QUASI-PERIODIC SOLUTIONS OF THE EQUATION $v_{tt} - v_{xx} + v^3 = f(v)$

PIETRO BALDI

Abstract. We consider 1D completely resonant nonlinear wave equations of the type $v_{tt} - v_{xx} = -v^3 + \mathcal{O}(v^4)$ with spatial periodic boundary conditions. We prove the existence of a new type of quasi-periodic small amplitude solutions with two frequencies, for more general nonlinearities. These solutions turn out to be, at the first order, the superposition of a traveling wave and a modulation of long period, depending only on time.

1. Introduction

This paper deals with a class of one-dimensional completely resonant nonlinear wave equations of the type

$$\begin{cases} v_{tt} - v_{xx} = -v^3 + f(v) \\ v(t,x) = v(t, x+2\pi), \qquad (t,x) \in \mathbb{R}^2 \end{cases} \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is analytic in a neighborhood of $v = 0$ and $f(v) = \mathcal{O}(v^4)$ as $v \to 0$.

In the recent paper [12], M. Procesi proved the existence of small-amplitude quasi-periodic solutions of (1) of the form

$$v(t,x) = u(\omega_1 t + x,\, \omega_2 t - x), \tag{2}$$

where u is an odd analytic function, 2π-periodic in both its arguments, and the frequencies $\omega_1, \omega_2 \sim 1$ belong to a Cantor-like set of zero Lebesgue measure. It is assumed that f is odd and $f(v) = \mathcal{O}(v^5)$, see Theorem 1 in [12].

These solutions $v(t,x)$ correspond — at the first order — to the superposition of two waves, traveling in opposite directions:

$$v(t,x) = \sqrt{\varepsilon}\big[r(\omega_1 t + x) + s(\omega_2 t - x) + h.o.t.\big]$$

where $\omega_1, \omega_2 = 1 + \mathcal{O}(\varepsilon)$.

Motivated by the previous result, we study in the present paper the existence of quasi-periodic solutions of (1) having a different form, namely

$$v(t,x) = u(\omega_1 t + x,\, \omega_2 t + x). \tag{3}$$

Moreover we do not assume f to be odd.

First of all, we have to consider different frequencies than in [12]. Precisely, the appropriate choice for the relationship between the amplitude ε and the frequencies ω_1, ω_2 turns out to be

$$\omega_1 = 1 + \varepsilon + b\varepsilon^2, \quad \omega_2 = 1 + b\varepsilon^2, \tag{4}$$

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where $b \sim 1/2,\ \varepsilon \sim 0$. This choice leads to look for quasi-periodic solutions $v(t,x)$ of (1) of the form

$$v(t,x) = u\big(\varepsilon t,\ (1+b\varepsilon^2)t + x\big), \tag{5}$$

where $(b,\varepsilon) \in \mathbb{R}^2$, $\frac{1+b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}$. On the contrary, taking in (3) frequencies $\omega_1 = 1+\varepsilon,\ \omega_2 = 1 + a\varepsilon$ as in [12], no quasi-periodic solutions can be found, see Remark in section 2. We show that there is no loss of generality passing from (3) to (5), because all the possible quasi-periodic solutions of (1) of the form (3) are of the form (5), see Appendix B.

Searching small amplitude quasi-periodic solutions of the form (5) by means of the Lyapunov-Schmidt method, leads to the usual system of a range equation and a bifurcation equation.

The former is solved, in a similar way as [12], by means of the standard Contraction Mapping Theorem, for a set of zero measure of the parameters. These arguments are carried out in section 4.

In section 5 we study the bifurcation equation, which is infinite-dimensional because we deal with a completely resonant equation. Here new difficulties have to be overcome. Since f is not supposed to be odd, we cannot search odd solutions as in [12], so we look for even solutions. In this way, the bifurcation equation contains a new scalar equation for the average of u, see [C-equation] in (12), and the other equations contain supplementary terms.

To solve the bifurcation equation we use an ODE analysis; we cannot directly use variational methods as in [3],[4],[6] because we have to ensure that both components r, s in (12) are non-trivial, in order to prove that the solution v is actually quasi-periodic.

First, we find an explicit solution of the bifurcation equation (Lemma 1) by means of Jacobi elliptic functions (following [11],[12],[9]).

Next we prove its non-degeneracy (Lemmas 2,3,4); these computations are the heart of the present work. Instead of using a computer assisted proof as in [12], we here employ purely analytic arguments, see also [11] (however, our problem requires much more involved computations than in [11]). In this way we prove the existence of quasi-periodic solutions of (1) of the form (5), see Theorem 1 (end of section 5).

¿From the physical point of view, this new class of solutions turns out to be, at the first order, the superposition of a traveling wave (with velocity greater than 1) and a modulation of long period, depending only on time:

$$v(t,x) = \varepsilon\big[r(\varepsilon t) + s((1+b\varepsilon^2)t + x) + h.o.t.\big].$$

Finally, in section 6 we show that our arguments can be also used to extend Procesi result to non-odd nonlinearities, see Theorem 2.

We also mention that recently existence of quasi-periodic solutions with n frequencies have been proved in [16]. The solutions found in [16] belong to a neighborhood of a solution $u_0(t)$ periodic in time, independent of x, so they are different from the ones found in the present paper.

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2. The functional setting

We consider nonlinear wave equation (1),

$$\begin{cases} v_{tt} - v_{xx} = -v^3 + f(v) \\ v(t,x) = v(t, x+2\pi) \end{cases}$$

where f is analytic in a neighborhood of $v = 0$ and $f(v) = \mathcal{O}(v^4)$ as $v \to 0$. We look for solutions of the form (3),

$$v(t,x) = u(\omega_1 t + x,\, \omega_2 t + x),$$

for $(\omega_1, \omega_2) \in \mathbb{R}^2$, $\omega_1, \omega_2 \sim 1$ and u 2π-periodic in both its arguments. Solutions $v(t,x)$ of the form (3) are *quasi-periodic* in time t when u actually depends on both its arguments and the ratio between the periods is irrational, $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$.

We set the problem in the space $\mathcal{H}_\sigma$ defined as follows. Denote $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the unitary circle, $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T}^2$. If u is doubly 2π-periodic, $u : \mathbb{T}^2 \to \mathbb{R}$, its Fourier series is

$$u(\varphi) = \sum_{(m,n)\in\mathbb{Z}^2} \hat{u}_{mn}\, e^{im\varphi_1} e^{in\varphi_2}. \tag{6}$$

Let $\sigma > 0$, $s \geq 0$. We define $\mathcal{H}_\sigma$ as the space of the even 2π-periodic functions $u : \mathbb{T}^2 \to \mathbb{R}$ which satisfy

$$\sum_{(m,n)\in\mathbb{Z}^2} |\hat{u}_{mn}|^2 \left[1 + (m^2+n^2)^s\right] e^{2\sqrt{m^2+n^2}\,\sigma} := \|u\|_\sigma^2 < \infty.$$

The elements of $\mathcal{H}_\sigma$ are even periodic functions which admit an analytic extension to the complex strip $\{z \in \mathbb{C} : |\mathrm{Im}(z)| < \sigma\}$.

$(\mathcal{H}_\sigma, \|\cdot\|_\sigma)$ is a Hilbert space; for $s > 1$ it is also an algebra, that is, there exists a constant $c > 0$ such that

$$\|uv\|_\sigma \leq c\, \|u\|_\sigma \, \|v\|_\sigma \quad \forall\, u, v \in \mathcal{H}_\sigma,$$

see Appendix A. Moreover the inclusion $\mathcal{H}_{\sigma,s+1} \hookrightarrow \mathcal{H}_{\sigma,s}$ is compact.

We fix $s > 1$ once and for all.

We note that all the possible quasi-periodic solutions of (1) of the form (3) are of the form (5) if we choose frequencies as in (4), see Appendix B. So we can look for solutions of (1) of the form (5), $v(t,x) = u\big(\varepsilon t,\, (1+b\varepsilon^2)t + x\big)$, without loss of generality. For functions of the form (5), problem (1) is written as

$$\begin{cases} \varepsilon \left[\varepsilon\, \partial^2_{\varphi_1} + 2(1+b\varepsilon^2)\, \partial^2_{\varphi_1\varphi_2} + b\varepsilon(2+b\varepsilon^2)\, \partial^2_{\varphi_2}\right](u) = -u^3 + f(u) \\ u \in \mathcal{H}_\sigma. \end{cases}$$

We define $M_{b,\varepsilon} = \varepsilon\, \partial^2_{\varphi_1} + 2(1+b\varepsilon^2)\, \partial^2_{\varphi_1\varphi_2} + b\varepsilon(2+b\varepsilon^2)\, \partial^2_{\varphi_2}$, rescale $u \to \varepsilon u$ and set $f_\varepsilon(u) = \varepsilon^{-3} f(\varepsilon u)$, so (1) can be written as

$$\begin{cases} M_{b,\varepsilon}[u] = -\varepsilon u^3 + \varepsilon f_\varepsilon(u) \\ u \in \mathcal{H}_\sigma. \end{cases} \tag{7}$$

The main result of the present paper is the existence of solutions $u_{(b,\varepsilon)}$ of (7) for (b, ε) in a suitable uncountable set (Theorem 1).

Remark. If we simply choose frequencies $\omega_1 = 1 + \varepsilon$, $\omega_2 = 1 + a\varepsilon$ as in [12], we obtain a bifurcation equation different than (12). Precisely, it appears 0 instead of $-b(2 + b\varepsilon^2)\, s''$ in the left-hand term of the Q_2-equation in (12); so we do not find solutions which are non-trivial in both its arguments, but only solutions depending on the variable φ_1. This is a problem because the quasi-periodicity condition requires dependence on both variables.

So we have to choose frequencies depending on ε in a more general way; a good choice is (4), $\omega_1 = 1 + \varepsilon + b\varepsilon^2$, $\omega_2 = 1 + b\varepsilon^2$.

3. Lyapunov-Schmidt reduction

The operator $M_{b,\varepsilon}$ is diagonal in the Fourier basis $e_{mn} = e^{im\varphi_1} e^{in\varphi_2}$ with eigenvalues $-D_{b,\varepsilon}(m,n)$, that is, if u is written in Fourier series as in (6),

$$M_{b,\varepsilon}[u] = - \sum_{(m,n)\in\mathbb{Z}^2} D_{b,\varepsilon}(m,n)\, \hat{u}_{mn}\, e^{im\varphi_1}\, e^{in\varphi_2}, \tag{8}$$

where the eigenvalues $D_{b,\varepsilon}(m,n)$ are given by

$$\begin{aligned} D_{b,\varepsilon}(m,n) &= \varepsilon\, m^2 + 2(1+b\varepsilon^2)\, mn + b\varepsilon(2+b\varepsilon^2)\, n^2 \\ &= (2+b\varepsilon^2) \Big(\frac{\varepsilon}{2+b\varepsilon^2}\, m + n \Big) \big(m + b\varepsilon\, n \big). \end{aligned} \tag{9}$$

For $\varepsilon = 0$ the operator is $M_{b,0} = 2\,\partial^2_{\varphi_1\varphi_2}$; its kernel Z is the subspace of functions of the form $u(\varphi_1,\varphi_2) = r(\varphi_1) + s(\varphi_2)$ for some r, $s \in \mathcal{H}_\sigma$ one-variable functions,

$$Z = \big\{ u \in \mathcal{H}_\sigma : \hat{u}_{mn} = 0 \ \ \forall\, (m,n) \in \mathbb{Z}^2,\ m,n \neq 0 \big\}.$$

We can decompose $\mathcal{H}_\sigma$ in four subspaces setting

$$\begin{aligned} &C = \{ u \in \mathcal{H}_\sigma : u(\varphi) = \hat{u}_{0,0} \} \cong \mathbb{R}, \\ &Q_1 = \{ u \in \mathcal{H}_\sigma : u(\varphi) = \textstyle\sum_{m\neq 0} \hat{u}_{m,0}\, e^{im\varphi_1} = r(\varphi_1) \}, \\ &Q_2 = \{ u \in \mathcal{H}_\sigma : u(\varphi) = \textstyle\sum_{n\neq 0} \hat{u}_{0,n}\, e^{in\varphi_2} = s(\varphi_2) \}, \\ &P = \{ u \in \mathcal{H}_\sigma : u(\varphi) = \textstyle\sum_{m,n\neq 0} \hat{u}_{mn}\, e^{im\varphi_1}\, e^{in\varphi_2} = p(\varphi_1,\varphi_2) \}. \end{aligned} \tag{10}$$

Thus the kernel is the direct sum $Z = C \oplus Q_1 \oplus Q_2$ and the whole space is $\mathcal{H}_\sigma = Z \oplus P$. Any element u can be decomposed as

$$\begin{aligned} u(\varphi) = &\ \hat{u}_{0,0} + r(\varphi_1) + s(\varphi_2) + p(\varphi_1,\varphi_2) \\ = &\ z(\varphi) + p(\varphi). \end{aligned} \tag{11}$$

We denote $\langle\,\cdot\,\rangle$ the integral average: given $g \in \mathcal{H}_\sigma$,

$$\langle g \rangle = \langle g \rangle_\varphi = \tfrac{1}{(2\pi)^2} \textstyle\int_0^{2\pi}\int_0^{2\pi} g(\varphi)\, d\varphi_1 d\varphi_2,$$

$$\langle g \rangle_{\varphi_1} = \tfrac{1}{2\pi} \textstyle\int_0^{2\pi} g(\varphi)\, d\varphi_1, \qquad \langle g \rangle_{\varphi_2} = \tfrac{1}{2\pi} \textstyle\int_0^{2\pi} g(\varphi)\, d\varphi_2.$$

Note that $\frac{1}{2\pi}\int_0^{2\pi} e^{ikt}\, dt = 0$ for all integers $k \neq 0$, so

$$\begin{aligned} &\langle r \rangle = \langle r \rangle_{\varphi_1} = 0 && \langle r \rangle_{\varphi_2} = r \\ &\langle s \rangle = \langle s \rangle_{\varphi_2} = 0 && \langle s \rangle_{\varphi_1} = s \\ &\langle p \rangle = \langle p \rangle_{\varphi_1} = \langle p \rangle_{\varphi_2} = 0 && \langle u \rangle = \hat{u}_{0,0} \end{aligned}$$

for all $r \in Q_1$, $s \in Q_2$, $p \in P$, $u \in \mathcal{H}_\sigma$, and by means of these averages we can construct the projections on the subspaces,

$$\Pi_C = \langle\,\cdot\,\rangle, \qquad \Pi_{Q_1} = \langle\,\cdot\,\rangle_{\varphi_2} - \langle\,\cdot\,\rangle, \qquad \Pi_{Q_2} = \langle\,\cdot\,\rangle_{\varphi_1} - \langle\,\cdot\,\rangle.$$

Let $u = z + p$ as in (11); we write u^3 as $u^3 = z^3 + (u^3 - z^3)$ and compute the cube $z^3 = (\hat{u}_{0,0} + r + s)^3$. The operator $M_{b,\varepsilon}$ maps every subspace of (10) in itself and it holds $M_{b,\varepsilon}[r] = \varepsilon r''$, $M_{b,\varepsilon}[s] = b\varepsilon(2+b\varepsilon^2)s''$, $M_{b,\varepsilon}[\hat{u}_{0,0}] = 0$. So we can project our problem (7) on the four subspaces:

$$\begin{aligned}
0 =&\ \hat{u}_{0,0}^3 + 3\hat{u}_{0,0}\left(\langle r^2\rangle + \langle s^2\rangle\right) + \langle r^3\rangle + \langle s^3\rangle + \\
&+\Pi_C\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad\qquad [C\text{-}equation]\\
-r'' =&\ 3\hat{u}_{0,0}^2 r + 3\hat{u}_{0,0}\left(r^2 - \langle r^2\rangle\right) + r^3 - \langle r^3\rangle + 3\langle s^2\rangle\, r + \\
&+\Pi_{Q_1}\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad\qquad [Q_1\text{-}equation]\\
-b(2+b\varepsilon^2)\, s'' =&\ 3\hat{u}_{0,0}^2 s + 3\hat{u}_{0,0}\left(s^2 - \langle s^2\rangle\right) + s^3 - \langle s^3\rangle + 3\langle r^2\rangle\, s + \\
&+\Pi_{Q_2}\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad\qquad [Q_2\text{-}equation]\\
M_{b,\varepsilon}[p] =&\ \varepsilon\,\Pi_P\left[-u^3 + f_\varepsilon(u)\right]. \qquad\qquad [P\text{-}equation]
\end{aligned} \tag{12}$$

Now we study separately the projected equations.

4. The range equation

We write the P-equation thinking p as variable and z as a "parameter",

$$M_{b,\varepsilon}[p] = \varepsilon\,\Pi_P\left[-(z+p)^3 + f_\varepsilon(z+p)\right].$$

We would like to invert the operator $M_{b,\varepsilon}$. In Appendix C we prove that, fixed any $\gamma \in (0, \frac{1}{4})$, there exists a non-empty uncountable set $\mathcal{B}_\gamma \subseteq \mathbb{R}^2$ such that, for all $(b,\varepsilon) \in \mathcal{B}_\gamma$, it holds

$$|D_{b,\varepsilon}(m,n)| > \gamma \qquad \forall\, m,n \in \mathbb{Z},\ m,n \neq 0.$$

Precisely, our Cantor set $\mathcal{B}_\gamma$ is

$$\mathcal{B}_\gamma = \left\{(b,\varepsilon) \in \mathbb{R}^2 :\ \frac{\varepsilon}{2+b\varepsilon^2}, b\varepsilon^2 \in \tilde{B}_\gamma,\ \ \left|\frac{\varepsilon}{2+b\varepsilon^2}\right|, |b\varepsilon^2| < \frac{1}{4},\ \ \frac{1+b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}\right\},$$

where $\tilde{B}_\gamma$ is a set of "badly approximable numbers" defined as

$$\tilde{B}_\gamma = \left\{x \in \mathbb{R} : |m + nx| > \frac{\gamma}{|n|}\ \ \forall\, m,n \in \mathbb{Z},\ m \neq 0,\ n \neq 0\right\}, \tag{13}$$

see Appendix C. Therefore $M_{b,\varepsilon|P}$ is invertible for $(b,\varepsilon) \in \mathcal{B}_\gamma$ and by (8) it follows

$$(M_{b,\varepsilon|P})^{-1}[h] = -\sum_{m,n\neq 0} \frac{\hat{h}_{mn}}{D_{b,\varepsilon}(m,n)}\, e^{im\varphi_1}\, e^{in\varphi_2}$$

for every $h = \sum_{m,n\neq 0} \hat{h}_{mn}\, e^{im\varphi_1}\, e^{in\varphi_2} \in P$. Thus we obtain a bound for the inverse operators, uniformely in $(b,\varepsilon) \in \mathcal{B}_\gamma$:

$$\left\|(M_{b,\varepsilon|P})^{-1}\right\| \leq \frac{1}{\gamma}.$$

Applying the inverse operator $(M_{b,\varepsilon|P})^{-1}$, the P-equation becomes

$$p + \varepsilon (M_{b,\varepsilon|P})^{-1}\Pi_P\left[(z+p)^3 - f_\varepsilon(z+p)\right] = 0. \tag{14}$$

We would like to apply the Implicit Function Theorem, but the inverse operator $(M_{b,\varepsilon|P})^{-1}$ is defined only for $(b,\varepsilon) \in \mathcal{B}_\gamma$ and in the set $\mathcal{B}_\gamma$ there are infinitely many holes, see Appendix C. So we fix $(b,\varepsilon) \in \mathcal{B}_\gamma$, introduce an auxiliary parameter μ and consider the auxiliary equation

$$p + \mu (M_{b,\varepsilon|P})^{-1}\Pi_P\left[(z+p)^3 - f_\mu(z+p)\right] = 0. \tag{15}$$

Following Lemma 2.2 in [12], we can prove, by the standard Contraction Mapping Theorem, that there exists a positive constant c_1 depending only on f such that, if

$$(\mu, z) \in \mathbb{R} \times Z, \qquad |\mu| \, \|z\|_\sigma^2 < c_1 \gamma, \tag{16}$$

equation (15) admits a solution $p_{(b,\varepsilon)}(\mu, z) \in P$. Moreover, there exists a positive constant c_2 such that the solution $p_{(b,\varepsilon)}(\mu, z)$ respects the bound

$$\left\| p_{(b,\varepsilon)}(\mu, z) \right\|_\sigma \le \frac{c_2}{\gamma} \, \|z\|_\sigma^3 \, |\mu|. \tag{17}$$

Than we can apply the Implicit Function Theorem to the operator

$$\begin{aligned} &\mathbb{R} \times Z \times P \longrightarrow P \\ &(\mu, z, p) \longmapsto p + \mu (M_{b,\varepsilon|P})^{-1} \Pi_P \left[(z+p)^3 - f_\mu(z+p) \right] \end{aligned}$$

at every point $(0, z, 0)$, so, by local uniqueness, we obtain the regularity: $p_{(b,\varepsilon)}$, as function of (μ, z), is at least of class $\mathcal{C}^1$.

Notice that the domain of any function $p_{(b,\varepsilon)}$ is defined by (16), so it does not depend on $(b, \varepsilon) \in \mathcal{B}_\gamma$.

In order to solve (14), we will need to evaluate $p_{(b,\varepsilon)}$ at $\mu = \varepsilon$; we will do it as last step, after the study of the bifurcation equation.

We observe that in these computations we have used the Hilbert algebra property of the space $\mathcal{H}_\sigma$, $\|uv\|_\sigma \le c \, \|u\|_\sigma \, \|v\|_\sigma \; \forall \, u, v \in \mathcal{H}_\sigma$.

5. The bifurcation equation

We consider auxiliary Z-equations: we put f_μ instead of f_ε in (12),

$$\begin{aligned} 0 =\ & \hat{u}_{0,0}^3 + 3\hat{u}_{0,0} \left(\langle r^2 \rangle + \langle s^2 \rangle \right) + \langle r^3 \rangle + \langle s^3 \rangle + \\ & + \Pi_C \left[(u^3 - z^3) - f_\mu(u) \right] \qquad [C - \mathit{equation}] \\[1ex] -r'' =\ & 3\hat{u}_{0,0}^2 r + 3\hat{u}_{0,0} \left(r^2 - \langle r^2 \rangle \right) + r^3 - \langle r^3 \rangle + 3\langle s^2 \rangle \, r + \\ & + \Pi_{Q_1} \left[(u^3 - z^3) - f_\mu(u) \right] \qquad [Q_1 - \mathit{equation}] \\[1ex] -b(2 + b\varepsilon^2) \, s'' =\ & 3\hat{u}_{0,0}^2 s + 3\hat{u}_{0,0} \left(s^2 - \langle s^2 \rangle \right) + s^3 - \langle s^3 \rangle + 3\langle r^2 \rangle \, s + \\ & + \Pi_{Q_2} \left[(u^3 - z^3) - f_\mu(u) \right]. \qquad [Q_2 - \mathit{equation}] \end{aligned} \tag{18}$$

We substitute the solution $p_{(b,\varepsilon)}(\mu, z)$ of the auxiliary P-equation (15) inside the auxiliary Z-equations (18), writing $u = z + p = z + p_{(b,\varepsilon)}(\mu, z)$, for (μ, z) in the domain (16) of $p_{(b,\varepsilon)}$.

We have $p_{(b,\varepsilon)}(\mu, z) = 0$ for $\mu = 0$, so the term $\left[(u^3 - z^3) - f_\mu(u) \right]$ vanishes for $\mu = 0$ and the bifurcation equations at $\mu = 0$ become

$$\begin{aligned} 0 =\ & \hat{u}_{0,0}^3 + 3\hat{u}_{0,0} \left(\langle r^2 \rangle + \langle s^2 \rangle \right) + \langle r^3 \rangle + \langle s^3 \rangle \\ -r'' =\ & 3\hat{u}_{0,0}^2 r + 3\hat{u}_{0,0} \left(r^2 - \langle r^2 \rangle \right) + r^3 - \langle r^3 \rangle + 3\langle s^2 \rangle \, r \\ -b(2 + b\varepsilon^2) \, s'' =\ & 3\hat{u}_{0,0}^2 s + 3\hat{u}_{0,0} \left(s^2 - \langle s^2 \rangle \right) + s^3 - \langle s^3 \rangle + 3\langle r^2 \rangle \, s. \end{aligned} \tag{19}$$

We look for non-trivial $z = \hat{u}_{0,0} + r(\varphi_1) + s(\varphi_2)$ solution of (19). We rescale setting

$$\begin{aligned} r &= x & \hat{u}_{0,0} &= c \\ s &= \sqrt{b(2 + b\varepsilon^2)} \; y & \lambda &= \lambda_{b,\varepsilon} = b(2 + b\varepsilon^2), \end{aligned} \tag{20}$$

so the equations become

$$\begin{aligned} & c^3 + 3c\big(\langle x^2\rangle + \lambda\langle y^2\rangle\big) + \langle x^3\rangle + \lambda^{3/2}\langle y^3\rangle = 0 \\ & x'' + 3c^2 x + 3c\big(x^2 - \langle x^2\rangle\big) + x^3 - \langle x^3\rangle + 3\lambda\langle y^2\rangle\, x = 0 \\ & y'' + 3c^2\tfrac{1}{\lambda}y + 3c\tfrac{1}{\sqrt{\lambda}}\big(y^2 - \langle y^2\rangle\big) + y^3 - \langle y^3\rangle + 3\tfrac{1}{\lambda}\langle x^2\rangle\, y = 0. \end{aligned} \tag{21}$$

In the following we show that, for $|\lambda - 1|$ sufficiently small, the system (21) admits a non-trivial non-degenerate solution. We consider λ as a free real parameter, recall that $Z = C \times Q_1 \times Q_2$ and define $G : \mathbb{R} \times Z \to Z$ setting $G(\lambda, c, x, y)$ as the set of three left-hand terms of (21).

Lemma 1. *There exist $\bar\sigma > 0$ and a non-trivial one-variable even analytic function β_0 belonging to $\mathcal{H}_\sigma$ for every $\sigma \in (0, \bar\sigma)$, such that $G(1, 0, \beta_0, \beta_0) = 0$, that is $(0, \beta_0, \beta_0)$ solves (21) for $\lambda = 1$.*

Proof. We prove the existence of a non-trivial even analytic function β_0 which satisfies

$$\beta_0'' + \beta_0^3 + 3\langle\beta_0^2\rangle\,\beta_0 = 0, \qquad \langle\beta_0\rangle = \langle\beta_0^3\rangle = 0. \tag{22}$$

For any $m \in (0,1)$ we consider the Jacobi amplitude $\mathrm{am}(\,\cdot\,, m) : \mathbb{R} \to \mathbb{R}$ as the inverse of the elliptic integral of the first kind

$$I(\,\cdot\,, m) : \mathbb{R} \to \mathbb{R}, \qquad I(\varphi, m) = \int_0^\varphi \frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}}.$$

We define the Jacobi elliptic cosine setting

$$\mathrm{cn}(\xi) = \mathrm{cn}(\xi, m) = \cos(\mathrm{am}(\xi, m)),$$

see [1] ch.16, [15]. It is a periodic function of period $4K$, where $K = K(m)$ is the complete elliptic integral of the first kind

$$K(m) = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m\sin^2\vartheta}}.$$

Jacobi cosine is even, and it is also odd-symmetric with respect to K on $[0, 2K]$, that is $\mathrm{cn}(\xi + K) = -\mathrm{cn}(\xi - K)$, just like the usual cosine. Then the averages on the period $4K$ are

$$\langle\mathrm{cn}\rangle = \langle\mathrm{cn}^3\rangle = 0.$$

Therefore it admits an analytic extension with a pole at iK', where $K' = K(1-m)$, and it satisfies $(\mathrm{cn}')^2 = -m\,\mathrm{cn}^4 + (2m-1)\,\mathrm{cn}^2 + (1-m)$, then cn is a solution of the ODE

$$\mathrm{cn}'' + 2m\,\mathrm{cn}^3 + (1 - 2m)\,\mathrm{cn} = 0.$$

We set $\beta_0(\xi) = V\mathrm{cn}(\Omega\xi, m)$ for some real parameters $V, \Omega > 0$, $m \in (0,1)$. β_0 has a pole at $i\frac{K'}{\Omega}$, so it belongs to $\mathcal{H}_\sigma$ for every $0 < \sigma < \frac{K'}{\Omega}$.

β_0 satisfies

$$\beta_0'' + \Big(2m\frac{\Omega^2}{V^2}\Big)\,\beta_0^3 + \Omega^2(1 - 2m)\,\beta_0 = 0.$$

If there holds the equality $2m\Omega^2 = V^2$, the equation becomes

$$\beta_0'' + \beta_0^3 + \Omega^2(1 - 2m)\,\beta_0 = 0.$$

β_0 is $\frac{4K(m)}{\Omega}$-periodic; it is 2π-periodic if $\Omega = \frac{2K(m)}{\pi}$. Hence we require

$$2m\Omega^2 = V^2, \qquad \Omega = \frac{2K(m)}{\pi}. \tag{23}$$

The other Jacobi elliptic functions we will use are

$$\mathrm{sn}(\xi) = \sin(\mathrm{am}(\xi, m)), \qquad \mathrm{dn}(\xi) = \sqrt{1 - m\,\mathrm{sn}^2(\xi)},$$

see [1],[15]. From the equality $m\,\mathrm{cn}^2(\xi) = \mathrm{dn}^2(\xi) - (1-m)$, with change of variable $x = \mathrm{am}(\xi)$ we obtain

$$\int_0^{K(m)} m\,\mathrm{cn}^2(\xi)\,d\xi = E(m) - (1-m)K(m),$$

where $E(m)$ is the complete elliptic integral of the second kind,

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2\vartheta}\,d\vartheta.$$

Thus the average on $[0, 2\pi]$ of β_0^2 is

$$\langle \beta_0^2 \rangle = \frac{V^2}{m\,K(m)}\big[E(m) - (1-m)K(m)\big].$$

We want the equality $3\langle\beta_0^2\rangle = \Omega^2(1-2m)$ and this is true if

$$E(m) + \frac{8m-7}{6}\,K(m) = 0. \tag{24}$$

The left-hand term $\psi(m) := E(m) + \frac{8m-7}{6}\,K(m)$ is continuous in m; its value at $m = 0$ is $-(\pi/12) < 0$, while at $m = 1/2$, by definition of E and K,

$$\psi\Big(\frac{1}{2}\Big) = \frac{1}{2}\int_0^{\pi/2} \frac{\cos^2\vartheta}{\big(1 - \frac{1}{2}\sin^2\vartheta\big)^{1/2}}\,d\vartheta > 0.$$

Moreover, its derivative is strictly positive for every $m \in \big[0, \frac{1}{2}\big]$,

$$\begin{aligned}\psi'(m) &= \int_0^{\pi/2} \tfrac{8 - \frac{5}{2}\sin^2\vartheta + 3m\sin^4\vartheta - 8m\sin^2\vartheta}{6(1-m\sin^2\vartheta)^{3/2}}\,d\vartheta \\ &\geq \int_0^{\pi/2} \tfrac{3+\cos^2\vartheta}{6}\,d\vartheta > 0,\end{aligned}$$

hence there exists a unique $\bar m \in \big(0, \frac{1}{2}\big)$ which solves (24). Thanks to the tables in [1], p. 608-609, we have $0.20 < \bar m < 0.21$.

By (23) the value $\bar m$ determines the parameters $\bar\Omega$ and $\bar V$, so the function $\beta_0(\xi) = \bar V\mathrm{cn}(\bar\Omega\xi, \bar m)$ satisfies (22) and $(0, \beta_0, \beta_0)$ is a solution of (21) for $\lambda = 1$. Therefore $\beta_0 \in \mathcal{H}_\sigma$ for every $\sigma \in (0, \bar\sigma)$, where $\bar\sigma = (\frac{K'}{\Omega})_{|m=\bar m}$. □

The next step will be to prove the non-degeneracy of the solution $(1, 0, \beta_0, \beta_0)$, that is to show that the partial derivative $\partial_Z G(1, 0, \beta_0, \beta_0)$ is an invertible operator. This is the heart of the present paper. We need some preliminary results.

Lemma 2. *Given h even 2π-periodic, there exists a unique even 2π-periodic w such that*

$$w'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)w = h.$$

This defines the Green operator $L : \mathcal{H}_\sigma \to \mathcal{H}_\sigma$, $L[h] = w$.

Proof. We fix a 2π-periodic even function h. We look for even 2π-periodic solutions of the non-homogeneous equation

$$x'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)x = h. \tag{25}$$

First of all, we construct two solutions of the homogeneous equation

$$x'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)x = 0. \tag{26}$$

We recall that β_0 satisfies $\beta_0'' + \beta_0^3 + 3\langle\beta_0^2\rangle\,\beta_0 = 0$, then deriving with respect to its argument ξ we obtain $\beta_0''' + 3\beta_0^2\beta_0' + 3\langle\beta_0^2\rangle\beta_0' = 0$, so β_0' satisfies (26). We set

$$\bar u(\xi) = -\frac{1}{\bar V\bar\Omega^2}\beta_0'(\xi) = -\frac{1}{\bar\Omega}\,\mathrm{cn}'(\bar\Omega\xi, \bar m), \tag{27}$$

thus $\bar u$ is the solution of the homogeneous equation such that $\bar u(0) = 0,\ \bar u'(0) = 1.$ It is odd and 2π-periodic.

Now we construct the other solution. We indicate c_0 the constant $c_0 = \langle \beta_0 \rangle$. We recall that, for any V, Ω, m the function $y(\xi) = V \mathrm{cn}(\Omega\xi, m)$ satisfies

$$y'' + \Big(2m\frac{\Omega^2}{V^2}\Big)\, y^3 + \Omega^2(1-2m)\, y = 0.$$

We consider m and V as functions of the parameter $\Omega,$ setting

$$m = m(\Omega) = \frac{1}{2} - \frac{3c_0}{2\Omega^2}, \qquad V = V(\Omega) = \sqrt{\Omega^2 - 3c_0}. \tag{28}$$

We indicate $y_\Omega(\xi) = V(\Omega)\,\mathrm{cn}\big(\Omega\xi, m(\Omega)\big),$ so $(y_\Omega)_\Omega$ is a one-parameter family of solutions of

$$y_\Omega'' + y_\Omega^3 + 3c_0 y_\Omega = 0.$$

We can derive this equation with respect to $\Omega,$ obtaining

$$(\partial_\Omega y_\Omega)'' + 3y_\Omega^2(\partial_\Omega y_\Omega) + 3c_0(\partial_\Omega y_\Omega) = 0.$$

Now we evaluate $(\partial_\Omega y_\Omega)$ at $\Omega = \bar\Omega,$ where $\bar\Omega$ correspond to the value $\bar m$ found in Lemma 1. For $\Omega = \bar\Omega$ it holds $y_{\bar\Omega} = \beta_0$, so $(\partial_\Omega y_\Omega)_{|\Omega=\bar\Omega}$ satisfy (26). In order to normalize this solution, we compute

$$(\partial_\Omega y_\Omega)(\xi) = (\partial_\Omega V)\mathrm{cn}(\Omega\xi, m) + V\xi\mathrm{cn}'(\Omega\xi, m) + V\partial_m \mathrm{cn}(\Omega\xi, m)(\partial_\Omega m).$$

Since $\mathrm{cn}(0,m) = 1\ \forall m,$ it holds $\partial_m\mathrm{cn}(0,m) = 0;$ therefore $\mathrm{cn}'(0,m) = 0\ \forall m.$ From (28) we have $\partial_\Omega V = \frac{\Omega}{V},$ so we can normalize setting

$$\bar v(\xi) = \frac{\bar V}{\bar\Omega}\,\big(\partial_\Omega y_\Omega\big)_{|\Omega=\bar\Omega}(\xi).$$

$\bar v$ is the solution of the homogeneous equation (26) such that $\bar v(0) = 1,\ \bar v'(0) = 0.$ We can write an explicit formula for $\bar v$. From the definitions it follows for any m

$$\partial_m \mathrm{am}(\xi, m) = -\mathrm{dn}(\xi, m)\frac{1}{2}\int_0^\xi \frac{\mathrm{sn}^2(t,m)}{\mathrm{dn}^2(t,m)}\,dt.$$

Therefore $\mathrm{cn}'(\xi) = -\mathrm{sn}(\xi)\,\mathrm{dn}(\xi);$ then we obtain for $(V,\Omega,m) = (\bar V, \bar\Omega, \bar m)$

$$\bar v(\xi) = \mathrm{cn}(\bar\Omega\xi) + \frac{\bar V^2}{\bar\Omega}\,\mathrm{cn}'(\bar\Omega\xi)\left[\xi + \frac{2\bar m - 1}{2}\int_0^\xi \frac{\mathrm{sn}^2(\bar\Omega t)}{\mathrm{dn}^2(\bar\Omega t)}\,dt\right]. \tag{29}$$

By formula (29) we can see that $\bar v$ is even; it is not periodic and there holds

$$\bar v(\xi + 2\pi) - \bar v(\xi) = \frac{\bar V^2 k}{\bar\Omega}\,\mathrm{cn}'(\bar\Omega\xi) = -\bar V^2 k\,\bar u(\xi), \tag{30}$$

where

$$k := 2\pi + \frac{2\bar m - 1}{2}\int_0^{2\pi} \frac{\mathrm{sn}^2(\bar\Omega t)}{\mathrm{dn}^2(\bar\Omega t)}\,dt. \tag{31}$$

¿From the equalities (L.1) and (L.2) of Lemma 3 we obtain

$$k = 2\pi\,\frac{-1 + 16\bar m - 16\bar m^2}{12\bar m(1-\bar m)}, \tag{32}$$

so $k > 0$ because $\bar m \in (0.20, 0.21)$.

We have constructed two solutions $\bar u, \bar v$ of the homogeneous equation; their wronskian $\bar u'\bar v - \bar u\bar v'$ is equal to 1, so we can write a particular solution $\bar w$ of the non-homogeneous equation (25) as

$$\bar w(\xi) = \left(\int_0^\xi h\bar v\right)\bar u(\xi) - \left(\int_0^\xi h\bar u\right)\bar v(\xi).$$

Every solution of (25) is of the form $w = A\bar{u} + B\bar{v} + \bar{w}$ for some $(A, B) \in \mathbb{R}^2$. Since h is even, $\bar{w}$ is also even, so w is even if and only if $A = 0$.

An even function $w = B\bar{v} + \bar{w}$ is 2π-periodic if and only if $w(\xi + 2\pi) - w(\xi) = 0$, that is, by (30),

$$\Big(\int_{\xi}^{\xi+2\pi} h\bar{v} \Big) \bar{u}(\xi) + \Big[\Big(\int_0^{\xi} h\bar{u} \Big) - B \Big] \bar{V}^2 k \, \bar{u}(\xi) = 0 \quad \forall \, \xi.$$

We remove $\bar{u}(\xi)$, derive the expression with respect to ξ and from (30) it results zero at any ξ. Then the expression is a constant; we compute it at $\xi = 0$ and obtain, since $h\bar{u}$ is odd and 2π-periodic, that w is 2π-periodic if and only if $B = \frac{1}{\bar{V}^2 k} \int_0^{2\pi} h\bar{v}$.

Thus, given h even 2π-periodic, there exists a unique even 2π-periodic w such that $w'' + \big(3\beta_0^2 + 3\langle \beta_0^2 \rangle\big) w = h$ and this defines the operator L,

$$L[h] = \Big(\int_0^{\xi} h\bar{v} \Big) \bar{u}(\xi) + \Big[\Big(\frac{1}{\bar{V}^2 k} \int_0^{2\pi} h\bar{v} \Big) - \int_0^{\xi} h\bar{u} \Big] \bar{v}(\xi). \tag{33}$$

L is linear and continuous with respect to $\| \cdot \|_{\sigma}$; it is the Green operator of the equation $x'' + \big(3\beta_0^2 + 3\langle \beta_0^2 \rangle\big) x = h$, so, by classical arguments, it is a bounded operator of $\mathcal{H}_{\sigma,s}$ into $\mathcal{H}_{\sigma,s+2}$; the inclusion $\mathcal{H}_{\sigma,s+2} \hookrightarrow \mathcal{H}_{\sigma,s}$ is compact, then $L : \mathcal{H}_{\sigma} \to \mathcal{H}_{\sigma}$ is compact. □

Lemma 3. *There holds the following equalities and inequalities:*

(L.1) $\langle \mathrm{cn}^2 \rangle = \frac{1-2\bar{m}}{6\bar{m}}$ for $m = \bar{m}$. (Recall: $\mathrm{cn} = \mathrm{cn}(\,\cdot\,, m)$)

(L.2) $\langle \frac{\mathrm{sn}^2}{\mathrm{dn}^2} \rangle = \frac{1}{1-m} \langle \mathrm{cn}^2 \rangle$ for any m.

(L.3) $m \langle \mathrm{cn}^2 \frac{\mathrm{sn}^2}{\mathrm{dn}^2} \rangle = 1 - 2 \langle \mathrm{cn}^2 \rangle$ for any m.

(L.4) *Exchange rule.* $\langle g L[h] \rangle = \langle h L[g] \rangle \quad \forall \, g, h$ even 2π-periodic.

(L.5) $1 - 3 \langle \beta_0^2 L[1] \rangle = 3 \langle \beta_0^2 \rangle \langle L[1] \rangle$.

(L.6) $\langle \beta_0^2 L[\beta_0] \rangle = - \langle \beta_0^2 \rangle \langle L[\beta_0] \rangle$.

(L.7) $3 \langle \beta_0^2 L[\beta_0^2] \rangle = \langle \beta_0^2 \rangle \Big(1 - 3 \langle L[\beta_0^2] \rangle \Big)$.

(L.8) $\langle \beta_0^2 L[\beta_0] \rangle = \langle \beta_0 L[\beta_0^2] \rangle = \langle L[\beta_0] \rangle = 0$.

(L.9) $A_0 := \; 1 - 3 \langle \beta_0^2 L[1] \rangle \neq 0$.

(L.10) $B_0 := \; 1 - 6 \langle \beta_0 L[\beta_0] \rangle \neq 0$.

(L.11) $C_0 := \; 1 + 6 \langle \beta_0 L[\beta_0] \rangle \neq 0$.

(L.12) $A_0 \neq 1, \;\; \langle L[\beta_0^2] \rangle \neq 0$.

Proof. (L.1) By construction of β_0 we have $\bar{\Omega}^2(1 - 2\bar{m}) = 3\langle \beta_0^2 \rangle = 3\bar{V}^2 \langle \mathrm{cn}^2(\cdot, \bar{m}) \rangle$ and $\bar{V}^2 = 2\bar{m}\bar{\Omega}^2$, see Proof of Lemma 1.

(L.2) We observe that

$$\frac{d}{d\xi} \left[\frac{\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)} \right] = \frac{(m-1)\mathrm{sn}(\xi)}{\mathrm{dn}^2(\xi)},$$

then we can integrate by parts

$$\int_0^{4K} \frac{\mathrm{sn}^2(\xi)}{\mathrm{dn}^2(\xi)}\, d\xi = \int_0^{4K} \frac{\mathrm{sn}(\xi)}{m-1} \frac{d}{d\xi}\left[\frac{\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)}\right] d\xi = \frac{1}{1-m}\int_0^{4K} \mathrm{cn}^2(\xi)\, d\xi.$$

(L.3) We compute the derivative

$$\frac{d}{d\xi}\left[\frac{\mathrm{cn}(\xi)\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)}\right] = 2\mathrm{cn}^2(\xi) - 1 + m\,\frac{\mathrm{sn}^2(\xi)\mathrm{cn}^2(\xi)}{\mathrm{dn}^2(\xi)}$$

and integrate on the period $[0, 4K]$.

(L.4) From the formula (33) of L we have

$$\begin{aligned}\langle gL[h]\rangle - \langle hL[g]\rangle = {} & \langle \tfrac{d}{d\xi}\big[\big(\textstyle\int_0^\xi h\bar v\big)\big(\int_0^\xi g\bar u\big)\big]\rangle - \langle \tfrac{d}{d\xi}\big[\big(\int_0^\xi h\bar u\big)\big(\int_0^\xi g\bar v\big)\big]\rangle + \\ & + \tfrac{1}{\bar V^2 k}\, 2\pi\big[\langle h\bar v\rangle\langle g\bar v\rangle - \langle g\bar v\rangle\langle h\bar v\rangle\big] = 0.\end{aligned}$$

(L.5) By definition, $L[1]$ satisfies $L[1]'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)L[1] = 1,$ so we integrate on the period $[0, 2\pi]$.

(L.6),(L.7) Similarly by definition of $L[\beta_0],\ L[\beta_0^2];$ recall that $\langle\beta_0\rangle = 0$.

(L.8) By (L.6) and (L.4), it is sufficient to show that $\langle L[\beta_0]\rangle = 0$. From the formula (33), integrating by parts we have

$$\langle L[\beta_0]\rangle = -\langle \beta_0 \bar v\Big(\int_0^\xi \bar u\Big)\rangle - \langle\Big(\int_0^\xi \beta_0\bar u\Big)\bar v\rangle + \frac{1}{\bar V^2 k}\Big(\int_0^{2\pi}\beta_0\bar v\Big)\langle\bar v\rangle.$$

From the formulas (27), (29) of $\bar u, \bar v,$ recalling that $\beta_0(\xi) = \bar V\mathrm{cn}(\bar\Omega\xi)$, we compute

$$\int_0^\xi \bar u = -\frac{1}{\bar\Omega^2}\big(\mathrm{cn}(\bar\Omega\xi) - 1\big), \qquad \int_0^\xi \beta_0\bar u = -\frac{\bar V}{2\bar\Omega^2}\big(\mathrm{cn}^2(\bar\Omega\xi) - 1\big). \tag{34}$$

Observe that $\int_0^{2\pi}\mathrm{cn}(\bar\Omega\xi)\frac{\mathrm{sn}^2(\bar\Omega\xi)}{\mathrm{dn}^2(\bar\Omega\xi)}\, d\xi = 0$ by odd-symmetry with respect to $\frac{\pi}{2}$ on $[0,\pi]$ and periodicity. So, recalling that $\bar V^2 = 2\bar m\bar\Omega^2$, we compute $\langle\bar v\rangle = \frac{\bar m k}{\pi}$. We can resume the computation of $\langle L[\beta_0]\rangle$ obtaining

$$\langle L[\beta_0]\rangle = \frac{3\bar V}{2\bar\Omega^2}\langle\bar v(\xi)\mathrm{cn}^2(\bar\Omega\xi)\rangle - \frac{\bar V\bar m k}{2\pi\bar\Omega^2}.$$

Since $\langle\mathrm{cn}^3\rangle = \langle\mathrm{cn}^3\frac{\mathrm{sn}^2}{\mathrm{dn}^2}\rangle = 0$ by the same odd-symmetry reason, by (29) we have $\langle\bar v(\xi)\mathrm{cn}^2(\bar\Omega\xi)\rangle = \frac{\bar m k}{3\pi}$, and so $\langle L[\beta_0]\rangle = 0$.

Moreover we can remark that by (L.4) there holds also $\langle\beta_0 L[1]\rangle = 0$.

(L.9) By (L.5), it is equivalent to show that $\langle L[1]\rangle \neq 0$. From the formula (33), integrating by parts we have

$$\langle L[1]\rangle = \frac{2\pi}{\bar V^2 k}\langle\bar v\rangle^2 - 2\langle\Big(\int_0^\xi \bar u\Big)\bar v\rangle.$$

We know that $\langle\bar v\rangle = \frac{\bar m k}{\pi}$, so by (34)

$$\langle L[1]\rangle = \frac{1}{\bar\Omega^2}\langle\bar v(\xi)\big(2\mathrm{cn}(\bar\Omega\xi) - 1\big)\rangle.$$

¿From the equalities (L.1) and (L.3) we have $\langle\bar v(\xi)\mathrm{cn}(\bar\Omega\xi)\rangle = \frac{2}{3}(1-2\bar m) + \frac{\bar m k}{2\pi}$, thus

$$\langle L[1]\rangle = \frac{4(1-2\bar m)}{3\bar\Omega^2} \tag{35}$$

and this is strictly positive because $\bar m < \frac{1}{2}$.

(L.10) From (33) integrating by parts we have

$$\langle \beta_0 L[\beta_0] \rangle = -2 \langle \beta_0 \bar{v} \Big(\int_0^\xi \beta_0 \bar{u} \Big) \rangle + \frac{2\pi}{\bar{V}^2 k} \langle \beta_0 \bar{v} \rangle^2.$$

Using (L.3), integrating by parts and recalling the definition (31) of k we compute

$$\langle \beta_0 \bar{v} \rangle = \bar{V} \bar{m} \langle \mathrm{cn}^2 \rangle + \frac{\bar{V} \bar{m} k}{2\pi} + \frac{\bar{V}(1 - 2\bar{m})}{2}$$

and, by (L.1) and (32),

$$\langle \beta_0 \bar{v} \rangle = \frac{\bar{V}(7 - 8\bar{m})}{12(1 - \bar{m})}. \tag{36}$$

By (34), $\langle \beta_0 \bar{v} \Big(\int_0^\xi \beta_0 \bar{u} \Big) \rangle = \frac{\bar{V}}{2\bar{\Omega}^2} \langle \beta_0 \bar{v}\, \mathrm{cn}^2 \rangle + \frac{\bar{V}}{2\bar{\Omega}^2} \langle \beta_0 \bar{v} \rangle$. The functions β_0 and $\bar{v}$ satisfy $\beta_0'' + \beta_0^3 + 3\langle \beta_0^2 \rangle \beta_0 = 0$ and $\bar{v}'' + 3\beta_0^2 \bar{v} + 3\langle \beta_0^2 \rangle \bar{v} = 0$, so that

$$\bar{v}'' \beta_0 - \bar{v} \beta_0'' + 2\beta_0^3 \bar{v} = 0. \tag{37}$$

Deriving (30) we have $\bar{v}'(2\pi) - \bar{v}'(0) = -\bar{V}^2 k$, so we can integrate (37) obtaining

$$\langle \beta_0^3 \bar{v} \rangle = \frac{\bar{V}^3 k}{4\pi};$$

since $\langle \beta_0^3 \bar{v} \rangle = \bar{V}^2 \langle \beta_0 \bar{v}\, \mathrm{cn}^2 \rangle$, we write

$$\langle \beta_0 \bar{v} \Big(\int_0^\xi \beta_0 \bar{u} \Big) \rangle = -\frac{\bar{m} k}{4\pi} + \frac{\bar{V}}{2\bar{\Omega}^2} \langle \beta_0 \bar{v} \rangle.$$

Thus, by (36) and (31), we can express $\langle \beta_0 L[\beta_0] \rangle$ in terms of $\bar{m}$ only,

$$\langle \beta_0 L[\beta_0] \rangle = \frac{32\bar{m}^2 - 32\bar{m} - 1}{12(16\bar{m}^2 - 16\bar{m} + 1)} = \frac{1}{6} - \frac{1}{4(16\bar{m}^2 - 16\bar{m} + 1)}. \tag{38}$$

The polynomial $p(m) = 16m^2 - 16m + 1$ is non-zero for $m \in \big(\frac{2-\sqrt{3}}{4}, \frac{2+\sqrt{3}}{4} \big)$ and $\bar{m} \in (0.20, 0.21)$; so $B_0 = \frac{6}{4p(\bar{m})} \neq 0$, in particular $B_0 \in (-1, -0.9)$.

(L.11) From (38) it follows that $C_0 \neq 0$, in particular $2.9 < C_0 = 2 - \frac{3}{2p(\bar{m})} < 3$.

(L.12) By Exchange rule (L.4) and (L.5), it is sufficient to show that $A_0 \neq 1$, that is $3\langle \beta_0^2 \rangle \langle L[1] \rangle \neq 1$. Recall that, by construction of $\bar{m}$, $3\langle \beta_0^2 \rangle = \bar{\Omega}^2 (1 - 2\bar{m})$. So from (35) it follows

$$3\langle \beta_0^2 \rangle \langle L[1] \rangle = \frac{4}{3} (1 - 2\bar{m})^2,$$

and $\frac{4}{3}(1 - 2m)^2 = 1$ if and only if $16m^2 - 16m + 1 = 0$, while $\bar{m} \in (0.20, 0.21)$, like above; in particular $0.4 < 3\langle \beta_0^2 \rangle \langle L[1] \rangle < 0.5$. □

Remark. Approximated computations give

$\bar{m} \in (0.20,\ 0.21)$	$\bar{\sigma} \in (2.10,\ 2.16)$	$\bar{\Omega} \in (1.05,\ 1.06)$
$\bar{V}^2 \in (0.44,\ 0.48)$	$\langle \mathrm{cn}^2 \rangle \in (2.85,\ 2.90)$	$\langle \beta_0^2 \rangle \in (1.27,\ 1.37)$.

Lemma 4. *The partial derivative* $\partial_Z G(1, 0, \beta_0, \beta_0)$ *is an invertible operator.*

Proof. Let $\partial_Z G(1, 0, \beta_0, \beta_0)[\eta, h, k] = (0, 0, 0)$ for some $(\eta, h, k) \in Z$, that is

$$\begin{aligned} &6\eta \langle \beta_0^2 \rangle + 3\langle \beta_0^2 h \rangle + 3\langle \beta_0^2 k \rangle = 0 \\ &3\eta \big(\beta_0^2 - \langle \beta_0^2 \rangle \big) + h'' + \big(3\beta_0^2 + 3\langle \beta_0^2 \rangle \big) h - 3\langle \beta_0^2 h \rangle + 6\langle \beta_0 k \rangle \beta_0 = 0 \\ &3\eta \big(\beta_0^2 - \langle \beta_0^2 \rangle \big) + k'' + \big(3\beta_0^2 + 3\langle \beta_0^2 \rangle \big) k - 3\langle \beta_0^2 k \rangle + 6\langle \beta_0 h \rangle \beta_0 = 0. \end{aligned} \tag{39}$$

We evaluate the second and the third equation at the same variable and subtract; $\rho = h - k$ satisfies

$$\rho'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)\rho - 3\langle\beta_0^2\rho\rangle - 6\langle\beta_0\rho\rangle\,\beta_0 = 0. \tag{40}$$

By definition of L, see Lemma 2, (40) can be written as

$$\rho = 3\langle\beta_0^2\rho\rangle\,L[1] + 6\langle\beta_0\rho\rangle\,L[\beta_0]. \tag{41}$$

Multiplying this equation by β_0^2 and integrating we obtain

$$\langle\beta_0^2\rho\rangle\,\big(1 - 3\langle\beta_0^2 L[1]\rangle\big) = 6\langle\beta_0\rho\rangle\,\langle\beta_0^2 L[\beta_0]\rangle.$$

In Lemma 3 we prove that $\big(1 - 3\langle\beta_0^2 L[1]\rangle\big) = A_0 \neq 0$ and $\langle\beta_0^2 L[\beta_0]\rangle = 0$, then $\langle\beta_0^2\rho\rangle = 0$.

On the other hand, multiplying (41) by β_0 and integrating we have

$$\langle\beta_0\rho\rangle\,\big(1 - 6\langle\beta_0 L[\beta_0]\rangle\big) = 3\langle\beta_0^2\rho\rangle\,\langle\beta_0 L[1]\rangle;$$

in Lemma 3 we show that $\big(1 - 6\langle\beta_0 L[\beta_0]\rangle\big) = B_0 \neq 0$ and $\langle\beta_0 L[1]\rangle = 0$, then $\langle\beta_0\rho\rangle = 0$. From (41) we have so $\rho = 0$. Thus $h = k$ and (39) becomes

$$\eta\langle\beta_0^2\rangle + \langle\beta_0^2 h\rangle = 0$$

$$3\eta\big(\beta_0^2 - \langle\beta_0^2\rangle\big) + h'' + \big(3\beta_0^2 + 3\langle\beta_0^2\rangle\big)h - 3\langle\beta_0^2 h\rangle + 6\langle\beta_0 h\rangle\,\beta_0 = 0.$$

By substitution we have

$$h = -3\eta L[\beta_0^2] - 6\langle\beta_0 h\rangle\,L[\beta_0].$$

Multiplying, as before, by β_0^2 and by β_0 and integrating, we obtain $\langle\beta_0 h\rangle = \langle\beta_0^2 h\rangle = 0$ because $\big(1 + 6\langle\beta_0 L[\beta_0]\rangle\big) = C_0 \neq 0$, $\langle\beta_0 L[\beta_0^2]\rangle = 0$, and $\langle\beta_0^2\rangle - 3\langle\beta_0^2 L[\beta_0^2]\rangle = 3\langle\beta_0^2\rangle\,\langle L[\beta_0^2]\rangle \neq 0$, see Lemma 3 again. Thus $h = 0$, $\eta = 0$ and the derivative $\partial_Z G(1, 0, \beta_0, \beta_0)$ is injective.

The operator $Z \to Z$, $(\eta, h, k) \mapsto ((6\langle\beta_0^2\rangle)^{-1}\eta,\, L[h],\, L[k])$ is compact because L is compact, see Lemma 2. So, by the Fredholm Alternative, the partial derivative $\partial_Z G(1, 0, \beta_0, \beta_0)$ is also surjective. □

By the Implicit Function Theorem and the regularity of G, using the rescaling (20) we obtain, for $|b - \frac{1}{2}|$ and ε small enough, the existence of a solution close to $(0, \beta_0, \beta_0)$ for the Z-equation (12).

More precisely: from Lemma 1 and 4 it follows the existence of a $\mathcal{C}^1$-function g defined on a neighborhood of $\lambda = 1$ such that

$$G\big(\lambda, g(\lambda)\big) = 0,$$

that is, $g(\lambda)$ solves (21), and $g(1) = (0, \beta_0, \beta_0)$. Moreover, for $|\lambda - 1|$ small, it holds

$$\|g(\lambda) - g(1)\|_\sigma \le \tilde{c}|\lambda - 1| \tag{42}$$

for some positive constant $\tilde{c}$. In the following, we denote several positive constants with the same symbol $\tilde{c}$.

We set $\Phi_{(b,\varepsilon)} : (\hat{u}_{0,0}, r, s) \mapsto (c, x, y)$ the rescaling map (20) and $H_{(b,\varepsilon)} : \mathbb{R}\times Z \to Z$ the operator corresponding to the auxiliary bifurcation equation (18), which so can be written as

$$H_{(b,\varepsilon)}(\mu, z) = 0.$$

We define

$$z^*_{(b,\varepsilon)} = \Phi^{-1}_{(b,\varepsilon)}[g(\lambda_{(b,\varepsilon)})],$$

thus it holds $H_{(b,\varepsilon)}(0, z^*_{(b,\varepsilon)}) = 0$, that is, $z^*_{(b,\varepsilon)}$ solves the bifurcation equation (18) for $\mu = 0$.

We observe that $p_{(b,\varepsilon)}(0,z) = \partial_z p_{(b,\varepsilon)}(0,z) = 0$ for every z and so, in particular, for $z = z^*_{(b,\varepsilon)}$; it follows that

$$\partial_z H_{(b,\varepsilon)}(0, z^*_{(b,\varepsilon)}) = (\Phi^{-1}_{(b,\varepsilon)})^3\, \partial_z G\big(\lambda_{(b,\varepsilon)},\, g(\lambda_{(b,\varepsilon)})\big)\, \Phi_{(b,\varepsilon)}. \tag{43}$$

G is of class $\mathcal{C}^1$, so $\partial_Z G(\lambda, g(\lambda))$ remains invertible for λ sufficiently close to 1. Notice that $\lambda_{(b,\varepsilon)}$ is sufficiently close to 1 if $|b-\frac{1}{2}|$ and ε are small enough. Then, by (43), the partial derivative $\partial_z H_{(b,\varepsilon)}(0, z^*_{(b,\varepsilon)})$ is invertible. By the Implicit Function Theorem, it follows that for every μ sufficiently small there exists a solution $z_{(b,\varepsilon)}(\mu)$ of equation (18), that is

$$H_{(b,\varepsilon)}(\mu, z_{(b,\varepsilon)}(\mu)) = 0.$$

We indicate $z_0 = (0, \beta_0, \beta_0)$. The operators $\big(\partial_z H_{(b,\varepsilon)}(\mu, z)\big)^{-1}$ and $\partial_\mu H_{(b,\varepsilon)}(\mu, z)$ are bounded by some constant for every (μ, z) in a neighborhood of $(0, z_0)$, uniformly in (b,ε), if $|b-1/2|, \varepsilon$ are small enough. So the implicit functions $z_{(b,\varepsilon)}$ are defined on some common interval $(-\mu_0, \mu_0)$ for $|b-1/2|$, ε small, and it holds

$$\|z_{(b,\varepsilon)}(\mu) - z^*_{(b,\varepsilon)}\|_\sigma \le \tilde{c}|\mu| \tag{44}$$

for some $\tilde{c}$ which does not depend on (b,ε).

Such a common interval $(-\mu_0, \mu_0)$ permits the evaluation $z_{(b,\varepsilon)}(\mu)$ at $\mu = \varepsilon$ for $\varepsilon < \mu_0$, obtaining a solution of the original bifurcation equation written in (12).

Moreover, $\|\Phi^{-1}_{(b,\varepsilon)} - \mathrm{Id}_Z\|_\sigma = |\sqrt{b(2+b\varepsilon^2)} - 1| \le |b - \frac{1}{2}| + \varepsilon^2$, so, by (42) and triangular inequality,

$$\|z^*_{(b,\varepsilon)} - z_0\|_\sigma \le \tilde{c}(|b - \tfrac{1}{2}| + \varepsilon^2). \tag{45}$$

Thus from (44) and (45) we have

$$\|z_{(b,\varepsilon)}(\varepsilon) - z_0\|_\sigma \le \tilde{c}(|b - \tfrac{1}{2}| + \varepsilon),$$

and, by (17),

$$\|p\big(\varepsilon,\, z_{(b,\varepsilon)}(\varepsilon)\big)\|_\sigma \le \tilde{c}\varepsilon.$$

Remark. Since the solutions $z_{(b,\varepsilon)}(\varepsilon)$ are close to $z_0 = (0, \beta_0, \beta_0)$, they actually depend on the two arguments (φ_1, φ_2); this is a necessary condition for the quasi-periodicity.

We define $u_{(b,\varepsilon)} = z_{(b,\varepsilon)}(\varepsilon) + p_{(b,\varepsilon)}\big(\varepsilon,\, z_{(b,\varepsilon)}(\varepsilon)\big)$. Renaming $\mu_0 = \varepsilon_0$, we have finally proved:

Theorem 1. *Let $\bar{\sigma} > 0$, β_0 as in Lemma 1, $\tilde{B}_\gamma$ as in (13) with $\gamma \in (0, \frac{1}{4})$. For every $\sigma \in (0, \bar{\sigma})$, there exist positive constants δ_0, ε_0, $\bar{c}_1$, $\bar{c}_2$ and the uncountable Cantor set*

$$\mathcal{B}_\gamma = \Big\{(b,\varepsilon) \in \Big(\frac{1}{2} - \delta_0,\, \frac{1}{2} + \delta_0\Big) \times (0, \varepsilon_0) :\ \frac{\varepsilon}{2 + b\varepsilon^2}, b\varepsilon^2 \in \tilde{B}_\gamma,\quad \frac{1 + b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}\Big\}$$

such that, for every $(b,\varepsilon) \in \mathcal{B}_\gamma$, there exists a solution $u_{(b,\varepsilon)} \in \mathcal{H}_\sigma$ of (7). According to decomposition (11), $u_{(b,\varepsilon)}$ can be written as

$$u_{(b,\varepsilon)}(\varphi_1, \varphi_2) = \hat{u}_{0,0} + r(\varphi_1) + s(\varphi_2) + p(\varphi_1, \varphi_2),$$

where its components satisfy

$$\|r - \beta_0\|_\sigma + \|s - \beta_0\|_\sigma + |\hat{u}_{0,0}| \le \bar{c}_1(|b - \tfrac{1}{2}| + \varepsilon), \qquad \|p\|_\sigma \le \bar{c}_2\varepsilon.$$

As a consequence, problem (1) admits uncountable many small amplitude, analytic, quasi-periodic solutions $v_{(b,\varepsilon)}$ with two frequencies, of the form (5):

$$\begin{aligned} v_{(b,\varepsilon)}(t,x) &= \varepsilon\, u_{(b,\varepsilon)}\big(\varepsilon t,\ (1+b\varepsilon^2)t + x\big) \\ &= \varepsilon\left[\hat{u}_{0,0} + r(\varepsilon t) + s((1+b\varepsilon^2)t + x) + \mathcal{O}(\varepsilon)\right] \\ &= \varepsilon\left[\beta_0(\varepsilon t) + \beta_0((1+b\varepsilon^2)t + x) + \mathcal{O}\big(|b - \tfrac{1}{2}| + \varepsilon\big)\right]. \end{aligned}$$

6. Waves traveling in opposite directions

In this section we look for solutions of (1) of the form (2),

$$v(t,x) = u(\omega_1 t + x,\, \omega_2 t - x),$$

for $u \in \mathcal{H}_\sigma$. We introduce two parameters $(a,\varepsilon) \in \mathbb{R}^2$ and set the frequencies as in [12],

$$\omega_1 = 1 + \varepsilon, \qquad \omega_2 = 1 + a\varepsilon.$$

For functions of the form (2), problem (1) is written as

$$L_{a,\varepsilon}[u] = -u^3 + f(u)$$

where

$$L_{a,\varepsilon} = \varepsilon(2+\varepsilon)\,\partial^2_{\varphi_1} + 2\big(2 + (a+1)\varepsilon + a\varepsilon^2\big)\,\partial^2_{\varphi_1\varphi_2} + a\varepsilon(2+a\varepsilon)\,\partial^2_{\varphi_2}.$$

We rescale $u \to \sqrt{\varepsilon}\, u$ and define $f_\varepsilon(u) = \varepsilon^{-3/2} f(\sqrt{\varepsilon}\, u)$. Thus the problem can be written as

$$L_{a,\varepsilon}[u] = -\varepsilon u^3 + \varepsilon f_\varepsilon(u). \tag{46}$$

For $\varepsilon = 0$, the operator is $L_{a,0} = 4\partial^2_{\varphi_1\varphi_2}$; its kernel is the direct sum $Z = C \oplus Q_1 \oplus Q_2$, see (10). Writing u in Fourier series we obtain an expression similar to (8),

$$L_{a,\varepsilon}[u] = -\sum_{(m,n)\in\mathbb{Z}^2} D_{a,\varepsilon}(m,n)\,\hat{u}_{mn}\, e^{im\varphi_1}\, e^{in\varphi_2},$$

where the eigenvalues $D_{a,\varepsilon}(m,n)$ are given by

$$\begin{aligned} D_{a,\varepsilon}(m,n) &= \varepsilon(2+\varepsilon)\,m^2 + a\varepsilon(2+a\varepsilon)\,n^2 + 2\left(2 + (a+1)\,\varepsilon + a\varepsilon^2\right) mn \\ &= (2+\varepsilon)\,(2+a\varepsilon)\left(m + \frac{a\varepsilon}{2+\varepsilon}\,n\right)\left(\frac{\varepsilon}{2+a\varepsilon}\,m + n\right). \end{aligned}$$

By Lyapunov-Schmidt reduction we project the equation (46) on the four subspaces,

$$\begin{aligned} 0 =&\ \hat{u}^3_{0,0} + 3\hat{u}_{0,0}\left(\langle r^2\rangle + \langle s^2\rangle\right) + \langle r^3\rangle + \langle s^3\rangle + \\ &+ \Pi_C\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad [C - equation] \end{aligned}$$

$$\begin{aligned} -(2+\varepsilon)\,r'' =&\ 3\hat{u}^2_{0,0} r + 3\hat{u}_{0,0}\left(r^2 - \langle r^2\rangle\right) + r^3 - \langle r^3\rangle + 3\langle s^2\rangle\, r + \\ &+ \Pi_{Q_1}\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad [Q_1 - equation] \end{aligned}$$

$$\begin{aligned} -a(2+a\varepsilon)\,s'' =&\ 3\hat{u}^2_{0,0} s + 3\hat{u}_{0,0}\left(s^2 - \langle s^2\rangle\right) + s^3 - \langle s^3\rangle + 3\langle r^2\rangle\, s + \\ &+ \Pi_{Q_2}\left[(u^3 - z^3) - f_\varepsilon(u)\right] \qquad [Q_2 - equation] \end{aligned}$$

$$L_{a,\varepsilon}[p] = \varepsilon\,\Pi_P\left[-u^3 + f_\varepsilon(u)\right]. \qquad [P - equation]$$

We repeat the arguments of Appendix C and find a Cantor set $\mathcal{A}_\gamma$ such that $|D_{a,\varepsilon}(m,n)| > \gamma$ for every $(a,\varepsilon) \in \mathcal{A}_\gamma$. Then $L_{a,\varepsilon}$ is invertible for $(a,\varepsilon) \in \mathcal{A}_\gamma$ and the P-equation can be solved as in the section 4.

We repeat the same procedure already shown in section 5 and solve the bifurcation equation. The only differences are:

- the parameter a tends to 1 instead of $b \to \frac{1}{2}$;
- the rescaling map is $\Psi_{(a,\varepsilon)} : (\hat{u}_{0,0}, r, s) \mapsto (c, x, y)$, where

$$r = \sqrt{2+\varepsilon}\; x \qquad \hat{u}_{0,0} = \sqrt{2+\varepsilon}\; c$$
$$s = \sqrt{a(2+a\varepsilon)}\; y \qquad \lambda = \lambda_{(a,\varepsilon)} = \tfrac{a(2+a\varepsilon)}{2+\varepsilon},$$

instead of $\Phi_{(b,\varepsilon)}$ defined in (20).

We note that by means of the rescaling map $\Psi_{(a,\varepsilon)}$ we obtain just the equation (21). Thus we conclude:

Theorem 2. *Let $\bar{\sigma} > 0$, β_0 as in Lemma 1, $\tilde{B}_\gamma$ as in (13) with $\gamma \in (0, \frac{1}{4})$. For every $\sigma \in (0, \bar{\sigma})$, there exist positive constants δ_0, ε_0, $\bar{c}_1$, $\bar{c}_2$ and the uncountable Cantor set*

$$\mathcal{A}_\gamma = \Big\{ (a,\varepsilon) \in (1-\delta_0,\, 1+\delta_0) \times (0, \varepsilon_0) : \ \frac{a\varepsilon}{2+\varepsilon}, \ \frac{\varepsilon}{2+a\varepsilon} \in \tilde{B}_\gamma, \ \ \frac{1+\varepsilon}{1+a\varepsilon} \notin \mathbb{Q} \Big\}$$

such that, for every $(a,\varepsilon) \in \mathcal{A}_\gamma$, there exists a solution $u_{(a,\varepsilon)} \in \mathcal{H}_\sigma$ of (46). According to decomposition (11), $u_{(a,\varepsilon)}$ can be written as

$$u_{(a,\varepsilon)}(\varphi_1, \varphi_2) = \hat{u}_{0,0} + r(\varphi_1) + s(\varphi_2) + p(\varphi_1, \varphi_2),$$

where its components satisfy

$$\|r - \beta_0\|_\sigma + \|s - \beta_0\|_\sigma + |\hat{u}_{0,0}| \le \bar{c}_1(|a-1| + \varepsilon), \qquad \|p\|_\sigma \le \bar{c}_2 \varepsilon.$$

As a consequence, problem (1) admits uncountable many small amplitude, analytic, quasi-periodic solutions $v_{(a,\varepsilon)}$ with two frequencies, of the form (2):

$$\begin{aligned} v_{(a,\varepsilon)}(t,x) &= \sqrt{\varepsilon}\, u_{(a,\varepsilon)}\big((1+\varepsilon)t + x,\ (1+a\varepsilon)t - x\big) \\ &= \sqrt{\varepsilon}\, \big[\hat{u}_{0,0} + r\big((1+\varepsilon)t + x\big) + s\big((1+a\varepsilon)t - x\big) + \mathcal{O}(\varepsilon)\big] \\ &= \sqrt{\varepsilon}\, \big[\beta_0\big((1+\varepsilon)t + x\big)\big) + \beta_0\big((1+a\varepsilon)t - x\big) + \mathcal{O}\big(|a-1| + \varepsilon\big)\big]. \end{aligned}$$

7. Appendix A. Hilbert algebra property of $\mathcal{H}_\sigma$

Let $u, v \in \mathcal{H}_\sigma$, $u = \sum_{m \in \mathbb{Z}^2} \hat{u}_m\, e^{im\cdot\varphi}$, $v = \sum_{m \in \mathbb{Z}^2} \hat{v}_m\, e^{im\cdot\varphi}$. The product uv is

$$uv = \sum_j \Big(\sum_k \hat{u}_{j-k} \hat{v}_k \Big) e^{ij\cdot\varphi},$$

so its $\mathcal{H}_\sigma$-norm, if it converges, is

$$\|uv\|_\sigma^2 = \sum_j \Big| \sum_k \hat{u}_{j-k} \hat{v}_k \Big|^2 \big(1 + |j|^{2s}\big)\, e^{2|j|\sigma}.$$

We define

$$a_{jk} = \left[\frac{\big(1 + |j-k|^{2s}\big)\big(1 + |k|^{2s}\big)}{\big(1 + |j|^{2s}\big)} \right]^{1/2}.$$

Given any $(x_k)_k$, it holds by Hölder inequality

$$\Big| \sum_k x_k \Big|^2 = \Big| \sum_k \frac{1}{a_{jk}}\, x_k a_{jk} \Big|^2 \le c_j^2 \sum_k |x_k a_{jk}|^2, \tag{47}$$

where
$$c_j^2 := \sum_k \frac{1}{a_{jk}^2}.$$

We show that there exists a constant $c > 0$ such that $c_j^2 \le c^2$ for every $j \in \mathbb{Z}^2$. We recall that, fixed $p \ge 1$, it holds
$$(a+b)^p \le 2^{p-1}\left(a^p + b^p\right) \quad \forall\, a, b \ge 0.$$

Then, for $s \ge \frac{1}{2}$, we have
$$\begin{aligned} 1 + |j|^{2s} \le 1 + \big(|j-k| + |k|\big)^{2s} &\le 1 + 2^{2s-1}\left(|j-k|^{2s} + |k|^{2s}\right) \\ &< 2^{2s-1}\left(1 + |j-k|^{2s} + 1 + |k|^{2s}\right), \end{aligned}$$

so
$$\frac{1}{a_{jk}^2} < 2^{2s-1}\Big(\frac{1}{1+|j-k|^{2s}} + \frac{1}{1+|k|^{2s}}\Big).$$

The series $\sum_{k\in\mathbb{Z}^2} \frac{1}{1+|k|^p}$ converges if $p > 2$, thus for $s > 1$
$$c_j^2 < 2^{2s-1}\Big(\sum_k \frac{1}{1+|j-k|^{2s}} + \sum_k \frac{1}{1+|k|^{2s}}\Big) = 2^{2s} \sum_{k\in\mathbb{Z}^2} \frac{1}{1+|k|^{2s}} := c^2 < \infty.$$

We put $x_k = \hat{u}_{j-k}\hat{v}_k$ in (47) and compute
$$\begin{aligned} \Big|\sum_k \hat{u}_{j-k}\hat{v}_k\Big|^2 \left(1+|j|^{2s}\right) &\le c^2 \sum_k |\hat{u}_{j-k}\hat{v}_k a_{jk}|^2 \left(1+|j|^{2s}\right) \\ &= c^2 \sum_k |\hat{u}_{j-k}\hat{v}_k|^2 \left(1+|j-k|^{2s}\right)\left(1+|k|^{2s}\right), \end{aligned}$$

$$\begin{aligned} \|uv\|_\sigma^2 &= \sum_j \Big|\sum_k \hat{u}_{j-k}\hat{v}_k\Big|^2 \left(1+|j|^{2s}\right) e^{2|j|\sigma} \\ &\le \sum_j c^2 \sum_k |\hat{u}_{j-k}|^2 |\hat{v}_k|^2 \left(1+|j-k|^{2s}\right)\left(1+|k|^{2s}\right) e^{2(|j-k|+|k|)\sigma} \\ &= c^2 \sum_k \Big(\sum_j |\hat{u}_{j-k}|^2 \left(1+|j-k|^{2s}\right) e^{2|j-k|\sigma}\Big) |\hat{v}_k|^2 \left(1+|k|^{2s}\right) e^{2|k|\sigma} \\ &= c^2 \|u\|_\sigma^2 \|v\|_\sigma^2 . \end{aligned}$$

So $\|uv\|_\sigma \le c\,\|u\|_\sigma \|v\|_\sigma$ for all $u, v \in \mathcal{H}_\sigma$. We notice that the constant c depends only on s,
$$c = 2^s \Big(\sum_{k\in\mathbb{Z}^2} \frac{1}{1+|k|^{2s}}\Big)^{1/2}.$$

8. Appendix B. Change of form for quasi-periodic functions

First an algebric proposition, then we show that one can pass from (3) to (5) without loss of generality.

Proposition. *Let $A, B \in \mathrm{Mat}_2(\mathbb{R})$ be invertible matrices such that AB^{-1} has integer coefficient. Then, given any $u \in \mathcal{H}_\sigma$, the function $v(t,x) = u\big(A(t,x)\big)$ can be written as $v(t,x) = w\big(B(t,x)\big)$ for some $w \in \mathcal{H}_\sigma$, that is $\{u \circ A : u \in \mathcal{H}_\sigma\} \subseteq \{w \circ B : w \in \mathcal{H}_\sigma\}$.*

Proof. Let $u \in \mathcal{H}_\sigma$. The function $u \circ A$ belongs to $\{w \circ B : w \in \mathcal{H}_\sigma\}$ if and only if $u \circ A \circ B^{-1} = w$ for some $w \in \mathcal{H}_\sigma$, and this is true if and only if $u \circ AB^{-1}$ is 2π periodic; since $AB^{-1} \in \mathrm{Mat}_2(\mathbb{Z})$, we can conclude. □

Lemma. *The set of the quasi-periodic functions of the form (3) is equal to the set of the quasi-periodic functions of the form (5), that is,*

$$\Big\{v : v(t,x) = u(\omega_1 t + x, \omega_2 t + x), (\omega_1,\omega_2) \in \mathbb{R}^2, \omega_1 \neq 0, \omega_2 \neq 0, \tfrac{\omega_1}{\omega_2} \notin \mathbb{Q}, u \in \mathcal{H}_\sigma\Big\}$$
$$= \Big\{v : v(t,x) = u\big(\varepsilon t, (1+b\varepsilon^2)t + x\big), (b,\varepsilon) \in \mathbb{R}^2, \varepsilon \neq 0, (1+b\varepsilon^2) \neq 0, \tfrac{1+b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}, u \in \mathcal{H}_\sigma\Big\}.$$

Proof. Given any $\omega_1, \omega_2, b, \varepsilon$, we define

$$A = \begin{pmatrix} \omega_1 & 1 \\ \omega_2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} \varepsilon & 0 \\ (1+b\varepsilon^2) & 1 \end{pmatrix}.$$

Let $v(t,x)$ be any element of the set of quasi-periodic functions of the form (3), that is $v = u \circ A$ for some fixed $\omega_1, \omega_2 \neq 0$ such that $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$ and $u \in \mathcal{H}_\sigma$. We observe that v belongs to the set of quasi-periodic functions of the form (5) if $v = w \circ B$ for some (b,ε) such that $\varepsilon \neq 0$, $\frac{1+b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}$ and some $w \in \mathcal{H}_\sigma$. By the Proposition, this is true if we find (b,ε) such that $AB^{-1} \in \mathrm{Mat}_2(\mathbb{Z})$. We can choose

$$b = \frac{\omega_2 - 1}{(\omega_1 - \omega_2)^2}, \qquad \varepsilon = \omega_1 - \omega_2,$$

so that $AB^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We notice that $\frac{1+b\varepsilon^2}{\varepsilon} \notin \mathbb{Q}$ if and only if $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$.

Conversely, we fix (b,ε) and look for (ω_1,ω_2) such that $BA^{-1} \in \mathrm{Mat}_2(\mathbb{Z})$. This condition is satisfied if we choose the inverse transformation, $\omega_1 = 1 + \varepsilon + b\varepsilon^2$, $\omega_2 = 1 + b\varepsilon^2$. □

9. Appendix C. Small divisors

Fixed $\gamma \in (0, \frac{1}{4})$, we have defined in (13) the set $\tilde{B}_\gamma$ of "badly approximable numbers" as

$$\tilde{B}_\gamma = \Big\{x \in \mathbb{R} : |m + nx| > \frac{\gamma}{|n|} \quad \forall\, m,n \in \mathbb{Z},\ m \neq 0,\ n \neq 0\Big\}.$$

$\tilde{B}_\gamma$ is non-empty, symmetric, it has zero Lebesgue-measure and it accumulates to 0. Moreover, for every $\delta > 0$, $\tilde{B}_\gamma \cap (-\delta,\delta)$ is uncountable.

In fact, $\tilde{B}_\gamma$ contains all the irrational numbers whose continued fractions expansion is of the form $[0, a_1, a_2, \dots]$, with $a_j < \gamma^{-1} - 2$ for every $j \geq 2$. Such a set is uncountable: since $\gamma^{-1} - 2 > 2$, for every $j \geq 1$ there are at least two choices for the value of a_j. Moreover, it accumulates to 0: if $y = [0, a_1, a_2, \dots]$, it holds $0 < y < a_1^{-1}$, and a_1 has no upper bound. See also Remark 2.4 in [2] and, for the inclusion of such a set in $\tilde{B}_\gamma$, the proof of Theorem 5F in [14], p. 22.

We prove the following estimate.

Proposition. *Let* $\gamma \in (0, \frac{1}{4})$, $\delta \in (0, \frac{1}{2})$. *Then for all* $x, y \in \tilde{B}_\gamma \cap (-\delta,\delta)$ *it holds*

$$\big|(m+nx)(my+n)\big| > \gamma(1 - \delta - \delta^2) \quad \forall\, m,n \in \mathbb{Z},\ m,n \neq 0.$$

Proof. We shortly set $D = |(m+nx)(my+n)|$. There are four cases.
Case 1. $|m+nx| > 1$, $|my+n| > 1$. Then $|D| > 1$.

Case 2. $|m+nx|<1,\ |my+n|>1$. Multiplying the first inequality by $|y|$,

$$\begin{aligned}|y| >\ & |my+nxy| = |my+n-n(1-xy)| \\ \geq\ & \big||n(1-xy)|-|my+n|\big| \geq |n(1-xy)|-|my+n|,\end{aligned}$$

so $|my+n|>|n|(1-xy)-|y|$ and

$$\begin{aligned}|D| >\ & \tfrac{\gamma}{|n|}\big[|n|(1-xy)-|y|\big] = \gamma\Big[(1-xy)-\tfrac{|y|}{|n|}\Big] \\ >\ & \gamma[(1-\delta^2)-\delta].\end{aligned}$$

Case 3. $|m+nx|>1,\ |my+n|<1$. Analogous to case 2.
Case 4. $|m+nx|<1,\ |my+n|<1$. Dividing the first inequality by $|n|$, for triangular inequality we have

$$\Big|\frac{m}{n}\Big| \leq \Big|\frac{m}{n}+x\Big|+|x| < \frac{1}{|n|}+\delta,$$

and similarly $\big|\frac{n}{m}\big|<\frac{1}{|m|}+\delta$. So

$$\Big(\frac{1}{|n|}+\delta\Big)\Big(\frac{1}{|m|}+\delta\Big) > \Big|\frac{n}{m}\cdot\frac{m}{n}\Big| = 1.$$

If $|n|,|m|\geq 2$, then $\left(\frac{1}{|n|}+\delta\right)\left(\frac{1}{|m|}+\delta\right)<1$, a contradiction. It follows that at least one between $|n|$ and $|m|$ is equal to 1. Suppose $|n|=1$. Then $|m+nx|=|m\pm x|\geq|m|-\delta$ and

$$|D| > \frac{\gamma}{|m|}\big(|m|-\delta\big) = \gamma\Big(1-\frac{\delta}{|m|}\Big) \geq \gamma(1-\delta).$$

If $|m|=1$ the conclusion is the same. □

Fixed $\gamma\in(0,\frac14)$ and $\delta\in(0,\frac12)$, we define the set

$$\begin{aligned}B(\gamma,\delta)=\Big\{(b,\varepsilon)\in\mathbb{R}^2:\ & \varepsilon\neq 0,\quad 1+b\varepsilon^2\neq 0,\quad 2+b\varepsilon^2\neq 0, \\ & \tfrac{1+b\varepsilon^2}{\varepsilon}\notin\mathbb{Q},\quad \tfrac{\varepsilon}{2+b\varepsilon^2},\, b\varepsilon^2\in\tilde{B}_\gamma\cap(-\delta,\delta)\Big\}\end{aligned}$$

and the map

$$g: B(\gamma,\delta)\to\mathbb{R}^2, \qquad g(b,\varepsilon)=\Big(\frac{\varepsilon}{2+b\varepsilon^2},\, b\varepsilon^2\Big).$$

$g'(b,\varepsilon)$ is invertible on $B(\gamma,\delta)$. Its imagine is the set $R(g)=\{(x,y)\in\mathbb{R}^2 : x,y\in\tilde{B}_\gamma\cap(-\delta,\delta),\ \frac{1}{x}-y\notin\mathbb{Q}\}$ and its inverse is

$$g^{-1}(x,y)=\left(\frac{y(1-xy)}{2x},\ \frac{2x}{1-xy}\right).$$

Thus $B(\gamma,\delta)$ is homeomorphic to $R(g)=\{(x,y)\in\tilde{B}_\gamma^2 : |x|,|y|<\delta,\ \frac{1}{x}-y\notin\mathbb{Q}\}$. We observe that, fixed any $\bar{x}\in\tilde{B}_\gamma\cap(-\delta,\delta)$, it occurs $\frac{1}{\bar{x}}-y\in\mathbb{Q}$ only for countably many numbers y. We know that $\tilde{B}_\gamma\cap(-\delta,\delta)$ is uncountable so, removing from $[\tilde{B}_\gamma\cap(-\delta,\delta)]^2$ the couples $\{(\bar{x},y): y=\frac{1}{\bar{x}}-q\ \exists q\in\mathbb{Q}\}$, it remains uncountably many other couples. Thus $R(g)$ is uncountable and so, through g, also $B(\gamma,\delta)$.

Moreover, if we consider couples $(x,y)\in[\tilde{B}_\gamma\cap(-\delta,\delta)]^2$ such that $x\to 0$ and $(x/y)\to 1$, applying g^{-1} we find couples $(b,\varepsilon)\in B(\gamma,\delta)$ which satisfy $\varepsilon\to 0$, $b\to 1/2$. In other words, the set $B(\gamma,\delta)$ accumulates to $(1/2,0)$.

Finally we estimate $D_{b,\varepsilon}(m,n)$ for $(b,\varepsilon)\in B(\gamma,\delta)$. We have

$$|2+b\varepsilon^2| = \frac{2}{|1-xy|} > \frac{2}{1+\delta^2},$$

so from the previous Proposition and (9) it follows

$$\big|D_{b,\varepsilon}(m,n)\big| = |D|\,|2+b\varepsilon^2| > \gamma(1-\delta-\delta^2)\frac{2}{1+\delta^2}.$$

The factor on the right of γ is greater than 1 if we choose, for example, $\delta = 1/4$; we define $\mathcal{B}_\gamma = B(\gamma,\delta)_{|\delta=\frac{1}{4}}$, so that there holds

$$\big|D_{b,\varepsilon}(m,n)\big| > \gamma \qquad \forall\,(b,\varepsilon)\in\mathcal{B}_\gamma.$$

We can observe that the condition $\frac{1+b\varepsilon^2}{\varepsilon}\notin\mathbb{Q}$ implies $1+b\varepsilon^2\neq 0$, that $\frac{\varepsilon}{2+b\varepsilon^2}\in\tilde{B}_\gamma$ implies $\varepsilon\neq 0$ and $|b\varepsilon^2|<\delta$ implies $2+b\varepsilon^2\neq 0$, so that we can write

$$\mathcal{B}_\gamma = \Big\{(b,\varepsilon)\in\mathbb{R}^2:\ \frac{\varepsilon}{2+b\varepsilon^2}, b\varepsilon^2\in\tilde{B}_\gamma,\ \ \Big|\frac{\varepsilon}{2+b\varepsilon^2}\Big|, |b\varepsilon^2| < \frac{1}{4},\ \ \frac{1+b\varepsilon^2}{\varepsilon}\notin\mathbb{Q}\Big\}.$$

We notice also that, for $|b-\frac{1}{2}|$ and ε small enough, there holds automatically $|\frac{\varepsilon}{2+b\varepsilon^2}|<\frac{1}{4}$, $|b\varepsilon^2|<\frac{1}{4}$. So, if we are interested to couples (b,ε) close to $\big(\frac{1}{2},0\big)$, say $|b-\frac{1}{2}|<\delta_0$, $|\varepsilon|<\varepsilon_0$, we can write

$$\mathcal{B}_\gamma = \Big\{(b,\varepsilon)\in\Big(\frac{1}{2}-\delta_0,\,\frac{1}{2}+\delta_0\Big)\times(0,\varepsilon_0):\ \frac{\varepsilon}{2+b\varepsilon^2}, b\varepsilon^2\in\tilde{B}_\gamma,\ \ \frac{1+b\varepsilon^2}{\varepsilon}\notin\mathbb{Q}\Big\}.$$

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Pietro Baldi - Sissa, via Beirut 2–4, 34014 Trieste, Italy.

E-mail address: `baldi@sissa.it`