RESTRICTION THEOREMS FOR μ-(SEMI)STABLE FRAMED SHEAVES

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Abstract. We provide a generalization of Mehta-Ramanathan theorems to framed sheaves: we prove that the restriction of a μ-semistable framed sheaf on a nonsingular projective irreducible variety of dimension $d \geq 2$ to a general hypersurface of sufficiently high degree is again μ-semistable. The same holds for μ-stability under some additional assumptions.

1. Introduction

In the last ten years moduli spaces of framed sheaves have been studied quite intensively for their connection with moduli spaces of framed instantons. In [3] Donaldson proved that the moduli space of framed $SU(r)$-instantons with instanton charge n on S^4 is isomorphic to the moduli space of vector bundles on $\mathbb{CP}^2$ that are trivial along the line at infinity l_∞ and with a fixed trivialization there. It is an open subset $M^{reg}_{\mathbb{CP}^2,l_\infty}(r,n)$ of the moduli space $M_{\mathbb{CP}^2,l_\infty}(r,n)$ of *framed sheaves* on $\mathbb{CP}^2$, that is, the moduli space parametrizing pairs (E,α), modulo isomorphism, where E is a torsion free sheaf on $\mathbb{CP}^2$ of rank r and $c_2(E)=n$, locally trivial in a neighborhood of l_∞ and $\alpha : E|_{l_\infty} \xrightarrow{\sim} O^{\oplus r}_{l_\infty}$ is the *framing at infinity.* $M_{\mathbb{CP}^2,l_\infty}(r,n)$ is a nonsingular quasi-projective variety of dimension $2rn$. Note that this moduli space also admits a description in terms of linear data, the so-called *ADHM* data (see, for example, chapter 2 in [15]). In some sense we can look at $M_{\mathbb{CP}^2,l_\infty}(r,n)$ as a partial compactification of $M^{reg}_{\mathbb{CP}^2,l_\infty}(r,n)$. There exists another type of partial compactification $M^{Uh}_{\mathbb{CP}^2,l_\infty}(r,n)$ of the latter moduli space, called *Uhlenbeck-Donaldson compactification*: using linear data and geometric invariant theory it is possible to construct a projective morphism

$$\pi_r : M_{\mathbb{CP}^2,l_\infty}(r,n) \to M^{Uh}_{\mathbb{CP}^2,l_\infty}(r,n) = \bigsqcup_{i=0}^{n} M^{reg}_{\mathbb{CP}^2,l_\infty}(r,n-i) \times S^i(\mathbb{C}^2)$$

such that the restriction to the "locally free" part is an isomorphism with its image (see chapter 3 in [15]).

The moduli spaces of framed sheaves can be regarded as higher-rank generalizations of Hilbert schemes of points on a quasi-projective surface. From this point of view, the previous

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morphism is a higher-rank generalization of Hilbert-Chow morphism for Hilbert schemes of points on the complex plane.

Thus the geometry of moduli spaces of framed sheaves on the complex projective plane is quite well known.

In [7] and [8] Huybrechts and Lehn laid the foundations of a systematic theory of framed sheaves on varieties of arbitrary dimension (they used different names to denote the same object like, e.g., *stable pairs*, *framed modules*, etc). Let X be a nonsingular, projective, irreducible variety of dimension d defined over an algebraically closed field k of characteristic zero. A *framed sheaf* is a pair (E, α) where E is a coherent sheaf on X and α is a morphism from E to a fixed coherent sheaf F. They define a generalization of Gieseker semistability (resp. μ-semistability) for framed sheaves that depends on a rational polynomial δ of degree $d-1$ with positive leading coefficient (resp. a rational number δ_1). It is possible to define a contravariant functor $\mathcal{M}^{ss}$ (resp. $\mathcal{M}^{s}$) associated to semistable objects (resp. stable objects). The main result in their papers is the following:

Theorem. *There exists a projective scheme $M^{ss}(F,\delta)$ that corepresents the functor $\mathcal{M}^{ss}$. Moreover there is an open subscheme $M^{s}(F,\delta)$ of $M^{ss}(F,\delta)$ that represents the functor $\mathcal{M}^{s}$, i.e., $M^{s}(F,\delta)$ is a fine moduli space for framed sheaves.*

If X is a surface, we can extend the original definition of framed sheaves on $\mathbb{CP}^2$ with framing along the line at infinity in the following way: let F be a coherent sheaf on X, supported on a big and nef curve D such that F is a semistable locally free O_D-module. A (D,F)*-framed sheaf* is a pair (E,α) where E is a coherent sheaf of positive rank with $\ker\alpha \neq 0$ torsion free, E is locally free in a neighborhood of D and $\alpha|_D$ is an isomorphism. It is possible to prove that there exists a rational polynomial δ such that there is an open subset in $M^{s}(F,\delta)$ parametrizing (D,F)-framed sheaves on X. Moreover if the surface X is rational and the genus of D is zero, then the moduli space of (D,F)-framed sheaves is a nonsingular quasi-projective variety (see [1]).

Leaving aside these results on the representability of the moduli functor, a complete theory of framed sheaves and a study of the geometry of the their moduli spaces is missing in the literature. In the present paper we fill one of the gaps of the theory, by providing a generalization of Mehta-Ramanathan theorems:

Theorem. *Let F be a coherent sheaf on X supported on a divisor D_F. Let $\mathcal{E} = (E, \alpha : E \to F)$ be a framed sheaf on X of positive rank with nontrivial framing. If $\mathcal{E}$ is μ-semistable with respect to δ_1, then there is a positive integer a_0 such that for all $a \geq a_0$ there is a dense open subset $U_a \subset |O_X(a)|$ such that for all $D \in U_a$ the divisor D is smooth, meets transversally the divisor D_F and $\mathcal{E}|_D$ is μ-semistable with respect to $a\delta_1$.*

Theorem. *Let F be a coherent sheaf on X supported on a divisor D_F, which is a locally free O_{D_F}-module. Let $\mathcal{E} = (E, \alpha : E \to F)$ be a (D_F, F)-framed sheaf on X. If $\mathcal{E}$ is μ-stable with respect to δ_1, then there is a positive integer a_0 such that for all $a \geq a_0$ there is a dense open subset $W_a \subset |O_X(a)|$ such that for all $D \in W_a$ the divisor D is smooth, meets transversally the divisor D_F and $\mathcal{E}|_D$ is μ-stable with respect to $a\delta_1$.*

Mehta-Ramanathan theorems are very useful as they often allow one to reduce a problem from a higher-dimensional variety to a surface, as for example happens with in the proof of Hitchin-Kobayashi correspondence (see chapter VI in [10]).

The classical Mehta-Ramanathan theorems, for instance, are used in the algebro-geometric construction of the Uhlenbeck-Donaldson compactification of moduli space of μ-stable vector bundles on a nonsingular projective surface ([12]). In the same way, our result is used in the corresponding work of Bruzzo, Markushevich and Tikhomirov for framed sheaves: in [2] they give a generalization to μ-stable framed sheaves of the Uhlenbeck-Donaldson compactification. In this way they provide a higher-rank generalization of Hilbert-Chow morphism for any quasi-projective surface.

The main difficulty in the generalization of Mehta-Ramanathan theorems was the lack in the theory of framed sheaves of some basic tools. These are, for the first theorem, the (relative) Harder-Narasimhan filtration and, for the second theorem, Jordan-Hölder filtrations. Moreover we want to stress the fact the we could not prove the second theorem in the same generality as the first one, because we have met some problems in the relation between μ-semistability for framed sheaves and duality. We have been able to circumvent these problem only by somehow strengthening the assumptions.

The article is structured as follows. In section 2 we define the notion of framed sheaves and morphisms of framed sheaves. In section 3 we introduce the definition of semistability for framed sheaves and in section 4 we give a characterization of this. In section 5 we construct the maximal framed destabilizing subsheaf (and the minimal framed destabilizing quotient) for a framed sheaf and in section 6 we define the Harder-Narasimhan filtration for framed sheaves. In section 7 we give the definition of Jordan-Hölder filtration for framed sheaves and characterize it for a special family of framed sheaves. In section 8 we construct the relative Harder-Narasimhan filtration. In section 9 we define the μ-semistability condition for framed sheaves and, finally, in section 10 we prove Mehta-Ramanathan theorems for framed sheaves.

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2. Preliminaries

Let X be a nonsingular, projective, irreducible variety of dimension d defined over an algebraically closed field k of characteristic zero, endowed with a very ample line bundle $O_X(1)$. The pair $(X, O_X(1))$ will be fixed throughout this paper.

Fix a coherent sheaf F on X and a polynomial

$$\delta(n) = \delta_1 \frac{n^{d-1}}{(d-1)!} + \delta_2 \frac{n^{d-2}}{(d-2)!} + \cdots + \delta_d \in \mathbb{Q}[n]$$

with $\delta_1 > 0$. We call F *framing sheaf* and δ *stability polynomial.*

Definition 1. A *framed sheaf* on X is a pair $\mathcal{E} := (E, \alpha)$, where E is a coherent sheaf on X and $\alpha : E \to F$ is a morphism of coherent sheaves. We call α *framing* of E.

For any framed sheaf $\mathcal{E} = (E, \alpha)$, we define the function $\epsilon(\alpha)$ by

$$\epsilon(\alpha) := \begin{cases} 1 & \text{if } \alpha \neq 0, \\ 0 & \text{if } \alpha = 0. \end{cases}$$

The *framed Hilbert polynomial of* $\mathcal{E}$ is

$$P_{\mathcal{E}}(n) := P_E(n) - \epsilon(\alpha)\delta(n),$$

and the *framed degree of* $\mathcal{E}$ is

$$\deg(\mathcal{E}) := \deg(E) - \epsilon(\alpha)\delta_1.$$

If E is a d-dimensional coherent sheaf, then we define the *rank of* $\mathcal{E}$ as the rank of E. The *reduced framed Hilbert polynomial of* $\mathcal{E}$ is

$$p_{\mathcal{E}}(n) := \frac{P_{\mathcal{E}}(n)}{\operatorname{rk}(\mathcal{E})}$$

and the *framed slope of* $\mathcal{E}$ is

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\operatorname{rk}(\mathcal{E})}.$$

If E' is a subsheaf of E with quotient $E'' = {}^{E}/_{E'}$, the framing α induces framings $\alpha' := \alpha|_{E'}$ on E' and α'' on E'', where the framing α'' is defined in the following way: $\alpha'' = 0$ if $\alpha' \neq 0$, else α'' is the induced morphism on E''. With this convention the framed Hilbert polynomial of $\mathcal{E}$ behaves additively:

$$P_{\mathcal{E}} = P_{(E',\alpha')} + P_{(E'',\alpha'')} \tag{1}$$

and the same happens for the framed degree:

$$\deg(\mathcal{E}) = \deg(E',\alpha') + \deg(E'',\alpha''). \tag{2}$$

Notation: If $\mathcal{E} := (E,\alpha)$ is a framed sheaf on X and E' is a subsheaf of E, then we denote by $\mathcal{E}'$ the framed sheaf (E',α') and by $\mathcal{E}''$ the framed sheaf (E'',α'').

Thus we have a canonical framing on subsheaves and on quotients. The same happens for subquotients, indeed we have the following result.

Lemma 2 (Lemma 1.12 in [8])**.** *Let $H \subset G \subset E$ be coherent sheaves and α a framing of E. Then the framings induced on ${}^{G}/_{H}$ as a quotient of G and as a subsheaf of ${}^{E}/_{H}$ agree. Moreover*

$$P_{\left(\frac{E/H}{G/H},\beta\right)} = P_{(E/G,\gamma)} \quad \textit{and} \quad \deg\left(\frac{E/H}{G/H},\beta\right) = \deg\left({}^{E}/_{G},\gamma\right),$$

where we denote by β and by γ the induced framings.

Now we introduce the notion of morphism of framed sheaves.

Definition 3. Let $\mathcal{E} = (E,\alpha)$ and $\mathcal{G} = (G,\beta)$ be framed sheaves. A *morphism of framed sheaves* $\varphi : \mathcal{E} \to \mathcal{G}$ between $\mathcal{E}$ and $\mathcal{G}$ is a morphism of the underlying coherent sheaves $\varphi : E \to G$ for which there is an element $\lambda \in k$ such that $\beta \circ \varphi = \lambda\alpha$. We say that $\varphi : \mathcal{E} \to \mathcal{G}$ is injective (surjective) if the morphism $\varphi : E \to G$ is injective (surjective).

Example. Let $\mathcal{E} = (E,\alpha)$ be a framed sheaf. If E' is a subsheaf of E with quotient $E'' = {}^{E}/_{E'}$, then we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r, "i"] \arrow[d, "\alpha'"] & E \arrow[r, "q"] \arrow[d, "\alpha"] & E'' \arrow[r] \arrow[d, "\alpha''"] & 0 \\
& F \arrow[r, "\cdot\lambda"] & F \arrow[r, "\cdot\mu"] & F &
\end{tikzcd}$$

where $\lambda = 0, \mu = 1$ if $\alpha' = 0$, $\lambda = 1, \mu = 0$ if $\alpha' \neq 0$. Thus the inclusion morphism i (the projection morphism q) induces a morphism of framed sheaves between $\mathcal{E}'$ and $\mathcal{E}$ ($\mathcal{E}$ and $\mathcal{E}''$).

Lemma 4 (Lemma 1.5 in [8])**.** *The set* $\mathrm{Hom}(\mathcal{E}, \mathcal{G})$ *of morphisms of framed sheaves is a linear subspace of* $\mathrm{Hom}(E, G)$. *If* $\varphi : \mathcal{E} \to \mathcal{G}$ *is an isomorphism, then the factor* λ *in the definition can be taken in* k^*. *Moreover, if* $\mathcal{E}$ *and* $\mathcal{G}$ *are isomorphic, then their framed Hilbert polynomials and their framed degrees coincide.*

Proposition 5. *Let* $\mathcal{E} = (E, \alpha)$ *and* $\mathcal{G} = (G, \beta)$ *be framed sheaves. If* φ *is a nontrivial morphism of framed sheaves between* $\mathcal{E}$ *and* $\mathcal{G}$, *then*

$$P_{({}^{E}\!/\ker\varphi, \alpha'')} \leq P_{(\mathrm{Im}\varphi, \beta')} \quad \textit{and} \quad \deg\left({}^{E}\!/\ker\varphi, \alpha''\right) \leq \deg(\mathrm{Im}\varphi, \beta').$$

Proof. Consider a morphism of framed sheaves $\varphi \in \mathrm{Hom}(\mathcal{E}, \mathcal{G})$, $\varphi \neq 0$. There exists $\lambda \in k$ such that $\beta \circ \varphi = \lambda\alpha$. Note that ${}^{E}\!/\ker\varphi \simeq \mathrm{Im}\varphi$ hence their Hilbert polynomials and their degree coincide. It remains to prove that $\epsilon(\alpha'') \geq \epsilon(\beta')$. If $\lambda = 0$, then $\beta' = 0$ and therefore $\epsilon(\beta') = 0 \leq \epsilon(\alpha'')$. Assume now $\lambda \neq 0$: $\alpha = 0$ if and only if $\beta|_{\mathrm{Im}\varphi} = 0$, hence $\epsilon(\beta') = 0 = \epsilon(\alpha'')$. If $\alpha \neq 0$, then also $\alpha'' \neq 0$. Indeed if $\alpha'' = 0$, then $\alpha|_{\ker\varphi} \neq 0$; this implies that $\lambda(\alpha|_{\ker\varphi}) = (\beta \circ \varphi)|_{\ker\varphi} = 0$ and therefore $\lambda = 0$, but this is in contradiction with our previous assumption. Thus, if $\lambda \neq 0$ and $\alpha \neq 0$ then we obtain $\epsilon(\beta') = 1 = \epsilon(\alpha'')$. □

3. Semistability

Recall that there is a natural ordering of rational polynomials given by the lexicographic order of their coefficients. Explicitly, $f \leq g$ if and only if $f(m) \leq g(m)$ for $m \gg 0$. Analogously, $f < g$ if and only if $f(m) < g(m)$ for $m \gg 0$.

We shall use the following convention: if the word "(semi)stable" occurs in any statement in combination with the symbol $(\leq)$, then two variants of the statement are asserted at the same time: a "semistable" one involving the relation "$\leq$" and a "stable" one involving the relation "$<$".

We give now a definition of semistability for framed sheaves $\mathcal{E} = (E, \alpha)$ of positive rank.

Definition 6. A framed sheaf $\mathcal{E} = (E, \alpha)$ of positive rank is said to be *(semi)stable* with respect to δ if and only if the following conditions are satisfied:

(i) $\mathrm{rk}(E) P_{E'} (\leq) \mathrm{rk}(E') P_{\mathcal{E}}$ for all subsheaves $E' \subset \ker\alpha$ with $0 \leq \mathrm{rk}(E') \leq \mathrm{rk}(\ker\alpha)$,
(ii) $\mathrm{rk}(E)(P_{E'} - \delta) (\leq) \mathrm{rk}(E') P_{\mathcal{E}}$ for all subsheaves $E' \subset E$ with $0 \leq \mathrm{rk}(E') < \mathrm{rk}(E)$.

Note that if α is zero, then this definition coincides with Gieseker (semi)stability for torsion free sheaves (see def. 1.2.4 in [9]).

Lemma 7 (Lemma 1.2 in [8])**.** *Let* $\mathcal{E} = (E, \alpha)$ *be a framed sheaf of positive rank. If* $\mathcal{E}$ *is semistable with respect to* δ, *then* $\ker\alpha$ *is torsion free.*

There are two different definition of (semi)stability for rank zero framed sheaves in Huybrechts and Lehn's works: in [7], they use the statement of definition 6 also for rank zero framed sheaves, in this way all framed sheaves of rank zero are semistable but not stable. In [8] they change a little bit the definition of (semi)stability (see def. 1.1 in [8]) and respect to this definition, no all rank zero framed sheaves are semistable, but all semistable framed sheaves of rank zero automatically are stable. Now we give a new definition of (semi)stability for framed sheaves of rank zero that will be - in my opinion - a good compromise between these two different definitions.

Definition 8. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\mathrm{rk}(E) = 0$. If α is injective, we say that $\mathcal{E}$ is *semistable* with respect to δ. Moreover, if $P_E = \delta$ we say that $\mathcal{E}$ is *stable* with respect to δ.

Remark 9. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\mathrm{rk}(E) = 0$. Assume that $\mathcal{E}$ is stable with respect to δ. If G is a subsheaf of E, then $P_G < P_E = \delta$; on the other side since the framing is injective, $\alpha|_G \neq 0$ and therefore $P_{(G,\alpha|_G)} = P_G - \delta < 0 = P_E - \delta = P_{\mathcal{E}}$. Hence also stable framed sheaves of rank zero satisfy an inequality condition.

Lemma 10 (Lemma 2.1 in [7])**.** *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker \alpha \neq 0$. If $\mathcal{E}$ is (semi)stable with respect to δ, then*

$$\delta (\leq) P_E - \frac{\mathrm{rk}(E)}{\mathrm{rk}(\ker \alpha)}(P_E - P_F).$$

If F is a torsion sheaf, then $\delta(\leq) P_F$ and in particular $\delta_1 (\leq) \deg(F)$.

4. Characterization of semistability

Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker \alpha \neq 0$ torsion free. In this section we would like to answer the following question: to verify if $\mathcal{E}$ is (semi)stable or not, do we need to check the inequalities (i) and (ii) in the definition 6 for all subsheaves of E? Or, can we restrict our attention to a smaller family of subsheaves of E?

Definition 11. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker \alpha \neq 0$ torsion free. Let E' be a subsheaf of E.

- If $E' \subset \ker \alpha$, then the *framed-saturation* $\bar{E}'$ of E' is the saturation of E' as subsheaf of $\ker \alpha$.
- If $E' \not\subset \ker \alpha$, then the *framed-saturation* $\bar{E}'$ of E' is the saturation of E' as subsheaf of E.

Remark 12. Let $\bar{E}'$ be the framed-saturation of $E' \subset E$. In the first case described in the definition, if $\mathrm{rk}(E') < \mathrm{rk}(\ker \alpha)$, then the quotient $Q = E/\bar{E}'$ is a coherent sheaf of positive rank, with nonzero induced framing, such that it fits into an exact sequence

$$0 \longrightarrow Q' \longrightarrow Q \longrightarrow \mathrm{Im}\alpha \longrightarrow 0, \tag{3}$$

where Q' is a torsion free quotient of $\ker \alpha$. If $\mathrm{rk}(E') = \mathrm{rk}(\ker \alpha)$, then $\bar{E}' = \ker \alpha$ and $Q = E/\ker \alpha$. In the second case, Q is a torsion free sheaf with zero induced framing. Moreover

- $\mathrm{rk}(E') = \mathrm{rk}(\bar{E}')$, $P_{E'} \leq P_{\bar{E}'}$ and $\deg(E') \leq \deg(\bar{E}')$,
- $P_{\mathcal{E}'} \leq P_{\bar{\mathcal{E}}'}$ and $\deg(\mathcal{E}') \leq \deg(\bar{\mathcal{E}}')$.

Proposition 13. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker \alpha \neq 0$ torsion free. Then the following conditions are equivalent:*

(a) *$\mathcal{E}$ is semistable with respect to δ.*
(b) *For any framed-saturated subsheaf $E' \subset E$ one has $P_{(E',\alpha')} \leq \mathrm{rk}(E') p_{\mathcal{E}}$.*
(c) *For any surjective morphism of framed sheaves $\varphi : \mathcal{E} \to (Q, \beta)$, one has $\mathrm{rk}(Q) p_{\mathcal{E}} \leq P_{(Q,\beta)}$.*

(d) *For any surjective morphism of framed sheaves $\varphi : \mathcal{E} \to (Q, \beta)$, where $\alpha = \beta \circ \varphi$ and Q has the form described in the remark 12, one has* $\mathrm{rk}(Q)p_{\mathcal{E}} \leq P_{(Q,\beta)}$.

Proof. The implications $(a) \Rightarrow (b)$ and $(c) \Rightarrow (d)$ are obvious.

Consider a framed sheaf $\mathcal{Q} = (Q, \beta)$ and a surjective morphism $\varphi : \mathcal{E} \to \mathcal{Q}$ of framed sheaves. Put $E' = \ker \varphi$ and $E'' = E/\ker\varphi$. Using formula (1) one has

$$\big(P_{\mathcal{E}'} - \mathrm{rk}(E')p_{\mathcal{E}}\big) = \big(\mathrm{rk}(E'')p_{\mathcal{E}} - P_{\mathcal{E}''}\big) \tag{4}$$

If $\mathcal{E}$ is semistable, then $\mathrm{rk}(E'')p_{\mathcal{E}} \leq P_{\mathcal{E}''}$ and using proposition 5 we have $(a) \Rightarrow (c)$. Moreover the framed sheaf $\mathcal{Q}$ has the properties asserted in condition (d) if and only if $\ker \varphi$ is a framed-saturated subsheaf, hence $(b) \iff (d)$. Finally by remark 12, $P_{\mathcal{E}'} \leq P_{\bar{\mathcal{E}}'} \leq \mathrm{rk}(\bar{E}')p_{\mathcal{E}} = \mathrm{rk}(E')p_{\mathcal{E}}$, where $\bar{E}'$ is the framed-saturation of E', thus $(b) \Rightarrow (a)$. □

Corollary 14. *Let $\mathcal{E} = (E, \alpha)$ and $\mathcal{G} = (G, \beta)$ be framed sheaves of positive rank with the same reduced framed Hilbert polynomial p.*

(1) *If $\mathcal{E}$ is semistable and $\mathcal{G}$ is stable, then all nontrivial morphisms $\varphi : \mathcal{E} \to \mathcal{G}$ are surjective.*
(2) *If $\mathcal{E}$ is stable and $\mathcal{G}$ is semistable, then all nontrivial morphisms $\varphi : \mathcal{E} \to \mathcal{G}$ are injective.*

Corollary 15. *Let $\mathcal{E} = (E, \alpha)$ and $\mathcal{G} = (G, \beta)$ be framed sheaves of positive rank. If $\mathcal{E}$ and $\mathcal{G}$ are stable with the same framed reduced Hilbert polynomial, then any nontrivial morphism $\varphi : \mathcal{E} \to \mathcal{G}$ is an isomorphism. Moreover, in this case*

$$\mathrm{Hom}(\mathcal{E}, \mathcal{G}) \simeq k.$$

Definition 16. Let $\mathcal{E}$ be a framed sheaf. We say that $\mathcal{E}$ is simple if $\mathrm{End}(\mathcal{E}) \simeq k$.

5. Maximal framed-destabilizing subsheaf

Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker \alpha \neq 0$ torsion free. If $\mathcal{E}$ is not semistable with respect to δ, then there exist destabilizing subsheaves of $\mathcal{E}$. In this section we would like to find the maximal one (with respect to the inclusion) and show that it has some interesting properties.

Proposition 17. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with* $\ker \alpha \neq 0$ *torsion free. If $\mathcal{E}$ is not semistable with respect to δ, then there is a subsheaf $G \subset E$ such that for any subsheaf $E' \subseteq E$ one has*

$$\mathrm{rk}(E')P_{\mathcal{G}} \geq \mathrm{rk}(G)P_{\mathcal{E}'}$$

and in case of equality, one has $E' \subset G$.

Moreover, the framed sheaf $\mathcal{G}$ is uniquely determined and is semistable with respect to δ.

Proof. On the set of nontrivial subsheaves of E we define the following order relation $\preceq$: let G_1 and G_2 be nontrivial subsheaves of E, $G_1 \preceq G_2$ if and only if $G_1 \subseteq G_2$ and $\mathrm{rk}(G_2)P_{\mathcal{G}_1} \leq \mathrm{rk}(G_1)P_{\mathcal{G}_2}$. Since any ascending chain of subsheaves stabilizes, for any subsheaf E', there is a subsheaf G' such that $E' \subseteq G' \subseteq E$ and G' is maximal with respect to $\preceq$.

First note that if there exists a $\preceq$-maximal subsheaf E' of rank zero, then $P_{\mathcal{E}'} > 0$, that is, $P_{E'} > \delta$. Let $T(E)$ be the torsion subsheaf of E. Then $P_{T(E)} \geq P_{E'} > \delta$. We choose $G := T(E)$. From now on assume that for every rank zero subsheaf $T \subset E$ we have $P_{(T,\alpha')} \leq 0$. Let $G \subset E$

be a $\preceq$-maximal subsheaf with minimal rank among all $\preceq$-maximal subsheaves. Note that $\mathrm{rk}(G) > 0$.

Suppose there exists a subsheaf $H \subset E$ with $\mathrm{rk}(H)p_{\mathcal{G}} < P_{\mathcal{H}}$. By hypothesis we have $\mathrm{rk}(H) > 0$. From $\preceq$-maximality of G we get $G \nsubseteq H$ (in particular $H \neq E$). First we show that we can assume $H \subset G$ by replacing H with $G \cap H$. If $H \nsubseteq G$, then the morphism $\varphi : H \to E \to {}^{E}/_{G}$ is nonzero. Moreover $\ker\varphi = G \cap H$. Put $J = \ker\varphi$ and $I = \mathrm{Im}\varphi$. The sheaf I is of the form ${}^{T}/_{G}$ with $G \subsetneq T \subset E$ and $\mathrm{rk}(T) > 0$. By the $\preceq$-maximality of G we have

$$p_{\mathcal{T}} < p_{\mathcal{G}}. \tag{5}$$

By lemma 2 one has $P_{\mathcal{I}} = P_{\mathcal{T}} - P_{\mathcal{G}}$, hence

$$\begin{aligned} \mathrm{rk}(G)P_{\mathcal{I}} &= \mathrm{rk}(G)(P_{\mathcal{T}} - P_{\mathcal{G}}) \\ &< \mathrm{rk}(T)P_{\mathcal{G}} - \mathrm{rk}(G)P_{\mathcal{G}} = \mathrm{rk}(I)P_{\mathcal{G}}, \end{aligned}$$

therefore

$$\mathrm{rk}(G)P_{\mathcal{I}} < \mathrm{rk}(I)P_{\mathcal{G}}. \tag{6}$$

First we show that $J \neq 0$. Indeed, if $J = 0$, then $\mathrm{rk}(I) = \mathrm{rk}(H)$ and we have the following commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & {}^{E}/_{G} \\ \Big\downarrow{\scriptstyle \alpha'} & & \Big\downarrow{\scriptstyle \beta} \\ F & \xrightarrow{\cdot\lambda} & F \end{array}$$

where β is the induced framing on the quotient; if $\alpha' \neq 0$ and $\beta = 0$, then we put $\lambda = 0$, for the other combinations of values of $\epsilon(\alpha')$ and $\epsilon(\beta)$, we put $\lambda = 1$. So we obtain a morphism of framed sheaves φ between $\mathcal{H}$ and $({}^{E}/_{G}, \beta)$. By proposition 5, $P_{\mathcal{H}} \leq P_{\mathcal{I}}$ and using formula 6 one has

$$\mathrm{rk}(H)P_{\mathcal{G}} < \mathrm{rk}(G)P_{\mathcal{H}} \leq \mathrm{rk}(G)P_{\mathcal{I}} < \mathrm{rk}(H)P_{\mathcal{G}},$$

which it is absurd. Moreover the rank of J is positive, in fact if we assume that $\mathrm{rk}(J) = 0$, then we have $\mathrm{rk}(I) = \mathrm{rk}(H)$ and

$$\begin{aligned} \mathrm{rk}(G)P_{\mathcal{J}} &= \mathrm{rk}(G)P_{\mathcal{H}} - \mathrm{rk}(G)P_{({}^{H}/_{J}, \alpha')} \\ &\geq \mathrm{rk}(G)P_{\mathcal{H}} - \mathrm{rk}(G)P_{\mathcal{I}} > \mathrm{rk}(G)P_{\mathcal{H}} - \mathrm{rk}(H)P_{\mathcal{G}} > 0 \end{aligned}$$

hence J is a rank zero subsheaf of E with $P_{\mathcal{J}} > 0$, but this is in contradiction with the hypothesis. By the following computation:

$$\begin{aligned} \mathrm{rk}(J)\left(p_{\mathcal{J}} - p_{\mathcal{H}}\right) &= \mathrm{rk}({}^{H}/_{J})\left(p_{\mathcal{H}} - p_{({}^{H}/_{J}, \alpha')}\right) \\ &> \mathrm{rk}(I)\left(p_{\mathcal{H}} - p_{\mathcal{I}}\right) > \mathrm{rk}(I)\left(p_{\mathcal{H}} - p_{\mathcal{G}}\right) > 0 \end{aligned}$$

we get $p_{\mathcal{H}} < p_{\mathcal{J}}$, hence we can consider a subsheaf $H \subset G$ such that H is $\preceq$-maximal in G, $\mathrm{rk}(H) > 0$ and

$$p_{\mathcal{G}} < p_{\mathcal{H}}.$$

Let $H' \subset E$ be a sheaf that contains H and is $\preceq$-maximal in E. In particular, one has

$$p_{\mathcal{G}} < p_{\mathcal{H}} \leq p_{\mathcal{H}'}.$$

By $\preceq$-maximality of H and G, we have $H' \nsubseteq G$. Then the morphism $\psi : H' \to E \to \frac{E}{G}$ is nonzero and $H \subset \ker\psi$. As before

$$p_{\mathcal{H}'} < p_{(\ker\psi,\alpha')}.$$

Thus we have $H \subset H' \cap G = \ker\psi$ and $p_{\mathcal{H}} < p_{(\ker\psi,\alpha')}$, hence $H \preceq \ker\psi$. This contradicts the $\preceq$-maximality of H in G. Thus for all subsheaves $H \subseteq E$, we have $\mathrm{rk}(H)p_{\mathcal{G}} \geq P_{\mathcal{H}}$.

If there is a subsheaf $H \subset E$ of rank zero such that $P_{\mathcal{H}} = 0$ and $H \nsubseteq G$, then by using the same argument than before, we get $P_{(H\cap G,\alpha')} > 0$, but this is in contradiction with the hypothesis. So there are no subsheaves $H \subset E$ of rank zero such that $P_{\mathcal{H}} = 0$ and $H \nsubseteq G$. If there is a subsheaf $H \subset E$ of positive rank such that $p_{\mathcal{G}} = p_{\mathcal{H}}$, then $H \subset G$. In fact, if $H \nsubseteq G$ then we can replace H by $G \cap H$ and using the same argument than before we obtain $p_{\mathcal{G}} = p_{\mathcal{H}} < p_{(H\cap G,\alpha')}$ and this is absurd. □

Definition 18. We call G *the maximal framed-destabilizing subsheaf of* $\mathcal{E}$.

Remark 19. Note that if G is a maximal framed-destabilizing sheaf of $\mathcal{E}$ and $G \neq \ker\alpha$, then the quotient sheaf ${}^{E}/_{G}$ is a torsion free sheaf or a coherent sheaf of positive rank which fits into a exact sequence of the form (3).

We give now a criterion to find the maximal framed-destabilizing subsheaf that will be useful later.

Proposition 20. *Let* $\mathcal{E} = (E,\alpha)$ *be a framed sheaf with* $\ker\alpha \neq 0$ *torsion free. Assume that* $\mathcal{E}$ *is not semistable with respect to* δ *but there are no rank zero framed-destabilizing subsheaves. If* $G \subset E$ *is a positive rank subsheaf with positive rank quotient* $G' = {}^{E}/_{G}$ *such that*

(i) *the framed sheaves* $\mathcal{G}$ *e* $\mathcal{G}'$ *are semistable with respect to* δ,
(ii) $p_{\mathcal{G}} > p_{\mathcal{G}'}$,

then G *is a maximal framed-destabilizing subsheaf of* $\mathcal{E}$.

Proof. Let H be a subsheaf of E.

Case 1: $H \subsetneq G$. By semistability we get $P_{\mathcal{H}} \leq \mathrm{rk}(H)p_{\mathcal{G}}$.

Case 2: $G \subsetneq H$. By properties (i) and (ii) one has $\mathrm{rk}({}^{H}/_{G})p_{\mathcal{G}} > \mathrm{rk}({}^{H}/_{G})p_{\mathcal{G}'} \geq P_{({}^{H}/_{G},\gamma)}$ where γ is the induced framing on ${}^{H}/_{G}$. By lemma 2 we have $P_{\mathcal{H}} = P_{({}^{H}/_{G},\gamma)} + P_{\mathcal{G}}$ and therefore

$$\begin{aligned} P_{\mathcal{H}} &= \left(P_{({}^{H}/_{G},\gamma)} + P_{\mathcal{G}}\right) \\ &< (\mathrm{rk}({}^{H}/_{G})p_{\mathcal{G}} + \mathrm{rk}(G)p_{\mathcal{G}}) = \mathrm{rk}(H)p_{\mathcal{G}}. \end{aligned}$$

Consider now the case in which $G \nsubseteq H$ and $H \nsubseteq G$. The morphism $\varphi : H \to E \to \frac{E}{G}$ is nonzero. Moreover $\ker\varphi = H \cap G$.

Case 3: $H \cap G = 0$. If $\mathrm{rk}(H) = 0$, then by hypothesis $P_{\mathcal{H}} \leq 0 = \mathrm{rk}(H)p_{\mathcal{G}}$. If $\mathrm{rk}(H) > 0$, then the morphism φ is injective and induces a morphism of framed sheaves $\varphi : \mathcal{H} \to \mathcal{G}'$, hence by proposition 5 we obtain $p_{\mathcal{H}} \leq p_{(\mathrm{Im}\varphi,\beta')} \leq p_{\mathcal{G}'} < p_{\mathcal{G}}$, where β is the induced framing on G'.

Case 4: $H \cap G \neq 0$. From the hypothesis follows that $P_{(H\cap G,\alpha')} \leq \mathrm{rk}(H \cap G)p_{\mathcal{G}}$ and $P_{({}^{H}/_{\ker\varphi},\alpha'')} \leq P_{(\mathrm{Im}\varphi,\beta')} \leq \mathrm{rk}({}^{H}/_{\ker\varphi})p_{\mathcal{G}'} < \mathrm{rk}({}^{H}/_{\ker\varphi})p_{\mathcal{G}}$, hence we get

$$\begin{aligned} P_{\mathcal{H}} &= P_{(\ker\varphi,\alpha')} + P_{({}^{H}/_{\ker\varphi},\alpha'')} \\ &< (\mathrm{rk}(\ker\varphi) + \mathrm{rk}({}^{H}/_{\ker\varphi}))p_{\mathcal{G}}, \end{aligned}$$

and therefore
$$P_{\mathcal{H}} < \mathrm{rk}(H) p_{\mathcal{G}}.$$
$\square$

If the rank of the framing sheaf F is zero, then we have this additional characterization:

Proposition 21. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker \alpha \neq 0$ torsion free. Assume that $\mathcal{E}$ is not semistable with respect to δ. Then $\ker \alpha$ is the maximal framed-destabilizing subsheaf of $\mathcal{E}$ if and only if it is Gieseker-semistable and $P_{(E/\ker \alpha, \beta)} < 0$, where β is the induced framing.*

Proof. It follows from the same arguments used in the previous proposition. $\square$

Minimal framed-destabilizing quotient. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker \alpha \neq 0$ torsion free. Suppose that $\mathcal{E}$ is not semistable with respect to δ.

Consider first the case $\mathrm{rk}(F) = 0$. In this situation, we have to assume that $\ker \alpha$ is not the maximal framed-destabilizing subsheaf.

Let T_1 be the set consisting of the quotients $E \xrightarrow{q} Q \to 0$ such that

- Q is torsion free,
- the induced framing on $\ker(q)$ is nonzero,
- $p_Q < p_{\mathcal{E}}$.

Let T_2 be the set consisting of the quotients $E \xrightarrow{q} Q \to 0$ such that

- Q has positive rank,
- the induced framing on $\ker(q)$ is zero,
- Q fits into a exact sequence of the form 3,
- $p_{\mathcal{Q}} < p_{\mathcal{E}}$.

By proposition 13, the set $T_1 \cup T_2$ is nonempty. For $i = 1, 2$ define an order relation on T_i as follows: if $Q_1, Q_2 \in T_i$, we say that $Q_1 \sqsubseteq Q_2$ if and only if $p_{\mathcal{Q}_1} \leq p_{\mathcal{Q}_2}$ and $\mathrm{rk}(Q_1) \leq \mathrm{rk}(Q_2)$ in the case $p_{\mathcal{Q}_1} = p_{\mathcal{Q}_2}$. Let Q^i_- be a $\sqsubseteq$-minimal element in T_i, for $i = 1, 2$. Define
$$Q_- := \begin{cases} Q^1_- & \text{if } p_{\mathcal{Q}_1} < p_{\mathcal{Q}_2} \text{ or if } p_{\mathcal{Q}_2} = p_{\mathcal{Q}_1} \text{ and } \mathrm{rk}(Q_1) \leq \mathrm{rk}(Q_2), \\ Q^2_- & \text{if } p_{\mathcal{Q}_2} < p_{\mathcal{Q}_1} \text{ or if } p_{\mathcal{Q}_2} = p_{\mathcal{Q}_1} \text{ and } \mathrm{rk}(Q_2) < \mathrm{rk}(Q_1). \end{cases}$$

Proposition 22. *The sheaf $G := \ker(E \to Q_-)$ is $\preceq$-maximal in $\mathcal{E}$.*

Proof. If $\mathrm{rk}(G) = 0$, then $G = T(E)$, indeed if $G \subsetneq T(E)$, then $P_{T(E)} > P_G$. So $P_E - P_{T(E)} < P_E - P_G$, hence $p_{E/T(E)} < p_{E/G}$, but this contradicts the $\sqsubseteq$-minimality of Q_-. Assume now that $\mathrm{rk}(G) > 0$. If G is not $\preceq$-maximal, then there exists $G \subsetneq G' \subseteq E$ such that $G \preceq G'$ and G' is $\preceq$-maximal in E, i.e. $p_{\mathcal{G}} \leq p_{\mathcal{G}'}$.

If $p_{\mathcal{G}} < p_{\mathcal{G}'}$, then using formula 4 we have $p_{(E/G', \gamma)} < p_{\mathcal{Q}_-}$, where γ is the induced framing on E/G', which is a contradiction with the hypothesis. If $p_{\mathcal{G}} = p_{\mathcal{G}'}$, by the same computation than before we have $p_{(E/G', \gamma)} = p_{\mathcal{Q}_-}$. Since $\mathrm{rk}(G) \leq \mathrm{rk}(G')$, we have $\mathrm{rk}(E/G') \leq \mathrm{rk}(Q)$, but by definition of Q_- we must have $\mathrm{rk}(E/G') = \mathrm{rk}(Q_-)$. Thus E/G' is $\sqsubseteq$-minimal. If $Q_- \in T_1$, then $\alpha|_G \neq 0$, hence $\alpha|_{G'} \neq 0$ and $E/G' \in T_1$. From $\mathrm{rk}(E/G') = \mathrm{rk}(Q_-)$ and $p_{(E/G', \gamma)} = p_{\mathcal{Q}_-}$ it follows $P_{E/G'} = P_{Q_-}$, hence $G = G'$. If $Q_- \in T_2$, by definition of Q_- we must have that E/G' is an element of T_2 and by using the same argument than before we get $G = G'$. $\square$

Remark 23. By proposition 17, we have that G is the maximal framed-destabilizing subsheaf of $\mathcal{E}$.

For the case $\mathrm{rk}(F) > 0$, we can use the same arguments than before without any assumption on $\ker\alpha$ and we obtain the same results.

6. Harder-Narasimhan filtration

In this section we construct the Harder-Narasimhan filtration for a framed sheaf. We adapt the techniques used by Harder and Narasimhan in the case of vector bundles on curves (see [6]).

Consider first the case $\mathrm{rk}(F) = 0$.

Definition 24. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker\alpha \neq 0$ torsion free. A *Harder-Narasimhan filtration* for $\mathcal{E}$ is an increasing filtration of framed-saturated subsheaves

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) \subset \cdots \subset \mathrm{HN}_l(\mathcal{E}) = E \tag{7}$$

which satisfies the following conditions

(A) the quotient sheaf $gr_i^{\mathrm{HN}}(\mathcal{E}) := {}^{\mathrm{HN}_i(\mathcal{E})}/_{\mathrm{HN}_{i-1}(\mathcal{E})}$ with the induced framing α_i is a semistable framed sheaf with respect to δ for $i = 1, 2, \ldots, l$.

(B) The quotient ${}^{E}/_{\mathrm{HN}_{i-1}(\mathcal{E})}$ has positive rank, the kernel of the induced framing is nonzero and torsion free and the subsheaf $gr_i^{\mathrm{HN}}(\mathcal{E})$ is the maximal framed-destabilizing subsheaf of $({}^{E}/_{\mathrm{HN}_{i-1}(\mathcal{E})}, \alpha'')$ for $i = 1, 2, \ldots, l-1$.

Lemma 25. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker\alpha \neq 0$ torsion free. Suppose that $\mathcal{E}$ is not semistable (with respect to δ). Let G be the maximal framed-destabilizing sheaf of $\mathcal{E}$. If $G \neq \ker\alpha$, then for every rank zero subsheaf T of ${}^{E}/_{G}$, we get $P_{(T,\beta')} \leq 0$, where β is the induced framing on ${}^{E}/_{G}$.*

Proof. If the quotient ${}^{E}/_{G}$ is torsion free then the condition is trivially satisfied. Otherwise let $T \subset {}^{E}/_{G}$ be a rank zero subsheaf with $P_{(T,\beta')} > 0$. The sheaf T is of the form ${}^{E'}/_{G}$, where $G \subset E'$ and $\mathrm{rk}(E') = \mathrm{rk}(G)$, hence we obtain $p_{\mathcal{E}'} > p_{\mathcal{G}}$, therefore E' contradicts the maximality of G. □

Theorem 26. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker\alpha \neq 0$ torsion free. Then there exists a unique Harder-Narasimhan filtration for $\mathcal{E}$.*

Proof. Existence. If $\mathcal{E}$ is a semistable framed sheaf with respect to δ, then we put $l = 1$ and a Harder-Narasimhan filtration is

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) = E$$

Else there exists a subsheaf $E_1 \subset E$ such that E_1 is the maximal framed-destabilizing subsheaf of $\mathcal{E}$. If $E_1 = \ker\alpha$, then a Harder-Narasimhan filtration is

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \ker\alpha \subset \mathrm{HN}_2(\mathcal{E}) = E$$

Otherwise, by lemma 25 $({}^{E}/_{E_1}, \alpha'')$ is a framed sheaf with $\ker\alpha'' \neq 0$ torsion free and no rank zero framed-destabilizing subsheaves. If $({}^{E}/_{E_1}, \alpha'')$ is a semistable framed sheaf, then a Harder-Narasimhan filtration is

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset E_1 \subset \mathrm{HN}_2(\mathcal{E}) = E$$

Else there exists a subsheaf $E_2' \subset {}^{E}/_{E_1}$ of positive rank such that E_2' is the maximal framed-destabilizing subsheaf of $({}^{E}/_{E_1}, \alpha'')$. We denote by E_2 its pre-image in E. Now we apply the previous argument to E_2 instead of E_1. Thus we can iterate this procedure and we obtain a finite length increasing filtration of framed-saturated subsheaves of E, which satisfies conditions (A) and (B).

Uniqueness. The uniqueness of the Harder-Narasimhan filtration follows from the uniqueness of the maximal framed-destabilizing subsheaf. □

By construction for $i > 0$ at most one of the framings α_i is nonzero and all but possibly one of the factors $gr_i^{\mathrm{HN}}(\mathcal{E})$ are torsion free. In particular if $\mathrm{rk}(gr_1^{\mathrm{HN}}(\mathcal{E})) = 0$, then $gr_1^{\mathrm{HN}}(\mathcal{E}) = T(E)$ and $\alpha_1 \neq 0$; if $\mathrm{rk}(gr_l^{\mathrm{HN}}(\mathcal{E})) = 0$, then $gr_l^{\mathrm{HN}}(\mathcal{E}) = {}^{E}/_{\ker\alpha}$ and $\alpha_l \neq 0$.

Proposition 27. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker\alpha \neq 0$ torsion free. Suppose there exists a filtration of the form 7 satisfying condition (A). Then condition (B) is equivalent to the following:*

(B') *the quotient $({}^{E}/_{\mathrm{HN}_J(\mathcal{E})}, \alpha'')$ is a framed sheaf with $\ker\alpha'' \neq 0$ torsion free for $j = 1, 2, \ldots, l-2$, it has no rank zero framed-destabilizing subsheaves, and*

$$\mathrm{rk}(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}))P_{(gr_i^{\mathrm{HN}}(\mathcal{E}),\alpha_i)} > \mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P_{(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}),\alpha_{i+1})} \tag{8}$$

for $i = 1, \ldots, l-1$.

Proof. The arguments used to prove this proposition are similar to that used in the proof of the analogous result for vector bundles on curves (see lemma 1.3.8 in [6]). □

Lemma 28. *Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank and B a torsion free sheaf with zero framing. Suppose that $p_{\mathcal{E}} > p_{max}(B)$. Then $\mathrm{Hom}(\mathcal{E}, (B, 0)) = 0$.*

Proof. Let $\varphi \in \mathrm{Hom}(\mathcal{E}, B)$, $\varphi \neq 0$. Let j be minimal such that $\varphi(E) \subset HN_j(B)$. Then there exists a nontrivial morphism of framed sheaves $\bar{\varphi} : \mathcal{E} \to gr_j^{HN}(B)$. By propositions 5 and 13 we get

$$p_{\mathcal{E}} \leq p_{({}^{E}/_{\ker\bar{\varphi}}, \alpha'')} \leq p_{\mathrm{Im}\bar{\varphi}} \leq p_{gr_j^{HN}(B)} \leq p_{max}(B)$$

and this is a contradiction with our assumption. □

Proposition 29. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker\alpha \neq 0$ torsion free. Assume that $\mathcal{E}$ is not semistable with respect to δ. Let G be the maximal framed-destabilizing subsheaf of $\mathcal{E}$. Then*

$$\mathrm{Hom}\left((G, \alpha'), ({}^{E}/_{G}, \alpha'')\right) = 0.$$

Proof. We have to consider separately three different cases.

Case 1: $G = \ker\alpha$. In this case $\mathrm{Hom}\left((G, \alpha'), ({}^{E}/_{G}, \alpha'')\right)$ coincides with the set of morphism $\varphi : \ker\alpha \to {}^{E}/_{\ker\alpha}$ such that $\beta\circ\varphi = 0$, where β is the framing induced by α. Since β is injective, we get $\mathrm{Hom}(\mathcal{G}, \mathcal{G}') = 0$.

Case 2: $\alpha|_G = 0$ and $\mathrm{rk}(G) < \ker\alpha$. In this case $\mathrm{Hom}\left((G, \alpha'), ({}^{E}/_{G}, \alpha'')\right) = \mathrm{Hom}(G, {}^{\ker\alpha}/_{G})$. Recall that G is a Gieseker-semistable sheaf and ${}^{\ker\alpha}/_{G}$ is a torsion free sheaf; moreover from the maximality of G follows that $p_G > p_{T/G}$ for all subsheaves ${}^{T}/_{G} \subset {}^{\ker\alpha}/_{G}$, hence $p_{min}(G) = p_G > p_{max}({}^{\ker\alpha}/_{G})$ and by lemma 1.3.3 in [9] we obtain the thesis.

Case 3: $\alpha|_G \neq 0$ and $\mathrm{rk}(G) > 0$. In this case ${}^{E}\!/_{G}$ is a torsion free sheaf and the induced framing is zero. From the maximality of $\mathcal{G}$ it follows that $p_{\mathcal{G}} > p_{T/G}$ for all subsheaves ${}^{T}\!/_{G} \subset {}^{E}\!/_{G}$, so we can apply lemma 28 and we get the thesis.

Case 4: $G = T(E)$. Let $\varphi : T(E) \to {}^{E}\!/_{T(E)}$. Since $\mathrm{rk}(\mathrm{Im}\varphi) = 0$ and ${}^{E}\!/_{T(E)}$ is torsion free, we have $\mathrm{Im}\varphi = 0$ and therefore we obtain the thesis. □

The case $\mathrm{rk}(F) > 0$ is simpler and we have the following:

Theorem 30. *Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\ker\alpha \neq 0$ torsion free. There exists a unique increasing filtration of framed-saturated subsheaves*

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) \subset \cdots \subset \mathrm{HN}_l(\mathcal{E}) = E$$

which satisfies the following conditions

(A) *the quotient sheaf $gr_i^{\mathrm{HN}}(\mathcal{E}) := {}^{\mathrm{HN}_i(\mathcal{E})}\!/_{\mathrm{HN}_{i-1}(\mathcal{E})}$ with the induced framing α_i is a semistable framed sheaf with respect to δ for $i = 1, 2, \ldots, l$.*

(B)
$$\mathrm{rk}(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}))P_{(gr_i^{\mathrm{HN}}(\mathcal{E}),\alpha_i)} > \mathrm{rk}(gr_i^{\mathrm{HN}}(\mathcal{E}))P_{(gr_{i+1}^{\mathrm{HN}}(\mathcal{E}),\alpha_{i+1})}$$
for $i = 1, \ldots, l-1$.

Base field extension. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf on X with $\ker\alpha \neq 0$ torsion free sheaf. Let K be an extension of k. Consider the following cartesian diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\phi}} & X \\ \Big\downarrow{\scriptstyle \tilde{f}} & & \Big\downarrow{\scriptstyle f} \\ \mathrm{Spec}(K) & \xrightarrow{\phi} & \mathrm{Spec}(k) \end{array}$$

Put $\tilde{E} := \tilde{\phi}^*(E)$, $\tilde{F} := \tilde{\phi}^*(F)$, $\tilde{\alpha} := \tilde{\phi}^*(\alpha)$ and $\tilde{\mathcal{E}} = (\tilde{E}, \tilde{\alpha})$. Now we give a generalization of proposition 3 in [11]:

Theorem 31. *A subsheaf $G \subset E$ is the maximal framed-destabilizing subsheaf of $\mathcal{E}$ if and only if $\tilde{\phi}^*(G)$ is so for $\tilde{\mathcal{E}}$.*

Proof. Note that since $\tilde{\phi}$ is a flat morphism, the sheaf $\ker\tilde{\alpha} = \tilde{\phi}^*(\ker\alpha)$ is torsion free. The Hilbert polynomial is preserved under base extensions, so the framed Hilbert polynomial is preserved. If $E' \subset E$ is a framed-destabilizing subsheaf, then so is $\tilde{\phi}^*(E') \subset \tilde{E}$. Hence if $\tilde{\mathcal{E}}$ is semistable, then $\mathcal{E}$ is semistable. So it suffices to prove that if G_K is the maximal framed-destabilizing subsheaf of $\tilde{\mathcal{E}}$, then there is $G \subset E$ such that $\tilde{\phi}^*(G) = G_K$.

Since G_K is finitely presented, it is defined over some field L, $k \subset L \subset K$, which is finitely generated over k, so we can suppose that $K = k(x)$ for some single element $x \in K$ and ${}^{K}\!/_{k}$ is a purely trascendental or separable extension. Note that there do not exist field extensions of k which are purely inseparable, because k is a perfect field. Let $\sigma \in \mathrm{Gal}({}^{K}\!/_{k})$, we denote by $\sigma_{\tilde{X}}$ the automorphism of $\tilde{X}$ over X induced by σ. Since $\sigma_{\tilde{X}}^*(G_K)$ has the same Hilbert polynomial of G_K and $\epsilon(\tilde{\alpha}|_{G_K}) = \epsilon(\tilde{\alpha}|_{\sigma_{\tilde{X}}^*(G_K)})$, we must have $\sigma_{\tilde{X}}^*(G_K) = G_K$. Hence by descent theory (see [4], p. 22) there exists a subsheaf $G \subset E$ such that $\tilde{\phi}^*(G) = G_K$. Since the framed Hilbert polynomial of G coincides with the one of G_K, we get that G is the maximal framed-destabilizing subsheaf of $\mathcal{E}$. □

7. Jordan-Hölder filtration

In analogy to the study of torsion free Gieseker-semistable coherent sheaves we will define Jordan-Hölder filtrations for framed sheaves.

Definition 32. Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank r. A Jordan-Hölder filtration of $\mathcal{E}$ is a filtration

$$E_\bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

such that all the factors ${}^{E_i}/_{E_{i-1}}$ together with the induced framings α_i are stable with framed Hilbert polynomial $P_{({}^{E_i}/_{E_{i-1}}, \alpha_i)} = \mathrm{rk}({}^{E_i}/_{E_{i-1}}) p_{\mathcal{E}}$.

Proposition 33 (Proposition 1.13 in [8])**.** *Jordan-Hölder filtrations always exist. The framed sheaf*

$$gr(\mathcal{E}) = (gr(E), gr(\alpha)) = \bigoplus_i ({}^{E_i}/_{E_{i-1}}, \alpha_i)$$

does not depend on the choice of the Jordan-Hölder filtration.

Remark 34. By construction for $i > 0$ all subsheaves E_i are framed-saturated and the framed sheaves (E_i, α') are semistable with framed Hilbert polynomial $\mathrm{rk}(E_i)p_{\mathcal{E}}$. In particular (E_1, α') is a stable framed sheaf. Moreover at most one of the framings α_i is nonzero and all but possibly one of the factors ${}^{E_i}/_{E_{i-1}}$ are torsion free.

Lemma 35. *Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank r. Then there exists at most one subsheaf $E' \subset E$ such that $\alpha|_{E'} \neq 0$, $\mathcal{E}'$ is a stable framed sheaf and $P_{\mathcal{E}'} = \mathrm{rk}(E')p_{\mathcal{E}}$.*

Proof. Suppose that there exist E_1 and E_2 subsheaves of E such that $\alpha|_{E_i} \neq 0$, the framed sheaf $\mathcal{E}_i$ is stable (with respect to δ) and $P_{\mathcal{E}_i} = r_i p_{\mathcal{E}}$, where $r_i = \mathrm{rk}(E_i)$, for $i = 1, 2$. So we have $P_{E_i} = r_i p_{\mathcal{E}} + \delta$ for $i = 1, 2$. Let $E_{12} = E_1 \cap E_2$. Suppose that $E_{12} \neq 0$ and $\alpha|_{E_{12}} \neq 0$. Denote by r_{12} the rank of E_{12}. Since $\mathcal{E}_i$ is stable, we have that $P_{E_{12}} - \delta < r_{12}p_{\mathcal{E}}$. Consider the exact sequence

$$0 \longrightarrow E_{12} \longrightarrow E_1 \oplus E_2 \longrightarrow E_1 + E_2 \longrightarrow 0.$$

The induced framing on $E_1 + E_2$ is nonzero; we denote it by β.

$$\begin{aligned} P_{E_1+E_2} &= P_{E_1} + P_{E_2} - P_{E_{12}} = r_1 p_{\mathcal{E}} + \delta + r_2 p_{\mathcal{E}} + \delta - P_{E_{12}} \\ &> \mathrm{rk}(E_1 + E_2)p_{\mathcal{E}} + \delta \end{aligned}$$

and therefore

$$P_{(E_1+E_2,\beta)} = P_{E_1+E_2} - \delta > \mathrm{rk}(E_1 + E_2)p_{\mathcal{E}},$$

but this is a contradiction, because $\mathcal{E}$ is semistable. Now consider the case $\alpha|_{E_{12}} = 0$. By similar computations, we obtain

$$P_{(E_1+E_2,\beta)} = P_{E_1+E_2} - \delta > \mathrm{rk}(E_1 + E_2)p_{\mathcal{E}} + \mathrm{rk}(E_1 + E_2)\delta > \mathrm{rk}(E_1 + E_2)p_{\mathcal{E}},$$

but this is absurd. Thus $E_{12} = 0$ and therefore $E_1 + E_2 = E_1 \oplus E_2$. In this case we get

$$\begin{aligned} P_{(E_1+E_2,\beta)} &= P_{E_1+E_2} - \delta = P_{E_1} + P_{E_2} - \delta \\ &= r_1 p_{\mathcal{E}} + \delta + r_2 p_{\mathcal{E}} + \delta - \delta \\ &= \mathrm{rk}(E_1 + E_2)p_{\mathcal{E}} + \delta > \mathrm{rk}(E_1 + E_2)p_{\mathcal{E}}, \end{aligned}$$

but this is not possible. □

Remark 36. Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank r. If there exists $E' \subset E$ such that $\mathrm{rk}(E') = 0$ and $P_{E'} = \delta$, then $E' = T(E)$, indeed from $P_{T(E)} \geq P'_E$ follows that $P_{T(E)} \geq \delta$. Since $\mathcal{E}$ is semistable, we have $P_{T(E)} = \delta$ and so $E' = T(E)$.

By using similar computations than before, one can prove:

Lemma 37. *Let $\mathcal{E} = (E, \alpha)$ be a semistable framed sheaf of positive rank. Let E_1 and E_2 be two different subsheaves of E such that $P_{\mathcal{E}_i} = \mathrm{rk}(E_i) p_{\mathcal{E}}$ for $i = 1, 2$. Then $P_{(E_1+E_2,\alpha')} = \mathrm{rk}(E_1 + E_2) p_{\mathcal{E}}$ and $P_{(E_1 \cap E_2,\alpha')} = \mathrm{rk}(E_1 \cap E_2) p_{\mathcal{E}}$.*

7.1. **Framed sheaves that are locally free along the support of the framing sheaf.** In this section we assume that F is supported on a divisor D_F and is a locally free O_{D_F}-module.

Definition 38. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf on X. We say that $\mathcal{E}$ is (D_F, F)-framable if E is locally free in a neighborhood of D_F and $\alpha|_D$ is an isomorphism. We call $\mathcal{E}$ also (D_F, F)-framed sheaf.

Recall that in general for a framed sheaf $\mathcal{E} = (E, \alpha)$, the torsion subsheaf of E is supported on D_F, therefore if $\mathcal{E}$ is (D_F, F)-framable then E is torsion free.

Lemma 39. *Let $\mathcal{E} = (E, \alpha)$ be a semistable (D_F, F)-framed sheaf. Let E_1 and E_2 be two different framed-saturated subsheaves of E such that $p_{\mathcal{E}_i} = p_{\mathcal{E}}$, for $i = 1, 2$. Assume that $\alpha|_{E_1} = 0$. Then $E_1 + E_2$ is a framed-saturated subsheaf of E such that $gr(E_1 + E_2, \alpha') = gr(\mathcal{E}_1) \oplus gr(\mathcal{E}_2)$.*

Proof. Since $\mathcal{E}$ is (D_F, F)-framable, the quotient E/E_i is torsion free for $i = 1, 2$, hence $E/(E_1 + E_2)$ is torsion free as well and therefore $E_1 + E_2$ is framed-saturated. By lemma 37, $p_{(E_1+E_2,\alpha')} = p_{\mathcal{E}}$. Moreover we can always start with a Jordan-Hölder filtration of $\mathcal{E}_i$ and complete it to one of $(E_1 + E_2, \alpha')$, hence we get $gr(\mathcal{E}_i) \subset gr(E_1 + E_2, \alpha')$ (as framed sheaves) for $i = 1, 2$. Let $G_\bullet : 0 = G_0 \subset G_1 \subset \cdots \subset G_{l-1} \subset G_l = E_1$ be a Jordan-Hölder filtration for $\mathcal{E}_1$ and $H_\bullet : 0 = H_0 \subset H_1 \subset \cdots \subset H_{s-1} \subset H_s = E_2$ a Jordan-Hölder filtration for $\mathcal{E}_2$. Consider the filtration

$$0 = G_0 \subset G_1 \subset \cdots \subset G_{l-1} \subset G_l = E_1 \subset E_1 + H_p \subset \cdots \subset E_1 + H_{t-1} \subset H_t = E_1 + E_2$$

where $p = \min\{i \,|\, H_i \not\subset E_1\}$. We want to prove that this is a Jordan-Hölder filtration for $(E_1 + E_2, \alpha')$. It suffices to prove that $E_1 + H_j/E_1 + H_{j-1}$ with its induced framing γ_j is stable for $j = p, \ldots, t$ (we put $H_{p-1} = 0$). First note that by lemma 37, we get $P_{(E_1+H_j,\alpha')} = \mathrm{rk}(E_1 + H_j) p_{\mathcal{E}}$ and $P_{(E_1+H_{j-1},\alpha')} = \mathrm{rk}(E_1 + H_{j-1}) p_{\mathcal{E}}$, hence

$$P_{(E_1 + H_j/E_1 + H_{j-1}, \gamma_j)} = \mathrm{rk}(E_1 + H_j/E_1 + H_{j-1}) p_{\mathcal{E}}.$$

Since $E/E_1 + H_{j-1}$ is torsion free, $\mathrm{rk}(E_1 + H_j/E_1 + H_{j-1}) > 0$. Let $T/E_1 + H_{j-1}$ be a subsheaf of $E_1 + H_j/E_1 + H_{j-1}$. We have

$$\begin{aligned} P_{(T/E_1 + H_{j-1}, \gamma'_j)} &= P_{(T,\alpha')} - P_{(E_1+H_{j-1},\alpha')} \leq \mathrm{rk}(T) p_{\mathcal{E}} - \mathrm{rk}(E_1 + H_{j-1}) p_{\mathcal{E}} \\ &= \mathrm{rk}(T/E_1 + H_{j-1}) p_{\mathcal{E}} = \mathrm{rk}(T/E_1 + H_{j-1}) p_{(E_1 + H_j/E_1 + H_{j-1}, \gamma_j)}, \end{aligned}$$

so the framed sheaf $(E_1 + H_j/E_1 + H_{j-1}, \gamma_j)$ is semistable. Moreover we can construct the following exact sequence of coherent sheaves

$$0 \longrightarrow E_1 \cap H_j/E_1 \cap H_{j-1} \longrightarrow H_j/H_{j-1} \xrightarrow{\varphi} E_1 + H_j/E_1 + H_{j-1} \longrightarrow 0.$$

Recall that the induced framing on E_1 is zero, hence the induced framing on ${}^{E_1 \cap H_j}/_{E_1 \cap H_{j-1}}$ is zero and therefore the morphism φ induces a surjective morphism between framed sheaves

$$\varphi : ({}^{H_j}/_{H_{j-1}}, \beta_j) \longrightarrow ({}^{E_1 + H_j}/_{E_1 + H_{j-1}}, \gamma_j).$$

Since $({}^{H_j}/_{H_{j-1}}, \beta_j)$ is stable, by proposition 14 the morphism φ is injective, hence it is an isomorphism. □

Definition 40. Let $\mathcal{E} = (E, \alpha)$ be a semistable (D_F, F)-framed sheaf. We call *framed socle* of $\mathcal{E}$ the framed sheaf given by the sum of all stable framed sheaves $\mathcal{E}' = (E', \alpha')$ with framed Hilbert polynomial $p_{\mathcal{E}'} = p_{\mathcal{E}}$, where $E' \subset E$ is a framed-saturated subsheaf and α' is the induced framing by α.

Let $\mathcal{E} = (E, \alpha)$ be a semistable (D_F, F)-framed sheaf of positive rank. Consider the following two conditions on framed-saturated subsheaves $E' \subset E$:

(a) $p_{\mathcal{E}'} = p_{\mathcal{E}}$,
(b) each component of $gr(\mathcal{E}')$ is isomorphic (as a framed sheaf) to a subsheaf of E.

Let E_1 and E_2 two different framed-saturated subsheaves of E satisfying conditions (a) and (b). By previous lemmas the subsheaf $E_1 + E_2$ is a framed-saturated subsheaf of E satisfying conditions (a) and (b) as well.

Definition 41. For a semistable (D_F, F)-framed sheaf $\mathcal{E} = (E, \alpha)$, we call *extended framed socle* the maximal framed-saturated subsheaf of E satisfying the above conditions (a) and (b).

Proposition 42. *Let G be the extended framed socle of a semistable (D_F, F)-framed sheaf $\mathcal{E} = (E, \alpha)$. Then*

(1) *G contains the framed socle of $\mathcal{E}$.*
(2) *If $\mathcal{E}$ is simple and not stable, then G is a proper subsheaf of E.*

Proof. (1) It follows directy from the definition.

(2) Let $E_\bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$ be a Jordan-Hölder filtration of $\mathcal{E}$. If $E = G$, then the framed sheaf $({}^{E}/_{E_{l-1}}, \alpha_l)$ is isomorphic (as framed sheaf) to a proper subsheaf $E' \subset E$ with induced framing α'. The composition of morphisms of framed sheaves

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[d, "\alpha"] & {}^{E}/_{E_{l-1}} \arrow[r, "\sim"] \arrow[d, "\alpha_l"] & E' \arrow[r, "i"] \arrow[d, "\alpha'"] & E \arrow[d, "\alpha"] \\ F \arrow[r, "\cdot\nu"] & F \arrow[r, "\cdot\lambda"] & F \arrow[r, "\cdot\mu"] & F \end{tikzcd}$$

induces a morphism $\varphi : \mathcal{E} \to \mathcal{E}$ that is not a scalar endomorphism of $\mathcal{E}$. □

8. Relative Harder-Narasimhan filtration

In this section we would like to construct a flat family of minimal framed-destabilizing quotients associated to a framed sheaf.

Notation: let $Y \to S$ be a morphism of finite type of Noetherian schemes. If $T \to S$ is a S-scheme, then we denote by Y_T the fibre product $T \times_S Y$ and by $p_T : Y_T \to T$ and $p_Y : Y_T \to Y$ the natural projections. If H is a coherent sheaf on Y, we denote by H_T its pull-back on Y_T. For $s \in S$ we denote by Y_s the fibre $\mathrm{Spec}(k(s)) \times_S Y$. For a coherent sheaf H on Y, we denote by H_s its pull-back on Y_s.

Definition 43. A *flat family of sheaves on Y parametrized by S* is a coherent sheaf G on Y, which is flat over S.

Henceforth S will be an integral k-scheme of finite type and $f : X \to S$ a projective flat morphism. We denote by $\bar{d}$ the dimension of the fibre X_s for $s \in S$. Note that $(O_X(1))_s$ is a f-ample line bundle on X. Moreover assume that F is a flat family of sheaves of rank zero.

Definition 44. A *flat family of framed sheaves of positive rank on X parametrized by S* consists of a framed sheaf $\mathcal{E} = (E, \alpha)$ on X, where $\alpha_s \neq 0$ for all $s \in S$, E and $\mathrm{Im}\alpha$ are flat families of coherent sheaves on X parametrized by S.

Remark 45. By flatness of E and $\mathrm{Im}\alpha$, we have that also $\ker \alpha$ is S-flat.

Fix a flat family $\mathcal{E} = (E, \alpha)$ of framed sheaves of positive rank r on X parametrized by S and a rational polynomial $\bar{\delta}$ of degree $\bar{d} - 1$ and positive leading coefficient $\bar{\delta}_1$, such that $P_{\mathrm{Im}\alpha_s} \geq \bar{\delta}$ for $s \in S$.

The direction we choose to obtain a flat family of minimal framed-destabilizing quotients is the following: first we construct a *universal quotient* (with fixed Hilbert polynomial) such that the induced framing is either nonzero at each fibre or zero at each fibre. In this way not only the Hilbert polynomial of that quotient is constant along the fibres, but also its framed Hilbert polynomial. Later we need to find a numerical polynomial such that the universal quotient with this polynomial as Hilbert polynomial gives the minimal framed-destabilizing quotient at each fibre.

Relative framed Quot functor. Let $P(n) \in \mathbb{Q}[n]$ be a numerical polynomial. Define the contravariant functor from the category of Noetherian S-schemes of finite type to the category of sets

$$\underline{FQuot}_{X/S}(E, \alpha, P) : ({}^{Sch}/_{S}) \longrightarrow (Sets)$$

in the following way:

- For an object $T \to S$, $\underline{FQuot}_{X/S}(E, \alpha, P)(T \to S)$ is the set consisting of the quotients (modulo isomorphism) $E_T \xrightarrow{q} Q \to 0$ such that
  (i) Q is T-flat,
  (ii) the Hilbert polynomial of Q_t is P for all $t \in T$,
  (iii) there is a induced morphism $\widetilde{\alpha} : Q \to F_T$ such that $\widetilde{\alpha} \circ q = \alpha_T$.
- For a S-morphism $\varphi : T' \to T$, $\underline{FQuot}_{X/S}(E, \alpha, P)(\varphi)$ is the pull-back of φ.

This functor is a subfunctor of the *relative Quot functor* $\underline{Quot}_{X/S}(E, P)$, that is representable by a projective S-scheme $\pi : Quot_{X/S}(E, P) \to S$. Since the property (iii) in the definition is closed, one can prove that the functor $\underline{FQuot}_{X/S}(E, \alpha, P)$ is representable by a closed subscheme $FQuot_{X/S}(E, \alpha, P) \subset Quot_{X/S}(E, P)$ (it suffices to use results (iii) and (iv) in the proof of theorem 1.6 in [16]). The *universal object* on $FQuot_{X/S}(E, \alpha, P) \times_S X$ is the pull-back of the universal object on $Quot_{X/S}(E, P) \times_S X$. We denote by π_{fr} the restriction of π to $FQuot_{X/S}(E, \alpha, P)$.

Let $s \in S$ and $q \in \pi_{fr}^{-1}(s)$ be k-rational points corresponding to the commutative diagram on X_s

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] & E_s \arrow[r, "q"] \arrow[d, "\alpha_s"] & Q \arrow[r] \arrow[dl, "\tilde{\alpha}"] & 0 \\ & & F_s & & \end{tikzcd}$$

One has the following result about the tangent space of $\pi_{fr}^{-1}(s)$ at q:

Proposition 46. *The kernel of the linear map* $(\mathrm{d}\pi_{fr})_q : T_q FQuot_{X/S}(E, \alpha, P) \to T_s S$ *is isomorphic to the linear space* $\mathrm{Hom}(K, {}^{\ker\alpha_s}/_{K}) = \mathrm{Hom}(\mathcal{K}, \mathcal{Q})$.

Proof. It suffices to use the same techniques of the proof of the corresponding result for π (see proposition 4.4.4 in [17]). $\square$

Now we have a tool for constructing a flat family of quotients (with a fixed Hilbert polynomial) of $\mathcal{E}$ such that the induced framing is nonzero in each fibre. Using the relative Quot scheme associated to E, one can construct a flat family of quotients such that the induced framing is *generically* zero.

Boundedness result. Recall the following result: let E be a flat family of coherent sheaves on X parametrized by S. Grothendieck proved that the family of torsion free quotients Q of E_s for $s \in S$ with hat-slopes bounded from above is a bounded family (see lemma 1.7.9 in [9]). From this result it follows that there are only a finite number of rational polynomials corresponding to Hilbert polynomials of destabilizing quotients of E. Thus it is possible to find the "minimal" polynomial that will be the Hilbert polynomial of the minimal destabilizing quotient. Now we want to use the same argument in the framed case.

First we recall the following technical result that we will use later:

Lemma 47 (Lemma 2.5 in [5])**.** *Let Y be a proper S-scheme of finite type. Let L be a coherent sheaf on Y and $\mathscr{E}$ the set of isomorphic classes of quotient sheaves G of L_s for some $s \in S$. Suppose that the dimension of Y_s is r and* $\dim(G) = r$ *for $G \in \mathscr{E}$. Then the coefficient $\beta_r(G)$ is bounded from above and from below and $\beta_{r-1}(G)$ is bounded from below.*

Let $\mathscr{F}_1$ be a family of quotients $E_s \xrightarrow{q} Q$, for $s \in S$, such that

- $\ker \alpha_s$ is torsion free,
- $\ker(q) \not\subseteq \ker \alpha_s$,
- Q is torsion free and $\hat{\mu}(Q) < \hat{\mu}(E_s)$.

Proposition 48. *The family $\mathscr{F}_1$ is bounded.*

Proof. The family $\mathscr{F}_1$ is contained in the family of torsion free quotients of E, with hat-slopes bounded from above. $\square$

Let $\mathscr{F}_2$ be a family of quotients

$$\begin{tikzcd} E_s \arrow[r, "q"] \arrow[d, "\alpha_s"'] & Q \arrow[r] \arrow[dl, "\tilde{\alpha}"] & 0 \\ F_s & & \end{tikzcd}$$

for which

- $\ker \alpha_s$ is a torsion free sheaf,
- Q fits into a exact sequence
$$0 \longrightarrow Q' \longrightarrow Q \longrightarrow \mathrm{Im}\alpha_s \longrightarrow 0$$
where Q' is a nonzero torsion free quotient of $\ker \alpha_s$,
- $\hat{\mu}(Q) < \hat{\mu}(E_s) + \bar{\delta}_1$.

Proposition 49. *The family $\mathscr{F}_2$ is bounded.*

Proof. It suffices to prove that every element in $\mathscr{F}_2$ is an extension of two elements from two bounded families (see proposition 1.2 in [5]). By definition of flat family of framed sheaves, the families $\{\ker \alpha_s\}_{s\in S}$ and $\{\mathrm{Im}\alpha_s\}_{s\in S}$ are bounded. So it remains to prove that the family of quotients Q' is bounded. Since the family $\{\ker \alpha_s\}$ is bounded, there exists a coherent sheaf T on X such that $\ker \alpha_s$ admits a surjective morphism $T_s \to \ker \alpha_s$ (see lemma 1.7.6 in [9]), hence the quotient Q' admits a surjective morphism $T_s \to Q'$. By proposition 47, the coefficient $\beta_{\bar{d}}(Q')$ is bounded from above and from below and the coefficient $\beta_{\bar{d}-1}(Q')$ is bounded from below. Moreover, since $\{E_s\}$ and $\{\mathrm{Im}\alpha_s\}$ are bounded families, the coefficients of their Hilbert polynomials are uniformly bounded from above and from below, hence $\hat{\mu}(E_s)$ is uniformly bounded from above and from below and since $\hat{\mu}(Q) < \hat{\mu}(E_s) + \bar{\delta}_1$, we obtain that $\hat{\mu}(Q)$ is uniformly bounded from above. By a simple computation we obtain that $\hat{\mu}(Q') \leq A\hat{\mu}(Q) + B$ for some constants A, B, hence we get that $\hat{\mu}(Q')$ is uniformly bounded from above and by lemma 1.7.9 in [9] we obtain the thesis. □

Relative minimal framed-destabilizing quotient and Harder-Narasimhan filtration.

Definition 50. Let P be a property of framed sheaves. P is said to be an *open property*, if for any nonsingular projective irreducible variety X defined over an algebraically closed field of characteristic zero, any integral k-scheme of finite type S, any projective flat morphism $f : X \to S$ and any flat family (E, α) of framed sheaves of positive rank on X parametrized by S, the set of points $s \in S$ such that (E_s, α_s) has P is an open subset in S.

Proposition 51. *Let X, S, f and (E, α) be as in the definition. Let $\bar{\delta}$ be a rational polynomial of degree $\bar{d} - 1$ and positive leading coefficient $\bar{\delta}_1$ such that $P_{\mathrm{Im}\alpha_s} \geq \bar{\delta}$ for $s \in S$. The set of points $s \in S$ such that (E_s, α_s) is semistable with respect to $\bar{\delta}$ is open in S.*

Proof. Let P denote the Hilbert polynomial of E. For $i = 1, 2$ let $A_i \subset \mathbb{Q}[n]$ be the set consisting of polynomials P'' such that there is a point $s \in S$ and a surjection $E_s \to E''$, where $P_{E''} = P''$ and $E'' \in \mathscr{F}_i$. Note that by propositions 48 and 49 the sets A_1 and A_2 are finite. Denote by p'' the reduced Hilbert polynomial associated to the rational polynomial P'' and by r'' its leading coefficient. Define the sets

$$\begin{aligned} T_1 &= \left\{ P'' \in A_1 \,|\, p'' < p - \frac{\bar{\delta}}{r} \right\}, \\ T_2 &= \left\{ P'' \in A_2 \,|\, p'' - \frac{\bar{\delta}}{r''} < p - \frac{\bar{\delta}}{r} \right\}. \end{aligned}$$

For any $P'' \in A_1$ we consider the relative Quot scheme $\pi : Quot_{X/S}(E, P'') \to S$. Since π is projective, the image $S(P'')$ is a closed subset of S. For any $P'' \in A_2$ the image $S_{fr}(P'')$ of $FQuot_{X/S}(E, \alpha, P'')$ through π_{fr} is closed in S. Thus

$$(E_s, \alpha_s) \text{ is semistable if and only if } s \notin \left(\bigcup_{P'' \in T_1} S(P'') \right) \cup \left(\bigcup_{P'' \in T_2} S_{fr}(P'') \right),$$

Note that these unions are finite, hence closed in S. □

Proposition 52. *Let X, S, f and (E, α) be as in the definition. Let $\bar{\delta}$ be a rational polynomial of degree $\bar{d} - 1$ and positive leading coefficient $\bar{\delta}_1$ such that $P_{\mathrm{Im}\alpha_s} > \bar{\delta}$ for $s \in S$. The set of points $s \in S$ such that (E_s, α_s) is stable with respect to $\bar{\delta}$ is open in S.*

Proof. The proof is similar to the previous one, using the sets

$$\begin{aligned} T_1' &= \left\{ P'' \in A_1 \,|\, p'' \leq p - \frac{\bar{\delta}}{r} \text{ and } P'' < P \right\}, \\ T_2' &= \left\{ P'' \in A_2 \,|\, p'' - \frac{\bar{\delta}}{r''} \leq p - \frac{\bar{\delta}}{r} \text{ and } P'' < P \right\}. \end{aligned}$$

□

Now we can prove the following generalization to the relative case of the theorem of minimal framed-destabilizing quotient.

Theorem 53. *Let S be an integral k-scheme of finite type and $f : X \to S$ a projective flat morphism. Denote by $\bar{d}$ the dimension of X_s for $s \in S$. Let F be a flat family of sheaves of rank zero on X parametrized by S and $\mathcal{E} = (E, \alpha : E \to F)$ a flat family of framed sheaves of positive rank on X parametrized by S. Fix a rational polynomial $\bar{\delta}$ of degree $\bar{d} - 1$ and positive leading coefficient $\bar{\delta}_1$, such that $P_{\mathrm{Im}\alpha_s} \geq \bar{\delta}$ for $s \in S$. Then there is an integral k-scheme T of finite type, a projective birational morphism $g : T \to S$, a dense open subscheme $U \subset T$ and a flat quotient Q of E_T such that for all points t in U, (E_t, α_t) is a framed sheaf of positive rank with $\ker \alpha_t \neq 0$ torsion free and $\mathcal{Q}_t$ is the minimal framed-destabilizing quotient of $\mathcal{E}_t$ with respect to $\bar{\delta}$ or $Q_t = E_t$.*

Moreover, the pair (g, Q) is universal in the sense that if $g' : T' \to S$ is any dominant morphism of k-integral schemes and Q' is a flat quotient of $E_{T'}$ satisfying the same property of Q, then there is an S-morphism $h : T' \to T$ such that $h_X^*(Q) = Q'$.

Proof. Let

$$\begin{aligned} B_1 &= \left\{ P'' \in A_1 \,|\, p'' < p - \frac{\bar{\delta}}{r} \right\}, \\ B_2 &= \left\{ P'' \in A_2 \,|\, p'' - \frac{\bar{\delta}}{r''} \leq p - \frac{\bar{\delta}}{r} \right\}. \end{aligned}$$

The set $B_1 \cup B_2$ is nonempty. We define an order relation on B_1: $P_1 \sqsubseteq P_2$ if and only if $p_1 \leq p_2$ and $r_1 \leq r_2$ in the case $p_1 = p_2$. We define an order relation on B_2: $P_1 \sqsubseteq P_2$ if and only if $p_1 - \frac{\bar{\delta}}{r_1} \leq p_2 - \frac{\bar{\delta}}{r_2}$ and $r_1 \leq r_2$ in the case of equality.

Let C_1 be the set of polynomials $P'' \in B_1$ such that $\pi(Quot_{X/S}(E, P'')) = S$ and for any $s \in S$ one has $\pi^{-1}(s) \not\subset FQuot_{X/S}(E, \alpha, P'')$. Let C_2 be the set of polynomials $P'' \in B_2$ such that $\pi_{fr}(FQuot_{X/S}(E, \alpha, P'')) = S$. Note that $C_1 \cup C_2$ is nonempty. Now we want to find a polynomial P_- in $C_1 \cup C_2$ that is the Hilbert polynomial of the minimal framed-destabilizing quotient of E_s for a general point $s \in S$. Let P_-^i be a $\sqsubseteq$-minimal polynomial among all polynomials of C_i for $i = 1, 2$. Consider the following cases:

- Case 1: $p_-^1 < p_-^2 - \frac{\bar{\delta}}{r_-^2}$. Put $P_- := P_-^1$.
- Case 2: $p_-^1 > p_-^2 - \frac{\bar{\delta}}{r_-^2}$. Put $P_- := P_-^2$.
- Case 3: $p_-^1 = p_-^2 - \frac{\bar{\delta}}{r_-^2}$. If $r_-^2 < r_-^1$, put $P_- := P_-^2$, otherwise $P_- := P_-^1$.

Note that the set

$$\left(\bigcup_{P'' \in B_1, P'' \sqsubset P_-^1} \pi(Quot_{X/S}(E, P'')) \right) \cup \left(\bigcup_{P'' \in B_2, P'' \sqsubset P_-^2} \pi_{fr}(FQuot_{X/S}(E, \alpha, P'')) \right)$$

is a proper closed subscheme of S. Let U_- be its complement. Let U_{tf} the dense open subscheme of S consisting of points s such that $\ker \alpha_s$ is torsion free. Put $V = U_- \cap U_{tf}$.

Suppose that $P_- \in C_2$, the other case is similar. By definition of P_- the projective morphism $\pi_{fr} : FQuot_{X/S}(E, \alpha, P_-) \to S$ is surjective. For any point $s \in S$ the fiber of π_{fr} at s parametrizes possible quotients of E_s with Hilbert polynomial P_-. If $s \in V$, then any such quotient is a minimal framed-destabilizing quotient by construction of V. Recall that the minimal framed-destabilizing quotient is unique by proposition 17: this implies that $\pi_{fr}|_U : U := \pi_{fr}^{-1}(V) \to V$ is bijective. Moreover by theorem 31, that quotient is defined over the residue field $k(s)$, hence for $t \in U$, $s = \pi_{fr}(t)$ one has $k(s) \simeq k(t)$. Let $t \in \pi_{fr}^{-1}(s)$ be a point corresponding to a diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] & E_t \arrow[r, "q"] \arrow[d, "\alpha_t"] & Q \arrow[r] \arrow[dl, "\tilde{\alpha}"] & 0 \\ & & F_t & & \end{tikzcd}$$

By proposition 46, the Zariski tangent space of $\pi_{fr}^{-1}(s)$ at t is $\mathrm{Hom}(\mathcal{K}, \mathcal{Q})$. Since K is the maximal framed-destabilizing subsheaf of $\mathcal{E}_t$, we have that $\mathrm{Hom}(\mathcal{K}, \mathcal{Q}) = 0$ by proposition 29 and therefore $\Omega_{U/V} = 0$, hence $\pi_{fr}|_U : U \to V$ is unramified. Since π_{fr} is projective, we have that $\pi_{fr}|_U$ is a proper morphism. Since V is integral, we obtain that $\pi_{fr}|_U$ is an isomorphism.

Now let T be the clousure of U in $FQuot_{X/S}(E, \alpha, P_-)$ with its reduced subscheme structure and $f := \pi|_T : T \to S$ is a projective birational morphism. We put Q equal to the pull-back on X_T of the universal quotient on $FQuot_{X/S}(E, \alpha, P_-) \times_S X$.

The proof of the universality of the pair (g, Q) is similar to that for the case of torsion free sheaves (second part of theorem 2.3.2 in [9]), since to prove this part of the theorem we need only the universal property of $FQuot_{X/S}(E, \alpha, P_-)$ or $Quot_{X/S}(E, P_-)$. □

We give now the relative version of the Harder-Narasimhan filtration.

Theorem 54. *Let S be an integral k-scheme of finite type and $f : X \to S$ a projective flat morphism. Denote by $\bar{d}$ the dimension of X_s for $s \in S$. Let F be a flat family F of sheaves of rank zero on X parametrized by S and $\mathcal{E} = (E, \alpha : E \to F)$ a flat family of framed sheaves of positive rank on X parametrized by S. Fix a rational polynomial $\bar{\delta}$ of degree $\bar{d}-1$ and positive leading coefficient $\bar{\delta}_1$, such that $P_{\mathrm{Im}\alpha_s} \geq \bar{\delta}$. Then there is an integral k-scheme T of finite type, a projective birational morphism $g : T \to S$ and a filtration*

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 = \mathrm{HN}_0(\mathcal{E}) \subset \mathrm{HN}_1(\mathcal{E}) \subset \cdots \subset \mathrm{HN}_l(\mathcal{E}) = E_T$$

such that the following holds:

- *The factors ${}^{\mathrm{HN}_i(\mathcal{E})}/{}_{\mathrm{HN}_{i-1}(\mathcal{E})}$ are T-flat for all $i = 1, \ldots, l$, and*
- *there is a dense open subscheme $U \subset T$ such that $(\mathrm{HN}_\bullet(\mathcal{E}))_t = g_X^* \mathrm{HN}_\bullet(\mathcal{E}_{g(t)})$ for all $t \in U$.*

Moreover, the pair $(g, \mathrm{HN}_\bullet(\mathcal{E}))$ is universal in the sense that if $g' : T' \to S$ is any dominant morphism of k-integral schemes and $E'_\bullet$ is a filtration of E_T satisfying these two properties, then there is an S-morphism $h : T' \to T$ such that $h_X^*(\mathrm{HN}_\bullet(\mathcal{E})) = E'_\bullet$.

Proof. Apply theorem 53 to the pair $(S, \mathcal{E})$ and get a projective birational morphism $g_1 : T_1 \to S$ of integral k-schemes of finite type, a dense open subsheme U_1 and a T_1-flat quotient Q with the properties asserted in that theorem. If $Q_t = E_t$ for all $t \in U_1$, then we obtain the trivial relative Harder-Narasimhan filtration:

$$\mathrm{HN}_\bullet(\mathcal{E}) : 0 \subset \mathrm{HN}_1(\mathcal{E}) = E_{T_1}$$

Otherwise, Q is a flat family of sheaves of positive rank parametrized by T. If the induced framings on the fibres of Q are nonzero, then Q with the induced framing by α is a flat family of framed sheaves of positive rank parametrized by T and we can apply theorem 53 to the pair $(T, \mathcal{Q})$. In this way we obtain a finite sequence of morphisms

$$T_l \longrightarrow T_{l-1} \longrightarrow \ldots \longrightarrow T_1 = T \to S$$

and an associated filtration such that the composition of these morphism and the filtration have the required properties. Moreover they are universal.

If on the contrary the framings are zero, then we can apply the nonframed version of the previous theorem (theorem 2.3.2 in [9]) to the pair $(T, \mathcal{Q})$ and, as before, by iterating this procedure we get the thesis. □

9. Slope-(semi)stability

We give now another definition of semistability for framed sheaves with d-dimensional underlying sheaves.

Definition 55. A framed sheaf $\mathcal{E} = (E, \alpha)$ of positive rank is μ-(semi)stable with respect to δ_1 if and only if $\ker \alpha$ is torsion free and the following conditions are satisfied:

(i) $\mathrm{rk}(E)\deg(E')(\leq)\mathrm{rk}(E')\deg(\mathcal{E})$ for all subsheaves $E' \subset \ker \alpha$ with $0 \leq \mathrm{rk}(E') \leq \mathrm{rk}(\ker \alpha)$,

(ii) $\mathrm{rk}(E)(\deg(E') - \delta_1)(\leq)\mathrm{rk}(E')\deg(\mathcal{E})$ for all subsheaves $E' \subset E$ with $0 \leq \mathrm{rk}(E') < \mathrm{rk}(E)$.

If $\alpha = 0$, then this definition coincides with Mumford-Takemoto (semi)stability condition for torsion free sheaves (see def. 1.2.12 in [9]). Obviously one has the following implications between different stability properties of a framed sheaf of positive rank:

$$\mu - \text{stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu - \text{semistable}.$$

Definition 56. Let $\mathcal{E} = (E, \alpha)$ be a framed sheaf with $\mathrm{rk}(E) = 0$. If α is injective, we say that $\mathcal{E}$ is μ-semistable with respect to δ_1. Moreover, if the degree of E is δ_1, we say that $\mathcal{E}$ is μ-stable with respect to δ_1.

All the previous results hold also for μ-(semi)stability. μ-Jordan-Hölder filtrations are constructed in the same way, but the associated graded object is not necessarily unique: two graded objects may differ by subsheaves supported in codimension greater or equal than two. To avoid this difficulty, we shall only consider saturated μ-Jordan-Hölder filtrations, in which every term is a maximal proper μ-semistable framed subsheaf of the next term. From now on we consider only saturated μ-Jordan-Hölder filtrations. Moreover the notions of framed socle and extended framed socle are given with respect to saturated μ-Jordan-Hölder filtrations.

10. Restriction theorems

10.1. **Slope-semistable case.** In this section we want to prove a generalization of Mehta-Ramanathan's theorem for μ-semistable torsion free sheaves ([13]):

Theorem 57. *Let F be a coherent sheaf on X supported on a divisor D_F. Let $\mathcal{E} = (E, \alpha : E \to F)$ be a framed sheaf on X of positive rank with nontrivial framing. If $\mathcal{E}$ is μ-semistable with respect to δ_1, then there is a positive integer a_0 such that for all $a \geq a_0$ there is a dense open subset $U_a \subset |O_X(a)|$ such that for all $D \in U_a$ the divisor D is smooth, meets transversally the divisor D_F and $\mathcal{E}|_D$ is μ-semistable with respect to $a\delta_1$.*

Before proving this theorem, we need some preliminar results: for a positive integer a, let $\Pi_a := |\mathcal{O}_X(a)|$ be the complete linear system of hypersurfaces of degree a and let $Z_a := \{(D, x) \in \Pi_a \times X | x \in D\}$ be the *incidence variety* with its natural projections

$$\begin{array}{ccc} Z_a & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ \Pi_a & & \end{array}$$

Remark 58. It is possible to give a schematic structure on Z_a so that p is a projective flat morphism (see section 3.1 in [9]). Moreover $\mathrm{Pic}(Z_a) = q^*(\mathrm{Pic}(X)) \oplus p^*(\mathrm{Pic}(\Pi_a))$ (see section 2 in [13]).

Note that for all $D \in \Pi_a$, the restrictions $E|_D$, $F|_D$ and $\mathrm{Im}\alpha|_D$ have the same Hilbert polynomials, indeed, e.g., the Hilbert polynomial of $E|_D$ is $P_{E|_D}(n) = P_E(n) - P_E(n-a)$. Since Π_a is a reduced scheme, by proposition 2.1.2 in [9] we have that q^*F is a flat family of sheaves of rank zero on Z_a parametrized by Π_a and $(q^*E, q^*\alpha)$ is a flat family of framed sheaves of positive rank on Z_a parametrized by Π_a. For any a and for general $D \in \Pi_a$ the restriction $\ker \alpha|_D$ is again torsion free (see corollary 1.1.14 in [9]), hence the set $\{C \in \Pi_a \mid \ker \alpha|_C \text{ is torsion free}\} \subset \Pi_a$ is a nonempty. Since $\mathcal{E}$ is μ-semistable with respect to δ_1, we have that $\deg(\mathrm{Im}\alpha) \geq \delta_1$, hence $\deg(\mathrm{Im}\alpha|_D) = a\deg(\mathrm{Im}\alpha) \geq a\delta_1$ for a positive integer. According to the theorem 53, which state s the existence of the relative minimal framed-destabilizing quotient with respect to $\bar{\delta}_1 = a\delta_1$, there are a dense open subset $V_a \subset \Pi_a$ and a V_a-flat quotient on $Z_{V_a} := V_a \times_{\Pi_a} Z_a$

$$\begin{array}{ccc} (q^*E)|_{Z_{V_a}} & \xrightarrow{q_a} & Q_a \\ \Big\downarrow {\scriptstyle (q^*\alpha)|_{Z_{V_a}}} & & \\ (q^*F)|_{Z_{V_a}} & & \end{array}$$

with a morphism $\tilde{\alpha}_a : Q_a \to (q^*F)|_{Z_{V_a}}$, such that for all $D \in V_a$ the framed sheaf $(E|_D, \alpha|_D)$ has positive rank, and $\ker \alpha|_D$ is torsion free; moreover, $Q_a|_D$ is a coherent sheaf of positive rank, $\tilde{\alpha}_a|_D$ is the induced framing by $\alpha|_D$ and $(Q_a|_D, \tilde{\alpha}_a|_D)$ is the minimal μ-framed-destabilizing quotient of $(E|_D, \alpha|_D)$. Let Q be an extension of $\det(Q_a)$ to some line bundle on all of Z_a. Then Q can be uniquely decomposed as $Q = q^*L_a \otimes p^*M$ with $L_a \in \mathrm{Pic}(X)$ and $M \in \mathrm{Pic}(\Pi_a)$. Note that $\deg(Q_a|_D) = a\deg(L_a)$ for $D \in V_a$.

Let $U_a \subset V_a$ be the dense open set of points $D \in V_a$ such that D is smooth and meets transversally the divisor D_F.

Let $\deg(a)$, $r(a)$ and $\mu_{fr}(a)$ denote the degree, the rank and the framed slope of the minimal μ-framed-destabilizing quotient of $(E|_D, \alpha|_D)$ for a general point $D \in \Pi_a$. By construction of the relative minimal μ-framed-destabilizing quotient, the quantity $\epsilon(\tilde{\alpha}_a|_D)$ is indipendent of $D \in V_a$, so we denote it by $\epsilon(a)$. Then we have $1 \leq r(a) \leq \mathrm{rk}(E)$ and

$$\frac{\mu_{fr}(a)}{a} = \frac{\deg L_a - \epsilon(a)\delta_1}{r(a)} \in \frac{\mathbb{Z}}{\delta_1''(\mathrm{rk}(E)!)} \subset \mathbb{Q},$$

where $\delta_1 = {}^{\delta_1'}/_{\delta_1''}$. Let $a_1, \ldots, a_l$ be positive integers and $a = \sum_i a_i$. We would like to compare $r(a)$ (resp. ${}^{\mu_{fr}(a)}/_a$) with $r(a_i)$ (resp. ${}^{\mu_{fr}(a_i)}/_{a_i}$) for all $i = 1, \ldots, l$. To do this, we use the following result, which allows us to compare the rank and the framed degree of Q_{a_i} in a generic fibre with the same invariants of a "special quotient" of $(q^*E)|_{Z_{V_a}}$.

Lemma 59 (Lemma 7.2.3 in [9])**.** *Let $a_1, \ldots, a_l$ be positive integers, $a = \sum_i a_i$, and let $D_i \in U_{a_i}$ be divisors such that $D = \sum_i D_i$ is a divisor with normal crossings. Then there is a smooth locally closed curve $C \subset \Pi_a$ containing the point $D \in \Pi_a$ such that $C \setminus \{D\} \subset U_a$ and $Z_C = C \times_{\Pi_a} Z_a$ is smooth in codimension 2.*

Remark 60. If $D_1 \in U_{a_1}$ is given, one can always find $D_i \in U_{a_i}$ for $i \geq 2$ such that $D = \sum_i D_i$ is a divisor with normal crossings.

Lemma 61. *Let $a_1, \ldots, a_l$ be positive integers, $a = \sum_i a_i$. Then $\mu_{fr}(a) \geq \sum_i \mu_{fr}(a_i)$ and in case of equality $r(a) \geq \max\{r(a_i)\}$.*

Proof. Let D_i be divisors satisfying the requirements of lemma 59 and let C be the curve with the properties of 59. Over V_a there is the quotient

$$\begin{tikzcd} (q^*E)|_{Z_{V_a}} \arrow[r, "q_a"] \arrow[d, "(q^*\alpha)|_{Z_{V_a}}"] & Q_a \\ (q^*F)|_{Z_{V_a}} & \end{tikzcd}$$

Now we have to consider two cases:

(1) there exists a nonzero framing $\tilde{\alpha}_a$ on Q_a such that $(q^*\alpha)|_{Z_{V_a}} = \tilde{\alpha}_a \circ q_a$,
(2) $\ker(q_a|_{D'}) \not\subset \ker(\alpha|_{D'})$ for all $D' \in V_a$.

Consider the first case: we have that $\tilde{\alpha}_a|_{D'} \neq 0$ for all $D' \in V_a$. Consider its restriction to $Z_{V_a \cap C}$:

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & (q^*E)|_{Z_{V_a\cap C}} \arrow[r, "q_a|_{Z_{V_a\cap C}}"] \arrow[d, "(q^*\alpha)|_{Z_{V_a\cap C}}"'] & Q_a|_{Z_{V_a\cap C}} \arrow[r] \arrow[dl, "\tilde{\alpha}_a|_{Z_{V_a\cap C}}"] & 0 \\ & & (q^*F)|_{Z_{V_a\cap C}} & & \end{tikzcd}$$

Since the morphism $Z_{V_a\cap C} \to Z_C$ is flat (because it is an open embedding), we have that $\ker(q^*\alpha|_{Z_{V_a\cap C}}) = (\ker(q^*\alpha|_{Z_C}))|_{Z_{V_a\cap C}}$ and we can extend the inclusion $K \subset \ker(q^*\alpha|_{Z_{V_a\cap C}})$ to an inclusion $K_C \subset \ker(q^*\alpha|_{Z_C})$ on Z_C. Since $V_a \cap C = C \setminus \{D\}$ in this way we extend $Q_a|_{Z_{V_a\cap C}}$ to a C-flat quotient Q_C of $q^*E|_{Z_C}$ and we get the following commutative diagram

$$\begin{tikzcd} (q^*E)|_{Z_C} \arrow[r, "q_C"] \arrow[d, "(q^*\alpha)|_{Z_C}"'] & Q_C \arrow[dl, "\tilde{\alpha}_C"] \\ (q^*F)|_{Z_C} & \end{tikzcd}$$

and therefore $\tilde{\alpha}_C|_c \neq 0$ for all $c \in C$. By flatness of Q_C we obtain that $P_{Q_C|_c}(n) = P_{Q_C|_D}(n)$ for all $c \in C \setminus \{D\}$, hence $\mathrm{rk}(Q_C|_D) = r(a)$ and $\deg(Q_C|_D) = \deg(a)$, therefore $\mu(Q_C|_D, \tilde{\alpha}_C|_D) = \mu_{fr}(a)$. Let $\bar{Q} = {}^{Q_C|_D}/_{T'(Q_C|_D)}$, where $T'(Q_C|_D)$ is the sheaf on D such that to every open subset U associates the set of sections f of $Q_C|_D$ in U such that there exists $n > 0$ for which $I_D^n \cdot f = 0$, where I_D is the ideal sheaf associated to D. Roughly speaking, $T'(Q_C|_D)$ is the part of the torsion subsheaf $T(Q_C|_D)$ of $Q_C|_D$ that is not supported in the intersection $D \cap D_F$. $T'(Q_C|_D) \subset \ker \tilde{\alpha}_C|_D$ by the transversality of D_i with respect to D_F, hence there is a nonzero induced framing $\bar{\alpha}$ on $\bar{Q}$. Moreover, $\mathrm{rk}(\bar{Q}|_{D_i}) = \mathrm{rk}(\bar{Q}) = \mathrm{rk}(Q_C|_D) = r(a)$. So

$$\mu_{fr}(a) = \mu(Q_C|_D, \tilde{\alpha}_C|_D) \geq \mu(\bar{Q}, \bar{\alpha}).$$

The sequence

$$0 \longrightarrow \bar{Q} \longrightarrow \bigoplus_i \bar{Q}|_{D_i} \longrightarrow \bigoplus_i \bigoplus_{i<j} \bar{Q}|_{D_i \cap D_j} \longrightarrow 0$$

is exact modulo sheaves of dimension $d-3$ (the kernel of the morphism $\bar{Q} \longrightarrow \bigoplus_i \bar{Q}|_{D_i}$ is zero because the divisors D_i are transversal with respect to the singular set of $\bar{Q}$). By the same computations as in the proof of lemma 7.2.5 in [9] we have

$$\mu(\bar{Q}) = \sum_i \left(\mu(\bar{Q}|_{D_i}) - \frac{1}{2} \sum_{j \neq i} \left(\frac{\mathrm{rk}\left(\bar{Q}|_{D_i \cap D_j}\right)}{r(a)} - 1 \right) a_i a_j \right).$$

For every i and $j \neq i$ we define also the sheaf $T_{ij}(\bar{Q}|_{D_i})$ as the sheaf on D_i that to every open subset U associates the set of sections f of $\bar{Q}|_{D_i}$ in U such that there exists $n > 0$ for which $I^n_{D_j} \cdot f = 0$. Note that $T_{ij}(\bar{Q}|_{D_i}) \subset \ker \bar{\alpha}|_{D_i}$. We define $Q_i = \frac{\bar{Q}|_{D_i}}{\bigoplus_{j \neq i} T_{ij}(\bar{Q}|_{D_i})}$. By construction $\mathrm{rk}(Q_i) = \mathrm{rk}(\bar{Q})$, there exists a nonzero induced framing α_i on Q_i, and

$$\mu(Q_i) = \mu(\bar{Q}|_{D_i}) - \sum_{j \neq i} \left(\frac{\mathrm{rk}\left(\bar{Q}|_{D_i \cap D_j}\right)}{r(a)} - 1 \right) a_i a_j.$$

Therefore $\mu(\bar{Q}) \geq \sum_i \mu(Q_i)$, and

$$\mu_{fr}(a) \geq \mu(\bar{Q}, \bar{\alpha}) \geq \sum_i \mu(Q_i, \alpha_i).$$

By definition of minimal framed μ-destabilizing quotient, we have $\mu(Q_i, \alpha_i) \geq \mu_{fr}(a_i)$, hence $\mu_{fr}(a) \geq \sum_i \mu_{fr}(a_i)$.

Consider the second case. On the restriction to $Z_{V_a \cap C}$ we have the quotient:

$$\begin{array}{ccc} (q^*E)|_{Z_{V_a \cap C}} & \xrightarrow{\quad q \quad} & Q_a|_{Z_{V_a \cap C}} \\ \Big\downarrow {\scriptstyle (q^*\alpha)|_{Z_{V_a \cap C}}} & & \\ (q^*F)|_{Z_{V_a \cap C}} & & \end{array}$$

By definition of Q_a we get $\ker(q|_{D'}) \not\subset \ker(\alpha|_{D'})$ for all points $D' \in V_a \cap C$, hence $\ker q \not\subset \ker((q^*\alpha)|_{Z_{V_a \cap C}})$. As before, we can extend $Q_a|_{Z_{V_a \cap C}}$ to a C-flat quotient

$$\begin{array}{ccc} (q^*E)|_{Z_C} & \xrightarrow{q_C} & Q_C \\ \Big\downarrow {\scriptstyle (q^*\alpha)|_{Z_C}} & & \\ (q^*F)|_{Z_C} & & \end{array}$$

Since $\ker q_C$ and $\ker((q^*\alpha)|_{Z_C})$ are C-flat, also $\ker q_C \cap \ker((q^*\alpha)|_{Z_C})$ is C-flat. Moreover for all points $D' \in V_a \cap C$ we have $(\ker q_C \cap \ker((q^*\alpha)|_{Z_C}))|_{D'} = \ker(q|_{D'}) \cap \ker(\alpha|_{D'})$, hence by flatness we get $\ker(q_C|_{D'}) \not\subset \ker(\alpha|_{D'})$ for all points $D' \in C$. As before, by flatness of Q_C we have that $\mathrm{rk}(Q_C|_D) = r(a)$ and $\deg(Q_C|_D) = \deg(a)$; moreover the induced framing on $Q_C|_D$ is zero, hence $\mu(Q_C|_D) = \mu_{fr}(a)$. Let $\bar{Q} = {}^{Q_C|_D}/_{T(Q_C|_D)}$ and $Q_i = {}^{\bar{Q}|_{D_i}}/_{T(\bar{Q}|_{D_i})}$. Using the same computations as in the proof of lemma 7.2.5 in [9], we obtain $\mu(\bar{Q}) \geq \sum_i \mu(Q_i)$. As before, we get $\mu_{fr}(a) = \mu(Q_C|_D) \geq \mu(\bar{Q}) \geq \sum_i \mu(Q_i) \geq \sum_i \mu_{fr}(a_i)$.

Now let us consider the case $\mu_{fr}(a) = \sum_i \mu_{fr}(a_i)$. In both two cases, if we denote by α_i the induced framing on Q_i, from this equality, follows that $\mu(Q_i, \alpha_i) = \mu_{fr}(a_i)$ and $\mathrm{rk}(\bar{Q}|_{D_i \cap D_j}) = r(a)$. Since $\mu_{fr}(a_i)$ is the framed-slope of the minimal framed μ-destabilizing quotient, we have that $r(a) = \mathrm{rk}(Q_i) \geq r(a_i)$ for all i. $\square$

Using the same arguments of corollary 7.2.6 in [9], we can prove:

Corollary 62. *$r(a)$ and $^{\mu_{fr}(a)}/_{a}$ are constant for $a \gg 0$.*

If $^{\mu_{fr}(a)}/_{a} = {}^{\mu_{fr}(a_i)}/_{a_i}$ and $r(a) = r(a_i)$ for all i, then Q_i is the minimal framed μ-destabilizing quotient of $E|_{D_i}$, hence $Q_C|_{D_i}$ differs from the minimal framed μ-destabilizing quotient of $E|_{D_i}$ only in dimension $d-3$, in particular their determinant line bundles as sheaves on D_i are equal. From this argument it follows:

Lemma 63. *There is a line bundle $L \in \mathrm{Pic}(X)$ such that $L_a \simeq L$ for all $a \gg 0$.*

Proof. The proof is similar to that of lemma 7.2.7 in [9]. $\square$

By corollary 62 and lemma 63 we have that $\epsilon(a)$ is constant for $a \gg 0$. We can now prove the theorem 57:

Proof. Suppose the theorem is false: we have to consider separately two cases: $\epsilon(a) = 1$ and $\epsilon(a) = 0$ for $a \gg 0$. In the first case we have

$$\frac{\deg(L) - \delta_1}{r} < \mu(\mathcal{E})$$

and $1 \leq r \leq \mathrm{rk}(E)$, where $r = r(a)$ for $a \gg 0$. We want to construct a rank r quotient Q of E, with nonzero induced framing β and $\det(Q) = L$. Thus

$$\mu(\mathcal{Q}) < \mu(\mathcal{E})$$

and therefore we obtain a contradiction with the hypothesis of μ-semistability of $\mathcal{E}$ with respect to δ_1. Let a be sufficiently large, $D \in U_a$ and the minimal framed μ-destabilizing quotient

$$\begin{tikzcd} E|_D \arrow[r, "q_D"] \arrow[d, "\alpha|_D"'] & Q_D \arrow[dl, "\beta_D"] \\ F|_D & \end{tikzcd}$$

Put $K_D = \ker \beta_D$ and $L_{K_D} = \det(K_D)$. By proposition 13 (for μ-semistability), Q_D fits into a exact sequence

$$0 \longrightarrow K_D \longrightarrow Q_D \longrightarrow \mathrm{Im}\alpha|_D \longrightarrow 0 \tag{9}$$

with K_D torsion free quotient of $\ker \alpha|_D$. So there exists a open subscheme $D' \subset D$ such that $K_D|_{D'}$ is locally free of rank r and $D \setminus D'$ is a closed subset of codimension two in D. Consider the restriction of the sequence 9 on D'

$$0 \longrightarrow K_D|_{D'} \longrightarrow Q_D|_{D'} \longrightarrow \mathrm{Im}\alpha|_{D'} \longrightarrow 0$$

By proposition V-6.9 in [10], we have a canonical isomorphism

$$L_{K_D}|_{D'} \otimes \det(\mathrm{Im}\alpha|_{D'}) = \det(Q_D|_{D'}) = L|_{D'}$$

If we denote by $\bar{L}$ the determinant bundle of Imα, we get

$$L_{K_D}|_{D'} = L|_{D'} \otimes \bar{L}^\vee|_{D'} = (L \otimes \bar{L}^\vee)|_{D'}$$

and therefore

$$L_{K_D} = (L \otimes \bar{L}^\vee)|_D$$

So we have a morphism $\sigma_D : \Lambda^r \ker \alpha|_D \to (L \otimes \bar{L}^\vee)|_D$ which is surjective on D' and morphisms

$$D' \longrightarrow Grass(\ker \alpha, r) \longrightarrow \mathbb{P}(\Lambda^r \ker \alpha)$$

By Serre's vanishing theorem and Serre duality, one has for $i = 0, 1$

$$\mathrm{Ext}^i(\Lambda^r \ker \alpha, (L \otimes \bar{L}^\vee)(-a)) = \mathrm{H}^{d-i}(X, \Lambda^r \ker \alpha \otimes (L \otimes \bar{L}^\vee)^\vee \otimes \omega_X^\vee(a))^\vee = 0$$

for all $a \gg 0$ (since $d \geq 2$), hence

$$\mathrm{Hom}(\Lambda^r \ker \alpha, L \otimes \bar{L}^\vee) = \mathrm{Hom}(\Lambda^r \ker \alpha|_D, (L \otimes \bar{L}^\vee)|_D).$$

So for a sufficiently large, the morphism σ_D extends to a morphism $\sigma : \Lambda^r \ker \alpha \to L \otimes \bar{L}^\vee$. The support of the cokernel of σ meets D in a closed subscheme of codimension two in D, hence there is an open subscheme $X' \subset X$ such that $\sigma|_{X'}$ is surjective, $X \setminus X'$ is a closed subscheme of codimension two and $D' = X' \cap D$. So we have a morphism $X' \to \mathbb{P}(\Lambda^r \ker \alpha)$ and we want that it factorizes through $Grass(\ker \alpha, r)$. Using the same arguments of the final part of proof of theorem 7.2.1 in [9], for sufficiently large a that morphism factorizes, hence we get a rank r locally free quotient

$$\ker \alpha|_{X'} \longrightarrow K_{X'}$$

such that $\det(K_{X'}) = (L \otimes \bar{L}^\vee)|_{X'}$. So we can extend $K_{X'}$ to a rank r coherent quotient K of $\ker \alpha$ such that $\det(K) = L \otimes \bar{L}^\vee$. Let $G = \ker(\ker \alpha \to K)$. We have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & \ker \alpha \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathrm{Im}\alpha \arrow[r] \arrow[d] & \mathrm{Im}\alpha \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We have that the determinant of Q is canonically isomorphic to $\det(K) \otimes \bar{L} = L$, so $\mathcal{Q}$ destabilizes $\mathcal{E}$ and this contradicts the hypothesis.

In the second case we have

$$\frac{\deg(L)}{r} < \mu(\mathcal{E}).$$

Let a be sufficiently large, $D \in U_a$ and the minimal framed μ-destabilizing quotient

$$\begin{array}{ccc} E|_D & \xrightarrow{q_D} & Q_D \\ \alpha|_D \downarrow & & \\ F|_D & & \end{array}$$

with $\ker q_D \not\subset \ker \alpha|_D$. By proposition 13 (for μ-semistability), Q_D is torsion free, hence there exists an open subscheme $D' \subset D$ such that $D \setminus D'$ is a closed set of codimension two in D and $Q_D|_{D'}$ is locally free of rank r. Moreover $\ker q_D|_{D'} \not\subset \ker \alpha|_{D'}$. Using the same techniques as in the last part of the proof of theorem 7.2.1 in [9], we extend $Q_D|_{D'}$ to a quotient $Q_{X'}$ of X' which is locally free of rank r with $\det(Q_{X'}) = L|_{X'}$. By construction we have $\ker(E|_{X'} \to Q_{X'}) \not\subset \ker \alpha|_{X'}$, hence in this way we obtain a quotient Q of E with $\det(Q) = L$ and zero induced framing, such that $\mathcal{Q}$ destabilizes $\mathcal{E}$. $\square$

10.2. **Slope-stable case.** Now we want to prove the following generalization of Mehta-Ramanathan's theorem for μ-stable torsion free sheaves ([14]):

Theorem 64. *Let F be a coherent sheaf on X supported on a divisor D_F, which is a locally free O_{D_F}-module. Let $\mathcal{E} = (E, \alpha : E \to F)$ be a (D_F, F)-framed sheaf on X. If $\mathcal{E}$ is μ-stable with respect to δ_1, then there is a positive integer a_0 such that for all $a \geq a_0$ there is a dense open subset $W_a \subset |O_X(a)|$ such that for all $D \in W_a$ the divisor D is smooth, meets transversally the divisor D_F and $\mathcal{E}|_D$ is μ-stable with respect to $a\delta_1$.*

The tecniques to prove this theorem are quite similar to the ones used before. By proposition 42 a μ-semistable (D_F, F)-framed sheaf which is simple but not μ-stable has a proper extended framed socle. Thus we first show that the restriction is simple and we use the extended framed socle (rather its quotient) as a replacement for the minimal framed μ-destabilizing quotient.

Proposition 65. *Let $\mathcal{E} = (E, \alpha)$ be a μ-stable (D_F, F)-framed sheaf. For $a \gg 0$ and general $D \in |O_X(a)|$ the restriction $\mathcal{E}|_D = (E|_D, \alpha|_D)$ is simple.*

To prove this result, we need to define the double dual of a framed sheaf. Let $\mathcal{E} = (E, \alpha)$ be a (D_F, F)-framed sheaf; we define a framing $\alpha^{\vee\vee}$ on the double dual of E in the following way: $\alpha^{\vee\vee}$ is the composition of morphisms

$$E^{\vee\vee} \longrightarrow E^{\vee\vee}|_{D_F} \simeq E|_{D_F} \xrightarrow{\alpha|_{D_F}} F|_{D_F}$$

α is then the framing induced on E by $\alpha^{\vee\vee}$ by means of the inclusion morphism $E \to E^{\vee\vee}$. We denote the framed sheaf $(E^{\vee\vee}, \alpha^{\vee\vee})$ by $\mathcal{E}^{\vee\vee}$. Note that also $\mathcal{E}^{\vee\vee}$ is a (D_F, F)-framed sheaf.

Lemma 66. *Let $\mathcal{E} = (E, \alpha)$ be a μ-stable (D_F, F)-framed sheaf. Then the framed sheaf $\mathcal{E}^{\vee\vee} = (E^{\vee\vee}, \alpha^{\vee\vee})$ is μ-stable.*

Proof. Consider the exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow A \longrightarrow 0$$

where A is a coherent sheaf supported on a closed subset of codimension at least two. Thus $\mathrm{rk}(E^{\vee\vee}) = \mathrm{rk}(E)$ and $\deg(E^{\vee\vee}) = \deg(E)$. Moreover, since $\alpha = \alpha^{\vee\vee}|_E$, we have $\mu(\mathcal{E}^{\vee\vee}) =$

$\mu(\mathcal{E})$. Let G be a subsheaf of $E^{\vee\vee}$ and denote by G' its intersection with E. So $\mathrm{rk}(G) = \mathrm{rk}(G')$, $\deg(G) = \deg(G')$ and $\alpha|_{G'} = \alpha^{\vee\vee}|_G$. Thus we obtain

$$\mu(G, \alpha^{\vee\vee}|_G) = \mu(G', \alpha|_{G'}) < \mu(\mathcal{E}) = \mu(\mathcal{E}^{\vee\vee}).$$

□

Lemma 67. *Let G be a reflexive sheaf. For $a \gg 0$ and $D \in |O_X(a)|$ the homomorphism* $\mathrm{End}(G) \to \mathrm{End}(G|_D)$ *is surjective.*

Proof. Let D be an element in $|O_X(a)|$. Consider the exact sequence

$$0 \longrightarrow G(-a) \longrightarrow G \longrightarrow G|_D \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}(G, \cdot)$ we obtain

$$0 \longrightarrow \mathrm{Hom}(G, G(-a)) \longrightarrow \mathrm{End}(G) \longrightarrow \mathrm{End}(G|_D) \longrightarrow \mathrm{Ext}^1(G, G(-a)) \to \cdots$$

Recall the Local-to-Global spectral sequence

$$\mathrm{H}^i(X, \mathcal{E}xt^j(G, G \otimes \omega_X(a))) \Rightarrow \mathrm{Ext}^{i+j}(G, G \otimes \omega_X(a)).$$

For sufficiently large $a \gg 0$ we get

$$\mathrm{Ext}^1(G, G(-a))^\vee \simeq \mathrm{Ext}^{n-1}(G, G \otimes \omega_X(a)) \simeq \mathrm{H}^0(X, \mathcal{E}xt^{n-1}(G, G) \otimes \omega_X(a)).$$

Since G is reflexive, $\mathrm{dh}(G) \leq n-2$ and therefore $\mathcal{E}xt^{n-1}(G, G) = 0$. Hence for a sufficiently large, $\mathrm{End}(G) \longrightarrow \mathrm{End}(G|_D)$ is surjective. □

Now we can prove proposition 65:

Proof. Recall that for arbitrary a and general $D \in |O_X(a)|$ $E|_D$ is torsion free and $E^{\vee\vee}|_D$ is reflexive, moreover the double dual of $E|_D$ is $E^{\vee\vee}|_D$. We have injective homomorphisms

$$\begin{aligned} \delta &: \mathrm{End}(E) \longrightarrow \mathrm{End}(E^{\vee\vee}), \\ \delta_D &: \mathrm{End}(E|_D) \longrightarrow \mathrm{End}(E^{\vee\vee}|_D). \end{aligned}$$

Let $\varphi \in \mathrm{End}(\mathcal{E})$: the image $\varphi^{\vee\vee} = \delta(\varphi)$ of φ is an element of $\mathrm{End}(E^{\vee\vee}, \alpha^{\vee\vee})$, indeed if $\alpha \circ \varphi = \lambda\alpha$, then we can define an endomorphism of $\mathcal{E}^{\vee\vee}$ in the following way:

$$\begin{tikzcd}
E^{\vee\vee} \arrow[r, "\varphi^{\vee\vee}"] \arrow[d] \arrow[ddd, bend right=60, "\alpha^{\vee\vee}"'] & E^{\vee\vee} \arrow[d] \arrow[ddd, bend left=60, "\alpha^{\vee\vee}"] \\
E^{\vee\vee}|_{D_F} \arrow[r, "\varphi^{\vee\vee}|_{D_F}"] \arrow[d, "\simeq"] & E^{\vee\vee}|_{D_F} \arrow[d, "\simeq"'] \\
E|_{D_F} \arrow[r, "\varphi|_{D_F}"] \arrow[d, "\alpha|_{D_F}"] & E|_{D_F} \arrow[d, "\alpha|_{D_F}"'] \\
F|_{D_F} \arrow[r, "\cdot\lambda"] & F|_{D_F}
\end{tikzcd}$$

In the same way it is possible to prove that for $\varphi \in \mathrm{End}(\mathcal{E}|_D)$, $\delta_D(\varphi)$ is an element of $\mathrm{End}(\mathcal{E}^{\vee\vee}|_D)$. So the homomorphisms

$$\begin{aligned} \delta &: \mathrm{End}(\mathcal{E}) \longrightarrow \mathrm{End}(\mathcal{E}^{\vee\vee}), \\ \delta_D &: \mathrm{End}(\mathcal{E}|_D) \longrightarrow \mathrm{End}(\mathcal{E}^{\vee\vee}|_D) \end{aligned}$$

are injective. Therefore it suffices to show that $\mathcal{E}^{\vee\vee}|_D$ is simple for $a \gg 0$ and general D. By lemma 66, $\mathcal{E}^{\vee\vee}$ is μ-stable, hence by lemma 15 it is simple. By lemma 67, the homomorphism $\chi : \mathrm{End}(E^{\vee\vee}) \to \mathrm{End}(E^{\vee\vee}|_D)$ is surjective for $a \gg 0$ and general D. Since for $\varphi \in \mathrm{End}(\mathcal{E}^{\vee\vee})$, $\chi(\varphi)$ is an element of $\mathrm{End}(\mathcal{E}^{\vee\vee}|_D)$, we have that the map

$$\chi : \mathrm{End}(\mathcal{E}^{\vee\vee}) \to \mathrm{End}(\mathcal{E}^{\vee\vee}|_D)$$

is also surjective. Thus $\mathrm{End}(\mathcal{E}|_D) = \mathrm{End}(\mathcal{E}^{\vee\vee}|_D) \simeq k$. □

Remark 68. Since $\mathcal{E}$ is μ-stable with respect to δ_1, we have $\deg(\mathrm{Im}\alpha) > \delta_1$, hence $\deg(\mathrm{Im}\alpha|_D) = a\deg(\mathrm{Im}\alpha) > a\delta_1$ for a positive integer, hence $\ker\alpha|_D$ is not framed μ-destabilizing for all $D \in \Pi_a$.

Let $a_0 \geq 3$ be an integer such that for all $a \geq a_0$ and a general $D \in \Pi_a$, the restriction $\mathcal{E}|_D$ is μ-semistable with respect to $a\delta_1$ and simple (cf. proposition 65). Suppose that for an integer $a \geq a_0$, the framed sheaf $\mathcal{E}|_D$ is not μ-stable with respect to $a\delta_1$ for a general divisor D. From the openness of stability (cf. proposition 52) it follows that $\mathcal{E}|_{D_\eta}$ is not μ-stable for the divisor D_η associated to the generic point $\eta \in \Pi_a$. Since $\mathcal{E}|_{D_\eta}$ is simple, by proposition 42 the extended socle of $\mathcal{E}|_{D_\eta}$ is a proper framed μ-destabilizing subsheaf. Consider the corresponding quotient sheaf Q_η, with induced framing β_η: we can extend it to a coherent quotient $q^*E \to Q_a$ over all of Z_a.

Let W_a be the dense open subset of points $D \in \Pi_a$ such that

- D is a smooth divisor, meets transversally the divisor D_F, $E|_D$ is torsion free,
- Q_a is flat over W_a and $\epsilon\left((\tilde{\alpha}_a)|_D\right) = \epsilon(\beta_\eta)$, where we denote by $\tilde{\alpha}_a$ the induced framing on Q_a.

Thus $Q_a|_D$ is a coherent sheaf of positive rank such that with the induced framing is a framed μ-destabilizing quotient for all $D \in W_a$.

Lemma 69. *If there exists a divisor $D_0 \in W_a$, $a \geq a_0$, such that $\mathcal{E}|_{D_0}$ is μ-stable with respect to $a\delta_1$, then for all $D' \in W_{a'}$ the framed sheaf $\mathcal{E}|_{D'}$ is μ-stable with respect to $a'\delta_1$ for all $a' \geq 2a$.*

Proof. If the lemma is false, then there exists $a' \geq 2a$ and a divisor $D \in W_{a'}$ such that $\mathcal{E}|_D$ is not μ-stable with respect to $a'\delta_1$. Choose a divisor $D_1 \in W_{a'-a}$ such that $D = D_0 + D_1$ is a divisor with normal crossings. Let $C \subset \Pi_{a'}$ be a curve with the properties asserted in the lemma 59. Using the same techniques as in the proof of lemma 61, we can extend $q^*E \to Q_{a'}$ to a C-flat quotient $(q^*E)|_{Z_C} \to Q_C$. Using the same computations than before, we have

$$a'\mu(E,\alpha) = \mu(E|_D, \alpha|_D) = \mu(Q_C|_D, \tilde{\alpha}_C|_D) \geq \mu(\bar{Q}, \bar{\alpha}) \geq \mu(Q_0, \alpha_0) + \mu(Q_1, \alpha_1).$$

Since $a' - a \geq a_0$, $(E|_{D_1}, \alpha|_{D_1})$ is μ-semistable, hence $\mu(Q_1, \alpha_1) \geq (a' - a)\mu(E, \alpha)$. Moreover by hypothesis $\mu(Q_0, \alpha_0) > a\mu(E, \alpha)$, hence we have a contradiction. □

Now we can prove the theorem 64:

Proof. Assume that the theorem is false: for all $a \geq a_0$ and general $D \in \Pi_a$, $\mathcal{E}|_D$ is not μ-stable with respect to $a\delta_1$. Thus one can construct for any $a \geq a_0$ a coherent quotient $q^*E \to Q_a$ and a dense open subset $W_a \subset \Pi_a$ such that $Q_a|_D$, with the induced framing, is a framed μ-destabilizing quotient for all $D \in W_a$. We denote by $\epsilon(a)$ the quantity $\epsilon(\tilde{\alpha}_a|_D)$

for $D \in W_a$. As before, there are line bundles $L_a \in \mathrm{Pic}(X)$ such that $\det(Q_a|_D) = L_a|_D$ for $D \in W_a$ and all $a \geq a_0$.

Let $N \subset \mathbb{Z}$ be an infinite subset consisting of integers $a \geq a_0$ such that $\mathrm{rk}(Q_a)$ is constant, say r. By remark 68 we have $0 < r < \mathrm{rk}(E)$. Using the same arguments of the proof of the lemma 61, one can prove that if $a_1, a_2, \ldots, a_l$ are integers in N, with $a_i \geq a_0$ for $i = 1, \ldots, l$ and $a = \sum a_i$, and D_i are divisors in W_{a_i} such that $D = \sum D_i$ is a divisor with normal crossings, then $L_a|_{D_i}$ is the determinant line bundle of some framed μ-destabilizing quotient of $\mathcal{E}|_{D_i}$.

Lemma 70. *Let $\mathcal{G} = (G, \beta)$ be a framed sheaf of positive rank. If $\mathcal{G}$ is μ-semistable with respect to $\bar{\delta}_1$, then the set T of determinant line bundles of framed μ-destabilizing quotients of $\mathcal{G}$ is finite and its cardinality is bounded by $2^{\mathrm{rk}(G)}$.*

Proof. Let $gr(\mathcal{G}) \simeq (G_1, \beta_1) \oplus (G_2, \beta_2) \oplus \cdots \oplus (G_l, \beta_l)$, with $\mathcal{G}_i = (G_i, \beta_i)$ μ-stable with respect to $\bar{\delta}_1$ and $\deg(\mathcal{G}_i) = \mathrm{rk}(G_i)\mu(\mathcal{G})$. Let G' be a subsheaf of G with $\deg(G', \beta') = \mathrm{rk}(G')\mu(\mathcal{G})$. We can start with a stable filtration of $\mathcal{G}'$ and complete it to one of $\mathcal{G}$:

$$0 = G'_0 \subset G'_1 \subset \cdots \subset G'_s = G' \subset \cdots \subset G_l = G.$$

Since $gr(\mathcal{G})$ is indipendent of the filtration, we have that $\det(G')$ has to be isomorphic to one of $\det(G_{i_1}) \otimes \cdots \otimes \det(G_{i_j})$. Thus the set T is finite and its cardinality is bounded by $2^{\mathrm{rk}(G)}$. □

Let $a \geq 2a_0$ and $D \in W_{a_0}$ be an arbitrary point. By lemma 69, we have that $\mathcal{E}|_D$ is not μ-stable with respect to $a_0\delta_1$. If we denote by T_D the set T of the previous lemma associated to $\mathcal{E}|_D$, then $L_a|_D \in T_D$. Consider the function

$$\begin{array}{rcl} \varphi : N_{\geq 2a_0} & \to & \prod_{D \in W_{a_0}} T_D \\ a & \mapsto & (L_a|_D)_D \end{array}$$

Let $\sim$ be the equivalence relation on $N_{\geq 2a_0}$ defined in the following way: $a \sim a'$ if and only if the set $\{s \in W_{a_0} \mid \varphi(a)(s) = \varphi(a')(s)\}$ is dense in W_{a_0}.

Lemma 71. *There are at most $2^{\mathrm{rk}(E)}$ distinct equivalence classes and, in particular, there is at least one infinite class $\tilde{N}$.*

Proof. Assume that there are distint classes $N_1, \ldots N_l$ with $l > 2^{\mathrm{rk}(E)}$. Choose representatives $a_i \in N_i$. For fixed $D \in W_{a_0}$, we have that $\varphi(a_1)([D]), \ldots \varphi(a_l)([D]) \in T_D$. Since $l > |T_D|$, at least two of these elements must be equal. In this way we can associate for any $D \in W_{a_0}$, a pair of indices (i, j). The set of all these pairs is finite, hence there at least one pair (i, j) which is associated to all points in a dense subset of W_{a_0}, but by definition this means $a_i \sim a_j$, hence the class N_i is equal to the class N_j and this is a contradiction. □

Lemma 72. *There is a line bundle $L \in \mathrm{Pic}(X)$ such that $L \simeq L_a$ for all $a \in \tilde{N}$. Moreover $\epsilon(a)$ is constant for $a \in \tilde{N}$.*

Proof. Let $a, a' \in \tilde{N}$. $a \sim a'$ means that $\varphi(a)$ and $\varphi(a')$ are equal on a dense subset of W_{a_0}, then $L_a|_D \simeq L_{a'}|_D$ for all D in a dense subset of Π_{a_0}. This suffices to prove that $L_a \simeq L_{a'}$ (see lemma 7.2.2 in [9]). □

Summing up, we have that there is a line bundle L on X and an integer $0 < r < \mathrm{rk}(E)$ such that for $a \in \tilde{N}$ and for general $D \in W_a$

$$\mu(Q_a|_D, \tilde{\alpha}_a|_D) = \frac{\deg(L|_D) - \epsilon(a)a\delta_1}{r} = a\left(\frac{\deg(L) - \epsilon(a)\delta_1}{r}\right) = \mu(E|_D, \alpha|_D) = a\mu(E, \alpha),$$

hence

$$\frac{\deg(L) - \epsilon(a)\delta_1}{r} = \mu(E, \alpha).$$

Using the arguments at the end of the proof of the restriction theorem for μ-semistable framed sheaves, one can show that this suffices to construct a framed μ-destabilizing quotient $E \to Q$ for sufficiently large a. This contradicts the assumptions of the theorem. □

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