Backreacting Flavors in the Klebanov-Strassler Background

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Abstract

In this paper we present new analytic solutions of pure Type IIB supergravity plus D7-branes describing the addition of an arbitrary number of flavors to the Klebanov-Tseytlin and Klebanov-Strassler backgrounds. We provide a precise field theory dual and a detailed analysis of the duality cascade which describes its RG flow. Matchings of beta functions and anomalies between the field theory and the string setup are presented. We give an understanding of Seiberg duality as a large gauge transformation on the RR and NSNS potentials. We also analyze the UV behavior of the field theory, that as suggested by the string background can be associated with a duality wall.

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1 Introduction

Early ideas of 't Hooft [1] and the experimental evidence for stringy behavior in hadronic physics suggested that aspects of the strong interaction can be described, predicted, and understood using a (not yet known) string theory. These ideas started to materialize when the Maldacena conjecture (also known as AdS/CFT correspondence) and some of its refinements were formulated [2, 3, 4, 5]. Indeed, a (3+1)-dimensional field theory was shown to contain strings that captured non-perturbative and perturbative physics. The downside was that the field theory in question ($\mathcal{N} = 4$ SYM) was not of immediate relevance to hadronic physics. The necessity of finding extensions of these ideas to phenomenologically more interesting field theories was then well motivated.

A very fruitful extension of the AdS/CFT correspondence stems from studying branes at conical singularities, of which the case of D3-branes on conifolds is a particular example presenting especially rich dynamics. The study of this system was developed in a list of many interesting papers: those that have been more influential to the present work are [6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

In this paper, we will consider the addition of new degrees of freedom to the Klebanov-Tseytlin(KT) [9] and Klebanov-Strassler (KS) [10] solutions. These new excitations will be incorporated in the form of D7 flavor branes, corresponding to fundamental matter in the dual field theory. Unlike the color branes, which disappear in the geometric transition and are substituted by closed string fluxes, the flavor branes appear explicitly in our solutions: they correspond to the open strings which are suggested by Veneziano's topological expansion of large N gauge theories [16]. The addition of flavors to these field theories was first considered in [17, 18, 19]. We will follow ideas introduced in [17], but will consider the case in which the number of fundamental fields is of the same order as the number of adjoint or bifundamental fields, that is $N_f \sim N_c$. This means that the new (strongly coupled) dynamics of the field theory is captured by a background that includes the backreaction of the flavor branes. In order to find the new solutions, we follow the ideas and techniques of [20, 21, 22, 23].

Let us describe the main achievements of this paper. We will present *analytic* solutions for the equations of motion of Type IIB supergravity coupled to the DBI+WZ action of the flavor D7-branes that preserve minimal SUSY in four dimensions; we show how to reduce these solutions to those found by Klebanov-Tseytlin/Strassler when the number of flavors is taken to zero. Using them, we make a precise matching between the field theory cascade (that, enriched by the presence of the fundamentals, is still self-similar) and the string predictions. We will also match anomalies and beta functions by using our new supergravity background. There is a variety of other field theory observables that can be predicted by our solutions, and will be presented in a future publication [24], where also many technical details and nice subtleties of the present paper will be explained.

The organization of this paper is as follows. In Section 2 we present the setup, the ansatz

and the strategy to find supersymmetric solutions of the Bianchi identities and equations of motion. We also introduce the notion of the so-called Page charges and we analyze their values in our particular ansatz. In Sections 3 and 4 we present two main solutions, which reflect the addition of flavors to the KS and KT backgrounds, respectively. In Section 5 we present the dual field theory and propose that its RG flow can be understood in terms of a cascade of Seiberg dualities. In Section 6 we show that the duality cascade is encoded in our supergravity solutions, by comparing ranks of the groups in field theory with effective charges in supergravity, and matching R-anomalies and beta functions of gauge couplings on both sides of the gauge/gravity duality. The behavior of the background in the UV of the gauge theory suggests that the field theory generates a 'duality wall'. We also give a nice picture of Seiberg duality as a large gauge transformation in supergravity. We close the paper with possible future directions in Section 7.

2   The setup and the ansatz

We are interested in adding to the KT/KS cascading gauge theory [9, 10] a number of flavors (fundamental fields) comparable with the number of colors (adjoint and bifundamental fields). Those supergravity backgrounds were obtained by considering a stack of regular and fractional D3-branes at the tip of the conifold. After the geometric transition, the color branes disappear from the geometry, but the closed string fluxes they sourced remain nontrivial. The addition of flavors in the field theory, in the large N limit considered by Veneziano [16], amounts to introduce mesonic currents and internal quark loops in the planar diagrams that survive 't Hooft's double scaling limit [1]: Feynman diagrams in Veneziano's limit arrange in a topological expansion reminiscent of the loop expansion of an open and closed string theory, where the new open string sector arises from loops of flavor fields in the field theory. The dual string picture of Veneziano's large N limit involves the addition of back-reacting 'flavor' branes to the backgrounds which describe the 't Hooft limit of some gauge theories.

Let us then consider a system of type IIB supergravity plus N_f D7-branes. The dynamics of the latter will be governed by the corresponding Dirac-Born-Infeld and Wess-Zumino actions. Our solution will have a non-trivial metric and dilaton ϕ and, as in any cascading background, non-vanishing RR three- and five-forms F_3 and F_5, as well as a non-trivial NSNS three-form H_3. In addition, the D7-branes act as a source for (the Hodge dual of) the RR one-form F_1 through the WZ coupling:

$$S_{WZ}^{D7} = T_7 \sum_{N_f} \int_{\mathcal{M}_8} \hat{C}_8 + \cdots , \qquad (2.1)$$

which generically induces a violation of the Bianchi identity $dF_1 = 0$. Therefore our configuration will also necessarily have a non-vanishing value of F_1. The ansatz we shall adopt for

the Einstein frame metric is the following:

$$ds^2 = \left[h(r) \right]^{-\frac{1}{2}} dx_{1,3}^2 + \left[h(r) \right]^{\frac{1}{2}} \left[dr^2 + e^{2G_1(r)} (\sigma_1^2 + \sigma_2^2) + \right.$$

$$\left. + e^{2G_2(r)} \left((\omega_1 + g(r)\,\sigma_1)^2 + (\omega_2 + g(r)\,\sigma_2)^2 \right) + \frac{e^{2G_3(r)}}{9} \left(\omega_3 + \sigma_3 \right)^2 \right], \quad (2.2)$$

where $dx_{1,3}^2$ denotes the four-dimensional Minkowski metric and σ_i and ω_i ($i = 1, 2, 3$) are one-forms that can be written in terms of five angular coordinates $(\theta_1, \varphi_1, \theta_2, \varphi_2, \psi)$ as follows:

$$\sigma_1 = d\theta_1, \qquad \sigma_2 = \sin\theta_1 \, d\varphi_1, \qquad \sigma_3 = \cos\theta_1 \, d\varphi_1,$$
$$\omega_1 = \sin\psi \sin\theta_2 \, d\varphi_2 + \cos\psi \, d\theta_2, \qquad \omega_2 = -\cos\psi \sin\theta_2 \, d\varphi_2 + \sin\psi \, d\theta_2,$$
$$\omega_3 = d\psi + \cos\theta_2 \, d\varphi_2. \quad (2.3)$$

Notice that our metric ansatz (2.2) depends on five unknown radial functions $G_i(r)$ ($i = 1, 2, 3$), $g(r)$ and $h(r)$. The ansatz for F_5 has the standard form, namely:

$$F_5 = dh^{-1}(r) \wedge dx^0 \wedge \cdots \wedge dx^3 + \text{Hodge dual}. \quad (2.4)$$

As expected for flavor branes, we will take D7-branes extended along the four Minkowski coordinates as well as other four internal coordinates. The κ-symmetric embedding of the D7-branes we start from will be discussed in Section 5. In order to simplify the computations, following the approach of [23], we will smear the D7-branes in their two transverse directions in such a way that the symmetries of the unflavored background are recovered. As explained in [23], this smearing amounts to the following generalization of the WZ term of the D7-brane action:

$$S_{WZ}^{D7} = T_7 \sum_{N_f} \int_{\mathcal{M}_8} \hat{C}_8 + \cdots \qquad \rightarrow \qquad T_7 \int_{\mathcal{M}_{10}} \Omega \wedge C_8 + \cdots, \quad (2.5)$$

where Ω is a two-form which determines the distribution of the RR charge of the D7-brane and $\mathcal{M}_{10}$ is the full ten-dimensional manifold. Notice that Ω acts as a magnetic charge source for F_1 which generates a violation of its Bianchi identity. Actually, from the equation of motion of C_8 one gets:

$$dF_1 = -\Omega. \quad (2.6)$$

In what follows we will assume that the flavors introduced by the D7-brane are massless, which is equivalent to require that the flavor brane worldvolume reaches the origin in the holographic direction. Under this condition one expects a radial coordinate independent D7-brane charge density. Moreover, the D7-brane embeddings that we will smear imply that Ω is symmetric under the exchange of the two S^2's parameterized by (θ_1, φ_1) and (θ_2, φ_2),

and independent of ψ (see Section 5). The smeared charge density distribution is the one already adopted in [23], namely:

$$dF_1 = -\frac{N_f}{4\pi}\left(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\right) = \frac{N_f}{4\pi}\left(\omega_1 \wedge \omega_2 - \sigma_1 \wedge \sigma_2\right), \qquad (2.7)$$

where the coefficient $N_f/4\pi$ is determined by normalization. With this ansatz for Ω the modified Bianchi identity (2.6) determines the value of F_1, namely:

$$F_1 = \frac{N_f}{4\pi}(\omega_3 + \sigma_3). \qquad (2.8)$$

The ansatz for the RR and NSNS 3-forms that we propose is an extension of the one given by Klebanov and Strassler and it is simply (in this paper we set for convenience $\alpha' = 1$):

$$B_2 = \frac{M}{2}\left[f\, g^1 \wedge g^2 + k\, g^3 \wedge g^4\right],$$

$$H_3 = dB_2 = \frac{M}{2}\left[dr \wedge (f'\, g^1 \wedge g^2 + k'\, g^3 \wedge g^4) + \frac{1}{2}(k-f)\, g^5 \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)\right],$$

$$F_3 = \frac{M}{2}\left\{g^5 \wedge \left[(F + \frac{N_f}{4\pi}f)g^1 \wedge g^2 + \left(1 - F + \frac{N_f}{4\pi}k\right)g^3 \wedge g^4\right] + F'dr \wedge \left(g^1 \wedge g^3 + g^2 \wedge g^4\right)\right\},$$

$$(2.9)$$

where M is a constant, $f(r)$, $k(r)$ and $F(r)$ are functions of the radial coordinate, and the g^i's are the set of one-forms

$$g^1 = \frac{1}{\sqrt{2}}(\omega_2 - \sigma_2), \qquad\qquad g^2 = \frac{1}{\sqrt{2}}(-\omega_1 + \sigma_1),$$

$$g^3 = \frac{-1}{\sqrt{2}}(\omega_2 + \sigma_2), \qquad\qquad g^4 = \frac{1}{\sqrt{2}}(\omega_1 + \sigma_1),$$

$$g^5 = \omega_3 + \sigma_3. \qquad (2.10)$$

The forms F_3, H_3 and F_5 must satisfy the following set of Bianchi identities:

$$dF_3 = H_3 \wedge F_1, \qquad\qquad dH_3 = 0, \qquad\qquad dF_5 = H_3 \wedge F_3. \qquad (2.11)$$

Notice that the equations for F_3 and H_3 are automatically satisfied by our ansatz (2.9). However, the Bianchi identity for F_5 gives rise to the following differential equation:

$$\frac{d}{dr}\left[h'\, e^{2G_1 + 2G_2 + G_3}\right] = -\frac{3}{4}M^2\left[(1 - F + \frac{N_f}{4\pi}k)f' + (F + \frac{N_f}{4\pi}f)k' + (k-f)F'\right], \qquad (2.12)$$

which can be integrated, with the result:

$$h'\, e^{2G_1 + 2G_2 + G_3} = -\frac{3}{4}M^2\left[f - (f-k)F + \frac{N_f}{4\pi}fk\right] + \text{constant}. \qquad (2.13)$$

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Let us now parameterize F_5 as

$$F_5 = \frac{\pi}{4} N_{eff}(r) \, g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 \, + \, \text{Hodge dual} , \qquad (2.14)$$

and let us define the five-manifold $\mathcal{M}_5$ as the one that is obtained by taking the Minkowski coordinates and r fixed to a constant value. As $\int_{\mathcal{M}_5} F_5 = (4\pi^2)^2 N_{eff}(r)$, it follows that $N_{eff}(r)$ can be interpreted as the effective D3-brane charge at the value r of the holographic coordinate. From our ansatz (2.4), it follows that:

$$N_{eff}(r) = -\frac{4}{3\pi} h' e^{2G_1 + 2G_2 + G_3} , \qquad (2.15)$$

and taking into account (2.13), we can write

$$N_{eff}(r) \equiv \frac{1}{(4\pi^2)^2} \int_{\mathcal{M}_5} F_5 = N_0 + \frac{M^2}{\pi} \left[f - (f-k)F + \frac{N_f}{4\pi} fk \right] , \qquad (2.16)$$

where N_0 is a constant. It follows from (2.16) that the RR five-form F_5 is determined once the functions F, f and k that parameterize the three-forms are known. Moreover, eq. (2.13) allows to compute the warp factor once the functions G_i and the three-forms are determined. Notice also that the effective D5-brane charge is obtained by integrating the gauge-invariant field strength F_3 over the 3-cycle S^3: $\theta_2 = \text{const.}$, $\varphi_2 = \text{const.}$. The result is:

$$M_{eff}(r) \equiv \frac{1}{4\pi^2} \int_{S^3} F_3 = M \left[1 + \frac{N_f}{4\pi} (f+k) \right] . \qquad (2.17)$$

The strategy to proceed further is to look at the conditions imposed by supersymmetry. We will smear, as in [23], D7-brane embeddings that are κ-symmetric and, therefore, the supersymmetry requirement is equivalent to the vanishing of the variations of the dilatino and gravitino of type IIB supergravity under supersymmetry transformations. These conditions give rise to a large number of BPS first order ordinary differential equations for the dilaton and the different functions that parameterize the metric and the forms. In the end, one can check that the first order differential equations imposed by supersymmetry imply the second order differential equations of motion. In particular, from the variation of the dilatino we get the following differential equation for the dilaton:

$$\phi' = \frac{3N_f}{4\pi} e^{\phi - G_3} . \qquad (2.18)$$

A detailed analysis of the conditions imposed by supersymmetry shows that the fibering function g in eq. (2.2) is subjected to the following algebraic constraint:

$$g \left[g^2 - 1 + e^{2(G_1 - G_2)} \right] = 0 , \qquad (2.19)$$

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which has obviously two solutions. The first of these solutions is $g = 0$ and, as it is clear from our metric ansatz (2.2), it corresponds to the cases of the flavored singular and resolved conifolds. In the second solution g is such that the term in brackets on the right-hand side of (2.19) vanishes. This solution gives rise to the flavored version of the warped deformed conifold. The flavored KT solution will be presented in Section 4, whereas the flavored KS solution will be analyzed in Section 3.

2.1 Maxwell and Page charges

Before presenting the explicit solutions for the metric and the forms of the supergravity equations, let us discuss the different charges carried out by our solutions. In theories, like type IIB supergravity, that have Chern-Simons terms in the action (which give rise to modified Bianchi identities), it is possible to define more than one notion of charge associated with a given gauge field. Let us discuss here, following the presentation of ref.[25], two particular definitions of this quantity, namely the so-called Maxwell and Page charges [26]. Given a gauge invariant field strength F_{8-p}, the (magnetic) Maxwell current associated to it is defined through the following relation:

$$d\, F_{8-p} \; = \; \star j^{Maxwell}_{D_p} \;\; , \tag{2.20}$$

or equivalently, the Maxwell charge in a volume V_{9-p} is given by:

$$Q^{Maxwell}_{D_p} \; \sim \; \int_{V_{9-p}} \star j^{Maxwell}_{D_p} \;\; , \tag{2.21}$$

with a suitable normalization. Taking $\partial V_{9-p} = M_{8-p}$ and using (2.20) and Stokes theorem, we can rewrite the previous expression as:

$$Q^{Maxwell}_{D_p} \; \sim \; \int_{M_{8-p}} F_{8-p} \;\; . \tag{2.22}$$

This notion of current is gauge invariant and conserved and it has other properties that are discussed in [25]. In particular, it is not "localized" in the sense that for a solution of pure supergravity (for which $d\, F_{8-p} = H_3 \wedge F_{6-p}$) this current does not vanish. These are the kind of charges we have calculated so far (2.16)-(2.17), namely:

$$Q^{Maxwell}_{D5} \; = \; M_{eff} \; = \; \frac{1}{4\pi^2} \int F_3 \;\; ,$$

$$Q^{Maxwell}_{D3} \; = \; N_{eff} \; = \; \frac{1}{(4\pi^2)^2} \int F_5 \;\; . \tag{2.23}$$

An important issue regarding these charges is that, in general, they are not quantized. Indeed, we have checked explicitly that $Q^{Maxwell}_{D5} = M_{eff}$ and $Q^{Maxwell}_{D3} = N_{eff}$ vary continuously with the holographic variable r (see eqs. (2.17) and (2.16)).

Let us move on to the notion of Page charge. The idea is first to write the Bianchi identities for F_3 and F_5 as the exterior derivatives of some differential form, which in general will not be gauge invariant, and then introduce the Page current as a source. In our case, we can define the following (magnetic) Page currents:

$$d(F_3 - B_2 \wedge F_1) = \star j_{D5}^{Page} \,,$$
$$d(F_5 - B_2 \wedge F_3 + \frac{1}{2} B_2 \wedge B_2 \wedge F_1) = \star j_{D3}^{Page} \,. \tag{2.24}$$

Notice that the currents defined by the previous expression are "localized" as a consequence of the Bianchi identities satisfied by F_3 and F_5, namely $dF_3 = H_3 \wedge F_1$ and $dF_5 = H_3 \wedge F_3$. The Page charges Q_{D5}^{Page} and Q_{D3}^{Page} are just defined as the integrals of $\star j_{D5}^{Page}$ and $\star j_{D3}^{Page}$ with the appropriate normalization, $i.e.$:

$$Q_{D5}^{Page} = \frac{1}{4\pi^2} \int_{V_4} \star j_{D5}^{Page} \,,$$
$$Q_{D3}^{Page} = \frac{1}{(4\pi^2)^2} \int_{V_6} \star j_{D3}^{Page} \,, \tag{2.25}$$

where V_4 and V_6 are submanifolds in the transverse space to the D5- and D3-branes respectively, which enclose the branes. By using the expressions of the currents $\star j_{D5}^{Page}$ and $\star j_{D3}^{Page}$ given in (2.24), and by applying Stokes theorem, we get:

$$Q_{D5}^{Page} = \frac{1}{4\pi^2} \int_{S^3} \left(F_3 - B_2 \wedge F_1 \right) \,,$$
$$Q_{D3}^{Page} = \frac{1}{(4\pi^2)^2} \int_{\mathcal{M}_5} \left(F_5 - B_2 \wedge F_3 + \frac{1}{2} B_2 \wedge B_2 \wedge F_1 \right) \,, \tag{2.26}$$

where S^3 and $\mathcal{M}_5$ are the same manifolds used to compute the Maxwell charges in eqs. (2.17) and (2.16). It is not difficult to establish the topological nature of these Page charges. Indeed, let us consider, for concreteness, the expression of Q_{D5}^{Page} in (2.26). Notice that the three-form under the integral can be *locally* represented as the exterior derivative of a two-form, since $F_3 - B_2 \wedge F_1 = dC_2$, with C_2 being the RR two-form potential. If C_2 were well-defined globally on the S^3, the Page charge Q_{D5}^{Page} would vanish identically as a consequence of Stokes theorem. Thus, Q_{D5}^{Page} can be naturally interpreted as a monopole number and it can be non-vanishing only in the case in which the gauge field is topologically non-trivial. For the D3-brane Page charge Q_{D3}^{Page} a similar conclusion can be reached.

Due to the topological nature of the Page charges defined above, one naturally expects that they are quantized and, as we shall shortly verify, they are independent of the holographic coordinate. This shows that they are the natural objects to compare with the numbers of branes that create the geometry in these backgrounds with varying flux. However, as it is manifest from the fact that Q_{D5}^{Page} and Q_{D3}^{Page} are given in (2.26) in terms of the B_2 field and

not in terms of its field strength H_3, the Page charges are not gauge invariant. In subsection 6.2 we will relate this non-invariance to the Seiberg duality of the field theory dual.

Let us now calculate the associated Page charges for our ansatz (2.9) . We shall start by computing the D5-brane Page charge for the three-sphere S^3 defined by $\theta_2, \varphi_2 = $ constant. We already know the value of the integral of F_3, which gives precisely M_{eff} (see eq. (2.17)). Taking into account that

$$\int_{S^3} g^5 \wedge g^1 \wedge g^2 = \int_{S^3} g^5 \wedge g^3 \wedge g^4 = 8\pi^2 , \tag{2.27}$$

we readily get:

$$\frac{1}{4\pi^2} \int_{S^3} B_2 \wedge F_1 = \frac{MN_f}{4\pi} (f+k) , \tag{2.28}$$

and therefore:

$$Q_{D5}^{Page} = M_{eff} - \frac{MN_f}{4\pi} (f+k) . \tag{2.29}$$

Using the expression of M_{eff} given in (2.17), we obtain:

$$Q_{D5}^{Page} = M , \tag{2.30}$$

which is certainly quantized and independent of the radial coordinate.

Let us now look at the D3-brane Page charge, which can be computed as an integral over the angular manifold $\mathcal{M}_5$. Taking into account that

$$\int_{\mathcal{M}_5} g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 = (4\pi)^3 , \tag{2.31}$$

we get that, for our ansatz (2.9):

$$\begin{aligned} \frac{1}{(4\pi^2)^2} \int_{\mathcal{M}_5} B_2 \wedge F_3 &= \frac{M^2}{\pi} \left[f - (f-k)F + \frac{N_f}{2\pi} fk \right] , \\ \frac{1}{(4\pi^2)^2} \int_{\mathcal{M}_5} B_2 \wedge B_2 \wedge F_1 &= \frac{M^2}{\pi} \frac{N_f}{4\pi} fk , \end{aligned} \tag{2.32}$$

and, thus

$$Q_{D3}^{Page} = N_{eff} - \frac{M^2}{\pi} \left[f - (f-k)F + \frac{N_f}{4\pi} fk \right] , \tag{2.33}$$

and, using the expression of N_{eff}, we obtain

$$Q_{D3}^{Page} = N_0 , \tag{2.34}$$

which is again independent of the holographic coordinate. Recall that these Page charges are not gauge invariant and we will study in subsection 6.2 how they change under a large gauge transformation.

We now proceed to present the solutions to the BPS equations of motion.

3   Flavored warped deformed conifold

Let us now consider the following solution of the algebraic constraint (2.19):

$$g^2 = 1 - e^{2(G_1 - G_2)} \ . \tag{3.1}$$

In order to write the equations for the metric and dilaton in this case, let us perform the following change of variable:

$$3 \, e^{-G_3} \, dr = d\tau \ . \tag{3.2}$$

In terms of this new variable, the differential equation for the dilaton is simply:

$$\dot{\phi} = \frac{N_f}{4\pi} \, e^\phi \ , \tag{3.3}$$

where the dot means derivative with respect to τ. This equation can be straightforwardly integrated, namely:

$$\frac{N_f}{4\pi} \, e^\phi = \frac{1}{\tau_0 - \tau} \ , \qquad 0 \le \tau \le \tau_0 \ , \tag{3.4}$$

where τ_0 is an integration constant. Let us now write the equations imposed by supersymmetry to the metric functions G_1, G_2 and G_3, which are:

$$\dot{G}_1 - \frac{1}{18} e^{2G_3 - G_1 - G_2} - \frac{1}{2} e^{G_2 - G_1} + \frac{1}{2} e^{G_1 - G_2} = 0 \ ,$$

$$\dot{G}_2 - \frac{1}{18} e^{2G_3 - G_1 - G_2} + \frac{1}{2} e^{G_2 - G_1} - \frac{1}{2} e^{G_1 - G_2} = 0 \ ,$$

$$\dot{G}_3 + \frac{1}{9} e^{2G_3 - G_1 - G_2} - e^{G_2 - G_1} + \frac{N_f}{8\pi} e^\phi = 0 \ . \tag{3.5}$$

In order to write the solution of this system of equations, let us define the following function

$$\Lambda(\tau) \equiv \frac{\left[2(\tau - \tau_0)(\tau - \sinh 2\tau) + \cosh(2\tau) - 2\tau\tau_0 - 1 \right]^{\frac{1}{3}}}{\sinh \tau} \ . \tag{3.6}$$

Then, the metric functions G_i are given by:

$$e^{2G_1} = \frac{1}{4} \, \mu^{\frac{4}{3}} \, \frac{\sinh^2 \tau}{\cosh \tau} \, \Lambda(\tau) \ , \qquad e^{2G_2} = \frac{1}{4} \, \mu^{\frac{4}{3}} \, \cosh \tau \, \Lambda(\tau) \ ,$$

$$e^{2G_3} = 6 \, \mu^{\frac{4}{3}} \, \frac{\tau_0 - \tau}{\left[\Lambda(\tau) \right]^2} \ , \tag{3.7}$$

where μ is an integration constant. Notice that the range of τ variable chosen in (3.4) is the one that makes the dilaton and the metric functions real. Moreover, for the solution we have found, the fibering function g is given by:

$$g = \frac{1}{\cosh \tau} \ . \tag{3.8}$$

By using this result, we can write the metric as:

$$ds^2 = \left[h(\tau) \right]^{-\frac{1}{2}} dx_{1,3}^2 + \left[h(\tau) \right]^{\frac{1}{2}} ds_6^2 , \qquad (3.9)$$

where ds_6^2 is the metric of the 'flavored' deformed conifold, namely

$$ds_6^2 = \frac{1}{2} \mu^{\frac{4}{3}} \Lambda(\tau) \left[\frac{4(\tau_0 - \tau)}{3\Lambda^3(\tau)} \left(d\tau^2 + (g^5)^2 \right) + \cosh^2\left(\frac{\tau}{2} \right) \left((g^3)^2 + (g^4)^2 \right) + \right.$$
$$\left. + \sinh^2\left(\frac{\tau}{2} \right) \left((g^1)^2 + (g^2)^2 \right) \right] . \qquad (3.10)$$

Notice the similarity between the metric (3.10) and the one corresponding to the 'unflavored' deformed conifold [10]. To further analyze this similarity, let us study the $N_f \to 0$ limit of our solution. By looking at the expression of the dilaton in (3.4), one realizes that this limit is only sensible if one also sends $\tau_0 \to +\infty$ with $N_f \tau_0$ fixed. Indeed, by performing this scaling and neglecting τ versus τ_0, one gets a constant value for the dilaton. Moreover, the function $\Lambda(\tau)$ reduces in this limit to $\Lambda(\tau) \approx (4\tau_0)^{\frac{1}{3}} K(\tau)$, where $K(\tau)$ is the function appearing in the metric of the deformed conifold, namely:

$$K(\tau) = \frac{\left[\sinh 2\tau - 2\tau \right]^{\frac{1}{3}}}{2^{\frac{1}{3}} \sinh \tau} . \qquad (3.11)$$

By using this result one easily verifies that, after redefining $\mu \to \mu/(4\tau_0)^{\frac{1}{4}}$, the metric (3.10) reduces to the one used in [10] for the unflavored system.

The requirement of supersymmetry imposes the following differential equations for the functions k, f and F appearing in the fluxes of our ansatz:

$$\dot{k} = e^\phi \left(F + \frac{N_f}{4\pi} f \right) \coth^2 \frac{\tau}{2} ,$$
$$\dot{f} = e^\phi \left(1 - F + \frac{N_f}{4\pi} k \right) \tanh^2 \frac{\tau}{2} ,$$
$$\dot{F} = \frac{1}{2} e^{-\phi}(k - f) . \qquad (3.12)$$

Notice, again, that for $N_f = 0$ the system (3.12) reduces to the one found in [10]. Moreover, for $N_f \neq 0$ this system can be solved as:

$$e^{-\phi} f = \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau - 1) , \qquad e^{-\phi} k = \frac{\tau \coth \tau - 1}{2 \sinh \tau} (\cosh \tau + 1) ,$$
$$F = \frac{\sinh \tau - \tau}{2 \sinh \tau} , \qquad (3.13)$$

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where e^ϕ is given in eq. (3.4). By using the solution given by (3.7) and (3.13) in the general eq. (2.13) we can immediately obtain the expression of the warp factor $h(\tau)$. Actually, if we require that h is regular at $\tau = 0$, the integration constant N_0 in (2.16) must be chosen to be zero. In this case, we get:

$$h(\tau) = -\frac{\pi M^2}{4\,\mu^{8/3}N_f} \int^\tau dx \frac{x\coth x - 1}{(x-\tau_0)^2 \sinh^2 x} \frac{-\cosh 2x + 4x^2 - 4x\tau_0 + 1 - (x-2\tau_0)\sinh 2x}{(\cosh 2x + 2x^2 - 4x\tau_0 - 1 - 2(x-\tau_0)\sinh 2x)^{2/3}} .$$
$$(3.14)$$

The integration constant can be fixed by requiring that the analytic continuation of $h(\tau)$ goes to zero as $\tau \to +\infty$, to connect with the Klebanov-Strassler solution in the unflavored (scaling) limit. Then, close to the tip of the geometry, $h(\tau) \sim h_0 - \mathcal{O}(\tau^2)$.

We should emphasize now an important point: even though at first sight this solution may look smooth in the IR ($\tau \sim 0$), where all the components of our metric approach the same limit as those of the KS solution (up to a suitable redefinition of parameters), there is actually a curvature singularity. Indeed, in Einstein frame the curvature scalar behaves as $R_E \sim 1/\tau$.[1] This singularity of course disappears when taking the unflavored limit, using the scaling described above.

The solution presented above is naturally interpreted as the addition of fundamentals to the KS background [10]. In the next section, we will present a solution that can be understood as the addition of flavors to the KT background [9].

4   Fractional branes in the singular conifold with flavor

Let us now consider the solutions with $g = 0$. First of all, let us change the radial variable from r to ρ, where the later is defined by the relation $dr = e^{G_3}d\rho$. The equation for the dilaton can be integrated trivially:

$$e^\phi = -\frac{4\pi}{3N_f}\frac{1}{\rho} , \qquad \rho < 0 . \qquad (4.1)$$

The supersymmetry requirement imposes now that the metric functions G_i satisfy in this case the following system of differential equations:

$$\dot{G}_i = \frac{1}{6}e^{2G_3 - 2G_i} , \qquad (i = 1, 2) ,$$

$$\dot{G}_3 = 3 - \frac{1}{6}e^{2G_3 - 2G_1} - \frac{1}{6}e^{2G_3 - 2G_2} - \frac{3N_f}{8\pi}e^\phi , \qquad (4.2)$$

where now the dot refers to the derivative with respect to ρ. This system is equivalent to the one analyzed in [23] for the Klebanov-Witten model with flavors. In what follows we

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[1]The simplest example of this kind of singularity appears at $r = 0$ in a 2-dimensional manifold whose metric is $ds^2 = dr^2 + r^2(1+r)d\varphi^2$.

will restrict ourselves to the particular solution with $G_1 = G_2$ given by:

$$e^{2G_1} = e^{2G_2} = \frac{1}{6}(1 - 6\rho)^{\frac{1}{3}} e^{2\rho} , \qquad e^{2G_3} = -6\rho (1 - 6\rho)^{-\frac{2}{3}} e^{2\rho} . \qquad (4.3)$$

Notice that, as in [23], the range of values of ρ for which the metric is well defined is $-\infty < \rho < 0$. The equations for the flux functions f, k and F are now:

$$\dot{f} - \dot{k} = 2e^{\phi}\dot{F} ,$$

$$\dot{f} + \dot{k} = 3e^{\phi}\left[1 + \frac{N_f}{4\pi}(f + k)\right] ,$$

$$F = \frac{1}{2}\left[1 + \left(e^{-\phi} - \frac{N_f}{4\pi}\right)(f - k)\right] . \qquad (4.4)$$

In this paper, we will focus on the particular solution of this system such that $f = k$ and F is constant, namely:

$$F = \frac{1}{2} , \qquad f = k = -\frac{2\pi}{N_f}\left(1 - \frac{\Gamma}{\rho}\right) , \qquad (4.5)$$

where Γ is an integration constant. By substituting these values of F, f and k in our ansatz (2.9) we obtain the form of F_3 and H. Notice that the constants M and Γ only appear in the combination $M\Gamma$. Accordingly, let us define $\mathcal{M}$ as $\mathcal{M} \equiv M\Gamma$. We will write the result in terms of the function:

$$M_{eff}(\rho) \equiv \frac{\mathcal{M}}{\rho} . \qquad (4.6)$$

One has:

$$F_3 = \frac{M_{eff}(\rho)}{4} g^5 \wedge \left(g^1 \wedge g^2 + g^3 \wedge g^4\right) ,$$

$$H_3 = -\frac{\pi}{N_f}\frac{M_{eff}(\rho)}{\rho} d\rho \wedge \left(g^1 \wedge g^2 + g^3 \wedge g^4\right) . \qquad (4.7)$$

Moreover, the RR five-form F_5 can be written as in (2.14) in terms of the effective D3-brane charge defined in (2.16). For the solution (4.5) one gets:

$$N_{eff}(\rho) = N + \frac{\mathcal{M}^2}{N_f}\frac{1}{\rho^2} , \qquad (4.8)$$

where $N \equiv N_0 - \frac{M^2}{N_f}$. By using eq. (2.13), one can obtain the expression of the warp factor, namely:

$$h(\rho) = -27\pi \int d\rho \left[N + \frac{\mathcal{M}^2}{N_f}\frac{1}{\rho^2}\right]\frac{e^{-4\rho}}{(1 - 6\rho)^{\frac{2}{3}}} . \qquad (4.9)$$

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To interpret the solution just presented, it is interesting to study it in the deep IR region $\rho \to -\infty$. Notice that in this limit the three-forms F_3 and H_3 vanish. Actually, it is easy to verify that for $\rho \to -\infty$ the solution obtained here reduces to the one studied in [23], corresponding to the Klebanov-Witten [6] model with flavors. Indeed, in this IR region it is convenient to go back to our original radial variable r. The relation between r and ρ for $\rho \to -\infty$ is $r \approx (-6\rho)^{\frac{1}{6}} e^{\rho}$. Moreover, one can prove that for $\rho \to -\infty$ (or equivalently $r \to 0$), the warp factor h and the metric functions G_i become:

$$h(r) \approx \frac{27\pi N}{4} \frac{1}{r^4} \ , \qquad e^{2G_1} = e^{2G_2} \approx \frac{r^2}{6} \ , \qquad e^{2G_3} \approx r^2 \ , \qquad (4.10)$$

which implies that the IR Einstein frame metric is $AdS_5 \times T^{1,1}$ plus logarithmic corrections, exactly as the solution found in [23]. The interpretation of the RG flow of the field theory dual to this solution will be explained in Sections 5 and 6.

Finally, let us stress that the UV behavior of this solution (coincident with that of the solution presented in Section 3) presents a divergent dilaton at the point $\rho = 0$ (or $\tau = \tau_0$ for the flavored warped deformed conifold). Hence the supergravity approximation fails at some value of the radial coordinate that we will associate in Section 6 with the presence of a duality wall [27] in the cascading field theory.

5 The field theory with flavors: a cascade of Seiberg dualities

The field theory dual to our supergravity solutions can be engineered by putting stacks of two kinds of fractional D3-branes (color branes) and two kinds of fractional D7-branes (flavor branes) on the singular conifold. The smeared charge distribution introduced in the previous sections can be realized by homogeneously distributing D7-branes among a class of localized κ-symmetric embeddings. The (complex structure of the) deformed conifold is described by one equation in $\mathbb{C}^4$: $z_1 z_2 - z_3 z_4 = \epsilon^2$. This has isometry group $SU(2)_A \times SU(2)_B$, where the non-abelian factors are realized through left and right multiplication on the matrix $\left(\begin{smallmatrix} z_1 & z_4 \\ z_3 & z_2 \end{smallmatrix} \right)$. We can also define a $U(1)_R$ action, which is a common phase rotation, that is broken to $\mathbb{Z}_2$ by the deformation parameter ϵ. Consider the embedding [19]:

$$z_3 + z_4 = 0 \ . \qquad (5.1)$$

This is invariant under $U(1)_R$ and a diagonal $SU(2)_D$ (and a $\mathbb{Z}_2$ which exchanges $z_3 \leftrightarrow z_4$). Moreover it is free of C_8 tadpoles and it was shown to be κ-symmetric in [19]. It could be useful to write it in the angular coordinates of the previous section: $\theta_1 = \theta_2$, $\varphi_1 = \varphi_2$, $\forall \psi, \forall \tau$. We can obtain other embeddings with the same properties by acting on it with the broken generators. One can show that the charge distribution obtained by homogeneously

spreading the D7-branes in this class is (2.7):

$$\Omega = \frac{N_f}{4\pi}\big(\sin\theta_1\, d\theta_1 \wedge d\varphi_1 + \sin\theta_2\, d\theta_2 \wedge d\varphi_2\big)\ , \qquad (5.2)$$

where N_f is the total number of D7-branes.

Notice that one could have considered the more general embedding: $z_3 + z_4 = m$, where m corresponds in field theory to a mass term for quarks. These embeddings and their corresponding supergravity solutions are not worked out in this paper.

Different techniques have been developed to identify the field theory dual to our Type IIB plus D7-branes background, which can be engineered by putting r_l fractional D3-branes of the first kind, r_s fractional D3-branes of the second kind, N_{fl} fractional D7-branes of the first kind, and N_{fs} fractional D7-branes of the second kind ($l, s = 1, 2$) on the singular conifold, before the deformation has dynamically taken place. The properties of the different kinds of fractional branes will be explained at the end of this Section and in Section 6; what matters for the time being is that this brane configuration will give rise to a field theory with gauge groups $SU(r_l) \times SU(r_s)$ and flavor groups $SU(N_{fl})$ and $SU(N_{fs})$ for the two gauge groups respectively, with the matter content displayed in Fig. 1. The most convenient technique for our purpose has been that of performing a T-duality along the isometry $(z_1, z_2) \to (e^{i\alpha}z_1, e^{-i\alpha}z_2)$ (one does not need the metric, only the complex structure). The system is mapped into Type IIA: neglecting the common spacetime directions, there is a NS5-brane along $x^{4,5}$, another orthogonal NS5 along $x^{8,9}$, r_l D4-branes along x^6 (which is a compact direction) connecting them on one side, other r_s D4 connecting them on the other side, N_{fl} D6-branes along x^7 and at a $\frac{\pi}{4}$ angle between $x^{4,5}$ and $x^{8,9}$, touching the stack of r_l D4-branes, and N_{fs} D6-branes along x^7 and at a $\frac{\pi}{4}$ angle between $x^{8,9}$ and $x^{4,5}$, touching the stack of r_s D4-branes. Then the spectrum is directly read off, and the superpotential comes from the analysis of the moduli space [28].

The field content of the gauge theory can be read from the quiver diagram of Figure 1: it is an extension of the Klebanov-Strassler field theory with nonchiral flavors for each gauge group. The superpotential is[2]

$$\begin{aligned} W = \lambda(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) + h_1\, \tilde{q}(A_1 B_1 + A_2 B_2)q + h_2\, \tilde{Q}(B_1 A_1 + B_2 A_2)Q + \\ + \alpha\, \tilde{q}q\tilde{q}q + \beta\, \tilde{Q}Q\tilde{Q}Q\ . \end{aligned} \qquad (5.3)$$

The factors $A_1 B_1 + A_2 B_2$ directly descend from the embedding equation (5.1), while the quartic term in the fundamental fields is derived from Type IIA. This superpotential explicitly breaks the $SU(2)_A \times SU(2)_B$ global symmetry of the unflavored theory to $SU(2)_D$, but this global symmetry is recovered after the smearing (see [23] for a more careful treatment of the smearing procedure). It's worth here to stress that the smearing procedure does not influence at all either the duality cascade, which is the main feature of our solutions that we

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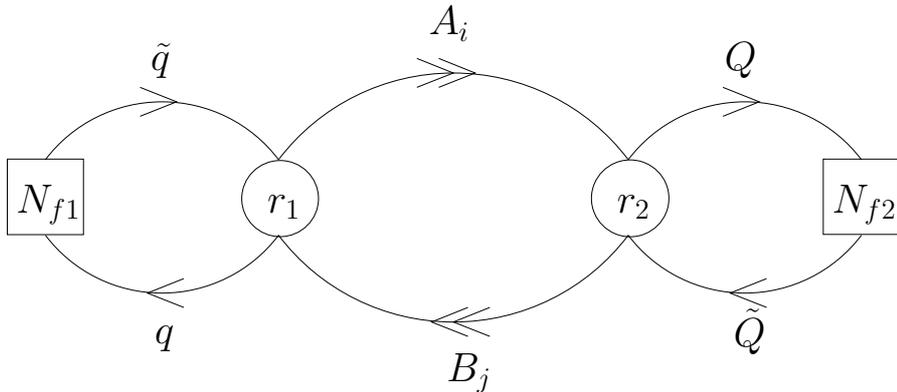[2]Sums over gauge and flavor indices are understood.

Figure 1: The quiver diagram of the gauge theory. Circles are gauge groups, squares are flavor groups, and arrows are bifundamental chiral superfields. N_{f1} and N_{f2} sum up to N_f.

want to address here, nor (presumably) the infrared dynamics which will be investigated in a forthcoming paper [24].

The N_f flavors are split into N_{fl} and N_{fs} groups, according to which gauge group they are charged under. Both sets come from D7-branes along the embedding (5.1).[3] The only feature that discriminates between these two kinds of (fractional) D7-branes is their coupling to the C_2 and C_4 gauge potentials. On the singular conifold, before the dynamical deformation, there is a vanishing 2-cycle, living at the singularity, which the D7-branes are wrapping.[4] According to the worldvolume flux on it, the D7-branes couple either to one or the other gauge group. Since this flux is stuck at the origin, far from the branes we can only measure the D3, D5 and D7-charges produced. Unfortunately three charges are not enough to fix four ranks. This curious ambiguity will show up again in Section 6.

5.1 The cascade

One can assume that, as in the unflavored case, the beta functions of the two gauge couplings have opposite sign. When the gauge coupling of the gauge group with larger rank is very large, one can go to a Seiberg-dual description [29]: remarkably, it's straightforward to see that the quartic superpotential is such that the field theory is self-similar, namely the field theory in the dual description is a quiver gauge theory with the same field content and superpotential, except for changes in the ranks of the groups.[5]

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[3]The embedding is in fact invariant under the $\mathbb{Z}_2$ ($z_3 \leftrightarrow z_4$) that exchanges the two gauge groups.

[4]Since the D7-brane has 4 internal directions, even if it wraps the two-cycle living only at the singularity, still out of it is four-dimensional.

[5]This is not the case for the chirally flavored version of Klebanov-Strassler's theory proposed by Ouyang [18], and for the flavored version of nonconformal theories obtained by putting branes at conical Calabi-Yau

Let us define the theory at some energy scale to be an $SU(r_l) \times SU(r_s)$ gauge theory (where l stands for the larger gauge group and s for the smaller: $r_l > r_s$), with flavor group $SU(N_{fl})$ $(SU(N_{fs}))$ for $SU(r_l)$ $(SU(r_s))$. In the beginning we can set, conventionally, $r_l = r_1$, $r_s = r_2$, $N_{fl} = N_{f1}$, $N_{fs} = N_{f2}$; after a Seiberg duality on the gauge group with the larger rank, the field theory is $SU(2r_2 - r_1 + N_{f1}) \times SU(r_2)$, with again N_{f1} and N_{f2} flavors respectively. In identifying which gauge group is now the larger and which is the smaller, we have to exchange the labelling of the groups, so that we get $r'_l = r_2$, $r'_s = 2r_2 - r_1 + N_{f1}$, $N'_{fl} = N_{f2}$ and $N'_{fs} = N_{f1}$. The assumption leads to an RG flow which is described by a cascade of Seiberg dualities, analogous to [9, 10]. In the UV the ranks of the gauge groups are much larger than their disbalance, which is much larger than the number of flavors. Hence the assumption of having beta functions with opposite sign is justified in the ultraviolet flow of the field theory.

The supergravity background of Section 3 is dual to a quiver gauge theory where the cascade goes on until the IR, with nonperturbative dynamics at the end, as in the Klebanov-Strassler solution.

In the background of Section 4, the cascade does not take place anymore below some value of the radial coordinate, and it asymptotes to the flavored Klebanov-Witten solution [23]. In the field theory, this reflects the fact that, because of a suitable choice of the ranks, the last step of the cascade leads to a theory where the beta functions of both gauge couplings are positive. The infrared dynamics is the one discussed in [23], but with a quartic superpotential for the flavors.

The description of the duality cascade in our solutions and its interesting ultraviolet behavior will be the content of the next section.

6   The cascade: supergravity side

We claim that our supergravity solutions are dual to the class of quiver gauge theories with backreacting fundamental flavors introduced in the previous section. Indeed we will show that the effective brane charges, the R-anomalies and the beta functions of the gauge couplings that we can read from the supergravity solutions precisely match the picture of a cascade of Seiberg dualities that we expect to describe the RG flow of the field theories, generalizing the results of [9, 10] to gauge theories which include dynamical flavors.

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singularities [30]. In those realizations the superpotential is cubic, and the theory is not self-similar under Seiberg duality: new gauge singlet fields appear or disappear after a Seiberg duality, making the cascade subtler.

6.1 Effective brane charges and ranks

By integrating fluxes over suitable compact cycles, we can compute three effective D-brane charges in our solutions, which are useful to pinpoint the changes in the ranks of gauge groups when the field theory undergoes a Seiberg duality: one of them (D7) is dual to a quantity which is constant along the RG flow, whereas two of them (D3, D5) are not independent of the holographic coordinate and are dual to the nontrivial part of the RG flow. Recall that the (Maxwell) charges of D3- and D5-brane (N_{eff} and M_{eff}) for our ansatz were already calculated in section 2 (see eqs. (2.16) and (2.17)). Let us now compute the D7-brane charge, integrating (2.7) on a 2-manifold with boundary which is intersected once by all the smeared D7-branes (e.g. $\mathcal{D}_2$: $\theta_2 = $ const., $\varphi_2 = $ const., $\psi = $ const.). This charge is conserved along the RG flow because no fluxes appear on the right hand side of (2.7). The D7-brane charge, which we interpret as the total number of flavors added to the Klebanov-Strassler gauge theory, is indeed:

$$N_{flav} \equiv \int_{\mathcal{D}_2} dF_1 = N_f \ . \tag{6.1}$$

Another important quantity is the integral of B_2 over the nontrivial 2-cycle S^2: $\theta_1 = \theta_2 \equiv \theta$, $\varphi_1 = 2\pi - \varphi_2 \equiv \varphi$, $\psi = $ const.:

$$b_0(\tau) \equiv \frac{1}{4\pi^2} \int_{S^2} B_2 = \frac{M}{\pi}\left(f \sin^2 \frac{\psi}{2} + k \cos^2 \frac{\psi}{2}\right) \ . \tag{6.2}$$

This quantity is important because string theory is invariant as it undergoes a shift of 1. For instance, in the KW background it amounts to a Seiberg duality, and the same happens here. So we will shift this last quantity by one unit, identify a shift in the radial variable τ that realizes the same effect, and see what happens to M_{eff} and N_{eff}. Actually, the cascade will not work along the whole flow down to the IR but only in the UV asymptotic (below the UV cut-off τ_0 obviously). Notice that the same happens for the unflavored solutions of [9] and [10]: in the KT solution one perfectly matches the cascade in field theory and supergravity, while in the KS solution close to the tip of the warped deformed conifold the matching is not so clean. On the other hand, this is expected, since the last step of the cascade is not a Seiberg duality. Thus we will not be worried and compute the cascade only in the UV asymptotic for large τ which also requires $\tau_0 \gg 1$ (we neglect $\mathcal{O}(e^{-\tau})$): in that regime the functions f and k become equal, and b_0 is ψ-independent.

Actually, we will not compute the explicit shift in τ but rather the shift in the functions f and k. We have:

$$b_0(\tau) \rightarrow b_0(\tau') = b_0(\tau) - 1 \quad \Longrightarrow \quad \begin{aligned} f(\tau) &\rightarrow f(\tau') = f(\tau) - \frac{\pi}{M} \\ k(\tau) &\rightarrow k(\tau') = k(\tau) - \frac{\pi}{M} \end{aligned} \tag{6.3}$$

Correspondingly, after a Seiberg duality step from τ to $\tau' < \tau$, that is going towards the IR, we have:

$$N_f \to N_f \tag{6.4}$$

$$M_{eff}(\tau) \to M_{eff}(\tau') = M_{eff}(\tau) - \frac{N_f}{2} \tag{6.5}$$

$$N_{eff}(\tau) \to N_{eff}(\tau') = N_{eff}(\tau) - M_{eff}(\tau) + \frac{N_f}{4} \tag{6.6}$$

This result is valid for all of our solutions.

We would like to compare this result with the action of Seiberg duality in field theory, as computed in Section 5. We need an identification between the brane charges computed in supergravity and the ranks of the gauge and flavor groups in the field theory.

The field theory of interest for us, with gauge groups $SU(r_l) \times SU(r_s)$ $(r_l > r_s)$, and flavor groups $SU(N_{fl})$ and $SU(N_{fs})$ for the gauge groups $SU(r_l)$ and $SU(r_s)$ respectively, is engineered, at least effectively at some radial distance, by the following objects: r_l fractional D3-branes of one kind (D5-branes wrapped on the shrinking 2-cycle), r_s fractional D3-branes of the other kind ($\overline{\text{D5}}$-branes wrapped on the shrinking cycle, supplied with -1 quanta of gauge field flux on the 2-cycle), N_{fs} fractional D7-branes without gauge field strength on the 2-cycle, and N_{fl} fractional D7-branes with -1 units of gauge field flux on the shrinking 2-cycle. This description is good for $b_0 \in [0, 1]$.

This construction can be checked explicitly in the case of the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold [31, 32], where one is able to quantize the open and closed string system for the case $b_0 = \frac{1}{2}$ that leads to a free CFT [33]. That is the $\mathcal{N} = 2$ field theory which flows to the field theory we are considering, when equal and opposite masses are given to the adjoint chiral superfields (the geometric description of this relevant deformation is a blowup of the orbifold singularity) [6, 34]. Fractional branes are those branes which couple to the twisted closed string sector.[6]

Here we will consider a general background value for B_2. In order to compute the charges, we will follow quite closely the computations in [35].

We will compute the charges of D7-branes and wrapped D5-branes on the singular conifold, described by $z_1 z_2 - z_3 z_4 = 0$. The D5 Wess-Zumino action is

$$S_{D5} = \mu_5 \int_{M^4 \times S^2} \left\{ C_6 + (2\pi F_2 + B_2) \wedge C_4 \right\}, \tag{6.7}$$

where S^2 is the only 2-cycle in the conifold, vanishing at the tip, that the D5-brane is

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[6]Notice that one can build, out of a fractional D3 of one kind and a fractional D3 of the other kind, a regular D3-brane (*i.e.* not coupled to the twisted sector) that can move outside the orbifold singularity; on the contrary, there is no regular D7-brane: the two kinds of fractional D7-branes, extending entirely along the orbifold, cannot bind into a regular D7-brane that does not touch the orbifold fixed locus and is not coupled to the twisted sector [32].

wrapping. We write also a world-volume gauge field F_2 on S^2. Then we expand:

$$B_2 = 2\pi\,\theta_B\,\omega_2 \qquad\qquad \theta_B = 2\pi\,b_0 \qquad\qquad F_2 = \Phi\,\omega_2 \ , \qquad (6.8)$$

where ω_2 is the 2-form on the 2-cycle, which satisfies $\int_{S^2}\omega_2 = 1$. In this conventions, b_0 has period 1, and Φ is quantized in $2\pi\,\mathbb{Z}$. We obtain (using $\mu_p(4\pi^2) = \mu_{p-2}$):

$$S_{D5} = \mu_5 \int_{M^4\times S^2} C_6 + \frac{\mu_3}{2\pi}\int_{M^4}(\Phi+\theta_B)\,C_4 \ . \qquad (6.9)$$

The first fractional D3-brane [36] is obtained with $\Phi = 0$ and has D3-charge b_0, D5-charge 1. The second fractional D3-brane is obtained either as the difference with a D3-brane, or as an anti-D5-brane (global $-$ sign in front) with $\Phi = -2\pi$, and has D3-charge $1 - b_0$, D5-charge -1. These charges are summarized in Table 1.

Now consider a D7-brane along the surface $z_3 + z_4 = 0$. It describes a $z_1 z_2 + z_3^2 = 0$ inside the conifold, which is a copy of $\mathbb{C}^2/\mathbb{Z}_2$. The D7 Wess-Zumino action is (up to a curvature term considered below)

$$S_{D7} = \mu_7 \int_{M^4\times\Sigma}\left\{ C_8 + (2\pi\,F_2 + B_2)\wedge C_6 + \frac{1}{2}(2\pi\,F_2 + B_2)\wedge(2\pi\,F_2 + B_2)\wedge C_4\right\} \ . \quad (6.10)$$

The surface $\Sigma = \mathbb{C}^2/\mathbb{Z}_2$ has a vanishing 2-cycle at the origin. Since the conifold has only one 2-cycle, these two must be one and the same and we can expand on Σ using ω_2 again. Moreover, being ω_2 the Poincaré dual to the 2-cycle on Σ,

$$\int_\Sigma \omega_2\wedge\alpha_2 = \frac{1}{2}\int_{S^2}\alpha_2 \qquad (6.11)$$

holds for any closed 2-form α_2, and $\frac{1}{2}$ arises from the self-intersection number of the S^2. There is another contribution of induced D3-charge coming from the curvature coupling [37]:

$$\frac{\mu_7}{96}(2\pi)^2\int_{M^4\times\Sigma} C_4\wedge\mathrm{Tr}\,\mathcal{R}_2\wedge\mathcal{R}_2 = -\mu_3\int_{M^4\times\Sigma} C_4\wedge\frac{p_1(\mathcal{R})}{48} \ . \qquad (6.12)$$

This can be computed in the following way. On K3 $p_1(\mathcal{R}) = 48$ and the induced D3-charge is -1. In the orbifold limit K3 becomes $T^4/\mathbb{Z}_2$ which has 16 orbifold singularities, thus on $\mathbb{C}^2/\mathbb{Z}_2$ the induced D3-charge is $-1/16$. Putting all together we get:

$$S_{D7} = \mu_7\int_{M^4\times\Sigma} C_8 + \frac{\mu_5}{4\pi}\int_{M^4\times S^2}(\Phi+\theta_B)\,C_6 + \frac{\mu_3}{16\pi^2}\int_{M^4}\left[(\Phi+\theta_B)^2 - \pi^2\right]C_4 \ . \qquad (6.13)$$

The second fractional D7-brane (the one that couples to the second gauge group) is obtained with $\Phi = 0$ and has D7-charge 1, D5-charge $\frac{b_0}{2}$ and D3-charge $(4b_0^2 - 1)/16$. With $\Phi = 2\pi$ we get a non-SUSY or non-minimal object (see [36] for some discussion of this). The first

Object	frac D3 (1)	frac D3 (2)	frac D7 (1)	frac D7 (2)
D3-charge	b_0	$1 - b_0$	$\dfrac{4(b_0 - 1)^2 - 1}{16}$	$\dfrac{4b_0^2 - 1}{16}$
D5-charge	1	-1	$\dfrac{b_0 - 1}{2}$	$\dfrac{b_0}{2}$
D7-charge	0	0	1	1
Number of objects	r_l	r_s	N_{fl}	N_{fs}

Table 1: Charges of fractional branes on the conifold

fractional D7-brane (coupled to the first gauge group) has $\Phi = -2\pi$ and has D7-charge 1, D5-charge $\frac{b_0-1}{2}$ and D3-charge $(4(b_0 - 1)^2 - 1)/16$. This is summarized in Table 1. Which fractional D7-brane provides flavors for the gauge group of which fractional D3-brane can be determined from the orbifold case with $b_0 = \frac{1}{2}$ (compare with [32]).

Given these charges, we can compare with the field theory cascade. First of all we construct the dictionary:

$$N_f = N_{fl} + N_{fs} \tag{6.14}$$

$$M_{eff} = r_l - r_s + \frac{b_0 - 1}{2} N_{fl} + \frac{b_0}{2} N_{fs} \tag{6.15}$$

$$N_{eff} = b_0\, r_l + (1 - b_0)\, r_s + \frac{4(1 - b_0)^2 - 1}{16} N_{fl} + \frac{4b_0^2 - 1}{16} N_{fs} \tag{6.16}$$

To derive this, we have only used that the brane configuration that engineers the field theory we consider consists of r_l fractional D3 of the 1st kind, r_s fractional D3 of the 2nd kind, N_{fl} fractional D7 of the 1st kind, and N_{fs} fractional D7 of the 2nd kind. Recall that, by convention, $r_l > r_s$ and N_{fl} (N_{fs}) are the flavors for $SU(r_l)$ ($SU(r_s)$).

It is important to remember that b_0 is defined modulo 1, and shifting b_0 by one unit amounts to go to a Seiberg dual description in the field theory. At any given energy scale in the cascading gauge theory, there are infinitely many Seiberg dual descriptions of the field theory, because Seiberg duality is exact along the RG flow [15]. Among these different pictures, there is one which gives the best effective description of the field theory degrees of freedom around that energy scale (this is also the description with positive squared gauge couplings): it is the one where b_0 has been redefined, by means of a large gauge transformation, so that $b_0 \in [0, 1]$ (see subsection 6.2). This is the description that we will use when we effectively engineer the field theory in terms of branes in some range of the RG flow that lies between two adjacent Seiberg dualities.

In field theory, as before, we start with gauge group $SU(r_1) \times SU(r_2)$ and N_{f1} flavors for $SU(r_1)$, N_{f2} flavors for $SU(r_2)$, with $r_1 > r_2$. The gauge group $SU(r_1)$ flows towards strong

coupling, and when its gauge coupling diverges we turn to a Seiberg dual description. After the Seiberg duality on the larger gauge group, we get $SU(2r_2 - r_1 + N_{f1}) \times SU(r_2)$, and the flavor groups are left untouched.

The effective D5- and D3-brane charges of a brane configuration that engineers this field theory *before* the duality are:

$$
\begin{aligned}
M_{eff} &= r_1 - r_2 + \frac{b_0 - 1}{2} N_{f1} + \frac{b_0}{2} N_{f2} \;, \\
N_{eff} &= b_0 r_1 + (1 - b_0) r_2 + \frac{4(1 - b_0)^2 - 1}{16} N_{f1} + \frac{4 b_0^2 - 1}{16} N_{f2} \;.
\end{aligned}
\tag{6.17}
$$

After the duality they become:

$$
\begin{aligned}
M'_{eff} &= -r_2 + r_1 - N_{f1} + \frac{b_0 - 1}{2} N_{f2} + \frac{b_0}{2} N_{f1} = M_{eff} - \frac{N_f}{2} \;, \\
N'_{eff} &= b_0 r_2 + (1 - b_0)(2r_2 - r_1 + N_{f1}) + \frac{4(1 - b_0)^2 - 1}{16} N_{f2} + \frac{4 b_0^2 - 1}{16} N_{f1} = \\
&= N_{eff} - M_{eff} + \frac{N_f}{4} \;.
\end{aligned}
\tag{6.18}
$$

They *exactly* reproduce the SUGRA behavior (6.4)-(6.6). Notice that the matching of the cascade between supergravity and field theory is there, irrespective of how we distribute the flavors between the two gauge groups; so, from the three charges and the cascade we are not able to determine how the flavors are distributed, but only their total number.

We conclude with some remarks. Even though the effective brane charges computed in supergravity are running and take integer values only at some values of the holographic coordinate, the ranks of gauge and flavor groups computed from them are constant and integer (for suitable choice of the integration constants) in the whole range of radial coordinate dual to the energy range where we use a specific field theory description. This range of scales is $b_0 \in [0, 1]$ mod 1. At the boundaries of this region, we perform a Seiberg duality and go into a new more effective description, and in particular if ranks are integer before the duality, they still are after it; meanwhile we shift b_0 by one unit. Hence the field theory description of the cascade is perfectly matched by the ranks that we can compute from our supergravity solution.

Notice also that the fact that M_{eff} shifts by $N_f/2$ instead of N_f confirms that the flavored version of the Klebanov-Strassler theory we are describing has nonchiral flavors (with a quartic superpotential) rather than chiral flavors (with a cubic superpotential) like in [18], where the shift goes with units of N_f.

Finally, we want to stress again that we are engineering a field theory with 4 objects, but we have only 3 charges to recognize them. The comparison of the cascade between sugra and field theory, surprisingly enough, does not help.

6.2 Seiberg duality as a large gauge transformation

We have argued in the previous subsection that a shift by a unit of the normalized flux b_0 as we move towards the IR along the holographic direction is equivalent to performing a Seiberg duality step on the field theory side (see equations (6.3)-(6.6)). Moreover, we have checked that, under this shift of b_0, the change of the effective (Maxwell) charges M_{eff} and N_{eff} of supergravity is exactly the same as the one computed in the field theory engineered with fractional branes on the singular conifold.

In this subsection we will present an alternative way of understanding in supergravity Seiberg duality at a *fixed* energy scale. As we know, for a given value of the holographic coordinate τ, the value of b_0 lies generically outside the interval $[0,1]$, where a good field theory description exists. However, the flux of the B_2 field is not a gauge invariant quantity in supergravity and can be changed under a large gauge transformation. Indeed, let us define Υ_2 as the following two-form:

$$\Upsilon_2 = \frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4) , \qquad (6.19)$$

and let us change B_2 as follows:

$$B_2 \rightarrow B_2 + \Delta B_2 , \qquad \Delta B_2 = -\pi n \Upsilon_2 , \qquad n \in \mathbb{Z} . \qquad (6.20)$$

As $d\Upsilon_2 = 0$, the field strength H_3 does not change and our transformation is a gauge transformation of the NSNS field. However the flux of B_2 does change as:

$$\int_{S^2} B_2 \rightarrow \int_{S^2} B_2 - 4\pi^2 n , \qquad (6.21)$$

or, equivalently $b_0 \rightarrow b_0 - n$. This non-invariance of the flux shows that this transformation of B_2 is a large gauge tranformation which cannot be globally written as $\Delta B_2 = d\Lambda$. Moreover, as always happens with large gauge transformations, it is quantized. If we want that our transformation (6.20) be a gauge transformation of supergravity, it should leave the RR field strength F_3 invariant. Defining the potential C_2 as $dC_2 = F_3 - B_2 \wedge F_1$, we see that dC_2 must change as:

$$dC_2 \rightarrow dC_2 + \frac{nN_f}{4} g^5 \wedge \Upsilon_2 . \qquad (6.22)$$

One can verify that this change of dC_2 can be obtained if the variation of C_2 is (see equations (2.9) and (4.7)):

$$\Delta C_2 = \frac{nN_f}{8}\left[(\psi-\psi^*)(\sin\theta_1 d\theta_1 \wedge d\varphi_1 - \sin\theta_2 d\theta_2 \wedge d\varphi_2) - \cos\theta_1\cos\theta_2 d\varphi_1 \wedge d\varphi_2\right], \quad (6.23)$$

where ψ^* is a constant. In the study of the R-symmetry anomaly of the next subsection it will be convenient to know the change of C_2 on the submanifold $\theta_1 = \theta_2 = \theta$, $\varphi_1 = 2\pi - \varphi_2 = \varphi$.

Denoting by C_2^{eff} the RR potential C_2 restricted to this cycle, we get from (6.23) that:

$$\Delta C_2^{eff} \; = \; \frac{nN_f}{4}(\psi - \psi^*)\,\sin\theta d\theta \wedge d\varphi \; . \tag{6.24}$$

Let us now study how the Page charges change under these large gauge transformations. From the expressions written in (2.26), we obtain:

$$\Delta Q_{D5}^{Page} = -\frac{1}{4\pi^2}\int_{S^3}\Delta B_2 \wedge F_1 \; ,$$
$$\Delta Q_{D3}^{Page} = \frac{1}{(4\pi^2)^2}\int_{M_5}\left(-\Delta B_2 \wedge F_3 + \Delta B_2 \wedge B_2 \wedge F_1 + \frac{1}{2}\Delta B_2 \wedge \Delta B_2 \wedge F_1\right). \tag{6.25}$$

By using in (6.25) our ansatz for F_3 and B_2 (2.9), together with the expression of ΔB_2 given in (6.20) as well as the relations (2.27) and (2.31), one readily gets:

$$\Delta Q_{D5}^{Page} \; = \; n\frac{N_f}{2} \; ,$$
$$\Delta Q_{D3}^{Page} \; = \; n\,M \; + \; n^2\frac{N_f}{4} \; . \tag{6.26}$$

Thus, under a large gauge transformation (6.20) with $n = 1$, the Page charges transform as:

$$Q_{D5}^{Page} \to Q_{D5}^{Page} \; + \; \frac{N_f}{2} \; ,$$
$$Q_{D3}^{Page} \to Q_{D3}^{Page} \; + \; M \; + \; \frac{N_f}{4} \; . \tag{6.27}$$

Recall that for our ansatz $Q_{D5}^{Page} = M$ and $Q_{D3}^{Page} = N_0$ (see eqs. (2.30) and (2.34)). Thus, eq. (6.27) gives how these constants change under a large gauge transformation. At a given holographic scale τ we should perform as many large transformations as needed to have $b_0 \in [0,1]$. Given that b_0 grows when the holographic coordinate increases, the transformation (6.27) should correspond to the change of ranks under a Seiberg duality when we flow towards the UV. By comparing (6.27) with our previous expressions one can show that this is the case. Actually, one can get an explicit expression of Q_{D5}^{Page} and Q_{D3}^{Page} in terms of the ranks r_l and r_s and the number of flavors N_{fl} and N_{fs}. In order to verify this fact, let us suppose that we are in a region of the holographic coordinate such that the two functions f and k of our ansatz are equal. Notice that for the flavored KS solution this happens in the UV, while for the flavored KT this condition holds for all values of the radial coordinate. If $f = k$ the normalized flux b_0 in (6.2) can be written as:

$$b_0(\tau) \; = \; \frac{M}{\pi}\,f(\tau) \; . \tag{6.28}$$

Using this expression we can write the D5-brane Page charge (2.29) as:

$$Q_{D5}^{Page} \; = \; M_{eff} \; - \; \frac{N_f}{2}\,b_0 \; . \tag{6.29}$$

Notice also that the supergravity expression (2.17) of M_{eff} can be written when $f = k$ as:

$$M_{eff} = M + \frac{N_f}{2} b_0 .$$ (6.30)

Let us next assume that we have chosen our gauge such that, at the given holographic scale, $b_0 \in [0, 1]$. In that case we can use the value of M_{eff} obtained by the field theory calculation of subsection 6.1 to evaluate the Page charge Q_{D5}^{Page}. Actually, by plugging the value of M_{eff} given in (6.15) on the right-hand side of (6.29) we readily get the following relation between Q_{D5}^{Page} and the field theory data:

$$Q_{D5}^{Page} = r_l - r_s - \frac{N_{fl}}{2} .$$ (6.31)

Similarly, for $f = k$, one can express the D3-brane Page charge (2.33) as:

$$Q_{D3}^{Page} = N_{eff} - b_0 M - \frac{b_0^2}{4} N_f ,$$ (6.32)

which, after using the relation (6.30), can be written in terms of M_{eff} as:

$$Q_{D3}^{Page} = N_{eff} - b_0 M_{eff} + \frac{N_f}{4} b_0^2 .$$ (6.33)

Again, if we assume that $b_0 \in [0, 1]$ and use the field theory expressions (6.16) and (6.15) of N_{eff} and M_{eff}, we get:

$$Q_{D3}^{Page} = r_s + \frac{3N_{fl} - N_{fs}}{16} .$$ (6.34)

Notice that, as it should, the expressions (6.31) and (6.34) of Q_{D5}^{Page} and Q_{D3}^{Page} that we have just found are independent of b_0, as far as $b_0 \in [0, 1]$. Moreover, one can verify that under a field theory Seiberg duality the right-hand sides of (6.31) and (6.34) transform as the left-hand sides do under a large gauge transformation of supergravity.

Finally, let us point out that in this approach Seiberg duality is performed at a fixed energy scale and M_{eff} and N_{eff} are left invariant (recall that Maxwell charges are gauge invariant). Indeed, by looking at our ansatz for B_2 one easily concludes that the change of B_2 written in (6.20) is equivalent to the following change in the functions f and k

$$f \to f - \frac{\pi}{M} n , \qquad k \to k - \frac{\pi}{M} n ,$$ (6.35)

and one can verify that the changes (6.26) and (6.35) leave the expressions of M_{eff} and N_{eff}, as written in eqs. (2.29) and (2.33), invariant. From eqs. (6.31) and (6.34) it is clear that the Page charges provide a clean way to extract the ranks and number of flavors of the corresponding (good) field theory dual at a given energy scale. Actually, the ranks of this good field theory description change as step-like functions along the RG flow, due to

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the fact that b_0 varies continuosly and needs to suffer a large gauge transformation every time that, flowing towards the IR, it reaches the value $b_0 = 0$ in the good gauge. This large gauge transformation changes Q_{D5}^{Page} and Q_{D3}^{Page} in the way described above, which realizes in supergravity the change of the ranks under a Seiberg duality in field theory.

Let us now focus on a different way of matching the behavior of the field theory and our solutions.

6.3   R-symmetry anomalies and β-functions

We can compute the β-functions (up to the energy-radius relation) and the R-symmetry anomalies for the two gauge groups both in supergravity and in field theory in the spirit of [32, 38, 39]. In the UV, where the cascade takes place, they nicely match. For the comparison we make use of the following holographic formulae, which can be derived in the $\mathcal{N} = 2$ orbifold case by looking at the Lagrangian of the low energy field theory living on probe (fractional) D3-branes:[7]

$$\frac{4\pi^2}{g_l^2} + \frac{4\pi^2}{g_s^2} = \pi\, e^{-\phi} \qquad\qquad \theta_l^{YM} + \theta_s^{YM} = -2\pi\, C_0$$

$$\frac{4\pi^2}{g_l^2} - \frac{4\pi^2}{g_s^2} = \frac{e^{-\phi}}{2\pi}\Big[\int_{S^2} B_2 - 2\pi^2 \ (\mathrm{mod}\ 4\pi^2)\Big] \qquad \theta_l^{YM} - \theta_s^{YM} = \frac{1}{\pi}\int_{S^2} C_2\ . \tag{6.36}$$

Strictly speaking, these formulae need to be corrected for small values of the gauge couplings and are only valid in the large 't Hooft coupling regime (see [40, 15, 12, 23]), which is the case under consideration. Moreover, they give positive squared couplings only if $b_0 = \frac{1}{4\pi^2}\int_{S^2} B_2$ is in the range $[0, 1]$. This is the physical content of the cascade: at a given energy scale we must perform a large gauge transformation on B_2 in supergravity to shift $\int B_2$ by a multiple of $4\pi^2$ to get a field theory description with positive squared couplings.

We have adapted the indices in (6.36) to the previous convention for the gauge group with the larger (the smaller) rank. Let us restrict our attention to an energy range, between two subsequent Seiberg dualities, where a field theory description in terms of specific ranks holds. In this energy range the gauge coupling g_l of the gauge group with larger rank flows towards strong coupling, while the gauge coupling g_s of the gauge group with smaller rank flows towards weak coupling. Indeed, as formulae (6.36) confirm, the coupling g_l was not touched by the previous Seiberg duality, starts different from zero and flows to ∞ at the end of this range, where a Seiberg duality on its gauge group is needed. The coupling g_s of the

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[7]We are not completely sure about the sign in the formula relating C_0 to the sum of the theta angles. At any rate, with the minus sign we can match the anomaly computations in field theory and in supergravity for the flavored version of Klebanov-Witten's theory proposed in [18] (with backreaction of the flavor branes only at first order) and in [23] (fully backreacting D7-branes). The field theory computation gives $\delta_\varepsilon(\theta_1 + \theta_2) = -N_f\varepsilon$, which can only be matched with the supergravity results of the two papers using this formula.

gauge group with smaller rank is the one which starts very large (actually divergent) after the previous Seiberg duality on its gauge group, and then flows toward weak coupling.

In supergravity, due to the presence of magnetic sources for F_1, we cannot define a potential C_0. Therefore we project our fluxes on the submanifold $\theta_1 = \theta_2 \equiv \theta$, $\varphi_1 = 2\pi - \varphi_2 \equiv \varphi$, $\forall \psi, \tau$ before integrating them. Recalling that $F_3 = dC_2 + B_2 \wedge F_1$, what we get from (2.8)-(2.9) (in the UV limit) are the effective potentials

$$C_0^{eff} = \frac{N_f}{4\pi}(\psi - \psi_0^*) \qquad \tilde{C}_2^{eff} = \left[\frac{M}{2} + \frac{nN_f}{4}\right](\psi - \psi_2^*)\sin\theta\, d\theta \wedge d\varphi\ . \qquad (6.37)$$

The integer n in $\tilde{C}_2^{eff}$ comes from a large gauge transformation on B_2 (Seiberg duality in field theory, see eq. 6.24) which shifts $b_0(\tau) \in [n, n+1]$ by n units - so that the gauge transformed $\tilde{b}_0(\tau) = b_0(\tau) - n$ is between 0 and 1 - and at the same time shifts $dC_2^{eff} \rightarrow d\tilde{C}_2^{eff} = dC_2^{eff} + \pi n \frac{N_f}{4\pi}\sin\theta\, d\theta \wedge d\varphi \wedge d\psi$, since F_3 is gauge-invariant, but leaves C_0 invariant.

The field theory possesses an anomalous R-symmetry which assigns charge $\frac{1}{2}$ to all chiral superfields.[8] The field theory R-anomalies are easily computed. Continuing to use r_l (r_s) for the larger (smaller) group rank and N_{fl} (N_{fs}) for the corresponding flavors (see Figure 1), the anomalies under a $U(1)_R$ rotation of parameter ε are:

$$\text{Field theory:} \qquad \begin{aligned} \delta_\varepsilon \theta_l &= [2(r_l - r_s) - N_{fl}]\,\varepsilon \\ \delta_\varepsilon \theta_s &= [-2(r_l - r_s) - N_{fs}]\,\varepsilon\ . \end{aligned} \qquad (6.38)$$

Along the cascade of Seiberg dualities, the coefficients of the anomalies for the two gauge groups change when we change the effective description; what does not change is the unbroken subgroup of the R-symmetry group. Because we want to match them with the supergravity computations, it will be convenient to rewrite the field theory anomalies in the following form:

$$\text{Field theory:} \qquad \begin{aligned} \delta_\varepsilon(\theta_l + \theta_s) &= -N_f\,\varepsilon \\ \delta_\varepsilon(\theta_l - \theta_s) &= [4(r_l - r_s) + N_{fs} - N_{fl}]\,\varepsilon\ . \end{aligned} \qquad (6.39)$$

An infinitesimal $U(1)_R$ rotation parameterized by ε in field theory corresponds to a shift $\psi \rightarrow \psi + 2\varepsilon$ in the geometry. Therefore, making use of (6.37), we find on the supergravity side:

$$\text{SUGRA:} \qquad \begin{aligned} \delta_\varepsilon(\theta_l + \theta_s) &= -N_f\,\varepsilon \\ \delta_\varepsilon(\theta_l - \theta_s) &= [4M + 2n\,N_f]\,\varepsilon\ . \end{aligned} \qquad (6.40)$$

These formulae agree with those computed in the field theory. For the difference of the anomalies, what we can compute and compare is its change after a step in the duality cascade. Notice indeed that the difference of the anomalies, as computed in (6.40), gives

---

[8]Although the R-charges of the chiral superfields are half-integer, an R-rotation of parameter $\varepsilon = 2\pi$ coincides with a baryonic rotation of parameter $\alpha = \pi$. It follows that $U(1)_R \times U(1)_B$ is parameterized by $\varepsilon \in [0, 2\pi]$, $\alpha \in [0, 2\pi]$.

a step function: as we flow towards the IR, after some energy scale (the scale of a Seiberg duality along the cascade) we need to perform a large gauge transformation in supergravity to turn to the correct Seiberg dual description of the field theory (the only one with positive squared gauge couplings). This corresponds to changing $n \to n - 1$ in (6.40), therefore the coefficient of the difference of the R-anomalies decreases by $2N_f$ units. This result is reproduced exactly by the field theory computation (6.39). In field theory the difference of the anomalies depends on the quantity $4(r_l - r_s) + N_{fs} - N_{fl}$. Keeping the same conventions adopted in subsection 5.1 and repeating the same reasoning, it's easy to see that after a step of the cascade towards the IR, this quantity decreases exactly by $2N_f$ units.

The dictionary (6.36) allows us also to compute the β-functions of the two gauge couplings and check further the picture of the duality cascade.

Since we will be concerned in the cascade, we will make use of the flavored Klebanov-Tseytlin solution of Section 4, to which the flavored Klebanov-Strassler solution of Section 3 reduces in the UV limit.

We shall keep in mind that, at a fixed value of the radial coordinate, we want to shift $b_0 = \frac{1}{4\pi^2} \int_{S^2} B_2$ by means of a large gauge transformation in supergravity in such a way that its gauge transformed $\tilde{b}_0 = b_0 - [b_0] \equiv b_0 - n$ [9] belongs to $[0, 1]$: in doing so, we are guaranteed to be using the good description in terms of a field theory with positive squared gauge couplings.

Recall that

$$e^{-\phi} = \frac{3N_f}{4\pi}(-\rho) \tag{6.41}$$

$$b_0 = \frac{2M}{N_f}\left(\frac{\Gamma}{\rho} - 1\right) , \tag{6.42}$$

and the dictionary (6.36), that we rewrite as:

$$\frac{8\pi^2}{g_+^2} \equiv \frac{8\pi^2}{g_l^2} + \frac{8\pi^2}{g_s^2} = 2\pi e^{-\phi} \tag{6.43}$$

$$\frac{8\pi^2}{g_-^2} \equiv \frac{8\pi^2}{g_l^2} - \frac{8\pi^2}{g_s^2} = 2\pi e^{-\phi}(2\tilde{b}_0 - 1) , \tag{6.44}$$

where $\tilde{b}_0 \equiv b_0 - [b_0] \in [0, 1]$ comes from integrating on the two-cycle the suitably gauge transformed Kalb-Ramond potential which must be used in order to describe the correct field theory effective degrees of freedom at the energy scale dual to the value of the radial coordinate.

---

[9] n is a step-like function of the radial coordinate.

Then we can compute the following 'radial' β-functions from the gravity dual:

$$\beta_+^{(\rho)} \equiv \beta_{\frac{8\pi^2}{g_+^2}}^{(\rho)} \equiv \frac{d}{d\rho} \frac{8\pi^2}{g_+^2} \tag{6.45}$$

$$\beta_-^{(\rho)} \equiv \beta_{\frac{8\pi^2}{g_-^2}}^{(\rho)} \equiv \frac{d}{d\rho} \frac{8\pi^2}{g_-^2} \ , \tag{6.46}$$

and we would like to match these with the field theory computations.
Using the expressions (6.43)-(6.44), we can conclude that

$$\beta_+^{(\rho)} = -3\frac{N_f}{2} \tag{6.47}$$

$$\beta_-^{(\rho)} = 3\left(\frac{N_f}{2} + Q\right) \ , \tag{6.48}$$

where $Q = N_f[b_0(\rho)] + 2M = N_f n(\rho) + 2M$ is a quantity which undergoes a change $Q \to Q - N_f$ as $b_0(\rho) \to b_0(\rho') = b_0(\rho) - 1$ (one Seiberg duality step along the cascade towards the IR), or equivalently $n(\rho) \to n(\rho') = n(\rho) - 1$. Up to an overall factor of 2, Q is the same quantity appearing in the difference of the R-anomalies in (6.40).

The field theory computations of the β-functions give:

$$\beta_l \equiv \beta_{\frac{8\pi^2}{g_l^2}} = 3r_l - 2r_s(1 - \gamma_A) - N_{fl}(1 - \gamma_q) \tag{6.49}$$

$$\beta_s \equiv \beta_{\frac{8\pi^2}{g_s^2}} = 3r_s - 2r_l(1 - \gamma_A) - N_{fs}(1 - \gamma_q) \ , \tag{6.50}$$

with the usual conventions. Hence

$$\beta_+ \equiv \beta_l + \beta_s = (r_l + r_s)(1 + 2\gamma_A) - N_f(1 - \gamma_q) \tag{6.51}$$

$$\beta_- \equiv \beta_l - \beta_s = (5 - 2\gamma_A)(r_l - r_s) + (N_{fs} - N_{fl})(1 - \gamma_q) \ . \tag{6.52}$$

In order to match the above quantities with the gravity computations (6.47)-(6.48), an energy-radius relation is required. This is something we miss here. Although it is not really needed to extract from our supergravity solutions the qualitative information on the running of the gauge couplings, we are going initially to make two assumptions, which can be viewed as an instructive simplification. Let us then assume that the radius-energy relation is $\rho = \ln\frac{\mu}{E_{UV}}$, where E_{UV} is the scale of the UV cutoff dual to the maximal value of the radial coordinate $\rho = 0$, and that the anomalous dimensions do not acquire subleading corrections. Matching β_+ implies $\gamma_A = \gamma_q = -\frac{1}{2}$. Matching β_-, once we insert these anomalous dimensions, implies that $Q = 2(r_l - r_s) - N_{fl}$. This quantity correctly shifts as $Q \to Q - N_f$ when $b_0 \to b_0 - 1$. This last observation allows us to check the consistency of the cascade of Seiberg dualities also against the running of the gauge couplings.

Actually, the qualitative picture of the RG flow in the UV can be extracted from our supergravity solution even without knowing the precise radius-energy relation, but simply recalling that the radius must be a monotonic function of the energy scale.

It is interesting to notice the following phenomenon: as we flow up in energy and approach the far UV $\rho \to 0^-$ in (6.42), a large number of Seiberg dualities is needed to keep b_0 varying in the interval $[0, 1]$. The Seiberg dualities pile up the more we approach the UV cut-off E_{UV}. Meanwhile, formula (6.48) reveals that, when going towards the UV cutoff E_{UV}, the 'slope' in the plots of $\frac{1}{g_i^2}$ versus the energy scale becomes larger and larger, and (6.47) reveals that the sum of the inverse squared gauge coupling goes to zero at this UV cutoff. At the energy scale E_{UV} the effective number of degrees of freedom needed for a weakly coupled description of the gauge theory becomes infinite. Since $\rho = 0$ is at finite proper radial distance from any point placed in the interior $\rho < 0$, E_{UV} is a finite energy scale.

The picture which stems from our flavored Klebanov-Tseytlin/Strassler solution is that E_{UV} is a so-called "Duality Wall", namely an accumulation point of energy scales at which a Seiberg duality is required in order to have a weakly coupled description of the gauge theory [27]. Above the duality wall, Seiberg duality does not proceed and a weakly coupled dual description of the field theory is not known. See Figure 2.
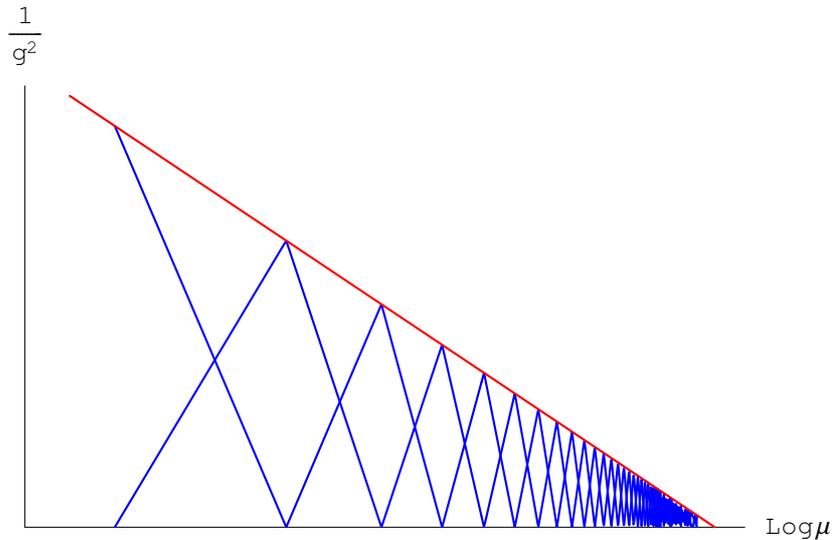

Figure 2: Qualitative plot of the running gauge couplings as functions of the logarithm of the energy scale in our cascading gauge theory. The blue lines are the inverse squared gauge couplings, while the red line is their sum.

Duality walls were studied in the context of quiver gauge theories first by Fiol [41] and later in a series of papers by Hanany and collaborators [42]. Their analysis of this phenomenon was in the framework of quiver gauge theories with only bifundamental chiral superfields,

and was restricted to the field theory.

To our knowledge, our solutions are the first explicit realizations of this exotic ultraviolet phenomenon on the supergravity side of the gauge/gravity correspondence.

7   Final remarks

In this paper we have presented a very precise example of the duality between field theories with flavors and string solutions that include the dynamics of (flavor) branes. We focused on the Klebanov-Tseytlin/Strassler case, providing a well defined dual field theory, together with different matchings that include the cascade of Seiberg dualities, beta functions and anomalies. Indeed, we have shown in detail how the ranks of the gauge groups change from a string theory viewpoint (in perfect agreement with the usual field theory prescription), providing also a rigorous definition of the gauge groups ranks in terms of Page charges. We have also shown how the runnings of the gauge couplings are matched by the string background and how global anomalies are also captured by our solution.

In a future publication [24], we will present many details concerning the brane embedding and dual field theory. We will also provide more general solutions to our system of BPS equations and analyze the details of their dual dynamics. The interesting IR dynamics (the last steps of the cascade, leading to a baryonic branch of the field theory, behavior of the Wilson loop, etc) will also be spelled out in detail in the forthcoming paper mentioned above.

Many other things can be done with the solution presented here. The study of implications of these new backgrounds to cosmology and D-brane inflation seems a natural project. On the supergravity side, finding new solutions describing the motion along the baryonic branch of this field theory [14], finding and studying the dynamics of the massless Goldstone mode (that should exist) [11]; what determines the dual to baryonic operators and their VEV [13] and of course, the possibility of softly breaking SUSY and studying the new dynamics [43], are some of the ideas that naturally come to mind given these new solutions.

Acknowledgments

References

[1] G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).

[2] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[3] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, Phys. Rev. D **58**, 046004 (1998) [arXiv:hep-th/9802042].

[4] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[5] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[6] I. R. Klebanov and E. Witten, Nucl. Phys. B **536**, 199 (1998) [arXiv:hep-th/9807080].

[7] S. S. Gubser and I. R. Klebanov, Phys. Rev. D **58**, 125025 (1998) [arXiv:hep-th/9808075].

[8] I. R. Klebanov and N. A. Nekrasov, Nucl. Phys. B **574**, 263 (2000) [arXiv:hep-th/9911096].

[9] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B **578**, 123 (2000) [arXiv:hep-th/0002159].

[10] I. R. Klebanov and M. J. Strassler, JHEP **0008**, 052 (2000) [arXiv:hep-th/0007191].

[11] S. S. Gubser, C. P. Herzog and I. R. Klebanov, JHEP **0409**, 036 (2004) [arXiv:hep-th/0405282].

[12] A. Dymarsky, I. R. Klebanov and N. Seiberg, JHEP **0601**, 155 (2006) [arXiv:hep-th/0511254].

[13] M. K. Benna, A. Dymarsky and I. R. Klebanov, arXiv:hep-th/0612136.

[14] A. Butti, M. Grana, R. Minasian, M. Petrini and A. Zaffaroni, JHEP **0503**, 069 (2005) [arXiv:hep-th/0412187].

[15] M. J. Strassler, arXiv:hep-th/0505153.

[16] G. Veneziano, Nucl. Phys. B **117**, 519 (1976).

[17] A. Karch and E. Katz, JHEP **0206**, 043 (2002) [arXiv:hep-th/0205236].

[18] P. Ouyang, Nucl. Phys. B **699**, 207 (2004) [arXiv:hep-th/0311084].

[19] S. Kuperstein, JHEP **0503**, 014 (2005) [arXiv:hep-th/0411097].

[20] R. Casero, C. Nunez and A. Paredes, Phys. Rev. D **73**, 086005 (2006) [arXiv:hep-th/0602027].

[21] A. Paredes, JHEP **0612**, 032 (2006) [arXiv:hep-th/0610270].

[22] R. Casero and A. Paredes, arXiv:hep-th/0701059.

[23] F. Benini, F. Canoura, S. Cremonesi, C. Nunez and A. V. Ramallo, JHEP **0702**, 090 (2007) [arXiv:hep-th/0612118].

[24] F. Benini, F. Canoura, S. Cremonesi, C. Nunez and A. V. Ramallo, in preparation.

[25] D. Marolf, arXiv:hep-th/0006117.

[26] D. D. Page, Phys. Rev. **28**, 2796 (1983).

[27] M. J. Strassler, *Prepared for International Workshop on Perspectives of Strong Coupling Gauge Theories (SCGT 96), Nagoya, Japan, 13-16 Nov 1996*

[28] F. Benini, S. Cremonesi, R. Tatar, in preparation.

[29] N. Seiberg, Nucl. Phys. B **435**, 129 (1995) [arXiv:hep-th/9411149].

[30] S. Franco and A. M. Uranga, JHEP **0606**, 031 (2006) [arXiv:hep-th/0604136].

[31] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and I. Pesando, JHEP **0102**, 014 (2001) [arXiv:hep-th/0011077].

[32] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, Nucl. Phys. B **621**, 157 (2002) [arXiv:hep-th/0107057].

[33] P. S. Aspinwall, Phys. Lett. B **357**, 329 (1995) [arXiv:hep-th/9507012].

[34] D. R. Morrison and M. R. Plesser, Adv. Theor. Math. Phys. **3**, 1 (1999) [arXiv:hep-th/9810201].

[35] M. Grana and J. Polchinski, Phys. Rev. D **65**, 126005 (2002) [arXiv:hep-th/0106014].

[36] J. Polchinski, Int. J. Mod. Phys. A **16**, 707 (2001) [arXiv:hep-th/0011193].

[37] M. Bershadsky, C. Vafa and V. Sadov, Nucl. Phys. B **463**, 420 (1996) [arXiv:hep-th/9511222].

[38] I. R. Klebanov, P. Ouyang and E. Witten, Phys. Rev. D **65**, 105007 (2002) [arXiv:hep-th/0202056].

[39] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, Phys. Lett. B **540**, 104 (2002) [arXiv:hep-th/0202195].

[40] S. Benvenuti and A. Hanany, JHEP **0508**, 024 (2005) [arXiv:hep-th/0502043].

[41] B. Fiol, JHEP **0207**, 058 (2002) [arXiv:hep-th/0205155].

[42] A. Hanany and J. Walcher, JHEP **0306**, 055 (2003) [arXiv:hep-th/0301231]. S. Franco, A. Hanany, Y. H. He and P. Kazakopoulos, [arXiv:hep-th/0306092].

[43] S. Kuperstein and J. Sonnenschein, JHEP **0402**, 015 (2004) [arXiv:hep-th/0309011]. M. Schvellinger, JHEP **0409**, 057 (2004) [arXiv:hep-th/0407152].