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Infinite Horizon Noncooperative Differential Games

Alberto Bressan$^{(*)}$ *and Fabio S. Priuli*$^{(**)}$

(*) Dept. of Mathematics, Penn State University, University Park 16802, U.S.A.
bressan@math.psu.edu

(**) S.I.S.S.A., Via Beirut 4, Trieste 34014, ITALY.
priuli@sissa.it

Abstract. For a non-cooperative differential game, the value functions of the various players satisfy a system of Hamilton-Jacobi equations. In the present paper, we consider a class of infinite-horizon games with nonlinear costs exponentially discounted in time. By the analysis of the value functions, we establish the existence of Nash equilibrium solutions in feedback form and provide results and counterexamples on their uniqueness and stability.

1 - Introduction

Problems of optimal control, or zero-sum differential games, have been the topic of an extensive literature. In both cases, an effective tool for the analysis of optimal solutions is provided by the *value function*, which satisfies a scalar Hamilton-Jacobi equation. Typically, this first order P.D.E. is highly non-linear and solutions may not be smooth. However, thanks to a very effective comparison principle, the existence and stability of solutions can be achieved in great generality by the theory of viscosity solutions, see [BC] and references therein.

In comparison, much less is known about non-cooperative differential games. In a Nash equilibrium solution, the value functions for the various players now satisfy not a scalar but a system of Hamilton-Jacobi equations [F]. For this type of nonlinear systems, no general theorems on the existence or uniqueness of solutions are yet known. A major portion of the literature is concerned with games having linear dynamics and quadratic costs, see [WSE],[EW],[AFJ] and [PMC]. In this case, solutions are sought among quadratic functions. This approach effectively reduces the P.D.E. problem to a finite dimensional O.D.E.. However, it does not provide insight on the stability (or instability) of the solutions w.r.t. small non-linear perturbations.

In [BS1] the first author studied a class of non-cooperative games with general terminal payoff, in one space dimension. Relying on recent advances in the theory of hyperbolic systems of conservation laws, some results on the existence and stability of Nash equilibrium solutions could be obtained. On the other hand, for games in several space dimensions and also in various one-dimensional cases, the analysis in [BS2] shows that the corresponding H-J system is not hyperbolic, hence ill posed.

In the present paper we begin exploring a class of non-cooperative differential games in infinite time horizon, with exponentially discounted costs. In one space dimension, the corresponding value functions satisfy a time-independent system of implicit O.D.E's. Global solutions are sought within a class of absolutely continuous functions, imposing certain growth conditions as $|x| \to \infty$, and suitable admissibility conditions at points where the gradient u_x has a jump.

The dynamics of our system is very elementary, and the cost functions that we consider are small perturbations of linear ones. However, already in this simple setting we find cases where the problem has unique solution, and cases where infinitely many solutions exist. This provides a glimpse of the extreme complexity of the problem, for general non-cooperative N-player games with non-linear cost functions.

The plan of the paper is as follows. In Section 2 we describe the differential game, introducing the basic notations and definitions. In Section 3 we prove that, from an admissible solution to the O.D.E. for the value function, one can always recover a Nash equilibrium solution to the differential game. The relevance of our admissibility conditions is then highlighted by two examples.

The existence and uniqueness of global admissible solutions to the H-J system for the value functions is then studied in Sections 3 and 4. We first consider the cooperative case, where both players wish to move the state of the system in the same direction. In the case with terminal payoff, this situation was leading to a well-posed hyperbolic Cauchy problem [BS1]. As expected, in the infinite-horizon case we still obtain an existence and uniqueness result. Subsequently, we consider the case of conflicting interests, where the players wish to steer the system in opposite directions. In the case with terminal payoff, this situation leads to an ill-posed Cauchy problem, as shown in [BS2]. Somewhat surprisingly, we find that the corresponding infinite-horizon case can have unique or multiple solutions, depending on the values of certain parameters.

2 - Basic definitions

Consider an m-persons non-cooperative differential game, with dynamics

$$\dot x = \sum_{i=1}^m f_i(x,\alpha_i), \qquad \alpha_i(t)\in A_i\,, \qquad x\in I\!\!R^n. \tag{2.1}$$

Here $t\mapsto \alpha_i(t)$ is the control chosen by the i-th player, within a set of admissible control values $A_i\subseteq I\!\!R^k$. We will study the **discounted, infinite horizon problem**, where the game takes place on an infinite interval of time $[0,\,\infty[$, and each player has only a running cost, discounted exponentially in time. More precisely, for a given initial data

$$x(0)=y\in I\!\!R^n\,, \tag{2.2}$$

the goal of the i-th player is to minimize the functional

$$J_i(\alpha)\doteq\int_0^\infty e^{-t}\,\psi_i\big(x(t),\,\alpha_i(t)\big)\,dt\,, \tag{2.3}$$

where $t\mapsto x(t)$ is the trajectory of (2.1). By definition, an m-tuple of feedback strategies $\alpha_i = \alpha_i^*(x)$, $i=1,\ldots,m$, represents a *Nash non-cooperative equilibrium solution* for the differential game (2.1)-(2.2) if the following holds. For every $i\in\{1,\ldots,m\}$, the feedback control $\alpha_i=\alpha^*(x)$ provides a solution to the the optimal control problem for the i-th player,

$$\min_{\alpha(\cdot)}\ J_i(\alpha)\,, \tag{2.4}$$

where the dynamics of the system is

$$\dot x = f_i(x,\alpha_i)+\sum_{j\neq i} f_j(x,\alpha_j^*(x)), \qquad \alpha_i(t)\in A_i\,. \tag{2.5}$$

More precisely, we require that, for every initial data $y \in I\!R$, the Cauchy problem

$$\dot{x} = \sum_{j=1}^{m} f_j\big(x, \alpha_j^*(x)\big), \qquad\qquad x(0) = y, \tag{2.6}$$

should have at least one Caratheodory solution $t \mapsto x(t)$, defined for all $t \in [0, \infty[$. Moreover, for every such solution and each $i = 1, \ldots, m$, the cost to the i-th player should provide the minimum for the optimal control problem (2.4)-(2.5). We recall that a Caratheodory solution is an absolutely continuous function $t \mapsto x(t)$ which satisfies the differential equation in (2.6) at almost every $t > 0$.

Nash equilibrium solutions in feedback form can be obtained by studying a related system of P.D.E's. Assume that a value function $u(y) = (u_1, \ldots, u_n)(y)$ exists, so that $u_i(y)$ represents the cost for the i-th player when the initial state of the system is $x(0) = y$ and the strategies $\alpha_1^*, \ldots, \alpha_m^*$ are implemented. By the theory of optimal control, see for example [BC], on regions where u is smooth, each component u_i should provide a solution to the corresponding scalar Hamilton-Jacobi-Bellman equation. The vector function u thus satisfies the stationary system of equations

$$u_i(x) = H_i(x, \nabla u_1, \ldots, \nabla u_m), \tag{2.7}$$

where the Hamiltonian functions H_i are defined as follows. For each $p_j \in I\!R^n$, assume that there exists an optimal control value $\alpha_j^*(x, p_j)$ such that

$$p_j \cdot f_j\big(x,\, \alpha_j^*(x, p_j)\big) + \psi_j\big(x,\, \alpha_j^*(x, p_j)\big) = \min_{a \in A_j} \big\{p_j \cdot f_j(x, a) + \psi_j(x, a)\big\}. \tag{2.8}$$

Then

$$H_i(x,\, p_1, \ldots, p_m) \doteq p_i \cdot \sum_{j=1}^{m} f_j\big(x,\, \alpha_j^*(x, p_j)\big) + \psi_i\big(x,\, \alpha_i^*(x, p_i)\big). \tag{2.9}$$

A rich literature is currently available on optimal control problems and on viscosity solutions to the corresponding scalar H-J equations. However, little is yet known about non-cooperative differential games, apart from the linear-quadratic case. In this paper we begin a study of this class of differential games, with two players in one space dimension. Our main interest is in the existence, uniqueness and stability of Nash equilibrium solutions in feedback form.

When x is a scalar variable, (2.7) reduces to a system of implicit O.D.E's:

$$u_i = H_i(x, u_1', \ldots, u_m'). \tag{2.10}$$

In general, this system will have infinitely many solutions. To single out a (hopefully unique) admissible solution, corresponding to a Nash equilibrium for the differential game, additional requirements must be imposed. These are of two types:

(i) Asymptotic growth conditions as $|x| \to \infty$.

(ii) Jump conditions, at points where the derivative u' is discontinuous.

To fix the ideas, consider a game with the simple dynamics

$$\dot{x}(t) = \alpha_1(t) + \cdots + \alpha_m(t), \tag{2.11}$$

and with cost functionals of the form

$$J_i(\alpha) \doteq \int_0^\infty e^{-t}\left[h_i\big(x(t)\big) + k_i\big(x(t)\big)\,\frac{\alpha_i^2(t)}{2}\right] dt. \tag{2.12}$$

We shall assume that the functions h_i, k_i are smooth and satisfy

$$\big|h_i'(x)\big| \leq C\,, \qquad\qquad \frac{1}{C} \leq k_i(x) \leq C\,, \tag{2.13}$$

for some constant $C > 0$. Notice that in this case (2.8) yields $\alpha_i^* = -p_i/k_i$, hence (2.10) becomes

$$u_i = \left(\frac{u_i'}{2k_i(x)} - \sum_{j=1}^m \frac{u_j'}{k_j(x)} \right) u_i' + h_i(x)\,. \tag{2.14}$$

For a solution to the system of H-J equations (2.14), a natural set of admissibility conditions is formulated below.

Definition 1. A function $u : I\!\!R \mapsto I\!\!R^m$ is called an *admissible solution* to the implicit system of O.D.E's (2.14) if the following holds.

(A1) u is absolutely continuous. Its derivative u' satisfies the equations (2.14) at a.e. point $x \in I\!\!R$.

(A2) u has sublinear growth at infinity. Namely, there exists a constant C such that, for all $x \in I\!\!R$,

$$\big|u(x)\big| \leq C\,\big(1 + |x|\big)\,. \tag{2.15}$$

(A3) At every point $y \in I\!\!R$, the derivative u' admits right and left limits $u'(y+)$, $u'(y-)$. At points where u' is discontinuous, these limits satisfy the admissibility conditions

$$\sum_{i=1}^m \frac{u_i'(y+)}{k_i(y)} \ \leq\ 0\ \leq\ \sum_{i=1}^m \frac{u_i'(y-)}{k_i(y)}\,. \tag{2.16}$$

Because of the assumption (2.13), the cost functions h_i are globally Lipschitz continuous. It is thus natural to require that the value functions u_i be absolutely continuous, with sub-linear growth as $x \to \pm\infty$. Call $p_i^\pm \doteq u_i'(y\pm)$. By the equations (2.14) and the continuity of the functions u_i, h_i, k_i, one obtains the identities

$$\frac{(p_i^+)^2}{2k_i(y)} + \sum_{j\neq i} \frac{p_i^+ p_j^+}{k_j(y)} \ =\ \frac{(p_i^-)^2}{2k_i(y)} + \sum_{j\neq i} \frac{p_i^- p_j^-}{k_j(y)} \qquad\qquad i = 1, \ldots, m\,. \tag{2.17}$$

Recalling that the feedback controls are $\alpha_i^* = -u_i'/k_i$, the condition (2.16) now becomes clear: it states that $\dot x(y-) \leq 0 \leq \dot x(y+)$, i.e., trajectories should move away from a point of discontinuity.

Notice that all of the above conditions are satisfied at a point y such that

$$\sum_j \frac{u_j'(y+)}{k_j(y)} \leq 0\,, \qquad\qquad u_i'(y+) + u_i'(y-) = 0 \qquad i = 1, \ldots, m\,. \tag{2.18}$$

By (A1), the derivatives $p_i = u_i'$ are defined at a. e. point $x \in I\!\!R$. The optimal feedback controls $\alpha_i^* = -p_i/k_i$ are thus defined almost everywhere. We can use the further assumption (A3) and extend these functions to the whole real line by taking limits from the right:

$$\alpha^*(x) \doteq -\frac{u_i'(x+)}{k_i(x)}\,. \tag{2.19}$$

In this way, all feedback control functions will be right-continuous.

The system of implicit differential equations (2.14) is highly nonlinear and difficult to study in full generality. In this paper we initiate the analysis by looking at some significant cases. Our main results can be roughly summarized as follows:

(i) If $u = (u_1, \dots u_m)$ provides an admissible solution to the system of Hamilton-Jacobi equations (2.14), then the feedback strategies (2.19) provide a Nash equilibrium solution to the differential game (2.11)-(2.12).

(ii) For games with two players, if the cost functions h_1, h_2 are both monotone increasing (or both monotone decreasing), then (2.14) has a unique admissible solution.

(iii) Still in the case of two players, one can give examples where the derivatives of the cost functions satisfy $h_i' + h_2' = 0$ and infinitely many admissible solutions of (2.14) are found. On the other hand, if the sum $h_1' + h_2'$ remains bounded away from zero, then under suitable assumptions the system (2.14) has a unique admissible solution.

3 - Solutions of the differential game

In this section we prove that admissible solutions to the H-J equations yield a solution to the differential game. Moreover, we give a couple of examples showing the relevance of the assumptions (A2) and (A3).

Theorem 1. *Consider the differential game (2.11)-(2.12), with the assumptions (2.13). Let $u : I\!R \mapsto I\!R^m$ be an admissible solution to the systems of H-J equations (2.14), so that the conditions (A1)–(A3) hold. Then the controls (2.19) provide a Nash equilibrium solution in feedback form.*

Proof. The theorem will be proved in several steps.

1. First of all, setting

$$g(x) \doteq \sum_i \alpha_i^*(x) = -\sum_i \frac{u_i'(x)}{k_i(x)}, \tag{3.1}$$

we need to prove that the Cauchy problem

$$\dot x(t) = g\big(x(t)\big), \qquad\qquad x(0) = y, \tag{3.2}$$

has a globally defined solution, for every initial data $y \in I\!R$. This is not entirely obvious, because the function g may be discontinuous. We start by proving the local existence of solutions.

CASE 1: $g(y) = 0$. In this trivial case $x(t) \equiv y$ is the required solution.

CASE 2: $g(y) > 0$. By right continuity, we then have $g(x) > 0$ for $x \in [y,\ y+\delta]$, for some $\delta > 0$. This implies the existence of a (unique) strictly increasing solution $x : [0, \varepsilon] \mapsto I\!R$, for some $\varepsilon > 0$.

CASE 3: $g(y) < 0$. By the admissibility conditions (2.16), this implies that g is continuous and negative in a neighborhood of y. Therefore the Cauchy problem (3.2) admits a (unique) strictly decreasing solution $x : [0, \varepsilon] \mapsto I\!R$, for some $\varepsilon > 0$.

2. Next, we prove that the local solution can be extended to all positive times. For this purpose, we need to rule out the possibility that $\big|x(t)\big| \to \infty$ in finite time. We first observe that each trajectory is monotone, i.e., either non-increasing, or non-decreasing, for $t \in [0,\infty[$. To fix the ideas, let $t \mapsto x(t)$ be strictly increasing, with $x(t) \to \infty$ as $t \to T-$. A contradiction is now obtained as follows. For each $\tau > 0$, using (2.14) we compute

$$\begin{aligned}
\sum_i u_i\big(x(\tau)\big) - \sum_i u_i\big(x(0)\big) &= \int_0^\tau \left\{ \frac{d}{dt} \sum_i u_i\big(x(t)\big) \right\} dt \\
&= \int_0^\tau - \left\{ \sum_i u_i'\big(x(t)\big) \cdot \sum_j \frac{u_j'\big(x(t)\big)}{k_j\big(x(t)\big)} \right\} dt \\
&= \int_0^\tau \sum_i \left\{ u_i\big(x(t)\big) - \frac{\big|u_i'(x(t))\big|^2}{2k_i(x(t))} - h_i\big(x(t)\big) \right\} dt
\end{aligned} \tag{3.3}$$

By assumptions, the functions u_i and h_i have sub-linear growth. Moreover, each k_i is uniformly positive and bounded above. Using the elementary inequality

$$\big|x(\tau) - x(0)\big| \le \int_0^\tau 1 \cdot \big|\dot x(t)\big|\, dt \le \left(\int_0^\tau 1\, dt \right)^{1/2} \cdot \left(\int_0^\tau \big|\dot x(t)\big|^2 dt \right)^{1/2},$$

from (3.3) we thus obtain

$$\begin{aligned}
\frac{\big|x(\tau) - x(0)\big|^2}{\tau} &\le \int_0^\tau \big|\dot x(t)\big|^2 dt \ \le 4C \int_0^\tau \sum_i \frac{\big|u_i'(x(t))\big|^2}{2k_i(x(t))}\, dt \\
&\le 4C \left\{ \left| \sum_i u_i\big(x(\tau)\big) - \sum_i u_i\big(x(0)\big) \right| + \int_0^\tau \sum_i \big|u_i(x(t))\big|\, dt + \int_0^\tau \sum_i \big|h_i(x(t))\big|\, dt \right\} \\
&\le C_0\,(1+\tau) \Big\{ 2 + \big|x(\tau)\big| + \big|x(0)\big| \Big\},
\end{aligned}$$

for some constant C_0. Therefore, either $\big|x(\tau)\big| \le 2 + 3\big|x(0)\big|$, or else

$$\big|x(\tau)\big| \le \big|x(0)\big| + 2\tau \cdot C_0\,(1+\tau)\,. \tag{3.4}$$

In any case, blow-up cannot occur at any finite time T.

3. To complete the proof, for each fixed $i \in \{1,\ldots,m\}$, we have to show that the feedback α_i^* in (2.19) provides solution to the optimal control problem for the i-th player:

$$\min_{\alpha_i(\cdot)} \int_0^\infty e^{-t} \left[h_i\big(x(t)\big) + k_i\big(x(t)\big)\, \frac{\alpha_i^2(t)}{2} \right] dt\,, \tag{3.5}$$

where the system has dynamics

$$\dot x = \alpha_i + \sum_{j \ne i} \alpha_j^*(x)\,. \tag{3.6}$$

Given an initial state $x(0) = y$, by the assumptions on u it follows that the feedback strategy $\alpha_i = \alpha_i^*(x)$ achieves a total cost given by $u_i(y)$. Now consider any absolutely continuous trajectory $t \mapsto x(t)$, with $x(0) = y$. Of course, this corresponds to the control

$$\alpha_i(t) \doteq \dot x(t) - \sum_{j \ne i} \alpha_j^*(x) \tag{3.7}$$

implemented by the i-th player. We claim that the corresponding cost satisfies

$$\int_0^\infty e^{-t}\Big[h_i\big(x(t)\big)+\frac{k_i}{2}\Big(\dot x(t)-\sum_{j\neq i}\alpha_j^*\big(x(t)\big)\Big)^2\Big]\,dt\ \geq\ u_i(y)\,. \tag{3.8}$$

To prove (3.8), we first observe that (3.4) implies

$$\lim_{t\to\infty} e^{-t}u_i\big(x(t)\big)=0 \qquad\qquad i=1,\ldots,n\,.$$

Hence

$$u_i(y)=u_i\big(x(0)\big)=-\int_0^\infty \frac{d}{dt}\Big[e^{-t}u_i\big(x(t)\big)\Big]\,dt\,.$$

The inequality (3.8) can now be established by checking that

$$e^{-t}\Big[h_i\big(x(t)\big)+\frac{k_i}{2}\Big(\dot x(t)-\sum_{j\neq i}\alpha_j^*\big(x(t)\big)\Big)^2\Big]\geq e^{-t}u_i\big(x(t)\big)-e^{-t}u_i'\big(x(t)\big)\cdot\dot x(t)\,. \tag{3.9}$$

Equivalently, letting α_i be as in (3.7),

$$u_i\leq\left(\alpha_i-\sum_{j\neq i}\frac{u_j'}{k_j}\right)u_i'+\frac{k_i}{2}\alpha_i^2+h_i\,.$$

This is clearly true because, by (2.8),

$$u_i(x)=\min_a\left\{\frac{k_i}{2}a^2+au_i'-\sum_{j\neq i}\frac{u_j'u_i'}{k_j}+h_i(x)\right\}\,.$$

□

We now give two examples showing that, if the growth assumptions (2.15) or if the jump conditions (2.16) are not satisfied, then the feedbacks (2.19) may not provide a Nash equilibrium solution. This situation is well known already in the context of control problem.

Examples. Consider the game for two players, with dynamics

$$\dot x=\alpha_1+\alpha_2\,, \tag{3.10}$$

and cost functionals

$$J_i=\int_0^\infty e^{-t}\cdot\frac{\alpha_i^2(t)}{2}\,dt\,.$$

In this case, if $u_i'=p_i$, the optimal control for the i-th player is

$$\alpha_i^*(p_i)\ =\ \arg\min_\omega\Big\{p_i\,\omega+\frac{\omega^2}{2}\Big\}\ =\ -p_i\,.$$

The system of H-J takes the simple form

$$\begin{cases} u_1 = -\left(\dfrac{u_1'}{2} + u_2'\right) u_1'\,, \\[2mm] u_2 = -\left(u_1' + \dfrac{u_2'}{2}\right) u_2'\,. \end{cases} \tag{3.11}$$

The obvious admissible solution is $u_1 \equiv u_2 \equiv 0$, corresponding to identically zero controls, and zero cost. We now observe that the functions

$$u_1(x) = \begin{cases} 0 & \text{if} \quad |x| \geq 1\,, \\ -\frac{1}{2}\big(1-|x|\big)^2 & \text{if} \quad |x| < 1\,, \end{cases} \qquad\qquad u_2(x) = 0\,,$$

provide a solution to (3.11), which is not admissible because the conditions (2.16) fail at $x = 0$.

Next, the functions

$$u_1(x) = -\frac{1}{2}x^2, \qquad\qquad u_2(x) = 0\,,$$

provide yet another another solution, which does not satisfy the growth conditions (2.15).

In the above two cases, the corresponding feedbacks $\alpha_i^*(x) = -u_i'(x)$ do not yield a solution to the differential game.

4 - Cooperative situations

We consider here a game for two players, with dynamics

$$\dot{x} = \alpha_1 + \alpha_2\,, \tag{4.1}$$

and cost functionals of the form

$$J_i(\alpha) \doteq \int_0^\infty e^{-t}\left[h_i\big(x(t)\big) + \frac{\alpha_i^2(t)}{2}\right] dt\,. \tag{4.2}$$

Notice that, for any positive constants k_1, k_2, λ, the more general case

$$J_i(\alpha) \doteq \int_0^\infty e^{-\lambda t}\left[\tilde{h}_i\big(x(t)\big) + \frac{\alpha_i^2(t)}{2k_i}\right] dt$$

can be reduced to (4.2) by a linear change of variables.

The system of H-J equations for the value functions now takes the form

$$\begin{cases} u_1(x) = h_1(x) - u_1'u_2' - (u_1')^2/2\,, \\ u_2(x) = h_2(x) - u_1'u_2' - (u_2')^2/2\,, \end{cases} \tag{4.3}$$

and the optimal feedback controls are given by

$$\alpha_i^*(x) = -u_i'(x)\,. \tag{4.4}$$

Differentiating (4.3) and setting $p_i = u_i'$ one obtains the system

$$\begin{cases} h_1' - p_1 = (p_1 + p_2)p_1' + p_1 p_2' \,, \\ h_2' - p_2 = p_2 p_1' + (p_1 + p_2)p_2' \,. \end{cases} \tag{4.5}$$

Set

$$\Lambda(p) \doteq \begin{pmatrix} p_1 + p_2 & p_1 \\ p_2 & p_1 + p_2 \end{pmatrix}, \qquad\qquad \Delta(p) \doteq \det \Lambda(p)\,.$$

From (4.5) we deduce

$$\begin{cases} p_1' = \Delta(p)^{-1}\big[- p_1^2 + (h_1' - h_2')p_1 + h_1' p_2\big]\,, \\ p_2' = \Delta(p)^{-1}\big[- p_2^2 + (h_2' - h_1')p_2 + h_2' p_1\big]\,. \end{cases} \tag{4.6}$$

Notice that

$$\frac{1}{2}\,(p_1^2 + p_2^2) \leq \Delta(p) \leq 2(p_1^2 + p_2^2)\,. \tag{4.7}$$

In particular, $\Delta(p) > 0$ for all $p = (p_1, p_2) \neq (0, 0)$. Up to a rescaling of the independent variable, we can thus study the equivalent system

$$\begin{cases} p_1' = (h_1' - h_2')p_1 + h_1' p_2 - p_1^2\,, \\ p_2' = (h_2' - h_1')p_2 + h_2' p_1 - p_2^2\,. \end{cases} \tag{4.8}$$

For piecewise smooth solutions, jumps are only allowed from any point (p_1^-, p_2^-) with

$$p_1^- + p_2^- \geq 0 \tag{4.9}$$

to the symmetric point

$$(p_1^+, p_2^+) = (-p_1^-,\ -p_2^-)\,. \tag{4.10}$$

Theorem 2. *Let the cost functions h_1, h_2 be smooth, and assume that their derivatives satisfy*

$$\frac{1}{C} \leq h_i'(x) \leq C \tag{4.11}$$

for some constant $C > 1$ and all $x \in I\!R$. Then the system (4.3) has an admissible solution. The corresponding functions α_i^* in (4.4) provide a Nash equilibrium solution to the non-cooperative game.

Proof. Write the O.D.E. (4.6) in the more compact form

$$\frac{dp}{dx} = f(p)\,. \tag{4.12}$$

To show the existence of at least one admissible solution of (4.3), for every $\nu \geq 1$ let $p^{(\nu)} : [-\nu, \infty[\to I\!R^2$ be the solution of the Cauchy problem

$$\frac{dp^{(\nu)}}{dx} = f\big(p^{(\nu)}\big)\,, \qquad\qquad p^{(\nu)}(-\nu) = (1, 1). \tag{4.13}$$

It is easy to check that the polygon

$$\Gamma \doteq \Big\{(p_1, p_2)\,;\quad p_1, p_2 \in [0, 2C]\,,\quad p_1 + p_2 \geq 1/2C\Big\}$$

is positively invariant for the flow of (4.6). Hence $p^{(\nu)}(x) \in \Gamma$ for all $\nu \geq 1$ and $x \geq -\nu$. We can extend each function $p^{(\nu)}$ to the whole real line by setting

$$p^{(\nu)}(x) = (1, 1) \qquad\qquad \text{for} \;\; x < -\nu\,.$$

By uniform boundedness and equicontinuity, the sequence $p^{(\nu)}$ admits a subsequence converging to a uniformly continuous function $p : I\!R \mapsto \Gamma$. Clearly this limit function provides a continuous, globally bounded solution of (4.6). We then define the controls $\alpha_i^*(x) \doteq -p_i(x)$ and the cost functions

$$u_i(y) \doteq \int_0^\infty e^{-t}\left[h_i\big(x(t,y)\big) + \frac{1}{2}\Big(\alpha_i^*\big(x(t,y)\big)\Big)^2\right] dt\,, \tag{4.14}$$

where $t \mapsto x(t, y)$ denotes the solution to the Cauchy problem

$$\dot{x} = \alpha_1^*(x) + \alpha_2^*(x)\,, \qquad\qquad x(0) = y\,. \tag{4.15}$$

This function provides a globally Lipschitz, smooth solution of the system (4.3). □

In the case where the oscillation of the derivatives h_i' is sufficiently small, we can also prove the uniqueness of the Nash feedback solution.

Theorem 3. *Let the cost functions be smooth, with derivatives satisfying (4.11), for some constant C. Assume that the oscillation of their derivatives satisfies*

$$\sup_{x,y\in I\!R} \big|h_i'(x) - h_i'(y)\big| \leq \delta \qquad i = 1, 2 \tag{4.16}$$

for some $\delta > 0$ sufficiently small (depending only on C). Then the admissible solution of the system (4.3) is unique.

Before giving details of the proof, we sketch the main ideas. In the case of linear cost functions, where $h_i(x) = \kappa_i\, x$, $h_i' \equiv \kappa_i$, the phase portrait of the planar O.D.E. (4.8) is depicted in Figure 1. We observe that

- Unbounded trajectories of (4.8), with $\big|p(s)\big| \to \infty$ as $s \to \bar{s}$, correspond to solutions $p = p(x)$ of (4.6) with $\big|p(x)\big| \to \infty$, $\big|p'(x)\big| \to \infty$ as $x \to \pm\infty$. Indeed, because of the rescaling (4.7), as the parameter s approaches s finite limit $\bar{s}$, we have $|x| \to \infty$. This yields a solution $u(x) = \int_*^x p(x)\, dx$ which does not satisfy the growth restrictions (2.15).

- The heteroclinic orbit, joining the origin with the point (κ_1, κ_2), corresponds to a trajectory of (4.16) defined on a half line, say $[\bar{x}, \infty[$. To prolong this solution for $x < \bar{x}$ one needs a trajectory of (4.16) which approaches the origin as $s \to \infty$. But the two available solutions are both unbounded, hence not acceptable.

- Finally, one must examine solutions whose gradient has one or more jumps, from a point $P = (p_1, p_2)$ with $p_1 + p_2 \geq 0$ to its symmetric point $-P = (-p_1, -p_2)$. However, a direct inspection shows that, even allowing these jumps, one still cannot construct any new globally bounded trajectory.

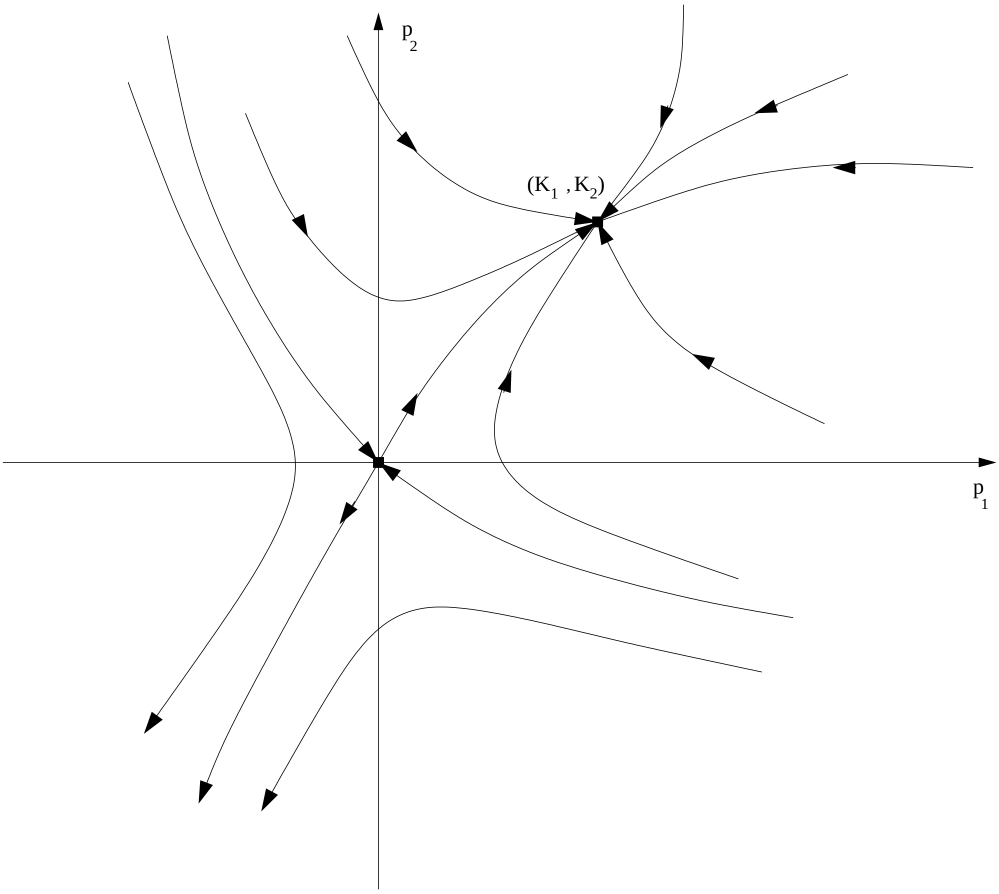

Figure 1

In the end, in linear case, one finds that the only admissible solution is $(p_1, p_2) \equiv (\kappa_1, \kappa_2)$. A perturbative argument shows that this conclusion remains valid if a small $\mathcal{C}^1$ perturbation is added to the cost functions.

Proof of Theorem 3. *First Step.* We begin with the case $h_i'(x) \equiv \kappa_i$ and assume, without any loss of generality, that $\kappa_1 \le \kappa_2$.

Let $\tilde{p}$ be a smooth solution of (4.8), as shown in Figure 1.
We observe that the following facts hold (see Figure 2):

1. Both sets $A = \{(p_1, p_2) \neq (0,0) \;:\; p_1 \le 0\,,\; p_2 \le 0\}$ and $\{(p_1, p_2) \neq (0,0) \;:\; p_1 \ge 0\,,\; p_2 \ge 0\}$ are positively invariant for the flow of (4.8) and both $B = \{(p_1, p_2) \;:\; p_1 > 0\,,\; p_2 < 0\}$ and $C = \{(p_1, p_2) \;:\; p_1 < 0\,,\; p_2 > 0\}$ are negatively invariant.

2. If $\tilde{p}(s_o) \in A = \{(p_1, p_2) \neq (0,0) \;:\; p_1 \le 0\,,\; p_2 \le 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as $s \to +\infty$. Indeed, since

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + \kappa_1 \tilde{p}_1 + \kappa_2 \tilde{p}_2 \le -\frac{1}{2}(\tilde{p}_1 + \tilde{p}_2)^2 < 0\,,$$

we can assume there exist $\bar{s} \ge s_o$ and $\varepsilon > 0$ such that $\tilde{p}_1(\bar{s}) + \tilde{p}_2(\bar{s}) < -\varepsilon$. Moreover, the following holds for any $\sigma > \bar{s}$:

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2)(\sigma) \leq -\frac{1}{2}(\tilde{p}_1(\sigma) + \tilde{p}_2(\sigma))^2 < -\frac{1}{2}(\tilde{p}_1(\bar{s}) + \tilde{p}_2(\bar{s}))(\tilde{p}_1(\sigma) + \tilde{p}_2(\sigma)) < \frac{\varepsilon}{2}(\tilde{p}_1(\sigma) + \tilde{p}_2(\sigma))\,.$$

Hence, an integration yields $(\tilde{p}_1 + \tilde{p}_2)(s) \leq -\eta e^{\frac{\varepsilon}{2}s}$ for $s > \bar{s}$ (and $\eta > 0$) and $(\tilde{p}_1 + \tilde{p}_2) \to -\infty$ as $s \to +\infty$.

3. If $\tilde{p}(s_o) \in B = \{(p_1, p_2) \,:\, p_1 > 0\,,\ p_2 < 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as $s \to -\infty$. Indeed, let $\varepsilon > 0$ such that $\tilde{p}_1(s_o) > \varepsilon$. Since

$$\frac{d}{ds}\tilde{p}_1 = -\tilde{p}_1^2 + (\kappa_1 - \kappa_2)\tilde{p}_1 + \kappa_1\tilde{p}_2 \leq -(\tilde{p}_1 + \kappa_2 - \kappa_1)\tilde{p}_1 < 0\,,$$

it is sufficient to observe that, for $\sigma < s_o$,

$$\frac{d}{ds}\tilde{p}_1(\sigma) < -(\tilde{p}_1(s_o) + \kappa_2 - \kappa_1)\tilde{p}_1(\sigma) \leq -(\varepsilon + \kappa_2 - \kappa_1)\tilde{p}_1(\sigma)\,.$$

Hence, an integration yields $\tilde{p}_1(s) \geq \eta e^{-(\varepsilon + \kappa_2 - \kappa_1)s}$ for $s < s_o$ (and $\eta > 0$) and $\tilde{p}_1 \to +\infty$ as $s \to -\infty$.

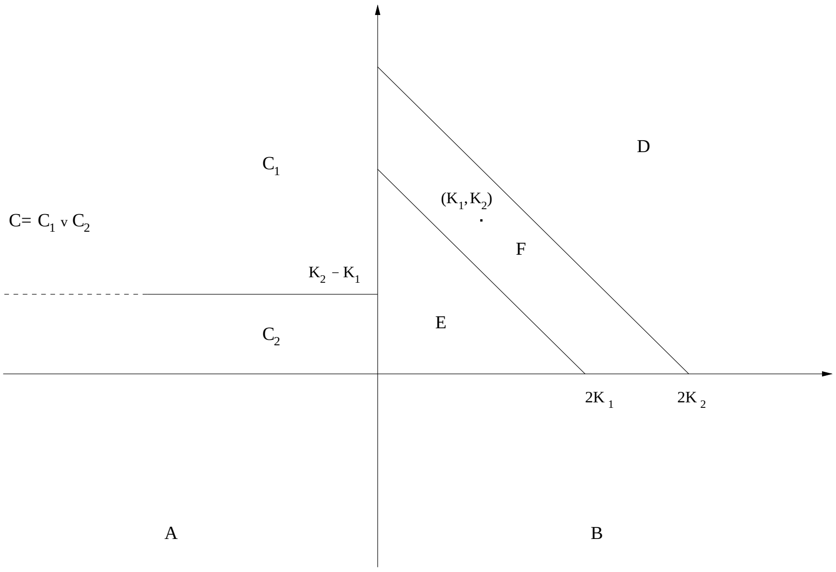

Figure 2

4. If $\tilde{p}(s_o) \in C_1 = \{(p_1, p_2) \,:\, p_1 < 0\,,\ p_2 > \kappa_2 - \kappa_1\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as $s \to -\infty$. Here the argument is exactly the same as in the previous case with $\tilde{p}_2$ in place of $\tilde{p}_1$.

5. If $\tilde{p}(s_o) \in C_2 = \{(p_1, p_2) \,:\, p_1 < 0\,,\ 0 < p_2 \leq \kappa_2 - \kappa_1\}$ then there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in C_1 as in case **4.** above. Indeed there could be only two situations.
If $-\tilde{p}_1(s_o)^2 + (\kappa_1 - \kappa_2)\tilde{p}_1(s_o) + \kappa_1\tilde{p}_2(s_o) \geq 0$, then, by negative invariance, $\tilde{p}$ could only have reached this region from C_1, hence there exists $\bar{s} \leq s_o$ such that $\tilde{p}(\bar{s})$ is as in case **4** above. Otherwise, using

again negative invariance and the fact that there are no equilibria in C_2, either there exists $\bar{s} \leq s_o$ such that $\tilde{p}(\bar{s})$ is in case **4** above, or there exists $s_1 < s_o$ such that $-\tilde{p}_1(s_1)^2 + (\kappa_1 - \kappa_2)\tilde{p}_1(s_1) + \kappa_1\tilde{p}_2(s_1) \geq 0$ and then, by the previous case, the existence of such a $\bar{s} < s_1 < s_o$ follows.

6. If $\tilde{p}(s_o) \in D = \{(p_1, p_2) \, : \, p_1 \geq 0 \, , \, p_2 \geq 0 \, , \, p_1 + p_2 \geq 2\kappa_2\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as $s \to -\infty$. Indeed, since

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + \kappa_1\tilde{p}_1 + \kappa_2\tilde{p}_2 \leq -\frac{1}{2}(\tilde{p}_1 + \tilde{p}_2 - 2\kappa_2)(\tilde{p}_1 + \tilde{p}_2) \leq 0$$

(and the inequality is actually strict when $p_1 + p_2 = 2\kappa_2$), we can assume that there exist $\bar{s} \leq s_o$ and $\varepsilon > 0$ such that $\tilde{p}_1(\bar{s}) + \tilde{p}_2(\bar{s}) > 2\kappa_2 + \varepsilon$. Moreover, the following holds for any $\sigma < \bar{s}$:

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2)(\sigma) < -\frac{1}{2}(\tilde{p}_1(\bar{s}) + \tilde{p}_2(\bar{s}) - 2\kappa_2)(\tilde{p}_1(\sigma) + \tilde{p}_2(\sigma)) < -\frac{\varepsilon}{2}(\tilde{p}_1(\sigma) + \tilde{p}_2(\sigma)) \, .$$

Hence by integrating we find $(\tilde{p}_1 + \tilde{p}_2)(s) \geq \eta e^{-\frac{\varepsilon}{2}s}$ for $s < \bar{s}$ and $\eta > 0$. Therefore $(\tilde{p}_1 + \tilde{p}_2) \to +\infty$ as $s \to -\infty$.

7. If $\tilde{p}(s_o) \in E = \{(p_1, p_2) \neq (0,0) \, : \, p_1 \geq 0 \, , \, p_2 \geq 0 \, , \, p_1 + p_2 \leq 2\kappa_1\}$ for some s_o, then from

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + \kappa_1\tilde{p}_1 + \kappa_2\tilde{p}_2 \geq -\frac{1}{2}(\tilde{p}_1 + \tilde{p}_2 - 2\kappa_1)(\tilde{p}_1 + \tilde{p}_2) \geq 0 \, ,$$

it follows, as above, that either $\tilde{p} \to 0$ for $s \to -\infty$ or there exists $\bar{s} \leq s_o$ such that $\tilde{p}(\bar{s})$ satisfies one of the previous cases **3-4-5**.

8. If $\tilde{p}(s_o) \in F = \{(p_1, p_2) \, : \, p_1 \geq 0 \, , \, p_2 \geq 0 \, , \, 2\kappa_1 < p_1 + p_2 < 2\kappa_2\}$ for some s_o and $p \neq \tilde{p}$, then there exists a small circle V (say with radius smaller than $|\tilde{p}(s_o) - p(s_o)|$) around the stable focus $p \equiv (\kappa_1, \kappa_2)$ such that $\tilde{p} \notin V$ for $s < s_o$. But then, looking at the signs of the derivatives of $\tilde{p}_i$, as $s \to -\infty$ our solution $\tilde{p}$ must go away from the whole region F and there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in one of the previous cases.

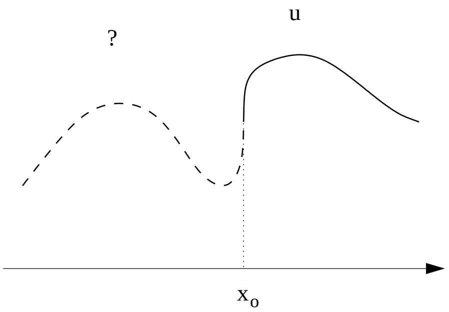

Figure 3a

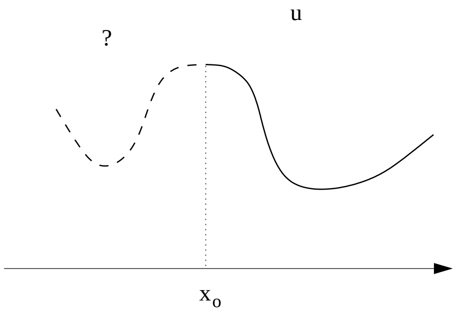

Figure 3b

9. We can now provide more accurate estimates on blow-up. Indeed by previous analysis, blow-up of $|\tilde{p}|$ can only occur when either $\tilde{p}_i \to -\infty$ as $s \to +\infty$ or $\tilde{p}_i \to +\infty$ as $s \to -\infty$, for some index $i \in \{1, 2\}$. To fix the ideas, assume $|\tilde{p}_1| \to \infty$. Then for s sufficiently large

$$-\frac{\tilde{p}_1^2}{2} + (\kappa_1 - \kappa_2)\tilde{p}_1 + \kappa_1\tilde{p}_2 < 0 \, .$$

Integrating the inequality $\frac{d}{ds}\tilde{p}_1 < -\frac{\tilde{p}_1^2}{2}$, one can conclude that $|\tilde{p}| \to \infty$ as $s \to s_o$, for some finite s_o in both cases. In particular for this $s_o \in I\!R$ (and $\eta > 0$) $\tilde{p}$ satisfies $|\tilde{p}(s)| \geq \frac{\eta}{|s-s_o|}$.

In terms of the original variable x, one may guess that the corresponding function $\tilde{p} = \tilde{p}(x)$ could be as in Figure 3a and that u may be continued beyond the point where $\tilde{p}$ blows-up (say $x_o = x(s_o)$). But this is not the case since such a trajectory yields a solution defined on the whole real line. Indeed by (4.7)

$$\left|\frac{dx}{ds}\right| = \Delta\big(\tilde{p}(s)\big) \geq \frac{c_o}{(s_o - s)^2}\,. \tag{4.17}$$

for some $c_o > 0$, and therefore either $x(s) \to +\infty$ as $s \to s_o-$ or $x(s) \to -\infty$ as $s \to s_o+$. Therefore, the solution $\tilde{u}(x)$, corresponding to $\tilde{p}(x)$, violates the growth assumptions (2.15) and is not admissible.

10. We remark that in case **7**, the solution $\tilde{p}$ can tend to 0 as $s \to -\infty$. But then for some $c_o > 0$

$$|\tilde{p}| \leq \tilde{p}_1 + \tilde{p}_2 \leq e^{c_o s}\,.$$

Recalling (4.7) we obtain, in terms of the variable x,

$$\left|\frac{dx}{ds}\right| = \Delta\big(\tilde{p}(s)\big) = \mathcal{O}(1)\cdot e^{2c_o s}\,, \tag{4.18}$$

$$\lim_{s\to-\infty} x(s) = x_o < \infty\,,$$

for some $x_o \in I\!R$. Therefore, to the entire trajectory $s \mapsto \tilde{p}(s)$, there corresponds only a portion of the trajectory $x \mapsto \tilde{p}(x)$, namely for $x > x_o$.

To prolong the solution $\tilde{u}$ for $x < x_o$, we need to construct another trajectory $s \mapsto p(s)$ such that $\lim_{s\to+\infty} p(s) = 0$. But this trajectory, by previous analysis, will be unbounded, hence the corresponding $\tilde{u}(x)$, will not be admissible.

11. Next, we consider the case where $\tilde{p}(s)$ is a discontinuous solution with admissible jumps. In this case, first of all we can say that $\tilde{p}$ has no more than 2 jumps. Indeed the set $\Xi_1 = \{(p_1,p_2)\,:\, p_2 < 0\,,\ p_1 + p_2 \leq 0\}$ is positively invariant and $\Xi_2 = \{(p_1,p_2)\,:\, p_2 < 0\,,\ p_1+p_2 > 0\}$ is negatively invariant. Hence if a jump occurs at s_o, either $\tilde{p}(s_o+) \in \Xi_1$ or $\tilde{p}(s_o-) \in \Xi_2$. In the former case $\tilde{p}(s)$ has no jumps for $s > s_o$; in the latter case $\tilde{p}$ has no jumps for $s < s_o$. This means that there could be at most two jumps when there exist $s_1 < s_2 \leq s_3$ such that

- a first jump occurs at s_1 and $\tilde{p}(s_1-) \in \Xi_2$,
- $\tilde{p}$ crosses the line $p_1 + p_2 = 0$ at s_2,
- a last jump occurs at s_3 and $\tilde{p}(s_3+) \in \Xi_1$.

In any case, the corresponding solution $\tilde{u}$ does not satisfy (2.15) and is not admissible. Indeed, we can have only three situations for a $\tilde{p}$ with an admissible jump at s_o:

(a) if $\tilde{p}(s_o-) \in \Xi_2$, then $|\tilde{p}| \to \infty$ as $s \to -\infty$;
(b) if $\tilde{p}(s_o+) \in \Xi_1$ and $\tilde{p}_1(s_o+) > 0$, then either $\tilde{p}(s)$ is continuous for $s < s_o$ (and therefore $|\tilde{p}| \to \infty$ as $s \to -\infty$) or $\tilde{p}$ has another jump at $\bar{s}$ such that $\tilde{p}(\bar{s}-) \in \Xi_2$ (and therefore again $|\tilde{p}| \to \infty$ as $s \to -\infty$);
(c) if $\tilde{p}(s_o+) \in \Xi_1$ and $\tilde{p}_1(s_o+) \leq 0$, then $|\tilde{p}| \to \infty$ as $s \to +\infty$.

Second Step. We now extend the proof, in the presence of a sufficiently small perturbation. By (4.16), there exist constants $\kappa_1, \kappa_2 > 0$ such that

$$\big|h_1'(x) - \kappa_1\big| \leq \delta\,, \qquad \big|h_2'(x) - \kappa_2\big| \leq \delta \qquad \text{for all } \ x \in I\!R\,. \tag{4.19}$$

Let $u(\cdot)$ be the solution constructed in Theorem 2, and let $\tilde{u}$ be any other smooth solution of (4.8). Call $p = u'$, $\tilde{p} = \tilde{u}'$ the corresponding gradients, rescaled as before, and let V be a small open bounded set containing the whole image of p and the point (κ_1, κ_2). Of course it is not restrictive to consider V as circular, say with radius $\rho > 0$.

Now we split the proof in three cases.

CASE 1: $\tilde{p}(s) \in V$ for every s. In this case we look at the difference $w(s) = \tilde{p}(s) - p(s)$. We can write a linear evolution equation for w:

$$\frac{dw}{ds} = A(s)\, w(s)\,, \tag{4.20}$$

where the matrix A is the "average" matrix

$$A(s) = \int_0^1 Df(\theta p(s) + (1-\theta)\tilde{p}(s))d\theta\,, \tag{4.21}$$

and f is the vector field at (4.12).

Since $p, \tilde{p} \in V$, the matrix $A(s)$ is very close to the Jacobian matrix $Df(\kappa_1, \kappa_2)$, therefore

$$\frac{d}{ds}\big|w(s)\big| \le -K\,\big|w(s)\big|\,, \tag{4.22}$$

for some constant $K > 0$. Indeed $Df(\kappa_1, \kappa_2)$ is negative definite and, provided δ (and then ρ) is small enough, $A(s)$ is negative definite too. Hence

$$2\big|w(s)\big|\frac{d}{ds}\big|w(s)\big| = \frac{d}{ds}\big|w(s)\big|^2 = 2\frac{d}{ds}w(s)\cdot w(s) = 2A(s)w(s)\cdot w(s) \le -2K\big|w(s)\big|^2\,.$$

Now integrating (4.22), we have for $s < 0$,

$$2\rho \ge \big|w(s)\big| \ge e^{-Ks}\big|w(0)\big|$$

and, letting $s \to -\infty$, find

$$\big|w(0)\big| \le \lim_{s\to-\infty}\big|w(s)\big|e^{Ks} \le \lim_{s\to-\infty} 2\rho e^{Ks} = 0\,. \tag{4.23}$$

This implies $p(0) = \tilde{p}(0)$, hence $p = \tilde{p}$ by the uniqueness of the Cauchy problem.

CASE 2: $\tilde{p}(s_o) \notin V$ for some s_o and, in particular, $\tilde{p}(s_o)$ in a small neighbourhood W of the origin. Consider the linearized system near $(0,0)$

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = H \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\,, \qquad H = \begin{pmatrix} h_1' - h_2' & h_1' \\ h_2' & h_2' - h_1' \end{pmatrix}\,,$$

and notice that the origin is a saddle point for this system. Indeed H has eigenvalues λ_1, λ_2 such that

$$0 < \sqrt{\frac{3}{4C^2}} \le |\lambda_i| = \sqrt{(h_1')^2 + (h_2')^2 - h_1'h_2'} \le \sqrt{2C^2 - \frac{1}{C^2}}\,, \tag{4.24}$$

where C is the constant in (4.11). Moreover its eigenvectors v_1, v_2 form angles α_1, α_2 with the positive direction of the p_1-axis such that

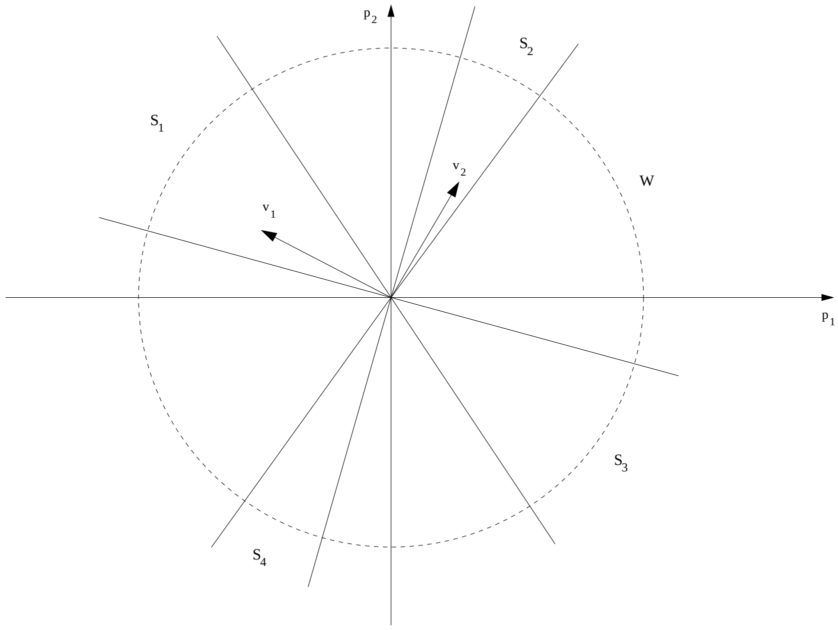

Figure 4

$$0 < \frac{1}{C}\left(\sqrt{d^2+\frac{1}{C^2}}-d\right) \leq |\tan\alpha_i| = \left|\frac{\lambda_i + (h_2' - h_1')}{h_1'}\right| \leq C\left(\sqrt{d^2+C^2}+d\right) , \tag{4.25}$$

where $d = (C - \frac{1}{C}) > 0$ and C is again from (4.11).
Hence, exactly as one can do with saddle points in the autonomous case, we can prove that there exist four sectors S_i, $i = 1, \dots, 4$ (see Figure 4), where the following facts hold:

(a) If $\tilde{p}(s_o)$ is in S_1 or S_3, then $|\tilde{p}(s)|$ grows for $s < s_o$ and the solution moves away from W;

(b) Both boundaries of S_2 and S_4 allow orbits to only exit from those sectors for $s < s_o$;

(c) If $\tilde{p}(s_o) \notin S_i$ for all $i = 1, \dots, 4$, then for $s < s_o$ the angle between the vector $(\tilde{p}_1, \tilde{p}_2)$ and the p_1-axis is strictly monotone, forcing the solution either to reach S_1 or S_3, or to move away from W;

(d) Finally, if $\tilde{p}(s_o)$ is in S_2 or S_4, then for $s < s_o$ the solution can tend to the origin. But, since

$$\frac{d}{ds}(\tilde{p}_1+\tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + h_1'\tilde{p}_1 + h_2'\tilde{p}_2 \geq -\frac{1}{2}(\tilde{p}_1+\tilde{p}_2-2C)(\tilde{p}_1+\tilde{p}_2) > 0 \,,$$

as in the constant case, one obtains an estimate of exponential type of the decay of $|\tilde{p}|$.

CASE 3: $\tilde{p}(s_o) \notin V$ and $\tilde{p}(s_o)$ not in a neighbourhood of the origin. In this case, combining (4.19) and the continuous dependence of solutions with the estimates of the constant case (indeed, using (4.11), they remain true), we can prove that $|\tilde{p}| \to \infty$ for finite s and that the rate of blow-up of $|\tilde{p}|$ can be estimated in the same way we did in the case of $h_i' \equiv \kappa_i$.

In any case either $\tilde{u} \equiv u$ or, in the original coordinates x, $\tilde{u}$ fails to satisfy (2.15).

It remains to prove what happens if $\tilde{u}$ is an admissible solution with discontinuous (rescaled) gradient $\tilde{p}(s)$. Then assume $\tilde{p}$ has an admissible jump at s_o. Using (4.19) it holds, for $\tilde{p}_1 > 0$,

$$\frac{d}{ds}(\tilde{p}_1 + \tilde{p}_2)\bigg|_{\tilde{p}_1+\tilde{p}_2=0} = -2\tilde{p}_1^2 + (h_1' - h_2')\tilde{p}_1 < -2\tilde{p}_1^2 + (\kappa_1 - \kappa_2 + 2\delta)\tilde{p}_1$$

and hence, provided δ small enough, the region Ξ_1 (resp. Ξ_2) defined in the *First Step* is positively (resp. negatively) invariant also in this setting. Then conclusions made in the constant case still hold and $\tilde{u}$ corresponding to $\tilde{p}$ is not admissible, since it violates (2.15). □

5 - Players with conflicting interests

We consider here a game for two players, with dynamics (4.1) and cost functionals as in (4.2). Contrary to the previous section, we now assume that the player have conflicting interest. Namely, their running costs h_i satisfy

$$h_1'(x) \leq 0 \leq h_2'(x). \tag{5.1}$$

We begin with an example showing that in this case the H-J system can have infinitely many admissible solutions. Each of these determines a different Nash equilibrium solution to the differential game.

Example 2. Consider the game (4.1)-(4.2), with

$$h_1(x) = -\kappa x\,, \qquad\qquad h_2(x) = \kappa x\,, \tag{5.2}$$

for some constant $\kappa > 0$ (see Figure 5).

In this special case, the equations (4.8) reduce to

$$\begin{cases} p_1' = -2\kappa p_1 - \kappa p_2 - p_1^2\,, \\ p_2' = \kappa p_1 + 2\kappa p_2 - p_2^2\,. \end{cases} \tag{5.3}$$

The point $\overline{P} \doteq (-\kappa, \kappa)$ is stationary for the flow of (5.3). Settinq $q_1 \doteq p_1 + \kappa$, $q_2 \doteq p_2 - \kappa$, the local behavior of the system near $\overline{P}$ is described by

$$\begin{cases} q_1' = -\kappa q_2 - q_1^2\,, \\ q_2' = \kappa q_1 - q_2^2\,. \end{cases} \tag{5.4}$$

Notice that

$$\frac{dp_2}{dp_1} = \frac{dq_2}{dq_1} = 0 \qquad \text{if} \qquad q_1 = \frac{q_2^2}{\kappa}\,,$$

$$\frac{dp_1}{dp_2} = \frac{dq_1}{dq_2} = 0 \qquad \text{if} \qquad q_2 = -\frac{q_1^2}{\kappa}\,,$$

$$\frac{dp_1}{dp_2} = \frac{dq_1}{dq_2} = 1 \qquad \text{if} \qquad p_1 = -p_2\,.$$

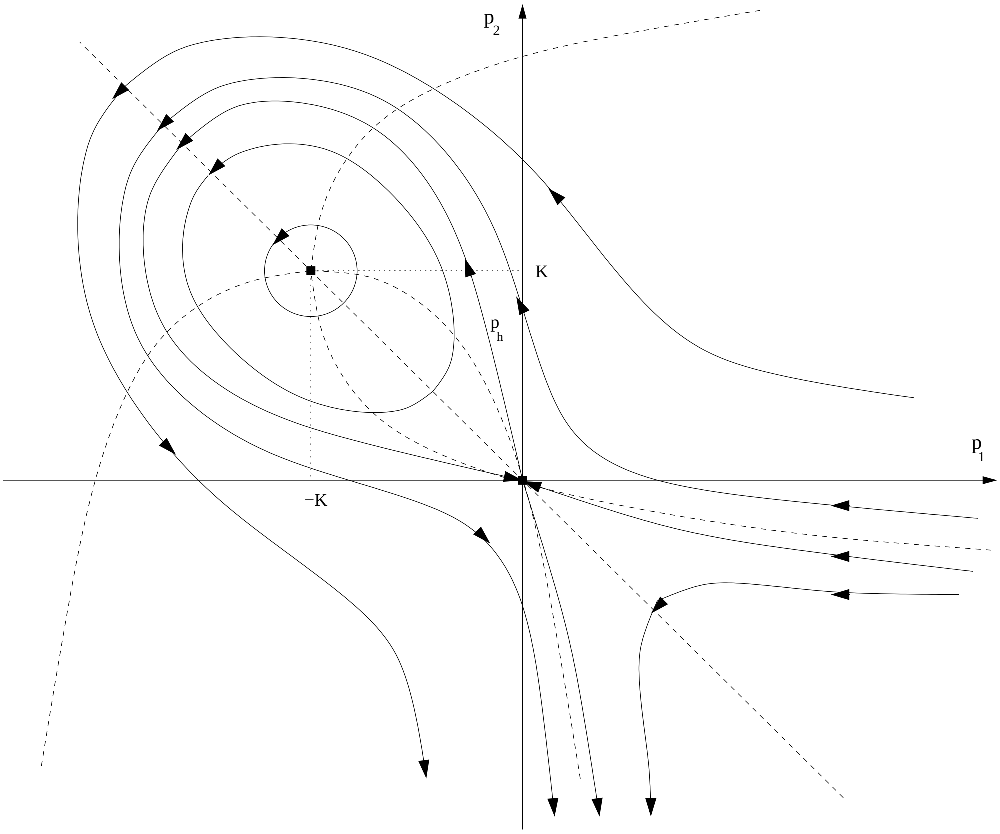

Figure 5

By symmetry across the line $p_1 + p_2 = 0$, any trajectory passing through a point $P_\alpha \doteq (-\alpha, \alpha)$ with $0 < \alpha < \kappa$ is a closed orbit. We thus have infinitely many solutions of the H-J equations (5.3), having bounded, periodic gradients. Therefore, all of these solutions are globally Lipschitz continuous and satisfy the growth condition (2.15). Notice that the homoclinic orbit $p_h(\cdot)$ starting and ending at the origin also yields a periodic solution to the original equation (4.6). Indeed, to a solution $p = p(s)$ of (5.3) with

$$\lim_{s\to-\infty} p(s) = \lim_{s\to+\infty} p(s) = 0\,,$$

through the reparametrization $x = x(s)$ there corresponds a solution $p = p(x)$ defined on some bounded interval $]\ell_o, \ell_1[$. This yields a periodic solution $p = p(x)$ with period $\ell = \ell_1 - \ell_o$.

The main result of this section is concerned with the existence and uniqueness of admissible solutions.

Theorem 4. *Let any two constants κ_1, κ_2 be given, with*

$$\kappa_1 < 0 < \kappa_2\,, \qquad\qquad \kappa_1 + \kappa_2 \neq 0\,. \tag{5.5}$$

Then there exists $\delta > 0$ such that the following holds. If h_1, h_2 are smooth functions whose derivatives satisfy

$$\big|h_1'(x) - \kappa_1\big| \leq \delta\,, \qquad\qquad \big|h_2'(x) - \kappa_2\big| \leq \delta\,, \tag{5.6}$$

for all $x \in I\!R$, then the system of H-J equations (4.3) has a unique admissible solution.

Proof. We will first consider the linear case, where $h_i' \equiv \kappa_i$ is constant. Then we recover the more general case by a perturbation argument.

Existence. Assume that $h_i(x) = \kappa_i x$ with $\kappa_1 + \kappa_2 > 0$, which is not rectrictive. The existence of an admissible solution for (4.8) is trivial, since we have the constant solution $p \equiv (\kappa_1, \kappa_2)$, which corresponds to

$$\big(u_1(x),\, u_2(x)\big) = \big(\kappa_1 x + \kappa_1\kappa_2 + \frac{\kappa_1^2}{2}\,,\ \kappa_2 x + \kappa_1\kappa_2 + \frac{\kappa_2^2}{2}\big)\,. \tag{5.7}$$

Consider now the case of h_1', h_2' small perturbations of the constants κ_1, κ_2. Notice that, in the previous case, every ball $B(\kappa, R)$ around $\kappa = (\kappa_1, \kappa_2)$ with radius $R < \frac{\sqrt{2}}{2}(\kappa_1 + \kappa_2)$ was positively invariant for the flow of (4.8).
Indeed, setting $q_i = p_i - \kappa_i$, the system becomes

$$\begin{cases} q_1' = -(\kappa_1 + \kappa_2)q_1 + \kappa_1 q_2 - q_1^2\,, \\ q_2' = \kappa_2 q_1 - (\kappa_1 + \kappa_2)q_2 - q_2^2\,. \end{cases} \tag{5.8}$$

and it holds

$$\frac{d}{ds}\frac{|q|^2}{2} = -q_1^3 - q_2^3 - (\kappa_1 + \kappa_2)(q_1^2 + q_2^2 - q_1q_2) = -(q_1^2 + q_2^2 - q_1q_2)(\kappa_1 + \kappa_2 + q_1 + q_2)\,.$$

Now, since $|q| \le R < \frac{\sqrt{2}}{2}(\kappa_1 + \kappa_2)$ ensures $\kappa_1 + \kappa_2 + q_1 + q_2 > 0$, one can conclude that

$$\frac{d}{ds}\frac{|q|^2}{2} = -(q_1^2 + q_2^2 - q_1q_2)(\kappa_1 + \kappa_2 + q_1 + q_2) \le -\frac{|q|^2}{2}(\kappa_1 + \kappa_2 + q_1 + q_2) < 0\,, \tag{5.9}$$

and this prove the positively invariance of such a ball B.

Then, provided δ is small enough, we can choose one of these balls as a neighborhood U of (κ_1, κ_2) positively invariant also for the perturbed system (i.e. $h_i' \not\equiv \kappa_i$). Once we found such a compact, positively invariant set $\overline{U}$, we can repeat the existence proof of Theorem 2:

a. Consider $p^{(\nu)}\colon [-\nu, \infty[\, \to I\!R^2$ solution of the Cauchy problem with initial datum $p^{(\nu)}(-\nu) = (\kappa_1, \kappa_2)$;

b. By positive invariance, $p^{(\nu)}(x) \in U$ for $x > -\nu$. We then extend the function $p^{(\nu)}$ to the whole real line by setting $p^{(\nu)}(x) \equiv (\kappa_1, \kappa_2)$ for $x < -\nu$;

c. By uniform boundedness and equicontinuity, the sequence $p^{(\nu)}$ admits a subsequence converging to a uniformly continuous function $p : I\!R \mapsto U$. Clearly, this limit function $p(\cdot)$ provides a global, bounded solution to the system (4.8). In turn, this yields an admissible solution $u(\cdot)$ to (4.6).

Uniqueness. *First Step.* Let $h_i' \equiv \kappa_i$ and $\kappa_1 + \kappa_2 > 0$. In order to prove that the previously found solution is the only one that satisfies (A1)-(A3), we assume that $\tilde{u}$ is another solution of the system (4.3), whose gradient will be denoted by $\tilde{p}$. Figure 6 depicts possible trajectories $s \mapsto \tilde{p}(s)$ of the planar system (4.8). We remark that:

1. The regions $\{(p_1, p_2) \neq (0,0)\ :\ p_1 \ge 0\,,\ p_2 \le 0\,,\ p_1 + p_2 \le 0\}$ and $\{(p_1, p_2) \neq (0,0)\ :\ p_1 \ge 0\,,\ p_2 \le 0\,,\ p_1 + p_2 \ge 0\}$ are positively and negatively invariant for the flow of (4.8), respectively.

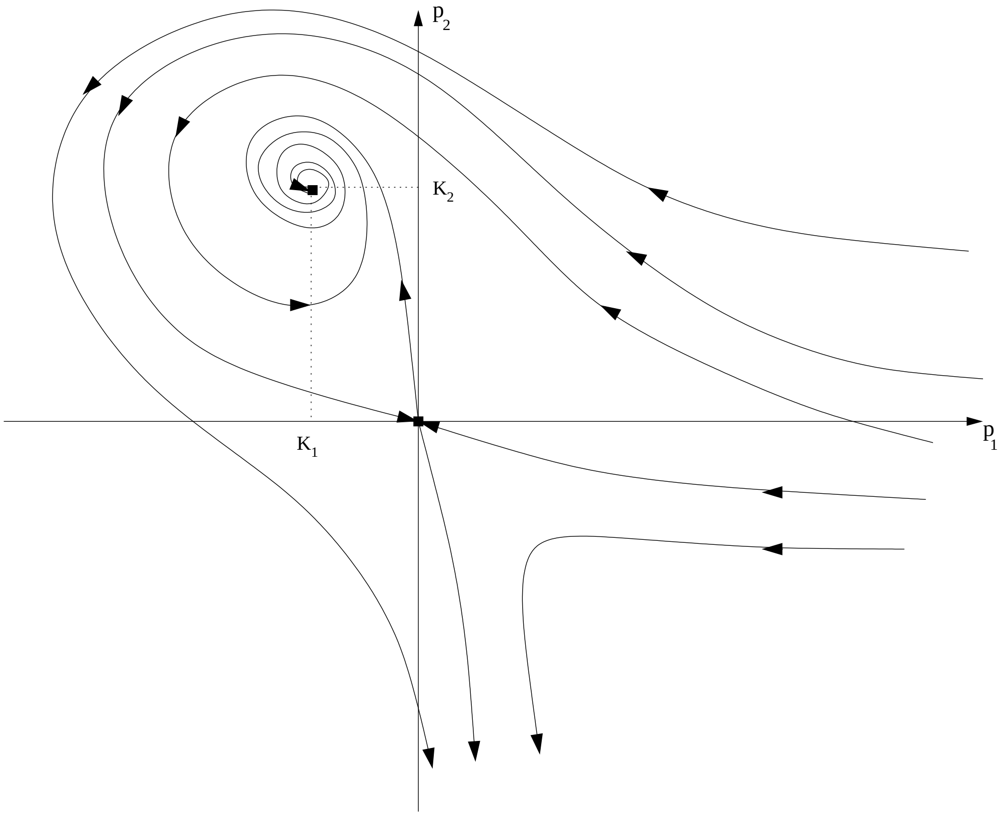

Figure 6

2. If $\tilde{p}(s_o) \in \{(p_1,p_2) \neq (0,0) \ : \ p_1 \geq 0\,,\ p_2 \leq 0\,,\ p_1+p_2 \leq 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as $s \to +\infty$. Indeed, since

$$\frac{d}{ds}(\tilde{p}_1+\tilde{p}_2) = -\tilde{p}_1^2 - \tilde{p}_2^2 + \kappa_1\tilde{p}_1 + \kappa_2\tilde{p}_2 < -(\tilde{p}_1+\tilde{p}_2)^2 < 0\,,$$

we can assume there exist $\bar{s} \geq s_o$ and $\varepsilon > 0$ such that $\tilde{p}_1(\bar{s})+\tilde{p}_2(\bar{s}) < -\varepsilon$. Moreover, for any $\sigma > \bar{s}$ we have

$$\frac{d}{ds}(\tilde{p}_1+\tilde{p}_2)(\sigma) \leq -(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma))^2 < -(\tilde{p}_1(\bar{s})+\tilde{p}_2(\bar{s}))(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma)) < \varepsilon(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma))\,.$$

After an integration, we find $(\tilde{p}_1+\tilde{p}_2)(s) \leq -\eta e^{\varepsilon s}$ for $s > \bar{s}$ (and $\eta > 0$) and hence $(\tilde{p}_1+\tilde{p}_2) \to -\infty$ as $s \to +\infty$.

3. If $\tilde{p}(s_o) \in \{(p_1,p_2) \neq (0,0) \ : \ p_1 \geq 0\,,\ p_2 \leq 0\,,\ p_1+p_2 \geq 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as $s \to -\infty$. Indeed, reasoning as above, we can assume there exist $\bar{s} \leq s_o$ and $\varepsilon > 0$ such that $\tilde{p}_1(\bar{s})+\tilde{p}_2(\bar{s}) > \varepsilon$ and the following holds for any $\sigma < \bar{s}$:

$$\frac{d}{ds}(\tilde{p}_1+\tilde{p}_2)(\sigma) \leq -(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma))^2 < -(\tilde{p}_1(\bar{s})+\tilde{p}_2(\bar{s}))(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma)) < -\varepsilon(\tilde{p}_1(\sigma)+\tilde{p}_2(\sigma))\,.$$

This implies $(\tilde{p}_1+\tilde{p}_2)(s) \geq \eta e^{-\varepsilon s}$ for $s < \bar{s}$ (and $\eta > 0$), hence $(\tilde{p}_1+\tilde{p}_2) \to +\infty$ as $s \to -\infty$.

4. If $\tilde{p}(s_o) \in \{(p_1,p_2) \, : \, p_1 > 0 \, , \; p_2 > 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as $s \to -\infty$. Indeed, let $\varepsilon > 0$ such that $\tilde{p}_1(s_o) > \varepsilon$. Since

$$\frac{d}{ds}\tilde{p}_1 = -\tilde{p}_1^2 + (\kappa_1 - \kappa_2)\tilde{p}_1 + \kappa_1\tilde{p}_2 < -\tilde{p}_1^2 < 0 \, ,$$

it is sufficient to observe that for $\sigma < s_o$

$$\frac{d}{ds}\tilde{p}_1(\sigma) \le -\tilde{p}_1(s_o)\tilde{p}_1(\sigma) < -\varepsilon\tilde{p}_1(\sigma) \, .$$

Hence, integrating, $\tilde{p}_1(s) \ge \eta e^{-\varepsilon s}$ for $s < s_o$ (and $\eta > 0$) and either $\tilde{p}_1 \to +\infty$ as $s \to -\infty$ or there exists $\bar{s} < s_o$ such that $\tilde{p}$ is in the previous case.

5. If $\tilde{p}(s_o) \in \{(p_1,p_2) \, : \, p_1 < 0 \, , \; p_2 < 0\}$ for some s_o, then $|\tilde{p}| \to +\infty$ as $s \to +\infty$. Here we can repeat the argument of **4.** with $\tilde{p}_2$ in place of $\tilde{p}_1$.

6. Let $\tilde{p}(s_o) \in \{(p_1,p_2) \neq (0,0) \, : \, p_1 \le 0 \, , \; p_2 \ge 0\}$ for some s_o and set $\hat{p}$ as the unique solution in this region that tends to the origin as $s \to +\infty$. Notice that, as $s \to -\infty$, either $\hat{p}(s)$ crosses the p_2-axis or $\hat{p}_2 \to +\infty$. Then:

- if $\tilde{p} = \hat{p}$, then as stated above either there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in the case **4**, or $\tilde{p}_2 \to \infty$ as $s \to -\infty$. In both cases $|\tilde{p}| \to \infty$ as $s \to -\infty$.
- if $\tilde{p}(s_o)$ belongs to the region between $\hat{p}$ and the p_2-axis, then there could be only three possibilities: either $\tilde{p}$ is the unique solution that tends to the origin as $s \to -\infty$, or $\tilde{p}_2 \to \infty$ as $s \to -\infty$ without $\tilde{p}$ crosses p_2-axis (and, of course, this can only happen if $\hat{p}$ does not cross it too), or there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in the case **4.** above. In the former case we will estimate the decay of $|\tilde{p}|$ in **8**; in the latter ones $|\tilde{p}| \to \infty$ as $s \to -\infty$.
- if $\tilde{p}(s_o)$ doesn't belong to the region between $\hat{p}$ and the p_2-axis, then either $\tilde{p}_2 \to \infty$ as $s \to -\infty$ or there exists $\bar{s} < s_o$ such that $\tilde{p}(\bar{s})$ is in case **5** above (and this is possible only if also $\hat{p}(s)$ crosses the p_2-axis). In both situations, again, $|\tilde{p}| \to \infty$ as $s \to -\infty$.

7. We can now provide more accurate estimates on the blow-up rate. Indeed by previous analysis, as in Theorem 3, a blow-up of $|\tilde{p}|$ can only occur when either $\tilde{p}_i \to -\infty$ as $s \to +\infty$ or $\tilde{p}_i \to +\infty$ as $s \to -\infty$ (for some index i). Hence, exactly as before, we can prove that there exists $s_o \in I\!R$ (and $\eta > 0$) such that $|\tilde{p}(s)| \ge \frac{\eta}{|s-s_o|}$. In terms of the original variable x, such a trajectory yields a solution defined on the whole real line, because by (4.7)

$$\left|\frac{dx}{ds}\right| = \Delta\big(\tilde{p}(s)\big) \ge \frac{c_o}{(s_o - s)^2} \, . \tag{5.10}$$

for some $c_o > 0$ and therefore either $x(s) \to +\infty$ as $s \to s_o-$ or $x(s) \to -\infty$ as $s \to s_o+$. In conclusion, the solution $\tilde{u}(x)$ which corresponds to $\tilde{p}(x)$ violates the growth condition (2.15), and hence it is not admissible.

8. Notice that only in case **6**-(ii), where $\tilde{p}$ is the unique solution that tends to 0 as $s \to -\infty$, we have a solution that could remain bounded in the whole $I\!R$. But in this case, we shall have as $s \to -\infty$

$$|\tilde{p}|(s) \le (\tilde{p}_2 - \tilde{p}_1)(s) \le \gamma e^{c_o s} \, , \tag{5.11}$$

for some $\gamma, c_o > 0$. Indeed studying the linearized system near the origin we see that $\tilde{p}$ tends to $(0,0)$ along the direction $(1, \frac{\kappa_2 - \kappa_1 + \sqrt{\kappa_1^2 + \kappa_2^2 - \kappa_1\kappa_2}}{\kappa_1})$. Then there exists $\bar{s}$ such that for $s < \bar{s}$ the following holds:

$$\tilde p_2(s) > (1+\frac{\sqrt{2}}{2})\frac{\kappa_2-\kappa_1}{\kappa_1}\,\tilde p_1(s) = \beta\,\frac{\kappa_2-\kappa_1+\sqrt{\kappa_1^2+\kappa_2^2-\kappa_1\kappa_2}}{\kappa_1}\,\tilde p_1(s)\,, \tag{5.12}$$

where

$$\beta = \frac{(1+\frac{\sqrt{2}}{2})(\kappa_2-\kappa_1)}{\kappa_2-\kappa_1+\sqrt{\kappa_1^2+\kappa_2^2-\kappa_1\kappa_2}}\,, \qquad \beta\in(0,1)\,. \tag{5.13}$$

Notice that, setting

$$\begin{aligned}\alpha &= \frac{\kappa_1-2\kappa_2-(\kappa_2-2\kappa_1)\,\beta\,\frac{\kappa_2-\kappa_1+\sqrt{\kappa_1^2+\kappa_2^2-\kappa_1\kappa_2}}{\kappa_1}}{1-\beta\,\frac{\kappa_2-\kappa_1+\sqrt{\kappa_1^2+\kappa_2^2-\kappa_1\kappa_2}}{\kappa_1}} = \\ &= \frac{\kappa_1-2\kappa_2-(\kappa_2-2\kappa_1)(1+\frac{\sqrt{2}}{2})\frac{\kappa_2-\kappa_1}{\kappa_1}}{1-(1+\frac{\sqrt{2}}{2})\frac{\kappa_2-\kappa_1}{\kappa_1}} > 0\,,\end{aligned} \tag{5.14}$$

we obtain exactly

$$\beta\,\frac{\kappa_2-\kappa_1+\sqrt{\kappa_1^2+\kappa_2^2-\kappa_1\kappa_2}}{\kappa_1} = \frac{\kappa_1-2\kappa_2-\alpha}{\kappa_2-2\kappa_1-\alpha}\,. \tag{5.15}$$

Hence, for $s<\bar s$,

$$\tilde p_2(s) > \frac{\kappa_1-2\kappa_2-\alpha}{\kappa_2-2\kappa_1-\alpha}\,\tilde p_1(s)\,, \tag{5.16}$$

i.e. $(\kappa_2-2\kappa_1)\tilde p_2-(\kappa_1-2\kappa_2)\tilde p_1 > \alpha(\tilde p_2-\tilde p_1)$. Recalling that $|\tilde p|\to 0$ as $s\to-\infty$, which implies the existence of $c_o>0$ and $\hat s$ such that $\alpha-\tilde p_1(s)-\tilde p_2(s)>c_o$ for any $s<\hat s$, we find

$$\frac{d}{ds}(\tilde p_2-\tilde p_1) = \tilde p_1^2-\tilde p_2^2+(\kappa_2-2\kappa_1)\tilde p_2-(\kappa_1-2\kappa_2)\tilde p_1 > (\alpha-\tilde p_1-\tilde p_2)(\tilde p_2-\tilde p_1) > c_o(\tilde p_2-\tilde p_1)\,,$$

for s small enough (namely $s<\min\{\bar s,\hat s\}$). Integrating we find $(\tilde p_2-\tilde p_1)(s)\le \gamma e^{c_o s}$ ($\gamma>0$) and hence (5.11) is proved. Next, recalling (4.7), in terms of the variable x we obtain

$$\left|\frac{dx}{ds}\right| = \Delta\big(\tilde p(s)\big) = \mathcal{O}(1)\cdot e^{2c_o s}\,, \tag{5.17}$$

$$\lim_{s\to-\infty} x(s) = x_o < \infty\,,$$

for some $x_o\in I\!R$. Therefore, to the entire trajectory $s\mapsto\tilde p(s)$, there corresponds only a portion of the trajectory $x\mapsto\tilde p(x)$, namely the values for $x>x_o$. To extend this trajectory also on the half line $]-\infty,x_o]$, we need to construct another trajectory $s\mapsto p(s)$ with $\lim_{s\to+\infty}p(s)=0$. But any such trajectory, by previous analysis, will yield a solution $\tilde u(x)$, which violates the sublinear growth condition (2.15) as $x\to-\infty$ and is not admissible.

Second Step. Next, we prove uniqueness of the admissible solution the case where h_i' is not constant. Let $u(\cdot)$ be the solution constructed before, with $p=u'$ remaining in a small disc V, centered at (κ_1,κ_2) with radius $\rho>0$, positively invariant for the flow of (4.8). Moreover, let $\tilde u$ be any other smooth solution of (4.3). We split the proof in three cases.

CASE 1: $\tilde{p}(s) \in V$ for every s. In this case, as in Theorem 3, we look at the difference $w(s) = \tilde{p}(s) - p(s)$ and at the linear evolution equation for w:

$$\frac{dw}{ds} = A(s)\, w(s)\,, \tag{5.18}$$

where A is the averaged matrix

$$A(s) = \int_0^1 Df(\theta p(s) + (1-\theta)\tilde{p}(s))d\theta\,, \tag{5.19}$$

and f is the vector field describing our system, as in (4.12). Since $p, \tilde{p} \in V$, the matrix $A(s)$ is very close to the Jacobian matrix $Df(\kappa_1, \kappa_2)$, therefore

$$\frac{d}{ds}\big|w(s)\big| \leq -K\,\big|w(s)\big|\,, \tag{5.20}$$

for some constant $K > 0$. Indeed,

$$Df(\kappa_1, \kappa_2)x \cdot x < -\frac{\kappa_1 + \kappa_2}{2}|x|^2. \tag{5.21}$$

Provided that $\delta, \rho > 0$ are small enough, there will exist $K > 0$ such that $A(s)x \cdot x < -K|x|^2$. But then, exactly as in Theorem 3, (5.20) implies $p(0) = \tilde{p}(0)$ and hence $p = \tilde{p}$ by the uniqueness of the Cauchy problem.

CASE 2: $\tilde{p}(s_o) \notin V$ for some s_o and, in particular, $\tilde{p}(s_o)$ in a small neighborhood W of the origin. Consider the linearized system

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = H \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\,, \qquad H = \begin{pmatrix} h_1' - h_2' & h_1' \\ h_2' & h_2' - h_1' \end{pmatrix}\,,$$

and notice that the origin is again a saddle point for this system. Indeed H has eigenvalues λ_1, λ_2 such that, recalling (5.6) and provided $\delta < \frac{1}{2}\min\{-\kappa_1, \kappa_2\}$,

$$0 < \frac{\sqrt{2}}{2}(\kappa_2 - \kappa_1 - 2\delta) \leq |\lambda_i| = \sqrt{(h_1')^2 + (h_2')^2 - h_1'h_2'} \leq \kappa_2 - \kappa_1 + 2\delta\,. \tag{5.22}$$

Moreover its eigenvectors v_1, v_2 form angles α_1, α_2 with the positive direction of the p_1-axis such that

$$0 \geq \tan\alpha_1 = \frac{\lambda_1 + (h_2' - h_1')}{h_1'} > \tan\alpha_2 = \frac{\lambda_2 + (h_2' - h_1')}{h_1'}\,.$$

More precisely, for δ small enough, we have

$$0 \geq \tan\alpha_1 > (1 - \frac{\sqrt{2}}{2})\,\frac{\kappa_2 - \kappa_1 + 2\delta}{\kappa_1 + \delta} > (1 + \frac{\sqrt{2}}{2})\,\frac{\kappa_2 - \kappa_1 - 2\delta}{\kappa_1 - \delta} > \tan\alpha_2\,. \tag{5.23}$$

Hence, as in the previous proof of Theorem 3, we show the existence of four sectors S_i, $i = 1, \ldots, 4$ (see Figure 7), where the following holds:

(a) If $\tilde{p}(s_o)$ is in S_1 or S_3, then $|\tilde{p}(s)|$ grows for $s < s_o$ and the solution moves away from W;

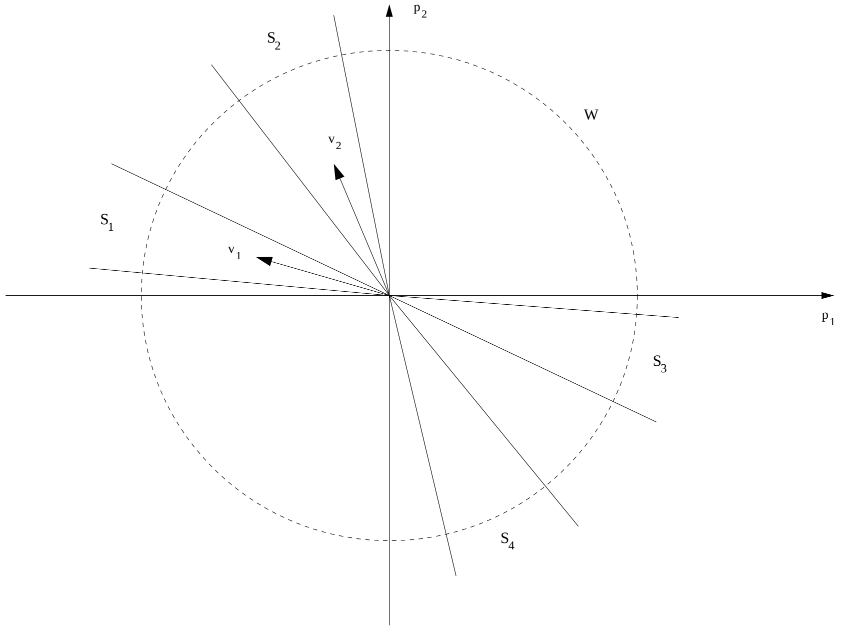

Figure 7

(b) Both boundaries of S_2 and S_4 allow orbits to only exit from those sectors for $s < s_o$;

(c) If $\tilde{p}(s_o) \notin S_i$ for all $i = 1, \ldots, 4$, then for $s < s_o$ the angle between the vector $(\tilde{p}_1, \tilde{p}_2)$ and the p_1-axis is strictly monotone, forcing the solution either to reach S_1 or S_3, or to move away from W;

(d) Finally, if $\tilde{p}(s_o)$ is in S_2 or S_4, then for $s < s_o$ the solution can tend to the origin. But

$$\begin{aligned}\frac{d}{ds}(\tilde{p}_2 - \tilde{p}_1) &= \tilde{p}_1^2 - \tilde{p}_2^2 + (h_2' - 2h_1')\tilde{p}_2 - (h_1' - 2h_2')\tilde{p}_1 > \\ &> \tilde{p}_1^2 - \tilde{p}_2^2 + (\kappa_2 - 2\kappa_1 - 3\delta)\tilde{p}_2 - (\kappa_1 - 2\kappa_2 - 3\delta)\tilde{p}_1 \,,\end{aligned}$$

and, provided δ is small enough, we can use (5.23) to find $\alpha > 0$ such that

$$\frac{d}{ds}(\tilde{p}_2 - \tilde{p}_1) > \tilde{p}_1^2 - \tilde{p}_2^2 + (\kappa_2 - 2\kappa_1 - 3\delta)\tilde{p}_2 - (\kappa_1 - 2\kappa_2 - 3\delta)\tilde{p}_1 > (\alpha - \tilde{p}_1 - \tilde{p}_2)(\tilde{p}_2 - \tilde{p}_1)\,. \quad (5.24)$$

Hence an estimate of exponential type of the decay of $|\tilde{p}|$ follows as in (5.11).

CASE 3: $\tilde{p}(s_o) \notin V$ and $\tilde{p}(s_o)$ not in a neighbourhood of the origin. In this case, combining (5.6) and the continuous dependence of solutions with the estimates of the constant case (indeed they are true also in this more general setting), we can prove that $|\tilde{p}| \to \infty$ for finite s and that the rate of blow-up of $|\tilde{p}|$ can be estimated in the same way we did in the *First Step*.

In any case either $\tilde{u} \equiv u$ or, in the original coordinates x, $\tilde{u}$ fails to satisfy (2.15).

Finally, we rule out the possibility that the gradient $\tilde{p} = \tilde{u}'$ has jumps. Looking at the phase portrait in Figure 6, we see that after a one or at most two admissible jump, the values of $\tilde{p}$ must fall within the positively invariant region $\{(p_1, p_2) \,:\, p_2 < 0\,,\ p_1 + p_2 < 0\}$. It follows that $\tilde{p}$ cannot have any more jumps, and the estimates in **2**, **5** and **7** (together with their analogs in the non-constant case) imply that $\tilde{u}(x)$ violates (2.15) as $x \to +\infty$. Therefore, $\tilde{u}$ is not an admissible solution. $\square$

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