

LOSS OF POLYCONVEXITY BY HOMOGENIZATION: A NEW EXAMPLE

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Abstract. This article is devoted to the study of the asymptotic behavior of the zero-energy deformations set of a periodic nonlinear composite material. We approach the problem using two-scale Young measures. We apply our analysis to show that polyconvex energies are not closed with respect to periodic homogenization. The counterexample is obtained through a rank-one laminated structure assembled by mixing two polyconvex functions with p -growth, where $p \geq 2$ can be fixed arbitrarily.

Keywords: composite materials, homogenization, quasiconvexity, polyconvexity, rank-one laminates, two-scale Young measures, Γ -convergence

2000 Mathematics Subject Classification: 28A20, 35B27, 35B40, 49J45, 73B27

CONTENTS

1. Introduction	1
2. Two-scale Young measures	3
3. Gamma-convergence	7
4. Polyconvexity	9
5. Proof of Theorem 1.3	9
References	12

1. INTRODUCTION

Many problems related to composite materials (see [16] and references therein) lead to the variational analysis of families of integral functionals.

In the case of a periodic composite, the energy density can be described by a family of the type $\{(\mathcal{W}_k, P_k)\}_{k=1, \dots, n}$, where $\{\mathcal{W}_k : \mathbb{M}^{s \times d} \rightarrow [0, +\infty)\}_{k=1, \dots, n}$ is a family of continuous functions describing the energy densities of the components and $\{P_k\}_{k=1, \dots, n}$ is a measurable partition of the unit cell $\square := [0, 1]^d$. In a mix of fineness ε , the functional E_ε that represents, at microscopic level, the stored energy of the composite is of the type

$$E_\varepsilon(u) = \int_{\Omega} \mathcal{W}\left(\left\langle \frac{x}{\varepsilon} \right\rangle, \nabla u(x)\right) dx,$$

where $\langle \cdot \rangle$ denotes the fractional part of a vector componentwise, Ω is an open bounded domain in \mathbb{R}^d and $\mathcal{W}(y, \Lambda) = \sum_{k=1}^n \chi_{P_k}(y) \mathcal{W}_k(\Lambda)$ with χ_{P_k} the characteristic function of P_k .

When the parameter ε tends to 0, the microscopic structure becomes finer and finer and the asymptotic behaviour of the composite is that of a homogeneous material. If the function \mathcal{W} satisfies coerciveness and growth conditions of order $p \in (1, +\infty)$, *i.e.*, there exist $c_1, c_2, c_3 > 0$ such that

$$c_1 |\Lambda|^p - c_2 \leq \mathcal{W}(y, \Lambda) \leq c_3 (1 + |\Lambda|^p) \quad \text{for all } (y, \Lambda) \in \square \times \mathbb{M}^{s \times d}, \quad (1.1)$$

then the stored energy E_{hom} of this material can be described efficiently by the Γ -limit of the family E_ε and is of the type

$$E_{hom}(u) = \int_{\Omega} \mathcal{W}_{hom}(\nabla u(x)) dx,$$

where \mathcal{W}_{hom} is a suitable quasiconvex function called *homogenized integrand*.

Question: Suppose that the sets $\mathfrak{A}_k := \mathcal{W}_k^{-1}(0)$ ($k = 1, \dots, n$) are not empty. Is it possible to characterise $\mathcal{W}_{hom}^{-1}(0)$ and its dependence on $\{(\mathfrak{A}_k, P_k)\}_{k=1, \dots, n}$?

Definition 1.1. Given a measurable partition $\{P_k\}_{k=1, \dots, n}$ of the unit cell \square and a family of compact sets $\{\mathfrak{A}_k\}_{k=1, \dots, n}$, we define \mathfrak{A}_{hom} as the set of matrices $A \in \mathbb{M}^{s \times d}$ such that for all $\varepsilon_h \rightarrow 0^+$ there exists a sequence $u_h \in W^{1, \infty}(\Omega, \mathbb{R}^s)$ satisfying

- i) $u_h \rightharpoonup Ax$ weakly* in $W^{1, \infty}(\Omega, \mathbb{R}^s)$;
- ii) $\sum_{k=1}^n \chi_{P_k}(\langle \frac{x}{\varepsilon_h} \rangle) \text{dist}(\nabla u_h(x), \mathfrak{A}_k) \rightarrow 0$ in measure.

The second condition is equivalent (see proposition 2.10) to the statement that the two-scale gradient Young measure $(\nu_{(x,y)})_{(x,y) \in \Omega \times \square}$ corresponding to (a subsequence of) ∇u_h satisfies

- ii)' for all $k \in \{1, \dots, n\}$ and for a.e. $(x, y) \in \Omega \times P_k$ we have $\text{supp } \nu_{(x,y)} \subseteq \mathfrak{A}_k$.

We call \mathfrak{A}_{hom} the *homogenized set* related to $\{(\mathfrak{A}_k, P_k)\}_{k=1, \dots, n}$.

The first result of this article gives an answer to the previous question, showing the link between homogenized integrands and homogenized sets. We were inspired by [4, section 4] and [5, section 4].

Theorem 1.2. Let $\{\mathcal{W}_k : \mathbb{M}^{s \times d} \rightarrow [0, +\infty)\}_{k=1, \dots, n}$ be a family of continuous functions, let $\{P_k\}_{k=1, \dots, n}$ be a measurable partition of the unit cell \square and let $\mathcal{W}(y, \Lambda) := \sum_{k=1}^n \chi_{P_k}(y) \mathcal{W}_k(\Lambda)$. Suppose that for $k \in \{1, \dots, n\}$

- i) the function \mathcal{W}_k satisfies coerciveness and growth conditions of order p ;
- ii) the set $\mathfrak{A}_k := \mathcal{W}_k^{-1}(0)$ is not empty.

Then

$$\mathfrak{A}_{hom} = \mathcal{W}_{hom}^{-1}(0),$$

i.e., the zero-level set of the homogenized integrand \mathcal{W}_{hom} related to \mathcal{W} coincide with the homogenized set \mathfrak{A}_{hom} related to $\{(\mathfrak{A}_k, P_k)\}_{k=1, \dots, n}$.

Let observe that in the simple case $n = 1$, the function \mathcal{W}_{hom} is the quasiconvexification of \mathcal{W}_1 and \mathfrak{A}_{hom} is the quasiconvex hull of \mathfrak{A}_1 .

The second result is to show that the polyconvexity of the sets \mathfrak{A}_k ($k = 1, \dots, n$) does not ensure the polyconvexity of \mathfrak{A}_{hom} . This aim is achieved with an explicit example in the case $d = s = n = 2$.

Theorem 1.3. (Loss of polyconvexity). Consider the sets

- $P_1 = [0, \frac{1}{2}) \times [0, 1)$ and $P_2 = [\frac{1}{2}, 1) \times [0, 1)$;
- $\mathfrak{A}_1 = \{O, A_1^1, A_1^2\}$ with $O = \text{diag}(0, 0)$, $A_1^1 = \text{diag}(-\frac{1}{2}, 1)$ and $A_1^2 = \text{diag}(-2, 2)$;
- $\mathfrak{A}_2 = \{O, A_2^1, A_2^2\}$ with $A_2^1 = \text{diag}(\frac{5}{2}, 1)$ and $A_2^2 = \text{diag}(3, 2)$;
- $\mathfrak{B} = \{O, B^1, B^2\}$ with $2B^j = A_1^j + A_2^j$ for $j = 1, 2$.

The following properties hold:

- i) \mathfrak{A}_1 and \mathfrak{A}_2 are both polyconvex sets;
- ii) fixed $p \geq 2$, there exist two polyconvex functions $\mathcal{W}_1, \mathcal{W}_2 : \mathbb{M}^{2 \times 2} \rightarrow [0, +\infty)$ p -coercive and with p -growth such that $\mathfrak{A}_1 = \mathcal{W}_1^{-1}(0)$ and $\mathfrak{A}_2 = \mathcal{W}_2^{-1}(0)$;
- iii) the homogenized set \mathfrak{A}_{hom} related to $\{(\mathfrak{A}_k, P_k)\}_{k=1, 2}$ is not polyconvex. More precisely

$$\mathfrak{B} \subseteq \mathfrak{A}_{hom} \quad \text{and} \quad \mathfrak{B}^{pc} \not\subseteq \mathfrak{A}_{hom},$$

where \mathfrak{B}^{pc} denotes the polyconvex hull of \mathfrak{B} ;

- iv) the homogenized integrand \mathcal{W}_{hom} related to $\mathcal{W}(y, \Lambda) := \chi_{P_1}(y) \mathcal{W}_1(\Lambda) + \chi_{P_2}(y) \mathcal{W}_2(\Lambda)$ is not polyconvex.

The example given in the result above shows that polyconvexity is not preserved by homogenization, unlike convexity and quasiconvexity.

The first example of loss of polyconvexity by homogenization is due to Braides [8]. He considers a function \mathcal{W} assembled by using two functions $\mathcal{W}_1, \mathcal{W}_2 : \mathbb{M}^{2 \times 2} \rightarrow [0, +\infty)$ with different growth conditions. More precisely

$$\mathcal{W}(y, \Lambda) := \begin{cases} \mathcal{W}_1(\Lambda) & \text{if } y \in \square \setminus [\frac{1}{4}, \frac{3}{4}]^2, \\ \mathcal{W}_2(\Lambda) & \text{if } y \in [\frac{1}{4}, \frac{3}{4}]^2, \end{cases}$$

where \mathcal{W}_1 is a convex function satisfying coerciveness and growth conditions of order $p < 2$, while \mathcal{W}_2 is a polyconvex function satisfying coerciveness and growth conditions of order 2. Since the homogenized integrand related to \mathcal{W} is not convex and satisfies a growth condition of order $p < 2$, it cannot be polyconvex. A suitable quadratic perturbation, considered in [9], permits to assume that also \mathcal{W}_1 satisfies coerciveness and growth conditions of order 2.

Our example is quite different. It is based only on the structure of $\{(\mathfrak{A}_k, P_k)\}_{k=1,2}$ and is independent of the growth condition. Indeed, we first construct two sets \mathfrak{A}_1 and \mathfrak{A}_2 such that the homogenized set \mathfrak{A}_{hom} related to $\{(\mathfrak{A}_k, P_k)\}_{k=1,2}$ is not polyconvex and then, fixed $p \geq 2$, we construct two polyconvex functions $\mathcal{W}_1, \mathcal{W}_2 : \mathbb{M}^{2 \times 2} \rightarrow [0, +\infty)$ p -coercive and with p -growth such that $\mathfrak{A}_1 = \mathcal{W}_1^{-1}(0)$ and $\mathfrak{A}_2 = \mathcal{W}_2^{-1}(0)$. The homogenized integrand \mathcal{W}_{hom} related to $\mathcal{W}(y, \Lambda) := \chi_{P_1}(y) \mathcal{W}_1(\Lambda) + \chi_{P_2}(y) \mathcal{W}_2(\Lambda)$ cannot be polyconvex because $\mathfrak{A}_{hom} = \mathcal{W}_{hom}^{-1}(0)$.

The idea behind our construction of $\{(\mathfrak{A}_k, P_k)\}_{k=1,2}$ is the following. The set \mathfrak{A}_1 (resp. \mathfrak{A}_2) is polyconvex because the difference of two elements of \mathfrak{A}_1 (resp. \mathfrak{A}_2) has negative (resp. positive) determinant. The set $\mathfrak{B} = \{O, I, \text{diag}(\frac{1}{2}, 2)\}$, composed of average of the correspondent matrices of \mathfrak{A}_1 and \mathfrak{A}_2 , does not have these properties and its polyconvex hull is not trivial. The fact that $\mathfrak{B}^{pc} \not\subseteq \mathfrak{A}_{hom}$ is proved by using the function \mathcal{V} defined in (5.1). This function was used originally by Šverák in [20] to prove the quasiconvexity of sets of the type $\{O, I, \text{diag}(b_1, b_2)\}$, where $0 < b_1 < 1$ and $b_2 > 1$. See also [2] for a different proof.

The plan of the paper is as follows. In section 2 we collect concepts and basic facts about Young measures and two-scale Young measures. In section 3 we recall some well known facts about Γ -convergence and prove Theorem 1.2. In section 4 we present some simple results about polyconvexity. Finally, in section 5 we provide a proof of Theorem 1.3.

2. TWO-SCALE YOUNG MEASURES

We start by collecting preliminary results about Young measures (see [3], [18] and [22]) and two-scale Young measures (see [7] and [23]).

Before we list the notation used in the following:

- Ω a bounded open subset of \mathbb{R}^d ;
- \square the unit cell $[0, 1)^d$;
- $\mathcal{L}(\Omega)$ the Lebesgue σ -algebra on Ω ;
- $|S|$ the Lebesgue measure of a measurable subset $S \subseteq \mathbb{R}^l$;
- $\mathcal{B}(S)$ the Borel σ -algebra on a subset $S \subseteq \mathbb{R}^l$;
- $\mathcal{P}(\mathbb{R}^m)$ the set of probability measures on \mathbb{R}^m ;
- $\mathcal{Y}(\Omega, \mathbb{R}^m)$ the family of all weakly* measurable maps $x \in \Omega \xrightarrow{\mu} \mu_x \in \mathcal{P}(\mathbb{R}^m)$;
- a corresponding definition holds for $\mathcal{Y}(\Omega \times \square, \mathbb{R}^m)$;
- $\langle x \rangle \in \square$ the fractional part of $x \in \mathbb{R}^d$ componentwise, *i.e.*,

$$\langle x \rangle_k = x_k - [x_k] \quad \text{for } k \in \{1, \dots, d\},$$

where $[x_k]$ stands for the largest integer less than or equal to x_k ;

- ε_h a sequence in $(0, +\infty)$.

Remark 2.1.

- i) The term *measurable* is tacitly understood as Lebesgue measurable, unless otherwise stated.
- ii) A map $\mu : \Omega \rightarrow \mathcal{P}(\mathbb{R}^m)$ is said to be *weakly* measurable* if $x \rightarrow \mu_x(S)$ is measurable for all $S \in \mathcal{B}(\mathbb{R}^m)$. By approximation, if $\mu \in \mathcal{Y}(\Omega, \mathbb{R}^m)$ and $\mathcal{W} : \Omega \times \mathbb{R}^m \rightarrow [0, +\infty)$ is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^m)$ -measurable, then $x \rightarrow \int_{\mathbb{R}^m} \mathcal{W}(x, \lambda) d\mu_x(\lambda)$ is measurable.
- iii) More precisely, the elements of $\mathcal{Y}(\Omega, \mathbb{R}^m)$ are equivalence classes of maps that agree *a.e.*; we usually do not distinguish these maps from their equivalence classes.

The following result is known as the Fundamental Theorem on Young Measures. It shows that the weak limit of a sequence of the type $\mathcal{W}(\cdot, u_h(\cdot))$ can be expressed through a suitable map $\mu \in \mathcal{Y}(\Omega, \mathbb{R}^m)$ associated to u_h . The proof can be found in [3] (see also [18] and [22]). We recall that a $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^m)$ -measurable function \mathcal{W} is a Carathéodory integrand if $\mathcal{W}(x, \cdot)$ is continuous for *a.e.* $x \in \Omega$.

Theorem 2.2. *Let u_h be a bounded sequence in $L^1(\Omega, \mathbb{R}^m)$. There exist a subsequence u_{h_i} and a map $\mu \in \mathcal{Y}(\Omega, \mathbb{R}^m)$ such that the following properties hold:*

i) *if $\mathcal{W} : \Omega \times \mathbb{R}^m \rightarrow [0, +\infty)$ is a Carathéodory integrand, then*

$$\liminf_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}(x, u_{h_i}(x)) dx \geq \int_{\Omega} \overline{\mathcal{W}}(x) dx$$

where

$$\overline{\mathcal{W}}(x) := \int_{\mathbb{R}^m} \mathcal{W}(x, \lambda) d\mu_x(\lambda);$$

ii) *if $\mathcal{W} : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a Caratheodory integrand and $\mathcal{W}(\cdot, u_h(\cdot))$ is equi-integrable, then $\mathcal{W}(x, \cdot)$ is μ_x -integrable for a.e. $x \in \Omega$, $\overline{\mathcal{W}}$ is in $L^1(\Omega)$ and*

$$\lim_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}(x, u_{h_i}(x)) dx = \int_{\Omega} \overline{\mathcal{W}}(x) dx;$$

iii) *if $\mathfrak{A} \subseteq \mathbb{R}^m$ is closed, then $\text{supp } \mu_x \subseteq \mathfrak{A}$ for a.e. $x \in \Omega$ if and only if $\text{dist}(u_{h_i}, \mathfrak{A}) \rightarrow 0$ in measure.*

Definition 2.3. The map μ is called the *Young measure* generated by the sequence u_{h_i} .

In homogenization processes, we are interested to asymptotic behaviour of sequences of the type $\mathcal{W}_h(\cdot) = \mathcal{W}(\cdot, \langle \frac{\cdot}{\varepsilon_h} \rangle, u_h(\cdot))$. Under technical hypothesis on \mathcal{W} , the weak limit of \mathcal{W}_h can be expressed through a suitable map $\nu \in \mathcal{Y}(\Omega \times \square, \mathbb{R}^m)$ associated to u_h .

Definition 2.4. A function $\mathcal{W} : \Omega \times \square \times \mathbb{R}^m \rightarrow \mathbb{R}$ is said to be an *admissible integrand* if there exist a family $\{X_\delta\}_{\delta>0}$ of compact subsets of Ω and a family $\{Y_\delta\}_{\delta>0}$ of compact subsets of \square such that $|\Omega \setminus X_\delta| \leq \delta$, $|\square \setminus Y_\delta| \leq \delta$ and $\mathcal{W}|_{X_\delta \times Y_\delta \times \mathbb{R}^m}$ is continuous for every $\delta > 0$.

Remark 2.5. It is not restrictive to suppose that the families $\{X_\delta\}_{\delta>0}$ and $\{Y_\delta\}_{\delta>0}$ are decreasing, i.e., $\delta' \leq \delta$ implies $X_\delta \subseteq X_{\delta'}$ and $Y_\delta \subseteq Y_{\delta'}$. Otherwise, it is sufficient to consider the new families $\{\tilde{X}_\delta\}_{\delta>0}$ and $\{\tilde{Y}_\delta\}_{\delta>0}$, where

$$\tilde{X}_\delta := \bigcap_{i \geq i_\delta} X_{2^{-i}}, \quad \tilde{Y}_\delta := \bigcap_{i \geq i_\delta} Y_{2^{-i}}$$

and i_δ is the minimum positive integer such that $2^{1-i_\delta} \leq \delta$.

Remark 2.6. Admissible integrands have good measurability properties: if \mathcal{W} is an admissible integrand, then there exist a Borel set $X \subseteq \Omega$ with $|\Omega \setminus X| = 0$ and a Borel set $Y \subseteq \square$ with $|\square \setminus Y| = 0$, such that $\mathcal{W}|_{X \times Y \times \mathbb{R}^m}$ is borelian. In particular, for every fixed ε , the function $(x, \lambda) \rightarrow \mathcal{W}(x, \langle \frac{x}{\varepsilon} \rangle, \lambda)$ is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R}^m)$ -measurable.

Remark 2.7. Let $\{\mathcal{W}_k : \Omega \times \mathbb{M}^{s \times d} \rightarrow [0, +\infty)\}_{k=1, \dots, n}$ be a family of Carathéodory integrands and let $\{P_k\}_{k=1, \dots, n}$ be a measurable partition of the unit cell \square . By applying Lusin theorem to each χ_{P_k} and Scorza-Dragoni theorem (see [13]) to each \mathcal{W}_k , we obtain that $\mathcal{W}(x, y, \lambda) := \sum_{k=1}^n \chi_{P_k}(y) \mathcal{W}_k(x, \lambda)$ is an admissible integrand.

The next two results are the equivalent of Theorem 2.2 in the case of two-scale Young measures.

Theorem 2.8. *Let $\varepsilon_h \rightarrow 0^+$, let u_h be a bounded sequence in $L^1(\Omega, \mathbb{R}^m)$ and let $w_h : \Omega \rightarrow \square \times \mathbb{R}^m$ be defined by $w_h(x) := (\langle \frac{x}{\varepsilon_h} \rangle, u_h(x))$. There exist a subsequence w_{h_i} and a map $\nu \in \mathcal{Y}(\Omega \times \square, \mathbb{R}^m)$ such that the following properties hold:*

i) *if $\mathcal{W} : \Omega \times \square \times \mathbb{R}^m \rightarrow [0, +\infty)$ is an admissible integrand, then*

$$\liminf_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}(x, w_{h_i}(x)) dx \geq \int_{\Omega \times \square} \overline{\mathcal{W}}(x, y) dx dy, \quad (2.1)$$

where

$$\overline{\mathcal{W}}(x, y) := \int_{\mathbb{R}^m} \mathcal{W}(x, y, \lambda) d\nu_{(x,y)}(\lambda);$$

ii) if $\mathcal{W} : \Omega \times \square \times \mathbb{R}^m \rightarrow \mathbb{R}$ is an admissible integrand and $\mathcal{W}(\cdot, w_h(\cdot))$ is equi-integrable, then $\mathcal{W}(x, y, \cdot)$ is $\nu_{(x,y)}$ -integrable for a.e. $(x, y) \in \Omega \times \square$, $\overline{\mathcal{W}}$ is in $L^1(\Omega \times \square)$ and

$$\lim_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}(x, w_{h_i}(x)) dx = \int_{\Omega \times \square} \overline{\mathcal{W}}(x, y) dx dy. \quad (2.2)$$

Definition 2.9. The map ν is called the *two-scale Young measure* generated by the sequence u_{h_i} with respect to ε_{h_i} . In the sequel we omit to specify the dependence on ε_{h_i} when it is clear from the context.

Proof of Theorem 2.8. For the sake of clarity, we divide the proof into three steps.

Step 1. Let $\mathcal{M}(\Omega \times \square \times \mathbb{R}^m)$ be the set of finite real valued Radon measures on $\Omega \times \square \times \mathbb{R}^m$ and let $\hat{\nu}_h$ ($h = 1, \dots$) be the measure canonically associated with w_h , i.e.,

$$\hat{\nu}_h(S) = \int_{\Omega} \chi_S(x, w_h(x)) dx \quad \text{for all } S \in \mathcal{B}(\Omega \times \square \times \mathbb{R}^m).$$

Since $\hat{\nu}_h$ is a bounded sequence of Radon measures, it has a weakly* convergent subsequence $\hat{\nu}_{h_i}$. Let $\hat{\nu}$ be the limit of $\hat{\nu}_{h_i}$. We claim that

$$\hat{\nu}(S \times \mathbb{R}^m) = |S| \quad \text{for all } S \in \mathcal{B}(\Omega \times \square). \quad (2.3)$$

We use a classical result about the convergence of $\hat{\nu}_h(S)$ when $S \subseteq \Omega \times \square \times \mathbb{R}^m$ is an open or compact set (see [1, Proposition 1.62]).

For $U \subseteq \Omega$ and $V \subseteq \square$ open

$$\begin{aligned} \hat{\nu}(U \times V \times \mathbb{R}^m) &\leq \liminf_{i \rightarrow +\infty} \hat{\nu}_{h_i}(U \times V \times \mathbb{R}^m) \\ &= \lim_{i \rightarrow +\infty} \int_{\Omega} \chi_U(x) \chi_V\left(\left\langle \frac{x}{\varepsilon_{h_i}} \right\rangle\right) dx = |U \times V|, \end{aligned} \quad (2.4)$$

by using Riemann-Lebesgue lemma (see [10, Theorem 1.5]) in the last equality.

Denoting with B_k the open ball with centre in the origin and radius $k \in \mathbb{N}^+$, we have for $X \subseteq \Omega$ and $Y \subseteq \square$ compact

$$\begin{aligned} \hat{\nu}(X \times Y \times \overline{B}_k) &\geq \limsup_{i \rightarrow +\infty} \hat{\nu}_{h_i}(X \times Y \times \overline{B}_k) \\ &= \limsup_{i \rightarrow +\infty} \left| \left\{ x \in X : \left\langle \frac{x}{\varepsilon_{h_i}} \right\rangle \in Y \text{ and } |u_{h_i}(x)| \leq k \right\} \right| \\ &\geq \lim_{i \rightarrow +\infty} \int_{\Omega} \chi_X(x) \chi_Y\left(\left\langle \frac{x}{\varepsilon_{h_i}} \right\rangle\right) dx - \frac{\sup_i \|u_{h_i}\|_{L^1}}{k} = |X \times Y| - \frac{\sup_i \|u_{h_i}\|_{L^1}}{k}. \end{aligned} \quad (2.5)$$

From (2.4) and (2.5), by using inner and outer regularity of the measure $\hat{\nu}$, we obtain equality (2.3).

As a consequence, from disintegration theorem (see [14, Theorem 1.5.1] and [1, Theorem 2.28]), we can infer that there exists a map $\nu \in \mathcal{Y}(\Omega \times \square, \mathbb{R}^m)$ such that

$$\int_{\Omega \times \square \times \mathbb{R}^m} \phi d\hat{\nu} = \int_{\Omega \times \square} \left(\int_{\mathbb{R}^m} \phi(x, y, \lambda) d\nu_{(x,y)}(\lambda) \right) dx dy$$

for each continuous and bounded $\phi : \Omega \times \square \times \mathbb{R}^m \rightarrow \mathbb{R}$.

Step 2. We prove property (i). Note that by Remark 2.6, both sides of inequality (2.1) are well defined. We first assume that there exists $b > 0$ such that

$$\mathcal{W}(x, y, \lambda) \leq b \quad \text{for all } (x, y, \lambda) \in \Omega \times \square \times \mathbb{R}^m. \quad (2.6)$$

By the admissibility condition, for every $\delta > 0$ there exist a compact set $X_\delta \subseteq \Omega$ and a compact set $Y_\delta \subseteq \square$ such that $|\Omega \setminus X_\delta| \leq \delta$, $|\square \setminus Y_\delta| \leq \delta$ and $W|_{X_\delta \times Y_\delta \times \mathbb{R}^m}$ is continuous. Let $\phi \in C(\Omega \times \square \times \mathbb{R}^m)$ be an extension of $\mathcal{W}|_{X_\delta \times Y_\delta \times \mathbb{R}^m}$ such that $0 \leq \phi(x, y, \lambda) \leq b$ for every $(x, y, \lambda) \in \Omega \times \square \times \mathbb{R}^m$. Since $\hat{\nu}_{h_i} \rightharpoonup \hat{\nu}$ weakly*, defining $\overline{\phi}(x, y) := \int_{\mathbb{R}^m} \phi(x, y, \lambda) d\nu_{(x,y)}(\lambda)$, we have by the previous step

$$\begin{aligned} \liminf_{i \rightarrow +\infty} \int_{\Omega} \phi(x, w_{h_i}(x)) dx &= \liminf_{i \rightarrow +\infty} \int_{\Omega \times \square \times \mathbb{R}^m} \phi d\hat{\nu}_{h_i} \\ &\geq \int_{\Omega \times \square \times \mathbb{R}^m} \phi d\hat{\nu} = \int_{\Omega \times \square} \overline{\phi}(x, y) dx dy. \end{aligned}$$

We can write

$$\begin{aligned}
& \int_{\Omega \times \square} \overline{\mathcal{W}}(x, y) \, dx \, dy - b \delta (1 + |\Omega|) \leq \int_{X_\delta \times Y_\delta} \overline{\mathcal{W}}(x, y) \, dx \, dy \\
& = \int_{X_\delta \times Y_\delta} \overline{\phi}(x, y) \, dx \, dy \leq \int_{\Omega \times \square} \overline{\phi}(x, y) \, dx \, dy \leq \liminf_{i \rightarrow +\infty} \int_{\Omega} \phi(x, w_{h_i}(x)) \, dx \\
& \leq \liminf_{i \rightarrow +\infty} \int_{\Omega} \left[\chi_{X_\delta}(x) \chi_{Y_\delta} \left(\left\langle \frac{x}{\varepsilon_{h_i}} \right\rangle \right) + \chi_{\Omega \setminus X_\delta}(x) + \chi_{\square \setminus Y_\delta} \left(\left\langle \frac{x}{\varepsilon_{h_i}} \right\rangle \right) \right] \phi(x, w_{h_i}(x)) \, dx \\
& \leq \liminf_{i \rightarrow +\infty} \int_{X_\delta} \chi_{Y_\delta} \left(\left\langle \frac{x}{\varepsilon_{h_i}} \right\rangle \right) \mathcal{W}(x, w_{h_i}(x)) \, dx + b \delta + b \lim_{i \rightarrow +\infty} \int_{\Omega} \chi_{\square \setminus Y_\delta} \left(\left\langle \frac{x}{\varepsilon_{h_i}} \right\rangle \right) \, dx \\
& \leq \liminf_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}(x, w_{h_i}(x)) \, dx + b \delta (1 + |\Omega|).
\end{aligned}$$

Being $\delta > 0$ arbitrary, inequality (2.1) follows.

In order to remove assumptions (2.6) we consider, for $k \in \mathbb{N}^+$, the functions

$$\mathcal{W}_k(x, y, \lambda) := \min\{k, \mathcal{W}(x, y, \lambda)\}.$$

By applying the first part of the step, we have

$$\liminf_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}(x, w_{h_i}(x)) \, dx \geq \liminf_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}_k(x, w_{h_i}(x)) \, dx \geq \int_{\Omega \times \square} \overline{\mathcal{W}}_k(x, y) \, dx \, dy.$$

By noting that \mathcal{W}_k is increasing and that $\mathcal{W}_k(x, y, \cdot) \rightarrow \mathcal{W}(x, y, \cdot)$ *a.e.* in \mathbb{R}^m for every fixed $(x, y) \in \Omega \times \square$, we deduce from the monotone convergence theorem that $\overline{\mathcal{W}}_k \rightarrow \overline{\mathcal{W}}$ *a.e.* in $\Omega \times \square$. The sequence $\overline{\mathcal{W}}_k$ is increasing so, again from monotone convergence theorem,

$$\int_{\Omega \times \square} \overline{\mathcal{W}}_k(x, y) \, dx \, dy \xrightarrow{k \rightarrow \infty} \int_{\Omega \times \square} \overline{\mathcal{W}}(x, y) \, dx \, dy.$$

Step 3. We prove property (ii). Let $\mathcal{W}^+ := \max\{0, \mathcal{W}\}$ and $\mathcal{W}^- := \max\{0, -\mathcal{W}\}$, so that $\mathcal{W} = \mathcal{W}^+ - \mathcal{W}^-$. Both the sequences $\mathcal{W}^+(\cdot, w_h(\cdot))$ and $\mathcal{W}^-(\cdot, w_h(\cdot))$ are equi-integrable, so it is enough to prove equality (2.2) when \mathcal{W} is non-negative.

If \mathcal{W} is bounded from above by a constant b , then (2.2) follows by applying (2.1) to \mathcal{W} and $b - \mathcal{W}$. For general non-negative \mathcal{W} , by an equivalent characterization of equi-integrability, for each $\eta > 0$ there exists $k \in \mathbb{N}^+$ such that $\sup_h \int_{\{x : \mathcal{W}(x, w_h(x)) \geq k\}} \mathcal{W}(x, w_h(x)) \, dx < \eta$. Hence

$$\begin{aligned}
& \limsup_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}(x, w_{h_i}(x)) \, dx - \eta \leq \lim_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}_k(x, w_{h_i}(x)) \, dx \\
& = \int_{\Omega \times \square} \overline{\mathcal{W}}_k(x, y) \, dx \, dy \leq \int_{\Omega \times \square} \overline{\mathcal{W}}(x, y) \, dx \, dy.
\end{aligned}$$

Being $\eta > 0$ arbitrary, the previous inequality completes the proof. \blacklozenge

Proposition 2.10. *Let u_h be a bounded sequence in $L^1(\Omega, \mathbb{R}^m)$ generating a two-scale Young measure ν . If $\{\mathfrak{A}_k\}_{k=1, \dots, n}$ is a family of closed subsets of \mathbb{R}^m and $\{P_k\}_{k=1, \dots, n}$ is a measurable partition of the unit cell \square , then the following conditions are equivalent:*

- i) for all $k \in \{1, \dots, n\}$ and for *a.e.* $(x, y) \in \Omega \times P_k$ $\text{supp } \nu_{(x, y)} \subseteq \mathfrak{A}_k$;
- ii) $\mathcal{D}(\langle \frac{x}{\varepsilon_h} \rangle, u_h(x)) \rightarrow 0$ in measure, where $\mathcal{D}(y, \lambda) := \sum_{k=1}^n \chi_{P_k}(y) \text{dist}(\lambda, \mathfrak{A}_k)$.

Proof. (i) \Rightarrow (ii) Fixed $\eta > 0$, we define $\mathcal{W}(y, \lambda) := \min\{\eta, \mathcal{D}(y, \lambda)\}$. By Remark 2.7, \mathcal{W} is an admissible integrand and therefore, by Theorem 2.8,

$$\lim_{h \rightarrow +\infty} \int_{\Omega} \mathcal{W}\left(\left\langle \frac{x}{\varepsilon_h} \right\rangle, u_h(x)\right) \, dx = \int_{\Omega \times \square} \left(\int_{\mathbb{R}^m} \mathcal{W}(y, \lambda) \, d\nu_{(x, y)}(\lambda) \right) \, dx \, dy = 0.$$

Since $\eta > 0$ is arbitrary, this proves that $\mathcal{D}(\langle \frac{x}{\varepsilon_h} \rangle, u_h(x)) \rightarrow 0$ in measure.

(ii) \Rightarrow (i) Let $\varphi \in C_k := \{\varphi \in C_c(\mathbb{R}^m) : \text{supp } \varphi \subseteq \mathbb{R}^m \setminus \mathfrak{A}_k\}$ and $\eta > 0$ such that $\varphi(\lambda) = 0$ if $\text{dist}(\lambda, \mathfrak{A}_k) \leq \eta$. Taking

$$S_h := \left\{ x \in \Omega : \left\langle \frac{x}{\varepsilon_h} \right\rangle \in P_k \text{ and } \text{dist}(u_h(x), \mathfrak{A}_k) > \eta \right\},$$

it follows from the hypothesis that $|S_h| \rightarrow 0$. Given $X \subseteq \Omega$ and $Y \subseteq P_k$ measurable, we define $\mathcal{W}(x, y, \lambda) := \chi_X(x)\chi_Y(y)\varphi(\lambda)$. By Lusin theorem, \mathcal{W} is an admissible integrand and therefore, by Theorem 2.8,

$$\int_{X \times Y} \left(\int_{\mathbb{R}^m} \varphi(\lambda) d\nu_{(x,y)}(\lambda) \right) dx dy = \lim_{h \rightarrow +\infty} \int_{S_h} \mathcal{W}\left(x, \left\langle \frac{x}{\varepsilon_h} \right\rangle, u_h(x)\right) dx = 0.$$

Since X and Y are arbitrary and C_k is separable, it remains proved that for *a.e.* $(x, y) \in \Omega \times P_k$ we have $\int_{\mathbb{R}^m} \varphi d\nu_{(x,y)} = 0$ for all $\varphi \in C_k$ or equivalently $\text{supp } \nu_{(x,y)} \subseteq \mathfrak{A}_k$. \blacklozenge

Definition 2.11. A map $\nu \in \mathcal{Y}(\Omega \times \square, \mathbb{M}^{d \times s})$ is a *two-scale gradient Young measure* if there exists a sequence u_h in $W^{1,1}(\Omega, \mathbb{R}^s)$ such that ∇u_h is bounded in $L^1(\Omega, \mathbb{M}^{d \times s})$ and generates ν in two-scale. For a complete characterization we refer to [6] (see also [19]).

3. GAMMA-CONVERGENCE

In this section we give a proof of Theorem 1.2. First we recall the definition of Γ -convergence, referring to [9] and [12] for a comprehensive treatment.

Definition 3.1. Let (U, τ) be a topological space satisfying the first countability axiom and E_h, E functionals from U to $[-\infty, +\infty]$; we say that E is the $\Gamma(\tau)$ -limit of the sequence E_h , or that E_h $\Gamma(\tau)$ -converges to E , and write

$$E = \Gamma(\tau)\text{-}\lim_{h \rightarrow +\infty} E_h,$$

if for every $u \in U$ the following conditions are satisfied:

$$E(u) \leq \inf \left\{ \liminf_{h \rightarrow +\infty} E_h(u_h) : u_h \xrightarrow{\tau} u \right\}$$

and

$$E(u) \geq \inf \left\{ \limsup_{h \rightarrow +\infty} E_h(u_h) : u_h \xrightarrow{\tau} u \right\}.$$

We can extend the definition of Γ -convergence to families depending on a parameter $\varepsilon > 0$.

Definition 3.2. For every $\varepsilon > 0$, let E_ε be a functional from U to $[-\infty, +\infty]$. We say that E is the $\Gamma(\tau)$ -limit of the family E_ε as $\varepsilon \rightarrow 0^+$, and write

$$E = \Gamma(\tau)\text{-}\lim_{\varepsilon \rightarrow 0^+} E_\varepsilon,$$

if we have for every sequence $\varepsilon_h \rightarrow 0^+$

$$E = \Gamma(\tau)\text{-}\lim_{h \rightarrow +\infty} E_{\varepsilon_h}.$$

We are interested in Γ -convergence for periodic homogenization of integral functionals. In the following, the space $L^p(\Omega, \mathbb{R}^s)$, $p \in (1, +\infty)$, is endowed with the strong topology. We consider the family of functionals $E_\varepsilon : L^p(\Omega, \mathbb{R}^s) \rightarrow [0, +\infty]$ defined by

$$E_\varepsilon(u) := \begin{cases} \int_{\Omega} \mathcal{W}\left(\left\langle \frac{x}{\varepsilon} \right\rangle, \nabla u(x)\right) dx & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^s), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{W} : \square \times \mathbb{M}^{s \times d} \rightarrow [0, +\infty)$ is a Carathéodory function satisfying coerciveness and growth conditions of order p as in (1.1).

The next theorem is a standard result (see [9, Theorem 14.5]).

Theorem 3.3. *The family E_ε $\Gamma(L^p)$ -converges and its $\Gamma(L^p)$ -limit $E_{hom} : W^{1,p}(\Omega, \mathbb{R}^s) \rightarrow [0, +\infty]$ can be written as*

$$E_{hom}(u) = \begin{cases} \int_{\Omega} \mathcal{W}_{hom}(\nabla u(x)) dx & \text{if } u \in W^{1,p}(\Omega, \mathbb{R}^s), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{W}_{hom} : \mathbb{M}^{s \times d} \rightarrow [0, +\infty)$ is a suitable quasiconvex function depending on \mathcal{W} .

Definition 3.4. We call \mathcal{W}_{hom} the homogenized integrand related to \mathcal{W} .

In the proof of Theorem 1.2, we will use the following lemma, which can be derived by [17, Theorem 4] (see also [24, Lemma 3.1]).

Lemma 3.5. *Assume that $\mathfrak{A} \subseteq \mathbb{M}^{s \times d}$ is a compact set. Let $A \in \mathbb{M}^{s \times d}$ and let u_h be a sequence converging weakly in $W^{1,1}(\Omega, \mathbb{R}^s)$ to Ax and such that $\text{dist}(\nabla u_h, \mathfrak{A}) \rightarrow 0$ in measure. Then there exists a sequence v_h in $W^{1,\infty}(\Omega, \mathbb{R}^s)$ such that*

- i) $v_h \rightharpoonup Ax$ weakly* in $W^{1,\infty}(\Omega, \mathbb{R}^s)$;
- ii) $|\{\nabla v_h \neq \nabla u_h\}| \rightarrow 0$.

Proof of Theorem 1.2. We begin to prove the inclusion $\mathfrak{A}_{hom} \subseteq \mathcal{W}_{hom}^{-1}(0)$. Note that, by coerciveness hypothesis, the sets $\mathfrak{A}_1, \dots, \mathfrak{A}_n$ are compact. Fix $A \in \mathfrak{A}_{hom}$ and $\varepsilon_h \rightarrow 0^+$. By definition of \mathfrak{A}_{hom} , there exists a sequence u_h such that $u_h \rightharpoonup Ax$ weakly* in $W^{1,\infty}(\Omega, \mathbb{R}^s)$ and $\mathcal{D}(\langle \frac{x}{\varepsilon_h} \rangle, \nabla u_h(x)) \rightarrow 0$ in measure, where $\mathcal{D}(y, \Lambda) := \sum_{k=1}^n \chi_{P_k}(y) \text{dist}(\Lambda, \mathfrak{A}_k)$. For a suitable subsequence h_i , ∇u_{h_i} generates a two-scale Young measure ν with respect to ε_{h_i} .

By Proposition 2.10, for all $k \in \{1, \dots, n\}$ and for a.e. $(x, y) \in \Omega \times P_k$ we have $\text{supp } \nu_{(x,y)} \subseteq \mathfrak{A}_k$, therefore $\int_{\mathbb{M}^{s \times d}} \mathcal{W}(y, \Lambda) d\nu_{(x,y)}(\Lambda) = 0$. Observed that \mathcal{W} is an admissible integrand, we obtain by Theorem 2.8

$$\lim_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}\left(\langle \frac{x}{\varepsilon_{h_i}} \rangle, \nabla u_{h_i}(x)\right) dx = \int_{\Omega \times \square} \left(\int_{\mathbb{M}^{s \times d}} \mathcal{W}(y, \Lambda) d\nu_{(x,y)}(\Lambda) \right) dx dy = 0.$$

By the definition of $\Gamma(L^p)$ -limit, $\mathcal{W}_{hom}(A) = 0$.

We prove now the opposite inclusion. Fix $A \in \mathcal{W}_{hom}^{-1}(0)$ and $\varepsilon_h \rightarrow 0^+$. By definition of \mathcal{W}_{hom} , there exists a sequence u_h such that $u_h \rightarrow Ax$ strongly in $L^p(\Omega, \mathbb{R}^s)$ and $E_{\varepsilon_h}(u_h) \rightarrow 0$. For h large enough, $E_{\varepsilon_h}(u_h)$ is finite and therefore $u_h \in W^{1,p}(\Omega, \mathbb{R}^s)$. Thanks to the p -coerciveness hypothesis on \mathcal{W} , we can suppose that $u_h \rightharpoonup Ax$ weakly in $W^{1,p}(\Omega, \mathbb{R}^s)$.

We claim that $\mathcal{D}(\langle \frac{x}{\varepsilon_h} \rangle, \nabla u_h(x)) \rightarrow 0$ in measure. If not, there exist $\delta, \eta > 0$ and a subsequence u_{h_i} such that

$$\inf_i \left| \left\{ x \in \Omega : \mathcal{D}\left(\langle \frac{x}{\varepsilon_{h_i}} \rangle, \nabla u_{h_i}(x)\right) > \eta \right\} \right| > \delta. \quad (3.1)$$

Refining the subsequence h_i if necessary, we can suppose that ∇u_{h_i} generates a two-scale Young measure ν with respect to ε_{h_i} . By Theorem 2.8

$$0 = \lim_{i \rightarrow +\infty} \int_{\Omega} \mathcal{W}\left(\langle \frac{x}{\varepsilon_{h_i}} \rangle, \nabla u_{h_i}(x)\right) dx \geq \int_{\Omega \times \square} \left(\int_{\mathbb{M}^{s \times d}} \mathcal{W}(y, \Lambda) d\nu_{(x,y)}(\Lambda) \right) dx dy$$

so that for all $k \in \{1, \dots, n\}$ and for a.e. $(x, y) \in \Omega \times P_k$ we have $\int_{\mathbb{M}^{s \times d}} \mathcal{W}_k(\Lambda) d\nu_{(x,y)}(\Lambda) = 0$. Since \mathcal{W}_k is continuous and non-negative, $\text{supp } \nu_{(x,y)} \subseteq \mathfrak{A}_k$ for a.e. $(x, y) \in \Omega \times P_k$. Therefore by Proposition 2.10 $\mathcal{D}(\langle \frac{x}{\varepsilon_{h_i}} \rangle, \nabla u_{h_i}(x)) \rightarrow 0$ in measure, in contradiction with (3.1).

Finally, noticed that $\text{dist}(\nabla u_h(x), \bigcup_{k=1}^n \mathfrak{A}_k) \rightarrow 0$ in measure, we can apply Lemma 3.5 to infer the existence of a sequence v_h such that $v_h \rightharpoonup Ax$ weakly* in $W^{1,\infty}(\Omega, \mathbb{R}^s)$ and $|\{\nabla v_h \neq \nabla u_h\}| \rightarrow 0$. In particular, $\mathcal{D}(\langle \frac{x}{\varepsilon_h} \rangle, \nabla v_h(x)) \rightarrow 0$ in measure. ◆

4. POLYCONVEXITY

In this section we recall some of the definitions and the results related to polyconvexity. General references are [10], [11] and [18]. In the following we always assume that $d, s \geq 2$ since otherwise polyconvexity agrees with ordinary convexity.

Definition 4.1. A function $\mathcal{W} : \mathbb{M}^{s \times d} \rightarrow \mathbb{R}$ is said to be *polyconvex* if there is a convex function $\mathcal{V} : \mathbb{R}^{\tau(d,s)} \rightarrow \mathbb{R}$ such that $\mathcal{W}(\Lambda) = \mathcal{V}(\widehat{\Lambda})$ for all $\Lambda \in \mathbb{M}^{s \times d}$. Here $\widehat{\Lambda}$ denotes the list of all minors (subdeterminants) of Λ and $\tau(d, s) = \binom{d+s}{d} - 1$. We can identify $\widehat{\Lambda}$ with a point of $\mathbb{R}^{\tau(d,s)}$. In the simple case $d = s = 2$ we have $\widehat{\Lambda} = (\Lambda, \det(\Lambda)) \in \mathbb{R}^5$.

Remark 4.2. The maximum of two polyconvex functions is still polyconvex.

Definition 4.3. We say that a set $\mathfrak{A} \subseteq \mathbb{M}^{s \times d}$ is *polyconvex* if there exists a convex set $\mathfrak{C} \subseteq \mathbb{R}^{\tau(d,s)}$ such that

$$\mathfrak{A} = \left\{ A \in \mathbb{M}^{s \times d} : \widehat{A} \in \mathfrak{C} \right\}. \quad (4.1)$$

Definition 4.4. The *polyconvex hull* of a set $\mathfrak{A} \subseteq \mathbb{M}^{s \times d}$ is the smallest polyconvex set containing \mathfrak{A} and is denoted by \mathfrak{A}^{pc} .

Lemma 4.5. *Let \mathfrak{A} be a compact set of $\mathbb{M}^{s \times d}$ and let $p \geq \max\{d, s\}$. If \mathfrak{A} is polyconvex, then there exists a polyconvex function $\mathcal{W} : \mathbb{M}^{s \times d} \rightarrow [0, +\infty)$ satisfying coerciveness and growth conditions of order p such that*

$$\mathfrak{A} = \mathcal{W}^{-1}(0).$$

Proof. Let $\mathfrak{C} \subseteq \mathbb{R}^{\tau(d,s)}$ be a compact and convex set such that (4.1) holds. Consider the functions

$$\mathcal{W}_1(\Lambda) := \text{dist}^p(\Lambda, \mathfrak{A}^{co}) \quad \text{and} \quad \mathcal{W}_2(\Lambda) := \text{dist}^{\frac{p}{q}}(\widehat{\Lambda}, \mathfrak{C}),$$

where $q = \max\{d, s\}$ and co denotes the convex hull. Both are polyconvex and with p -growth, moreover \mathcal{W}_1 is p -coercive and $\mathcal{W}_2^{-1}(0) = \mathfrak{A}$. The function $\mathcal{W} := \max\{\mathcal{W}_1, \mathcal{W}_2\}$ does the job. \blacklozenge

5. PROOF OF THEOREM 1.3

This section is devoted to the proof of Theorem 1.3. Assuming in the following that $d = 2$, we begin by a technical lemma.

Lemma 5.1. *Let \mathfrak{A}_{hom} be the homogenized set related to $\{(\mathfrak{A}_k, P_k)\}_{k=1,2}$, where $\mathfrak{A}_1, \mathfrak{A}_2, P_1$ and P_2 are defined as in Theorem 1.3. If $A \in \mathfrak{A}_{hom}$, then $A - \text{diag}(\frac{1}{2}, 1)$ is not positive definite.*

This lemma will be proved by using a remarkable result, due to Šverák.

Lemma 5.2. (See [20, Corollary 1] and also [15]). *Let $\mathcal{V} : \mathbb{M}^{2 \times 2} \rightarrow [0, +\infty)$ be the continuous function defined by*

$$\mathcal{V}(\Lambda) = \begin{cases} \det \mathcal{E}(\Lambda) & \text{if } \Lambda \text{ is positive definite,} \\ 0 & \text{otherwise,} \end{cases} \quad (5.1)$$

where $\mathcal{E}(\Lambda)$ denotes the symmetric part of Λ . Let $v_h \in W^{1,\infty}(\Omega, \mathbb{R}^2)$ be a sequence such that

- i) $v_h \rightharpoonup Ax$ weakly* in $W^{1,\infty}(\Omega, \mathbb{R}^2)$;
- ii) $\text{dist}(\nabla v_h, \mathbb{M}_{sym}^{2 \times 2}) \rightarrow 0$ in measure;
- ii) ∇v_h generates a Young measure $\mu \in \mathcal{Y}(\Omega, \mathbb{M}^{2 \times 2})$.

Then $\int_{\mathbb{M}^{2 \times 2}} \mathcal{V}(\Lambda) d\mu_x(\Lambda) \geq \mathcal{V}(A)$ for a.e. $x \in \Omega$.

Proof of Lemma 5.1. Define as usual $\mathcal{D}(y, \Lambda) := \chi_{P_1}(y) \text{dist}(\Lambda, \mathfrak{A}_1) + \chi_{P_2}(y) \text{dist}(\Lambda, \mathfrak{A}_2)$. Let $\varepsilon_h \rightarrow 0^+$ and let u_h be a sequence such that $u_h \rightharpoonup Ax$ weakly* in $W^{1,\infty}(\Omega, \mathbb{R}^2)$ and $\mathcal{D}(\langle \frac{x}{\varepsilon_h}, \nabla u_h(x) \rangle) \rightarrow 0$

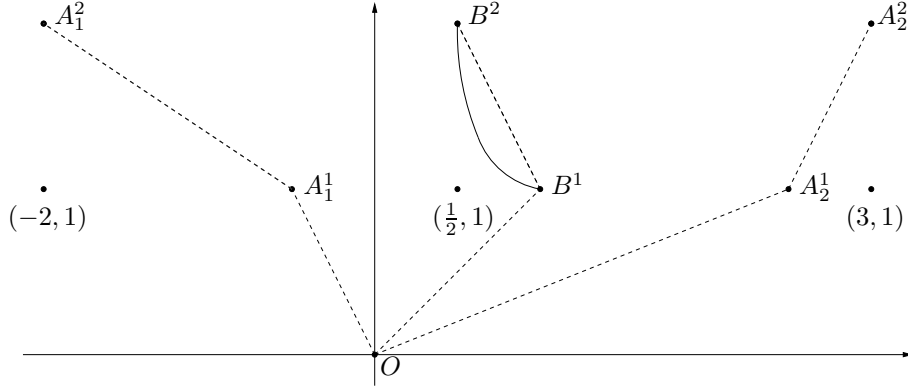


FIGURE 1. A representation of \mathfrak{A}_1 , \mathfrak{A}_2 and \mathfrak{B} in \mathbb{R}^2 , identified with the set of the diagonal matrices. The arc joining B^1 and B^2 represents the set $\{\text{diag}(b_1(t), b_2(t)) : t \in (0, 1)\}$.

in measure. Passing to a subsequence if necessary, we can assume that ∇u_h generates a two-scale Young measure ν . By Proposition 2.10, we have $\text{supp } \nu_{(x,y)} \subseteq \mathfrak{A}_k$ for *a.e.* $(x, y) \in \Omega \times P_k$ ($k = 1, 2$).

Let observe now that $\text{diag}(\frac{1}{2}, 1) = \frac{1}{2} [\text{diag}(-2, 1) + \text{diag}(3, 1)]$ and that the matrices belonging to $\mathfrak{A}_1 - \text{diag}(-2, 1)$ and $\mathfrak{A}_2 - \text{diag}(3, 1)$ are not positive definite. In particular, if \mathcal{V} is defined as in (5.1), then

$$\begin{aligned} \int_{\mathbb{M}^{2 \times 2}} \mathcal{V}(\Lambda - \text{diag}(-2, 1)) d\nu_{(x,y)}(\Lambda) &= 0 \quad \text{for } a.e. (x, y) \in \Omega \times P_1 \\ \int_{\mathbb{M}^{2 \times 2}} \mathcal{V}(\Lambda - \text{diag}(3, 1)) d\nu_{(x,y)}(\Lambda) &= 0 \quad \text{for } a.e. (x, y) \in \Omega \times P_2. \end{aligned} \quad (5.2)$$

Let $v_h \in W^{1,\infty}(\square, \mathbb{R}^2)$ be the sequence defined by

$$\begin{aligned} v_h(x)_1 &:= u_h(x)_1 - \frac{5}{2} \varepsilon_h \left| \left\langle \frac{x}{\varepsilon_h} \right\rangle - \frac{1}{2} \right| - \frac{1}{2} x_1 \\ v_h(x)_2 &:= u_h(x)_2 - x_2. \end{aligned} \quad (5.3)$$

It is trivial to check that $\nabla v_h(x) = \nabla u_h(x) - \chi_{P_1}(\langle \frac{x}{\varepsilon_h} \rangle) \text{diag}(-2, 1) - \chi_{P_2}(\langle \frac{x}{\varepsilon_h} \rangle) \text{diag}(3, 1)$ and $\text{dist}(\nabla v_h(x), \mathbb{M}_{sym}^{2 \times 2}) \rightarrow 0$ in measure. Moreover by Riemann-Lebesgue lemma follows that $v_h \rightharpoonup (A - \text{diag}(\frac{1}{2}, 1))x$ weakly* in $W^{1,\infty}(\Omega, \mathbb{R}^2)$. Passing to a subsequence if necessary, we can suppose that ∇v_h generates a Young measure μ .

Consider the function

$$\mathcal{W}(y, \Lambda) := \chi_{P_1}(y) \mathcal{V}(\Lambda - \text{diag}(-2, 1)) + \chi_{P_2}(y) \mathcal{V}(\Lambda - \text{diag}(3, 1)).$$

By construction of the sequence v_h , we have $\mathcal{W}(\langle \frac{x}{\varepsilon_h} \rangle, \nabla u_h(x)) = \mathcal{V}(\nabla v_h(x))$. Moreover, by Theorem 2.8 and equalities (5.2)

$$\lim_{h \rightarrow +\infty} \int_{\Omega} \mathcal{W}(\langle \frac{x}{\varepsilon_h} \rangle, \nabla u_h(x)) dx = \int_{\Omega \times \square} \left(\int_{\mathbb{M}^{2 \times 2}} \mathcal{W}(y, \Lambda) d\nu_{(x,y)}(\Lambda) \right) dx dy = 0$$

while by Theorem 2.2 and Lemma 5.2

$$\lim_{h \rightarrow +\infty} \int_{\Omega} \mathcal{V}(\nabla v_h(x)) dx = \int_{\Omega} \left(\int_{\mathbb{M}^{2 \times 2}} \mathcal{V}(\Lambda) d\mu_x(\Lambda) \right) dx \geq \int_{\Omega} \mathcal{V}(A - \text{diag}(\frac{1}{2}, 1)) dx.$$

Since the last integrand vanishes, $\mathcal{V}(A - \text{diag}(\frac{1}{2}, 1)) = 0$, *i.e.*, the matrix $A - \text{diag}(\frac{1}{2}, 1)$ is not positive definite. \blacklozenge

Proof of Theorem 1.3. Let $\mathfrak{C} = \{C^1, C^2, C^3\} \subseteq \mathbb{M}^{2 \times 2}$ and let ψ be the symmetric bilinear form on $\mathbb{M}^{2 \times 2}$ determined by $\det \Lambda = \frac{1}{2} \psi(\Lambda, \Lambda)$. We can write for all $(t_1, t_2, t_3) \in \mathbb{R}^3$

$$\begin{aligned} \sum_{k,j=1}^3 t_k t_j \det(C_k - C_j) &= \sum_{k,j=1}^3 t_k t_j \psi(C_k, C_k) - \sum_{k,j=1}^3 t_k t_j \psi(C_k, C_j) \\ &= 2 \sum_{k,j=1}^3 t_k t_j \det(C_k) - 2 \det\left(\sum_{k=1}^3 t_k C_k\right). \end{aligned}$$

In particular

$$\begin{aligned} \mathfrak{C}^{pc} &= \left\{ \sum_{k=1}^3 t_k C_k : t_k \geq 0, \sum_{k=1}^3 t_k = 1 \text{ and } \det\left(\sum_{k=1}^3 t_k C_k\right) = \sum_{k=1}^3 t_k \det C_k \right\} \\ &= \left\{ \sum_{k=1}^3 t_k C_k : t_k \geq 0, \sum_{k=1}^3 t_k = 1 \text{ and } \sum_{k,j=1}^3 t_k t_j \det(C_k - C_j) = 0 \right\}. \end{aligned}$$

By the formula above, since $\det(\Lambda - \Lambda')$ does not change sign on $\mathfrak{A}_1 \times \mathfrak{A}_1$ and $\mathfrak{A}_2 \times \mathfrak{A}_2$, it follows that

$$\mathfrak{A}_1 = \mathfrak{A}_1^{pc} \quad \text{and} \quad \mathfrak{A}_2 = \mathfrak{A}_2^{pc}.$$

Moreover, an elementary computation shows that $\mathfrak{B}^{pc} = \mathfrak{B} \cup \{\text{diag}(b_1(t), b_2(t)) : t \in (0, 1)\}$, where b_1, b_2 are defined by

$$b_1(t) := \frac{-3t + 2 + \sqrt{9t^2 - 4t + 4}}{4} \quad \text{and} \quad b_2(t) := \frac{3t + 2 + \sqrt{9t^2 - 4t + 4}}{4}.$$

By Lemma 4.5, there exist two polyconvex functions $\mathcal{W}_1, \mathcal{W}_2 : \mathbb{M}^{2 \times 2} \rightarrow [0, +\infty)$ p -coercive and with p -growth such that

$$\mathfrak{A}_1 = \mathcal{W}_1^{-1}(0) \quad \text{and} \quad \mathfrak{A}_2 = \mathcal{W}_2^{-1}(0).$$

Fixed $\varepsilon_h \rightarrow 0^+$ and $k \in \{1, 2\}$, it is easy to build as in (5.3) a sequence such that $u_h \rightharpoonup B^k x$ weakly* in $W^{1,\infty}(\Omega, \mathbb{R}^2)$ and $\nabla u_h(x) = \chi_{P_1}(\langle \frac{x}{\varepsilon_h} \rangle) A_1^k + \chi_{P_2}(\langle \frac{x}{\varepsilon_h} \rangle) A_2^k$. Therefore

$$\mathfrak{B} \subseteq \mathfrak{A}_{hom}.$$

On the other hand, noted that $b_1(t) > \frac{1}{2}$ and $b_2(t) > 1$ for all $t \in (0, 1)$, we have from Lemma 5.1

$$\mathfrak{B}^{pc} \not\subseteq \mathfrak{A}_{hom}.$$

Finally, the homogenized integrand \mathcal{W}_{hom} related to $\mathcal{W}(y, \Lambda) := \chi_{P_1}(y) \mathcal{W}_1(\Lambda) + \chi_{P_2}(y) \mathcal{W}_2(\Lambda)$ cannot be polyconvex because by Theorem 1.2 $\mathfrak{A}_{hom} = \mathcal{W}_{hom}^{-1}(0)$. \blacklozenge

Remark 5.3. The polyconvex hull of a compact set $\mathfrak{C} \subseteq \mathbb{M}^{2 \times 2}$ can be characterized as

$$\mathfrak{C}^{pc} = \left\{ \int_{\mathbb{M}^{2 \times 2}} \Lambda \, d\nu : \nu \in \mathcal{P}(\mathbb{M}^{2 \times 2}), \text{ supp } \nu \subseteq \mathfrak{C} \text{ and } \int_{\mathbb{M}^{2 \times 2}} \det \Lambda \, d\nu = \det \int_{\mathbb{M}^{2 \times 2}} \Lambda \, d\nu \right\}.$$

Thanks to this characterization, the polyconvexity of \mathfrak{A}_1 and \mathfrak{A}_2 can be derived by [21, Lemma 3]. Actually, our proof is a simple adaptation.

Remark 5.4. Since the matrices in $\mathfrak{A}_2^{co} - \text{diag}(3, 1)$ are not positive definite, by a straightforward modification of Lemma 5.1 we infer that the homogenized set related to $\{(\mathfrak{A}_1, P_1), (\mathfrak{A}_2^{co}, P_2)\}$ is not polyconvex. Consequently, the homogenized integrand related to $\chi_{P_1}(y) \mathcal{W}_1(\Lambda) + \chi_{P_2}(y) \text{dist}^p(\Lambda, \mathfrak{A}_2^{co})$ cannot be polyconvex. This proves that also by mixing a polyconvex function and a convex function both p -coercive and with p -growth, loss of polyconvexity can occur in the homogenization process.

ACKNOWLEDGMENTS

I wish to thank Gianni Dal Maso, Antonio DeSimone and Enzo Nesi for several stimulating discussions on the topic.

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