WHERE BEST TO PLACE A DIRICHLET CONDITION IN AN ANISOTROPIC MEMBRANE?

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ABSTRACT. We study a shape optimization problem for the first eigenvalue of an elliptic operator in divergence form, with non constant coefficients, over a fixed domain Ω. Dirichlet conditions are imposed along $\partial\Omega$ and, in addition, along a set Σ of prescribed length (1-dimensional Hausdorff measure). We look for the best shape and position for the supplementary Dirichlet region Σ in order to maximize the first eigenvalue. The limit distribution of the optimal sets, as their prescribed length tends to infinity, is characterized via Γ-convergence of suitable functionals defined over varifolds: the use of varifolds, as opposed to probability measures, allows one to keep track of the local orientation of the optimal sets (which comply with the anisotropy of the problem), and not just of their limit distribution.

1. INTRODUCTION

We investigate an optimization problem for the first Dirichlet eigenvalue of an elliptic operator in divergence form, where the unknown is a compact set Σ of prescribed length L, along which a homogeneous Dirichlet condition is imposed. Our interest is focused on the behavior of the optimal sets, when the prescribed length $L \to \infty$.

The case of the Laplacian was studied in [19]. Here, on the contrary, the elliptic operator is anisotropic, and this requires the introduction of new ideas and techniques: among them, the main novelty is the use of *varifolds* (instead of probability measures), as a tool to properly handle the anisotropy of the problem.

Throughout we shall assume, without further reference, that we are given:

- (i) a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$ (we do not assume Ω to be simply connected);
- (ii) a 2×2 symmetric, positive definite, matrix-valued function $A(x) = [a_{ij}(x)]$, defined for $x \in \overline{\Omega}$ and continuous there.

Observe that $A(x)$ is *uniformly elliptic*, i.e. for some constant $C > 0$

$$(1) \qquad C^{-1}|y|^2 \leq \langle A(x)y, y \rangle \leq C|y|^2 \quad \forall y \in \mathbb{R}^2, \quad \forall x \in \overline{\Omega}.$$

For every $L > 0$, the class of admissible sets (the "supplementary Dirichlet regions") is defined as follows:

$$(2) \qquad \mathcal{A}_L(\Omega) := \left\{ \Sigma \subset \overline{\Omega} \ : \ \Sigma \text{ is a continuum and } \mathcal{H}^1(\Sigma) \leq L \right\}$$

(throughout, "continuum" stands for "non-empty, connected, compact set" while $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure). We are interested in the

maximization problem

$$(3) \qquad\qquad \max\left\{\lambda_1^A(\Omega \setminus \Sigma): \ \Sigma \in \mathcal{A}_L(\Omega)\right\},$$

where the first eigenvalue $\lambda_1^A(\Omega \setminus \Sigma)$ is given by

$$(4) \qquad\qquad \lambda_1^A(\Omega \setminus \Sigma) = \min_{\substack{u \in H_0^1(\Omega \setminus \Sigma) \\ u \neq 0}} \frac{\int_\Omega \langle A(x)\nabla u(x), \nabla u(x)\rangle dx}{\int_\Omega u(x)^2 dx}$$

(since any admissible Σ has zero Lebesgue measure, integration over Ω is equivalent to integration over $\Omega \setminus \Sigma$). For fixed $\Sigma \in \mathcal{A}_L(\Omega)$, the coefficient matrix $A(x)$ gives rise, in the open set $\Omega \setminus \Sigma$, to the elliptic operator $-\operatorname{div} A(x)\nabla u$: the number $\lambda_1^A(\Omega \setminus \Sigma)$ is then the first eigenvalue relative to this operator, with a homogeneous Dirichlet condition along $\Sigma \cup \partial\Omega$. Every set $\Sigma \in \mathcal{A}_L(\Omega)$ is rectifiable and (except for the uninteresting case where it is a singleton) it has a *positive capacity*: therefore, the condition $u \in H_0^1(\Omega \setminus \Sigma)$ is in fact equivalent to $u \in H_0^1(\Omega)$ with the *additional* condition that $u = 0$ along Σ, a.e. with respect to the Hausdorff measure $\mathcal{H}^1$. Thus, if Ω is a membrane fixed along its boundary $\partial\Omega$, Σ can be interpreted as a stiffening rib, that will increase the fundamental frequency of the membrane. Observe that, by *monotonicity* with respect to the domain, the larger Σ, the higher the first eigenvalue.

In (2), the *length constraint* $\mathcal{H}^1(\Sigma) \leq L$, combined with the connectedness assumption, prevent Σ to become too large and spread out over Ω: these constraints can also be seen as a bound on the total resources available, in order to raise the membrane frequency (for related problems in the same setting, see e.g. [8, 14]). Without any length constraint, the best choice would be $\Sigma = \overline{\Omega}$, and the problem would be trivial: in (3), however, the optimal sets Σ_L still tend to *saturate* Ω as much as possible, compatibly with the constraint that $\Sigma_L \in \mathcal{A}_L(\Omega)$. In this paper we shall investigate *how* this saturation occurs as $L \to \infty$, answering questions such as: with what limit density (length per unit area) will Σ_L saturate Ω? With what local orientation?

Due to Gołab and Blaschke theorems (see [4]), $\mathcal{A}_L(\Omega)$ is compact with respect to the Hausdorff convergence, while the first eigenvalue λ_1^A is continuous (see [17, 13]). Therefore, the existence of an optimal set Σ_L for problem (3) is immediate:

Theorem 1. *For every $L > 0$ there exists a maximizer in* (3). *Moreover, every maximizer Σ_L satisfies $\mathcal{H}^1(\Sigma_L) = L$.*

The last part of the claim is typical of constrained problems that obey monotonicity with respect to domain inclusion: if $\mathcal{H}^1(\Sigma) < L$, then $\lambda_1^A(\Omega \setminus \Sigma)$ could be increased, e.g. by attaching to Σ some short segment (see [7, 8, 19]). Observe, en passant, that the corresponding *minimization* problem is trivial: the optimal sets are all singletons $\Sigma = \{x_0\}$ and all sets $\Sigma \subseteq \partial\Omega$ (in both cases, one has $\lambda_1^A(\Omega \setminus \Sigma) = \lambda_1^A(\Omega)$ which is clearly optimal).

Our goal is to study the limit behavior of the optimal sets Σ_L as $L \to \infty$, via Γ-convergence. If $A(x) = \sigma(x)I$ is a multiple of the identity matrix, i.e. if the membrane Ω is *isotropic* (though not necessarily uniform), this has been investigated in a previous paper [19]. Passing from $A(x) = \sigma(x)I$ to an anisotropic matrix $A(x)$, however, is not a merely technical extension: indeed, several new ideas are required, and even the final expression for the Γ-limit functional is not easy to figure out.

In [19], along lines typical of this kind of problems (see [5, 6, 8, 9, 14]), the main idea was to associate, with every admissible set $\Sigma_L \in \mathcal{A}_L(\Omega)$, the *probability measure* μ_Σ defined as

$$(5) \qquad \mu_\Sigma := \frac{\mathcal{H}^1 \llcorner \Sigma}{\mathcal{H}^1(\Sigma)},$$

where $\mathcal{H}^1 \llcorner \Sigma$ is the one-dimensional Hausdorff measure restricted to Σ. If Σ_L is a solution to (3) and $L \to \infty$, knowledge of the weak-* limits μ of the corresponding measures μ_{Σ_L} in the space of probability measures $\mathcal{P}(\overline{\Omega})$, provides information on *how* the optimal sets Σ_L tend to saturate Ω in the limit. In concrete terms, one can define the functionals $G_L : \mathcal{P}(\overline{\Omega}) \to [0, +\infty]$

$$(6) \qquad G_L(\mu) = \begin{cases} \dfrac{L^2}{\lambda_1^A(\Omega \setminus \Sigma)} & \text{if } \mu = \mu_\Sigma \text{ for some } \Sigma \in \mathcal{A}_L(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

and investigate their Γ-limit in $\mathcal{P}(\overline{\Omega})$ as $L \to \infty$. Indeed, for fixed L, the *maximization* problem (3) is equivalent to the *minimization* of G_L, while the factor L^2 is a renormalization that allows for a nontrivial Γ-limit. If the coefficient matrix $A(x) = \sigma(x)I$ is isotropic and $\sigma(x)$ is continuous, it was proved in [19] that the functionals G_L Γ-converge (with respect to the weak-* topology on $\mathcal{P}(\overline{\Omega})$) to the functional

$$G_\infty(\mu) := \frac{1}{\pi^2} \operatorname*{ess\,sup}_{x \in \Omega} \frac{1}{f_\mu(x)^2 \sigma(x)},$$

where $f_\mu(x)$ is the density of μ w.r.to the Lebesgue measure (note the interaction between $f_\mu(x)$ and $\sigma(x)$, the Young modulus of the membrane). This functional is minimized by a *unique* probability measure, an L^1 function proportional to $\sigma(x)^{-1/2}$: as a consequence ([19]), one infers that for large L the optimal sets Σ_L have, at small scales, a comb-shaped structure, essentially made up of parallel lines whose spacing is locally proportional to $\sigma(x)^{1/2}$. The local orientation of the comb teeth is, on the other hand, *irrelevant*, and rotating an optimal pattern preserves optimality: this clearly reflects the isotropy of the coefficient matrix $A(x) = \sigma(x)I$.

When the matrix $A(x)$ is *anisotropic*, the previous analysis is no longer valid. One still expects optimal sets to look like comb-shaped patterns at small scales, but their local orientation will now depend on $A(x)$: heuristically, since in the smooth case the gradient $\nabla u(x)$ of the first eigenfunction $u \in H_0^1(\Omega \setminus \Sigma)$ is roughly directed as the normal unit vector $\xi(x)$ at $x \in \Sigma$, one expects Σ to be locally directed in such a way that $\xi(x)$ is parallel to an eigenvector of $A(x)$, namely the one relative to its *largest* eigenvalue, so that the quadratic form $\langle A(x)\nabla u, \nabla u \rangle$ is (locally) maximized in (4). On the other hand, the probability measures μ in (5) and their weak-* limits keep *no track* of the orientation of the optimal Σ_L: they just keep memory of the density (length per unit area) of Σ_L inside Ω. In other words, to fully understand the anisotropic case we have to associate, with every admissible $\Sigma \in \mathcal{A}_L(\Omega)$, a more complex object than a measure $\mu \in \mathcal{P}(\overline{\Omega})$, possibly one that carries information also on the *unit normal* to Σ, and study Γ-convergence in that setting.

Roughly speaking, the main idea is to associate, with every $\Sigma \in \mathcal{A}_L(\Omega)$, a probability measure θ_Σ in the *product space* $\overline{\Omega} \times \mathbb{S}^1$, defined via integration by

$$(7) \quad \int_{\overline{\Omega} \times \mathbb{S}^1} \varphi(x, y) d\theta_\Sigma(x, y) := \frac{1}{\mathcal{H}^1(\Sigma)} \int_\Sigma \varphi(x, \xi_\Sigma(x)) \, d\mathcal{H}^1(x), \quad \varphi \in C_{\mathrm{sym}}(\overline{\Omega} \times \mathbb{S}^1),$$

where $\xi_\Sigma(x)$ is the unit normal vector to Σ at $x \in \Sigma$, while

$$C_{\mathrm{sym}}(\overline{\Omega} \times \mathbb{S}^1) := \left\{ \varphi \in C(\overline{\Omega} \times \mathbb{S}^1) \mid \varphi(x, y) = \varphi(x, -y) \quad \forall x \in \overline{\Omega}, \forall y \in \mathbb{S}^1 \right\}$$

is the space of continuous functions with *antipodal* symmetry. Since every set $\Sigma \in \mathcal{A}_L(\Omega)$ is 1-rectifiable, a unit normal $\pm\xi_\Sigma(x)$ is well defined, up to the orientation, at $\mathcal{H}^1$-a.e. $x \in \Sigma$, and $\xi_\Sigma(x)$ in (7) denotes any *measurable selection* of this normal field: due to the antipodal symmetry of φ, the second integral in (7) does *not* depend on the particular selection.

In fact, (7) defines θ_Σ as an element of the dual space of $C_{\mathrm{sym}}(\overline{\Omega} \times \mathbb{S}^1)$, that is, as a (probability) measure over $\overline{\Omega} \times \mathbb{P}^1$, where $\mathbb{P}^1$ is the *projective space*. In other words, (7) defines θ_Σ as the 1-dimensional *varifold* induced by Σ (see [1, 16]), with the proviso that we work with the *normal* instead of the more usual tangent space.

We denote by $\mathcal{V}_1(\overline{\Omega})$ the space of 1-dimensional varifolds with unit mass, that is, probability measures over $\overline{\Omega} \times \mathbb{P}^1$, endowed with the usual weak-* topology. Throughout, however, we shall always consider a varifold $\theta \in \mathcal{V}_1(\overline{\Omega})$ as an *equivalence class* of probability measures on $\overline{\Omega} \times \mathbb{S}^1$, two measures being equivalent $\iff$ they induce the same linear functional on $C_{\mathrm{sym}}(\overline{\Omega} \times \mathbb{S}^1)$: thus, by choosing a representative in its equivalence class, we can still treat $\theta \in \mathcal{V}_1(\overline{\Omega})$ as a probability measure over $\overline{\Omega} \times \mathbb{S}^1$. With this agreement (useful to avoid the technical language of varifolds), the weak-* convergence of a sequence $\{\theta_L\}$ in $\mathcal{V}_1(\overline{\Omega})$ to a varifold $\theta \in \mathcal{V}_1(\overline{\Omega})$ takes the concrete form

$$(8) \qquad \lim_{L \to \infty} \int_{\overline{\Omega} \times \mathbb{S}^1} \varphi(x, y) \, d\theta_L = \int_{\overline{\Omega} \times \mathbb{S}^1} \varphi(x, y) \, d\theta \quad \forall \varphi \in C_{\mathrm{sym}}(\overline{\Omega} \times \mathbb{S}^1).$$

Now we are ready to state our main result. By analogy with (6), for $L > 0$ we define the functional $F_L : \mathcal{V}_1(\overline{\Omega}) \to [0, \infty]$ as

$$(9) \qquad F_L(\theta) = \begin{cases} \dfrac{L^2}{\lambda_1^A(\Omega \setminus \Sigma)} & \text{if } \theta = \theta_\Sigma \text{ for some } \Sigma \in \mathcal{A}_L(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where θ_Σ is as in (7). The definition of the Γ-limit functional F_∞, which of course should depend on the coefficient matrix $A(x)$, is more involved.

Definition 2. *Given $\theta \in \mathcal{V}_1(\overline{\Omega})$, by choosing a representative in its equivalence class we can regard θ as an element of $\mathcal{P}(\overline{\Omega} \times \mathbb{S}^1)$, and we can consider its* first *marginal μ, i.e. the probability measure over $\overline{\Omega}$ defined by*

$$\mu(E) := \theta(E \times \mathbb{S}^1), \quad \text{for every Borel set } E \subseteq \overline{\Omega}.$$

Then, by well known results (see [2]), one can disintegrate *θ as $\mu \otimes \nu_x$, where $\{\nu_x\}$ is a μ-measurable family of probability measures over $\mathbb{S}^1$ defined for μ-a.e. $x \in \overline{\Omega}$. This means that*

$$\int_{\overline{\Omega} \times \mathbb{S}^1} \varphi(x, y) d\theta = \int_{\overline{\Omega}} \left(\int_{\mathbb{S}^1} \varphi(x, y) \, d\nu_x(y) \right) d\mu(x), \quad \forall \varphi \in C(\overline{\Omega} \times \mathbb{S}^1).$$

The measure μ depends only on θ as an element of $\mathcal{V}_1(\overline{\Omega})$ (not on the choice of its representative in $\mathcal{P}(\overline{\Omega} \times \mathbb{S}^1)$), while the measures ν_x may well depend on the particular representative. However, the integrals

$$x \mapsto \int_{\mathbb{S}^1} \sqrt{\langle A(x)y, y \rangle} \, d\nu_x(y) \quad \text{(defined for } \mu\text{-a.e. } x \in \overline{\Omega})$$

are independent of the particular representative of θ, since the function $(x, y) \mapsto \langle A(x)y, y \rangle^{1/2}$ belongs to $C_{sym}(\overline{\Omega} \times \mathbb{S}^1)$. As a consequence, if $f_\mu(x)$ denotes the density of μ (w.r.to the Lebesgue measure on $\overline{\Omega}$), the functional

$$(10) \qquad F_\infty(\theta) := \frac{1}{\pi^2} \operatorname*{ess\,sup}_{x \in \Omega} \frac{1}{\left(f_\mu(x) \int_{\mathbb{S}^1} \sqrt{\langle A(x)y, y \rangle} \, d\nu_x(y) \right)^2}, \qquad \theta \in \mathcal{V}_1(\overline{\Omega})$$

is well defined over $\mathcal{V}_1(\overline{\Omega})$ (with values in $[0, +\infty]$), since it only depends on θ as an element of $\mathcal{V}_1(\overline{\Omega})$.

The main result of the paper is then the following.

Theorem 3. *As $L \to \infty$, the functionals F_L defined in (9) Γ-converge, with respect to the weak-* topology on $\mathcal{V}_1(\overline{\Omega})$, to the functional F_∞ defined in (10).*

This has several consequences, as soon as we single out all those varifolds $\theta \in \mathcal{V}_1(\overline{\Omega})$ that minimize the functional F_∞. For $x \in \overline{\Omega}$, we let $\sigma_{\max}(x)$ denote the *largest eigenvalue* of $A(x)$, while $\sigma_{\min}(x)$ denotes its smallest eigenvalue.

Theorem 4. *A varifold $\theta_\infty \in \mathcal{V}_1(\overline{\Omega})$ is a minimizer of F_∞ if and only if it can be disintegrated as $\theta_\infty = f_\infty(x) \, dx \otimes \nu_x^\infty$, where f_∞ is the L^1 function defined as*

$$(11) \qquad f_\infty(x) = \frac{1/\sqrt{\sigma_{\max}(x)}}{\int_\Omega 1/\sqrt{\sigma_{\max}(x)} dx}$$

while, for a.e. $x \in \Omega$, $\nu_x^\infty \in \mathcal{P}(\mathbb{S}^1)$ is any probability measure supported on a set made up of (normalized) eigenvectors of $A(x)$, relative to $\sigma_{\max}(x)$.

The minimizer is unique if and only if $A(x)$ is purely anisotropic, *that is, if and only if $\sigma_{\min}(x) < \sigma_{\max}(x)$ occurs at a.e. $x \in \Omega$. In any case, the first marginal μ of a minimizer θ_∞ is necessarily the function in (11).*

As $\mathcal{V}_1(\overline{\Omega})$ is compact in the weak-* topology, from standard Γ-convergence theory (see [11]) we have

Corollary 5. *For $L > 0$, let Σ_L be a maximizer of problem (3) and let θ_{Σ_L} be the associated varifolds, according to (7). Then, as $L \to \infty$, $\theta_{\Sigma_L} \rightharpoonup \theta_\infty$ (up to subsequences) in the weak-* topology of $\mathcal{V}_1(\overline{\Omega})$, where θ_∞ is a minimizer of F_∞.*

In spite of its abstract formulation, this corollary has interesting consequences in the applications. In particular, it enables us to determine the asymptotic density (length per unit area) of the optimal sets Σ_L, as well as their *local* direction (distribution of the unit normal on the projective space), when $L \to \infty$:

Theorem 6. *For $L > 0$, let Σ_L be a maximizer of problem (3). For every square $Q \subset \Omega$,*

$$(12) \qquad \lim_{L \to \infty} \frac{\mathcal{H}^1(\Sigma_L \cap Q)}{\mathcal{H}^1(\Sigma_L)} = \int_Q f_\infty(x) \, dx \qquad \text{(optimal density of length)}$$

where f_∞ is as in (11). Moreover, if Q is contained in the anisotropy region where $\sigma_{\min}(x) < \sigma_{\max}(x)$, for every $\psi \in C(\mathbb{S}^1)$ such that $\psi(-y) = \psi(y)$ one has

$$(13) \qquad \lim_{L \to \infty} \frac{1}{\mathcal{H}^1(Q \cap \Sigma_L)} \int_{Q \cap \Sigma_L} \psi\big(\xi_L(x)\big) \, d\mathcal{H}^1(x) = \frac{\int_Q f_\infty(x) \psi\big(\xi(x)\big) \, dx}{\int_Q f_\infty(x) \, dx},$$

where $\xi_L(x)$ is the unit normal to Σ_L at $x \in \Sigma_L$, while $\xi(x)$ is the (unique up to the orientation) eigenvector of $A(x)$ relative to $\sigma_{\max}(x)$. Finally,

$$(14) \qquad \lim_{L \to \infty} \frac{L^2}{\lambda_1^A(\Omega \setminus \Sigma_L)} = \frac{\left(\int_\Omega 1/\sqrt{\sigma_{\max}(x)} dx\right)^2}{\pi^2}.$$

The meaning of (12) is that, if L is large, in order to solve (3) one should spread Σ over Ω with a density (length per unit area) roughly proportional to $f_\infty(x)$. Moreover, by (13), the optimal set Σ_L should be directed in such a way that its unit normal $\xi_L(x)$ is essentially parallel to the eigenvector $\xi(x)$ relative to the largest eigenvalue of $A(x)$ (this is uniquely determined only in the region where $A(x)$ is anisotropic).

The initial assumption on the continuity of $A(x)$ plays an important role in our proofs, since it enables us to localize our estimates, as if $A(x)$ were constant at small scales (the analog problem when $A(x)$ is merely measurable and satisfies (1) is entirely open). On the other hand, all our results are easily extended to cover the case where $A(x)$ is *piecewise continuous*, i.e. continuous over disjoint open sets Ω_i with smooth boundaries, such that $\overline{\Omega} = \bigcup \overline{\Omega}_i$. This extension would be a merely technical one and is not pursued in detail.

Finally, we point out that regularity of the optimal sets Σ_L is an open problem. In particular, it is not known whether Σ_L satisfies no-loop or blow-up properties, similar to those of the minimizers of the irrigation problem (see [10, 15]).

Notation For a Borel set $E \subset \mathbb{R}^2$ we denote by $|E|$ the two-dimensional Lebesgue measure of E and by $\mathcal{H}^1(E)$ its one-dimensional Hausdorff measure. If X is a compact space, $\mathcal{P}(X)$ denotes the space of all (Borel) probability measures over X. If μ is a measure, "μ-a.e." is an abbreviation of "μ-almost everywhere" (if μ is omitted, Lebesgue measure is understood), while "w.r.to μ" stands for "with respect to μ".

2. Some auxiliary estimates

In this section we prove some preliminary results on the first Dirichlet eigenvalue, when the coefficient matrix $A(x)$ is constant and the Dirichlet condition is prescribed along a generic compact set D.

More precisely, let $D \subset \overline{\Omega}$ be a compact set with finitely many connected components, such that $0 < \mathcal{H}^1(D) < \infty$: such a set is 1-rectifiable (see [4]) and has positive capacity, therefore given $u \in H^1(\Omega)$ the Dirichlet condition

$$u(x) = 0 \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in D$$

is meaningful and defines a nontrivial closed subspace of $H^1(\Omega)$ (it can be interpreted as a trace condition, since by rectifiability every connected component of D can be covered by a Lipschitz curve). Then, one can define the first Dirichlet eigenvalue of the domain Ω (relative to the coefficient matrix $A(x)$) with Dirichlet condition along D, as follows:

$$(15) \qquad \lambda_1^A(\Omega; D) := \min_{\substack{u \in H^1(\Omega) \setminus \{0\} \\ u \equiv 0 \text{ on } D}} \frac{\int_\Omega \langle A(x) \nabla u(x), \nabla u(x) \rangle dx}{\int_\Omega u(x)^2 dx}.$$

Observe that this is more general than (4), where the minimization takes place in a H_0^1 environment: given Σ, the eigenvalue $\lambda_1^A(\Omega \setminus \Sigma)$ in (4) is a particular case of

$\lambda_1^A(\Omega; D)$, namely when $D = \Sigma \cup \partial\Omega$ (as Σ is connected while $\partial\Omega$, being Lipschitz, has finitely many connected components, the same is true of D).

When $A(x) \equiv M$ is constant, equal to a 2×2 positive definite matrix M independent of x, it is possible to give an upper bound to $\lambda_1^M(\Omega; D)$ in terms of some geometric quantities, among which an important role is played by the "Riemannian" length

$$(16) \qquad \mathcal{H}_M^1(D) := \int_D \sqrt{\langle M\xi(x), \xi(x)\rangle}\, d\mathcal{H}^1(x),$$

where $\xi(x)$ is (a measurable selection of) the unit *normal* to D at the point $x \in D$ (by rectifiability, this normal is well defined, up to the orientation, at $\mathcal{H}^1$-a.e. $x \in D$).

Theorem 7. *Let M be a positive definite 2×2 matrix, and let $D \subset \overline{\Omega}$ be a compact connected set with κ connected components, such that $0 < \mathcal{H}^1(D) < \infty$. Then*

$$(17) \qquad \lambda_1^M(\Omega; D) \leq \frac{\pi^2}{4t^2}\left(1 + \frac{\kappa\pi t \det M^{1/2}}{\mathcal{H}_M^1(D)}\right),$$

where the number t is defined as

$$(18) \qquad t = \frac{|\Omega|}{\mathcal{H}_M^1(D) + \sqrt{\mathcal{H}_M^1(D)^2 + \kappa\pi|\Omega|\det M^{1/2}}}.$$

Proof. Set $N = M^{-1/2}$. Recalling (15), by the change of variable $z = Nx$ in the two integrals we have

$$\lambda_1^M(\Omega; D) = \min_{\substack{u \in H^1(\Omega) \\ u \equiv 0 \text{ on } D}} \frac{\int_\Omega \langle M\nabla u(x), \nabla u(x)\rangle dx}{\int_\Omega u(x)^2 dx} = \min_{\substack{v \in H^1(N\Omega) \\ v \equiv 0 \text{ on } ND}} \frac{\int_{N\Omega} |\nabla v(z)|^2 dz}{\int_{N\Omega} v(z)^2 dz},$$

so that $\lambda_1^M(\Omega; D)$ is the first eigenvalue of the Laplacian on the deformed domain $N\Omega$ (the image of Ω through the linear map N), with Dirichlet conditions along ND (observe that also ND has κ connected components). Therefore, applying Theorem 2.4 of [19] combined with Remark 2.5 therein, we obtain

$$(19) \qquad \lambda_1^M(\Omega; D) \leq \frac{\pi^2}{4T^2}\left(1 + \frac{\kappa\pi T}{\mathcal{H}^1(ND)}\right),$$

where

$$(20) \qquad T := \frac{|N\Omega|}{\mathcal{H}^1(ND) + \sqrt{\mathcal{H}^1(ND)^2 + \kappa\pi|N\Omega|}}.$$

Therefore, to prove (17), it suffices to relate T and t.

Using the *area formula* (see for example [2, lemma 2.91]), we have

$$(21) \qquad \mathcal{H}^1(ND) = \int_D |N\tau(x)| d\mathcal{H}^1(x),$$

where $\tau(x)$ is the unit tangent vector to D at the point $x \in D$. Now, for $\mathcal{H}^1$-a.e. $x \in D$, $\tau = \tau(x)$ is orthogonal to the unit normal $\xi = \xi(x)$ that appears in (16): therefore, since $N^2 = M^{-1}$, we have

$$|N\tau|^2 = \langle M^{-1}\tau, \tau\rangle = \det M^{-1}\langle \widehat{M}\tau, \tau\rangle = \det M^{-1}\langle M\xi, \xi\rangle$$

where $\widehat{M}$ is the cofactor matrix of M (the last equality is typical of dimension two). Taking square roots and integrating, we find from (21) and the preceding formula

$$\mathcal{H}^1(ND) = \det M^{-1/2} \int_D \sqrt{\langle M\xi(x), \xi(x)\rangle}\, d\mathcal{H}^1(x) = \det M^{-1/2}\mathcal{H}^1_M(D).$$

Moreover, $|N\Omega| = \det M^{-1/2}|\Omega|$. Plugging the last two identities in (20), we see that (19) is equivalent to (17). $\square$

The upper bound (17) will be at the basis of the Γ-liminf inequality (24), and is therefore asymptotically optimal when $\mathcal{H}^1(D) \to \infty$ (see [18] for a similar estimate for the compliance functional).

We shall also need the following result.

Theorem 8 (The first eigenvalue of thin domains). *Let $E \subset \mathbb{R}^2$ be a bounded open set contained in a strip*

$$S = \{x \in \mathbb{R}^2 \mid 0 < \langle x, \xi\rangle < h\}$$

of width $h > 0$, where ξ is the direction orthogonal to S $(|\xi| = 1|)$. If $M \in \mathbb{R}^{2\times 2}$ is positive definite, then

$$(22) \qquad \lambda_1^M(E) > \pi^2 \frac{\langle M\xi, \xi\rangle}{h^2}.$$

Proof. Performing the linear change of variable $x = M^{1/2}y$ in the integrals of the Rayleigh quotient, the anisotropic eigenvalue $\lambda_1^M(E)$ is easily seen to coincide with the first Laplace eigenvalue of a new domain $\widehat{E}$, namely

$$\lambda_1^M(E) = \min_{\substack{u \in H_0^1(E) \\ u \not\equiv 0}} \frac{\int_E \langle M\nabla u(x), \nabla u(x)\rangle\, dx}{\int_E u(x)^2\, dx} = \min_{\substack{v \in H_0^1(\widehat{E}) \\ v \not\equiv 0}} \frac{\int_{\widehat{E}} |\nabla v(y)|^2\, dy}{\int_{\widehat{E}} v(y)^2\, dy} = \lambda_1^I(\widehat{E})$$

where $\widehat{E} = M^{-1/2}E$ is now contained in the new strip $\widehat{S} = M^{-1/2}S$. Observe that

$$\widehat{S} = \{y \in \mathbb{R}^2 \mid 0 < \langle y, M^{1/2}\xi\rangle < h\},$$

so that the width of the strip $\widehat{S}$ is given by

$$(23) \qquad \widehat{h} = \frac{h}{|M^{1/2}\xi|} = \frac{h}{\sqrt{\langle M\xi, \xi\rangle}}.$$

Therefore, since $\widehat{E}$ is bounded, it can be boxed in a thin rectangle R of size $\widehat{h} \times k$ for some $k > 0$ (e.g. $k = \operatorname{diam}(\widehat{E})$). Then, by monotonicity and the explicit expression for the first Laplace eigenvalue of a rectangle, we obtain

$$\lambda_1^I(\widehat{E}) \geq \lambda_1^I(R) = \frac{\pi^2}{\widehat{h}^2} + \frac{\pi^2}{k^2} > \frac{\pi^2}{\widehat{h}^2}$$

which combined with (23) proves (22). $\square$

3. The Γ-liminf inequality

This section is devoted to proving that the Γ-liminf functional is not smaller than the functional F_∞ defined in (10).

Proposition 9 (Γ-liminf inequality). *For every varifold $\theta \in \mathcal{V}_1(\overline{\Omega})$ and every sequence $\{\theta_L\} \subset \mathcal{V}_1(\overline{\Omega})$ such that $\theta_L \rightharpoonup \theta$, it holds*

$$(24) \qquad \liminf_{L\to\infty} F_L(\theta_L) \geq F_\infty(\theta).$$

Proof. Passing if necessary to a subsequence (not relabelled), we may assume that the liminf is a *finite* limit. By (9) this implies that, for L large enough, every θ_L is of the kind $\theta_L = \theta_{\Sigma_L}$ as defined in (7), for a suitable set $\Sigma_L \in \mathcal{A}_L(\Omega)$. Therefore, by (7) and (8), the weak-* convergence $\theta_L \rightharpoonup \theta$ takes the concrete form

$$(25) \quad \lim_{L \to \infty} \frac{1}{\mathcal{H}^1(\Sigma_L)} \int_{\Sigma_L} \varphi(x, \xi_{\Sigma_L}(x)) d\mathcal{H}^1(x) = \int_{\overline{\Omega} \times \mathbb{S}^1} \varphi \, d\theta \quad \forall \varphi \in C_{\mathrm{sym}}(\overline{\Omega} \times \mathbb{S}^1),$$

where ξ_{Σ_L} is the unit normal to Σ_L. Similarly, using (9) and (10), the inequality in (24) is, after taking square roots, equivalent to

$$(26) \quad \lim_{L \to \infty} \frac{L}{\lambda_1^A(\Omega \setminus \Sigma_L)^{1/2}} \geq \frac{1}{\pi} \operatorname*{ess\,sup}_{x \in \Omega} \frac{1}{f_\mu(x) \int_{\mathbb{S}^1} \langle A(x)y, y \rangle \, d\nu_x(y)},$$

where the function f_μ and the measures $\{\nu_x\}$ are as in Definition 2.

The finiteness of the limit in (26) entails that $\lambda_1^A(\Omega \setminus \Sigma_L) \to \infty$, and this in turn forces the sets Σ_L to converge to $\overline{\Omega}$ in the Hausdorff metric (otherwise, a subsequence among the open sets $\Omega \setminus \Sigma_L$ would contain a ball $B_r(x_L)$ of fixed radius $r > 0$, and by monotonicity we would have $\lambda_1^A(\Omega \setminus \Sigma_L) \leq \lambda_1^A(B_r(x_L))$: by (1) this bound would be uniform in L, a contradiction).

Now choose an arbitrary open square Q such that $\overline{Q} \subset \Omega$. Since Σ_L is connected, the convergence $\Sigma_L \to \overline{\Omega}$ implies (see e.g. [14]) that

$$(27) \qquad \qquad \lim_{L \to \infty} \mathcal{H}^1(\Sigma_L \cap Q) = \infty$$

and forces Σ_L to cross the boundary ∂Q for large L, so that

$$(28) \qquad \qquad (\Sigma_L \cap Q) \cup \partial Q \qquad \text{is connected (if } L \text{ is large enough).}$$

Now fix a number $\varepsilon > 0$, and consider the matrix $M_Q := A(x_0)$, where x_0 is the center of the square Q: since $A(x)$ is uniformly continuous, the conditions

$$(29) \qquad \qquad \langle A(x)y, y \rangle \leq (1 + \varepsilon)^2 \langle M_Q y, y \rangle \qquad \forall x \in Q, \forall y \in \mathbb{S}^1$$

are satisfied as soon as the diameter of Q is small enough (depending only on ε). Therefore, using the inequality $L \geq \mathcal{H}^1(\Sigma_L)$, the monotonicity of the first eigenvalue with respect to domain inclusion, and (29), it follows that

$$(30) \qquad \frac{L}{\lambda_1^A(\Omega \setminus \Sigma_L)^{1/2}} \geq \frac{\mathcal{H}^1(\Sigma_L)}{\lambda_1^A(Q \setminus \Sigma_L)^{1/2}} \geq \frac{\mathcal{H}^1(\Sigma_L)}{(1 + \varepsilon)\lambda_1^{M_Q}(Q \setminus \Sigma_L)^{1/2}},$$

provided that, as we shall assume, $\mathrm{diam}(Q)$ is small enough.

Now $\lambda_1^{M_Q}(Q \setminus \Sigma_L)$ is the first eigenvalue in $H^1(Q)$, with M_Q as coefficient matrix and Dirichlet condition along $(\Sigma_L \cap Q) \cup \partial Q$: therefore, it can be estimated using Theorem 7, applied with $\Omega = Q$, $M = M_Q$ and $D = (\Sigma_L \cap Q) \cup \partial Q$ (by (28), we also have $\kappa = 1$ if L is large). Then (17), taking square roots, reads

$$(31) \qquad \qquad \lambda_1^{M_Q}(Q \setminus \Sigma_L)^{1/2} \leq \frac{\pi}{2t_L} \left(1 + \frac{\pi t_L \det M_Q^{1/2}}{\ell_L} \right)^{1/2},$$

where t_L, according to (18), is given by

$$(32) \qquad \qquad t_L = \frac{|Q|}{\ell_L + \sqrt{\ell_L^2 + \pi |Q| \det M_Q^{1/2}}},$$

while ℓ_L, according to (16), is the Riemannian length

$$(33) \qquad \ell_L = \int_{(\Sigma_L \cap Q) \cup \partial Q} \sqrt{\langle M_Q \xi_L(x), \xi_L(x) \rangle} \, d\mathcal{H}^1(x),$$

$\xi_L(x)$ being the unit normal to $(\Sigma_L \cap Q) \cup \partial Q$. We shall let $L \to \infty$ in (30) and use (31), hence we are interested in the asymptotics (as $L \to \infty$) of both t_L and ℓ_L. As $M_Q = A(x_0)$, the last equation and (1) give

$$\ell_L \geq C^{-1/2} \mathcal{H}^1\big((\Sigma_L \cap Q) \cup \partial Q\big) > C^{-1/2} \mathcal{H}^1(\Sigma_L \cap Q),$$

so that $\ell_L \to \infty$ by (27). Therefore, from (32) we find the asymptotics

$$(34) \qquad t_L \sim \frac{|Q|}{2\ell_L} \quad \text{as } L \to \infty$$

so that, letting $L \to \infty$ in (30), and using (31), (34), we obtain

$$(35) \qquad \begin{aligned} \lim_{L \to \infty} \frac{L}{\lambda_1^A(\Omega \setminus \Sigma_L)^{1/2}} &\geq \frac{1}{1+\varepsilon} \liminf_{L \to \infty} \frac{\mathcal{H}^1(\Sigma_L) \, 2t_L}{\pi} \\ &= \frac{|Q|}{(1+\varepsilon)\pi} \liminf_{L \to \infty} \frac{\mathcal{H}^1(\Sigma_L)}{\ell_L}. \end{aligned}$$

To estimate the last liminf, we get back to (33) and observe that

$$(36) \qquad \ell_L \sim \int_{\Sigma_L \cap Q} \sqrt{\langle M_Q \xi_{\Sigma_L}(x), \xi_{\Sigma_L}(x) \rangle} \, d\mathcal{H}^1(x) \quad \text{as } L \to \infty,$$

since the contribution of ∂Q to the integral in (33) is fixed, while that of $\Sigma_L \cap Q$ is dominant by (27). Now if $\eta \in C(\overline{\Omega})$ is a cutoff function such that $0 \leq \eta \leq 1$ and $\eta \equiv 1$ over $\overline{Q}$, letting

$$(37) \qquad \widehat{\varphi}(x,y) := \eta(x) \sqrt{\langle M_Q y, y \rangle}, \quad x \in \overline{\Omega}, \quad y \in \mathbb{S}^1,$$

we clearly have $\varphi \in C_{\text{sym}}(\overline{\Omega} \times \mathbb{S}^1)$. Therefore, using (36) and (25) we infer that

$$\begin{aligned} \limsup_{L \to \infty} \frac{\ell_L}{\mathcal{H}^1(\Sigma_L)} &= \limsup_{L \to \infty} \frac{1}{\mathcal{H}^1(\Sigma_L)} \int_{\Sigma_L \cap Q} \sqrt{\langle M_Q \xi_{\Sigma_L}(x), \xi_{\Sigma_L}(x) \rangle} \, d\mathcal{H}^1(x) \\ &\leq \lim_{L \to \infty} \frac{1}{\mathcal{H}^1(\Sigma_L)} \int_{\Sigma_L} \widehat{\varphi}(x, \xi_{\Sigma_L}(x)) \, d\mathcal{H}^1(x) = \int_{\overline{\Omega} \times \mathbb{S}^1} \widehat{\varphi}(x,y) \, d\theta. \end{aligned}$$

Now, recalling the properties of the cutoff function η in (37), it is possible to let $\eta(x) \downarrow \chi_{\overline{Q}}(x)$ pointwise in the last integral, and obtain by dominated convergence

$$\limsup_{L \to \infty} \frac{\ell_L}{\mathcal{H}^1(\Sigma_L)} \leq \int_{\overline{Q} \times \mathbb{S}^1} \sqrt{\langle M_Q y, y \rangle} \, d\theta = \int_{\overline{Q}} \left(\int_{\mathbb{S}^1} \sqrt{\langle M_Q y, y \rangle} \, d\nu_x(y) \right) d\mu(x),$$

where $\theta = \mu \otimes \nu_x$ is the slicing of θ as in Definition 2. Passing to reciprocals in the previous inequality, we obtain a *lower* bound for the last liminf in (35) which, plugged into (35), yields

$$(38) \qquad \lim_{L \to \infty} \frac{L}{\lambda_1^A(\Omega \setminus \Sigma_L)^{1/2}} \geq \frac{|Q|}{(1+\varepsilon)\pi \int_{\overline{Q}} \left(\int_{\mathbb{S}^1} \sqrt{\langle M_Q y, y \rangle} \, d\nu_x(y) \right) d\mu(x)}.$$

This lower bound holds for every square Q such that $\overline{Q} \subset \Omega$, and having a sufficiently small side length (depending only on ε, as to guarantee the validity of (29)). Moreover, the matrix $M_Q = A(x_0)$ where x_0 is the center of Q, so that M_Q is independent of the side length of Q. Thus, given $\varepsilon > 0$, we can first choose an arbitrary

$x_0 \in \Omega$, and then rely on the previous inequality for every Q (of sufficiently small side length) centered at x_0: it is therefore possible, for fixed x_0, to let Q shrink around x_0. On the other hand, from Lebesgue Differentiation Theorem we have

$$\lim_{Q \downarrow x_0} \frac{1}{|Q|} \int_{\overline{Q}} \left(\int_{\mathbb{S}^1} \sqrt{\langle M_Q y, y \rangle} \, d\nu_x(y) \right) d\mu(x) = f_\mu(x_0) \int_{\mathbb{S}^1} \sqrt{\langle A(x_0) y, y \rangle} \, d\nu_x(y)$$

for a.e. $x_0 \in \Omega$ (w.r.to the Lebesgue measure), where f_μ is as in Definition 2. Thus, to obtain (26), we can first shrink Q around its center x_0 in (38), then take the essential supremum over $x_0 \in \Omega$ on the right, and finally use the arbitrariness of ε. $\qquad\square$

4. The Γ-limsup inequality

In the following proposition we construct a fundamental pattern, whose periodic homogenization inside Ω will be the main ingredient in the construction of the recovery sequence, for the Γ-limsup inequality.

Proposition 10 (tile construction for elementary varifolds)**.** *Let $Q \subset \mathbb{R}^2$ be an open square with sides parallel to the coordinate axes, and let ν be a probability measure over $\mathbb{S}^1$. Moreover, let $M \in \mathbb{R}^{2 \times 2}$ be a positive definite matrix. Then, for every length ℓ large enough, there exists a continuum $\Sigma_\ell \in \mathcal{A}_\ell(Q)$ with the following properties.*

(i) *Boundary-covering and length matching properties:*

$$(39) \qquad \partial Q \subset \Sigma_\ell, \quad and \quad \lim_{\ell \to \infty} \frac{\mathcal{H}^1(\Sigma_\ell)}{\ell} = 1.$$

(ii) *Estimate on the anisotropic eigenvalue:*

$$(40) \qquad \limsup_{\ell \to \infty} \frac{\ell^2}{\lambda_1^M(Q \setminus \Sigma_\ell)} \leq \frac{|Q|^2}{\left(\pi \int_{\mathbb{S}^1} \sqrt{\langle My, y \rangle} \, d\nu(y) \right)^2}.$$

(iii) *The varifolds associated with Σ_ℓ converge to $|Q|^{-1} \chi_Q \otimes \nu$, that is, for every function $\varphi \in C(\overline{Q} \times \mathbb{S}^1)$ such that $\varphi(x, y) = \varphi(x, -y)$, one has*

$$(41) \qquad \lim_{\ell \to \infty} \frac{1}{\mathcal{H}^1(\Sigma_\ell)} \int_{\Sigma_\ell} \varphi(x, \xi_\ell(x)) \, d\mathcal{H}^1(x) = \frac{1}{|Q|} \int_Q \int_{\mathbb{S}^1} \varphi(x, y) \, d\nu(y) \, dx$$

where ξ_ℓ is any measurable selection of the unit normal to Σ_ℓ.

Proof. It is not restrictive to assume that Q has a side length of 1 (the general case, with a side length of s, is recovered by a scaling argument, replacing Q with sQ and the constructed Σ_ℓ with the rescaled $s\Sigma_{\ell/s}$).

Moreover, we initially assume that the measure ν is *purely atomic*, that is

$$(42) \qquad \nu = \sum_{j=1}^n \beta_j \delta_{\xi_j}, \quad \sum_{j=1}^n \beta_j = 1 \qquad (n \geq 1)$$

for suitable weights $\beta_j > 0$ and unit vectors $\xi_j \in \mathbb{S}^1$. With this notation, we have

$$(43) \qquad I := \int_{\mathbb{S}^1} \sqrt{\langle M\xi, \xi \rangle} \, d\nu(\xi) = \sum_{j=1}^n \beta_j \sqrt{\langle M\xi_j, \xi_j \rangle}$$

(the first equality is the definition of I) and, more generally,

$$(44) \qquad \int_{\mathbb{S}^1} v(\xi)\, d\nu(\xi) = \sum_{j=1}^{n} \beta_j v(\xi_j) \quad \forall v \in C(\mathbb{S}^1).$$

The sets Σ_ℓ will be obtained as the periodic homogenization, inside the square Q, of suitably rescaled fundamental tiles Γ_ε, initially constructed inside a unit square. Our construction consists of three main steps.

Step 1: tile construction. We first slice the unit square $Y := (0,1) \times (0,1)$ into n stacked rectangles Y_j $(1 \le j \le n)$ of size $1 \times h_j$, the height h_j being defined as

$$(45) \qquad h_j := \frac{\beta_j \sqrt{\langle M\xi_j, \xi_j \rangle}}{I}.$$

Note that, by (43), $\sum h_j = 1$ so that the heights of the n rectangles match the height of Y. Then, by drawing inside every Y_j a maximal family of parallel line segments, orthogonal to ξ_j and equally spaced a distance of ε_j apart from one another ($\varepsilon_j \ll 1$ to be chosen later), we further slice every rectangle Y_j into several thin polygons of width ε_j. For fixed j, the number of polygons inside Y_j is $O(1/\varepsilon_j)$: every polygon is a trapezoid of height ε_j, with the exception of *at most four polygons* that, due to the corners of Y_j, may degenerate into a triangle or a pentagon (one hexagon may also occur, if the line segments are almost parallel to a diagonal of Y_j and one polygon touches two opposite corners of Y_j).

Let K_j denote the union of all these line segments, orthogonal to ξ_j, drawn inside Y_j. Since every polygon (with at most four exceptions) is a trapezoid of height ε_j, summing the areas of the polygons we see that

$$\varepsilon_j \mathcal{H}^1(K_j) + O(\varepsilon_j) = |Y_j| = h_j,$$

where $O(\varepsilon_j)$ is the correction due to the exceptional polygons, whose total area is at most $4\sqrt{2}\varepsilon_j$. Thus, for the total length of K_j we have the asymptotics

$$\mathcal{H}^1(K_j) \sim \frac{h_j}{\varepsilon_j} \quad \text{as } \varepsilon_j \to 0.$$

From now on, the values of the ε_js shall be fixed according to

$$(46) \qquad \varepsilon_j := \varepsilon \sqrt{\langle M\xi_j, \xi_j \rangle}, \quad 1 \le j \le n$$

(where $\varepsilon \ll 1$ is a scale parameter to be tuned later) so that, using (45), the previous asymptotics take the concrete form
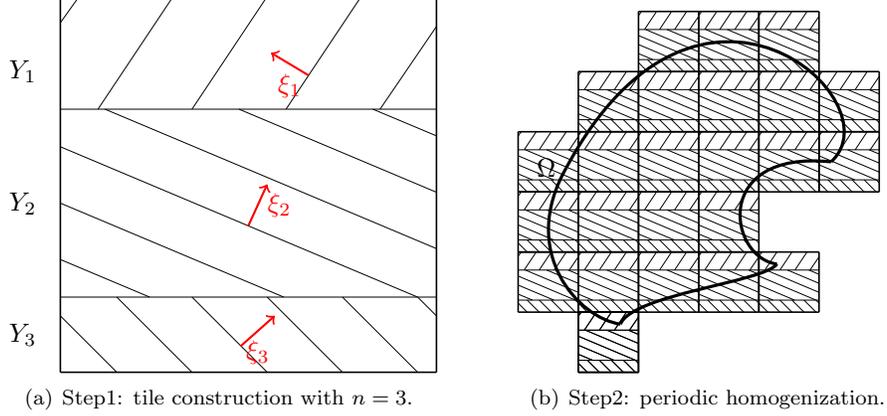
$$(47) \qquad \mathcal{H}^1(K_j) \sim \frac{\beta_j}{\varepsilon I} \quad \text{as } \varepsilon \to 0.$$

Our fundamental object, the *tile* Γ_ε, is defined as

$$(48) \qquad \Gamma_\varepsilon := R \cup S_\varepsilon, \quad R := \bigcup_{j=1}^{n} \partial Y_j, \quad S_\varepsilon := \bigcup_{j=1}^{n} K_j.$$

The set R (which is *independent* of ε) consists of the boundaries of the n rectangles Y_j and acts as a frame, while S_ε is the union of all the oblique line segments inside the rectangles. Clearly Γ_ε is compact and connected, and moreover

$$(49) \qquad \partial Y \subset \Gamma_\varepsilon$$

(a) Step1: tile construction with $n = 3$.   (b) Step2: periodic homogenization.

since $\partial Y \subset R$. We also have

$$(50) \qquad \mathcal{H}^1(R) = n + 3, \qquad \mathcal{H}^1(S_\varepsilon) \sim \frac{1}{\varepsilon I} \quad \text{as } \varepsilon \to 0$$

having used, in the last expansion, (47) and the fact that $\sum \beta_j = 1$.

In view of (41), let $\xi_\varepsilon(x)$ denote the unit normal to the set S_ε at $x \in S_\varepsilon$ (as usual, any measurable selection of $\pm\xi_\varepsilon(x)$ will do). By our construction, $\xi_\varepsilon(x) = \pm\xi_j$ for every $x \in K_j$; therefore, if $v \in C(\mathbb{S}^1)$ is such that $v(y) = v(-y)$, we have

$$\frac{1}{\mathcal{H}^1(S_\varepsilon)} \int_{S_\varepsilon} v(\xi_\varepsilon(t)) \, d\mathcal{H}^1(t) = \sum_{j=1}^n \frac{\mathcal{H}^1(K_j)}{\mathcal{H}^1(S_\varepsilon)} v(\xi_j).$$

Therefore, using (47) and (50) we have

$$(51) \qquad \lim_{\varepsilon \to 0} \frac{1}{\mathcal{H}^1(S_\varepsilon)} \int_{S_\varepsilon} v(\xi_\varepsilon(t)) \, d\mathcal{H}^1(t) = \sum_{j=1}^n \beta_j v(\xi_j) = \int_{\mathbb{S}^1} v(\xi) \, d\nu(\xi)$$

by (44). Finally, since every connected component of $Y \setminus \Gamma_\varepsilon$ is, by construction, a polygon of width ε_j in the direction ξ_j for some $j \in \{1, \ldots, n\}$, Theorem 8 gives

$$(52) \qquad \lambda_1^M(Y \setminus \Gamma_\varepsilon) \geq \min_j \pi^2 \frac{\langle M\xi_j, \xi_j \rangle}{\varepsilon_j^2} = \frac{\pi^2}{\varepsilon^2}$$

having used (46) in the last passage.

Step 2 (periodic homogenization). Since the square Q has a side length of 1, given an integer $m \geq 1$ we may fit m^2 copies of the rescaled tile $m^{-1}\Gamma_\varepsilon$ inside $\overline{Q}$ as in an $m \times m$ checkerboard: the resulting tiling is then $1/m$–periodic in the two directions parallel to the sides of Q.

We denote by $\Gamma_{m,\varepsilon}$ the union of these m^2 rescaled tiles: this set is connected, because so is Γ_ε and, by (49), each tile shares a side of length $1/m$ with each neighbor. The sets Σ_ℓ we want to construct are defined, for large ℓ, as follows:

$$\Sigma_\ell := \Gamma_{m,\varepsilon} \qquad m = m(\ell) \text{ and } \varepsilon = \varepsilon(\ell),$$

where

$$(53) \qquad \varepsilon(\ell) := \ell^{-2/3}, \qquad m(\ell) = \left\lceil \ell^{\frac{1}{3}} I \right\rceil$$

(observe that $\varepsilon \to 0$ and $m \to \infty$ when $\ell \to \infty$).

According to (48), Σ_ℓ is the union of m^2 disjoint copies of the rescaled set $m^{-1} S_\varepsilon$ (hereafter denoted by $S^i_{m,\varepsilon}$, $1 \le i \le m^2$) of total length $\sim m/(\varepsilon I)$ according to (50), plus the union of m^2 copies of the rescaled frame $m^{-1} R$: these are not disjoint, since adjacent tiles share a side, but their total length does not exceed $m(n+3)$ according to (50), and is therefore negligible compared to the total length of the $S^i_{m,\varepsilon}$s. As a consequence,

$$(54) \qquad \mathcal{H}^1(\Sigma_\ell) \sim \mathcal{H}^1 \left(\bigcup_{i=1}^{m^2} S^i_{m,\varepsilon} \right) \sim \frac{m}{\varepsilon I} \quad \text{as } \ell \to \infty,$$

and (39) follows from (53) (the first part of (39) is immediate, hence claim (i) is proved).

Similarly, the limit in (41) is unchanged if we restrict integration to the union of the $S^i_{m,\varepsilon}$s, and therefore (41) is equivalent to

$$\lim_{\ell \to \infty} \sum_{i=1}^{m^2} \frac{1}{m^2 \mathcal{H}^1(S^i_{m,\varepsilon})} \int_{S^i_{m,\varepsilon}} \varphi(x, \xi_\ell(x)) \, d\mathcal{H}^1(x) = \frac{1}{|Q|} \int_Q \int_{\mathbb{S}^1} \varphi(x,y) \, d\nu(y) \, dx,$$

having used the first equivalence in (54) and the fact that the $S^i_{m,\varepsilon}$s are disjoint and congruent. In addition, by a density argument, it suffices to prove (41) when $\varphi(x,y) = u(x)v(y)$ with $u \in C(\overline{Q})$ and $v \in C(\mathbb{S}^1)$ such that $v(y) = v(-y)$, so that the last equation reduces to

$$(55) \qquad \lim_{\ell \to \infty} \sum_{i=1}^{m^2} \frac{1}{m^2 \mathcal{H}^1(S^i_{m,\varepsilon})} \int_{S^i_{m,\varepsilon}} u(x)v(\xi_\ell(x)) \, d\mathcal{H}^1(x) = I_u I_v,$$

where

$$I_u = \int_Q u(x) \, dx, \qquad I_v = \int_{\mathbb{S}^1} v(y) \, d\nu(y)$$

(as $1/|Q| = 1$, this factor can be omitted). Now fix functions u, v as above, and let

$$(56) \qquad U^i_m := \frac{1}{|Q^i_m|} \int_{Q^i_m} u(x) \, dx = m^2 \int_{Q^i_m} u(x) \, dx$$

denote the integral average of u over the square Q^i_m, of side-length $1/m$, whose boundary frames $S^i_{m,\varepsilon}$. If $\omega(\delta) = \sup_{|z-x| \le \delta} |u(z) - u(x)|$ is the modulus of continuity of u over $\overline{Q}$, since $S^i_{m,\varepsilon} \subset \overline{Q^i_m}$ and $\operatorname{diam}(Q^i_m) = \sqrt{2}/m$ we have

$$\frac{1}{\mathcal{H}^1(S^i_{m,\varepsilon})} \int_{S^i_{m,\varepsilon}} \left| u(x) - U^i_m \right| d\mathcal{H}^1(x) \le \omega(\sqrt{2}/m), \quad 1 \le i \le m^2.$$

From these estimates, we immediately see that

$$\left| \sum_{i=1}^{m^2} \frac{1}{m^2 \mathcal{H}^1(S^i_{m,\varepsilon})} \int_{S^i_{m,\varepsilon}} \left(u(x) - U^i_m \right) v(\xi_\ell(x)) \, d\mathcal{H}^1(x) \right| \le \omega(\sqrt{2}/m) \|v\|_{L^\infty},$$

which tends to zero as ℓ (hence m) tends to infinity. Thus, to prove (55), we can freeze $u(x)$ in each integral, and replace it with the constant U_m^i. On the other hand, since each $S_{m,\varepsilon}^i$ is a rescaled copy of the set S_ε,

$$\frac{1}{\mathcal{H}^1(S_{m,\varepsilon}^i)} \int_{S_{m,\varepsilon}^i} v(\xi_\ell(x))\, d\mathcal{H}^1(x) = \frac{1}{\mathcal{H}^1(S_\varepsilon)} \int_{S_\varepsilon} v(\xi_\varepsilon(x))\, d\mathcal{H}^1(x), \quad 1 \le i \le m^2,$$

where ξ_ε, as before, is the unit normal to S_ε. Therefore, if we replace $u(x)$ with U_m^i in (55) and take it out of the integral, (55) simplifies to

$$\lim_{\ell \to \infty} \left(\sum_{i=1}^{m^2} \frac{1}{m^2} U_m^i \right) \left(\frac{1}{\mathcal{H}^1(S_\varepsilon)} \int_{S_\varepsilon} v(\xi_\varepsilon(x))\, d\mathcal{H}^1(x) \right) = I_u I_v$$

which is now obvious, since the sum coincides with I_u by (56), while the second factor tends to I_v by (51). Summing up, (41) is proved.

Finally, to prove (40), observe that $Q \setminus \Sigma_\ell$ is highly disconnected due to the checkerboard structure of Σ_ℓ, and each of its connected component is, by construction, congruent to a connected component of $Y \setminus \Gamma_\varepsilon$ scaled down by a factor $1/m$ (which amplifies the first eigenvalue by a factor m^2). Therefore, we have using (52) and (53)

$$\lambda_1^M(Q \setminus \Sigma_\ell) = m^2 \lambda_1^M(Y \setminus \Gamma_\varepsilon) \ge \frac{\pi^2 m^2}{\varepsilon^2} \sim \pi^2 I^2 \ell^2 \quad \text{as } \ell \to \infty,$$

so that (40) follows immediately from (43).

Step 3 (general ν). By a diagonal argument, we can now remove the restriction (42) and consider an arbitrary probability measure ν over $\mathbb{S}^1$.

Let $\{\nu_k\}$ be a sequence of atomic probability measures over $\mathbb{S}^1$, such that $\nu_k \rightharpoonup \nu$. For each k, we can apply Proposition 10 in the form just proved (with ν_k in place of ν), thus obtaining continua $\Sigma_\ell^k \in \mathcal{A}_\ell(Q)$ satisfying claims (i)–(iii) relative to ν_k. Then, since the weak-* topology of probability measures over $\mathbb{S}^1$ is metrizable, by a standard diagonal argument one can find an increasing sequence of lengths $\ell_1 < \ell_2 < \cdots$ such that, letting $\Sigma_\ell := \Sigma_\ell^k$ whenever $\ell_k \le \ell < \ell_{k+1}$, the resulting continua $\{\Sigma_\ell\}$ satisfy claims (i)–(iii) relative to ν. □

In view of extending Proposition 10 to a more general class of varifolds, the following terminology is needed.

Definition 11. *For $s > 0$, let $\mathcal{Q}_s$ denote the collection of all those open squares $Q_i \subset \mathbb{R}^2$, with side-length s and corners on the lattice $(s\mathbb{Z})^2$, such that $Q_i \cap \Omega \ne \emptyset$. We say that a varifold $\theta \in \mathcal{V}_1(\overline{\Omega})$ is fitted to $\mathcal{Q}_s$ if it can be represented as*

$$(57) \qquad \theta = \sum_{Q_i \in \mathcal{Q}_s} \alpha_i \chi_{\Omega \cap Q_i} \otimes \nu_i$$

for suitable constants $\alpha_i \ge 0$ and probability measures ν_i over $\mathbb{S}^1$, satisfying

$$(58) \qquad \sum_{Q_i \in \mathcal{Q}_s} \alpha_i |\Omega \cap Q_i| = 1.$$

Roughly speaking, a varifold is *fitted to $\mathcal{Q}_s$* if its restriction to a cylinder of the form $\overline{\Omega \cap Q_i} \times \mathbb{S}^1$ is the *product measure* $\alpha_i \otimes \nu_i$. In particular, for such a varifold its first marginal (i.e. its projection over $\overline{\Omega}$) is absolutely continuous with respect

to the Lebesgue measure, and is *piecewise constant* over Ω: as a consequence, (58) is equivalent to the request that the varifold has unit mass.

We observe, for future reference, that if θ is as in (57) the functional F_∞ defined in (10) takes the form

$$\text{(59)} \qquad F_\infty(\theta) = \max_{Q_i \in \mathcal{Q}_s} \sup_{x \in \Omega \cap Q_i} \frac{1}{\alpha_i^2 \left(\pi \int_{\mathbb{S}^1} \sqrt{\langle A(x)y, y \rangle} \, d\nu_i(y) \right)^2}.$$

Proposition 12. *Let $\theta \in \mathcal{V}_1(\overline{\Omega})$ be a varifold fitted to $\mathcal{Q}_s$. Then there exists a sequence of continua $\Sigma_L \subset \overline{\Omega}$ such that (25) holds true,*

$$\text{(60)} \qquad \lim_{L \to \infty} \frac{\mathcal{H}^1(\Sigma_L)}{L} = 1,$$

and

$$\text{(61)} \qquad \limsup_{L \to \infty} \frac{L^2}{\lambda_1^A(\Omega \setminus \Sigma_L)} \leq F_\infty(\theta).$$

Proof. We keep for θ the same notation as in Definition 11 (in particular (57) and (58)), and fix a number $\eta > 1$. By replacing s with $s/2^n$ for some large $n \geq 1$ and relabelling the α_is (thus keeping ν fitted to $\mathcal{Q}_s$), we may assume that s is so small that the following two conditions hold:

(C1) no connected component of $\partial\Omega$ is strictly contained in any square $Q_i \in \mathcal{Q}_s$;
(C2) for every square $Q_i \in \mathcal{Q}_s$, there exists a positive definite matrix M_i such that

$$\text{(62)} \qquad \langle A(x)y, y \rangle \geq \frac{1}{\eta} \langle M_i y, y \rangle \qquad \forall x \in \Omega \cap Q_i, \quad \forall y \in \mathbb{S}_1.$$

Condition C1 holds, for s small enough, since $\partial\Omega$, being Lipschitz, has finitely many connected components whose diameters have a positive lower bound. On the other hand, C2 is guaranteed, for small s, by the uniform continuity of the function $A(x)$ (one can set e.g. $M_i = A(x_i)$, for some $x_i \in \Omega \cap Q_i$).

The sets Σ_L we want to construct will be obtained as a patchwork of sets Σ_ℓ^i, one for each square $Q_i \in \mathcal{Q}_s$, obtained from Proposition 10. More precisely, for every square $Q_i \in \mathcal{Q}_s$ we apply Proposition 10 when the square $Q = Q_i$, the measure $\nu = \nu_i$ and the matrix $M = M_i$. This yields, for large enough ℓ, sets Σ_ℓ^i such that:

(i) $\partial Q_i \subset \Sigma_\ell^i$ and $\mathcal{H}^1(\Sigma_\ell^i) \sim \ell$ as $\ell \to \infty$;
(ii) $\displaystyle \limsup_{\ell \to \infty} \frac{\ell^2}{\lambda_1^{M_i}(Q_i \setminus \Sigma_\ell^i)} \leq \frac{s^4}{\left(\pi \int_{\mathbb{S}^1} \sqrt{\langle M_i y, y \rangle} \, d\nu_i(y) \right)^2}$;
(iii) for every $\varphi \in C(\overline{Q_i} \times \mathbb{S}^1)$ with $\varphi(x, y) = \varphi(x, -y)$,

$$\text{(63)} \qquad \lim_{\ell \to \infty} \frac{1}{\mathcal{H}^1(\Sigma_\ell^i)} \int_{\Sigma_\ell^i} \varphi(x, \xi_\ell^i(x)) \, d\mathcal{H}^1(x) = \frac{1}{|Q_i|} \int_{Q_i} \int_{\mathbb{S}^1} \varphi(x, y) \, d\nu_i(y) \, dx$$

where ξ_ℓ^i is any measurable selection of the unit normal to Σ_ℓ^i.

Observe that the domain Ω plays no role in this construction, and this is natural for those squares $Q_i \in \mathcal{Q}_s$ such that $Q_i \subset \Omega$. If, however, $Q_i \in \mathcal{Q}_s$ is such that $Q_i \cap \partial\Omega \neq \emptyset$, the weak-* convergence in (iii) (that occurs in the whole $\overline{Q_i} \times \mathbb{S}^1$) can still be *localized* to $\overline{\Omega \cap Q_i} \times \mathbb{S}^1$. More precisely, since $\partial(\Omega \cap Q_i)$ has null

Lebesgue measure, and the limit measure $|Q_i|^{-1}\chi_{Q_i}\otimes\nu_i$ does not charge the cylinder $\partial(\Omega\cap Q_i)\times\mathbb{S}^1$, from (63) we infer that

$$(64)\quad \lim_{\ell\to\infty}\frac{1}{\mathcal{H}^1(\Sigma_\ell^i)}\int_{\overline{\Omega}\cap\Sigma_\ell^i}\varphi(x,\xi_\ell^i(x))\,d\mathcal{H}^1(x)=\frac{1}{|Q_i|}\int_{\Omega\cap Q_i}\int_{\mathbb{S}^1}\varphi(x,y)\,d\nu_i(y)\,dx,$$

for every $\varphi\in C(\overline{\Omega\cap Q_i}\times\mathbb{S}^1)$ with $\varphi(x,y)=\varphi(x,-y)$ (in the first integral, a localization to $\overline{Q_i}$ is implicit since $\Sigma_\ell^i\subset\overline{Q_i}$ by assumption). In particular, testing with $\varphi\equiv 1$, we find

$$\lim_{\ell\to\infty}\frac{\mathcal{H}^1(\overline{\Omega}\cap\Sigma_\ell^i)}{\mathcal{H}^1(\Sigma_\ell^i)}=\frac{|\Omega\cap Q_i|}{|Q_i|}$$

i.e., using the second property in (i) above,

$$(65)\qquad \lim_{\ell\to\infty}\frac{\mathcal{H}^1(\overline{\Omega}\cap\Sigma_\ell^i)}{\ell}=\frac{|\Omega\cap Q_i|}{|Q_i|}$$

(when $Q_i\subset\Omega$ this is obvious since, in this case, it is contained in (i) above).

In (57) we may assume that $\alpha_i>0$ for all i, since if $\alpha_i=0$ for some i then the right-hand side of (61) is $+\infty$. Then, for large L we can define the sets

$$(66)\qquad \Sigma_L:=(\partial\Omega)\cup\Big(\bigcup_{Q_i\in\mathcal{Q}_s}\Sigma_{s^2\alpha_i L}^i\cap\Omega\Big),$$

where $\Sigma_{s^2\alpha_i L}^i$ denotes the set Σ_ℓ^i defined above, when $\ell=s^2\alpha_i L$ (recall that $s^2=|Q_i|$, the area of Q_i).

Some remarks are in order. First, since each set $\Sigma_{s^2\alpha_i L}^i$ is connected and covers ∂Q_i, building on (C1) above one can check that Σ_L is connected (to this purpose, the fact that $\Sigma_L\supset\partial\Omega$ is essential). Then, setting $\ell=s^2\alpha_i L$ in (65) we obtain

$$(67)\qquad \lim_{L\to\infty}\frac{\mathcal{H}^1(\overline{\Omega}\cap\Sigma_{s^2\alpha_i L}^i)}{L}=\frac{s^2\alpha_i|\Omega\cap Q_i|}{|Q_i|}=\alpha_i|\Omega\cap Q_i|\quad\forall Q_i\in\mathcal{Q}_s.$$

Now, observing that $\mathcal{H}^1(\partial\Omega)$ is finite, and that the sets $\Sigma_{s^2\alpha_i L}^i$ are pairwise disjoint, except for their overlapping along $\bigcup\partial Q_i$ whose total length is independent of L, from (66) and (67) we obtain

$$(68)\qquad \lim_{L\to\infty}\frac{\mathcal{H}^1(\Sigma_L)}{L}=\sum_{Q_i\in\mathcal{Q}_s}\lim_{L\to\infty}\frac{\mathcal{H}^1(\overline{\Omega}\cap\Sigma_{s^2\alpha_i L}^i)}{L}=\sum_{Q_i\in\mathcal{Q}_s}\alpha_i|\Omega\cap Q_i|=1$$

according to (58). Along the same lines, given $\varphi\in C(\overline{\Omega}\times\mathbb{S}^1)$ with $\varphi(x,y)=\varphi(x,-y)$, denoting by ξ_L the unit normal to Σ_L, we may split

$$(69)\qquad\begin{aligned}&\lim_{L\to\infty}\frac{1}{\mathcal{H}^1(\Sigma_L)}\int_{\Sigma_L}\varphi(x,\xi_L(x))\,d\mathcal{H}^1(x)\\&=\sum_{Q_i\in\mathcal{Q}_s}\lim_{L\to\infty}\frac{\mathcal{H}^1(\Sigma_{s^2\alpha_i L}^i)}{\mathcal{H}^1(\Sigma_L)}\cdot\frac{1}{\mathcal{H}^1(\Sigma_{s^2\alpha_i L}^i)}\int_{\overline{\Omega}\cap\Sigma_{s^2\alpha_i L}^i}\varphi(x,\xi_\ell^i(x))\,d\mathcal{H}^1(x).\end{aligned}$$

Now for every $Q_i\in\mathcal{Q}_s$, using (68) and condition (i) above with $\ell=s^2\alpha_i L$, we have

$$\lim_{L\to\infty}\frac{\mathcal{H}^1(\Sigma_{s^2\alpha_i L}^i)}{\mathcal{H}^1(\Sigma_L)}=s^2\alpha_i,$$

while putting $\ell = s^2 \alpha_i L$ in (64) gives

$$\lim_{L \to \infty} \frac{1}{\mathcal{H}^1(\Sigma^i_{s^2 \alpha_i L})} \int_{\overline{\Omega} \cap \Sigma^i_{s^2 \alpha_i L}} \varphi(x, \xi^i_\ell(x)) \, d\mathcal{H}^1(x) = \frac{1}{|Q_i|} \int_{\Omega \cap Q_i} \int_{\mathbb{S}^1} \varphi(x, y) \, d\nu_i(y) \, dx.$$

Plugging the last two limits into (69), since $s^2 = |Q_i|$ we obtain

$$(70) \quad \lim_{L \to \infty} \frac{1}{\mathcal{H}^1(\Sigma_L)} \int_{\Sigma_L} \varphi(x, \xi_L(x)) \, d\mathcal{H}^1(x) = \sum_{Q_i \in \mathcal{Q}_s} \alpha_i \int_{\Omega \cap Q_i} \int_{\mathbb{S}^1} \varphi(x, y) \, d\nu_i(y) \, dx,$$

that is (25), due to the structure of θ as established in (57).

Finally, to prove (61), observe that every connected component of $\Omega \setminus \Sigma_L$ is contained inside some square $Q_j \in \mathcal{Q}_s$ and moreover, by construction, $\Sigma_L \supset \partial(\Omega \cap Q_i)$ for *every* square $Q_i \in \mathcal{Q}_s$. As a consequence, we have

$$\lambda_1^A(\Omega \setminus \Sigma_L) = \min_{Q_i \in \mathcal{Q}_s} \lambda_1^A((\Omega \cap Q_i) \setminus \Sigma_L) = \min_{Q_i \in \mathcal{Q}_s} \lambda_1^A((\Omega \cap Q_i) \setminus \Sigma^i_{s^2 \alpha_i L})$$

$$\geq \eta^{-1} \min_{Q_i \in \mathcal{Q}_s} \lambda_1^{M_i}((\Omega \cap Q_i) \setminus \Sigma^i_{s^2 \alpha_i L}) \geq \eta^{-1} \min_{Q_i \in \mathcal{Q}_s} \lambda_1^{M_i}(Q_i \setminus \Sigma^i_{s^2 \alpha_i L})$$

(the last two inequalities follow from (62) and monotonicity of λ_1 with respect to set inclusion, respectively: observe that, since M_i is a constant matrix, the eigenvalue $\lambda_1^{M_i}(Q_i \setminus \Sigma^i_{s^2 \alpha_i L})$ makes sense for every Q_i, while $\lambda_1^A(Q_i \setminus \Sigma^i_{s^2 \alpha_i L})$ would make no sense if $Q_i \cap \partial\Omega \neq \emptyset$, since $A(x)$ is defined only for $x \in \overline{\Omega}$. Passing to reciprocals, swapping lim sup and max, and using (ii) with $\ell = s^2 \alpha_i L$, we obtain

$$\limsup_{L \to \infty} \frac{L^2}{\lambda_1^A(\Omega \setminus \Sigma_L)} \leq \eta \limsup_{L \to \infty} \left(\max_{Q_i \in \mathcal{Q}_s} \frac{L^2}{\lambda_1^{M_i}(Q_i \setminus \Sigma^i_{s^2 \alpha_i L})} \right)$$

$$\leq \eta \max_{Q_i \in \mathcal{Q}_s} \left(\limsup_{L \to \infty} \frac{L^2}{\lambda_1^{M_i}(Q_i \setminus \Sigma^i_{s^2 \alpha_i L})} \right)$$

$$\leq \frac{\eta}{s^4 \alpha_i^2} \max_{Q_i \in \mathcal{Q}_s} \frac{s^4}{\left(\pi \int_{\mathbb{S}^1} \sqrt{\langle M_i y, y \rangle} \, d\nu_i(y) \right)^2} \leq \eta F_\infty(\theta),$$

having used (59) in the last passage. This proves (61) up to the multiplicative constant $\eta > 1$ which, being arbitrary, can easily be remove by a diagonal argument, as in the final part of the proof of Proposition 10. $\qquad\square$

The passage to general ν is standard: by the classical Γ-convergence theory the Γ-limsup inequality for every varifolds in $\mathcal{V}_1(\overline{\Omega})$ holds if one proves the density in energy of those varifold considered in (57) in the space of all varifolds $\mathcal{V}_1(\overline{\Omega})$.

Lemma 13 (renormalization). *For some $a > 0$, let $p : [a, \infty) \to \mathbb{R}^+$ be a function such that $p(L) \sim L$ as $L \to \infty$. Then, for some $b > 0$, there exists another function $q : [b, \infty) \to [a, \infty)$ such that*

$$p(q(L)) \leq L \quad \forall L \geq b, \quad and \quad p(q(L)) \sim L \quad as \ L \to \infty.$$

Proof. For large L define the set $E(L) := \{x \geq a \mid p(x) \leq L\}$, and let $s(L) := \sup E(L)$. Since $p(L) \sim L$ as $L \to \infty$, one can easily check that also $s(L) \sim L$ as $L \to \infty$. Then, any function $q(L)$ satisfying

$$q(L) \in E(L) \quad \text{and} \quad s(L) - 1 < q(L) < s(L) \qquad \text{(for every L large enough)}$$

will satisfy the claim of the Lemma, since clearly $q(L) \sim s(L) \sim L$ as $L \to \infty$, while $q(L) \in E(L)$ guarantees that $p(q(L)) \leq L$. $\qquad\square$

Remark 14. Using the previous lemma, one can strengthen the claim of Proposition 12 with the *additional* requirement that

$$(71) \qquad \mathcal{H}^1(\Sigma_L) \le L, \quad \text{that is,} \quad \Sigma_L \in \mathcal{A}_L(\Omega).$$

Indeed, it suffices to apply the lemma to the function $p(L) := \mathcal{H}^1(\Sigma_L)$, where the sets Σ_L are those initially yielded by Proposition 12 (note that $p(L) \sim L$ by (60)). Then, one can define the new sets $\Sigma'_L := \Sigma_{q(L)}$, thus gaining that $\mathcal{H}^1(\Sigma'_L) = p(q(L)) \le L$ while keeping $\mathcal{H}^1(\Sigma'_L) \sim L$. Then, replacing each Σ_L with the corresponding Σ'_L proves the claim.

Proposition 15 (Γ-limsup inequality)**.** *For every varifold $\theta \in \mathcal{V}_1(\overline{\Omega})$, there exists a sequence of varifolds $\{\theta_L\} \subset \mathcal{V}_1(\overline{\Omega})$ such that $\theta_L \rightharpoonup \theta$ and, moreover,*

$$(72) \qquad \limsup_{L \to \infty} F_L(\theta_L) \le F_\infty(\theta).$$

Proof. Given $\theta \in \mathcal{V}_1(\overline{\Omega})$, if θ is fitted to $\mathcal{Q}_s$ for some $s > 0$, then the claim follows from Proposition 12, strengthened according to (71). Indeed, in this case, is suffices to define θ_L as the varifold associated with the set Σ_L, so that the left hand side of (72) coincides with that of (61) (the fact that $\theta_L \rightharpoonup \theta$ follows from (70)).

Then, by well known properties of Γ-convergence, it suffices to prove that the class of varifolds fitted to $\mathcal{Q}_s$ (for some $s > 0$) is *dense in energy*. More precisely, it suffices to prove that for every varifold $\theta \in \mathcal{V}_1(\overline{\Omega})$ there exist varifolds $\theta_L \in \mathcal{V}_1(\overline{\Omega})$, each fitted to $\mathcal{Q}_s$ for some $s > 0$ that may depend on L, such that

$$(73) \qquad \theta_L \rightharpoonup \theta \quad \text{in } \mathcal{V}_1(\overline{\Omega}), \text{ as } L \to \infty$$

and, at the same time,

$$(74) \qquad \limsup_{L \to \infty} F_\infty(\theta_L) \le F_\infty(\theta).$$

Now fix an arbitrary $L > 0$, set e.g. $s = 1/L$, and consider the family of squares $\mathcal{Q}_s$ as in Definition 11. The main idea is to define θ_L, fitted to $\mathcal{Q}_s$, by averaging θ over a partition of $\overline{\Omega}$ essentially based on $\mathcal{Q}_s$. Observe that the open squares in $\mathcal{Q}_s$ cover $\overline{\Omega}$ only up to a set C of measure zero, while the measure θ may well charge the set $C \times \mathbb{S}^1$. We overcome this difficulty by choosing *intermediate* Borel sets Ω_i such that

$$\Omega \cap Q_i \subseteq \Omega_i \subseteq \overline{\Omega \cap Q_i}, \qquad \overline{\Omega} = \bigcup_i \Omega_i, \qquad \Omega_i \cap \Omega_j = \emptyset \quad \forall i \ne j$$

(the actual choice of the Ω_is is irrelevant) and define

$$\theta_L := \sum_{Q_i \in \mathcal{Q}_s} \alpha_i \chi_{\Omega \cap Q_i} \otimes \nu_i,$$

where the constants α_i and the probability measures ν_i are given by

$$\alpha_i := \frac{\theta(\Omega_i \times \mathbb{S}^1)}{|\Omega_i|} \quad \text{and} \quad \nu_i(E) := \frac{\theta(\Omega_i \times E)}{\theta(\Omega_i \times \mathbb{S}^1)} \quad \text{(for every Borel set } E \subseteq \mathbb{S}^1\text{).}$$

It is clear that θ_L is a varifold fitted to $\mathcal{Q}_s$ (note that (58) is satisfied, since Ω_i is equivalent to $\Omega \cap Q_i$ up to a Lebesgue-negligible set). This definition makes sense only if $\theta(\Omega_i \times \mathbb{S}^1) > 0$ for every i: on the other hand, if $\theta(\Omega_i \times \mathbb{S}^1) = 0$ for some i, then $\alpha_i = 0$ and ν_i becomes irrelevant (for definiteness, one can define ν_i as *any* probability measure). Observe that, in this case, one has $F_\infty(\theta) = +\infty$, and (74) becomes trivial, regardless of θ_L.

The measure θ_L so defined is a discretization of θ over $\mathcal{Q}_s$, and in concrete terms its action on a Borel function $\varphi(x, y) \geq 0$ is given by

$$\int_{\overline{\Omega} \times \mathbb{S}^1} \varphi(x, y) \, d\theta_L(x, y) = \sum_{Q_i \in \mathcal{Q}_s} \frac{1}{|\Omega_i|} \int_{\Omega \cap Q_i} \left(\int_{\Omega_i \times \mathbb{S}^1} \varphi(x, y) \, d\theta(z, y) \right) dx$$

It is routine to check that (73) holds: e.g., the Wasserstein distance (see [3]) $W^1(\theta, \theta_L)$ does not exceed $\sqrt{2}/L$, the diameter of each Q_i. As a consequence, we only focus on (74) (assuming all $\alpha_i > 0$, otherwise (74) is trivial as already observed).

Now let $\theta = \mu \otimes \nu_x$ be the disintegration of θ (as discussed in Definition 2), and let $f \in L^1(\Omega)$ be the density of μ with respect to the Lebesgue measure (clearly, $\mu \geq f$ as measures). For every $Q_i \in \mathcal{Q}_s$, and for every $x \in \Omega \cap Q_i$, we have

$$\pi \alpha_i \int_{\mathbb{S}^1} \sqrt{\langle A(x)y, y \rangle} \, d\nu_i(y) = \frac{\pi}{|\Omega_i|} \int_{\Omega_i \times \mathbb{S}^1} \sqrt{\langle A(x)y, y \rangle} \, d\theta(z, y)$$

$$= \frac{\pi}{|\Omega_i|} \int_{\Omega_i} \left(\int_{\mathbb{S}^1} \sqrt{\langle A(x)y, y \rangle} \, d\nu_z(y) \right) d\mu(z)$$

$$\geq \frac{\pi}{|\Omega_i|} \int_{\Omega_i} \left(\int_{\mathbb{S}^1} \sqrt{\langle A(x)y, y \rangle} \, d\nu_z(y) \right) f(z) \, dz.$$

Now, for every small $\varepsilon > 0$, since the matrices $A(x)$ are uniformly continuous and positive definite over $\overline{\Omega}$, if L is large enough (and consequently the side length $s = 1/L$ of every $Q_j \in \mathcal{Q}_s$ is small enough) we have

$$x, z \in \Omega_i \Rightarrow \sqrt{\langle A(x)y, y \rangle} \geq (1 - \varepsilon) \sqrt{\langle A(z)y, y \rangle} \quad \forall y \in \mathbb{S}^1.$$

This allows us to replace, up to a factor $1 - \varepsilon$, $A(x)$ with $A(z)$ in the previous estimate: thus, recalling (10), we obtain

$$\pi \alpha_i \int_{\mathbb{S}^1} \sqrt{\langle A(x)y, y \rangle} \, d\nu_i(y) \geq \frac{(1 - \varepsilon)\pi}{|\Omega_i|} \int_{\Omega_i} f(z) \left(\int_{\mathbb{S}^1} \sqrt{\langle A(z)y, y \rangle} \, d\nu_z(y) \right) dz$$

$$\geq (1 - \varepsilon)\pi \operatorname*{ess\,inf}_{z \in \Omega} \left(f(z) \int_{\mathbb{S}^1} \sqrt{\langle A(z)y, y \rangle} \, d\nu_z(y) \right) = \frac{1 - \varepsilon}{\sqrt{F_\infty(\theta)}},$$

valid for L large enough (depending only on ε). Squaring and passing to reciprocals, the arbitrariness of x gives

$$\operatorname*{ess\,sup}_{x \in \Omega \cap Q_i} \frac{1}{\pi^2 \alpha_i^2 \left(\int_{\mathbb{S}^1} \sqrt{\langle A(x)y, y \rangle} \, d\nu_i(y) \right)^2} \leq \frac{F_\infty(\theta)}{(1 - \varepsilon)^2}, \quad \forall Q_i \in \mathcal{Q}_s.$$

Taking the maximum over Q_i and using (59), one obtains (72) up to a multiplicative factor $(1 - \varepsilon)^{-2}$, which can then removed by the usual diagonal argument. $\square$

5. Proofs of the main results

We are in a position to prove all the results stated in the introduction.

Proof of Theorem 3. The claim follows immediately as a consequence of Proposition 9 combined with Proposition 15. $\square$

Proof of Theorem 4. Consider a generic varifold $\theta \in \mathcal{V}_1(\overline{\Omega})$, disintegrated as $\theta = \mu \otimes \nu_x$ as in Definition 2. A look at (10) reveals that the following conditions are in any case *necessary*, for θ to be a (candidate) minimizer of F_∞:

(a) the density f_μ must satisfy $f_\mu(x) > 0$ for a.e. $x \in \Omega$, otherwise the denominator in (10) vanishes on a set of positive measure, and $F_\infty(\theta) = +\infty$;

(b) the first marginal $\mu \in \mathcal{P}(\overline{\Omega})$ must be *absolutely continuous* w.r.to the Lebesgue measure, i.e. $\mu = f_\mu(x)\,dx$, in such a way that $\int_\Omega f_\mu = 1$: otherwise, the varifold $\widehat{\theta} := \widehat{f}(x) \otimes \nu_x$, where $\widehat{f} := f_\mu / \|f_\mu\|_{L^1}$ is the renormalization of f_μ, would be such that $F_\infty(\widehat{\theta}) < F_\infty(\theta)$;

(c) for a.e. $x \in \Omega$, the measure $\nu_x \in \mathcal{P}(\mathbb{S}^1)$ must be supported on a set of normalized eigenvectors of $A(x)$ relative to its *largest eigenvalue* $\sigma_{\max}(x)$. Indeed, for any probability measure $\nu \in \mathcal{P}(\mathbb{S}^1)$ one has the double inequality

$$(75) \qquad \sqrt{\sigma_{\min}(x)} \leq \int_{\mathbb{S}^1} \sqrt{\langle A(x)y, y\rangle}\, d\nu(y) \leq \sqrt{\sigma_{\max}(x)},$$

where $\sigma_{\min}(x)$ is the smallest eigenvalue of $A(x)$. If $\sigma_{\min}(x) = \sigma_{\max}(x)$, i.e. if $A(x)$ is a multiple of the identity matrix, then both inequalities are in fact equalities, and in this case the condition that ν_x be supported on a set of eigenvectors relative to $\sigma_{\max}(x)$ is *trivial*, since any $y \in \mathbb{S}^1$ is such an eigenvector. On the other hand, if $\sigma_{\min}(x) < \sigma_{\max}(x)$ (i.e. if $A(x)$ is *anisotropic*) then the second inequality in (75) becomes an equality only when the measure ν is supported in $\{\xi, -\xi\}$, where $\xi \in \mathbb{S}^1$ is such that $A(x)\xi = \sigma_{\max}(x)\xi$ (this eigenvector is now unique up to the orientation, and this completely determines ν as a probability measure on the projective space). In order to minimize the essential supremum in (10), it is necessary that the second inequality in (75) becomes an equality for a.e. $x \in \Omega$: this is the claimed condition on the support of ν_x.

Thus, in the light of these necessary conditions, any candidate minimizer has the form $\theta = f_\mu(x)\,dx \otimes \nu_x$, with $f_\mu > 0$ such that $\int_\Omega f_\mu = 1$ and the ν_x as in (c). For any such varifold θ, from (10) we find

$$F_\infty(\theta) = \frac{1}{\pi^2} \operatorname*{ess\,sup}_{x \in \Omega} \frac{1}{\left(f_\mu(x)\sqrt{\sigma_{\max}(x)}\right)^2} = \frac{1}{\pi^2} \operatorname*{ess\,sup}_{x \in \Omega} \frac{1}{f_\mu(x)^2 \sigma_{\max}(x)},$$

and by Cauchy-Schwarz it is immediate to check that the only optimal choice for $f_\mu(x)$, under the constraints $f_\mu > 0$ and $\int_\Omega f_\mu = 1$, is that in (11).

Combining with condition (b) above, we see that the first marginal μ of any minimizer is necessarily given by $f_\infty(x)\,dx$. If, moreover, $A(x)$ is purely anisotropic, condition (c) forces the measures ν_x to be uniquely determined (as measures on the projective space): as a consequence, the minimizer θ is uniquely determined as an element of $\mathcal{V}_1(\overline{\Omega})$. $\qquad\square$

Proof of Theorem 6. By Corollary 5, up to a subsequence (not relabelled) we have $\theta_{\Sigma_L} \rightharpoonup \theta_\infty$ in $\mathcal{V}_1(\overline{\Omega})$, where θ_∞ is an absolute minimizer of F_∞. Recalling (8), this means that

$$(76) \qquad \lim_{L \to \infty} \int_{\overline{\Omega} \times \mathbb{S}^1} \varphi(x, y)\, d\theta_{\Sigma_L} = \int_{\overline{\Omega} \times \mathbb{S}^1} \varphi(x, y)\, d\theta_\infty \quad \forall \varphi \in C_{\mathrm{sym}}(\overline{\Omega} \times \mathbb{S}^1).$$

Moreover, by the last part of Theorem 4, the first marginal μ of θ_∞ is *uniquely* determined as a probability measure over $\overline{\Omega}$: it is the absolutely continuous measure with density $f_\infty(x)$ defined in (11). Now fix a square $Q \subset \Omega$. Since μ is absolutely continuous w.r.to the Lebesgue measure in Ω and $|\partial Q| = 0$, we have $\theta_\infty(\partial Q \times \mathbb{S}^1) = 0$, that is, the measure θ does not charge the set $(\partial Q) \times \mathbb{S}^1$. This set is the boundary

(relative to $\overline{\Omega} \times \mathbb{S}^1$) of the cylinder $Q \times \mathbb{S}^1$: therefore, from standard results in measure theory, the weak convergence in (76) can be localized to the cylinder $Q \times \mathbb{S}^1$, that is, we have

$$\lim_{L\to\infty} \int_{Q\times\mathbb{S}^1} \varphi(x,y)\, d\theta_{\Sigma_L} = \int_{Q\times\mathbb{S}^1} \varphi(x,y)\, d\theta_\infty \quad \forall \varphi \in C_{\text{sym}}(\overline{\Omega} \times \mathbb{S}^1).$$

More explicitly, according to (7) written with Σ_L in place of Σ, this means that

$$\lim_{L\to\infty} \frac{1}{\mathcal{H}^1(\Sigma_L)} \int_{Q\cap\Sigma_L} \varphi(x, \xi_{\Sigma_L}(x))\, d\mathcal{H}^1(x) = \int_Q \left(\int_{\mathbb{S}^1} \varphi(x,y)\, d\nu_x(y) \right) f_\infty(x)\, dx,$$

having used the slicing $\theta_\infty = f_\infty(x)dx \otimes \nu_x$, provided by Theorem 4, in the right hand side. In particular, (12) follows if we choose $\varphi \equiv 1$.

Similarly, choosing $\varphi(x,y) = \psi(y)$ where $\psi \in C(\mathbb{S}^1)$ satisfies $\psi(-y) = \psi(y)$, we have

$$\lim_{L\to\infty} \frac{1}{\mathcal{H}^1(\Sigma_L)} \int_{Q\cap\Sigma_L} \psi(\xi_{\Sigma_L}(x))\, d\mathcal{H}^1(x) = \int_Q \left(\int_{\mathbb{S}^1} \psi(y)\, d\nu_x(y) \right) f_\infty(x)\, dx.$$

If Q is contained in the anisotropy region where $\sigma_{\min}(x) < \sigma_{\max}(x)$, then by Theorem 4 for a.e. $x \in Q$ the probability measure ν_x is supported on the set $\{\xi(x), -\xi(x)\}$, where $\xi(x)$ is the eigenvector of $A(x)$ relative to its largest eigenvalue $\sigma_{\max}(x)$. Therefore, since $\psi(y) = \psi(-y)$, the last equation simplifies to

$$\lim_{L\to\infty} \frac{1}{\mathcal{H}^1(\Sigma_L)} \int_{Q\cap\Sigma_L} \psi(\xi_{\Sigma_L}(x))\, d\mathcal{H}^1(x) = \int_Q \psi\big(\xi(x)\big) f_\infty(x)\, dx.$$

If we multiply and divide the left hand side by $\mathcal{H}^1(Q\cap\Sigma_L)$, then (13) follows, using (12).

Finally, using the explicit structure of the minimizers θ_∞ provided by Theorem 4, from (11) we find

$$\min_{\theta \in \mathcal{V}_1(\overline{\Omega})} F_\infty(\theta) = F_\infty(\theta_\infty) = \frac{\left(\int_\Omega 1/\sqrt{\sigma_{\max}(x)}dx \right)^2}{\pi^2}.$$

Then (14) follows immediately, from basic Γ-convergence theory. $\square$

References

[1] F.J. ALMGREN, *The theory of varifolds: A variational calculus in the large for the k-dimensional area integrand*, Princeton: Princeton University Library, 1965.

[2] L. AMBROSIO, N. FUSCO AND D. PALLARA, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[3] L. AMBROSIO, N. GIGLI AND G. SAVARÉ, *Gradient flows in metric spaces and in the space of probability measures*, Second Edition, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.

[4] L. AMBROSIO AND P. TILLI, *Topics on analysis in metric spaces*, Oxford Lecture Series in Mathematics and its Applications, 25, Oxford University Press, Oxford, 2004.

[5] G. BOUCHITTÉ, C. JIMENEZ AND R. MAHADEVAN, *Asymptotic analysis of a class of optimal location problems*, J. Math. Pures Appl. (9), 95 (2011), pp. 382–419.

[6] G. BOUCHITTÉ, C. JIMENEZ AND R. MAHADEVAN, *Asymptotique d'un problème de positionnement optimal*, C. R. Math. Acad. Sci. Paris, 335 (2002), pp. 853–858.

[7] G. Buttazzo, E. Oudet, and E. Stepanov, *Optimal transportation problems with free Dirichlet regions*, in Variational Methods for Discontinuous Structures, Birkhäuser, Basel, pp. 41–65, 2002.

[8] G. Buttazzo and F. Santambrogio, *Asymptotical compliance optimization for connected networks*, Netw. Heterog. Media, 2 (2007), pp. 761–777.

[9] G. Buttazzo, F. Santambrogio and E. Stepanov, *Asymptotic optimal location of facilities in a competition between population and industries*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 12 (2013), pp. 239–273.

[10] G. Buttazzo and E. Stepanov, *Optimal transportation networks as free Dirichlet regions for the Monge-Kantorovich problem*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2 (2003), pp. 631–678.

[11] G. Dal Maso, *An introduction to Γ-convergence*, Progress in Nonlinear Differential Equations and their Applications, 8, Birkhäuser, Boston, 1993.

[12] E. M. Harrell II, P. Kröger and K. Kurata, *On the placement of an obstacle or a well so as to optimize the fundamental eigenvalue*, SIAM J. Math. Anal., 33 (2001), pp. 240–259.

[13] A. Henrot, *Extremum problems for eigenvalues of elliptic operators*, Frontiers in Mathematics, Birkhäuser, Basel, 2006.

[14] S. J. N. Mosconi and P. Tilli, *Γ-convergence for the irrigation problem*, J. Convex Anal., 12 (2005), pp. 145–158.

[15] F. Santambrogio and P. Tilli, *Blow-up of optimal sets in the irrigation problem*, J. Geom. Anal., 15 (2005), pp. 343–362.

[16] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[17] V. Šverák, *On optimal shape design*, J. Math. Pures Appl. (9), 72 (1993), pp. 537–551.

[18] P. Tilli, *Compliance estimates for two-dimensional problems with Dirichlet region of prescribed length*, Netw. Heterog. Media, 7 (2012), pp. 127–136.

[19] P. Tilli and D. Zucco, *Asymptotics of the first Laplace eigenvalue with Dirichlet regions of prescribed length*, SIAM J. Math. Anal. (6), 45 (2013), pp. 3266–3282.

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