim SU(2) \times SU(2)$ while the $U(1)$ factor corresponds to the rotations in the 56 plane (see Appendix A). The semiclassical monopoles of high-energy theory are [15]

(i) a triplet of degenerate monopoles of mass $2 |v_0|/g$ (they arise as in the $SO(5)$ theory discussed above);

(ii) a doublet of degenerate monopoles of mass $|v_0 - v_1|/g$: they arise from the breaking of $SU_+(2) \subset SO(4)_{1256}$ and $SU_+(2) \subset SO(4)_{3456}$ (see Appendix A);

(iii) a doublet of degenerate monopoles of mass $|v_0 + v_1|/g$: they also arise from the breaking of $SU_-(2) \subset SO(4)_{1256}$ and $SU_-(2) \subset SO(4)_{3456}$;

(iv) a singlet monopole of mass $2 |v_1|/g$ arising from the breaking of $SO(3)_{567}$.

Which of these semiclassical monopoles are the lightest and which of them are stable against decay into lighter monopole pairs, depend on the various VEVs. It is possible that the monopoles (ii) or (iii) are the lightest of all. Of course more detailed issues such as which of the degeneracies survives quantum effects, are questions which go beyond the semiclassical approximations.

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[15]Within the softly broken $\mathcal{N} = 2$, the quantum mechanical vacua with $SU(2) \times U(1)^2$ gauge symmetry, in the limit $m_i = m \simeq \Lambda$, appears to arise from the semiclassical vacua of the form of Eq. (5.8), with $v_0 = m/\sqrt{2} \gg \Lambda$, $v_1 = 0$, with classical symmetry $SU(2) \times U(1) \times SO(3)_{567}$. The $SO(3)_{567}$ gauge sector (pure $\mathcal{N} = 2$ theory) becomes strongly-coupled at low energies and breaks itself to $U(1)$. Thus it would be more correct to say $v_1 \sim \Lambda$, but then the discussion about semiclassical monopole masses $\sim v_1/g$, etc., should not be taken too literally. If one wishes, one could consider a larger gauge group, *e.g.*, $SO(9)$, to do a straightforward semiclassical analysis for an unbroken $SU(2)$ group. In general, the relation between the classical vacua and the fully quantum mechanical vacua is a rather subtle issue. See for instance the discussions in [16].

In fact, when $v_0, v_1 \sim \Lambda$ the standard semi-classical reasoning fails to give any reliable answer: a fully quantum-mechanical analysis is needed. Fortunately, in the softly broken $\mathcal{N} = 2$ theory such analyses have been performed [13] and we do know that the light monopoles in the fundamental representation ($\underline{2}$) of $SU(2)$ appear in an appropriate vacuum.

Knowing this, we might try to understand how such a result may follow from our definition of the dual group. At low energies the gauge group $U(2) \times U(1)$ is completely broken, leaving a color-flavor diagonal $SU(2)_{C+F}$ symmetry unbroken. The theory possesses vortices of

$$\pi_1(U(2) \times U(1)) = \mathbb{Z} \times \mathbb{Z}. \tag{5.9}$$

The minimal vortices corresponding to $\pi_1(U(2)) = \mathbb{Z}$ transform as a $\underline{2}$ of $SU(2)_{C+F}$.

A minimum element of $\pi_1(U(2) \times U(1))$ such as an angle 2π rotation in the $U(1)_{56}$ factor, or the minimal $U(2)$ loop, corresponds to vortices stable in the full theory. They would confine Dirac monopoles associated with $\pi_1(SO(7)) = \mathbb{Z}_2$, if the latter were introduced in the theory.

The regular monopoles in which we are interested in, are instead confined by some non-minimal ($k = 2$) vortices of the low-energy theory. However, in contrast to the $SO(5)$ theory discussed in the preceding subsection, this *does not* necessarily imply a second-rank tensor representation of $SU(2)_{C+F}$ of these monopoles. In fact, the monopoles of the (ii) group, for instance, carry the minimum charge of $U(2)$ and an unit charge of $U(1)$. Therefore, the relevant $k = 2$ vortex corresponds to the minimum element both of $\pi_1(U(2))$ and of $\pi_1(U(1))$, generated by a 2π rotation in the 56 plane together with a minimal loop of $\pi_1(U(2))$, analogous to the one discussed in the preceding subsection. As a consequence the monopoles confined by such vortices, by our discussion of Section 3, transform as a *doublet* of the dual group $\widetilde{SU}(2) \sim SU(2)_{C+F}$.

This discussion naturally generalizes to all other cases with symmetry breaking, $SO(2N+1) \to SU(r) \times U(1)^{N-r+1} \to \emptyset$, $r < N$. The dual magnetic $SU(r)$ group observed in the low-energy effective theory [13], under which the light monopoles transform as a fundamental multiplet, thus matches nicely with the properties of the dual $\widetilde{SU}(r) \sim SU(r)_{C+F}$ group.

The cases of $SO(2N) \to SU(r) \times U(1)^{N-r+1} \to \emptyset$, $r < N$ are similar. We expect that there is a qualitative difference between the breaking with the maximum rank SU factor and smaller $SU(r)$ unbroken groups. Such a difference is indeed observed in the fully quantum mechanical analysis of $SO(N)$ theory [13].

The behavior of monopoles in asymptotic-free $USp(2N)$ theories ($N_f < 2N + 2$) is more similar to those appearing in the $SU(N)$ theories, because of the property, $\pi_1(USp(2N)) = \emptyset$. All monopoles are regular monopoles due to the partial symmetry breaking, $USp(2N) \to SU(r) \times U(1)^{N-r+1}$, $r \leq N$. The transformation property of these monopoles, in the theory with exact unbroken $SU(r)_{C+F}$ global symmetry, is deduced from the transformation properties among the non-Abelian vortices of the low-energy system $SU(r) \times U(1)^{N-r+1} \to \emptyset$: they transform as $\underline{r}$ of $SU(r)_{C+F}$. Such a result is consistent dynamically, as long as $r \leq N_f/2$. It is comfortable that these are precisely what is found from the quantum mechanical analysis [12, 11].

6   Conclusion

In this paper we have examined an idea about the "non-Abelian monopoles", put forward some time ago [20], more systematically and by using some recent results on the non-Abelian *vortices*.

According to this idea, the dual transformation of non-Abelian monopoles occurring in a system with gauge symmetry breaking $G \longrightarrow H$ is to be defined by setting the low-energy H system in Higgs phase, so that the dual system is in confinement phase. The transformation law of the monopoles follows from that of monopole-vortex mixed configurations in the system

$$G \xrightarrow{v_1} H \xrightarrow{v_2} \emptyset, \qquad (v_1 \gg v_2)$$

under an unbroken, exact color-flavor diagonal symmetry $H_{C+F} \sim \tilde{H}$. The transformation properties of the regular monopoles (classified by $\pi_2(G/H)$) follow from those among the non-Abelian vortices (classified by $\pi_1(H)$), via the isomorphism $\pi_1(G) \sim \pi_1(H)/\pi_2(G/H)$. Our results, obtained in the semiclassical approximation (reliable at $v_1 \gg v_2 \gg \Lambda$) of softly-broken $\mathcal{N} = 2$ supersymmetric $SU(N)$ and $SO(N)$ theories, are – very non-trivially – found to be consistent with the fully quantum-mechanical low-energy effective action description (valid at $v_1, v_2 \sim \Lambda$), available in these theories.

For $G = SU(N+1)$, $H = U(N)$, $G_F = SU(N_f)$, $N_f \geq 2N$, this argument proves that the monopoles induced by the G/H breaking transform as $\underline{N}$ of $\tilde{H} = SU(N)$. Analogous result holds for $G = SU(N+1)$, $H = U(r)$, $G_F = SU(N_f)$, $r < N_f/2$, where the semi-classical monopoles transform as in the fundamental multiplets ($\underline{r}$) (as well as some singlets) of $SU(r)$. These results are in agreement with what was found in the fully quantum mechanical treatment of the system [11, 12].

For $G = SO(2N + 1)$, $H = U(r) \times U(1)^{N-r}$, $G_F = SU(N_f)$ (with $r < N_f/2$, $r < N$) we find monopoles which transform in the fundamental representation of the dual $\widetilde{SU}(r) = SU(r)_{C+F}$ group. This result is again consistent with the fully quantum mechanical analysis of $\mathcal{N} = 2$ supersymmetric $SO(N)$ models [13] and in agreement with the universality of certain superconformal theories discovered in this context by Eguchi et. al. [47].

In the case of maximal-rank SU subgroup, such as $G = SO(5)$, $H = U(2)$, there is a qualitative difference both in our duality argument and in the full quantum results. For instance the set of monopoles found earlier by E. Weinberg is shown to belong to a singlet and a triplet representations of the dual $SU(2)$ group, but their quantum fate is not known. In supersymmetric models a renormalization-group argument suggests (and the explicit analysis of softly broken $\mathcal{N} = 2$ theory shows) that the triplet does not survive the quantum effects, as long as the underlying $SO(5)$ theory is asymptotically free.

For $G = SO(2N)$, $H = U(r) \times U(1)^{N-r}$, $G_F = SU(N_f)$ the situation is similar. When $r < N$, $r < N_f/2$ we find monopoles transforming in the $\underline{r}$ representation of the dual $\widetilde{SU}(r) = SU(r)_{C+F}$, whereas the maximal case, $r = N$, encounters the same renormalization-group constraint as in $SO(2N + 1)$.

Finally for $G = USp(2N)$, $H = U(r) \times U(1)^{N-r}$, $G_F = SU(N_f)$ the picture is very much like in $SU(N+1)$. We have monopoles in the fundamental representation of the dual $\widetilde{SU}(r) = SU(r)_{C+F}$ as long as $N_f > 2r$.

Summarizing, in the context of softly-broken $\mathcal{N} = 2$ supersymmetric gauge theories with SU, SO and USp groups, where fully quantum mechanical results are available by combining the various knowledges such as the Seiberg-Witten curves, decoupling theorem, Nambu-Goldstone theorem, non-renormalization of Higgs branches, $\mathcal{N} = 1$ ADS instanton superpotential, vacuum counting, universality of conformal theories, etc., our idea on non-Abelian monopoles is in agreement with these known exact results. Although such an agreement is comfortable, our

argument, based on the homotopy-map-stability argument on almost BPS solitons and on some exact symmetries, should be of more general validity.

Acknowledgement

A Monopoles in $SO(N)$ theories

Here are some formulae useful for the discussion of Section 5. The minimal $SU(2)$ embeddings (*i.e.*, with the smallest Dynkin index, $\text{Tr}\, T^a T^b$) in $SO(N)$ groups are obtained through various $SO(4) \subset SO(N)$ subgroups. For instance the $SU(2) \times SU(2) \subset SO(5)$ subgroups are generated by

$$T_1^{\pm} = -\frac{i}{2}(\Sigma_{23} \pm \Sigma_{41}), \quad T_2^{\pm} = -\frac{i}{2}(\Sigma_{31} \pm \Sigma_{42}), \quad T_3^{\pm} = -\frac{i}{2}(\Sigma_{12} \pm \Sigma_{43}), \tag{A.1}$$

where *e.g.*

$$\Sigma_{23} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

is a rotation in the 23 plane. Non-minimal embeddings correspond to various $SO(3)$ subgroups, acting in 125 and 345 subspaces, for instance, in the $SO(5)$ example.

The VEV Eq. (5.2) is proportional to T_3^+: it leaves $SU_-(2) \times U_+(1)$ unbroken. An $SO(5)$ solution can be obtained [3, 4] by embedding the 't Hooft-Polyakov monopoles [1] in the broken $SU(2)$ as $(S_a \equiv T_a^+)$

$$A_i(\mathbf{r}) = A_i^a(\mathbf{r}, \mathbf{h} \cdot \alpha)\, S_a; \qquad \phi(\mathbf{r}) = \chi^a(\mathbf{r}, \mathbf{h} \cdot \alpha)\, S_a + [\,\mathbf{h} - (\mathbf{h} \cdot \alpha)\, \alpha^*\,] \cdot \mathbf{H}, \tag{A.2}$$

where

$$A_i^a(\mathbf{r}) = \epsilon_{aij}\frac{r^j}{r^2}A(r); \qquad \chi^a(\mathbf{r}) = \frac{r^a}{r}\chi(r), \qquad \chi(\infty) = \mathbf{h} \cdot \alpha. \tag{A.3}$$

Note that $\phi(\mathbf{r} = (0, 0, \infty)) = \phi_0$. In the above formula the Higgs field vacuum expectation value (VEV) has been parametrized in the form

$$\phi_0 = \mathbf{h} \cdot \mathbf{H}, \tag{A.4}$$

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where $\mathbf{h} = (h_1, \ldots, h_{\mathrm{rank}(G)})$ is a constant vector representing the VEV. The root vectors orthogonal to $\mathbf{h}$ ($\propto \alpha$ in Fig. 4) belong to the unbroken subgroup H (γ in Fig. 4).

The above consideration is basically group-theoretic and is valid in any types of theories, supersymmetric or not. Now we specialize to the concrete dynamical models we are working on: $\mathcal{N} = 2$ supersymmetric gauge theories. Under the symmetry breaking $SO(5) \to U(2)$ the quark superfields Q and $\tilde{Q}$ in the first four components of the vector representation rearrange themselves as follows. Recall that the relevant superpotential terms have the form, $Q(m\mathbf{1} + \sqrt{2}\Phi)\tilde{Q}$, summed over diagonal flavor indices, $A = 1, 2, \ldots, N_f$ left implicit. For each flavor, the adjoint scalar VEV of the form Eq. (5.8), with $v = m/\sqrt{2}$, gives rise to a 2×2 block-diagonal mass matrix

$$m\mathbf{1} + \sqrt{2}\Phi = \begin{pmatrix} \mathbf{V} & 0 \\ 0 & \mathbf{V} \end{pmatrix} \qquad \mathbf{V} = m \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{A.5}$$

in color. $\mathbf{V}$ has one vanishing and one massive eigenvalues. Thus the four fields

$$\hat{Q}^1 = \frac{1}{\sqrt{2}}(Q^1 + iQ^2), \quad \hat{Q}^3 = \frac{1}{\sqrt{2}}(Q^3 + iQ^4), \quad \hat{\tilde{Q}}^1 = \frac{1}{\sqrt{2}}(\tilde{Q}^1 + i\tilde{Q}^2), \quad \hat{\tilde{Q}}^3 = \frac{1}{\sqrt{2}}(\tilde{Q}^3 + i\tilde{Q}^4),$$
$$\tag{A.6}$$

are massless. The orthogonal combinations such as $\frac{1}{\sqrt{2}}(Q^1 - iQ^2)$ become massive and decouple from the low-energy theory.

The massless quark superfields of the low-energy $SU(2)$ theory are the combinations

$$q^1 = \frac{1}{\sqrt{2}}(\hat{Q}^1 + i\hat{Q}^3); \qquad q^2 = \frac{1}{\sqrt{2}}(i\hat{Q}^1 + \hat{Q}^3), \tag{A.7}$$

which form a $\underline{2}$, and

$$\tilde{q}^1 = \frac{1}{\sqrt{2}}(\hat{\tilde{Q}}^1 - i\hat{\tilde{Q}}^3); \qquad \tilde{q}^2 = \frac{1}{\sqrt{2}}(-i\hat{\tilde{Q}}^1 + \hat{\tilde{Q}}^3), \tag{A.8}$$

which form a $\underline{2}^*$. [16]

It is straightforward to generalize the above construction to $SO(2N{+}1) \to SU(r) \times U(1)^{N-r+1}$, $r < N$. N_f quark hypermultiplets in the $SO(2N + 1)$ vector representation yield precisely N_f flavors of massless quarks in $\underline{r}$ of $SU(r)$ plus a number of singlets.

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