Solutions to the nonlinear Schrödinger equation carrying momentum along a curve. Part II: proof of the existence result

Fethi MAHMOUDI[a]   Andrea MALCHIODI[a]

[a]SISSA, Sector of Mathematical Analysis
Via Beirut 2-4, 34014 Trieste, Italy

ABSTRACT. We prove existence of a special class of solutions to the (elliptic) Nonlinear Schrödinger Equation $-\varepsilon^2 \Delta \psi + V(x)\psi = |\psi|^{p-1}\psi$ on a manifold or in the Euclidean space. Here V represents the potential, p is an exponent greater than 1 and ε a small parameter corresponding to the Planck constant. As ε tends to zero (namely in the semiclassical limit) we prove existence of complex-valued solutions which concentrate along closed curves, and whose phase in highly oscillatory. Physically, these solutions carry quantum-mechanical momentum along the limit curves. In the first part of this work we identified the limit set and constructed approximate solutions, while here we give the complete proof of our main existence result Theorem 1.1.

1   Introduction

In this paper we continue our study of [10], concerning concentration phenomena for solutions of the singularly-perturbed elliptic problem

$$(1) \qquad -\varepsilon^2 \Delta_g \psi + V(x)\psi = |\psi|^{p-1}\psi \qquad \text{on } M,$$

where M is an n-dimensional compact manifold (or the flat Euclidean space $\mathbb{R}^n$), V a smooth positive function on M satisfying the properties

$$(2) \qquad 0 < V_1 \le V \le V_2; \qquad \|V\|_{C^3} \le V_3,$$

ψ a complex-valued function, $\varepsilon > 0$ a small parameter and p is an exponent greater than 1. Here Δ_g stands for the Laplace-Beltrami operator on (M, g).

Solutions to (1) represent *standing waves* of the Nonlinear Schrödinger Equation, and here we are interested in the *semiclassical limit*, namely the asymptotics of solutions when the parameter ε (representing the Planck constant) tends to zero. Typically, if concentration occurs near some point $x_0 \in M$, such solutions behave like $\psi_\varepsilon(x) \simeq u\left(\frac{dist(x,x_0)}{\varepsilon}\right)$, where $dist(\cdot, \cdot)$ denotes the distance on M and where u solves the equation

$$(3) \qquad -\Delta u + V(x_0)u = u^p \qquad \text{in } \mathbb{R}^n.$$

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[1]E-mail addresses: mahmoudi@ssissa.it (F.Mahmoudi), malchiod@sissa.it (A. Malchiodi)

We refer the reader to the introduction of [10] for the motivation of the study and for a brief description of the existing results in the literature about this topic. There are several works concerning standing waves concentrating at a single (or multiple) points of M, and for which the corresponding solutions of (3) decay to zero at infinity. On the other hand, only very recently it has been proven existence of solutions concentrating at higher dimensional sets, like curves or manifolds. In all these results (except for [2]), the profile is given by (real) solutions to (3) which are independent of some of the variables and hence do not tend to zero at infinity: if concentration occurs near a k-dimensional set, then the profile in the directions orthogonal to the limit set will be given by a soliton in $\mathbb{R}^{n-k}$.

In this paper we are going to construct a different type of solutions. These still concentrate along curves in M, but their phase is highly oscillatory along the limit set. More precisely, we consider standing waves (namely the solution of (3)) whose profile has the following expression

$$(4) \qquad \phi(x', x_n) = e^{-i\hat{f}x_n}\hat{U}(x'), \qquad x' = (x_1, \ldots, x_{n-1}),$$

where $\hat{f}$ is some constant and $\hat{U}(x')$ a real function. With this choice of ϕ, if concentration occurs near some point x_0, then the function $\hat{U}$ satisfies the equation

$$(5) \qquad -\Delta\hat{U} + \left(\hat{f}^2 + V(x_0)\right)\hat{U} = |\hat{U}|^{p-1}\hat{U} \qquad \text{in } \mathbb{R}^{n-1},$$

and decays to zero at infinity. Solutions to (5) can be found by considering the radial function $U : \mathbb{R}^{n-1} \to \mathbb{R}$ which solves

$$(6) \qquad -\Delta U + U = U^p \qquad \text{in } \mathbb{R}^{n-1}.$$

It is known that U (and its derivatives) behaves at infinity like

$$(7) \qquad U(r) \simeq e^{-r}r^{-\frac{n-2}{2}} \qquad \text{as } r \to +\infty.$$

Using the scaling

$$(8) \qquad \hat{U}(x') = \hat{h}U(\hat{k}x'), \qquad \hat{h} = \left(\hat{f}^2 + V(x_0)\right)^{\frac{1}{p-1}}, \quad \hat{k} = \left(\hat{f}^2 + V(x_0)\right)^{\frac{1}{2}},$$

in (4) the constant $\hat{f}$ can be chosen arbitrarily, and then $\hat{h}, \hat{k}$ are determined according to the last formula, depending on $V(x_0)$. Indeed $\hat{f}$ represents the speed of the phase oscillation, and is physically related to the velocity of the quantum-mechanical particle associated to the wave function. If concentration occurs near some closed curve $\gamma = \gamma(\overline{s})$ in M, and if we allow the parameter $\hat{f}$ to depend on the variable s, then the solution ψ will be of the form

$$(9) \qquad \psi(\overline{s}, \zeta) \simeq e^{-i\frac{f(\overline{s})}{\varepsilon}}h(\overline{s})U\left(\frac{k(\overline{s})y}{\varepsilon}\right),$$

where $\overline{s}$ stands for the arc-length parameter of γ, and y for a system geodesic coordinates normal to γ. Here the functions $h(\overline{s})$ and $k(\overline{s})$ are chosen so that

$$(10) \qquad h(\overline{s}) = \left((f'(\overline{s}))^2 + V(\overline{s})\right)^{\frac{1}{p-1}}, \qquad k(\overline{s}) = \left((f'(\overline{s}))^2 + V(\overline{s})\right)^{\frac{1}{2}}.$$

basically replacing $\hat{f}$ with $f'(\overline{s})$ in (8).

If γ is given, it was shown in [10] (using formal expansions in ε) that the corresponding function f should satisfy the following condition

$$(11) \qquad f'(\overline{s}) \simeq \mathcal{A}h^\sigma(\overline{s}) \qquad \text{with} \quad \sigma = \frac{(n-1)(p-1)}{2} - 2,$$

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where $\mathcal{A}$ is an arbitrary constant. At this point, only the limit curve γ should be determined. Since we require the function f to be periodic, if we consider variations of γ (for all of which (11) holds true) then it is natural to work in the restricted class

$$\Gamma := \left\{ \gamma : \mathbb{R} \to M \text{ periodic } : \ \mathcal{A} \int_{\gamma} h(\overline{s})^{\sigma} d\overline{s} = \int_{\gamma} f'(\overline{s})^{\sigma} d\overline{s} = \text{constant} \right\},$$

where, as before, $\overline{s}$ stands for the arc-length parameter. It is shown in [10] that the candidate limit curves are critical points of the functional $\gamma \mapsto \int_{\gamma} h^{\theta}(\overline{s})d\overline{s}$, where

$$\theta = p + 1 - \frac{1}{2}(p - 1)(n - 1).$$

With a direct computation one can prove that the extremality condition is the following

$$(12) \qquad \nabla^N V = \left(\frac{p - 1}{\theta} h^{p-1} - 2\mathcal{A}^2 h^{2\sigma} \right) \mathbf{H},$$

where $\nabla^N V$ represents the normal gradient of V and $\mathbf{H}$ is the curvature vector of γ.

Similarly, via some long but straightforward calculation, one can find a natural non-degeneracy condition for stationary points, which is expressed by the invertibility of the operator $\mathfrak{J}$, acting on normal sections $\mathcal{V}$ to γ, which in components is given by

$$
\begin{aligned}
(\mathfrak{J}\mathcal{V})^m \ = \ & -\left(h^{\theta} - \frac{2\mathcal{A}^2\theta}{p-1}h^{\sigma} \right) \ddot{\mathcal{V}}^m - \theta \left(h^{\theta-1} - \frac{2\mathcal{A}^2\sigma}{p-1}h^{\sigma-1} \right) h'\dot{\mathcal{V}}^m + \frac{\theta}{p-1}h^{-\sigma}((\nabla^N)^2 V)[\mathcal{V}, E_m] \\
(13) \qquad & + \ \frac{1}{2}\left(h^{\theta} - \frac{2\mathcal{A}^2\theta}{p-1}h^{\sigma} \right)\left(\sum_j \left(\partial^2_{jm}g_{11} \right)\mathcal{V}^j \right) - 2\mathcal{A}\mathcal{A}'_1 \frac{(\theta-\sigma)h^{p-1}}{[(p-1)h^{\theta} - 2\sigma\mathcal{A}^2h^{\sigma}]}H^m \\
& + \ H^m\langle \mathbf{H}, \mathcal{V}\rangle \left[\frac{-(p-1)\left(3 + \frac{\sigma}{\theta}\right)h^{2\theta} - \frac{16\sigma\theta\mathcal{A}^4}{p-1}h^{2\sigma} + 2\mathcal{A}^2(5\sigma + 3\theta)h^{\theta+\sigma}}{(p-1)h^{\theta} - 2\mathcal{A}^2\sigma h^{\sigma}} \right],
\end{aligned}
$$

$m = 2, \ldots, n$. We refer to Section 2 in [10] for the notation used in this formula. We point out that, since (11) determines only the derivative of the phase, to obtain periodicity we need to introduce a nonlocal term, denoted here with $\mathcal{A}'_1$. Letting $L(\gamma)$ be the length of the curve γ, our main result is the following.

Theorem 1.1 *Let M be a compact n-dimensional manifold and let $V : M \to \mathbb{R}$ be a smooth positive function (or let $M = \mathbb{R}^n$ and let V satisfy (2)) and $1 < p < \frac{n+1}{n-3}$. Let γ be a simple closed curve in M: then there exists a positive constant $\mathcal{A}_0$, depending on $V|_{\gamma}$ and p for which the following holds. If $0 \leq \mathcal{A} < \mathcal{A}_0$, if γ satisfies (12) and the operator in (13) is invertible on normal sections of γ, there is a sequence $\varepsilon_k \to 0$ such that problem (NLS_{ε_k}) possesses solutions ψ_{ε_k} having the asymptotics in (9), with f satisfying (11).*

As a consequence of this theorem, see Corollary 1.3 in [10], we prove a conjecture posed in [1] for the case of one-dimensional limit sets. We also improve the result in [2], in the sense that we characterize explicitly the limit set and we do not require any symmetry on the potential V: indeed in [2] V is assumed cylindrically symmetric in $\mathbb{R}^3$, and solutions are found via separation of variables. The restriction on the exponent p is natural since it is a necessary condition for the solutions of (6) to vanish at infinity by the *Pohozaev's identity*. The smallness condition on the constant $\mathcal{A}$ and the fact that concentration is not proved for all the values of (small) ε are discussed below in the introduction. For the latter issue and for the main difficulties caused by removing the symmetries see also the introduction of [10].

The main goal of Part I, [10], was to show that the condition (12) and the non-degeneracy of the operator in (13), arising from the *reduced functional* $\gamma \mapsto \int_{\gamma} h^{\theta}(\overline{s})d\overline{s}$, appear naturally when considering (1), and

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in particular when we try to solve it formally with an expansion in power series of ε. To explain this fact, it is convenient to scale problem (1) in the following way

$$(14) \qquad -\Delta_{g_\varepsilon}\psi + V(\varepsilon x)\psi = |\psi|^{p-1}\psi \qquad \text{in } M_\varepsilon,$$

where M_ε denotes the manifold M endowed with the scaled metric $g_\varepsilon = \frac{1}{\varepsilon^2}g$ (with an abuse of notation we might often write $M_\varepsilon = \frac{1}{\varepsilon}M$). We are now looking for a solution concentrated near the dilated curve $\gamma_\varepsilon := \frac{1}{\varepsilon}\gamma$. We let s be the arc-length parameter of γ_ε, so that $\overline{s} = \varepsilon s$, and we let $(E_j)_{j=2,\ldots,n}$ denote an orthonormal frame in $N\gamma$ (the normal bundle of γ) transported parallely with respect to the normal connection, see Section 2 in [10]: we also let $(y_j)_j$ be a corresponding set of normal coordinates. Since we want to allow some flexibility both in the choice of the phase and of the curve of concentration, we define $\tilde{f}_0(\overline{s}) = f(\overline{s}) + \varepsilon f_1(\overline{s})$, and we set $z_j = y_j - \Phi_j(\overline{s})$, where $(\Phi_j)_{j=2,\ldots,n}$ are the components (with respect to the above coordinates y) of a section Φ in $N\gamma$. Then, with a formal expansion of ψ in powers of ε up to the second order, in the coordinates (s,z) near γ_ε, we set

$$\psi_{2,\varepsilon}(s,z) = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left\{h(\varepsilon s)U\left(k(\varepsilon s)z\right) + \varepsilon\left[w_r + iw_i\right] + \varepsilon^2\left[v_r + iv_i\right]\right\}, \qquad s \in [0, L/\varepsilon], z \in \mathbb{R}^{n-1},$$

$(L = L(\gamma))$ for some corrections w_r, w_i, v_r, v_i (which have to be determined) to the above approximate solutions. We saw in Section 3 of [10] that these terms solve equations of the form $\mathcal{L}_r w_r = \mathcal{F}_r$, $\mathcal{L}_i w_i = \mathcal{F}_i$, $\mathcal{L}_r v_r = \tilde{\mathcal{F}}_r$, $\mathcal{L}_i v_i = \tilde{\mathcal{F}}_i$ where

$$(15) \qquad \begin{cases} \mathcal{L}_r v = -\Delta_z v + V(\overline{s})v - ph(\overline{s})^{p-1}U(k(\overline{s})z)^{p-1}v; \\ \mathcal{L}_i v = -\Delta_z v + V(\overline{s})v - h(\overline{s})^{p-1}U(k(\overline{s})z)^{p-1}v \end{cases} \qquad \text{in } \mathbb{R}^{n-1},$$

and where $\mathcal{F}_r$, $\mathcal{F}_i$, $\tilde{\mathcal{F}}_r$, $\tilde{\mathcal{F}}_i$ are given data which depend on V, γ, $\overline{s}$, $\mathcal{A}$, Φ and f_1. The operators $\mathcal{L}_r$ and $\mathcal{L}_i$ are Fredholm (and symmetric) from $H^2(\mathbb{R}^{n-1})$ into $L^2(\mathbb{R}^{n-1})$, and the above equations for the corrections can be solved provided the right-hand sides are orthogonal to the kernels of these operators. As explained in [10], the condition (12) and the non-degeneracy of the operator $\mathfrak{J}$ allow us to determine w_r, w_i and v_r, v_i respectively, namely to solve (14) at order ε first, and then at order ε^2.

To make the above arguments rigorous, we can start with an approximate solution $\psi_{0,\varepsilon}$ behaving like $\psi_{0,\varepsilon} \simeq e^{-i\frac{f(\varepsilon s)}{\varepsilon}}h(\varepsilon s)U(k(\varepsilon s)z)$, and try to find a true solution of the form $e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}[h(\varepsilon s)U(k(\varepsilon s)z) + \tilde{w}]$, with $\tilde{w}$ suitably small and $\tilde{f}$ close to f, via some local inversion arguments. From a linearization of the equation near $\psi_{0,\varepsilon}$, the operator L_ε acting on $\tilde{w}$ in the coordinates (s,z) is then the following

$$(16) \qquad L_\varepsilon\tilde{w} := -\partial_{ss}^2\tilde{w} - \Delta_z\tilde{w} + V(\varepsilon x) - |\psi_{0,\varepsilon}|^{p-1}\tilde{w} - (p-1)|\psi_{0,\varepsilon}|^{p-3}\psi_{0,\varepsilon}\Re(\psi_{0,\varepsilon}\overline{\tilde{w}}).$$

Here $\Re$ denotes the real part. Decomposing first $\tilde{w}$ into its real and imaginary parts, and then in Fourier modes with respect to the variable εs, we can write $\tilde{w} = \tilde{w}_r + i\tilde{w}_i = \sum_j \sin(j\varepsilon s)\tilde{w}_{r,j}(z) + i\sum_j \sin(j\varepsilon s)\tilde{w}_{i,j}(z)$ (forgetting for simplicity about the cosine functions). If we take (as a model problem) $V \equiv 1$, then the operators (in the z variables) acting on the real and imaginary components are respectively $\mathcal{L}_r + \varepsilon^2 j^2$ and $\mathcal{L}_i + \varepsilon^2 j^2$. It is well-known, see for example [6], that $\mathcal{L}_r$ has a single negative eigenvalue, a kernel with multiplicity $n-1$ spanned by the functions $\partial_l U(k(\overline{s})z)$, $l = 2,\ldots,n$ (the generators of the normal translations), while all the remaining eigenvalues are positive. The operator $\mathcal{L}_i$ instead has one zero eigenvalue with eigenfunction $U(k(\overline{s})z)$ (the generator of complex rotations) and all the remaining eigenvalues positive.

As a consequence, the kernels of $\mathcal{L}_r$ and $\mathcal{L}_i$ produce a sequence of eigenvalues for L_ε which behave *qualitatively* like $\varepsilon^2 j^2$, and for small values of j these become resonant. With an accurate expansion of these eigenvalues, one finds that the non-degeneracy assumption on (13) prevents each of them to vanish: anyway, a direct application of the implicit function theorem is not possible since a further resonance phenomenon occurs. This arises from the fact that $\mathcal{L}_r$ possesses a negative eigenvalue as well, which generates an extra sequence of eigenvalues of L_ε, qualitatively of the form $-1 + \varepsilon^2 j^2$, $j \in \mathbb{N}$. This resonance is typical of concentration for (1) along sets of positive dimension, and the only hope to get invertibility is to choose the values of ε appropriately. Indeed, differently from the previous sequence of

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eigenvalues, this new one causes a divergence of the Morse index when ε tends to zero, and the presence of a kernel for some epsilon's is unavoidable. The eigenfunctions corresponding to these eigenvalues will have faster and faster oscillations along the limit curve γ.

This phenomenon is also present when one looks for solutions of the singularly perturbed problem $-\varepsilon^2 \Delta u + u = u^p$ in bounded domains of $\mathbb{R}^n$, when Neumann boundary conditions are imposed. In the papers [9], [12], [13], [14] concentration along sets of dimension $k = 1, \ldots, n-1$ has been proved, and analogous spectral properties hold true. By the Weyl's asymptotic formula, if solutions concentrate along a set of dimension d, the counterpart of the latter sequence of eigenvalues behaves like $-1 + \varepsilon^2 j^{\frac{2}{d}}$, and the average distance between those close to zero is of order ε^d. The resonance phenomenon was taken care of using a theorem by T. Kato, see [5], page 445, which allows to differentiate eigenvalues with respect to ε. In the aforementioned papers it was shown that when varying the parameter ε the spectral gaps near zero almost do not shrink, and invertibility can be obtained for a large family of epsilon's.

However when the concentration set is one-dimensional the spectral gaps of the resonant eigenvalues (with fast-oscillating eigenfunctions) are relatively large, of order ε, and the profile of the corresponding eigenfunctions can be analyzed by means of a scalar function on $[0, L]$ (see below in the introduction and in Subsection 4.2.3). This fact indeed allows sometimes to bypass Kato's theorem and to use a more direct approach, employed in [15] to study existence of constant mean curvature surfaces of cylindrical type embedded in manifolds, and in [3] for studying solutions of (1) in $\mathbb{R}^2$. We can partially take advantage of these techniques, see the comments in Section 2, but some new difficulties arise due to the fast phase oscillations in (9). We describe them below, together with the strategy of the proof.

By the above discussion, we expect to find three possible resonances: two of them for small values of the index j (with eigenvectors roughly of the form $e^{-i\frac{f(\varepsilon s)}{\varepsilon}} \partial_l U(k(\overline{s})z) \sin(\varepsilon j s)$, $l = 2, \ldots, n$, and $i e^{-i\frac{f(\varepsilon s)}{\varepsilon}} U(k(\overline{s})z) \sin(\varepsilon j s)$ respectively) and a third one for j of order $\frac{1}{\varepsilon}$, precisely when $-\varepsilon^2 j^2$ coincides with the first eigenvalue of $\mathcal{L}_r$.

To understand this behavior, we first study the spectrum of a *model* operator similar to (16), where we assume $V \equiv \hat{V} > 0$ and $\psi_{0,\varepsilon}$ to coincide with the function in (4). For this case we characterize completely the spectrum of the operator and the properties of the eigenfunctions, see Subsection 2.3 and in particular Proposition 2.5. The condition on the smallness of $\mathcal{A}$ appears precisely here (and only here), and is used to show that the resonant eigenvalues are only of the forms described above. Removing the smallness assumption might indeed lead to further resonance phenomena, see Remark 2.7 for further comments.

We next consider the case of non-constant potential V: since this has a slow dependence in s along γ_ε, one might guess that the approximate kernel of L_ε (see (16)) might be obtained from that for constant V, introducing also a slow dependence in s of the profile of these functions. With this criterion, given a small positive parameter δ, we introduce a set K_δ (see (67) and the previous formulas) consisting of *candidate* approximate eigenfunctions on L_ε, once multiplied by the phase factor $e^{-i\frac{f(\varepsilon s)}{\varepsilon}}$. More comments on the specific construction of this set can be found in Subsection 2.3, especially before (67).

In Proposition 2.9 we show that this guess is indeed correct: in fact, we prove that the operator L_ε is invertible provided we restrict ourselves to the subset $\overline{H}_\varepsilon$ of functions which are orthogonal to $e^{-i\frac{f(\varepsilon s)}{\varepsilon}} K_\delta$. This property allows us to solve the equation up to a lagrange multiplier in K_δ, see Proposition 2.14. For technical reasons, we prove invertibility of L_ε in suitable weighted norms, which are convenient to deal with functions decaying exponentially away from γ_ε. As done in [3], [9] and [12], this decay allows us to shift the problem from the whole manifold M_ε to the normal bundle $N\gamma_\varepsilon$ via a localization method, see Subsection 2.2.

Compared to the other results in the literature which deal with this kind of resonance phenomena, the approximate kernel here depends *genuinely* on the variable s (in [9], [12], [13], [14], [15] the problem is basically homogeneous along the limit set, while in [3] it can be made such through a change of variables). To deal with this feature, which mostly causes difficulties in Proposition 2.9, we localize the problem in the variable s as well. Multiplying by a cutoff function depending on s, we show that orthogonality to K_δ implies approximate orthogonality to the set $\hat{K}_\delta$, see (70) and the previous formulas, which is the counterpart of K_δ for a potential freezed at some point in γ_ε: once this is shown, we use the spectral analysis of Proposition 2.5.

Section 3 is devoted to choose a family of approximate solutions to (14): since we have many small eigenvalues appearing, it is natural trying to look for functions which solve (14) as accurately as possible. Our final goal is to annihilate the Lagrange multiplier in Proposition 2.14, and to do this we choose approximate solutions $\tilde{\Psi}_{2,\varepsilon}$ (in the notation of Section 3) which depend on suitable parameters: a normal section Φ, a phase factor f_2 and a real function β. The latter parameters correspond to different components of K_δ, and are related to the kernels of $\mathcal{L}_r(+\varepsilon^2 j^2)$ and $\mathcal{L}_i(+\varepsilon^2 j^2)$, see the above comments. The function β in particular is highly oscillatory, and takes care of the resonances due to the fast Fourier modes.

Differently from [10] (see in particular Section 4 there), where the expansions were only formal, we need here to derive rigorous estimates on the error terms, and to study in particular their Lipschitz dependence on the data Φ, f_2 and β. Proposition 3.2 collects the final expression for $-\Delta_{g_\varepsilon}\psi + V(\varepsilon x)\psi - |\psi|^{p-1}\psi$ on the approximate solutions $\tilde{\Psi}_{2,\varepsilon}$: the error terms $\tilde{\mathfrak{A}}$'s are listed (and estimated) before in that section, together with their Lipschitz dependence on the parameters.

Finally, after performing a Lyapunov-Schmidt reduction onto the set K_δ, see Proposition 4.1, we study the bifurcation equation in order to annihilate the Lagrange multiplier. In doing this we use crucially the computations in Part I, [10], together with the error estimates in Section 3. In particular, for Φ and f_2 we find as main terms respectively the operator $\mathfrak{J}$ in (13) and the one in the left-hand side of (113), both appearing when we performed formal expansions: these operators are both invertible by our assumptions, and therefore we are able to determine Φ and f_2 without difficulties.

The operator acting on β instead is more delicate, since it is <u>qualitatively</u> of the form

$$(17) \qquad\qquad -\varepsilon^2\beta''(\overline{s}) + \lambda(\overline{s})\beta \qquad\qquad \text{on } [0, L],$$

with periodic boundary conditions, where λ is a negative function. This operator is precisely the one related to the peculiar resonances described above. In particular it is resonant on frequencies of order $\frac{1}{\varepsilon}$, and this requires to choose a norm for β which is weighted in the Fourier modes, see (125) and Subsection 4.2.3. For operators like that in (17) there is in general a sequence of epsilon's for which a non-trivial kernel exists. Using Kato's theorem though, as in [12], [13], [14], [9] and [7], we provide estimates on the derivatives of the eigenvalues with respect to ε, showing that for several values of this parameter the operator acting on β is invertible. In this operation also the value of the constant $\mathcal{A}$, see (11), has to be suitably modified (depending on ε), in order to preserve the periodicity of our functions. Once we have this, we apply the contraction mapping theorem to solve the bifurcation equation as well.

The results in this paper and in [10] are briefly summarized in the note [11].

Notation and conventions

Dealing with coordinates, capital letters like $A, B, \ldots$ will vary between 1 and n while indices like $j, l, \ldots$ will run between 2 and n. The symbol i will stands for the imaginary unit.

For summations, we use the standard convention of summing terms where repeated indices appear.

We will often work with coordinates $(\overline{s}, \overline{y}_1, \ldots, \overline{y}_{n-1})$ near the curve γ, where $\overline{s}$ is the arc-length parameter of γ and the $\overline{y}_j$'s are Fermi normal coordinates (see the next section). The dilated curve $\gamma_\varepsilon := \frac{1}{\varepsilon}\gamma$ will be parameterized by $s = \frac{1}{\varepsilon}\overline{s}$, and we also denote by y the scaled normal coordinates. The length of γ is denoted by L. When dealing with functions (or normal sections) depending on the variables s or $\overline{s}$, the notation *prime* will always denote the derivative with respect to $\overline{s}$. When we differentiate with respect to s, we usually adopt the symbol ∂_s.

For simplicity, a constant C is allowed to vary from one formula to another, also within the same line.

For a real positive variable r and an integer m, $O(r^m)$ (resp. $o(r^m)$) will denote a complex-valued function for which $\left|\frac{O(r^m)}{r^m}\right|$ remains bounded (resp. $\left|\frac{o(r^m)}{r^m}\right|$ tends to zero) when r tends to zero. We might also write $o_\varepsilon(1)$ for a quantity which tends to zero as ε tends to zero.

Sometimes we shall need to work with integer indices j which belong to sets depending on ε. e.g. $\{0, 1, \ldots, [1/\varepsilon]\}$, where the latter square brackets stand for the integer part. For convenience, we will often omit the to add the square brackets, assuming that this convention is understood.

2 Lyapunov-Schmidt reduction of the problem

In this section we show how to reduce problem (14) to a system of three ordinary (integro-)differential equations on $\mathbb{R}/[0, L]$. We first introduce a metric on the normal bundle $N\gamma_\varepsilon$ of γ_ε and then study operators which mimic the properties of the linearization of (14) near an approximate solution. Next, we turn to the reduction procedure: this follows basically from a localization method, since the functions we are dealing with have an exponential decay away from γ_ε. We introduce a set K_δ consisting of approximate (resonant) eigenfunctions of the linearized operator L_ε: calling $\overline{H}_\varepsilon$ the orthogonal complement of this set (which has to be multiplied by a phase factor close to $e^{-i\frac{f(\varepsilon s)}{\varepsilon}}$) we show in Proposition 2.14 that L_ε is invertible on the projection onto this set, once suitable weighted norms are introduced.

2.1 A metric structure on $N\gamma_\varepsilon$

In this subsection we define a metric $\hat{g}_\varepsilon$ on $N\gamma_\varepsilon$, the normal bundle to γ_ε, and then introduce some basic tools which are useful for working in local coordinates on this set.

First of all, we choose a local orthonormal frame $(E_i)_i$ in $N\gamma$ and, using the notation of Subsection 2.2 in [9], we set $\nabla^N_{\partial_{\overline{s}}} E_j = \beta_j^l(\partial_{\overline{s}})E_l$, $j, l = 1, \ldots, n-1$. If we impose that the E_j's are transported parallely via the normal connection ∇^N, as in Subsection 2.1 of [10], we find that $\beta_j^l(\partial_{\overline{s}}) \equiv 0$ for all j, l. As a consequence, see formula (18) in [9], we have that if $(V^j)_j$, $j = 1, \ldots, n-1$ is a normal section to γ, then the components of the normal Laplacian $\Delta^N V$ are simply given by

$$(18) \qquad (\Delta^N V)^j = \Delta_\gamma(V^j) = \partial^2_{ss} V^j, \qquad j = 1, \ldots, n-1.$$

We next define a metric $\hat{g}$ on $N\gamma$ as follows. Given $v \in N\gamma$, a tangent vector $W \in T_v N\gamma$ can be identified with the velocity of a curve $w(t)$ in $N\gamma$ which is equal to v at time $t = 0$. The metric $\hat{g}$ on $N\gamma$ acts on an arbitrary couple $(W, \tilde{W}) \in (T_v N\gamma)^2$ in the following way (see [4], page 79)

$$\hat{g}(W, \tilde{W}) = g\left(\pi_* W, \pi_* \tilde{W}\right) + \left\langle \frac{D^N w}{dt}\Big|_{t=0}, \frac{D^N \tilde{w}}{dt}\Big|_{t=0} \right\rangle_N.$$

In this formula π denotes the natural projection from $N\gamma$ onto γ, $\frac{D^N w}{dt}$ the (normal) covariant derivative of the vector field $w(t)$ along the curve $\pi w(t)$, and $\tilde{w}(t)$ stands for a curve in $N\gamma$ with initial value v and initial velocity equal to $\tilde{W}$.

Following the notation in Subsection 2.1 of [10] we have that, if $w(t) = w^j(t)E_j(t)$, then

$$\frac{D^N w}{dt} = \frac{dw^j(t)}{dt}E_j(t).$$

Therefore, if we choose a system of coordinates $(\overline{s}, \overline{y})$ on $N\gamma$ defined by

$$(\overline{s}, \overline{y}) \in \mathbb{R} \times \mathbb{R}^{n-1} \qquad \mapsto \qquad \overline{y}_j E_j(\gamma(\overline{s})),$$

we get that

$$\hat{g}_{11}(\overline{s}, \overline{y}) = g_{11}(\overline{s}) + \overline{y}_l \overline{y}_j \left\langle \nabla^N_{E_1} E_l, \nabla^N_{E_1} E_j \right\rangle_N = g_{11}(\overline{s}) \equiv 1,$$

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and

$$\hat{g}_{1\bar{l}}(\overline{s},\overline{y}) \equiv 0; \qquad \hat{g}_{\bar{l}\bar{j}}(\overline{s},\overline{y}) = \delta_{\bar{l}\bar{j}},$$

where we have set $\partial_{\bar{l}} = \frac{\partial}{\partial\overline{y}_l}$. We notice also that the following co-area type formula holds, for any smooth compactly supported function $f : N\gamma \to \mathbb{R}$

$$(19) \qquad \int_{N\gamma} f\, dV_{\hat{g}} = \int_{\gamma} \left(\int_{N\gamma(\overline{s})} f(\overline{y})d\overline{y} \right) d\overline{s}.$$

This follows immediately from the fact that $\det \hat{g} = \det g$, and by our choice of $(\overline{s},\overline{y})$.

Since in the above coordinates the metric $\hat{g}$ is diagonal, the Laplacian of any (real or complex-valued) function ϕ defined on $N\gamma$ with respect to this metric is

$$\Delta_{\hat{g}}\phi = \partial^2_{\overline{s}\overline{s}}\phi + \partial^2_{\bar{j}\bar{j}}\phi \qquad \text{in } N\gamma.$$

We endow next $N\gamma_\varepsilon$ with a natural metric, inherited by $\hat{g}$ through a scaling. If T_ε denotes the dilation $x \mapsto \varepsilon x$, we define a metric $\hat{g}_\varepsilon$ on $N\gamma_\varepsilon$ simply by

$$\hat{g}_\varepsilon = \frac{1}{\varepsilon^2}[(T_\varepsilon)_* \hat{g}].$$

In particular, choosing coordinates (s,y) on $N\gamma_\varepsilon$ via the scaling $(\overline{s},\overline{y}) = \varepsilon(s,y)$, one easily checks that the components of $\hat{g}_\varepsilon$ are given by

$$(\hat{g}_\varepsilon)_{11}(s,y) = g_{11}(\overline{s}) \equiv 1; \qquad (\hat{g}_\varepsilon)_{1l}(s,y) \equiv 0; \qquad (\hat{g}_\varepsilon)_{lj}(s,y) = \delta_{lj}.$$

Therefore, if ψ is a smooth function in $N\gamma_\varepsilon$, it follows that in the above coordinates (s,y)

$$\Delta_{\hat{g}_\varepsilon}\psi = \partial^2_{ss}\psi + \partial^2_{jj}\psi, \qquad \text{in } N\gamma_\varepsilon.$$

In the case $\psi(s,y) = e^{-i\hat{f}s}u(s,y)$, for $\hat{f} = \mathcal{A}\hat{h}^\sigma$ (see (8)) and for u real, we have clearly that

$$\Delta_{\hat{g}_\varepsilon}\psi = e^{-i\hat{f}s}\partial^2_{ss}u - 2i\hat{f}e^{-i\hat{f}s}\partial_s u - \hat{f}^2 e^{-i\hat{f}s}u + e^{-i\hat{f}s}\partial^2_{jj}u.$$

Similarly to (19) one easily finds

$$(20) \qquad \int_{N\gamma_\varepsilon} f\, dV_{\hat{g}_\varepsilon} = \int_{\gamma_\varepsilon} \left(\int_{N\gamma_\varepsilon(s)} f(y)dy \right) ds.$$

2.2 Localizing the problem to a subset of the normal bundle $N\gamma_\varepsilon$

We next exploit the exponential decay of solutions (or approximate solutions) away from γ_ε to reduce (14) from the whole scaled manifold M_ε to the normal bundle $N\gamma_\varepsilon$: this step of the proof follows closely a procedure in [3]. We first define a smooth non-increasing cutoff function $\overline{\eta} : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} \overline{\eta}(t) = 1, & \text{for } t \leq 0; \\ \overline{\eta}(t) = 0, & \text{for } t \geq 1; \\ \overline{\eta}(t) \in [0,1], & \text{for every } t \in \mathbb{R}. \end{cases}$$

Next, if (s,y) are the coordinates introduced above in $N\gamma_\varepsilon$, and if $\Phi(\varepsilon s)$ is a section of $N\gamma$, using the notation of Subsection 3.1 in [10] we define

$$z = y - \Phi(\varepsilon s).$$

We will assume throughout the paper that Φ satisfies the following bounds

$$(21) \qquad \|\Phi\|_\infty + \|\Phi'\|_\infty + \varepsilon\|\Phi''\|_\infty \leq C\varepsilon$$

for some fixed constant $C > 0$. Next, for a small $\overline{\delta} > 0$ and for a smooth function $K(\varepsilon s) > 0$, both to be determined below and letting $h, k : [0, L] \to \mathbb{R}$ be as in the introduction, we set

$$(22) \qquad \tilde{\psi}_{0,\varepsilon} = \overline{\eta}_\varepsilon(s, z)\psi_{0,\varepsilon} := \overline{\eta}\left(K(\varepsilon s)\left(|z| - \frac{\varepsilon^{-\overline{\delta}}}{K(\varepsilon s)}\right)\right) e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} h(\varepsilon s) U(k(\varepsilon s)z),$$

where $\tilde{f}$ (to be defined later) is close to the function f (also defined in the introduction). For $\tau \in (0, 1)$, we let $S_\varepsilon : C^{2,\tau}(M_\varepsilon) \to C^\tau(M_\varepsilon)$ be the operator

$$(23) \qquad S_\varepsilon(\psi) = -\Delta_{g_\varepsilon}\psi + V(\varepsilon x)\psi - |\psi|^{p-1}\psi \qquad \text{in } M_\varepsilon.$$

If we let $\tilde{\psi}_\varepsilon$ denote an approximate solution of (14) (we will take later $\tilde{\psi}_\varepsilon$ equal to $\tilde{\psi}_{0,\varepsilon}$, with some small correction), then setting $\psi = \tilde{\psi}_\varepsilon + \tilde{\phi}$, we have $S_\varepsilon(\psi) = 0$ if and only if

$$L_\varepsilon(\tilde{\phi}) = S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\tilde{\phi}) \qquad \text{in } M_\varepsilon,$$

where $L_\varepsilon(\tilde{\phi})$ stands for the linear correction in $\tilde{\phi}$, namely

$$(24) \qquad L_\varepsilon(\tilde{\phi}) = -\Delta_{g_\varepsilon}\tilde{\phi} + V(\varepsilon x)\tilde{\phi} - |\tilde{\psi}_\varepsilon|^{p-1}\tilde{\phi} - (p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\overline{\tilde{\phi}}) \qquad \text{in } M_\varepsilon,$$

and where the nonlinear operator $N_\varepsilon(\tilde{\phi})$ is defined as

$$(25) \qquad N_\varepsilon(\tilde{\phi}) = |\tilde{\psi}_\varepsilon + \tilde{\phi}|^{p-1}(\tilde{\psi}_\varepsilon + \tilde{\phi}) - |\tilde{\psi}_\varepsilon|^{p-1}\tilde{\psi}_\varepsilon - |\tilde{\psi}_\varepsilon|^{p-1}\tilde{\phi} - (p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\overline{\tilde{\phi}}).$$

Then, in the coordinates (s, z) we can write

$$\tilde{\phi} = \overline{\eta}_\varepsilon(z)\phi + \varphi$$

where, with an abuse of notation, we assume ϕ defined on $N\gamma_\varepsilon$ (through the exponential map normal to γ_ε) and where the correction φ is defined on the whole M_ε. In this way we need to solve the equation

$$(26) \qquad L_\varepsilon(\overline{\eta}_\varepsilon(z)\phi) + L_\varepsilon(\varphi) = S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\overline{\eta}_\varepsilon(z)\phi + \varphi) \qquad \text{in } M_\varepsilon.$$

We will require ϕ to be supported in a cylindrical-shaped region in $N\gamma_\varepsilon$ centered around the zero section. For technical reasons, convenient for proving the results in the next subsection, we define

$$(27) \qquad \tilde{D}_\varepsilon = \left\{(s, z) \in N\gamma_\varepsilon \;:\; |z| \leq \frac{\varepsilon^{-\overline{\delta}} + 1}{K(\varepsilon s)}\right\},$$

and then the subspace of functions in $N\gamma_\varepsilon$

$$H_{\tilde{D}_\varepsilon} = \left\{u \in L^2(N\gamma_\varepsilon; \mathbb{C}) \;:\; u \text{ is supported in } \tilde{D}_\varepsilon\right\}.$$

Using elementary computations, we see that (26) is satisfied if (tautologically) the following two conditions are imposed

$$(28) \qquad \begin{aligned} L_\varepsilon(\phi) &= \left[S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\overline{\eta}_\varepsilon(z)\phi + \varphi)\right] \\ &+ |\tilde{\psi}_\varepsilon|^{p-1}\varphi + (p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\overline{\varphi}) \quad \text{in } \tilde{D}_\varepsilon, \qquad \phi \in H_{\tilde{D}_\varepsilon}; \end{aligned}$$

$$(29) \qquad \mathcal{L}_{\tilde{\psi}_\varepsilon}\varphi = (1 - \overline{\eta}_\varepsilon(z))\left[S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\overline{\eta}_\varepsilon(z)\phi + \varphi)\right] + 2\nabla_{g_\varepsilon}\overline{\eta}_\varepsilon(z)\cdot\nabla_{g_\varepsilon}\phi + \Delta_{g_\varepsilon}\overline{\eta}_\varepsilon(z)\phi \qquad \text{in } M_\varepsilon,$$

where

$$(30) \qquad \mathcal{L}_{\tilde{\psi}_\varepsilon}\varphi = -\Delta_{g_\varepsilon}\varphi + V(\varepsilon x)\varphi - (1 - \overline{\eta}_\varepsilon(z))\left[|\tilde{\psi}_\varepsilon|^{p-1}\varphi + (p-1)|\tilde{\psi}_\varepsilon|^{p-2}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\overline{\varphi})\right].$$

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We have next an existence result for equation (29): in order to state it we need to introduce some notation. For a regular periodic function $\mathfrak{p} : [0, L] \to \mathbb{R}$, for $m \in \mathbb{N}$ and $\tau \in (0, 1)$ we define the weighted norms

$$(31) \qquad \|\varphi\|_{C^{m,\tau}_{\mathfrak{p}}} = \sup_{x \in \tilde{D}_\varepsilon} \left[e^{\mathfrak{p}(\varepsilon s)|z|} \|\varphi\|_{C^{m,\tau}(B_1(x))} \right], \qquad x = (s, z).$$

We also recall the definition of $k(\varepsilon s)$ in (10).

Proposition 2.1 *Let $k_2(\overline{s}) < k_1(\overline{s}) < k_0(\overline{s}), K(\overline{s})$ be smooth positive L-periodic functions in $\overline{s}$, and $\tau \in (0, 1)$. Then, if $V(\overline{s}), K^2(\overline{s}) > k_2^2(\overline{s})$ and if $\|\tilde{\psi}_\varepsilon\|_{C^\tau_{k_0}}, \|S_\varepsilon(\tilde{\psi}_\varepsilon)\|_{C^\tau_{k_0}} \le 1$, there exists a positive constant C depending on $\overline{\delta}, \tau, k, k_0, k_1$ and k_2 such that given any ϕ with $\|\phi\|_{C^{1,\tau}_{k_1}} \le 1$ problem (29) has a unique solution $\varphi(\phi)$ whose restriction to $\tilde{D}_\varepsilon$ satisfies*

$$(32) \qquad \|\varphi(\phi)\|_{C^\tau_{-k_2}} \le C \left(e^{-\varepsilon^{-\overline{\delta}} \inf \frac{k_2+k_0}{K}} \|S_\varepsilon(\tilde{\psi}_\varepsilon)\|_{C^\tau_{k_0}} + e^{-\varepsilon^{-\overline{\delta}} \inf \frac{k_1+k_2}{K}} \|\phi\|_{C^{1,\tau}_{k_1}} \right).$$

Moreover, if $\tilde{\psi}^1_\varepsilon, \tilde{\psi}^2_\varepsilon$ satisfy $\|S_\varepsilon(\tilde{\psi}^j_\varepsilon)\|_{C^\tau_{k_0}} \le 1$, $j = 1, 2$, if $\|\phi_j\|_{C^{1,\tau}_{k_1}} \le 1$, $j = 1, 2$, and if $\varphi_j(\phi_j)$, $j = 1, 2$, are the corresponding solutions, for the restrictions to $\tilde{D}_\varepsilon$ we also have

$$(33) \quad \|\varphi(\phi_1) - \varphi(\phi_2)\|_{C^\tau_{-k_2}} \le C \left(e^{-\varepsilon^{-\overline{\delta}} \inf \frac{k_2+k_0}{K}} \|S_\varepsilon(\tilde{\psi}^1_\varepsilon) - S_\varepsilon(\tilde{\psi}^2_\varepsilon)\|_{C^\tau_{k_0}} + e^{-\varepsilon^{-\overline{\delta}} \inf \frac{k_1+k_2}{K}} \|\phi_1 - \phi_2\|_{C^{1,\tau}_{k_1}} \right).$$

Remark 2.2 *(a) The choice of the norm in (31) is done for considering functions which grow at most like $e^{-\mathfrak{p}(\varepsilon s)|z|}$, and in particular functions which decay at infinity if $\mathfrak{p}$ is positive. In the left-hand side of (32) we have a negative exponent, representing the fact that φ can grow as $|z|$ increases. However (we will take later k_0, k_1, k_2, K very close), the coefficients in the right-hand side are so tiny that φ is everywhere small in $\tilde{D}_\varepsilon$, and indeed with an even smaller bound for $|z|$ close to zero. This reflects the fact that the support of the right-hand side in (29) is $\frac{\varepsilon^{-\overline{\delta}}}{K(\varepsilon s)} < |z| < \frac{\varepsilon^{-\overline{\delta}}+1}{K(\varepsilon s)}$, so φ should decay away from this set.*
(b) We introduced the functions k_0, k_1 and k_2 for technical reasons, since we want to allow some flexibility for the (exponential) decay rate in $|z|$.

PROOF. We prove the result only when the manifold M in (1) is compact. For the modifications needed for $M = \mathbb{R}^n$ see Remark 2.3 *(b)*.

Consider a smooth non-decreasing cutoff function $\chi : [0, 1] \to [0, 1]$ satisfying

$$\begin{cases} \chi(t) = 0 & \text{for } t \le \frac{1}{4}; \\ \chi(t) = t & \text{for } t \ge \frac{3}{4}; \\ 0 \le \chi'(t) \le 4 & \text{for all } t; \\ 0 \le \chi''(t) \le 16 & \text{for all } t. \end{cases}$$

Next, given a large constant $\mathcal{B}$ (to be specified later) depending only on V and k_2, we define $\tilde{\chi}(\overline{s}, |z|)$ as

$$\tilde{\chi}(\overline{s}, |z|) = \begin{cases} \mathcal{B}\chi\left(\frac{|z|}{\mathcal{B}}\right) & \text{for } |z| \le \mathcal{B}; \\ |z| & \text{for } \mathcal{B} \le |z| \le \frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} - 1; \\ |z| - \frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} - \frac{1}{2} - 2\chi\left(|z| - \frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} - \frac{1}{2}\right) & \text{for } \frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} - 1 \le |z| \le \frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} + 1; \\ 2\frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} - |z| & \text{for } \frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} + 1 \le |z| \le 2\frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} - \mathcal{B}; \\ \mathcal{B}\chi\left(\frac{2\frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} - |z|}{\mathcal{B}}\right) & \text{for } 2\frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})} - \mathcal{B} \le |z| \le 2\frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})}; \\ 0 & \text{for } |z| \ge 2\frac{\varepsilon^{-\overline{\delta}}}{k_2(\overline{s})}. \end{cases}$$

By our choice of χ, the function $\tilde{\chi}$ satisfies the following inequalities (where, here, the gradient and the Laplacian are taken with respect to the Euclidean metric)

$$|\nabla_z \tilde{\chi}| \le 1; \qquad \Delta_z \tilde{\chi} \le \frac{16 + 4(n-2)}{\mathcal{B}}.$$

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Using the above coordinates (s, z), we define next the barrier function $\mathfrak{u} : M_\varepsilon \to \mathbb{R}$ as

$$\mathfrak{u}(s, z) = e^{k_2(\varepsilon s)\tilde{\chi}(\varepsilon s, z)} \qquad \text{for } |z| \leq 2\frac{\varepsilon^{-\bar{\delta}}}{k_2(\varepsilon s)},$$

and we extend $\mathfrak{u}$ identically equal to 1 elsewhere. By our choice of $\tilde{\chi}$, this function is indeed smooth and strictly positive on the whole M_ε. We consider next the linear equation (motivated by (30))

$$\mathcal{L}_{\tilde{\psi}_\varepsilon}\varphi = \vartheta \qquad \text{on } M_\varepsilon,$$

where $\vartheta : M_\varepsilon \to \mathbb{R}$ is Hölder continuous (with $supp(\vartheta) \subset\subset \tilde{D}_\varepsilon$, see (35) below). Since the operator $\mathcal{L}_{\tilde{\psi}_\varepsilon}$ is uniformly elliptic, the latter equation is (uniquely) solvable, and we would like next to derive some pointwise estimates on its solutions. To this aim we define

$$v(x) = \frac{\varphi(x)}{\mathfrak{u}(x)}, \qquad x \in M_\varepsilon.$$

With this notation, we have that

$$\mathfrak{u}\mathcal{L}_{\tilde{\psi}_\varepsilon}v - v\Delta_{g_\varepsilon}\mathfrak{u} - 2\nabla_{g_\varepsilon}v \cdot \nabla_{g_\varepsilon}\mathfrak{u} = \vartheta \qquad \text{on } M_\varepsilon.$$

Using the expression of the metric coefficients in the coordinates (s, z), see Lemma 2.1 in [10], (21) and the properties of the cutoff function $\tilde{\chi}$, one easily checks that

$$\Delta_{g_\varepsilon}\mathfrak{u} \begin{cases} \leq (k_2(\bar{s})^2 + o_\mathcal{B}(1) + o_\varepsilon(1))\mathfrak{u} & \text{for } |z| \leq \frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})}, \\ = 0 & \text{elsewhere,} \end{cases}$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$ and $o_\mathcal{B}(1) \to 0$ as $\mathcal{B} \to +\infty$. Therefore we obtain that the function v satisfies

$$\begin{cases} \left|(\mathcal{L}_{\tilde{\psi}_\varepsilon} - k_2^2(\bar{s}) + o_\mathcal{B}(1) + o_\varepsilon(1))v\right| \leq \frac{|\vartheta|}{\mathfrak{u}} & \text{for } |z| \leq 2\frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})}, \\ \mathcal{L}_{\tilde{\psi}_\varepsilon}v = \frac{\vartheta}{\mathfrak{u}} & \text{elsewhere.} \end{cases}$$

Since we assumed $V(\bar{s}) > k_2(\bar{s})^2$, we obtain that $V - k_2(\bar{s})^2 + o_\mathcal{B}(1) + o_\varepsilon(1)$ is strictly positive (provided $\mathcal{B}$ is sufficiently large and ε sufficiently small) for $|z| \leq 2\frac{\varepsilon^{-\bar{\delta}}}{k_2(\bar{s})}$, and hence the function v satisfies a uniformly elliptic equation with a non-negative coefficient in the zero-th order term with right-hand side given by $\frac{\vartheta}{\mathfrak{u}}$. Therefore from the maximum principle we derive the estimate

$$\max_{M_\varepsilon} |v| \leq C\max_{M_\varepsilon} \frac{|\vartheta|}{\mathfrak{u}},$$

where C depends on the uniform lower bound of the above coefficient. The latter estimate clearly implies

$$|\varphi(x)| \leq C\mathfrak{u}(x)\max_{M_\varepsilon} \frac{\vartheta}{\mathfrak{u}}, \qquad \text{for every } x \in M_\varepsilon.$$

We define next the weighted norm

$$\|\varphi\|_{m,\tau,\mathfrak{u}} := \sup_{x \in M_\varepsilon} \left\|\frac{\varphi}{\mathfrak{u}}\right\|_{C^{m,\tau}(B_1(x))},$$

which is equivalent (with constants depending on $\mathcal{B}$ only) to $\|\cdot\|_{C^{m,\tau}_{-k_2}}$ on the set $\tilde{D}_\varepsilon$. Using the explicit form of the function $\mathfrak{u}$ and standard elliptic regularity estimates one can improve the latter inequality to

$$(34) \qquad \qquad \|\varphi\|_{2,\tau,\mathfrak{u}} \leq C\|\vartheta\|_{0,\tau,\mathfrak{u}}.$$

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The proof of the proposition will now follow from this linear estimate and the contraction mapping theorem: in fact, defining

$$G_{\phi,\varepsilon}(\varphi) = (1 - \overline{\eta}_\varepsilon(s,z))\left[S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\overline{\eta}_\varepsilon(z)\phi + \varphi)\right] + 2\,\nabla_{g_\varepsilon}\overline{\eta}_\varepsilon(z) \cdot \nabla_{g_\varepsilon}\phi + \Delta_{g_\varepsilon}\overline{\eta}_\varepsilon(z)\phi,$$

equation (29) is equivalent to

$$(35) \qquad\qquad\qquad\qquad \varphi = \mathcal{L}_{\tilde{\psi}_\varepsilon}^{-1} G_{\phi,\varepsilon}(\varphi).$$

First of all, notice that $\mathcal{L}_{\tilde{\psi}_\varepsilon}$ is invertible since we are assuming M (and hence M_ε) to be compact, see the beginning of the proof. Secondly, to apply (34), we need to estimate $\|G_{\phi,\varepsilon}(\varphi)\|_{0,\tau,\mathfrak{u}}$, together with its Lipschitz dependence in φ: our goal indeed is to apply the contraction mapping theorem.

Let us consider for instance the term $(1 - \overline{\eta}_\varepsilon(s,z))S_\varepsilon(\tilde{\psi}_\varepsilon)$. Using the fact that $(1 - \overline{\eta}_\varepsilon)$ is zero for $|z| \leq \frac{\varepsilon^{-\overline{\delta}}}{K(\varepsilon s)}$, that $S_\varepsilon(\tilde{\psi}_\varepsilon)$ is zero for $|z| \geq \frac{\varepsilon^{-\overline{\delta}}+1}{K(\varepsilon s)}$ and that $k_2 < K$, we obtain

$$(36) \quad \left\|(1 - \overline{\eta}_\varepsilon)S_\varepsilon(\tilde{\psi}_\varepsilon)\right\|_{0,\tau,\mathfrak{u}} \leq C \sup_{\{\frac{\varepsilon^{-\overline{\delta}}}{K(\varepsilon s)} \leq |z| \leq \frac{\varepsilon^{-\overline{\delta}}+1}{K(\varepsilon s)}\}} \left\|\frac{S_\varepsilon(\tilde{\psi}_\varepsilon)}{\mathfrak{u}}\right\|_{C^\tau(B_1(z))} \leq C e^{-\varepsilon^{-\overline{\delta}} \inf \frac{k_0+k_2}{K}} \|S_\varepsilon(\tilde{\psi}_\varepsilon)\|_{C_{k_0}^\tau}.$$

Now to estimate the remaining terms of $G_{\phi,\varepsilon}$ we notice that

$$|N_\varepsilon(\overline{\eta}_\varepsilon(z)\phi + \varphi)| \leq \begin{cases} C|\tilde{\psi}_\varepsilon|^{p-2}|\overline{\eta}_\varepsilon(z)\phi + \varphi|^2 & \text{if } |\overline{\eta}_\varepsilon(z)\phi + \varphi| \leq |\tilde{\psi}_\varepsilon|; \\ |\overline{\eta}_\varepsilon(z)\phi + \varphi|^p & \text{otherwise.} \end{cases}$$

Since $p > 1$, we can find a number $\zeta \in (0,1)$ such that $p - 2 + 1 - \zeta > 0$, so the last formula implies

$$(37) \qquad |N_\varepsilon(\overline{\eta}_\varepsilon(z)\phi + \varphi)| \leq C\left(|\tilde{\psi}_\varepsilon|^{p-1-\zeta}(|\overline{\eta}_\varepsilon(z)\phi|^\zeta + |\varphi|^\zeta)(|\overline{\eta}_\varepsilon(z)\phi| + |\varphi|) + |\overline{\eta}_\varepsilon(z)\phi|^p + |\varphi|^p\right).$$

Using the fact that $\|\tilde{\psi}_\varepsilon\|_{C_{k_0}^\tau}, \|\phi\|_{C_{k_1}^{1,\tau}} \leq 1$ and reasoning as for (36), after some computations we deduce (assuming $\|\varphi\|_\infty \leq 1$, which will be verified later)

$$
\begin{aligned}
(38) \quad \|G_{\phi,\varepsilon}(\varphi)\|_{0,\tau,\mathfrak{u}} \leq\ & C\left(e^{-\varepsilon^{-\overline{\delta}} \inf \frac{k_2+k_0}{K}} \|S_\varepsilon(\tilde{\psi}_\varepsilon)\|_{C_{k_0}^\tau} + e^{-\varepsilon^{-\overline{\delta}} \inf \frac{k_2+k_1}{K}} \|\phi\|_{C_{k_1}^{1,\tau}}\right. \\
& + \left. e^{-\varepsilon^{-\overline{\delta}} \inf \frac{pk_1+k_2}{K}} \|\phi\|_{C_{k_1}^{0,\tau}} + e^{-\varepsilon^{-\overline{\delta}} \inf \frac{(p-1)k_0+k_1+k_2}{K}} \|\phi\|_{C_{k_1}^{0,\tau}}\right) \\
& + C\left(e^{-\varepsilon^{-\overline{\delta}}(p-1-\zeta) \inf \frac{k_0}{K}} + \|\varphi\|_\infty^{p-1}\right) \|\varphi\|_{0,\tau,\mathfrak{u}}.
\end{aligned}
$$

Similarly, for two functions φ_1, φ_2 with $\|\varphi_1\|_\infty, \|\varphi_2\|_\infty \leq 1$ and with finite $\|\cdot\|_{0,\tau,\mathfrak{u}}$ norm we have

$$(39) \quad \|G_{\phi,\varepsilon}(\varphi_1) - G_{\phi,\varepsilon}(\varphi_2)\|_{0,\tau,\mathfrak{u}} \leq \left(e^{-\varepsilon^{-\overline{\delta}}(p-1-\zeta) \inf \frac{k_0}{K}} + \|\varphi_1\|_\infty^{p-1} + \|\varphi_2\|_\infty^{p-1}\right) \|\varphi_1 - \varphi_2\|_{0,\tau,\mathfrak{u}}.$$

We now consider the map $\varphi \mapsto G_{\phi,\varepsilon}(\varphi)$ in the set

$$\mathfrak{B} = \left\{\varphi\ :\ \|\varphi\|_{0,\tau,\mathfrak{u}} \leq C_1\left(e^{-\varepsilon^{-\overline{\delta}} \inf \frac{k_2+k_0}{K}} \|S_\varepsilon(\tilde{\psi}_\varepsilon)\|_{C_{k_0}^\tau} + e^{-\varepsilon^{-\overline{\delta}} \inf \frac{k_2+k_1}{K}} \|\phi\|_{C_{k_1}^{1,\tau}}\right)\right\},$$

where C_1 is a sufficiently large positive constant: notice that if $\varphi \in \mathfrak{B}$ then $\|\varphi\|_\infty = o_\varepsilon(1)$. From (38), (39) it then follows that this map is a contraction from $\mathfrak{B}$ into itself, endowed with the above norm, and therefore a solution φ exists as a fixed point of $G_{\phi,\varepsilon}$. The fact that $k_2 < K$ implies that the norm $\|\cdot\|_{C_{-k_2}^\tau}$ is equivalent to $\|\cdot\|_{0,\tau,\mathfrak{u}}$ in $\tilde{D}_\varepsilon$ (see also the comments in Remark 2.2), so we obtain (32). A similar reasoning, still based on regularity theory and elementary inequalities, also yields (33). ∎

Remark 2.3 (a) From elliptic regularity theory it follows that in (32)-(33) the norm $\| \cdot \|_{C^{\tau}_{-k_2}}$ can be replaced by the stronger one $\| \cdot \|_{C^{2,\tau}_{-k_2}}$, yielding an estimate in the norm $\| \cdot \|_{C^{\tau'}_{-k_2}}$ for any $\tau' \in (\tau, 1)$.

(b) In the case $M = \mathbb{R}^n$ the above proof needs to be slightly modified: in fact the invertibility of $\mathcal{L}_{\tilde{\psi}_\varepsilon}$ will be guaranteed provided we work in an appropriate class of functions Y decaying exponentially at infinity. To guarantee this condition, we can vary the form of the barrier function $\mathfrak{u}$ in order that it both remains a super-solution of $\mathcal{L}_{\tilde{\psi}_\varepsilon} = 0$ and decays exponentially to zero at infinity. This is indeed possible using the uniform positive lower bound on V, see (2): we omit the details of this construction.

As a consequence of Proposition 2.1, we obtain that solvability of (1) is equivalent to that of (28).

Proposition 2.4 *Suppose the assumptions of Proposition 2.1 hold, and consider the corresponding $\varphi = \varphi(\phi)$. Then $\psi = \tilde{\psi}_\varepsilon + \overline{\eta}_\varepsilon(z)\phi + \varphi(\phi)$ solves (14) if and only if $\phi \in H_{\tilde{D}_\varepsilon}$ satisfies*

$$(40) \qquad\qquad L_\varepsilon(\phi) = \tilde{S}_\varepsilon(\phi) \qquad\qquad in \ \tilde{D}_\varepsilon,$$

with

$$(41) \qquad \tilde{S}_\varepsilon(\phi) = S_\varepsilon(\tilde{\psi}_\varepsilon) + N_\varepsilon(\overline{\eta}_\varepsilon(z)\phi + \varphi(\phi)) + |\tilde{\psi}_\varepsilon|^{p-1}\varphi(\phi) + (p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\overline{\varphi(\phi)}) \quad in \ \tilde{D}_\varepsilon,$$

where $\overline{\eta}_\varepsilon, S_\varepsilon$ and N_ε are given in (22), (23) and (25) respectively.

2.3 Construction of an approximate kernel for L_ε

We perform here some preliminary analysis useful to understand the spectral properties of L_ε. More precisely in this subsection we consider a *model* case, when the domain $\tilde{D}_\varepsilon$ (see (27)) is replaced by $[0, L/\varepsilon] \times \mathbb{R}^{n-1}$ and the profile of approximate solutions is independent of the variable s (only the phase varies, periodically in s). As in formula (8), we consider positive constants $\hat{V}, \hat{h}, \hat{k}$ satisfying

$$(42) \qquad\qquad \hat{h} = \left(\hat{f}^2 + \hat{V}\right)^{\frac{1}{p-1}}; \qquad\qquad \hat{k} = \left(\hat{f}^2 + \hat{V}\right)^{\frac{1}{2}}.$$

Our goal is to study the following eigenvalue problem, which models our linearized equation

$$\hat{L}_\varepsilon u = \lambda u \quad in \ [0, L/\varepsilon] \times \mathbb{R}^{n-1};$$

$$(43) \qquad \hat{L}_\varepsilon u = -\Delta_{\hat{g}_\varepsilon} u + \hat{V}u - \hat{h}^{p-1}U(\hat{k}y)^{p-1}u - (p-1)\hat{h}^{p-1}U(\hat{k}y)^{p-1}e^{-i\hat{f}s}\Re(e^{-i\hat{f}s}\overline{u}),$$

and in particular we would like to characterize the small eigenvalues and the corresponding eigenfunctions. First of all we can write u as

$$u = e^{-i\hat{f}s}(u_r + iu_i),$$

for some real u_r and u_i. With this notation, we are reduced to study the coupled system

$$\begin{cases} -\Delta_{\hat{g}_\varepsilon} u_r + (\hat{V} + \hat{f}^2)u_r - p\hat{h}^{p-1}U(\hat{k}y)^{p-1}u_r - 2\hat{f}\frac{\partial u_i}{\partial s} = \lambda u_r & in \ [0, L/\varepsilon] \times \mathbb{R}^{n-1}; \\ -\Delta_{\hat{g}_\varepsilon} u_i + (\hat{V} + \hat{f}^2)u_i - \hat{h}^{p-1}U(\hat{k}y)^{p-1}u_i + 2\hat{f}\frac{\partial u_r}{\partial s} = \lambda u_i & in \ [0, L/\varepsilon] \times \mathbb{R}^{n-1}. \end{cases}$$

Making the change of variables $y \mapsto \hat{k}y$ and using (42), we are reduced to

$$(44) \qquad \begin{cases} -\frac{1}{\hat{k}^2}\frac{\partial^2 u_r}{\partial s^2} - \Delta_y u_r + u_r - pU(y)^{p-1}u_r - \frac{2\hat{f}}{\hat{k}^2}\frac{\partial u_i}{\partial s} = \frac{\lambda}{\hat{k}^2}u_r & in \ [0, L/\varepsilon] \times \mathbb{R}^{n-1}; \\ -\frac{1}{\hat{k}^2}\frac{\partial^2 u_i}{\partial s^2} - \Delta_y u_i + u_i - U(y)^{p-1}u_i + \frac{2\hat{f}}{\hat{k}^2}\frac{\partial u_r}{\partial s} = \frac{\lambda}{\hat{k}^2}u_i & in \ [0, L/\varepsilon] \times \mathbb{R}^{n-1}. \end{cases}$$

It is now convenient to use a Fourier decomposition in s of u_r and u_i, writing

$$u_r = \sum_j \left(\cos\left(\frac{2\pi\varepsilon js}{L}\right)u_{r,c,j}(y) + \sin\left(\frac{2\pi\varepsilon js}{L}\right)u_{r,s,j}(y)\right), \qquad s \in [0, L/\varepsilon], y \in \mathbb{R}^{n-1};$$

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$$u_i = \sum_j \left(\cos\left(\frac{2\pi\varepsilon js}{L}\right) u_{i,c,j}(y) + \sin\left(\frac{2\pi\varepsilon js}{L}\right) u_{i,s,j}(y) \right), \qquad s \in [0, L/\varepsilon], y \in \mathbb{R}^{n-1}.$$

In this way the functions $u_{r,c,j}, u_{r,s,j}, u_{i,c,j}, u_{i,s,j}$ satisfy the following systems of equations

$$\begin{cases} -\Delta_y u_{r,c,j} + \left(1 + \frac{4\pi^2 \varepsilon^2 j^2}{L^2 \hat{k}^2}\right) u_{r,c,j} - pU(y)^{p-1} u_{r,c,j} - \frac{4\pi \hat{f}\varepsilon j}{L\hat{k}^2} u_{i,s,j} = \frac{\lambda}{\hat{k}^2} u_{r,c,j} & \text{in } \mathbb{R}^{n-1}; \\ -\Delta_y u_{i,s,j} + \left(1 + \frac{4\pi^2 \varepsilon^2 j^2}{L^2 \hat{k}^2}\right) u_{i,s,j} - U(y)^{p-1} u_{i,s,j} - \frac{4\pi \hat{f}\varepsilon j}{L\hat{k}^2} u_{r,c,j} = \frac{\lambda}{\hat{k}^2} u_{i,s,j} & \text{in } \mathbb{R}^{n-1}, \end{cases}$$

$$\begin{cases} -\Delta_y u_{r,s,j} + \left(1 + \frac{4\pi^2 \varepsilon^2 j^2}{L^2 \hat{k}^2}\right) u_{r,s,j} - pU(y)^{p-1} u_{r,s,j} + \frac{4\pi \hat{f}\varepsilon j}{L\hat{k}^2} u_{i,c,j} = \frac{\lambda}{\hat{k}^2} u_{r,s,j} & \text{in } \mathbb{R}^{n-1}; \\ -\Delta_y u_{i,c,j} + \left(1 + \frac{4\pi^2 \varepsilon^2 j^2}{L^2 \hat{k}^2}\right) u_{i,c,j} - U(y)^{p-1} u_{i,c,j} + \frac{4\pi \hat{f}\varepsilon j}{L\hat{k}^2} u_{r,s,j} = \frac{\lambda}{\hat{k}^2} u_{i,c,j} & \text{in } \mathbb{R}^{n-1}. \end{cases}$$

If we set $\frac{2\pi\varepsilon j}{L\hat{k}} = \alpha$, $\frac{2\hat{f}}{\hat{k}} = \mu$ and $\tilde{\lambda} = \frac{\lambda}{\hat{k}^2}$ then the latter two systems are equivalent to the following one

$$(45) \qquad \begin{cases} -\Delta_y u + (1 + \alpha^2)u - pU(y)^{p-1}u + \mu\alpha v = \tilde{\lambda} u & \text{in } \mathbb{R}^{n-1}; \\ -\Delta_y v + (1 + \alpha^2)v - U(y)^{p-1}v + \mu\alpha u = \tilde{\lambda} v & \text{in } \mathbb{R}^{n-1}. \end{cases}$$

The equivalence with the second system is obvious: for the first one it is sufficient to switch the sign of the second component. We characterize the spectrum of the last system in the next proposition: the value of μ is fixed, while α is allowed to vary. We remark that it is irrelevant for our purposes to take α positive or negative, since we can still switch the sign of one of the two components.

Proposition 2.5 *Let $\eta_\alpha, \sigma_\alpha$ and τ_α denote the first three eigenvalues of (45). Then there exists $\mu_0 > 0$ such that for $\mu \in [0, \mu_0]$ the following properties hold*

a *there exists $\alpha_0 > 0$ such that η_α is simple, increasing and differentiable in α for $\alpha \in [0, \alpha_0]$, $\frac{\partial \eta_\alpha}{\partial \alpha} > 0$ for $\alpha \in (0, \alpha_0]$, $\eta_0 < 0$ and $\eta_{\alpha_0} > 0$;*

b *the eigenvalue σ_α is zero for $\alpha = 0$ with multiplicity n, it satisfies $\frac{\partial \sigma_\alpha}{\partial \alpha} > 0$ for α small positive and stays uniformly bounded away from zero if α stays bounded away from zero;*

c *τ_α is strictly positive and stays uniformly bounded away from zero for all α's;*

d *the eigenfunction u_α corresponding to η_α is simple, radial in y, radially decreasing and depends smoothly on α; for $\alpha = 0$ the eigenfunction of (45) corresponding to $\eta_0 < 0$ is of the form $(\tilde{Z}, 0)$ with $\tilde{Z}$ radial and radially decreasing, while those corresponding to $\sigma_0 = 0$ are linear combinations of $(\nabla_{y_j} U, 0)$, $j = 1, \ldots, n-1$, and $(0, U)$;*

e *let $\overline{\alpha}$ be the unique α for which $\eta_{\overline{\alpha}} = 0$ (see **a**): then the corresponding eigenfunction is of the form (Z, W) for some radial functions Z, W satisfying the following decay $|Z| + |W| \leq Ce^{-(1+\hat{\eta})|y|}$ for some constants $C, \hat{\eta} > 0$.*

PROOF. This result is known for $\mu = 0$, see e.g. Proposition 4.2 in [9] and Proposition 2.9 in [12].

For $\mu \neq 0$ sufficiently small the functions $\alpha \mapsto \eta_\alpha$, $\alpha \mapsto \sigma_\alpha$ and $\alpha \mapsto \tau_\alpha$ will be C^1-close to those corresponding to $\mu = 0$: therefore, to prove **a-d** it is sufficient to show that $\eta_\alpha, \sigma_\alpha$ are twice differentiable in α for α small, that $\frac{\partial \eta_\alpha}{\partial \alpha} = \frac{\partial \sigma_\alpha}{\partial \alpha} = 0$, and that $\frac{\partial^2 \eta_\alpha}{\partial \alpha^2}, \frac{\partial^2 \sigma_\alpha}{\partial \alpha^2} > 0$.

We prove this statement only formally, but a rigorous proof can be easily derived. Differentiating

$$(46) \qquad \begin{cases} -\Delta_y u_\alpha + (1 + \alpha^2)u_\alpha - pU(y)^{p-1}u_\alpha + \mu\alpha v_\alpha = \eta_\alpha u_\alpha & \text{in } \mathbb{R}^{n-1}; \\ -\Delta_y v_\alpha + (1 + \alpha^2)v_\alpha - U(y)^{p-1}v_\alpha + \mu\alpha u_\alpha = \eta_\alpha v_\alpha & \text{in } \mathbb{R}^{n-1} \end{cases}$$

with respect to α we find
(47)
$$\begin{cases} -\Delta_y \frac{\partial u_\alpha}{\partial \alpha} + (1 + \alpha^2)\frac{\partial u_\alpha}{\partial \alpha} - pU(y)^{p-1}\frac{\partial u_\alpha}{\partial \alpha} + \mu\alpha\frac{\partial v_\alpha}{\partial \alpha} + 2\alpha u_\alpha + \mu v_\alpha = \eta_\alpha \frac{\partial u_\alpha}{\partial \alpha} + \frac{\partial \eta_\alpha}{\partial \alpha} u_\alpha & \text{in } \mathbb{R}^{n-1}; \\ -\Delta_y \frac{\partial v_\alpha}{\partial \alpha} + (1 + \alpha^2)\frac{\partial v_\alpha}{\partial \alpha} - U(y)^{p-1}\frac{\partial v_\alpha}{\partial \alpha} + \mu\alpha\frac{\partial u_\alpha}{\partial \alpha} + 2\alpha v_\alpha + \mu u_\alpha = \eta_\alpha \frac{\partial v_\alpha}{\partial \alpha} + \frac{\partial \eta_\alpha}{\partial \alpha} v_\alpha & \text{in } \mathbb{R}^{n-1}. \end{cases}$$

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To compute $\frac{\partial \eta_\alpha}{\partial \alpha}$ at $\alpha = 0$ it is sufficient to multiply the first equation by u_α, the second by v_α, to take the sum and integrate: if we choose $\frac{\partial u_\alpha}{\partial \alpha}$ and $\frac{\partial v_\alpha}{\partial \alpha}$ so that $\int_{\mathbb{R}^{n-1}} u_\alpha \frac{\partial u_\alpha}{\partial \alpha} + v_\alpha \frac{\partial v_\alpha}{\partial \alpha} = 0$ (choosing for example $\int (u_\alpha^2 + v_\alpha^2) = 1$ for all α's), then with an integration by parts we find that

$$\frac{\partial \eta_\alpha}{\partial \alpha}|_{\alpha=0} \int_{\mathbb{R}^{n-1}} u_\alpha^2 + v_\alpha^2 = 2\mu \int_{\mathbb{R}^{n-1}} u_0 v_0.$$

Using the fact that $v_0 = 0$, see **d**, we then obtain $\frac{\partial \eta_\alpha}{\partial \alpha}|_{\alpha=0} = 0$. The same argument applies for evaluating $\frac{\partial \sigma_\alpha}{\partial \alpha}|_{\alpha=0}$, since the eigenfunctions corresponding to $\sigma_0 = 0$ always have one component vanishing.

To compute the second derivative with respect to α we differentiate (47) once more at $\alpha = 0$, obtaining

$$\begin{cases} -\Delta_y \frac{\partial^2 u_\alpha}{\partial \alpha^2} + \frac{\partial^2 u_\alpha}{\partial \alpha^2} - pU(y)^{p-1} \frac{\partial^2 u_\alpha}{\partial \alpha^2} + 2\mu \frac{\partial v_\alpha}{\partial \alpha} + 2u_0 = \frac{\partial^2 \eta_\alpha}{\partial \alpha^2} u_0 & \text{in } \mathbb{R}^{n-1}; \\ -\Delta_y \frac{\partial^2 v_\alpha}{\partial \alpha^2} + \frac{\partial^2 v_\alpha}{\partial \alpha^2} - U(y)^{p-1} \frac{\partial^2 v_\alpha}{\partial \alpha^2} + 2\mu \frac{\partial u_\alpha}{\partial \alpha} + 2v_0 = \frac{\partial^2 \eta_\alpha}{\partial \alpha^2} v_0 & \text{in } \mathbb{R}^{n-1}. \end{cases}$$

As for the previous case we get

$$\frac{\partial^2 \eta_\alpha}{\partial \alpha^2}|_{\alpha=0} = 2 + 2\mu \int_{\mathbb{R}^{n-1}} \left(u_0 \frac{\partial v_\alpha}{\partial \alpha}|_{\alpha=0} + v_0 \frac{\partial u_\alpha}{\partial \alpha}|_{\alpha=0} \right)$$

so, using the smallness of μ, the claim follows.

For the second derivative of σ_α the procedure is similar, but notice that in this case we might obtain a multivalued function, due to the multiplicity (n) of σ_0, see **b**. However, if in the last formula we plug in the corresponding eigenfunctions, see **d**, we still obtain a sign condition for each of the two branches of σ_α (one of them will have multiplicity $n-1$ by the rotation invariance of the equations). $\blacksquare$

Remark 2.6 *Using the same argument in the previous proof one can show that*

$$\frac{\partial \eta}{\partial \alpha}|_{\alpha=\overline{\alpha}} = 2\overline{\alpha} + 2\mu \int_{\mathbb{R}^{n-1}} Z_{\overline{\alpha}} W_{\overline{\alpha}}.$$

Remark 2.7 *Proposition 2.5 is the only result where the smallness of the constant $\mathcal{A}$ is used, see Theorem 1.1. Remark that $V \equiv \hat{V}$ implies $\hat{f} = \mathcal{A}\hat{h}^\sigma$, and that $\mu = 2\frac{\hat{f}}{\hat{k}}$, so the smallness of $\mathcal{A}$ is equivalent to that of μ. Notice that by (2), (8), when $\mathcal{A} \to 0$, $\hat{h}$ and $\hat{k}$ stay uniformly bounded and bounded away from zero.*

We believe that dropping this smallness condition might lead to further resonance phenomena in addition to these encountered here (see the introduction and the last section).

Remark 2.8 *Considering (47) with σ_α replacing η_α and for $\alpha = 0$ one finds that $\mathcal{L}_r^0 \frac{\partial u_\alpha}{\partial \alpha}|_{\alpha=0} = -\mu v_0$ and $\mathcal{L}_i^0 \frac{\partial v_\alpha}{\partial \alpha}|_{\alpha=0} = -\mu u_0$, where $\mathcal{L}_r^0 v = -\Delta_y v + v - pU(y)^{p-1} v$, $\mathcal{L}_i^0 v = -\Delta_y v + v - U(y)^{p-1} v$. Since for $\alpha = 0$ we have $(u_0, v_0) = (\partial_j U, 0)$ or $(u_0, v_0) = (0, U)$ (see **d**) and*

$$(48) \qquad \mathcal{L}_r^0 \left(-\frac{1}{p-1} U - \frac{1}{2} \nabla U(y) \cdot y \right) = U; \qquad \mathcal{L}_i^0 (y_j U(y)) = -2\partial_j U,$$

see Subsection 3.2 in [10], one finds respectively that

$$\frac{\partial v_\alpha}{\partial \alpha}|_{\alpha=0} = \frac{\mu}{2} y_j U(y); \qquad \frac{\partial u_\alpha}{\partial \alpha}|_{\alpha=0} = \mu \left(\frac{1}{p-1} U + \frac{1}{2} \nabla U(y) \cdot y \right).$$

These expressions, together with (59) in [8] and some integration by parts allow us to compute the explicitly $\frac{\partial^2 \sigma_\alpha}{\partial \alpha^2}$, whose values along the two branches are

$$\frac{\partial^2 \sigma_\alpha}{\partial \alpha^2} = \frac{2}{(p-1)} \left((p-1) - 2\mathcal{A}^2 \theta \hat{h}^{2\sigma-p+1} \right); \qquad \frac{\partial^2 \sigma_\alpha}{\partial \alpha^2} = \frac{2}{(p-1)} \left((p-1) - 2\mathcal{A}^2 \sigma \hat{h}^{2\sigma-p+1} \right).$$

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Therefore, we find that the second derivatives of the eigenfunctions satisfy respectively the equations

$$(49) \qquad \mathcal{L}_r^0 \frac{\partial^2 u_\alpha}{\partial \alpha^2} = \frac{2}{p-1}\left((p-1) - 2\mathcal{A}^2\theta\hat{h}^{2\sigma-p+1}\right)\nabla_j U - 2\nabla_j U - 4\mathcal{A}^2\hat{h}^{2\sigma-p+1}y_j U;$$

$$(50) \qquad \mathcal{L}_i^0 \frac{\partial^2 v_\alpha}{\partial \alpha^2} = \frac{2}{p-1}\left((p-1) - 2\mathcal{A}^2\sigma\hat{h}^{2\sigma-p+1}\right)U - 2U - 8\mathcal{A}^2\hat{h}^{2\sigma-p+1}\tilde{U}.$$

These formulas will be used crucially later on. Below, we will denote for brevity

$$(51) \qquad \hat{\mathfrak{V}}_j := \frac{1}{2}\frac{\partial^2 u_\alpha}{\partial \alpha^2}, \quad j = 1, \ldots, n-1; \qquad \hat{\mathfrak{W}} := \frac{1}{2}\frac{\partial^2 v_\alpha}{\partial \alpha^2}.$$

The factor $\frac{1}{2}$ arises in the Taylor expansion of the eigenfunctions in α, and j is the index in (49).

We next consider the case of variable coefficients, which can be reduced to the previous one through a localization argument in s. To have a more accurate model for L_ε the constants $\hat{k}$ and $\hat{f}$ in (44) have to be substituted with the functions $k(\varepsilon s)$ and $f(\varepsilon s)$ satisfying (10). Precisely, in $N\gamma_\varepsilon$ we define

$$(52) \quad L_\varepsilon^1 u = -\Delta_{\hat{g}_\varepsilon} u + V(\varepsilon s)u - h(\varepsilon s)^{p-1}U(k(\varepsilon s)y)^{p-1}u - (p-1)h(\varepsilon s)^{p-1}U(k(\varepsilon s)y)^{p-1}e^{-i\frac{f(\varepsilon s)}{\varepsilon}}\Re(e^{-i\frac{f(\varepsilon s)}{\varepsilon}}\overline{u})$$

(recall the definition of $\hat{g}_\varepsilon$ in Subsection 2.1: in particular, working with the coordinates (s, y) integrals will be computed using the co-area formula (20)). Before proving rigorous results, we first discuss heuristically what the approximate kernel of L_ε^1 should look like. Using Fourier expansions as above (freezing the coefficients at some $\overline{s}$), the profile of the functions which lie in an approximate kernel of L_ε^1 will be given by the solution of (recall (45))

$$(53) \qquad \begin{cases} -\Delta_y u + (1+\alpha^2)u - pU(y)^{p-1}u + 2\frac{f'(\overline{s})}{k(\overline{s})}\alpha v = \tilde{\lambda}u & \text{in } \mathbb{R}^{n-1}; \\ -\Delta_y v + (1+\alpha^2)v - U(y)^{p-1}v + 2\frac{f'(\overline{s})}{k(\overline{s})}\alpha u = \tilde{\lambda}v & \text{in } \mathbb{R}^{n-1}, \end{cases}$$

where $\tilde{\lambda}$ is close to zero. For α small (low Fourier modes), Proposition 2.5 **d** gives the profile $\nabla_y U(k(\overline{s})y)$ or $iU(k(\overline{s})y)$ (recall also the scaling in y before (44)). The remaining part of the approximate kernel is the counterpart of that given in Proposition 2.5 **e**: for variable coefficients it is uniquely defined a function $\alpha(\overline{s})$ such that

$$(54) \qquad \eta_{\alpha(\overline{s})} = 0,$$

where η_α here stands for the first eigenvalue of (53). We denote by $\left(Z_{\alpha(\overline{s})}(k(\overline{s})y), W_{\alpha(\overline{s})}(k(\overline{s})y)\right)$ the components of the relative eigenfunction.

We next consider two bases of eigenfunctions for the weighted eigenvalue problems (the operators $\mathfrak{J}$ and T are defined in (13), (169) and are self-adjoint)

$$(55) \qquad \mathfrak{J}\varphi_j(\overline{s}) = h(\overline{s})^\theta\lambda_j\varphi_j(\overline{s}); \qquad T\omega_j = h(\overline{s})^{-\sigma}\rho_j\omega_j.$$

Because of the weights on the right-hand sides, we can choose these eigenfunctions to be normalized so that $\int_0^L h^\theta \varphi_j\varphi_l = \delta_{jl}$ and $\int_0^L h^{-\sigma}\omega_j\omega_l = \delta_{jl}$: this choice will be useful in Subsections 4.2.1, 4.2.2.

These heuristic arguments suggest that the following subspaces $K_{1,\delta}, K_{2,\delta}$, where δ is a small positive constant, once multiplied by $e^{-\frac{if(\varepsilon s)}{\varepsilon}}$ consist of approximate eigenfunctions for L_ε^1 with eigenvalues close to zero (this will be verified below, in the proof of Proposition 2.9, see also Remark 2.11)

$$(56) \qquad K_{1,\delta} = span\left\{h(\varepsilon s)^{\frac{p+1}{4}}\left(\langle\varphi_j(\varepsilon s), \nabla_y U(ky)\rangle + i\varepsilon\langle\varphi_j'(\varepsilon s), y\rangle\frac{f'}{k}U(ky) - \frac{\varepsilon^2}{k^2}\langle\varphi_j''(\varepsilon s), \mathfrak{V}(ky)\rangle\right)\right\};$$

$$(57) \qquad K_{2,\delta} = span\left\{h(\varepsilon s)^{\frac{1}{2}}\left(\omega_j(\varepsilon s)iU(ky) + \frac{2\varepsilon f'(\varepsilon s)}{k}\omega_j'(\varepsilon s)\tilde{U}(y) - \frac{\varepsilon^2}{k^2}\omega_j''(\varepsilon s)\mathfrak{W}(ky)\right)\right\},$$

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$j = 0, \ldots, \frac{\delta}{\varepsilon}$, where

$$\tilde{U} = \left(\frac{1}{h^{p-1}(p-1)} U(ky) + \frac{1}{2k} \nabla U(ky) \cdot y \right).$$

Here $\mathfrak{V} = (\mathfrak{V}_j)_{j=1,\ldots,n-1}$ is the counterpart of $\hat{\mathfrak{V}}$ in (51) substituting $\hat{h}$ with $h(\overline{s})$ (the same holds for $\mathfrak{W}$). The choice of the weights (as powers of h) in (55) and in $K_{1,\delta}$, $K_{2,\delta}$ are again done for technical reasons, and will be convenient below, see in particular Subsections 4.2.1 and 4.2.2.

We also need to construct an approximate kernel with the profile (Z, W), see Proposition 2.5 **e**. To this aim we introduce the functions (recall (54))

$$(58) \qquad Q_{1,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} Z^2_{\alpha(\overline{s})}; \qquad Q_{2,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} W^2_{\alpha(\overline{s})}; \qquad Q_{3,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} Z_{\alpha(\overline{s})} W_{\alpha(\overline{s})},$$

and consider the following eigenvalue problem (with periodic boundary conditions)

$$-\varepsilon^2 \xi'' - k^2 \alpha^2 \xi = \frac{\tilde{\nu}}{1 + 2f' \frac{Q_{3,\alpha}}{k\alpha}} \xi \qquad \text{on } [0, L].$$

By the Weyl's asymptotic formula we have that the eigenvalues $\tilde{\nu}_j$ (counted with multiplicity) have the qualitative behavior $\tilde{\nu}_j \simeq -1 + \varepsilon^2 j^2$ as $j \to +\infty$. Hence, there is a first index j_ε (of order $\frac{1}{\varepsilon}$) for which $\tilde{\nu}_{j_\varepsilon} \geq 0$. Setting $\nu_j = \tilde{\nu}_{j_\varepsilon + j}$ and denoting by ξ_j the eigenfunctions corresponding to ν_j, then we have

$$(59) \qquad -\varepsilon^2 \xi''_j - k^2 \alpha^2 \xi_j = \frac{\nu_j}{1 + 2f' \frac{Q_{3,\alpha}}{k\alpha}} \xi_j; \qquad \nu_j = \hat{C}_0 \varepsilon j + O(\varepsilon^2 j^2) + O(\varepsilon) \quad \text{if } |j| \leq \frac{\delta^2}{\varepsilon},$$

where $\delta > 0$ is any given positive (small) constant. Notice that the family $(\xi_j)_j$ can be chosen normalized in L^2 with the weight $\frac{1}{1 + 2f' \frac{Q_{3,\alpha}}{k\alpha}}$ (this follows from (59) and the Courant-Fischer formula). Next we set

$$(60) \qquad \beta_j = -\frac{1}{k\alpha} \left(1 - \frac{Q_{1,\alpha}}{k^2 \alpha^2 + 2f' k\alpha Q_{3,\alpha}} \nu_j \right) \varepsilon \xi'_j.$$

By our choices, the functions β_j and ξ_j satisfy (this system will be useful in Subsection 4.3)

$$(61) \qquad \begin{cases} -\varepsilon^2 \beta''_j - k^2 \alpha^2 \beta_j - 2f' \frac{Q_{3,\alpha}}{Q_{1,\alpha}} (\varepsilon \xi'_j + k\alpha \beta_j) = \nu_j \beta_j + (O(\nu_j^2) + O(\varepsilon)) \beta_j; \\ -\varepsilon^2 \xi''_j - k^2 \alpha^2 \xi_j + 2f' \frac{Q_{3,\alpha}}{Q_{2,\alpha}} (\varepsilon \beta'_j - k\alpha \xi_j) = \nu_j \xi_j + (O(\nu_j^2) + O(\varepsilon)) \xi_j \end{cases} \qquad \text{for } |j| \leq \frac{\delta^2}{\varepsilon}.$$

Our next goal is to introduce a family of approximate eigenfunctions of L^1_ε which have the profile (from now on, we might denote $(Z_{\alpha(\overline{s})}, W_{\alpha(\overline{s})})$, see (54), simply by (Z_α, W_α))

$$(62) \qquad v_{3,j} := (\beta_j + q_j) Z_\alpha + \gamma_j \frac{\partial Z_\alpha}{\partial \alpha} + i \xi_j W_\alpha + i \kappa_j \frac{\partial W_\alpha}{\partial \alpha} :$$

the functions β_j, ξ_j are as in (59)-(60), while q_j, γ_j and κ_j are small corrections to be chosen properly so that $L^1_\varepsilon(e^{-i \frac{f(\varepsilon s)}{\varepsilon}} v_{3,j}) = \nu_j e^{-i \frac{f(\varepsilon s)}{\varepsilon}} v_{3,j}$, up to an error $o(\nu_j) + O(\varepsilon)$.

With simple computations, using (46), (47), Remark 2.6 and (61), one finds that

$$e^{i \frac{f(\varepsilon s)}{\varepsilon}} \left(L^1_\varepsilon(e^{-i \frac{f(\varepsilon s)}{\varepsilon}} v_{3,j}) - \nu_j e^{-i \frac{f(\varepsilon s)}{\varepsilon}} v_{3,j} \right)$$

$$= Z_\alpha 2f' \frac{Q_{3,\alpha}}{Q_{1,\alpha}} \left[2Q_{1,\alpha} \gamma_j k + (\varepsilon \xi'_j + k\alpha \beta_j) \right] + W_\alpha 2f' \left(-\gamma_j k - (\varepsilon \xi'_j + \beta_j k\alpha) - k\alpha q_j \right)$$

$$+ \frac{\partial Z_\alpha}{\partial \alpha} (-\varepsilon^2 \gamma''_j - \alpha^2 k^2 \gamma_j) + \frac{\partial W_\alpha}{\partial \alpha} 2f'(-\varepsilon \kappa'_j - k\alpha \gamma_j)$$

$$+ i Z_\alpha 2f'(-\kappa_j k - k\alpha \xi_j + \varepsilon \beta'_j + \varepsilon q'_j) + i W_\alpha 2f' \frac{Q_{3,\alpha}}{Q_{2,\alpha}} (2Q_{2,\alpha} k\kappa_j + k\alpha \xi_j - \varepsilon \beta'_j)$$

$$+ i \frac{\partial Z_\alpha}{\partial \alpha} 2f'(-k\alpha \kappa_j + \varepsilon \gamma'_j) + i \frac{\partial W_\alpha}{\partial \alpha} (-\varepsilon^2 \kappa''_j - \alpha^2 k^2 \kappa_j) + Z_\alpha(-\varepsilon^2 q''_j - \alpha^2 k^2 q_j)$$

$$+ (O(\nu_j^2) + O(\varepsilon)) \xi_j + (O(\nu_j^2) + O(\varepsilon)) \beta_j.$$

To make the coefficients of the terms Z_α, W_α and iW_α in the second and fourth lines vanish we choose

$$\gamma_j = -\frac{1}{2kQ_{1,\alpha}}(\varepsilon\xi'_j + k\alpha\beta_j); \qquad \kappa_j = -\frac{1}{2kQ_{2,\alpha}}(k\alpha\xi_j - \varepsilon\beta'_j); \qquad q_j = -\frac{1}{k\alpha}(\varepsilon\xi'_j + \beta_j k\alpha + k\gamma_j).$$

Using (60) we get

$$(63) \qquad \gamma_j = -\varepsilon\xi'_j \frac{1}{2k} \frac{1}{k^2\alpha^2 + 2f'k\alpha Q_{3,\alpha}}\nu_j; \qquad \kappa_j = \frac{1}{2k} \frac{\nu_j}{1 + 2f'\frac{Q_{3,\alpha}}{k\alpha}}\xi_j + O(\nu_j^2)\xi_j :$$

these equations and (59) imply the relations $-\varepsilon^2\gamma''_j - \alpha^2k^2\gamma_j = O(\nu_j^2)\beta_j$, $-\varepsilon\kappa'_j - k\alpha\gamma_j = O(\nu_j^2)\beta_j$, $-k\alpha\kappa_j + \varepsilon\gamma'_j = O(\nu_j^2)\xi_j$ and $-\varepsilon^2\kappa''_j - \alpha^2k^2\kappa_j = O(\nu_j^2)\xi_j$. Similarly one finds

$$(64) \qquad q_j = \xi_j \frac{1}{2k} \frac{1}{k\alpha + 2f'Q_{3,\alpha}}\nu_j.$$

This also yields $-\gamma_j k - (\varepsilon\xi'_j + \beta_j k\alpha) - k\alpha q_j = O(\nu_j^2)\beta_j$, $-\kappa_j k - k\alpha\xi_j + \varepsilon\beta'_j + \varepsilon q'_j = O(\nu_j^2)\xi_j$ and $-\varepsilon^2 q''_j - \alpha^2k^2 q_j = O(\nu_j^2)\beta_j$, so we obtain

$$(65) \qquad L^1_\varepsilon(e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v_{3,j}) = \nu_j e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v_{3,j} + (O(\nu_j^2) + O(\varepsilon))\xi_j + (O(\nu_j^2) + O(\varepsilon))\beta_j \qquad \text{for } |j| \leq \frac{\delta^2}{\varepsilon},$$

which was our claim. We next define

$$(66) \qquad K_{3,\delta} = span\left\{(\beta_j + q_j)Z_\alpha + \gamma_j\frac{\partial Z_\alpha}{\partial\alpha} + i\xi_j W_j + i\kappa_j\frac{\partial W_\alpha}{\partial\alpha} \; : \; j = -\frac{\delta^2}{\varepsilon}, \ldots, \frac{\delta^2}{\varepsilon}\right\}.$$

In the $K_{l,\delta}$'s we added some corrections to the approximate eigenfunctions which take into account the variation of the profile with the frequency, see the derivation of (45) and Remark 2.8. In $K_{1,\delta}$ and $K_{2,\delta}$ the corrections are up to the second order (in εj), while in $K_{3,\delta}$ only up to the first: the reason is that the corresponding eigenvalues have a *quadratic* dependence in εj for $K_{1,\delta}$ and $K_{2,\delta}$ (they correspond to η_α in Proposition 2.5), and an *affine* dependence in εj for $K_{3,\delta}$ (corresponding to μ_α in Proposition 2.5). Since the former dependence is more delicate in the indices, we need a more accurate expansion of the eigenfunctions. We finally set

$$(67) \qquad K_\delta = span\{K_{1,\delta}, K_{2,\delta}, K_{3,\delta}\}.$$

2.4 Invertibility of L_ε in the orthogonal complement of K_δ

Since K_δ (multiplied by $e^{-i\frac{f(\varepsilon s)}{\varepsilon}}$) is a good candidate for the span of the eigenfunctions of L^1_ε with small eigenvalues, it seems plausible to invert L^1_ε on the orthogonal complement to $e^{-i\frac{f(\varepsilon s)}{\varepsilon}}K_\delta$: this is the content of the next result. We recall the definition of the constant $\mathcal{A}$ in the introduction.

Proposition 2.9 *There exists $\mathcal{A}_0$ sufficiently small such that for any $\mathcal{A} \in [0, \mathcal{A}_0]$ the following property holds. For $\delta > 0$ small enough there exists $C > 0$ (independent of δ) such that if*

$$(68) \qquad \Re \int_{N\gamma_\varepsilon} e^{-i\frac{f(\varepsilon s)}{\varepsilon}} v\,\overline{\phi}\,dV_{\hat{g}_\varepsilon} = 0 \qquad \text{for all } v \in K_\delta,$$

one has $\|\Pi_\varepsilon L^1_\varepsilon(\phi)\|_{L^2(N\gamma_\varepsilon)} \geq C^{-1}\delta^2\|\phi\|_{L^2(N\gamma_\varepsilon)}$. Here Π_ε denotes the projection in $L^2(N\gamma_\varepsilon)$ onto the orthogonal complement of the set $\left\{e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v \; : \; v \in K_\delta\right\}$.

Before starting with the proof, which relies on a localization argument and the spectral analysis of Proposition 2.5, we introduce some notation and a preliminary Lemma. We fix $\hat{s} \in [0, L]$ and we denote

by $\hat{f}$, $\hat{h}$, $\hat{k}$, $\hat{\alpha}$ the values of $f'(\hat{s}), h(\hat{s}), k(\hat{s}), \alpha(\hat{s})$ respectively, so that the counterpart of (10) holds true. For a large constant $\tilde{C}_0$ to be fixed later, we also define

$$\hat{K}_{1,\delta} = span\left\{ \langle \hat{\varphi}_j(\varepsilon s), \nabla_y U(\hat{k}y) \rangle + i\varepsilon \langle \hat{\varphi}'_j(\varepsilon s), y \rangle \frac{\hat{f}}{\hat{k}} U(\hat{k}y) - \frac{\varepsilon^2}{\hat{k}^2} \langle \varphi''_j(\varepsilon s), \hat{\mathfrak{V}}(\hat{k}y) \rangle \right\};$$

$$\hat{K}_{2,\delta} = span\left\{ \left(\hat{\omega}_j(\varepsilon s) iU(\hat{k}y) + \frac{2\varepsilon f'(\varepsilon s)}{\hat{k}} \hat{\omega}'_j(\varepsilon s) \hat{\tilde{U}}(y) - \frac{\varepsilon^2}{\hat{k}^2} \tilde{\omega}''_j(\varepsilon s) \hat{\mathfrak{W}}(\hat{k}y) \right) \right\},$$

$j = 0, \ldots, \frac{\delta}{\tilde{C}_0 \varepsilon}$, and

$$\hat{K}_{3,1,\delta} = span\left\{ \cos(\hat{\alpha}_j s) Z_{\hat{\alpha}_j}(\hat{k}y) - i\sin(\hat{\alpha}_j s) W_{\hat{\alpha}_j}(\hat{k}y) \; : \; j = -\frac{\delta^2}{\tilde{C}_0 \varepsilon}, \ldots, \frac{\delta^2}{\tilde{C}_0 \varepsilon} \right\};$$

$$\hat{K}_{3,2,\delta} = span\left\{ \sin(\hat{\alpha}_j s) Z_{\hat{\alpha}_j}(\hat{k}y) + i\cos(\hat{\alpha}_j s) W_{\hat{\alpha}_j}(\hat{k}y) \; : \; j = -\frac{\delta^2}{\tilde{C}_0 \varepsilon}, \ldots, \frac{\delta^2}{\tilde{C}_0 \varepsilon} \right\},$$

where

$$(69) \qquad \hat{\tilde{U}} = \left(\frac{1}{\hat{h}^{p-1}(p-1)} U(\hat{k}y) + \frac{1}{2\hat{k}} \nabla U(\hat{k}y) \cdot y \right); \qquad \hat{\alpha}_j = \left[\frac{\hat{\alpha}\hat{k}L}{2\pi\varepsilon} + j \right] \frac{2\pi\varepsilon}{L},$$

(again, the latter square bracket stands for the integer part, and this choice makes the functions L/ε-periodic). In the above formulas $(\hat{\varphi}_j)_j$ are the eigenfunctions of the normal Laplacian with the flat metric on γ, and $\hat{\omega}_j$ those of ∂^2_{ss} on $[0, L]$: the symbols $Z_{\hat{\alpha}_j}, W_{\hat{\alpha}_j}$ stand for the components of the eigenfunctions of (45) corresponding to $\eta_{\hat{\alpha}_j}$. In analogy with (67) we also define

$$(70) \qquad \hat{K}_\delta = span\left\{ \hat{K}_{1,\delta}, \hat{K}_{2,\delta}, \hat{K}_{3,1,\delta}, \hat{K}_{3,2,\delta} \right\}.$$

Given a small constant $\eta > 0$ to be chosen later (of order $\sqrt{\varepsilon}$), we consider then a smooth cutoff function χ_η (depending only on s) with support near $\frac{\hat{s}}{\varepsilon}$ and with length of order $\frac{\eta}{\varepsilon}$. For example, one can take $\chi_\eta(s) = \chi(\frac{\varepsilon}{\eta}(s - \hat{s}/\varepsilon))$ for a fixed compactly supported cutoff χ which is 1 in a neighborhood of 0. The next result uses Fourier cancelation, and is related to Lemma 2.7 in [12].

Lemma 2.10 *There exists $\tilde{C}_0$ sufficiently large (depending only on V, L and $\mathcal{A}_0$) with the following property. For any given integer number m there exists $C_m > 0$ depending on m and χ_η such that for $|j| \leq \frac{\delta^2}{\tilde{C}_0 \varepsilon}$ and for $|l| \geq \frac{\delta^2}{\varepsilon}$ one has*

$$\left| \int \chi_\eta(s) \xi_l(s) \cos(\hat{\alpha}_j s) ds \right| + \left| \int \chi_\eta(s) \xi_l(s) \sin(\hat{\alpha}_j s) ds \right| \leq \frac{1}{\varepsilon} \frac{C_m}{|\nu_l|^m} \left[\eta(1 + |\nu_l|) + \frac{\varepsilon}{\eta} \right]^m.$$

PROOF. We clearly have that $(\cos(\hat{\alpha}_j s))'' = -\hat{\alpha}_j^2 \cos(\hat{\alpha}_j s)$: therefore, integrating by parts, after some manipulation we obtain that

$$\int \chi_\eta \xi_l(s) \cos(\hat{\alpha}_j s) ds = \frac{1}{\hat{\alpha}_j^2 - \hat{k}^2 \hat{\alpha}^2 + \frac{\nu_l}{1 + 2\hat{f} \frac{Q_{3,\hat{\alpha}}}{\hat{k}\hat{\alpha}}}} \times$$

$$(71) \qquad \times \left\{ \int \chi_\eta \xi_l(s) \cos(\hat{\alpha}_j s) \left(k^2 \alpha^2 - \hat{k}^2 \hat{\alpha}^2 + \frac{\nu_l}{1 + 2\hat{f} \frac{Q_{3,\hat{\alpha}}}{\hat{k}\hat{\alpha}}} - \frac{\nu_l}{1 + 2f' \frac{Q_{3,\alpha}}{k\alpha}} \right) \right.$$

$$\left. - \int \cos(\hat{\alpha}_j s) \left[\chi''_\eta \xi_l(s) + 2\chi'_\eta \xi'_l(s) \right] \right\}.$$

By (69) the numbers $\hat{\alpha}_j$ satisfy $\hat{\alpha}_j \simeq \hat{k}\hat{\alpha} + \frac{2\pi\varepsilon j}{L}$ for $|j| \leq \frac{\delta^2}{\hat{C}_0\varepsilon}$, while by (59) $|\nu_l| \geq \frac{1}{2}\hat{C}_0\delta^2$ for $|l| \geq \frac{\delta^2}{\varepsilon}$. Notice also that $1 + 2f'\frac{Q_{3,\alpha}}{k\alpha}$ is uniformly bounded above and below by positive constants when $\mathcal{A}_0$ tends to zero (see for example the comments in Remark 2.7). By these facts and the properties of χ_η we find

$$\left| \int \chi_\eta \xi_l(s) \cos(\hat{\alpha}_j s) ds \right| \leq \frac{1}{\varepsilon} \frac{C}{|\nu_l|} \left[\eta(1 + |\nu_l|) + \frac{\varepsilon}{\eta} \right].$$

which yields the statement for $m = 1$ (similar computations can be performed to deal with the sin function). The factor $\frac{1}{\varepsilon}$ inside the brackets arises from the fact that we are integrating over the interval $[0, L/\varepsilon]$, and by the normalization of ξ_j (see the comments before (59)). To obtain it for general m, it is sufficient to iterate the procedure for (71) m times and integrate by parts. ∎

PROOF OF PROPOSITION 2.9 The proof mainly relies on a localization argument and Lemma 2.10. If $\eta = \sqrt{\varepsilon}$ and χ_η is as in Lemma 2.10, we show next that the function $\chi_\eta \phi$ is *almost* orthogonal to $e^{-i\hat{f}s}\hat{K}_\delta$ if ε is sufficiently small. We consider for example a function $\hat{v} \in \hat{K}_{3,1,\delta}$ of the form

$$\hat{v} = \sum_{j=-\frac{\delta^2}{\hat{C}_0\varepsilon}}^{\frac{\delta^2}{\hat{C}_0\varepsilon}} \hat{b}_j \left[\cos(\hat{\alpha}_j s) Z_{\hat{\alpha}_j}(\hat{k}y) - i \sin(\hat{\alpha}_j s) W_{\hat{\alpha}_j}(\hat{k}y) \right],$$

for some arbitrary coefficients $(\hat{b}_j)_j$, and we also set

$$\tilde{v} = \sum_{j=-\frac{\delta^2}{\hat{C}_0\varepsilon}}^{\frac{\delta^2}{\hat{C}_0\varepsilon}} \hat{b}_j \left[\cos(\hat{\alpha}_j s) Z_{\alpha(\varepsilon s)}(k(\overline{s})y) - i \sin(\hat{\alpha}_j s) W_{\alpha(\varepsilon s)}(k(\overline{s})y) \right].$$

We are going to evaluate the real part of the integral $\int_{N_{\gamma_\varepsilon}} e^{-i\hat{f}s}\hat{v}\chi_\eta\overline{\phi}$: first of all, since $|k(\overline{s}) - \hat{k}| \leq C\eta$ and $|\hat{\alpha}_j - \alpha(\overline{s})| \leq C(\eta + \delta^2)$ on the support of χ_η we have that

$$\|Z_{\alpha(\varepsilon s)}(k(\overline{s})y) - Z_{\hat{\alpha}_j}(\hat{k}y)\|_{L^2(\mathbb{R}^{n-1})} = O(\eta + \delta^2) \qquad \text{in } supp(\chi_\eta),$$

so, clearly

$$(72) \qquad \Re \int_{N_{\gamma_\varepsilon}} e^{-i\hat{f}s}\hat{v}\chi_\eta\overline{\phi} = \Re \int_{N_{\gamma_\varepsilon}} e^{-i\hat{f}s}\tilde{v}\chi_\eta\overline{\phi} + O(\eta + \delta^2)\|\chi_\eta\phi\|_{L^2(N_{\gamma_\varepsilon})}\|\hat{v}\|_{L^2(N_{\gamma_\varepsilon})}.$$

We next write $\hat{w}(s) = \chi_\eta(s) \sum_{j=-\frac{\delta^2}{\hat{C}_0\varepsilon}}^{\frac{\delta^2}{\hat{C}_0\varepsilon}} \hat{b}_j \sin(\hat{\alpha}_j s)$, and notice that

$$(73) \qquad \hat{w}'(s) = \chi_\eta'(s) \sum_{j=-\frac{\delta^2}{\hat{C}_0\varepsilon}}^{\frac{\delta^2}{\hat{C}_0\varepsilon}} \hat{b}_j \sin(\hat{\alpha}_j s) + \chi_\eta(s) \sum_{j=-\frac{\delta^2}{\hat{C}_0\varepsilon}}^{\frac{\delta^2}{\hat{C}_0\varepsilon}} \hat{b}_j \hat{\alpha}_j \cos(\hat{\alpha}_j s)$$

$$= k\alpha \chi_\eta(s) \sum_{j=-\frac{\delta^2}{\hat{C}_0\varepsilon}}^{\frac{\delta^2}{\hat{C}_0\varepsilon}} \hat{b}_j \cos(\hat{\alpha}_j s) + \chi_\eta'(s) \sum_{j=-\frac{\delta^2}{\hat{C}_0\varepsilon}}^{\frac{\delta^2}{\hat{C}_0\varepsilon}} \hat{b}_j \sin(\hat{\alpha}_j s) - \chi_\eta(s) \sum_{j=-\frac{\delta^2}{\hat{C}_0\varepsilon}}^{\frac{\delta^2}{\hat{C}_0\varepsilon}} \hat{b}_j(k\alpha - \hat{\alpha}_j) \sin(\hat{\alpha}_j s).$$

Using this formula and the same argument as for (72) we get (recall our notation before (62))

$$(74) \qquad \chi_\eta\tilde{v} = \frac{1}{k\alpha}\hat{w}'(s)Z_\alpha - i\hat{w}(s)W_\alpha + O(\eta + \delta^2)\|\chi_\eta\hat{v}\|_{L^2(N_{\gamma_\varepsilon})}.$$

Notice that, by the explicit form of $\hat{w}$ and $\hat{\alpha}_j$, for any integer m one has $\|\hat{w}\|^2_{H^m([0,L/\varepsilon])} \leq C_m\|\hat{w}\|^2_{L^2([0,L/\varepsilon])}$: therefore, if we write $\hat{w}$ with respect to the basis ξ_l as (notice the shift of index before (59))

$$(75) \qquad \hat{w}(s) = \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l \xi_l(\varepsilon s),$$

we also find that

$$(76) \qquad \sum_{l=-j_\varepsilon}^{\infty} (1+|\nu_l|)^m \check{b}_l^2 \leq C_m\|\hat{w}\|^2_{H^m([0,L/\varepsilon])} \leq C_m\|\hat{w}\|^2_{L^2([0,L/\varepsilon])} \leq C_m\|\hat{v}\|^2_{L^2(N\gamma_\varepsilon)}.$$

Differentiating (75) with respect to s and using the definition of ξ_j together with (60) we find that

$$\hat{w}'(s) = \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l \varepsilon \xi_l'(\varepsilon s) = \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l(-k\alpha + O(\nu_l))\beta_l(\varepsilon s).$$

The last formula and (74) imply

$$\chi_\eta \tilde{v} = -\sum_{l=-j_\varepsilon}^{\infty} \check{b}_l(\beta_l Z_\alpha + i\xi_l W_\alpha) + \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l O(\nu_l)\beta_l Z_\alpha + O(\eta + \delta^2)\|\chi_\eta \hat{v}\|_{L^2(N\gamma_\varepsilon)}$$

$$(77) \qquad = -\sum_{l=-j_\varepsilon}^{\infty} \check{b}_l v_{3,l} + \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l(v_{3,l} - \beta_l Z_\alpha - i\xi_l W_\alpha) + \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l O(\nu_l)\beta_l Z_\alpha + O(\eta + \delta^2)\|\chi_\eta \hat{v}\|_{L^2(N\gamma_\varepsilon)}.$$

In the support of χ_η there exists $\hat{\theta} \in \mathbb{R}$ such that $\frac{f(\varepsilon s)}{\varepsilon} = \hat{f}s + \hat{\theta} + O(\eta)$, which yields

$$\int_{N\gamma_\varepsilon} \tilde{v}e^{-i\hat{f}s}\chi_\eta \overline{\phi} = \int_{N\gamma_\varepsilon} \tilde{v}e^{-i\frac{f(\varepsilon s)}{\varepsilon}}\chi_\eta \overline{\phi} + O(\eta)\|\phi\|_{L^2(supp(\chi_\eta))}\|\hat{v}\|_{L^2(N\gamma_\varepsilon)}.$$

Now, recalling that $\eta = \sqrt{\varepsilon}$ and that we have orthogonality between ϕ and $e^{-i\frac{f(\varepsilon s)}{\varepsilon}}K_\delta$, from the last two formulas we obtain that

$$(78) \qquad \int_{N\gamma_\varepsilon} \tilde{v}e^{-i\hat{f}s}\chi_\eta \overline{\phi} = A_1 + A_2 + A_3 + O(\eta + \delta^2)\|\phi\|_{L^2(supp(\chi_\eta))}\|\hat{v}\|_{L^2(N\gamma_\varepsilon)};$$

$$A_1 = -\int_{supp(\chi_\eta)} e^{-i\hat{f}s}\overline{\phi} \sum_{|l| \geq \frac{\delta^2}{\varepsilon}} \check{b}_l v_{3,l}; \qquad A_2 = \int_{supp(\chi_\eta)} e^{-i\hat{f}s}\overline{\phi} \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l(v_{3,l} - \beta_l Z_\alpha - i\xi_l W_\alpha);$$

$$A_3 = \int_{supp(\chi_\eta)} e^{-i\hat{f}s}\overline{\phi} \sum_{l=-j_\varepsilon}^{\infty} \check{b}_l O(\nu_l)\beta_l Z_\alpha.$$

To estimate these terms we notice first that, by the normalization of ξ_j before (60), the coefficients $\check{b}_l$ in (75) can be computed as

$$\check{b}_l = \varepsilon \int_0^{L/\varepsilon} \frac{\hat{w}(s)}{1 + 2f'\frac{Q_{3,\alpha}}{k\alpha}}\xi_l(\varepsilon s)ds = \varepsilon \sum_{j=-\frac{\delta^2}{C_0\varepsilon}}^{\frac{\delta^2}{C_0\varepsilon}} \hat{b}_j \int_0^{L/\varepsilon} \chi_\eta(s)\frac{\sin(\hat{\alpha}_j s)}{1 + 2f'\frac{Q_{3,\alpha}}{k\alpha}}\xi_l(\varepsilon s)ds.$$

Using this formula, Lemma 2.10 and the Hölder inequality we find that for any integer m

$$(79) \qquad \check{b}_l^2 \leq C_m\varepsilon^{2m+2}\left(\sum_{j=-\frac{\delta^2}{C_0\varepsilon}}^{\frac{\delta^2}{C_0\varepsilon}} |\hat{b}_j|\right)^2 \leq C_m\varepsilon^{2m+1} \sum_{j=-\frac{\delta^2}{C_0\varepsilon}}^{\frac{\delta^2}{C_0\varepsilon}} \hat{b}_j^2 \leq C_m\varepsilon^{2m+2}\|\hat{v}\|^2_{L^2(N\gamma_\varepsilon)}; \qquad |l| \geq \frac{\delta^2}{\varepsilon}.$$

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From the explicit expression for the functions $v_{3,\delta}$ the above term A_1 can be estimated as

$$A_1 \leq C\Big(\frac{1}{\varepsilon} \sum_{|l| \geq \frac{\delta^2}{\varepsilon}} (1 + |\nu_l|^2)^2 \check{b}_l^2\Big)^{\frac{1}{2}} \|\phi\|_{L^2(supp(\chi_\eta))}.$$

As before, the factor $\frac{1}{\varepsilon}$ inside the brackets arises from the fact that we are integrating over $[0, L/\varepsilon]$.

Using the fact that $C^{-1}|\varepsilon l| \leq |\nu_l| \leq C(|\varepsilon l| + \varepsilon^2 l^2)$ for $|l| \geq \frac{\delta^2}{\varepsilon}$ (which follows from the Weyl's asymptotic formula for the eigenvalue problem in (59)), (76) and (79), one finds that for any large integer m and any $d \in (1, m/8)$

$$\sum_{\frac{\delta^2}{\varepsilon} \leq |l| \leq \varepsilon^{-d}} (1 + |\nu_l|^2)^2 \check{b}_l^2 \leq C_m \varepsilon^{11+2m-8d} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)}^2; \qquad \sum_{|l| \geq \varepsilon^{-d}} (1 + |\nu_l|^2)^2 \check{b}_l^2 \leq C_m \varepsilon^{(d-1)(m-4)} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)}^2.$$

By the arbitrarity of m it follows that for any $m' \in \mathbb{N}$

$$|A_1| \leq C_{m'} \varepsilon^{m'} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)} \|\phi\|_{L^2(supp(\chi_\eta))}.$$

Dividing the set of indices l into $\{|l| \leq \frac{\delta^2}{\varepsilon}\}$ and $\{|l| \geq \frac{\delta^2}{\varepsilon}\}$ and using similar arguments (taking also into account (63) and (64)) we get

$$|A_2| + |A_3| \leq C\|\phi\|_{L^2(supp(\chi_\eta))} \left(\frac{1}{\varepsilon} \sum_{l=-j_\varepsilon}^{\infty} (\nu_l^2 + \nu_l^4) \check{b}_l^2\right)^{\frac{1}{2}} \leq C\delta^2 \|\phi\|_{L^2(supp(\chi_\eta))} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)}.$$

Therefore, using (72) and (78) one finds

$$\Re \int_{N\gamma_\varepsilon} e^{-i\hat{f}s} \hat{v} \chi_\eta \overline{\phi} = O(\eta + \delta^2) \|\phi\|_{L^2(supp(\chi_\eta))} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)}; \qquad \hat{v} \in \hat{K}_{3,1,\delta}.$$

Similar estimates hold for $\hat{v} \in span\{\hat{K}_{1,\delta}, \hat{K}_{2,\delta}, \hat{K}_{3,2,\delta}\}$, so we obtain

$$(80) \qquad \int_{N\gamma_\varepsilon} e^{-i\hat{f}s} \hat{v} \chi_\eta \overline{\phi} = O(\delta^2 + \eta) \|\phi\|_{L^2(supp(\chi_\eta))} \|\hat{v}\|_{L^2(N\gamma_\varepsilon)} \quad \text{for every } \hat{v} \in \hat{K}_\delta.$$

Next we let $\hat{L}_\varepsilon$ denote the operator in (43) with coefficients frozen at $\hat{s}$. Since $e^{-i\hat{f}s}\hat{K}_\delta$ consist of all the eigenfunctions of $\hat{L}_\varepsilon$ (up to an error $o(\delta^2)$) with eigenvalues smaller in absolute value than δ^2, see Proposition 2.5 and Remark 2.8, from (80) we then deduce

$$(81) \qquad \|\hat{L}_\varepsilon(e^{-i\hat{f}s}\chi_\eta\phi)\|_{L^2(N\gamma_\varepsilon)} \geq \frac{\delta^2}{C} \|\chi_\eta\phi\|_{L^2(N\gamma_\varepsilon)} + O(\delta^4 + \delta^2\eta)\|\phi\|_{L^2(supp(\chi_\eta))}$$

for some fixed constant C independent of δ.

It is now possible to choose the cutoff function χ (see the comments before Lemma 2.10) so that it is even, compactly supported in $[-2, 2]$, $\chi \equiv 1$ in $[-1, 1]$, and so that $\chi(2 - t) + \chi(t) \equiv 1$ for $t \in [1, 2]$. With this choice, we can find a partition of unity $(\chi_{\eta,j})_j$ of $[0, L/\varepsilon]$ consisting of translates of χ_η (plus a negligible scaling), with j running between 1 and a number of order $\frac{L}{\sqrt{\varepsilon}}$. For each index j we choose a point $\hat{s}_j$ in the support of $\chi_{\eta,j}$ and we denote by $\hat{L}_j$ the operator corresponding to (43) with coefficients frozen at $\hat{s}_j$. Then, using (81), with easy computations one finds

$$\begin{aligned}
\|L_\varepsilon^1 \phi\|_{L^2(N\gamma_\varepsilon)}^2 &= \Big\|L_\varepsilon^1 \sum_j \chi_{\eta,j}\phi\Big\|_{L^2(N\gamma_\varepsilon)}^2 = \Big\|\sum_j \hat{L}_j(\chi_{\eta,j}\phi)\Big\|_{L^2(N\gamma_\varepsilon)}^2 + O(\sqrt{\varepsilon})\|\phi\|_{L^2(N\gamma_\varepsilon)}^2 \\
&\geq \frac{\delta^4}{C} \sum_j \|\chi_{\eta,j}\phi\|_{L^2(N\gamma_\varepsilon)}^2 + O(\sqrt{\varepsilon})\|\phi\|_{L^2(N\gamma_\varepsilon)}^2 = \frac{\delta^4}{C}\|\phi\|_{L^2(N\gamma_\varepsilon)}^2 + O(\sqrt{\varepsilon})\|\phi\|_{L^2(N\gamma_\varepsilon)}^2
\end{aligned}$$

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for some C independent of δ. To complete the proof we need to bound from below the norm of $\Pi_\varepsilon L^1_\varepsilon \phi$, showing that $\|\Pi_\varepsilon L^1_\varepsilon \phi\|^2_{L^2(N\gamma_\varepsilon)} \geq \frac{\delta^2}{C}\|\phi\|^2_{L^2(N\gamma_\varepsilon)}$. To see this, by the last formula it is sufficient to have

$$(82) \qquad (L^1_\varepsilon \phi, e^{-i\frac{f(\varepsilon s)}{\varepsilon}} v)_{L^2(N\gamma_\varepsilon)} = o(\delta^2)\|\phi\|_{L^2(N\gamma_\varepsilon)}\|v\|_{L^2(N\gamma_\varepsilon)} \qquad \text{for any } v \in K_\delta.$$

We prove this claim for $v \in K_{1,\delta}$ only: for the other $K_{j,\delta}$'s the arguments are similar, see Remark 2.11 below for more details. Setting $v = v_r + iv_i$ one finds (see (15))

$$L^1_\varepsilon(e^{-\frac{if(\varepsilon s)}{\varepsilon}}v) = e^{-\frac{if(\varepsilon s)}{\varepsilon}}(\mathcal{L}_r v_r + i\mathcal{L}_i v_i) - e^{-\frac{if(\varepsilon s)}{\varepsilon}}\left(\frac{\partial^2 v_r}{\partial s^2} + i\frac{\partial^2 v_i}{\partial s^2}\right) + 2if'e^{-\frac{if(\varepsilon s)}{\varepsilon}}\left(\frac{\partial v_r}{\partial s} + i\frac{\partial v_i}{\partial s}\right).$$

When differentiating v with respect to s, we either hit the functions φ_j's (and their derivatives) or other functions like $k(\varepsilon s)$ or $f'(\varepsilon s)$ (see the definition of $K_{1,\delta}$ above). The latter ones have a *slow* dependence in s and therefore these terms can be collected within an error of the form $O(\varepsilon)\|v\|_{L^2(N\gamma_\varepsilon)}$.

However, by our choices of the second and the third parts of the elements in $K_{1,\delta}$ (see Remark 2.8, in particular formulas (48), (49) and (50)), terms containing zero-th or first order derivatives of φ_j will have coefficients bounded by ε, while the only term containing second derivatives of φ_j will be a linear combination (in j) of the expressions

$$-\varepsilon^2 h(\varepsilon s)^{\frac{p+1}{4}}\left(1 - \frac{2\mathcal{A}^2\theta}{p-1}h(\overline{s})^{\sigma-\theta}\right)\langle \varphi''_j(\varepsilon s), \nabla_y U(k(\overline{s})y)\rangle, \qquad j = 0, \dots, \frac{\delta}{\varepsilon}.$$

The remaining terms will contain third and the fourth derivatives of φ_j only (multiplied respectively by ε^3 and ε^4). Therefore, if we set

$$v_{1,j} = h(\varepsilon s)^{\frac{p+1}{4}}\left(\langle \varphi_j(\varepsilon s), \nabla_y U(ky)\rangle + i\varepsilon\langle \varphi'_j(\varepsilon s), y\rangle\frac{f'}{k}U(ky) - \frac{\varepsilon^2}{k^2}\langle \varphi''_j(\varepsilon s), \mathfrak{V}(ky)\rangle\right),$$

by the above comments and the fact that $\mathfrak{J}\varphi_j = h(\varepsilon s)^\theta \lambda_j \varphi_j$ (see (55)) we have

$$(83) \qquad L^1_\varepsilon\left(e^{-i\frac{f(\varepsilon s)}{\varepsilon}}\sum_{j=0}^{\frac{\delta}{\varepsilon}} a_j v_{1,j}\right) = e^{-i\frac{f(\varepsilon s)}{\varepsilon}}\sum_{j=0}^{\frac{\delta}{\varepsilon}} \lambda_j a_j v_{1,j} + R(v); \qquad v = \sum_{j=0}^{\frac{\delta}{\varepsilon}} a_j v_{1,j},$$

where $R(v)$ contains terms of order ε or linear combinations of third and fourth derivatives of $\varphi_j(\varepsilon s)$, so using Fourier analysis one can derive the estimate

$$(84) \qquad \|R(v)\|_{L^2(N\gamma_\varepsilon)} \leq C\left(\frac{1}{\varepsilon}\sum_{j=0}^{\frac{\delta}{\varepsilon}} a_j^2(\varepsilon + \varepsilon^3 j^3)^2\right)^{\frac{1}{2}} \leq C(\varepsilon + \delta^3)\|v\|_{L^2(N\gamma_\varepsilon)},$$

for some constant $C > 0$. Therefore, using (83) and (84) we obtain

$$(L^1_\varepsilon \phi, e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v)_{L^2(N\gamma_\varepsilon)} = O(\varepsilon + \delta^3)\|\phi\|_{L^2(N\gamma_\varepsilon)}\|v\|_{L^2(N\gamma_\varepsilon)},$$

which yields (82) and concludes the proof. ∎

Remark 2.11 *The last step in the proof of Proposition 2.9 is nearly identical for $v \in K_{2,\delta}$ except that, still by the computations in Remark 2.8, in the counterpart of (83) we will obtain ρ_j instead of λ_j (see (55)). When considering $K_{3,\delta}$, setting $v = \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} a_j v_{3,j}$ (see (65)) one finds*

$$(85) \qquad L^1_\varepsilon\left(e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v\right) = e^{-i\frac{f(\varepsilon s)}{\varepsilon}}\sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \nu_j a_j v_{3,j} + \tilde{R}(v);$$

$$\|\tilde{R}(v)\|_{L^2} \leq C\left(\frac{1}{\varepsilon}\sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} a_j^2(\varepsilon + \varepsilon^2 j^2)^2\right)^{\frac{1}{2}} \leq C(\varepsilon + \delta^4)\|v\|_{L^2(N\gamma_\varepsilon)}.$$

2.5 Invertibility of L_ε in weighed spaces

Our goal is to show that the linearized operator L_ε (see (24)) at approximate solutions is invertible on spaces of functions satisfying suitable constraints. We begin with some preliminary notation and lemmas: we first collect a decay properties of Green's kernels in Euclidean space. Let us consider the equation

$$(86) \qquad\qquad\qquad -\Delta u + u = f \qquad\quad \text{in } \mathbb{R}^{n-1},$$

where f decays to zero at infinity. The solution of the above equation can be represented as

$$u(x) = \int_{\mathbb{R}^{n-1}} G_0(|x - y|) f(y) dy,$$

where $G_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a function singular at 0 which decays exponentially to zero at infinity. Using the notation of Subsection 2.2 and standard elliptic regularity theory, one can prove the following result (the choice $\alpha \geq \frac{1}{2}$ for the Hölder exponent is technical, and is used in the proof of Lemma 2.13).

Lemma 2.12 *Let $\overline{Q} > 0$, $\alpha \geq \frac{1}{2}$, let $0 < \tau < 1$, $0 < \varsigma < 1$ and let $f \in C_\tau^\alpha$. Then equation (86) has a (unique) solution u of class $C_{\varsigma\tau}^{2,\alpha}$ which vanishes on $\partial B_{\overline{Q}}(0)$. Moreover, there exist $\varsigma_0 > 0$ sufficiently close to 1 and $\check{C}_0$ sufficiently large (depending only on n, α, $\min\{\overline{Q}, 1\}$ and ς) such that for $\varsigma_0 \leq \varsigma < 1$*

$$\|u\|_{C_{\varsigma\tau}^{2,\alpha}} \leq \check{C}_0 \|f\|_{C_\varsigma^\alpha}.$$

Let now $\tau, \varsigma \in (0, 1)$ (to be fixed later). For any integer m we let $\overline{C}_{\varsigma,}^{m,\tau}$ denote the weighted Hölder space

$$(87) \qquad\qquad \overline{C}_\varsigma^{m,\tau} = \left\{ u : \mathbb{R}^{n-1} \to \mathbb{C} \ : \ \sup_{y \in \mathbb{R}^{n-1}} e^{\varsigma|y|} \|u\|_{C^{m,\tau}(B_1(y))} < +\infty \right\}.$$

We also consider the following set of functions L/ε-periodic in s

$$(88) \qquad \overline{L}^2(\overline{C}_\varsigma^{m,\tau}) = \left\{ u : [0, L/\varepsilon] \times \mathbb{R}^{n-1} \to \mathbb{C} \ : \ s \mapsto u(s, \cdot) \in L^2([0, L/\varepsilon]; \overline{C}_\varsigma^{m,\tau}) \right\},$$

and for $l \in \mathbb{N}$, we define similarly the functional space

$$(89) \qquad \overline{H}^l(\overline{C}_\varsigma^{m,\tau}) = \left\{ u : [0, L/\varepsilon] \times \mathbb{R}^{n-1} \to \mathbb{C} \ : \ s \mapsto u(s, \cdot) \in H^l([0, L/\varepsilon]; C_\varsigma^{m,\tau}) \right\}.$$

The weights here are suited for studying functions which decay in y like $e^{-|y|}$, as the fundamental solution of $-\Delta_{\mathbb{R}^{n-1}} u + u = 0$. The parameter $\varsigma < 1$ has been introduced to allow some flexibility in the decay rate. When dealing with functions belonging to the above three spaces, the symbols $\|\cdot\|_{\overline{C}_\varsigma^{m,\tau}}$, $\|\cdot\|_{\overline{L}^2(\overline{C}_\varsigma^{m,\tau})}$, $\|\cdot\|_{\overline{H}^l(\overline{C}_\varsigma^{m,\tau})}$ will denote norms induced by formulas (87), (88) and (89). Also, we keep the same notation for the norms when considering functions defined on subsets of $[0, L/\varepsilon] \times \mathbb{R}^{n-1}$.

We next consider some positive constants $\hat{V}, \hat{f}, \hat{h}, \hat{k}$ which satisfy the relations in (8). If δ and $\hat{K}_\delta$ are as in the previous subsection and $\overline{\delta}$ as in Section 2, letting

$$D_{L,\varepsilon} = [0, L/\varepsilon] \times B_{\varepsilon^{-\overline{\delta}}+1}(0) \subseteq [0, L/\varepsilon] \times \mathbb{R}^{n-1},$$

we define the space of functions

$$\hat{H}_\varepsilon := \left\{ \phi \ : \ \Re \int_{D_{L,\varepsilon}} \overline{\phi}(s, y) e^{-i\hat{f}s} v\left(s, y/\sqrt{\hat{V}}\right) = 0 \qquad \text{for all } v \in \hat{K}_\delta \right\}.$$

This conditions represents, basically, orthogonality with respect to $\hat{K}_\delta$ (multiplied by the phase factor), when the function ϕ is scaled in y by $\sqrt{V}$. This is a choice made for technical reasons, which will be helpful in Proposition 2.14. We next have the following result, related to Proposition 2.9 once we scale y.

Lemma 2.13 *Let $\frac{1}{2} \leq \tau < 1$ and $\varsigma \in (0,1)$. Then, for δ small there exists a positive constant C, depending only on p, τ, ς, L, $\hat{V}$ and $\hat{f}$, such that the following property holds: for $\hat{f}$ small, for $\varepsilon \to 0$ and for any function $\mathfrak{b} \in \overline{L}^2(\overline{C}_\varsigma^\tau)$ there exist $u \in \hat{H}_\varepsilon$, and $\underline{v} \in \hat{K}_\delta$ such that, in $D_{L,\varepsilon}$*

$$
(90) \quad
\begin{cases}
-\frac{1}{\hat{V}}\partial_{ss}^2 u - \Delta_y u + u - \frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}u - (p-1)\frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}e^{-i\hat{f}s}\Re(e^{-i\hat{f}s}\overline{u}) \\
= \mathfrak{b} + e^{-i\hat{f}s}\underline{v}; \\[2mm]
u = 0 \qquad \text{on } \partial D_{L,\varepsilon},
\end{cases}
$$

(notice that $\underline{v}$ above is intended scaled in y) and such that we have the estimates

$$
(91) \qquad \|u\|_{\overline{L}^2(\overline{C}_\varsigma^{2,\tau})} + \|u\|_{\overline{H}^1(\overline{C}_\varsigma^{1,\tau})} + \|u\|_{\overline{H}^2(\overline{C}_\varsigma^\tau)} \leq \frac{C}{\delta^2} \inf_{v \in \hat{K}_\delta} \|\mathfrak{b} + e^{-i\hat{f}s}v\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)};
$$

$$
(92) \qquad \|\underline{v}\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)} \leq C\|\mathfrak{b}\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)}.
$$

PROOF.   First of all we observe that a solution to (90) of class L^2 exists. In fact, replacing $D_{L,\varepsilon}$ with $[0, L/\varepsilon] \times \mathbb{R}^{n-1}$, this would simply follow from Proposition 2.9 with $V \equiv \hat{V}$. However, since the functions in $\hat{K}_\delta$ decay exponentially to zero as $|y| \to +\infty$ the Dirichlet boundary conditions do not affect the solvability property: for more details see for example [13], Lemma 5.5. Notice that indeed, by (7) and Proposition 2.5 **e**, the elements of K_δ decay at the rate $e^{-\hat{k}|y|}$, and by (8) $\hat{k} > \sqrt{\hat{V}}$. In particular $\|\underline{v}\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)}$ is finite and (92) holds. We also have (91) replacing the left-hand side by the L^2 norm of u. We divide the rest of the proof into two steps.

Step 1: $u \in \overline{L}^2(\overline{C}_\varsigma^\tau)$ **and** $\|u\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)} \leq \frac{C}{\delta^2}\|\mathfrak{b}\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)}.$ We set $u = e^{-i\hat{f}s}v$ and $\mathfrak{c} = e^{-i\hat{f}s}(\mathfrak{b} + \underline{v})$, so v satisfies

$$
\begin{cases}
-\Delta v + (1 + \hat{f}^2/\hat{V})v + 2i\hat{f}/\hat{V}\partial_s v - \frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}v \\
-(p-1)\frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}\Re(\overline{v}) = \mathfrak{c} & \text{in } B_{\varepsilon^{-\overline{\delta}}+1}(0), \\
v = 0 & \text{on } \partial B_{\varepsilon^{-\overline{\delta}}+1}(0).
\end{cases}
$$

We now use a Fourier decomposition in the variable s: setting

$$
\mathfrak{c}(s,y) = \sum_j \mathfrak{c}_j(y)e^{ij\varepsilon s}; \qquad v(s,y) = \sum_j v_j(y)e^{ij\varepsilon s},
$$

(here we are assuming for simplicity that $L = 2\pi$) we see that each $\mathfrak{c}_j$ belongs to $C_{\varsigma,\hat{V}}^\tau$, that

$$
(93) \qquad \sum_j \|\mathfrak{c}_j\|_{\overline{C}_\varsigma^\tau}^2 = \frac{1}{\varepsilon}\|\mathfrak{c}\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)}^2; \qquad \sum_j \|v_j\|_{\overline{C}_\varsigma^\tau}^2 = \frac{1}{\varepsilon}\|v\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)}^2 \leq \frac{C}{\varepsilon\delta^2}\|\mathfrak{b}\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)},
$$

and that each v_j solves

$$
(94) \quad
\begin{cases}
-\Delta_y v_j + \left(1 + \frac{\hat{f}^2 + \varepsilon^2 j^2 - 2\hat{f}\varepsilon j}{\hat{V}}\right)v_j - \frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}v_j \\
-(p-1)\frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}\Re(\overline{v}_j) = \mathfrak{c}_j & \text{in } B_{\varepsilon^{-\overline{\delta}}+1}(0), \\
v_j = 0 & \text{on } \partial B_{\varepsilon^{-\overline{\delta}}+1}(0).
\end{cases}
$$

From elliptic regularity theory, we find that for any $R > 0$ there exists a constant C depending only on R, p and τ such that $\|v_j\|_{C^\tau(B_R)} \leq C\|\mathfrak{c}_j\|_{\overline{C}_\varsigma^\tau} + C\|v_j\|_{L^2}$. Now we choose R (depending on p and ς) so

large that $p\frac{\hat{h}^{p-1}}{\hat{V}}U^{p-1}(y\hat{k}/\sqrt{\hat{V}}) < \frac{1}{4}(1-\varsigma)$ for $|y| \geq \frac{R}{2}$, and a smooth radial cutoff function $\hat{\chi}$ such that $\hat{\chi}(y) = 1$ for $|y| \leq \frac{R}{2}$, and $\hat{\chi}(y) = 0$ for $|y| \geq R$. Next, we write equation (94) as

$$\begin{cases} -\Delta_y v_j + \left(1 + \frac{\hat{f}^2 + \varepsilon^2 j^2 - 2\hat{f}\varepsilon j}{\hat{V}}\right)v_j - (1-\hat{\chi})p\frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}v_j = \tilde{\mathfrak{c}}_j + \hat{\chi}p\frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}v_j, \\ v_j = 0 \qquad \text{on } \partial B_{\varepsilon^{-\delta}+1}(0). \end{cases}$$

We notice that the first linear coefficient of v_j is bounded below (uniformly in j) by 1. Therefore, using the Green's representation formula, the maximum principle and our choice of R (see Lemma 2.12) for any $\varsigma' < \varsigma$ we then have the estimate

$$\|v_j\|_{\overline{C}_{\varsigma'}^\tau} \leq C(\|\mathfrak{c}_j\|_{\overline{C}_\varsigma^\tau} + \|v_j\|_{L^2})$$

for some fixed constant C depending only on p, ς and τ. Taking the square and summing over j we get

$$\|u\|_{L^2(\overline{C}_{\varsigma'}^\tau)}^2 = \|v\|_{L^2(\overline{C}_{\varsigma'}^\tau)}^2 \leq C\|\mathfrak{b}\|_{L^2(\overline{C}_\varsigma^\tau)}^2 + C\|v\|_{L^2}^2 \leq \|\mathfrak{b}\|_{L^2(\overline{C}_\varsigma^\tau)}^2.$$

We next want to replace in the last formula ς' with ς. Rewrite (90) as

$$\begin{cases} -\frac{1}{\hat{V}}\partial_{ss}^2 u - \Delta_y u + u = \hat{\mathfrak{c}} := \frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}u \\ +(p-1)\frac{\hat{h}^{p-1}}{\hat{V}}U(y\hat{k}/\sqrt{\hat{V}})^{p-1}e^{-i\hat{f}s}\Re(e^{-i\hat{f}s}\overline{u}) + \mathfrak{b} + e^{-i\hat{f}s}\underline{v}; \\ \\ u = 0 \qquad \text{on } \partial D_{L,\varepsilon}. \end{cases}$$

Using the same procedure as above, write $\hat{\mathfrak{c}}(s,y) = \sum_j \hat{\mathfrak{c}}_j(y)e^{ij\varepsilon s}$ and $u(s,y) = \sum_j u_j(y)e^{ij\varepsilon s}$.

We consider now the function $U^{p-1}u_j$: by (7), if we choose $\varsigma' + \frac{(p-1)\hat{k}}{\sqrt{\hat{V}}} > \varsigma$, it follows from the above estimates that $\|\hat{\mathfrak{c}}\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)}$ is finite and that

$$\sum_j \|\hat{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau}^2 \leq \frac{C}{\varepsilon}\|\mathfrak{b}\|_{L^2(\overline{C}_\varsigma^\tau)}^2.$$

Moreover u_j satisfies

$$\begin{cases} -\Delta u_j + \left(1 + \frac{\varepsilon^2 j^2}{\hat{V}}\right)u_j = \hat{\mathfrak{c}}_j & \text{in } B_{\varepsilon^{-\delta}+1}(0), \\ u_j = 0 & \text{on } \partial B_{\varepsilon^{-\delta}+1}(0). \end{cases}$$

Also, it is easy to show

$$\begin{aligned} &\|u\|_{L^2(\overline{C}_\varsigma^{2,\tau})}^2 + \|u\|_{H^1(\overline{C}_\varsigma^{1,\tau})}^2 + \|u\|_{H^2(\overline{C}_\varsigma^\tau)}^2 \\ (95) \quad &= \frac{1}{\varepsilon}\sum_j \left[\|u_j\|_{\overline{C}_\varsigma^{2,\tau}}^2 + (1+\varepsilon^2 j^2)\|u_j\|_{\overline{C}_\varsigma^{1,\tau}}^2 + (1+\varepsilon^2 j^2 + \varepsilon^4 j^4)\|u_j\|_{\overline{C}_\varsigma^\tau}^2\right], \end{aligned}$$

and therefore we are reduced to find estimate $\|u_j\|_{\overline{C}_\varsigma^{2,\tau}}^2$, $\|u_j\|_{\overline{C}_\varsigma^{1,\tau}}^2$ and $\|u_j\|_{\overline{C}_\varsigma^\tau}^2$, done in the next step.

Step 2: proof concluded. We now set $a_j = 1 + \frac{\varepsilon^2 j^2}{\hat{V}}$, and $v_j(y) = u_j\left(\frac{y}{\sqrt{a_j}}\right)$. Then, from a change of variables we have the equation

$$\begin{cases} -\Delta v_j(y) + v_j(y) = \hat{F}_j(y) := \frac{1}{a_j}\hat{\mathfrak{c}}_j\left(\frac{y}{\sqrt{a_j}}\right) & \text{in } B_{\sqrt{a_j}(\varepsilon^{-\delta}+1)}(0), \\ v_j = 0 & \text{on } \partial B_{\sqrt{a_j}(\varepsilon^{-\delta}+1)}(0). \end{cases}$$

Notice that $a_j > 0$ stays bounded from below independently of j, and therefore by a scaling argument (and some elementary inequalities) one finds

$$
\sup_{y,z \in B_1(x)} |\hat{F}_j(y) - \hat{F}_j(z)| = \frac{1}{a_j} \sup_{y,z \in B_{1/\sqrt{a_j}}(x)} \left| \hat{\mathfrak{c}}_j(y/\sqrt{a_j}) - \hat{\mathfrak{c}}_j(z/\sqrt{a_j}) \right|
$$

$$
(96) \qquad \leq \frac{C}{a_j} \sup_{y,z \in B_1(x)} \left| \hat{\mathfrak{c}}_j(y/\sqrt{a_j}) - \hat{\mathfrak{c}}_j(z/\sqrt{a_j}) \right|
$$

$$
\leq \frac{C \|\hat{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau}}{a_j^{1+\frac{\tau}{2}}} \sup_{y,z \in B_1(x)} |y - z|^\tau e^{-\frac{\varsigma |x|}{\sqrt{a_j}}}
$$

where C depends on τ only, and hence we get

$$
(97) \qquad \|\hat{F}_j\|_{\overline{C}_{\frac{\varsigma}{\sqrt{a_j}}}^\tau} \leq \frac{C}{a_j^{1+\frac{\tau}{2}}} \|\hat{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau}.
$$

Now Lemma 2.12 implies that $\|v_j\|_{\overline{C}_{\varsigma/\sqrt{a_j}}^{2,\tau}} \leq \frac{C}{a_j^{1+\frac{\tau}{2}}} \|\hat{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau}$. From this estimate, we will obtain next some control on u_j by scaling back the variables.

We consider an arbitrary $x \in \mathbb{R}^{n-1}$: similarly as before we have

$$
\sup_{y,z \in B_1(x)} \frac{|u_j(y) - u_j(z)|}{|y - z|^\tau} = \sup_{y,z \in B_1(x)} \frac{|v_j(\sqrt{a_j} y) - v_j(\sqrt{a_j} z)|}{|y - z|^\tau}.
$$

Since a_j can be arbitrarily large, we cannot evaluate the difference $v_j(\sqrt{a_j} y) - v_j(\sqrt{a_j} z)$ directly using the weighted norm in the definition (87) (as we did for the first inequality in (96)), since the two points $\sqrt{a_j} y$ and $\sqrt{a_j} z$ might not belong to the same unit ball. We avoid this problem choosing $[\sqrt{a_j}]$ (the integer part of $\sqrt{a_j}$) points $(y^l)_l$ lying on the segment $[\sqrt{a_j} y, \sqrt{a_j} z]$ at equal distance one from each other, and using the triangular inequality. Now the distance of two consecutive points y^l and y^{l+1} will stay uniformly bounded from above, and the minimal norm of the y^l's is bounded from below by $C^{-1} \sqrt{a_j}(|x| - 1)$. Therefore, adding $[\sqrt{a_j}]$ times the inequality and using (97) we obtain

$$
\sup_{y,z \in B_1(x)} \frac{|u_j(y) - u_j(z)|}{|y - z|^\tau} \leq \frac{C \sqrt{a_j}}{|y - z|^\tau} \frac{C}{a_j^{1+\frac{\tau}{2}}} \left| \frac{y - z}{\sqrt{a_j}} \right|^\tau e^{-\sqrt{a_j} \varsigma \frac{(|x|-1)}{\sqrt{a_j}}} \|\hat{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau}
$$

$$
\leq \frac{C}{a_j^{1+\tau-\frac{1}{2}}} e^{-\varsigma |x|} \|\hat{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau} \leq \frac{C}{a_j} e^{-\varsigma |x|} \|\hat{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau} \leq \frac{C}{a_j} e^{-\varsigma |x|} \|\tilde{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau},
$$

since we chose $\tau \geq \frac{1}{2}$ and since a_j is uniformly bounded from below. Similarly, taking first and second derivatives we find that

$$
\sup_{y,z \in B_1(x)} \frac{|\nabla u_j(y) - \nabla u_j(z)|}{|y - z|^\tau} \leq \frac{C}{\sqrt{a_j}} e^{-\varsigma |x|} \|\tilde{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau};
$$

$$
\sup_{y,z \in B_1(x)} \frac{|\nabla^2 u_j(y) - \nabla^2 u_j(z)|}{|y - z|^\tau} \leq C e^{-\varsigma |x|} \|\tilde{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau}
$$

where, again, C depends only on τ. Recalling that $a_j = \hat{V} + \varepsilon^2 j^2$, we have in this way proved that

$$
\|u_j\|_{\overline{C}_\varsigma^{2,\tau}}^2 \leq C \|\tilde{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau}^2; \qquad \|u_j\|_{\overline{C}_\varsigma^{1,\tau}}^2 \leq \frac{C}{1 + \varepsilon^2 j^2} \|\tilde{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau}^2; \qquad \|u_j\|_{\overline{C}_\varsigma^\tau}^2 \leq \frac{C}{(1 + \varepsilon^2 j^2)^2} \|\tilde{\mathfrak{c}}_j\|_{\overline{C}_\varsigma^\tau}^2.
$$

Now the conclusion follows from (93), (95), the last formula and the fact that

$$
\|\tilde{\mathfrak{c}}\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)} \leq C \inf_{v \in \hat{K}_\delta} \|\mathfrak{b} + v\|_{\overline{L}^2(\overline{C}_\varsigma^\tau)},
$$

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see the beginning of Step 1. ∎

We next consider the operator L_ε in $\tilde{D}_\varepsilon$, see (24), acting on a suitable subset of $H_{\tilde{D}_\varepsilon}$ (verifying an orthogonality condition similar to (68)). We want to allow some flexibility in the choice of approximate solutions: to do this we consider a normal section Φ to γ which verifies the following two conditions

$$(98) \qquad \Phi \in span\left\{ h^{\frac{p+1}{4}}\varphi_j \;:\; j = 0, \dots, \frac{\delta}{\varepsilon} \right\}; \qquad \|\Phi\|_{H^2(0,L)} \le c_1 \varepsilon.$$

Here $(\varphi_j)_j$ are as in (55), while c_1 is a large constant to be determined later. Notice that by (98) we have $\|\Phi'''\|_{L^2[0,L]} \le C$, which implies $\|\Phi''\|_{L^\infty} \le C$, so also (21) holds true. This will allow us, in the next section, to apply Proposition 2.1. Next, we define the variables

$$(99) \qquad z = y - \Phi(\varepsilon s).$$

In the above coordinates (s, z), we will consider the approximate solution

$$(100) \qquad \tilde{\psi}_\varepsilon = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \eta_\varepsilon \left(h(\varepsilon s) U(k(\varepsilon s) z) + U_1(s, z) \right) := \tilde{\psi}_{0,\varepsilon} + \hat{\psi}_\varepsilon,$$

where $\varepsilon s = \overline{s}$, and where $\tilde{f}, U_1$ satisfy, for some fixed $C > 0$ and $\tau \in (0, 1)$

$$(101) \qquad \|\tilde{f} - f\|_{H^2([0,L])} \le C\varepsilon^2; \qquad |U_1|(s, z) \le C\varepsilon(1 + |z|^C) e^{-k(\overline{s})|z|};$$

$$(102) \qquad \| \,|hU(k\cdot) + U_1|^{p-1} - |hU(k\cdot)|^{p-1}\|_{C^\tau} \le C\varepsilon \qquad \text{in } \tilde{D}_\varepsilon.$$

With this choice of $\tilde{\psi}_\varepsilon$, we are going to study the analogue of Lemma 2.13 for L_ε, see (24), using a perturbation method.

To state our final result we need to introduce some more notation. Recalling the definition in (87), still using the coordinates (s, z), for $\tau \in (0, 1)$ and $\varsigma > 0$ we define the function space

$$(103) \qquad L^2(C^{m,\tau}_{\varsigma,V}) = \left\{ u : \tilde{D}_\varepsilon \to \mathbb{C} \;:\; s \mapsto u(s, \cdot/\sqrt{V(\varepsilon s)}) \in L^2([0, L/\varepsilon]; C^{m,\tau}_{\varsigma,1}) \right\}.$$

Also, for $m \in \mathbb{N}$, we define similarly

$$(104) \qquad H^l(C^{m,\tau}_{\varsigma,V}) = \left\{ u : \tilde{D}_\varepsilon \to \mathbb{C} \;:\; s \mapsto u(s, \cdot/\sqrt{V(\varepsilon s)}) \in H^l([0, L/\varepsilon]; C^{m,\tau}_{\varsigma,1}) \right\}.$$

We next let $\tilde{K}_\delta$ be the counterpart of K_δ (see (67)), when we replace the coordinates y by z. Finally, we denote by $\overline{H}_\varepsilon$ the following subspace of functions

$$(105) \qquad \overline{H}_\varepsilon := \left\{ \phi \in H_{\tilde{D}_\varepsilon} \;:\; \Re \int_{\tilde{D}_\varepsilon} e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} v\,\overline{\phi} = 0 \qquad \text{for all } v \in \tilde{K}_\delta \right\}.$$

Defining

$$(106) \qquad \| \cdot \|_{\varsigma,V} := \| \cdot \|_{L^2(C^{2,\tau}_{\varsigma,V})} + \| \cdot \|_{H^1(C^{1,\tau}_{\varsigma,V})} + \| \cdot \|_{H^2(C^\tau_{\varsigma,V})},$$

we have then the following result (recall the definition of $\tilde{D}_\varepsilon$ in (27)).

Proposition 2.14 *Suppose $0 < \varsigma < 1$ and $\frac{1}{2} \le \tau < 1$. Suppose $\tilde{\psi}_\varepsilon$ is as in (100), with $\tilde{f}, U_1$ satisfying (101). Then, if $K^2(\varepsilon s) = V(\varepsilon s)$, if $\mathcal{A}$ and δ are sufficiently small, in the limit $\varepsilon \to 0$ the following property holds: for any function $b \in L^2(C^\tau_{\varsigma,V})$ there exist $\tilde{u} \in \overline{H}_\varepsilon$, and $\underline{\tilde{v}} \in \tilde{K}_\delta$ such that*

$$(107) \qquad \begin{cases} -\Delta_{g_\varepsilon}\tilde{u} + V(\varepsilon x)\tilde{u} - |\tilde{\psi}_\varepsilon|^{p-1}\tilde{u} - (p-1)|\tilde{\psi}_\varepsilon|^{p-3}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\overline{\tilde{u}}) = b + e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\underline{\tilde{v}} & \text{in } \tilde{D}_\varepsilon \\ \tilde{u} = 0 & \text{on } \partial\tilde{D}_\varepsilon \end{cases}$$

is solvable, and such that for every $\varsigma' < \varsigma$ there exists some $C > 0$ for which we have the estimates

$$(108) \qquad \|\tilde{u}\|_{\varsigma',V} \le \frac{C}{\delta^2} \inf_{\tilde{v} \in \tilde{K}_\delta} \|b + e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\tilde{v}\|_{L^2(C^\tau_{\varsigma,V})}; \qquad \|\underline{\tilde{v}}\|_{L^2(C^\tau_{\varsigma,V})} \le C\|b\|_{L^2(C^\tau_{\varsigma,V})}.$$

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PROOF. We divide the proof into two steps.

Step 1: solvability of (107). First of all we notice that, from Proposition 2.9 and from elliptic regularity results, if H_ε denotes the subspace of function in $H^2(N\gamma_\varepsilon)$ satisfying (68), then the operator L_ε^1 is invertible from $(H_\varepsilon, \|\cdot\|_{H^2(N\gamma_\varepsilon)})$ onto $(\Pi_\varepsilon L^2(N\gamma_\varepsilon), \|\cdot\|_{L^2(N\gamma_\varepsilon)})$; moreover the norm of the inverse operator is bounded by $\frac{C}{\delta^2}$.

By the comments at the beginning of the proof of Lemma 2.13, we also deduce the following property. Given $b \in L^2\left(\{|y| \leq (\varepsilon^{-\overline{\delta}}+1)/K(\varepsilon s)\}\right)$ there exist $u \in H^2\left(\{|y| \leq (\varepsilon^{-\overline{\delta}}+1)/K(\varepsilon s)\}\right)$ and $\underline{v} \in K_\delta$ such that

$$
\begin{aligned}
\tilde{L}_\varepsilon u \quad := \quad & -\Delta_{\hat{g}_\varepsilon} u + V(\varepsilon s) u - h(\varepsilon s)^{p-1} U(k(\varepsilon s)y)^{p-1} u \\
(109) \quad - \quad & (p-1)h(\varepsilon s)^{p-1} U(k(\varepsilon s)y)^{p-1} e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \Re(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\overline{u}) = b + e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\underline{v} \qquad \text{in } \{|y| \leq (\varepsilon^{-\overline{\delta}}+1)/K\};
\end{aligned}
$$

$$
\Re \int_{\{|y| \leq (\varepsilon^{-\overline{\delta}}+1)/K\}} \overline{u} e^{-i\frac{f(\varepsilon s)}{\varepsilon}} v \, dV_{\hat{g}_\varepsilon} = 0 \qquad \text{for every } v \in K_\delta.
$$

Again, we have the estimates
(110)
$$
\|u\|_{H^2(\{|y|\leq(\varepsilon^{-\overline{\delta}}+1)/K\})} \leq \frac{C}{\delta^2}\|b\|_{L^2(\{|y|\leq(\varepsilon^{-\overline{\delta}}+1)/K\})}; \quad \|\underline{v}\|_{L^2(\{|y|\leq(\varepsilon^{-\overline{\delta}}+1)/K\})} \leq C\|b\|_{L^2(\{|y|\leq(\varepsilon^{-\overline{\delta}}+1)/K\})}.
$$

Using a perturbative argument, we show that we can recover the same invertibility result for (107) where, compared to (109), we need to substitute y with z, $\Delta_{\hat{g}_\varepsilon}$ with $\Delta_{\tilde{g}_\varepsilon}$, f with $\tilde{f}$ and $e^{-i\frac{f(\varepsilon s)}{\varepsilon}}hU$ with $\tilde{\psi}_\varepsilon$.

In fact, let us denote by Π_y and Π_z the orthogonal projections in L^2 onto the orthogonal complements of the sets $\{e^{-i\frac{f(\varepsilon s)}{\varepsilon}}v \ : \ v \in K_\delta\}$, $\{e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}v \ : \ v \in \tilde{K}_\delta\}$ with respect to the scalar products induced by the metrics $\hat{g}_\varepsilon$ and $\tilde{g}_\varepsilon$ respectively. By (98), Lemma 3.1 in [10] and (101) for every $u \in H^2(\tilde{D}_\varepsilon)$ and every $b \in L^2(\tilde{D}_\varepsilon)$ one has

$$
\|L_\varepsilon u - \tilde{L}_\varepsilon u\|_{L^2(\tilde{D}_\varepsilon)} \leq C(c_1)\varepsilon\|u\|_{H^2(\tilde{D}_\varepsilon)}; \qquad \|\Pi_y b - \Pi_z b\|_{L^2(\tilde{D}_\varepsilon)} \leq C(c_1)\varepsilon\|b\|_{L^2(\tilde{D}_\varepsilon)},
$$

where $C(c_1)$ is a positive constant which depends on γ, V and the constant c_1 in (98).

From (110) and the last formula we deduce the solvability of (107), together with the estimates $\|\tilde{u}\|_{H^2(\tilde{D}_\varepsilon)} \leq \frac{C}{\delta^2}\|b\|_{L^2(\tilde{D}_\varepsilon)}$ and $\|\tilde{\underline{v}}\|_{L^2(\tilde{D}_\varepsilon)} \leq C\|b\|_{L^2(\tilde{D}_\varepsilon)}$.

Step 2: proof of (108). Recall that the coordinates y (see the beginning of this section) are not global, since they are defined locally in s by normal parallel transport: the same holds of course for the coordinates z. Therefore, if we prolong the z's along γ_ε, there will be a discontinuity between 0 and L/ε.

To reduce ourselves to the periodic case, as in Lemma 2.13, we apply a rotation $R_\varepsilon = R_\varepsilon(\varepsilon s)$ to the z axes which makes the coordinates $\tilde{z} := R(\varepsilon s)z$ periodic in s. To compute the Laplace-Beltrami operator in the new coordinates $\tilde{z}$ one should apply the chain rule in this way

$$
\partial_{z_j} u = (R_\varepsilon)_{jl}\partial_{\tilde{z}_l} u; \qquad \partial^2_{sz_j} u = \varepsilon\partial_{\overline{s}}(R_\varepsilon)_{jl}\partial_{\tilde{z}_l} u + (R_\varepsilon)_{jl}\partial^2_{s\tilde{z}_l} u; \qquad \partial^2_{z_j z_l} u = R_{mj}R_{tl}\partial^2_{\tilde{z}_m\tilde{z}_t} u.
$$

In particular, since R_ε is orthogonal, $\partial^2_{z_j z_j} u = R_{mj}R_{tj}\partial^2_{\tilde{z}_m\tilde{z}_t} u = (R_\varepsilon)_{mj}(R_\varepsilon^{-1})_{jt}\partial^2_{\tilde{z}_m\tilde{z}_t} u = \partial^2_{\tilde{z}_m\tilde{z}_m} u$, namely the main term in the Laplacian stays invariant. Taking into account Lemma 3.1 in [10] and the last formulas, for $\varsigma'' \in (\varsigma', \varsigma)$ one finds

$$
(111) \qquad \|\Delta^{\tilde{z}}_{\tilde{g}_\varepsilon} u - \Delta^z_{\tilde{g}_\varepsilon} u\|_{L^2(C^\tau_{\varsigma'',k})} \leq C(c_1)\varepsilon\|u\|_{\varsigma'',V}.
$$

We use next a localization argument as in the proof of Proposition 2.9. If $\hat{s}_j$ and $\chi_{\eta,j}$ are as in that proof, by (101) we can find $\hat{\theta}_j \in \mathbb{R}$ such that $\frac{\tilde{f}(\varepsilon s)}{\varepsilon} - \hat{f}_j s - \hat{\theta}_j = O(\sqrt{\varepsilon})$ in the support of $\chi_{\eta,j}$. If we set

$\Delta_{\mathbb{R}^n}^{(s,\tilde{z})} = \Delta_{\mathbb{R}^{n-1}}^{\tilde{z}} + \partial_{ss}^2$, and if we scale the $\tilde{z}$ variables by $K(\varepsilon s) = \sqrt{V(\varepsilon s)}$, the function $\chi_{\eta,j}(s)u(s,\tilde{z})$ (which is now periodic in s) satisfies the equation

$$
\begin{cases}
-\frac{1}{\hat{V}_j}\partial_{ss}^2 \chi_{\eta,j}u - \Delta_{\mathbb{R}^{n-1}}^{\tilde{z}}(\chi_{\eta,j}u) + \chi_{\eta,j}u - \frac{\hat{h}_j^{p-1}}{\hat{V}_j}U\big(\tilde{z}\hat{k}_j/\sqrt{\hat{V}_j}\big)^{p-1}\chi_{\eta,j}u \\
-(p-1)\frac{\hat{h}_j^{p-1}}{\hat{V}_j}U\big(\tilde{z}\hat{k}_j/\sqrt{\hat{V}_j}\big)^{p-1}e^{-i(\hat{f}s+\hat{\theta}_j)}\Re(e^{-i(\hat{f}s+\hat{\theta}_j)}\chi_{\eta,j}\overline{u}) = \mathcal{F}_j & \text{in } \{|\tilde{z}| \leq (\varepsilon^{-\overline{\delta}}+1)/K\}; \\
\chi_{\eta,j}u = 0 & \text{on } \{|\tilde{z}| = (\varepsilon^{-\overline{\delta}}+1)/K\},
\end{cases}
$$

where

$$
\begin{aligned}
\mathcal{F}_j &= \frac{1}{V(\overline{s})}\chi_{\eta,j}e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}(b+\underline{v}) + \frac{1}{V(\overline{s})}(\Delta_{\tilde{g}_\varepsilon}^{\tilde{z}} - \Delta_{\mathbb{R}^n}^{(s,\tilde{z})})\chi_{\eta,j}u - \frac{1}{V(\overline{s})}(2\nabla_{\tilde{g}_\varepsilon}^{\tilde{z}}u\cdot\nabla_{\tilde{g}_\varepsilon}^{\tilde{z}}\chi_{\eta,j} + u\Delta_{\tilde{g}_\varepsilon}^{\tilde{z}}\chi_{\eta,j}) \\
&+ \frac{1}{V(\overline{s})}(\hat{V}_j - V)\chi_{\eta,j}u + \frac{1}{V(\overline{s})}|\tilde{\psi}_\varepsilon|^{p-1}\chi_{\eta,j}u + \frac{1}{V(\overline{s})}(p-1)|\tilde{\psi}_\varepsilon|^{p-2}\tilde{\psi}_\varepsilon\Re(\tilde{\psi}_\varepsilon\chi_{\eta,j}\overline{u}) \\
&- \frac{\hat{h}_j^{p-1}}{\hat{V}_j}U\big(\tilde{z}\hat{k}_j/\sqrt{\hat{V}_j}\big)^{p-1}\chi_{\eta,j}u - (p-1)\frac{\hat{h}_j^{p-1}}{\hat{V}_j}U\big(\tilde{z}\hat{k}_j/\sqrt{\hat{V}_j}\big)^{p-1}e^{-i(\hat{f}s+\hat{\theta}_j)}\Re(e^{-i(\hat{f}s+\hat{\theta}_j)}\chi_{\eta,j}\overline{u}).
\end{aligned}
$$

In the last formula, the functions $b, \underline{v}$, V and $\tilde{\psi}_\varepsilon$ are intended scaled in $\tilde{z}$ by $\sqrt{V(\varepsilon s)}$. Reasoning as for (80), from (105) one finds that $\int_{\tilde{D}_\varepsilon}e^{-i\hat{f}_js-\hat{\theta}_j}\hat{v}\chi_{\eta,j}\overline{\phi} = O(\delta^2 + \sqrt{\varepsilon})\|\phi\|_{L^2(supp(\chi_{\eta,j}))}\|\hat{v}\|_{L^2(\tilde{D}_\varepsilon)}$ for every $\hat{v} \in \hat{K}_\delta$. Moreover, as for (111) one can show that

$$
\|(\Delta_{\tilde{g}_\varepsilon}^{\tilde{z}} - \Delta_{\mathbb{R}^n}^{(s,\tilde{z})})\chi_{\eta,j}u\|_{\overline{L}^2(\overline{C}_{\varsigma''}^\tau)} \leq C(c_1)\varepsilon\left(\|\chi_{\eta,j}u\|_{\overline{L}^2(\overline{C}_{\varsigma''}^{2,\tau})} + \|\chi_{\eta,j}u\|_{\overline{H}^1(\overline{C}_{\varsigma''}^{1,\tau})} + \|\chi_{\eta,j}u\|_{\overline{H}^2(\overline{C}_{\varsigma''}^\tau)}\right).
$$

Therefore, using Lemma 2.13, (101) and (102) we obtain the estimate

$$
\begin{aligned}
\|\chi_{\eta,j}u\|_{\overline{L}^2(\overline{C}_{\varsigma''}^{2,\tau})} + \|\chi_{\eta,j}u\|_{\overline{H}^1(\overline{C}_{\varsigma''}^{1,\tau})} + \|\chi_{\eta,j}u\|_{\overline{H}^2(\overline{C}_{\varsigma''}^\tau)} &\leq \frac{C}{\delta^2}\|\mathcal{F}_j\|_{\overline{L}^2(\overline{C}_{\varsigma''}^\tau)} \\
&\leq \frac{C}{\delta^2}\|\chi_{\eta,j}b\|_{\overline{L}^2(\overline{C}_{\varsigma''}^\tau)} + \frac{C}{\delta^2}\|\chi_{\eta,j}\underline{v}\|_{\overline{L}^2(\overline{C}_{\varsigma''}^\tau)}
\end{aligned}
$$

$$
\text{(112)} \qquad +C\sqrt{\varepsilon}\left(\|u\|_{\overline{L}^2(\overline{C}_{\varsigma''}^{2,\tau},supp(\chi_{\eta,j}))} + \|u\|_{\overline{H}^1(\overline{C}_{\varsigma''}^{1,\tau},supp(\chi_{\eta,j}))} + \|u\|_{\overline{H}^2(\overline{C}_{\varsigma''}^\tau,supp(\chi_{\eta,j}))}\right),
$$

where the last symbols denote the restrictions of the weighted norms to $supp(\chi_{\eta,j})$. Recall that the functions in the previous formula have been scaled in $\tilde{z}$ by $\sqrt{V(\varepsilon s)}$: therefore, from the uniform continuity of $V(\overline{s})$, for some $C > 0$ we have (recall that $\varsigma'' \in (\varsigma', \varsigma)$)

$$
\frac{1}{C}\|\chi_{\eta,j}u\|_{\varsigma',V} \leq \|\chi_{\eta,j}u(\cdot,\sqrt{V(\overline{s})}\cdot)\|_{\overline{L}^2(\overline{C}_{\varsigma''}^{2,\tau})} + \|\chi_{\eta,j}u(\cdot,\sqrt{V(\overline{s})}\cdot)\|_{\overline{H}^1(\overline{C}_{\varsigma''}^{1,\tau})} + \|\chi_{\eta,j}u(\cdot,\sqrt{V(\overline{s})}\cdot)\|_{\overline{H}^2(\overline{C}_{\varsigma''}^\tau)}.
$$

A similar inequality holds for the restriction of u to the support of $\chi_{\eta,j}$, together with

$$
\|\chi_{\eta,j}b(\cdot,\sqrt{V(\overline{s})}\cdot)\|_{\overline{L}^2(\overline{C}_{\varsigma''}^\tau)} + \|\chi_{\eta,j}\underline{v}(\cdot,\sqrt{V(\overline{s})}\cdot)\|_{\overline{L}^2(\overline{C}_{\varsigma''}^\tau)} \leq C\left(\|\chi_{\eta,j}b\|_{L^2(C_{\varsigma,V}^\tau)} + \|\chi_{\eta,j}\underline{v}\|_{L^2(C_{\varsigma,V}^\tau)}\right).
$$

Using the last two inequalities, taking the square of (112) and summing over j, we can bring the last term in the right-hand side to the left, so we get (108). ∎

3 Approximate solutions

In this section we construct some approximate solutions to (14) which depend on suitable parameters, and find rigorous estimates on the error terms.

As in the previous subsection, we let y be a system of Fermi coordinates in $N\gamma_\varepsilon$, and for a normal section Φ of $N\gamma_\varepsilon$ of class H^2 we define the coordinates (see (99))

$$
z = y - \Phi(\varepsilon s), \qquad z \in \mathbb{R}^{n-1}.
$$

By the results in Subsection 2.2, we will restrict our attention to the set $\tilde{D}_\varepsilon$.

Remark 3.1 *In the spirit of Proposition 2.4, we will work with approximate solutions $\tilde{\psi}_\varepsilon$ supported in $\tilde{D}_\varepsilon$. Therefore, using the above coordinates, $\tilde{\psi}_\varepsilon(s,z)$ has to vanish for $|z|$ sufficiently large. This can be achieved by defining formally $\tilde{\psi}_\varepsilon(s,z)$ on $N\gamma_\varepsilon$, and multiplying it by a cutoff function η_ε as in Subsection 2.2. However, since the functions we are dealing with decay exponentially to zero as $|z| \to \infty$, the effect of this cutoff on the expansions below is exponentially small in ε, and it will turn out to be negligible for our purposes. Therefore, for reasons of brevity and clarity, we will tacitely assume that $\tilde{\psi}_\varepsilon(s,z)$ is multiplied by such a cutoff, without writing it explicitly.*

Recall that in (23) we defined

$$S_\varepsilon(\psi) = -\Delta_g \psi + V(\varepsilon x)\psi - |\psi|^{p-1}\psi.$$

We set $\tilde{f}_0(\overline{s}) = f(\overline{s}) + \varepsilon f_1(\overline{s})$, where f is given in (11) and f_1 (depending on Φ and $\mathcal{A}$) was defined at the end of Subsection 4.1 in [10]), in order to satisfy the equation

$$(113) \qquad \partial_{\overline{s}} \left(\frac{h^2 f_1'}{(p-1)k^{n+1}} \left[(p-1)h^{p-1} - 2\sigma \mathcal{A}^2 h^{2\sigma} \right] \right) = 2\mathcal{A} \left(\frac{p-1}{2\theta} - 1 \right) \partial_{\overline{s}} \langle \mathbf{H}, \Phi \rangle.$$

This equation is indeed solvable explicitly and the solution is given by

$$(114) \qquad f_1' = \frac{2\mathcal{A}(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma \mathcal{A}^2 h^{2\sigma+2}} \left(\frac{p-1}{2\theta} - 1 \right) \langle \mathbf{H}, \Phi \rangle + \mathcal{A}' \frac{(p-1)k^{n+1}}{(p-1)h^{p+1} - 2\sigma \mathcal{A}^2 h^{2\sigma+2}},$$

(we refer to [10] for the definition of $\mathcal{A}'$). If w_r and w_i are smooth functions of $\overline{s}$ and z we have, formally

$$S_\varepsilon \left(e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} (hU(kz) + \varepsilon(w_r + iw_i)) \right) = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} \left(\varepsilon(\mathcal{R}_r + i\mathcal{R}_i) \right) + o(\varepsilon),$$

for some quantities $\mathcal{R}_r, \mathcal{R}_i$ given in Subsection 3.2 of [10] (where we refer to also for the derivation of the last formula). $\mathcal{R}_r$ and $\mathcal{R}_i$ can be written as $\mathcal{R}_r = \mathcal{L}_r w_r - \mathcal{F}_r$, $\mathcal{R}_i = \mathcal{L}_i w_i - \mathcal{F}_i$, where

$$(115) \qquad \mathcal{F}_r = -2f' f_1' hU - 2f'^2 hU(kz)\langle \mathbf{H}, z + \Phi \rangle - hk\langle \mathbf{H}, \nabla U(kz) \rangle - \langle \nabla^N V, z + \Phi \rangle hU(kz);$$

$$(116) \qquad \mathcal{F}_i = -\left[f'' hU(kz) + 2f' h' U(kz) + 2f' hk' \nabla U(kz) \cdot z \right] + 2\sum_j [\Phi_j' f' hk \partial_j U(kz)],$$

and where the operators $\mathcal{L}_r$, $\mathcal{L}_i$ are defined in (15). Therefore, for canceling the errors of order ε we require w_r and w_i to be formally determined by the equations $\mathcal{L}_r w_r = \mathcal{F}_r$, $\mathcal{L}_i w_i = \mathcal{F}_i$.

Dividing the right-hand sides of (115) and (116) into their even and odd parts (in the variables z), we obtained that $w_r = w_{r,e} + w_{r,o}$, and $w_i = w_{i,e} + w_{i,o}$, where

$$(117) \qquad w_{r,e} = \left[\frac{p-1}{\theta} h^p \langle \mathbf{H}, \Phi \rangle + 2f' f_1' h \right] \left(\frac{1}{(p-1)h^{p-1}} U(kz) + \frac{1}{2k} \nabla U(kz) \cdot z \right);$$

$$(118) \qquad w_{i,e} = \frac{p-1}{4} f' h' |z|^2 U(kz); \qquad w_{i,o} = -\sum_j \Phi_j' f' h z_j U(kz),$$

and where $w_{r,o}$ is given implicitly by the equation

$$(119) \qquad \mathcal{L}_r w_{r,o} = -2(f')^2 hU(kz)\langle \mathbf{H}, z \rangle - hk\sum_j H^j \partial_j U(kz) - \langle \nabla^N V, z \rangle hU(kz).$$

As noticed in Subsection 3.2 in [10] (see also the introduction here), the solvability of the last equation is guaranteed by the stationarity condition (12). Moreover, it is standard to check that $w_{r,o}$ has exponential decay in z, as for the other correction terms. Defining

$$\tilde{\psi}_{1,\varepsilon} = e^{-i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} (hU(kz) + \varepsilon w_r + i\varepsilon w_i),$$

from the expansions in Subsection 3.3 of [10] we can write that

$$
(120) \qquad
\begin{aligned}
e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}} S_\varepsilon(\psi_{1,\varepsilon}) \;=\;& \varepsilon^2(\tilde{R}_{r,e} + \tilde{R}_{r,o}) + \varepsilon^2(\tilde{R}_{r,e,f_1} + \tilde{R}_{r,o,f_1}) \\
+\;& \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}) + \varepsilon^2 i(\tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1}) + o(\varepsilon^2).
\end{aligned}
$$

In the last formula, the $\tilde{R}$'s represent the terms of order ε^2 appearing in the expansion, see Subsection 3.3 of [10], while $o(\varepsilon^2)$ stands for the terms which are formally of higher order. Here indeed we want to prove rigorous estimates, so we want to be careful in treating the latter term.

To allow some more flexibility in the choice of approximate solutions, we substitute the phase $\tilde{f}_0$ with the function $\tilde{f} = f + \varepsilon f_1 + \varepsilon^2 f_2$, where f_2 is some function of class H^2. On Φ and f_2 we assume the following conditions for some constants c_1, c_2 to be determined later

$$
(121) \qquad \|\Phi\|_{H^2} \leq c_1 \varepsilon; \qquad \|f_2\|_{H^2} \leq c_2.
$$

Moreover, letting δ be as in Subsections 2.3 and φ_j, ω_j as in (55), we also assume that

$$
(122) \qquad \Phi \in span\left\{ h^{\frac{p+1}{4}}\varphi_j \;:\; j = 0, \dots, \frac{\delta}{\varepsilon} \right\}; \qquad f_2 \in span\left\{ h^{\frac{1}{2}}\omega_j \;:\; j = 0, \dots, \frac{\delta}{\varepsilon} \right\}.
$$

To deal with the resonance phenomenon mentioned in the introduction, related to the components in $K_{3,\delta}$ of the approximate kernel, we add to the approximate solutions a function v_δ like

$$
(123) \qquad v_\delta = \beta(\varepsilon s)Z_{\alpha(\varepsilon s)} + i\xi(\varepsilon s)W_{\alpha(\varepsilon s)}
$$

(see (54) and the lines after), with β, ξ given by

$$
(124) \qquad \beta(\varepsilon s) = \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j \beta_j(\varepsilon s); \qquad \xi = \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j \xi_j(\varepsilon s),
$$

where, we recall, ξ_j solves (59) and is related to β_j by (60). Below, we will regard β as an independent variable, and ξ as a function of β. Introducing the norm

$$
(125) \qquad \|\beta\|_\sharp := \left(\sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j^2 (1 + |j|)^2 \right)^{\frac{1}{2}},
$$

we will assume later on that

$$
(126) \qquad \|\beta\|_\sharp \leq c_3 \varepsilon^2
$$

for some constant $c_3 > 0$ to be specified later.

We will look for approximate solutions of the form

$$
(127) \qquad \tilde{\Psi}_{2,\varepsilon}(s,z) := e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \left\{ h(\varepsilon s) U\left(k(\varepsilon s)z\right) + \varepsilon \left[w_r + i w_i\right] + \varepsilon^2 \tilde{v} + \varepsilon^2 v_0 + v_\delta \right\}.
$$

In this formula $\tilde{f}$ is as above, while $\tilde{v}$ and v_0 are corrections whose choice is given below, in order to improve the accuracy of the approximate solutions.

Our goal is to estimate with some accuracy the quantity $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$: for simplicity, to treat separately some terms in this expression, we will write $\tilde{\Psi}_{2,\varepsilon}$ as

$$
(128) \qquad \tilde{\Psi}_{2,\varepsilon}(s,z) = \tilde{\Psi}_{1,\varepsilon}(s,z) + E(s,z) + F(s,z) + G(s,z),
$$

where $\tilde{\Psi}_{1,\varepsilon}$, E, F and G are respectively defined by

$$\tilde{\Psi}_{1,\varepsilon}(s,z) := e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left\{ h(\varepsilon s)U\left(k(\varepsilon s)z\right) + \varepsilon\left[w_r + iw_i\right]\right\} := e^{-i\frac{\varepsilon^2 f_2(\varepsilon s))}{\varepsilon}}\tilde{\psi}_{1,\varepsilon};$$

$$E(s,z) := \varepsilon^2 e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\tilde{v}; \qquad F(s,z) := \varepsilon^2 e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}v_0; \qquad G(s,z) := e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}v_\delta.$$

To expand $S_\varepsilon(\Psi_{2,\varepsilon})$ conveniently, we can write

$$S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) = S_\varepsilon(\tilde{\Psi}_{1,\varepsilon}) + \mathfrak{A}_3 + \mathfrak{A}_4 + \mathfrak{A}_5 + \mathfrak{A}_6,$$

where $\mathfrak{A}_3, \ldots, \mathfrak{A}_6$ are respectively the linear terms in the equation which involve E, F and G (see (128)):

(129)
$$\mathfrak{A}_3 = -\Delta_g E + V(\varepsilon x)E - |\tilde{\Psi}_{1,\varepsilon}|^{p-1}E - (p-1)|\tilde{\Psi}_{1,\varepsilon}|^{p-3}\tilde{\Psi}_{1,\varepsilon}\Re(\tilde{\Psi}_{1,\varepsilon}\overline{E});$$

(130)
$$\mathfrak{A}_4 = -\Delta_g F + V(\varepsilon x)F - |\tilde{\Psi}_{1,\varepsilon}|^{p-1}F - (p-1)|\tilde{\Psi}_{1,\varepsilon}|^{p-3}\tilde{\psi}_{1,\varepsilon}\Re(\tilde{\Psi}_{1,\varepsilon}\overline{F});$$

(131)
$$\mathfrak{A}_5 = -\Delta_g G + V(\varepsilon x)G - |\tilde{\Psi}_{1,\varepsilon}|^{p-1}G - (p-1)|\tilde{\Psi}_{1,\varepsilon}|^{p-3}\tilde{\Psi}_{1,\varepsilon}\Re(\tilde{\Psi}_{1,\varepsilon}\overline{G}),$$

and where $\mathfrak{A}_6$ contains the contribution of the nonlinear part

(132)
$$\mathfrak{A}_6 = -|\tilde{\Psi}_{2,\varepsilon}|^{p-1}\tilde{\Psi}_{2,\varepsilon} + |\tilde{\Psi}_{1,\varepsilon}|^{p-1}(E+F+G) + (p-1)|\tilde{\Psi}_{1,\varepsilon}|^{p-3}\tilde{\Psi}_{1,\varepsilon}\Re(\tilde{\Psi}_{1,\varepsilon}(\overline{E}+\overline{F}+\overline{G})).$$

Next we also write (tautologically)

(133)
$$S_\varepsilon(\tilde{\Psi}_{1,\varepsilon}) = e^{-i\frac{\varepsilon^2 f_2(\varepsilon s)}{\varepsilon}}S_\varepsilon(\tilde{\psi}_{1,\varepsilon}) + \mathfrak{A}_1; \qquad \mathfrak{A}_1 = S_\varepsilon(\tilde{\Psi}_{1,\varepsilon}) - e^{-i\frac{\varepsilon^2 f_2(\varepsilon s)}{\varepsilon}}S_\varepsilon(\tilde{\psi}_{1,\varepsilon}),$$

and set

$$\mathfrak{A}_2 = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left(e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}S_\varepsilon(\tilde{\psi}_{1,\varepsilon}) - \varepsilon^2(\tilde{R}_{r,o} + \tilde{R}_{r,e}) - \varepsilon^2(\tilde{R}_{r,o,f_1} + \tilde{R}_{r,e,f_1})\right.$$

(134)
$$\left. - \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}) - \varepsilon^2 i(\tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1})\right),$$

so that $\mathfrak{A}_2$ represents the terms which are formally of order ε^3 and higher in $S_\varepsilon(\tilde{\psi}_{1,\varepsilon})$ (multiplied by a phase factor). Therefore, from the definitions (129)-(134) we find that

$$S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left(\varepsilon^2(\tilde{R}_{r,o} + \tilde{R}_{r,e}) + \varepsilon^2(\tilde{R}_{r,o,f_1} + \tilde{R}_{r,e,f_1})\right.$$

(135)
$$\left. + \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}) + \varepsilon^2 i(\tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1})\right) + \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3 + \mathfrak{A}_4 + \mathfrak{A}_5 + \mathfrak{A}_6.$$

To estimate rigorously the $\mathfrak{A}_i$'s, we display the first and second order derivatives of $\tilde{\Psi}_{2,\varepsilon}$

$$\partial_s\tilde{\Psi}_{2,\varepsilon} = -i\widetilde{f'}(\varepsilon s)e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left[h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon\left[w_r + iw_i\right] + \varepsilon^2\tilde{v} + \varepsilon^2 v_0 + v_\delta\right]$$

$$+ e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left[\varepsilon h'U(kz) + \varepsilon hk'\nabla U \cdot z + \varepsilon^2\partial_s w_r + i\varepsilon^2\partial_s w_i + \varepsilon^3\partial_s\tilde{v} + \varepsilon^3\partial_s v_0 + \partial_s v_\delta\right];$$

$$\partial_j\tilde{\Psi}_{2,\varepsilon} = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left[h(\varepsilon s)k(\varepsilon s)\partial_j U(k(\varepsilon s)z) + \varepsilon\left[\partial_j w_r + i\partial_j w_i\right] + \varepsilon^2\partial_j\tilde{v} + \varepsilon^2\partial_j v_0 + \partial_j v_\delta\right];$$

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$$\partial_{ss}^2 \tilde{\Psi}_{2,\varepsilon} = (-\tilde{f}'^2 - i\varepsilon\tilde{f}'')(\varepsilon s)e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left[h(\varepsilon s)U(k(\varepsilon s)z) + \varepsilon\left[w_r + iw_i\right] + \varepsilon^2\tilde{v} + \varepsilon^2 v_0 + a(\varepsilon s)Z(kz)\right]$$

$$+ \quad e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left[\varepsilon^2 h''U(kz) + 2\varepsilon^2 h'k'\nabla U \cdot z + \varepsilon^2 hk''\nabla U \cdot z + \varepsilon^2 hk'^2\nabla^2 U[z,z] + \varepsilon^3\partial_{ss}^2 w_r + i\varepsilon^3\partial_{ss}^2 w_i\right.$$

$$+ \quad \left.\varepsilon^4\partial_{ss}^2\tilde{v} + \varepsilon^4\partial_{ss}^2 v_0 + \partial_{ss}^2 v_\delta\right]$$

$$- \quad 2i\tilde{f}'(\varepsilon s)e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left[\varepsilon h'U(kz) + \varepsilon hk'\nabla U \cdot z + \varepsilon^2\partial_s w_r + i\varepsilon^2\partial_s w_i + \varepsilon^3\partial_s\tilde{v} + \varepsilon^3\partial_s v_0 + \partial_s v_\delta\right];$$

$$\partial_{jl}^2\tilde{\Psi}_{2,\varepsilon} = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left[h(\varepsilon s)k^2\partial_{jl}^2 U(k(\varepsilon s)z) + \varepsilon\left[\partial_{jl}^2 w_r + i\partial_{jl}^2 w_i\right] + \varepsilon^2\partial_{jl}^2\tilde{v} + \varepsilon^2\partial_{jl}^2 v_0 + \partial_{jl}^2 v_\delta\right];$$

$$\partial_{sj}^2\tilde{\Psi}_{2,\varepsilon} = -i\tilde{f}'(\varepsilon s)e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left[h(\varepsilon s)k\partial_j U(k(\varepsilon s)z) + \varepsilon\left[\partial_j w_r + i\partial_j w_i\right] + \varepsilon^2\partial_j\tilde{v} + \varepsilon^2\partial_j v_0\right.$$

$$+ \quad \left.a(\varepsilon s)k\partial_j Z(kz)\right] + e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\left[\varepsilon h'k\partial_j U(kz) + \varepsilon hk'\partial_j U + \varepsilon hkk'z_l\partial_{jl}^2 U + \varepsilon^2\partial_{sj}^2 w_r\right.$$

$$+ \quad \left.i\varepsilon^2\partial_{sj}^2 w_i + \varepsilon^3\partial_{sj}^2\tilde{v} + \varepsilon^3\partial_{sj}^2 v_0 + \partial_{sj}^2 v_\delta\right].$$

To simplify the expressions of the error terms, we introduce some convenient notation. For any positive integer q, the two symbols $\mathfrak{R}_q(\Phi,\Phi')$ and $\mathfrak{R}_q(\Phi,\Phi',\Phi'')$ will denote error terms satisfying the following bounds, for some fixed constants C, d (which depend on q, c_1, c_2, c_3 but not on ε, s and δ)

$$\begin{cases} |\mathfrak{R}_q(\Phi,\Phi')| \le C\varepsilon^q(1+|z|^d)e^{-k|z|}; \\ |\mathfrak{R}_q(\Phi,\Phi') - \mathfrak{R}_q(\tilde{\Phi},\tilde{\Phi}')| \le C\varepsilon^q(1+|z|^d)[|\Phi - \tilde{\Phi}| + |\Phi' - \tilde{\Phi}'|]e^{-k|z|}, \end{cases}$$

while the term $\mathfrak{R}_q(\Phi,\Phi',\Phi'')$ (which involves also second derivatives of Φ) stands for a quantity for which

$$|\mathfrak{R}_q(\Phi,\Phi',\Phi'')| \le C\varepsilon^q(1+|z|^d)e^{-k|z|} + C\varepsilon^{q+1}(1+|z|^d)e^{-k|z|}|\Phi''|;$$

$$|\mathfrak{R}_q(\Phi,\Phi',\Phi'') - \mathfrak{R}_q(\tilde{\Phi},\tilde{\Phi}',\tilde{\Phi}'')| \le C\varepsilon^q(1+|z|^d)[|\Phi - \tilde{\Phi}| + |\Phi' - \tilde{\Phi}'|]e^{-k|z|}$$

$$+ \quad C\varepsilon^{q+1}(1+|z|^d)\left(|\Phi'' + \tilde{\Phi}''|(|\Phi - \tilde{\Phi}| + |\Phi' - \tilde{\Phi}'|) + |\Phi'' - \tilde{\Phi}''|\right)e^{-k|z|}.$$

Similarly, we will let $\mathfrak{R}_q(\overline{s})$ denote a quantity (depending only on $\overline{s}$ and z) such that

$$|\mathfrak{R}_q(\overline{s})| \le C\varepsilon^q(1+|z|^d)e^{-k|z|},$$

and which depends smoothly on $\overline{s}$. In the estimates below, the assumptions (121)-(122) will be used: one hand by (121) we have L^∞ estimates on Φ, f_2 and their first derivatives; one the other by (122) we have L^2 estimates on the higher order derivatives, of the type $\|\Phi^{(l)}\|_{L^2} \le C_l\frac{\delta^l}{\varepsilon^l}\|\Phi\|_{L^2}$, for $l \in \mathbb{N}$.

We will also use notations like $\Phi\mathfrak{R}_q(\Phi,\Phi')$, $f_2''\mathfrak{R}_q(\Phi,\Phi')$, etc., to denote error terms which are products of functions of $\overline{s}$, like Φ or f_2', and the above $\mathfrak{R}_q$'s.

Having defined this notation, we can compute (and estimate) $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$ term by term.

● **Estimate of $\mathfrak{A}_1$**

From the expression of the Laplace-Beltrami operator (see Subsection 3.1 in [10]) it follows that

$$e^{i\frac{\varepsilon^2 f_2(\varepsilon s)}{\varepsilon}}S_\varepsilon(\tilde{\Psi}_{1,\varepsilon}) - S_\varepsilon(\tilde{\psi}_{1,\varepsilon}) = \tilde{g}^{11}\left[\varepsilon^4(f_2')^2\tilde{\psi}_{1,\varepsilon} + i\varepsilon^3 f_2''\tilde{\psi}_{1,\varepsilon} + 2i\varepsilon^2 f_2'\partial_s\tilde{\psi}_{1,\varepsilon}\right]$$

$$+ \quad 2i\sum_l \tilde{g}^{1l}\varepsilon^2 f_2'\partial_l\tilde{\psi}_{1,\varepsilon} + \frac{i}{\sqrt{\det\tilde{g}}}\partial_A\left(g^{A1}\sqrt{\det\tilde{g}}\right)\varepsilon^2 f_2'\tilde{\psi}_{1,\varepsilon}.$$

Using the expressions of w_r, w_i and the expansions of the metric coefficients in Subsection 3.1 of [10], multiplying the last equation by $e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}$ one obtains

$$e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\mathfrak{A}_1 = e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}\left(e^{i\frac{\varepsilon^2 f_2(\varepsilon s)}{\varepsilon}}S_\varepsilon(\tilde{\Psi}_{1,\varepsilon}) - S_\varepsilon(\tilde{\psi}_{1,\varepsilon})\right) = e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}S_\varepsilon(\tilde{\Psi}_{1,\varepsilon}) - e^{i\frac{\tilde{f}_0(\varepsilon s)}{\varepsilon}}S_\varepsilon(\tilde{\psi}_{1,\varepsilon})$$

(136)
$$= \mathfrak{A}_{1,0} + \tilde{\mathfrak{A}}_1 := \mathfrak{A}_{1,0} + \mathfrak{A}_{1,r,e} + \mathfrak{A}_{1,r,o} + \mathfrak{A}_{1,i,e} + \mathfrak{A}_{1,i,o} + \mathfrak{A}_{1,1},$$

where

$$\mathfrak{A}_{1,0} = 2\varepsilon^2 f' f_2' hU; \qquad \mathfrak{A}_{1,r,e} = \varepsilon^3 f_2' \left[2f_1' hU + 4\langle \mathbf{H}, \Phi \rangle f' hU + 2f' w_{r,e}\right],$$

$$\mathfrak{A}_{1,r,o} = \varepsilon^3 f_2' \left[4\langle \mathbf{H}, z \rangle f' hU + 2f' w_{r,o}\right];$$

(137) $\qquad \mathfrak{A}_{1,i,e} = i\varepsilon^3 f_2'' hU + 2i\varepsilon^3 f_2' \left[f' w_{i,e} + h' U + hk'\nabla U \cdot z\right]; \qquad \mathfrak{A}_{1,i,o} = 2i\varepsilon^3 f' f_2' w_{i,o}$

$$\mathfrak{A}_{1,1} = (f_2')^2 \mathfrak{R}_4(\Phi, \Phi') + f_2'' \mathfrak{R}_5(\Phi, \Phi') + f_2' \Phi'' \mathfrak{R}_4(\Phi, \Phi') + f_2' \mathfrak{R}_4(\Phi, \Phi').$$

● **Estimate of $\mathfrak{A}_2$**

Reasoning as for the previous estimate, collecting the terms of order ε^3 and higher in $S_\varepsilon(\tilde{\psi}_{1,\varepsilon})$, we obtain

(138) $$e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\mathfrak{A}_2 = \mathfrak{A}_{2,0} + \tilde{\mathfrak{A}}_2 := \mathfrak{A}_{2,r,e} + \mathfrak{A}_{2,r,o} + \mathfrak{A}_{2,i,e} + \mathfrak{A}_{2,i,o} + \mathfrak{A}_{2,1},$$

where $\mathfrak{A}_{2,0} = 0$ and where the remaining terms are given by

$$\mathfrak{A}_{2,r,e} = \varepsilon^3 \Phi'' F_e(\overline{s});$$

$$\mathfrak{A}_{2,r,o} = 2\varepsilon^3 hk\langle \mathbf{H}, \Phi \rangle \sum_j \Phi_j'' \partial_j U + \varepsilon^3 f' h f_1' \sum_j z_j \Phi_j'' U + 2\varepsilon^3 f'^2 h\langle \mathbf{H}, \Phi \rangle \sum_j \Phi_j'' z_j U;$$

$$\mathfrak{A}_{2,i,e} = -2i\varepsilon^3 f' h \sum_{j,l} \Phi_l' \Phi_j'' \partial_l(z_j U) + 2i\varepsilon^4 f' h\langle \mathbf{H}, z \rangle \sum_j \Phi_j''' z_j U;$$

$$\mathfrak{A}_{2,i,o} = i\varepsilon^3 \sum_j \Phi_j'' z_j \left(f'' hU + f' h' U + f' hk'\nabla U \cdot z\right) + i\varepsilon^3 \Phi''' F_o(\overline{s}) + 2i\varepsilon^4 f' h\langle \mathbf{H}, \Phi \rangle \sum_j \Phi_j''' z_j U;$$

$$\mathfrak{A}_{2,1} = \mathfrak{R}_3(\overline{s}) + (\Phi + \Phi')\mathfrak{R}_3(\Phi, \Phi', \Phi'') + \mathfrak{R}_4(\Phi, \Phi', \Phi''),$$

where $F_e(\overline{s})$ and $F_o(\overline{s})$ are respectively an even real function and an odd real function in the variables z, with smooth coefficients in $\overline{s} = \varepsilon s$, and satisfying the decay property $|F_e(\overline{s})| + |F_o(\overline{s})| \leq C(1 + |z|^d)e^{-k|z|}$.

● **Choice of $\tilde{v}$ and estimate of $\mathfrak{A}_3$**

We choose the function $\tilde{v}$ in such a way to annihilate (roughly) one of the main terms in (136), namely $2\varepsilon^2 f' f_2' hU(kz)$. Hence we define $\tilde{v}$ so that it solves

(139) $$\mathcal{L}_r \tilde{v} = -2f' f_2' hU(kz).$$

Reasoning as for the definition of w_r (see [10], Subsection 3.2), $\tilde{v}$ can be explicitly determined as

$$\tilde{v} = 2f' f_2' h\tilde{U}(kz).$$

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With this definition, using the above estimates on the metric coefficients and the expressions of error terms, the linear terms involving E in $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$ can be written as

$$(140) \qquad e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\mathfrak{A}_3 = \mathfrak{A}_{3,0} + \tilde{\mathfrak{A}}_3 := \mathfrak{A}_{3,0} + \mathfrak{A}_{3,r,e} + \mathfrak{A}_{3,r,o} + \mathfrak{A}_{3,i,e} + \mathfrak{A}_{3,i,o} + \mathfrak{A}_{3,1},$$

where

$$\mathfrak{A}_{3,0} = \varepsilon^2 \mathcal{L}_r \tilde{v};$$

$$\begin{aligned}
\mathfrak{A}_{3,r,e} &= 2\varepsilon^3 f' h f_2' \left(2(f')^2\langle \mathbf{H}, \Phi\rangle + \langle\nabla^N V, \Phi\rangle - p(p-1)h^{p-2}U^{p-2}w_{r,e}\right)\tilde{U} \\
&+ 4\varepsilon^3(f')^2 f_1' h f_2' \tilde{U}(kz) - 2\varepsilon^4 f' f_2''' h\tilde{U}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{A}_{3,r,o} &= 2\varepsilon^3 f' h f_2' \left(2(f')^2\langle \mathbf{H}, z\rangle + \langle\nabla^N V, z\rangle - p(p-1)h^{p-2}U^{p-2}w_{r,o}\right)\tilde{U} \\
&+ 2\varepsilon^3 f' f_2' hk \sum_j H^j \partial_j \tilde{U}(kz) - 2\varepsilon^4 f' hk f_2' \sum_j \Phi_j'' \partial_j \tilde{U}
\end{aligned}$$

$$(141) \qquad \mathfrak{A}_{3,i,e} = 4if'\varepsilon^3\partial_{\bar{s}}\left(hf'f_2'\tilde{U}\right) + 2i\varepsilon^3 f'f_2'\left(f''h - (p-1)h^{p-1}U^{p-1}w_{i,e}\right)\tilde{U}$$

$$\mathfrak{A}_{3,i,o} = -2(p-1)i\varepsilon^3 f'f_2' h^{p-1}U^{p-1}w_{i,o}\tilde{U} - 4i\varepsilon^3(f')^2 f_2' hk \sum_j \Phi_j' \partial_j \tilde{U}$$

$$\begin{aligned}
\mathfrak{A}_{3,1} &= f_2'\left[\mathfrak{R}_4(\Phi, \Phi', \Phi'')\right] + f_2''\left[\mathfrak{R}_4(\Phi, \Phi') + \mathfrak{R}_6(\Phi, \Phi', \Phi'')\right] + \varepsilon^4 f_2'''\left[\mathfrak{R}_1(\Phi, \Phi')\right] \\
&+ \mathfrak{R}_4(\Phi, \Phi')f_2'\left[f_2'(1 + \varepsilon^2 f_2') + \varepsilon f_2''\right] + \mathfrak{R}_5(\Phi, \Phi')f_2''f_2' + \mathfrak{R}_6(\Phi, \Phi', \Phi'')(f_2')^2.
\end{aligned}$$

• Choice of v_0 and estimate of $\mathfrak{A}_4$

In order to make the approximate solution as accurate as possible, we add a correction $\varepsilon^2 v_0$ in such a way to compensate (most of) the terms $\varepsilon^2(\tilde{R}_{r,e} + i\tilde{R}_{i,o})$, see Subsection 3.3 in [10]. We notice that these terms contain parts which are independent of Φ, which we denote them by $\tilde{R}_{r,e}^0$ and $\tilde{R}_{i,o}^0$ respectively, and parts which are quadratic in Φ or its derivatives, $\tilde{R}_{r,e}^\Phi$ and $\tilde{R}_{i,o}^\Phi$ respectively. Since we will take Φ of order ε, we regard the latter ones as higher order terms, and we add corrections to cancel $\tilde{R}_{r,e}^0$ and $\tilde{R}_{i,o}^0$. Precisely we define $v_{r,e}^0$ and $v_{i,o}^0$ by

$$\begin{aligned}
-\mathcal{L}_r v_{r,e}^0 &= -\tfrac{1}{2}(f')^2 hU(kz)\sum_{l,m}\partial_{lm}^2 g_{11}z_m z_l + 2(f')^2\langle\mathbf{H}, +w_{r,o}z\rangle + 4(f')^2 hU(kz)\langle\mathbf{H}, z\rangle^2 + 2f'\partial_s w_{i,e} \\
&+ f''w_{i,e} - \left[h''U(kz) + 2h'k'\nabla U(kz)\cdot z + hk''\nabla U(kz)\cdot z + h(k')^2\nabla^2 U(kz)[z,z]\right] \\
(142) \quad &+ \tfrac{1}{2}\sum_{l,m}\partial_{lm}^2 g_{tj}z_m z_l hk^2\partial_{tj}^2 U(kz) + kh[\langle\mathbf{H}, z\rangle H^m - \tfrac{1}{2}\sum_l \partial_{ml}^2 g_{11}z_l]\partial_m U(kz) \\
&+ hk\sum_l \partial_{lm}^2 g_{mj}z_l\partial_j U(kz) + \sum_l H^l\partial_l w_{r,o} + hk\langle\mathbf{H}, z\rangle H^l\partial_l U(kz) + \langle\nabla^N V, w_{r,o}z\rangle \\
&- \tfrac{1}{2}(p-1)h^{p-2}U(kz)^{p-2}w_{i,e}^2 - \tfrac{1}{2}p(p-1)h^{p-2}U(kz)^{p-2}w_{r,o}^2 + \tfrac{1}{2}\sum_{m,j}\partial_{mj}^2 V z_m z_j hU(kz);
\end{aligned}$$

$$\begin{aligned}
-\mathcal{L}_i v_{i,o}^0 &= 2\left[f''hU(kz) + 2f'h'U(kz) + 2f'hk'\nabla U(kz)\cdot z\right]\langle\mathbf{H}, z\rangle + \sum_i H^j\partial_j w_{i,e} \\
(143) \quad &+ 2(f')^2\langle\mathbf{H}, w_{i,e}z\rangle + 2f'\partial_s w_{r,o} + f''w_{r,o} - f'hk\sum_j \partial_j U(kz)\sum_{l,m}\partial_{lm}^2 g_{1j}z_m z_l \\
&- f'hU(kz)\left(\sum_m \partial_{1m}^2 g_{11}z_m\right) - f'h\left(\sum_{j,l}\partial_{lj}^2 g_{1j}z_l\right)U(kz) + \tfrac{1}{2}f'h\left(\sum_l \partial_{1l}^2 g_{11}z_l\right)U(kz) \\
&- (p-1)h^{p-2}U(kz)^{p-2}w_{r,o}w_{i,e} + \langle\nabla^N V, w_{i,e}z\rangle.
\end{aligned}$$

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We notice that the right-hand side of (142) is even in z, and hence orthogonal to the kernel of $\mathcal{L}_r$. As a consequence the equation is indeed solvable in $v_{r,e}^0$, see the comments after (15). The same comment applies to (143), where the right-hand side which is odd in z. Furthermore the right-hand sides decay at infinity at most like $(1 + |z|^d)e^{-k|z|}$ for some integer d, so the same holds true for $v_{r,e}^0$ and $v_{i,o}^0$. In conclusion, after some computations one finds

$$e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\mathfrak{A}_4 = \mathfrak{A}_{4,0} + \tilde{\mathfrak{A}}_4 := \mathfrak{A}_{4,0} + \mathfrak{A}_{4,r,e} + \mathfrak{A}_{4,r,o} + \mathfrak{A}_{4,i,e} + \mathfrak{A}_{4,i,o} + \mathfrak{A}_{4,1},$$

where

$$\mathfrak{A}_{4,0} = \varepsilon^2 \mathcal{L}_r v_{r,e}^0 + i\varepsilon^2 \mathcal{L}_i v_{i,o}^0;$$

(144) $\qquad \mathfrak{A}_{4,r,e} = \varepsilon^3 F_{4,r,e}(\overline{s}); \qquad \mathfrak{A}_{4,r,o} = \varepsilon^3 F_{4,r,o}(\overline{s}); \qquad \mathfrak{A}_{4,i,e} = \varepsilon^3 F_{4,i,e}(\overline{s}); \qquad \mathfrak{A}_{4,i,o} = \varepsilon^3 F_{4,i,o}(\overline{s});$

(145) $\qquad \mathfrak{A}_{4,1} = \mathfrak{R}_4(\Phi, \Phi') + (\Phi + \Phi')(1 + f_2')\mathfrak{R}_3(\Phi, \Phi') + f_2''\mathfrak{R}_5(\Phi, \Phi') + (f_2')^2\mathfrak{R}_6(\overline{s}) + \Phi''\mathfrak{R}_4(\Phi, \Phi').$

As for $F_e(\overline{s})$ and $F_o(\overline{s})$ in $\mathfrak{A}_2$, the F_4's depend only on V, γ, M, and are bounded above by $C(1 + |z|^d)e^{-k|z|}$.

• Estimate of $\mathfrak{A}_5$

The term involving v_δ in $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$ is given by

(146) $\qquad \mathfrak{A}_5 = \mathfrak{A}_{5,0} + \tilde{\mathfrak{A}}_5 := \mathfrak{A}_{5,0} + \mathfrak{A}_{5,r,e} + \mathfrak{A}_{5,r,o} + \mathfrak{A}_{5,i,e} + \mathfrak{A}_{5,i,o} + \mathfrak{A}_{5,1}$

where

(147) $\qquad \mathfrak{A}_{5,0} = \beta\mathcal{L}_r Z_\alpha(kz) - \varepsilon^2\beta'' Z_\alpha(kz) - 2\varepsilon\xi' W_\alpha f' + i\xi\mathcal{L}_i W_\alpha - i\varepsilon^2\xi'' W_\alpha + 2i\varepsilon\beta' f' Z_\alpha;$

$$\begin{aligned}
\mathfrak{A}_{5,r,e} &= -\varepsilon f''\xi W_\alpha - 2\varepsilon^2\beta'\left(\frac{\partial Z_\alpha}{\partial\alpha}\alpha' + k'\nabla Z_\alpha(kz)\cdot z\right) \\
&\quad - 2\varepsilon f'\xi\left(\frac{\partial W_\alpha}{\partial\alpha}\alpha' + k'\nabla W_\alpha(kz)\cdot z\right) - (p-1)\varepsilon h^{p-2}U^{p-2}\xi w_{i,e}W_\alpha;
\end{aligned}$$

$$\begin{aligned}
\mathfrak{A}_{5,r,o} &= \varepsilon\beta\sum_j H^j\partial_j Z_\alpha + 2\varepsilon\langle\mathbf{H}, z\rangle\left[(f')^2\beta Z_\alpha + \varepsilon^2\beta'' Z_\alpha - 2\varepsilon\xi' f' W_\alpha\right] \\
&\quad + \varepsilon\langle\nabla^N V, z\rangle\beta Z_\alpha - p(p-1)\varepsilon h^{p-2}U^{p-2}w_{r,o}\beta Z_\alpha;
\end{aligned}$$

$$\begin{aligned}
\mathfrak{A}_{5,i,e} &= \varepsilon f''\beta Z_\alpha - 2\varepsilon^2\xi'\left(\frac{\partial W_\alpha}{\partial\alpha}\alpha' + k'\nabla W_\alpha(kz)\cdot z\right) \\
&\quad + 2\varepsilon f'\beta\left(\frac{\partial Z_\alpha}{\partial\alpha}\alpha' + k'\nabla Z_\alpha(kz)\cdot z\right) - (p-1)\varepsilon h^{p-2}U^{p-2}\beta w_{i,e}Z_\alpha;
\end{aligned}$$

$$\begin{aligned}
\mathfrak{A}_{5,i,o} &= \varepsilon\xi\sum_j H^j\partial_j W_\alpha + 2\varepsilon\langle\mathbf{H}, z\rangle\left[(f')^2\xi W_\alpha + \varepsilon^2\xi'' W_\alpha + 2\varepsilon\beta' f' Z_\alpha\right] \\
&\quad + \varepsilon\langle\nabla^N V, z\rangle\xi W_\alpha - (p-1)\varepsilon h^{p-2}U^{p-2}w_{r,o}\xi W_\alpha.
\end{aligned}$$

The error term $\mathfrak{A}_{5,1} = \mathfrak{A}_{5,1}(\beta, \Phi, f_2)$ satisfies the following estimates

$$|\mathfrak{A}_{5,1}(\beta, \Phi, f_2)| \leq C(\varepsilon^2 + \varepsilon^2|f_2'| + \varepsilon^3|f_2''| + \varepsilon^3|\Phi''|)(1 + |z|^d)e^{-k|z|}(|\beta| + \varepsilon|\beta'| + \varepsilon^2|\beta''| + \varepsilon^3|\beta'''|);$$

$$\left|\mathfrak{A}_{5,1}(\beta,\Phi,f_2) - \mathfrak{A}_{5,1}(\tilde\beta,\tilde\Phi,\tilde f_2)\right| \leq C(\varepsilon|\Phi-\tilde\Phi| + \varepsilon|\Phi'-\tilde\Phi'| + \varepsilon^3|\Phi''-\tilde\Phi''| + \varepsilon^2|f_2'-\tilde f_2'| + \varepsilon^3|f_2''-\tilde f_2''|)$$
$$\times \quad (1+|z|^d)e^{-k|z|}(|\beta| + \varepsilon|\beta'| + \varepsilon^2|\beta''| + \varepsilon^3|\beta'''|)$$
$$+ \quad C(\varepsilon^2 + \varepsilon^2|f_2'| + \varepsilon^3|f_2''| + \varepsilon^3|\Phi''|)$$
$$\times \quad (1+|z|^d)e^{-k|z|}(|\beta-\tilde\beta| + \varepsilon|\beta'-\tilde\beta'| + \varepsilon^2|\beta''-\tilde\beta''| + \varepsilon^3|\beta'''-\tilde\beta'''|).$$

By the form of the function β, see (59), (60) and (124), its Fourier modes are naively concentrated around indices of order $\frac{1}{\varepsilon}$. As a consequence, L^2 norms of functions like $\varepsilon\beta, \varepsilon^2\beta'', \varepsilon^3\beta'''$, etc. can be controlled with the L^2 norm of β, see also the comments before (75).

• Estimate of $\mathfrak{A}_6$

First of all we notice that we are taking Φ' and f_2' in $H^1([0,L])$, and hence they belong to $L^\infty([0,L])$. As a consequence, since we have the bound $\|\beta\|_{L^\infty([0,L])} + \varepsilon\|\beta'\|_{L^\infty([0,L])} + \varepsilon^2\|\beta''\|_{L^\infty([0,L])} + \varepsilon^3\|\beta'''\|_{L^\infty([0,L])} \leq C\varepsilon^2$ (which follows from (126) and the above comments), one has the estimate

$$(148) \qquad\qquad |E| + |F| + |G| \leq C\varepsilon^2(1+|z|^d)e^{-k|z|}.$$

If then we choose δ sufficiently small (recall also the expressions of w_r, w_i and (128)), we deduce that

$$|\tilde\Psi_{2,\varepsilon} - \tilde\Psi_{1,\varepsilon}| \leq |\tilde\Psi_{1,\varepsilon}| \qquad \text{in } \tilde D_\varepsilon.$$

This estimate implies that $\mathfrak{A}_6$ admits a uniform quadratic Taylor expansion in $|\tilde\Psi_{2,1}-\tilde\Psi_{1,\varepsilon}|$ and is bounded by $|\tilde\Psi_{1,\varepsilon}|^{p-2}|\tilde\Psi_{2,\varepsilon}-\tilde\Psi_{1,\varepsilon}|^2$. Precisely, we can write

$$(149) \qquad\qquad \mathfrak{A}_6 = \mathfrak{A}_{6,0} + \tilde{\mathfrak{A}}_6 := \mathfrak{A}_{6,0} + \mathfrak{A}_{6,r,e} + \mathfrak{A}_{6,r,o} + \mathfrak{A}_{6,i,e} + \mathfrak{A}_{6,i,o} + \mathfrak{A}_{6,1},$$

where

$$(150) \qquad\qquad \mathfrak{A}_{6,0} = \mathfrak{A}_{6,r,e} = \mathfrak{A}_{6,r,o} = \mathfrak{A}_{6,i,e} = \mathfrak{A}_{6,i,o} = 0; \qquad \mathfrak{A}_{6,1} = R_4(f_2',\Phi,\Phi',\beta),$$

where $R_4(f_2',\Phi,\Phi',\beta)$ is a quantity satisfying the estimates

$$|R_4(f_2',\Phi,\Phi',\beta)| \leq C\left[\varepsilon^4 + (\varepsilon^2 + \|\beta\|_{L^\infty} + \varepsilon\|\beta'\|_{L^\infty})(|\beta| + \varepsilon|\beta'|)\right](1+|z|^d)e^{-k|z|};$$

$$|R_4(f_2',\Phi,\Phi',\beta) - R_4(\tilde f_2',\tilde\Phi,\tilde\Phi',\tilde\beta)| \leq C\left(\varepsilon^2 + |\beta| + \varepsilon|\beta'| + |\tilde\beta| + \varepsilon|\tilde\beta'|\right)(1+|z|^d)e^{-k|z|}$$
$$\times \quad \left(\varepsilon|\Phi-\tilde\Phi| + \varepsilon|\Phi'-\tilde\Phi'| + \varepsilon^2|f_2'-\tilde f_2'| + |\beta-\tilde\beta| + \varepsilon|\beta'-\tilde\beta'|\right).$$

• Final estimate of $S_\varepsilon(\tilde\Psi_{2,\varepsilon})$

By (135), in the above notation we have

$$e^{i\frac{\tilde f(\varepsilon s)}{\varepsilon}}S_\varepsilon(\tilde\Psi_{2,\varepsilon}) = \varepsilon^2(\tilde R_{r,o} + \tilde R_{r,e}) + \varepsilon^2(\tilde R_{r,o,f_1} + \tilde R_{r,e,f_1})$$
$$+ \quad \varepsilon^2 i(\tilde R_{i,e} + \tilde R_{i,o}) + \varepsilon^2 i(\tilde R_{i,e,f_1} + \tilde R_{i,o,f_1}) + \sum_{i=1}^{6}\mathfrak{A}_{i,0} + \sum_{i=1}^{6}\tilde{\mathfrak{A}}_i.$$

Recalling the choices of $\tilde v, v_{r,e}^0$ and $v_{i,o}^0$ in (139) and (142), (143) (and recalling the notation for the R's after (120)) we finally obtain the following result.

Proposition 3.2 *Suppose Φ, f_2 and β satisfy (121), (122) and (126) for some $c_1, c_2, c_3 > 0$. Let $\tilde f = f + \varepsilon f_1 + \varepsilon^2 f_2$, where f is given in (11) and f_1 in (114). Let also $w_r = w_{r,e} + w_{r,o}$, with $w_{r,e}, w_{r,o}$*

given respectively in (117), (119), and $w_i = w_{i,e} + w_{i,o}$, where $w_{i,e}$ and $w_{i,o}$ are given respectively in (118). Let $\tilde{\Psi}_{2,\varepsilon}$ be defined in (127). Then, as ε tends to zero, we have that

$$
\begin{aligned}
e^{i\frac{\tilde{f}}{\varepsilon}} S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) &= \varepsilon^2(\tilde{R}_{r,e}^\Phi + \tilde{R}_{r,o} + \tilde{R}_{r,e,f_1} + \tilde{R}_{r,o,f_1}) + \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}_{i,o}^\Phi + \tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1}) \\
&\quad + \beta\mathcal{L}_r Z_\alpha(kz) + \varepsilon^2 \beta'' Z_\alpha(kz) - 2\varepsilon\xi' W_\alpha f' + i\xi\mathcal{L}_i W_\alpha \\
&\quad + i\varepsilon^2\xi'' W_\alpha + 2i\varepsilon\beta' f' Z_\alpha + e^{i\frac{\tilde{f}}{\varepsilon}}\sum_{j=1}^{6}\tilde{\mathfrak{A}}_j,
\end{aligned}
$$
(151)

where the R's are as in (120), where $\tilde{R}_{r,e}^\Phi$, $\tilde{R}_{i,o}^\Phi$ are the terms quadratic in Φ, Φ' within $\tilde{R}_{r,e}$, $\tilde{R}_{i,o}$, and where the latter error terms are given in (136), (138), (140), (145), (146), (149) respectively.

4 Proof of Theorem 1.1

In this section we prove our main theorem. First we solve the equation in the $\overline{H}_\varepsilon$ components, see (105), using a Lyapunov-Schmidt reduction. Then we turn to the components in $\tilde{K}_\delta$ and solve the bifurcation equation as well: in this last step we use crucially the non-degeneracy assumption on γ and an accurate choice for the values of the parameter ε.

4.1 Solvability in the component of $\overline{H}_\varepsilon$

In Proposition 2.4 we showed that problem (14) is reduced to finding a solution of $L_\varepsilon(\phi) = \tilde{S}_\varepsilon(\phi)$ in $\tilde{D}_\varepsilon$, see (24), (40) and (41), if we take $K^2(\varepsilon s) = V(\varepsilon s)$. Choosing in Proposition 2.14 as approximate solution $\tilde{\psi}_\varepsilon = \tilde{\Psi}_{2,\varepsilon}$ (the function constructed in the previous subsection), we have the following result where, as usual, δ is sufficiently small. We recall Proposition 2.4, formulas (103)-(106) and the definition of $\tilde{K}_\delta$ after (104): also, we denote by $\tilde{\Pi}_\varepsilon$ the orthogonal projection onto the set $\{e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\tilde{v} : \tilde{v} \in \tilde{K}_\delta\}$.

Proposition 4.1 Let $\tilde{\Psi}_{2,\varepsilon}$ be as in Proposition 3.2. Then there exists $\check{v}_\delta \in \tilde{K}_\delta$, depending on the parameters Φ, f_2, β, such that the following problem admits a solution

$$
(152) \qquad \begin{cases} -\Delta_{g_\varepsilon}\hat{\phi} + V(\varepsilon x)\hat{\phi} - |\tilde{\Psi}_{2,\varepsilon}|^{p-1}\hat{\phi} - (p-1)|\tilde{\Psi}_{2,\varepsilon}|^{p-3}\tilde{\Psi}_{2,\varepsilon}\Re(\tilde{\Psi}_{2,\varepsilon}\overline{\hat{\phi}}) = \tilde{S}_\varepsilon(\hat{\phi}) + e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}}\check{v}; \\ \hat{\phi} \in \overline{H}_\varepsilon, \quad \check{v} \in \tilde{K}_\delta. \end{cases}
$$

Furthermore, if $m \in N$, if $\tilde{\tilde{\Psi}}_{2,\varepsilon}$ is an approximate solution corresponding to different Φ, f_2, β, for a fixed constant C independent of ε and δ, for $\tau = \frac{1}{2}$ and $0 < \varsigma' < \varsigma < 1$ sufficiently small, we have

$$
(153) \qquad \|\hat{\phi}\|_{\varsigma',V} \le \frac{C}{\delta^2}\|\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C_{\varsigma,V}^\tau)} + C\varepsilon^m; \qquad \|\check{v}\|_{L^2(C_{\varsigma,V}^\tau)} \le C\|S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C_{\varsigma,V}^\tau)}.
$$

$$
(154) \qquad \|\hat{\phi} - \hat{\tilde{\phi}}\|_{\varsigma',V} \le \frac{C}{\delta^2}\|\tilde{\Pi}_\varepsilon(S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) - S_\varepsilon(\tilde{\tilde{\Psi}}_{2,\varepsilon}))\|_{L^2(C_{\varsigma,V}^\tau)}.
$$

PROOF. The proof relies on Proposition 2.1, Proposition 2.14 and the contraction mapping theorem. By Proposition 2.14, the operator L_ε (see (24)) is invertible from $(\overline{H}_\varepsilon, \|\cdot\|_{\varsigma',V})$ into $L^2(C_{\varsigma,V}^\tau)$, and the norm of the inverse is uniformly bounded by C/δ^2. By this invertibility, (152) is satisfied if and only if $\hat{\phi}$ is a fixed point of the operator $\check{F}_\varepsilon : (\overline{H}_\varepsilon, \|\cdot\|_{\varsigma',V}) \to (\overline{H}_\varepsilon, \|\cdot\|_{\varsigma',V})$ defined by

$$
\begin{aligned}
\check{F}_\varepsilon(\hat{\phi}) &= L_\varepsilon^{-1}\left[\tilde{\Pi}_\varepsilon\left(\tilde{S}_\varepsilon(\hat{\phi})\right)\right] := L_\varepsilon^{-1}\left[\tilde{\Pi}_\varepsilon\left(S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) + N_\varepsilon(\eta_\varepsilon\hat{\phi} + \varphi(\hat{\phi}))\right.\right. \\
&\quad + \left.\left. |\tilde{\Psi}_{2,\varepsilon}|^{p-1}\varphi(\hat{\phi}) + (p-1)|\tilde{\Psi}_{2,\varepsilon}|^{p-3}\tilde{\Psi}_{2,\varepsilon}\Re(\tilde{\Psi}_{2,\varepsilon}\overline{\varphi(\hat{\phi})})\right)\right].
\end{aligned}
$$

We recall that, in the last formula, $\varphi(\hat{\phi})$ is given by Proposition 2.1, while N_ε is defined in (25).

Our next goal is to show that $\check{F}_\varepsilon$ is a contraction on a metric ball (in the $\|\cdot\|_{\varsigma',V}$ norm) of radius $\frac{C}{\delta^2}\|\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})} + C\varepsilon^m$ for C large enough and m arbitrary integer. Setting for simplicity

$$\check{G}_\varepsilon(\hat{\phi}) = N_\varepsilon(\eta_\varepsilon\hat{\phi} + \varphi(\hat{\phi})) + |\tilde{\Psi}_{2,\varepsilon}|^{p-1}\varphi(\hat{\phi}) + (p-1)|\tilde{\Psi}_{2,\varepsilon}|^{p-3}\tilde{\Psi}_{2,\varepsilon}\Re(\tilde{\Psi}_{2,\varepsilon}\overline{\varphi(\hat{\phi})}),$$

one clearly finds

$$(155) \qquad \begin{cases} \|\check{F}_\varepsilon(\hat{\phi})\|_{L^2(C^\tau_{\varsigma',V})} \leq \frac{C}{\delta^2}\left(\|\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})} + \|\check{G}_\varepsilon(\hat{\phi})\|_{L^2(C^\tau_{\varsigma,V})}\right); \\ \|\check{F}_\varepsilon(\hat{\phi}_1) - \check{F}_\varepsilon(\hat{\phi}_2)\|_{L^2(C^\tau_{\varsigma',V})} \leq \frac{C}{\delta^2}\|\check{G}_\varepsilon(\hat{\phi}_1) - \check{G}_\varepsilon(\hat{\phi}_2)\|_{L^2(C^\tau_{\varsigma,V})}. \end{cases}$$

We next evaluate $\|\check{G}_\varepsilon(\hat{\phi})\|_{L^2(C^\tau_{\varsigma,V})}$, and show that it is *superlinear* in $\|\hat{\phi}\|_{L^2(C^\tau_{\varsigma',V})}$ up to negligible terms: we make first the following claim.

Claim: in the notation (31), letting $k_1(\overline{s}) = (\varsigma')^2\sqrt{V(\overline{s})}$ we have $\|\hat{\phi}\|_{C^{1,\frac{1}{2}}_{k_1}} \leq C\|\hat{\phi}\|_{\varsigma',V}$ for some $C > 0$.

Assuming the claim true and choosing $\varsigma'' < (\varsigma')^2$, we can apply Proposition 2.1 with $\tau = \frac{1}{2}$, $k_0(\overline{s}) = \varsigma\sqrt{V(\overline{s})}$, $k_1(\overline{s}) = (\varsigma')^2\sqrt{V(\overline{s})}$ and $k_2(\overline{s}) = \varsigma''\sqrt{V(\overline{s})}$, to find

$$(156) \qquad \|\varphi(\hat{\phi})\|_{C^{\frac{1}{2}}_{-k_2}} \leq C\left(e^{-\inf\frac{k_2+k_0}{K}\varepsilon^{-\overline{\delta}}}\|S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{C^{\frac{1}{2}}_{k_0}} + e^{-\inf\frac{k_2+k_1}{K}\varepsilon^{-\overline{\delta}}}\|\hat{\phi}\|_{C^{1,\frac{1}{2}}_{k_1}}\right).$$

From the expression of $w_r, w_i, \tilde{v}, v_0$ and formula (148), one can deduce that $|\tilde{\Psi}_{2,\varepsilon}| \leq Ce^{-k_0|z|}$: moreover, from the estimates in the proof of Proposition 3.2 one also finds that $\|S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})} \to 0$ as $\varepsilon \to 0$. By (37) (recall that $\zeta > 0$), the latter bounds on $\tilde{\Psi}_{2,\varepsilon}$, the previous claim and (156), if m is an arbitrary integer and if ς'' is sufficiently close to 1 after some elementary computations we deduce

$$\|\check{G}_\varepsilon(\hat{\phi})\|_{L^2(C^\tau_{\varsigma,V})} \leq C\left(\|\hat{\phi}\|^{1+\zeta}_{L^2(C^\tau_{\varsigma,V})} + \|\hat{\phi}\|^p_{L^2(C^\tau_{\varsigma,V})} + \varepsilon^m\left(1 + \|\hat{\phi}\|_{L^2(C^\tau_{\varsigma,V})}\right)\right).$$

Similarly, if $\|\hat{\phi}_1\|_{L^2(C^\tau_{\varsigma,V})}$, $\|\hat{\phi}_2\|_{L^2(C^\tau_{\varsigma,V})}$ are finite one also finds

$$\|\check{G}_\varepsilon(\hat{\phi}_1) - \check{G}_\varepsilon(\hat{\phi}_2)\|_{L^2(C^\tau_{\varsigma,V})} \leq C\left[\max_{l=1,2}\left\{\|\hat{\phi}_l\|^{\zeta\wedge(p-1)}_{L^2(C^\tau_{\varsigma,V})}\right\} + \varepsilon^m\right]\|\hat{\phi}_1 - \hat{\phi}_2\|_{L^2(C^\tau_{\varsigma,V})},$$

where the symbol $\wedge$ stands for the minimum. Formula (155) and the latter one show that $\check{F}_\varepsilon$ is a contraction, and we obtain (153); (154) follows similarly.

Proof of the claim. According to our previous notation, the norm $\|\cdot\|_{\varsigma',V}$ is evaluated using the variables (s, z), where the z's are defined in (99). If we want to estimate the $\|\cdot\|_{C^{1,\frac{1}{2}}_{k_1}}$ norm instead, we should use lipschitzianity with respect to s and y.

Given $s_1, s_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}^{n-1}$ we want to consider the difference $\nabla\hat{\phi}(s_1, y_1) - \nabla\hat{\phi}(s_2, y_2)$. Recalling (99) we can write that

$$\begin{aligned} \partial_s\hat{\phi}(s_1, y_1) - \partial_s\hat{\phi}(s_2, y_2) &= \partial_s\hat{\phi}(s_1, z_1 + \Phi(\varepsilon s_1)) - \partial_s\hat{\phi}(s_2, z_1 + \Phi(\varepsilon s_1)) \\ &+ \partial_s\hat{\phi}(s_2, z_1 + \Phi(\varepsilon s_1)) - \partial_s\hat{\phi}(s_2, z_2 + \Phi(\varepsilon s_2)). \end{aligned}$$

By the definition of $\|\cdot\|_{\varsigma',V}$, $\partial_s\hat{\phi} \in H^1(C^\tau_{\varsigma',V}) \subseteq C^{\frac{1}{2}}(C^\tau_{\varsigma',V})$. This fact, the smoothness of $V(\overline{s})$ and $\|\Phi\|_\infty + \|\Phi'\|_\infty \leq C(c_1)\varepsilon$ (which follows from (121)) imply that if $(s_1, y_1), (s_2, y_2) \in B_1(s, y)$ then

$$e^{(\varsigma')^2\sqrt{V(\overline{s})}|z|}|\partial_s\hat{\phi}(s_1, y_1) - \partial_s\hat{\phi}(s_2, y_2)| \leq C(c_1)\|\hat{\phi}\|_{\varsigma',V}\left(|s_1 - s_2|^{\frac{1}{2}} + |z_1 - z_2|^{\frac{1}{2}} + \varepsilon|s_1 - s_2|\right).$$

A similar estimate holds for the derivatives of $\hat{\phi}$ with respect to y, so from (31) we get the conclusion. ∎

To apply Proposition 4.1 we establish explicit estimates on $\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$ and $\tilde{\Pi}_\varepsilon(S_\varepsilon(\tilde{\Psi}_{2,\varepsilon}) - S_\varepsilon(\tilde{\tilde{\Psi}}_{2,\varepsilon}))$. Precisely, assuming from now on $\tau = \frac{1}{2}$, we have the following result.

Proposition 4.2 *Assume* $\Phi, f_2, \beta, \tilde{\Phi}, \tilde{f}_2, \tilde{\beta}$ *satisfy conditions* (121), (122) *and* (126). *Then, if* $\hat{\phi}$ *is defined as in Proposition 4.1 we have the estimates*

$$(157) \qquad \sqrt{\varepsilon}\delta^2 \|\hat{\phi}(\beta, \Phi, f_2)\|_* \le C(c_1, c_2, c_3)\varepsilon^3;$$

$$(158) \qquad \sqrt{\varepsilon}\delta^2 \|\hat{\phi}(\beta, \Phi, f_2) - \hat{\phi}(\tilde{\beta}, \tilde{\Phi}, \tilde{f}_2)\|_* \le C(c_1, c_2, c_3) \left[\varepsilon^2 \|\Phi - \tilde{\Phi}\|_{H^2} + \varepsilon^3 \|f_2 - \tilde{f}_2\|_{H^2} + \varepsilon \|\beta - \tilde{\beta}\|_\sharp \right],$$

where $C(c_1, c_2, c_3)$ *is a positive constant depending on* c_1, c_2, c_3 *but independent of* ε *and* δ.

PROOF. We prove (157) only: (158) will follow from similar considerations. To show (157) we use Proposition 4.1, so we are reduced to estimate $\|\tilde{\Pi}_\varepsilon S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})\|_{L^2(C^\tau_{\varsigma,V})}$, for which we can employ (151).

By our assumptions on Φ, f_2, β and by the estimates of the previous subsection, it is easy to see that

$$\left\| \varepsilon^2(\tilde{R}^\Phi_{r,e} + \tilde{R}_{r,o} + \tilde{R}_{r,e,f_1} + \tilde{R}_{r,o,f_1}) + \varepsilon^2 i(\tilde{R}_{i,e} + \tilde{R}^\Phi_{i,o} + \tilde{R}_{i,e,f_1} + \tilde{R}_{i,o,f_1}) \right\|_{L^2(C^\tau_{\varsigma,V})} \le \frac{C(c_1, c_2, c_3)\varepsilon^3}{\sqrt{\varepsilon}};$$

$$\left\| e^{i\frac{\tilde{f}}{\varepsilon}} \sum_{i=1}^{6} \tilde{\mathfrak{A}}_i \right\|_{L^2(C^\tau_{\varsigma,V})} \le \frac{C(c_1, c_2, c_3)\varepsilon^3}{\sqrt{\varepsilon}}.$$

Recall that in the choice of approximate solutions we have formally corrected all the terms of order up to ε^2, so we are left with terms of order ε^3 and higher. The factor $\sqrt{\varepsilon}$ in the denominator arises from the fact that the length of γ_ε is L/ε: this gives a factor $\frac{1}{\varepsilon}$ when computing the L^2 norm squared, and we need then to take the square root. For the estimates in $\tilde{\mathfrak{A}}_6$, which also require the L^∞ norm of β, we can use the interpolation inequalities

$$\|\beta\|_{L^\infty([0,L])} \le C\|\beta\|^{\frac{1}{2}}_{L^2([0,L])} \|\beta'\|^{\frac{1}{2}}_{L^2([0,L])} \le C\varepsilon^{\frac{3}{2}}; \qquad \|\beta'\|_{L^\infty([0,L])} \le C\|\beta'\|^{\frac{1}{2}}_{L^2([0,L])} \|\beta''\|^{\frac{1}{2}}_{L^2([0,L])} \le C\varepsilon^{\frac{1}{2}}.$$

It remains to consider now the other terms in the right-hand side of (151), involving the functions Z_α and W_α. Let us call $\tilde{L}^1_\varepsilon$ the operator obtained from L^1_ε (see (52)) by replacing the variables y with z and f with $\tilde{f}$. Let us first notice that the terms under interest, with this notation, are nothing but $\tilde{\Pi}_\varepsilon \tilde{L}^1_\varepsilon v_\delta$.

Let us now recall the expression of β in (124) and v_δ in (123): if $\tilde{v}_{3,j}$ stand for the functions in $K_{3,\delta}$ (see (66)) replacing y with z, we define the function

$$\tilde{v}_\delta = \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j \tilde{v}_{3,j}.$$

From the expression of $\tilde{v}_{3,j}$, see (62), one finds that

$$(159) \qquad \|v_\delta - \tilde{v}_\delta\|_{\varsigma,V} \le \frac{C}{\sqrt{\varepsilon}} \Big(\sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j^2 \varepsilon^2 (1+j^2) \Big)^{\frac{1}{2}};$$

$$(160) \quad \tilde{L}^1_\varepsilon(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} v_\delta) = \tilde{L}^1_\varepsilon(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \tilde{v}_\delta) + O\left(\frac{1}{\sqrt{\varepsilon}}\right) \Big(\sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j^2 \varepsilon^2 (1+j^2) \Big)^{\frac{1}{2}} \quad \text{(in the } \|\cdot\|_{L^2(C^\tau_{\varsigma,V})} \text{ norm)}.$$

Similarly to (85), recalling the asymptotic of ν_j (see (59) and the lines before) one finds that

$$(161) \qquad \tilde{L}^1_\varepsilon(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \tilde{v}_\delta) = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \nu_j b_j \tilde{v}_{3,j} + R_1,$$

where $\|R_1\|_{L^2(C^\tau_{\varsigma,V})} \le \frac{C}{\sqrt{\varepsilon}} \Big(\sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j^2 \varepsilon^2 (1+j^2) \Big)^{\frac{1}{2}} \le C\sqrt{\varepsilon}\|\beta\|_\sharp$. This implies the conclusion, by (126). ∎

4.2 Projections onto $\tilde{K}_\delta$

In this section we estimate the projections of the equation onto the components of $\tilde{K}_\delta$. We estimate first their size and their Lipschitz dependence in the data Φ, f_2 and β. Then we use the contraction mapping theorem to annihilate the function $\check{v}$ in Proposition 4.1, which implies the solvability of (14).

4.2.1 Projection onto $\tilde{K}_{1,\delta}$

We want to evaluate the $\tilde{K}_{1,\delta}$ component of the function $\check{v}_\delta$ in (152). To do this we consider a normal section $\underline{\Phi}$ to γ which satisfies the first relation in (122), and the function

$$v_{\underline{\Phi}} := h(\varepsilon s)^{\frac{p+1}{4}} \left(\langle \underline{\Phi}(\varepsilon s), \nabla_z U(kz) \rangle + i\varepsilon \langle \underline{\Phi}'(\varepsilon s), z \rangle \frac{f'}{k} U(kz) - \frac{\varepsilon^2}{k^2} \langle \underline{\Phi}''(\varepsilon s), \mathfrak{V}(kz) \rangle \right).$$

We then multiply both the left-hand side of (152) and $\tilde{S}_\varepsilon(\hat{\phi})$ (see (41)) by the conjugate of $e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} v_{\underline{\Phi}}$, integrate over $\tilde{D}_\varepsilon$ and take the real part. When multiplying the left-hand side, we can integrate by parts and let the operator L_ε act on $e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} v_{\underline{\Phi}}$: using the arguments in the proofs of Proposition 2.9 (see in particular (83) and (84)) and of Proposition 4.2 one finds that

$$L_\varepsilon(e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} v_{\underline{\Phi}}) = e^{-i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \tilde{v}_{\underline{\Phi}} + R(v_{\underline{\Phi}}),$$

where $\tilde{v}_{\underline{\Phi}} \in \tilde{K}_{1,\delta}$, and where $\|R(v_{\underline{\Phi}})\|_{L^2(C_{\varsigma,V}^\tau)} \le C(\varepsilon+\delta^3)\|v_{\underline{\Phi}}\|_{L^2(C_{\varsigma,V}^\tau)} \le \frac{C}{\sqrt{\varepsilon}}(\varepsilon+\delta^3)\|\underline{\Phi}\|_{L^2([0,L])}$. Therefore, since $\hat{\phi}$ is orthogonal to $\tilde{K}_\delta$, from (157) we deduce that

$$(162) \qquad \left| \Re \int_{\tilde{D}_\varepsilon} e^{i\frac{\tilde{f}(\varepsilon s)}{\varepsilon}} \overline{v_{\underline{\Phi}}} L_\varepsilon \hat{\phi} \right| dV_{\tilde{g}_\varepsilon} \le \frac{C}{\sqrt{\varepsilon}}(\varepsilon+\delta^3)\|\underline{\Phi}\|_{L^2([0,L])}\|\hat{\phi}\|_{L^2(C_{\varsigma,V}^\tau)} \le C(c_1,c_2,c_3)\delta\varepsilon^2\|\underline{\Phi}\|_{L^2([0,L])}.$$

We next have to consider $\tilde{S}_\varepsilon(\hat{\phi})$, whose main term is $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$: for this we use formula (151). Here we have three kinds of terms: the $\tilde{R}$'s, those involving Z_α, W_α (which coincide with $\mathfrak{A}_{5,0}$, with our notation in (147)) and the $\tilde{\mathfrak{A}}$'s.

For the $\tilde{R}$'s, since $v_{\underline{\Phi}}$ is odd in z, the products with the even terms will vanish. The products of the odd terms (notice that the two phases cancel and we use the change of variables $s \mapsto \varepsilon s$) instead give us

$$\varepsilon^2 \Re \int_{\tilde{D}_\varepsilon} (\tilde{R}_{r,o} + \tilde{R}_{r,o,f_1})\overline{v_{\underline{\Phi}}} dV_{\tilde{g}_\varepsilon} + \varepsilon^2 \Re \int_{\tilde{D}_\varepsilon} i(\tilde{R}_{i,o}^\Phi + \tilde{R}_{i,o,f_1})\overline{v_{\underline{\Phi}}} dV_{\tilde{g}_\varepsilon} = -\varepsilon\frac{p-1}{2\theta}C_0 \int_0^L \langle \mathfrak{J}(\Phi), \underline{\Phi} \rangle \, d\overline{s} + \tilde{R}_0,$$

where $C_0 = \int_{\mathbb{R}^{n-1}} U(y)^2 dy$ and $|\tilde{R}_0| \le C\delta\varepsilon\|\underline{\Phi}\|_{L^2([0,L])}$. To explain why this estimate holds, we notice first that $-\frac{p-1}{2\theta}C_0\langle\mathfrak{J}(\Phi), \underline{\Phi}\rangle$ is exactly the first term of $v_{\underline{\Phi}}$ multiplied by $\tilde{R}_{r,o} + \tilde{R}_{r,o,f_1}$, as shown in Subsections 4.1 and 4.2 in [10] (the factor $h^{\frac{p+1}{4}}$ in (56) is needed precisely to cancel the factor $\frac{1}{hk}$ in the last formula of Subsection 4.2 in [10]). The remaining terms in the last equation are given either by products of the imaginary part of $v_{\underline{\Phi}}$ and the imaginary $\tilde{R}$'s or that of $\tilde{R}_{r,o} + \tilde{R}_{r,o,f_1}$ and the last term in $v_{\underline{\Phi}}$. In the latter case for example, we obtain a quantity bounded by

$$C\varepsilon^2 \int_0^{L/\varepsilon} (|\Phi| + |\Phi'| + |\Phi''|)\varepsilon^2 \underline{\Phi}'' ds \le C\varepsilon^2\delta^2\|\underline{\Phi}\|_{L^2([0,L])}.$$

The last inequality follows from (121) and the fact that $\underline{\Phi}$ satisfies the first condition in (122). On the other hand, the terms involving $\underline{\Phi}'$ once integrated will be bounded by $C\varepsilon^2\delta\|\underline{\Phi}\|_{L^2([0,L])}$, still by (122).

Concerning $\mathfrak{A}_{5,0}$, we next claim that for any $m \in \mathbb{N}$ one has

$$(163) \qquad \left| \Re \int_{\tilde{D}_\varepsilon} \overline{v_{\underline{\Phi}}} \, \mathfrak{A}_{5,0} dV_{\tilde{g}_\varepsilon} \right| \le C\varepsilon^m\|\underline{\Phi}\|_{L^2([0,L])} \qquad \text{as } \varepsilon \to 0.$$

To see this, notice that Φ satisfies (122) while $\mathfrak{A}_{5,0}$ arises from functions involving v_δ (in particular β, see (124)): since j ranges between $-\frac{\delta^2}{\varepsilon}$ and $\frac{\delta^2}{\varepsilon}$, the main modes of β are much higher than the ones of Φ. Hence, using Fourier cancelation as in Lemma 2.10, one can deduce (163). It is also easy to see that

$$(164) \qquad \left| \Re \int_{\tilde{D}_\varepsilon} \overline{v_{\underline{\Phi}}} \sum_{j=1}^{6} \tilde{\mathfrak{A}}_j \, dV_{\tilde{g}_\varepsilon} \right| \leq C(c_1, c_2, c_3) \varepsilon^2 \|\underline{\Phi}\|_{L^2([0,L])}.$$

It remains finally to consider the product of $v_{\underline{\Phi}}$ and the last three terms in (41). Indeed, since these are either superlinear in $\hat{\phi}$ (see (37)) or contain $\varphi(\hat{\phi})$ (see (32)), they are of lower order compared to (162).

Using (162)-(164) and the above arguments we finally obtain that, if $\check{v}$ is as in Proposition 4.1, then

$$(165) \qquad \int_{\tilde{D}_\varepsilon} \check{v}\, \overline{v_{\underline{\Phi}}} dV_{\tilde{g}_\varepsilon} = -\varepsilon \frac{p-1}{2\theta} C_0 \int_0^L \langle \mathfrak{J}(\Phi), \underline{\Phi} \rangle \, d\overline{s} + R_1; \qquad |R_1| \leq C(c_1, c_2, c_3)\varepsilon^2 \|\underline{\Phi}\|_{L^2([0,L])}.$$

Similarly, using the estimates in Section 3 one finds that if $\tilde{v}$ corresponds to the triple $(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})$, then

$$(166) \qquad \int_{\tilde{D}_\varepsilon} (\check{v} - \tilde{v})\, \overline{v_{\underline{\Phi}}} dV_{\tilde{g}_\varepsilon} = -\varepsilon \frac{p-1}{2\theta} C_0 \int_0^L \langle \mathfrak{J}(\Phi - \tilde{\Phi}), \underline{\Phi} \rangle \, d\overline{s} + \tilde{R}_1,$$

where $\tilde{R}_1$ satisfies

$$(167) \qquad |\tilde{R}_1| \leq C(c_1, c_2, c_3) \left(\delta\varepsilon \|\Phi - \tilde{\Phi}\|_{H^2([0,L])} + \varepsilon^2 \|f_2 - \tilde{f}_2\|_{H^2([0,L])} + \delta \|\beta - \tilde{\beta}\|_\sharp \right) \|\underline{\Phi}\|_{L^2}.$$

4.2.2 Projection onto $\tilde{K}_{2,\delta}$

For this projection we will be more sketchy since most of the arguments of the previous one can be applied. If $\underline{f}_2$ satisfies the second condition in (122), we consider the function

$$v_{\underline{f}_2} = h(\varepsilon s)^{\frac{1}{2}} \left(i\underline{f}_2(\varepsilon s) U(kz) + 2\varepsilon \frac{f' \underline{f}_2'(\varepsilon s)}{k} \tilde{U}(kz) - i\varepsilon^2 \frac{\underline{f}_2''(\varepsilon s)}{k^2} \mathfrak{W}(kz) \right).$$

As for the previous case, the main contribution to the projection is given by the product of the first term in $v_{\underline{f}_2}$ and the imaginary parts of $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$ listed in (151) which are even in z.

We denote by $\hat{R}_{i,e,f_2}$ the sum of all imaginary even terms of order ε^3 appearing in the equation, namely $\mathfrak{A}_{1,i,e}$, $\mathfrak{A}_{3,i,e}$ and $\mathfrak{A}_{4,i,e} = F_{4,i,e}(\overline{s})$, see (137), (141) and (144)

$$\begin{aligned}
\hat{R}_{i,e,f_2} = \quad & 2h' f_2' U + 2h f_2' k' \nabla U \cdot z + 2f' f_2' w_{i,e} + f_2'' h U + 4f' \partial_s(h f' f_2' \tilde{U}) \\
+ \quad & 2f'' h f' f_2' \tilde{U} - 2(p-1)h^{p-1}|U|^{p-2} f' f_2' \tilde{U} w_{i,e} + F_{4,i,e}(\overline{s}) := \tilde{R}_{i,e,f_2} + F_{4,i,e}(\overline{s}).
\end{aligned}$$

Notice that $\tilde{R}_{i,e,f_2}$ coincides with the function $\tilde{R}_{i,e,f_1}$ in (120) (see Subsection 3.3 in [10] for the precise expression) if we replace f_1 with f_2. Therefore, from estimates similar to the previous ones (which use especially the computations in Subsection 4.1 in [10]) we find

$$(168) \qquad \int_{\tilde{D}_\varepsilon} \check{v}\, \overline{v_{\underline{f}_2}} dV_{\tilde{g}_\varepsilon} = \varepsilon^2 C_0 \int_0^L T(f_2) \underline{f}_2 \, d\overline{s} + \varepsilon^2 \int_0^L \left(\int_{\mathbb{R}^{n-1}} F_{4,i,e} U(k(\overline{s})) \right) \underline{f}_2 \, d\overline{s} + R_2,$$

where $C_0 = \int_{\mathbb{R}^{n-1}} U(y)^2 dy$, where

$$(169) \qquad T(f_2) = \partial_{\overline{s}} \left(\frac{h^2 f_2'}{(p-1)k^{n+1}} \left[(p-1)h^{p-1} - 2\sigma \mathcal{A}^2 h^{2\sigma} \right] \right),$$

and where R_2 satisfies

$$(170) \qquad |R_2| \leq C(c_1, c_2, c_3)\delta\varepsilon^2 \|\underline{f}_2\|_{L^2([0,L])}.$$

Moreover, if $\check{\tilde{v}}$ corresponds to the triple $(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})$, then

$$(171) \qquad \int_{\tilde{D}_\varepsilon} (\check{v} - \check{\tilde{v}}) \, \overline{v_{\underline{f}_2}} dV_{\tilde{g}_\varepsilon} = \varepsilon^2 C_0 \int_0^L T(f_2 - \tilde{f}_2) \underline{f}_2 \, d\overline{s} + \tilde{R}_2,$$

with

$$(172) \qquad |\tilde{R}_2| \le C(c_1, c_2, c_3) \left(\delta\varepsilon^2 \|f_2 - \tilde{f}_2\|_{H^2([0,L])} + \delta\varepsilon \|\Phi - \tilde{\Phi}\|_{H^2([0,L])} + \delta\|\beta - \tilde{\beta}\|_\sharp \right) \|\underline{f}_2\|_{L^2([0,L])}.$$

4.2.3 Projection onto $\tilde{K}_{3,\delta}$

To compute the last components of the projection we recall our notation in Subsection 2.3, and define

$$\underline{\beta}(\varepsilon s) = \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \underline{b}_j \beta_j(\varepsilon s); \qquad\qquad v_{\underline{\beta}} = \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \underline{b}_j \tilde{v}_{3,j}.$$

As for the previous cases, the main contribution to the projection comes here still from $S_\varepsilon(\tilde{\Psi}_{2,\varepsilon})$. In particular, following the arguments for $\tilde{K}_{1,\delta}$, when testing on v_β, by Fourier cancelation and parity the major terms are indeed $\mathfrak{A}_{5,0}$, $\mathfrak{A}_{5,r,e}$ and $\mathfrak{A}_{5,i,e}$. With straightforward computations one finds that

$$(173) \qquad \int_{\tilde{D}_\varepsilon} \check{v} \, \overline{v_{\underline{\beta}}} dV_{\tilde{g}_\varepsilon} = \frac{1}{\varepsilon} \int_0^L \Lambda(\beta, \xi, \underline{\beta}, \underline{\xi}) \, d\overline{s} + \hat{R}_3,$$

where

$$\begin{aligned}
\Lambda(\beta, \xi, \underline{\beta}, \underline{\xi}) &= \beta\underline{\beta}Q_{4,\alpha} - \varepsilon^2 \beta'' \underline{\beta} Q_{1,\alpha} - 2\varepsilon\xi' f' \underline{\beta} Q_{3,\alpha} + \xi\underline{\xi}Q_{5,\alpha} - \varepsilon^2\xi''\underline{\xi}Q_{2,\alpha} - \varepsilon f'' \left(\xi\underline{\beta} - \underline{\xi}\beta\right) Q_{3,\alpha} \\
&+ \; 2\varepsilon\beta' f'\underline{\xi}Q_{3,\alpha} - 2\varepsilon^2\alpha' \left(\beta'\underline{\beta}Q_{6,\alpha} + \underline{\xi}\xi'Q_{7,\alpha}\right) - 2\varepsilon^2 k' \left(\beta'\underline{\beta}Q_{10,\alpha} + \underline{\xi}\xi'Q_{11,\alpha}\right) \\
&- \; 2\varepsilon f' k' \left(\xi\underline{\beta}Q_{12,\alpha} - \beta\underline{\xi}Q_{13,\alpha}\right) - 2\varepsilon f'\alpha' \left(\xi\underline{\beta}Q_{8,\alpha} - \underline{\xi}\beta Q_{9,\alpha}\right) - (p-1)\varepsilon h^{p-2} \left(\underline{\xi}\beta + \xi\underline{\beta}\right) Q_{14,\alpha};
\end{aligned}$$

$$Q_{4,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} Z_{\alpha(\overline{s})} \mathcal{L}_r Z_{\alpha(\overline{s})}; \qquad Q_{5,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} W_{\alpha(\overline{s})} \mathcal{L}_i W_{\alpha(\overline{s})}; \qquad Q_{6,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} Z_{\alpha(\overline{s})} \frac{\partial Z_{\alpha(\overline{s})}}{\partial\alpha};$$

$$Q_{7,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} W_{\alpha(\overline{s})} \frac{\partial W_{\alpha(\overline{s})}}{\partial\alpha} \qquad Q_{8,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} Z_{\alpha(\overline{s})} \frac{\partial W_{\alpha(\overline{s})}}{\partial\alpha}; \qquad Q_{9,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} W_{\alpha(\overline{s})} \frac{\partial Z_{\alpha(\overline{s})}}{\partial\alpha};$$

$$Q_{10,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} Z_{\alpha(\overline{s})} \nabla_z Z_{\alpha(\overline{s})} \cdot z; \qquad Q_{11,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} W_{\alpha(\overline{s})} \nabla_z W_{\alpha(\overline{s})} \cdot z;$$

$$Q_{12,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} Z_{\alpha(\overline{s})} \nabla_z W_{\alpha(\overline{s})} \cdot z; \qquad Q_{13,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} W_{\alpha(\overline{s})} \nabla_z Z_{\alpha(\overline{s})} \cdot z;$$

$$Q_{14,\alpha}(\overline{s}) = \int_{\mathbb{R}^{n-1}} U(kz)^{p-2} w_{i,e} W_{\alpha(\overline{s})} Z_{\alpha(\overline{s})}.$$

$$(174) \qquad |\hat{R}_3| \le C(c_1, c_2, c_3) \delta\varepsilon^2 \|\underline{\beta}\|_{L^2([0,L])}.$$

After some manipulation using the fact that (Z_α, W_α) solve (46) with $\eta_\alpha = 0$, the normalization $\int_{\mathbb{R}^{n-1}} (Z_\alpha^2 + W_\alpha^2) = 1$ and some integration by parts in z we find that

$$(175) \qquad \frac{1}{\varepsilon} \int_0^L \Lambda(\beta, \xi, \underline{\beta}, \underline{\xi}) \, d\overline{s} = \frac{1}{\varepsilon} \int_0^L \Lambda_0(\beta, \xi, \underline{\beta}, \underline{\xi}) \, d\overline{s} + \int_0^L \Lambda_1(\beta, \xi, \underline{\beta}, \underline{\xi}) \, d\overline{s},$$

where

$$\begin{aligned}
(176) \qquad \Lambda_0(\beta, \xi, \underline{\beta}, \underline{\xi}) &= Q_{1,\alpha} \left(\varepsilon^2\beta'\underline{\beta}' - k^2\alpha^2\beta\underline{\beta}\right) + Q_{2,\alpha} \left(\varepsilon^2\xi'\underline{\xi}' - \alpha^2 k^2\xi\underline{\xi}\right) \\
&+ \; 2f'Q_{3,\alpha} \left(\varepsilon\beta'\underline{\xi} - \varepsilon\xi'\underline{\beta} - k\alpha\beta\underline{\beta} - k\alpha\xi\underline{\xi}\right);
\end{aligned}$$

$$\Lambda_1 = (\beta\underline{\xi} + \xi\underline{\beta})\mathfrak{g}(\overline{s}); \qquad \mathfrak{g}(\overline{s}) = \left[f''Q_{3,\alpha} + 2f'k'Q_{13,\alpha} + 2f'\alpha'Q_{9,\alpha} - (p-1)h^{p-2}Q_{14,\alpha}\right].$$

Now we notice that, by (60), one has

$$\beta\underline{\xi} + \xi\underline{\beta} = -\frac{\varepsilon}{k\alpha}\left[\sum_{j,l=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j\underline{b_l}(\xi'_j\xi_l + \xi_j\xi'_l) - F_1 \sum_{j,l=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j\underline{b_l}(\nu_j\xi_l\xi'_j + \nu_l\xi_j\xi'_l)\right],$$

where $F_1 = \frac{Q_{1,\alpha}}{k^2\alpha^2 + 2f'k\alpha Q_{3,\alpha}}$. Integrating by parts in $\overline{s}$ and using (59) we find that

$$(177) \qquad \int_0^L \Lambda_1(\beta,\xi,\underline{\beta},\underline{\xi})\,d\overline{s} = \varepsilon\int_0^L \xi\underline{\xi}\left(\frac{\mathfrak{g}}{k\alpha}\right)'(\overline{s})d\overline{s} + O(\delta^2)\|\beta\|_{L^2([0,L])}\|\underline{\beta}\|_{L^2([0,L])}.$$

Finally, combining (173), (174), (175) and (177) we deduce

$$(178) \qquad \int_{\tilde{D}_\varepsilon} \check{v}\,\overline{v_{\underline{\beta}}}dV_{\tilde{g}_\varepsilon} = \frac{1}{\varepsilon}\int_0^L \Lambda_0(\beta,\xi,\underline{\beta},\underline{\xi})\,d\overline{s} + R_3,$$

where

$$(179) \qquad |R_3| \le C(c_1,c_2,c_3)(\delta\varepsilon^2 + (\varepsilon + \delta^2)\|\beta\|_{L^2([0,L])})\|\underline{\beta}\|_{L^2([0,L])}.$$

Analogously we obtain

$$(180) \qquad \int_{\tilde{D}_\varepsilon} (\check{v} - \tilde{v})\,\overline{v_{\underline{\beta}}}dV_{\tilde{g}_\varepsilon} = \frac{1}{\varepsilon}\int_0^L \Lambda_0(\beta - \tilde{\beta}, \xi - \tilde{\xi}, \underline{\beta}, \underline{\xi}) + \tilde{R}_3,$$

where $\tilde{R}_3$ satisfies

$$(181) \qquad |\tilde{R}_3| \le C(c_1,c_2,c_3)\delta\left(\varepsilon\|\Phi - \tilde{\Phi}\|_{H^2([0,L])} + \varepsilon^2\|f_2 - \tilde{f}_2\|_{H^2([0,L])} + \|\beta - \tilde{\beta}\|_\sharp\right)\|\underline{\beta}\|_{L^2([0,L])}.$$

Remark 4.3 *Let us consider the eigenvalue problem in* (β,ξ)

$$\int_0^L \Lambda_0(\beta,\xi,\underline{\beta},\underline{\xi}) = \nu\int_0^L (Q_{1,\alpha}\beta\underline{\beta} + Q_{2,\alpha}\xi\underline{\xi}) \qquad \text{for all } (\underline{\beta},\underline{\xi})$$

where Q_1, Q_2 *are defined in* (58). *Then the eigenvalue equation is the following*

$$(182) \qquad \begin{cases} -\varepsilon^2\frac{(Q_{1,\alpha}\beta')'}{Q_{1,\alpha}} - k^2\alpha^2\beta - 2f'\frac{Q_{3,\alpha}}{Q_{1,\alpha}}(\varepsilon\xi' + k\alpha\beta) = \nu\beta; \\ -\varepsilon^2\frac{(Q_{2,\alpha}\xi')'}{Q_{2,\alpha}} - k^2\alpha^2\xi + 2f'\frac{Q_{3,\alpha}}{Q_{2,\alpha}}(\varepsilon\beta' - k\alpha\xi) = \nu\xi. \end{cases}$$

By (61), *the couple of functions* (β_j, ξ_j) *constructed in Subsection 2.3 represents a family of approximate eigenfunctions corresponding to* $\nu = \nu_j$.

4.3 The contraction argument

The usual procedure in performing a fixed point argument is to apply to the equation an invertible linear operator first. From the expansions in the last subsection, we showed that the main terms in the projections onto $\tilde{K}_\delta$ are the operators $\mathfrak{J}$, T and Λ_0 (the latter is identified by duality with the associated quadratic form), see (165), (168) and (178). By our non-degeneracy assumption on γ, $\mathfrak{J}$ is invertible and the same holds also for T, since it is coercive (and in divergence form). It remains then to invert Λ_0, which is the content of the next result: before stating it we introduce some notation. Using the symbology of Subsection 2.3 we define the spaces

$$X_{1,\delta} = span\left\{\varphi_j \;:\; j = 0,\ldots,\frac{\delta}{\varepsilon}\right\}; \qquad X_{2,\delta} = span\left\{\omega_j \;:\; j = 0,\ldots,\frac{\delta}{\varepsilon}\right\};$$

$$X_{3,\delta} = span\left\{\beta_j \ : \ j = -\frac{\delta^2}{\varepsilon}, \dots, \frac{\delta^2}{\varepsilon}\right\},$$

with $X_{1,\delta}$, $X_{2,\delta}$ endowed with the H^2 norm on $[0, L]$, and $X_{3,\delta}$ with the $\|\cdot\|_\sharp$ norm.

We also call $Y_{1,\delta}, Y_{2,\delta}, Y_{3,\delta}$ the same spaces of functions, but endowed with weighted L^2 norms: by the normalization after (55) it is natural to put the weights h^θ and $h^{-\sigma}$ on $Y_{1,\delta}$ and $Y_{2,\delta}$ respectively. Concerning $Y_{3,\delta}$, by Remark 4.3, we will endow it with the product $(\beta, \underline{\beta})_{Y_{3,\delta}} = \int_0^L (Q_{1,\alpha}\beta\underline{\beta} + Q_{2,\alpha}\xi\underline{\xi})d\overline{s}$ where, as above, ξ is related to β by (60) and (124). Notice that by (55) $\mathfrak{J}$ and T are exactly diagonal from $X_{1,\delta}$ to $Y_{1,\delta}$ and from $X_{2,\delta}$ to $Y_{2,\delta}$ respectively, while Λ_0 is nearly diagonal (see also (61)).

Lemma 4.4 *Letting $\Pi_{Y_{3,\delta}}$ denote the orthogonal projection onto $Y_{3,\delta}$, there exists a sequence $\varepsilon_k \to 0$ such that Λ_0 is invertible from $X_{3,\delta}$ into $Y_{3,\delta}$ and such that its inverse satisfies $\|(\Pi_{Y_{3,\delta}}\Lambda_0)^{-1}\| \le \frac{C}{\varepsilon_k}$ for some fixed constant C.*

PROOF. First of all we show that there exists $\varepsilon_k \to 0$ such that $\Pi_{Y_{3,\delta}}\Lambda_0$ cannot have eigenvalues in $Y_{3,\delta}$ smaller in absolute value than $C^{-1}\varepsilon_k$: after this, we estimate the (stronger) $X_{3,\delta}$ norm of its inverse.

To prove the claim we apply Kato's theorem (see [5], page 445): the latter allows to compute the derivative of an eigenvalue $\nu(\varepsilon)$ of $\Pi_{Y_{3,\delta}}\Lambda_0$ with respect to ε. The (possibly multiple) value of this derivative is given by the eigenvalues of $\Pi_{Y_{3,\delta}}\partial_\varepsilon\Lambda_0$, restricted to the $\nu(\varepsilon)$-eigenspace of $\Pi_{Y_{3,\delta}}\Lambda_0$.

Suppose that β satisfies the eigenvalue equation $\Pi_{Y_{3,\delta}}\Lambda_0\beta = \nu\beta$, which is equivalent to

$$(183) \qquad \int_0^L \Lambda_0(\beta, \xi, \underline{\beta}, \underline{\xi}) = \nu \int_0^L (Q_{1,\alpha}\beta\underline{\beta} + Q_{2,\alpha}\xi\underline{\xi}) \qquad \text{for all } (\underline{\beta}, \underline{\xi}) \text{ with } \underline{\beta} \in Y_{3,\delta}.$$

Looking at the powers of ε in Λ_0, see (176), we write $\Lambda_0 = \Lambda_{0,0} + \varepsilon\Lambda_{0,1} + \varepsilon^2\Lambda_{0,2}$: notice that $\Lambda_{0,0}$ is negative-definite and $\Lambda_{0,2}$ positive-definite. We also point out that, since f' satisfies (11), for $f(\varepsilon s)/\varepsilon$ to be L/ε-periodic, when we vary ε also $\mathcal{A}$ needs to be adjusted. Precisely, since the total variation of phase in (9) is

$$\mathcal{A}\int_0^{L/\varepsilon} h(\varepsilon s)^\sigma ds = \frac{\mathcal{A}}{\varepsilon}\int_0^L h(\overline{s})d\overline{s} = const.,$$

when differentiating with respect to ε we find that $\frac{\partial\mathcal{A}}{\partial\varepsilon} = \frac{\mathcal{A}}{\varepsilon}$. Hence, applying Kato's theorem we find

$$(184) \qquad \frac{\partial\nu}{\partial\varepsilon} \in \left[\min_{\beta_1,\beta_2\neq 0}\Theta(\beta_1,\beta_2), \max_{\beta_1,\beta_2\neq 0}\Theta(\beta_1,\beta_2)\right],$$

where

$$\Theta(\beta_1,\beta_2) = \frac{\int_0^L(\Lambda_{0,1} + 2\varepsilon\Lambda_{0,2})(\beta_1,\xi_1,\beta_2,\xi_2)}{\int_0^L(Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)} + \frac{1}{\varepsilon}\frac{\int_0^L(2f'Q_{3,\alpha}(\varepsilon\beta_1'\xi_2 - \varepsilon\xi_1'\beta_2 - k\alpha\beta_1\beta_2 - k\alpha\xi_1\xi_2)}{\int_0^L(Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)},$$

and where $(\beta_1, \xi_1), (\beta_2, \xi_2)$ are functions satisfying (183): using this, $\Theta(\beta_1, \beta_2)$ can be written as

$$\frac{\frac{1}{\varepsilon}\int_0^L(\Lambda_0 - \Lambda_{0,0})(\beta_1,\xi_1,\beta_2,\xi_2) + \varepsilon\int_0^L\Lambda_{0,2}(\beta_1,\xi_1,\beta_2,\xi_2)}{\int_0^L(Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)}$$

$$+ \ \frac{1}{\varepsilon}\frac{\int_0^L(2f'Q_{3,\alpha}(\varepsilon\beta_1'\xi_2 - \varepsilon\xi_1'\beta_2 - k\alpha\beta_1\beta_2 - k\alpha\xi_1\xi_2)}{\int_0^L(Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)}$$

$$= \ \frac{\nu}{\varepsilon} + \frac{\varepsilon\int_0^L(Q_{1,\alpha}\beta_1'\beta_2' + Q_{2,\alpha}\xi_1'\xi_2') + \frac{1}{\varepsilon}\int_0^L[k^2\alpha^2(Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2) + 2f'k\alpha Q_{3,\alpha}(\beta_1\beta_2 + \xi_1\xi_2)]}{\int_0^L(Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)}$$

$$+ \ \frac{1}{\varepsilon}\frac{\int_0^L(2f'Q_{3,\alpha}(\varepsilon\beta_1'\xi_2 - \varepsilon\xi_1'\beta_2 - k\alpha\beta_1\beta_2 - k\alpha\xi_1\xi_2)}{\int_0^L(Q_{1,\alpha}\beta_1\beta_2 + Q_{2,\alpha}\xi_1\xi_2)}.$$

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Applying (59), (60) and $Q_{1,\alpha} + Q_{2,\alpha} = 1$ (see (58) and the lines after (47)), the last expression simplifies as

$$\frac{\nu}{\varepsilon} + \frac{1}{\varepsilon} \frac{\int_0^L (2\alpha^2 k^2 + 4f'\alpha k Q_{3,\alpha})\xi_1 \xi_2}{\int_0^L \xi_1 \xi_2} + O(\delta^2)\frac{1}{\varepsilon}.$$

Since the numerator is symmetric in ξ_1, ξ_2, the infimum of the above ratio is realized by some ξ_0, so by (184) and the latter formula we find

$$(185) \quad \frac{\partial \nu}{\partial \varepsilon} \geq \frac{\nu}{\varepsilon} + \frac{1}{\varepsilon} \frac{\int_0^L (2\alpha^2 k^2 + 4f'\alpha k Q_{3,\alpha})\xi_0^2}{\int_0^L \xi_0^2} + O(\delta^2)\frac{1}{\varepsilon} \geq \frac{1}{\varepsilon}\left[\nu + \inf_{[0,L]} \left(2\alpha^2 k^2 + 4f'\alpha k Q_{3,\alpha}\right) - C\delta^2\right].$$

Notice that for ν and δ sufficiently small, the coefficient of $\frac{1}{\varepsilon}$ in the above formula is positive and uniformly bounded away from zero. From (61) and the asymptotics in (59) (which follows from the Weyl's formula), one can show that $\Pi_{Y_{3,\delta}}\Lambda_0$ has a number of negative eigenvalues of order $\frac{\delta^2}{\varepsilon}$. This fact and (185) yield the desired claim, which can be obtained as in [14], Proposition 4.5: since the argument is quite similar, we omit the details.

The above claim provides invertibility of $\Pi_{Y_{3,\delta}}\Lambda_0$ in $Y_{3,\delta}$, and gives

$$(186) \quad \|(\Pi_{Y_{3,\delta}}\Lambda_0)^{-1}\beta\|_{Y_{3,\delta}} \leq \frac{C}{\varepsilon}\|\beta\|_{Y_{3,\delta}} \qquad \text{for any } \beta \in Y_{3,\delta}:$$

we want next to estimate the $X_{3,\delta}$ norm of $(\Pi_{Y_{3,\delta}}\Lambda_0)^{-1}\beta$. Let $\beta = \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} b_j \beta_j$ and suppose $\hat{\beta} = \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \hat{b}_j \beta_j$ is such that $\Pi_{Y_{3,\delta}}\Lambda_0 \hat{\beta} = \beta$, in the sense that

$$\int_0^L \Lambda_0(\hat{\beta}, \hat{\xi}, \underline{\beta}, \underline{\xi}) = \int_0^L (Q_{1,\alpha}\beta\underline{\beta} + Q_{2,\alpha}\xi\underline{\xi}) \qquad \text{for all } (\underline{\beta}, \underline{\xi}) \text{ with } \underline{\beta} \in Y_{3,\delta}.$$

If $\underline{\beta} = \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \underline{b}_j \beta_j$, then by (61) integrating one finds

$$\sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \nu_j \hat{b}_j \underline{b}_j + O\left(\left(\sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} (\nu_j^2 + \varepsilon)^2 \hat{b}_j^2\right)^{\frac{1}{2}} \left(\sum_{l=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \underline{b}_l^2\right)^{\frac{1}{2}}\right) \leq C\left(\sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \hat{b}_j^2\right)^{\frac{1}{2}} \left(\sum_{l=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \underline{b}_l^2\right)^{\frac{1}{2}}.$$

Choosing $\underline{b}_j = \hat{b}_j$ for $j > 0$ and $\underline{b}_j = -\hat{b}_j$ for $j < 0$, from the asymptotics of ν_j in (59) we obtain for $\tilde{C}_1 > 0$ sufficiently large that

$$\varepsilon \sum_{\tilde{C}_1 \leq j \leq \frac{\delta^2}{\varepsilon}} |j|\, \hat{b}_j^2 \leq C \sum_{j=-\frac{\delta^2}{\varepsilon}}^{\frac{\delta^2}{\varepsilon}} \hat{b}_j^2 \leq C\|\hat{\beta}\|_{Y_{3,\delta}}^2.$$

By (186) we have $\|\hat{\beta}\|_{Y_{3,\delta}} \leq \frac{C}{\varepsilon}\|\beta\|_{Y_{3,\delta}}$, so recalling (125) we get $\|\hat{\beta}\|_{X_{3,\delta}}^2 := \|\hat{\beta}\|_{\sharp}^2 \leq C\|\hat{\beta}\|_{Y_{3,\delta}}^2 \leq \frac{C^2}{\varepsilon^2}\|\beta\|_{Y_{3,\delta}}^2$, which yields the conclusion. ∎

PROOF OF THEOREM 1.1 Let us introduce the operators

$$G_l : X_{1,\delta} \times X_{2,\delta} \times X_{3,\delta} \to Y_{l,\delta}, \qquad l = 1, 2, 3,$$

defined by duality as

$$(G_1(\Phi, f_2, \beta), \underline{\Phi})_{Y_{1,\delta}} = \int_{\tilde{D}_\varepsilon} \check{v}\, \overline{v_{\underline{\Phi}}}\, dV_{\tilde{g}_\varepsilon}; \qquad (G_2(\Phi, f_2, \beta), \underline{f}_2)_{Y_{2,\delta}} = \int_{\tilde{D}_\varepsilon} \check{v}\, \overline{v_{\underline{f}_2}}\, dV_{\tilde{g}_\varepsilon};$$

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$$(G_3(\Phi, f_2, \beta), \underline{\beta})_{Y_{3,\delta}} = \int_{\tilde{D}_\varepsilon} \check{v}\,\overline{v_{\underline{\beta}}}\, dV_{\tilde{g}_\varepsilon},$$

where $\check{v} = \check{v}(\Phi, f_2, \beta)$ is the function appearing in Proposition 4.1.

By Proposition 2.4, equation (14) (or (1)) is solved if and only if $\check{v} = 0$: in the above notations, this is equivalent to finding (Φ, f_2, β) such that $G_l(\Phi, f_2, \beta) = 0$ for every $l = 1, 2, 3$. If ε_k is the sequence given in Lemma 4.4 then Λ_0 is invertible, and the condition $\check{v} = 0$ is equivalent to the system (we set $\varepsilon = \varepsilon_k$)

$$(187) \quad \begin{cases} \Phi = \mathfrak{G}_1(\Phi, f_2, \beta) := -\frac{1}{\varepsilon}\tilde{\mathfrak{J}}^{-1}[G_1(\Phi, f_2, \beta) - \varepsilon\tilde{\mathfrak{J}}(\Phi)] \\[2mm] f_2 - \check{f}_2 = \mathfrak{G}_2(\Phi, f_2, \beta) := -\frac{1}{\varepsilon^2}\tilde{T}^{-1}\left[G_2(\Phi, f_2, \beta) - \varepsilon^2\tilde{T}f_2 - \varepsilon^2\int_{\mathbb{R}^{n-1}} F_{4,i,e}U(k(\overline{s})z)dz\right] \\[2mm] \beta = \mathfrak{G}_3(\Phi, f_2, \beta) := -\varepsilon(\Pi_{Y_{3,\delta}}\Lambda_0)^{-1}\left[G_3(\Phi, f_2, \beta) - \frac{1}{\varepsilon}\Pi_{Y_{3,\delta}}\Lambda_0\beta\right], \end{cases}$$

where $\tilde{\mathfrak{J}} = -\frac{p-1}{2\theta}C_0\mathfrak{J}$, $\tilde{T} = C_0 T$ ($C_0 = \int_{\mathbb{R}^{n-1}} U(y)^2 dy$) and where $\check{f}_2 = -\tilde{T}^{-1}(\int_{\mathbb{R}^{n-1}} F_{4,i,e}U(k(\overline{s})z)dz)$. By (165)-(172), (178)-(180) and (181) one finds

$$\|\mathfrak{G}_1(0,0,0)\|_{X_{1,\delta}} \le C\varepsilon; \qquad \|\mathfrak{G}_2(0,0,0)\|_{X_{2,\delta}} \le C\delta; \qquad \|\mathfrak{G}_3(0,0,0)\|_{X_{3,\delta}} \le C\delta\varepsilon^2;$$

moreover if Φ, f_2, β satisfy the bounds (121), (126), then

$$\|\mathfrak{G}_1(\Phi, f_2, \beta) - \mathfrak{G}_1(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})\|_{X_{1,\delta}} \le C(c_1, c_2, c_3)\left(\delta\|\Phi - \tilde{\Phi}\|_{X_{1,\delta}} + \varepsilon\|f_2 - \tilde{f}_2\|_{X_{2,\delta}} + \frac{\delta}{\varepsilon}\|\beta - \tilde{\beta}\|_{X_{3,\delta}}\right);$$

$$\|\mathfrak{G}_2(\Phi, f_2, \beta) - \mathfrak{G}_2(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})\|_{X_{2,\delta}} \le C(c_1, c_2, c_3)\left(\frac{\delta}{\varepsilon}\|\Phi - \tilde{\Phi}\|_{X_{1,\delta}} + \delta\|f_2 - \tilde{f}_2\|_{X_{2,\delta}} + \frac{\delta}{\varepsilon^2}\|\beta - \tilde{\beta}\|_{X_{3,\delta}}\right);$$

$$\|\mathfrak{G}_3(\Phi, f_2, \beta) - \mathfrak{G}_3(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})\|_{X_{3,\delta}} \le C(c_1, c_2, c_3)\left(\delta\varepsilon\|\Phi - \tilde{\Phi}\|_{X_{1,\delta}} + \delta\varepsilon^2\|f_2 - \tilde{f}_2\|_{X_{2,\delta}} + \delta\|\beta - \tilde{\beta}\|_{X_{3,\delta}}\right).$$

We now consider the scaled norms $\varepsilon\|\cdot\|_{\hat{X}_{1,\delta}} = \|\cdot\|_{X_{1,\delta}}$, $\delta^{\frac{1}{2}}\|\cdot\|_{\hat{X}_{2,\delta}} = \|\cdot\|_{X_{2,\delta}}$, $\varepsilon^2\|\cdot\|_{\hat{X}_{3,\delta}} = \|\cdot\|_{X_{3,\delta}}$: with this new notation the last formulas become

$$(188) \qquad \|\mathfrak{G}_1(0,0,0)\|_{\hat{X}_{1,\delta}} \le C; \qquad \|\mathfrak{G}_2(0,0,0)\|_{\hat{X}_{2,\delta}} \le C\delta^{\frac{1}{2}}; \qquad \|\mathfrak{G}_3(0,0,0)\|_{\hat{X}_{3,\delta}} \le C\delta;$$

$$\|\mathfrak{G}_1(\Phi, f_2, \beta) - \mathfrak{G}_1(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})\|_{\hat{X}_{1,\delta}} \le C(c_1, c_2, c_3)\left(\delta\|\Phi - \tilde{\Phi}\|_{\hat{X}_{1,\delta}} + \delta^{\frac{1}{2}}\|f_2 - \tilde{f}_2\|_{\hat{X}_{2,\delta}} + \delta\|\beta - \tilde{\beta}\|_{\hat{X}_{3,\delta}}\right);$$

$$\|\mathfrak{G}_2(\Phi, f_2, \beta) - \mathfrak{G}_2(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})\|_{\hat{X}_{2,\delta}} \le C(c_1, c_2, c_3)\left(\delta^{\frac{1}{2}}\|\Phi - \tilde{\Phi}\|_{\hat{X}_{1,\delta}} + \delta\|f_2 - \tilde{f}_2\|_{\hat{X}_{2,\delta}} + \delta^{\frac{1}{2}}\|\beta - \tilde{\beta}\|_{\hat{X}_{3,\delta}}\right);$$

$$\|\mathfrak{G}_3(\Phi, f_2, \beta) - \mathfrak{G}_3(\tilde{\Phi}, \tilde{f}_2, \tilde{\beta})\|_{\hat{X}_{3,\delta}} \le C(c_1, c_2, c_3)\left(\delta\|\Phi - \tilde{\Phi}\|_{\hat{X}_{1,\delta}} + \delta^{\frac{3}{2}}\|f_2 - \tilde{f}_2\|_{\hat{X}_{2,\delta}} + \delta\|\beta - \tilde{\beta}\|_{\hat{X}_{3,\delta}}\right).$$

If C is the constant appearing in (188), from the last four formulas we deduce that if δ is sufficiently small then $(\mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{G}_3)$ has a fixed point in $\{\|\cdot\|_{\hat{X}_{1,\delta}} \le 2C\} \cap \{\|\cdot - \check{f}_2\|_{\hat{X}_{2,\delta}} \le 2C\delta^{\frac{1}{2}}\} \cap \{\|\cdot\|_{\hat{X}_{3,\delta}} \le 2C\delta\}$. This, by the comments before (187), leads to a solution of (1) with the desired asymptotics. $\blacksquare$

Acknowledgments

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