Liouville and Toda field theories
on Riemann surfaces

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Abstract

We study the Liouville theory on a Riemann surface of genus g by means of their associated Drinfeld–Sokolov linear systems. We discuss the cohomological properties of the monodromies of these systems. We identify the space of solutions of the equations of motion which are single–valued and local and explicitly represent them in terms of Krichever–Novikov oscillators. Then we discuss the operator structure of the quantum theory, in particular we determine the quantum exchange algebras and find the quantum conditions for univalence and locality. We show that we can extend the above discussion to sl_n Toda theories.

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1   Introduction

A link between the Liouville equation and Riemann surfaces was found first in mathematics as a clue to the uniformization theory of Riemann surfaces. Much more recently the Liouville equation has appeared in the theoretical physics literature in connection with string theory, 2D gravity and conformal field theory (for reviews of various aspects of Liouville theory in physics, see [1]).

The Liouville action appears in the Polyakov string theory path integral when, in order to perform the functional integration over the metrics of a given Riemann surface, one fixes the gauge freedom by choosing the so–called conformal gauge. What one is supposed to do next, for off–critical string theory, is to integrate over the Liouville field (i.e. quantize the Liouville theory) and the other relevant modes (ghosts, matter), and finally integrate over the moduli space. It is evident that uniformization theory must play a very important role here [2]. This ambitious program meets with formidable obstacles, and, in any case, the approach based on matrix integrals and on topological field theories is certainly more effective, for the time being. However Polyakov's path integral remains a basic suggestion and a basic challenge in string theory. Eventually one should be able to reconcile the different approaches.

Another context in which the Liouville theory appears is the Coulomb gas representation of conformal field theories [3]. For it is well–known that the Coulomb gas is a set of practical recipes to construct conformal field theories of a certain type, typically minimal models, which is nothing but a manifestation of an underlying Liouville [4],[5] or conformal Toda theory. In this case the Liouville field does not play the role of a metric, as it does in string theory; the metric is a fixed one on a fixed Riemann surface and the most frequently studied case is the one of a flat metric in genus 0. Although general theorems have been formulated and partial results have been obtained for conformal field theories on Riemann surfaces, almost nothing is known concerning the approach to conformal field theories in higher genus by means of the Liouville or Toda theories, which, we recall, provide a systematic method to calculate correlation functions.

Whatever the context we consider, it is clear that a better knowledge of Liouville and Toda theories on Riemann surfaces would be most welcome. With this motivation we set out to study, in this paper, the quantization of a Liouville theory on a Riemann surface of fixed genus. By quantization we mean canonical quantization and to avoid misunderstandings we recall the quantization procedure in genus zero, more precisely on a cylindrical topology, presented in [6], [7], [8], [9]. There the classical phase space was defined as the space of solutions of the Liouville equation endowed with the canonical Poisson bracket. It was shown that this phase space can be represented by means of free bosonic oscillators, and, at this point, it was elementary, at least in principle, to quantize it by transforming the free bosonic oscillators into free bosonic creation and annihilation operators. In this construction a crucial role is played by the appropriate Drinfeld–Sokolov (DS) linear system [10]. The quantum construction on the other hand hinges upon the quantum group symmetry.

In section 2 of this paper we introduce the two (chiral and antichiral) DS systems appropriate for a Liouville theory on a generic genus g Riemann surface with the help of the concept of analytic connection. In section 3 we set out to study the properties of the solutions of the DS systems. In this context we found it very helpful to be able to explicitly express the connections $\mathfrak{p}$ and $\bar{\mathfrak{p}}$, which specify the DS systems, in terms of bases of meromorphic differentials on the Riemann surface with two punctures. The properties of these bases (the Krichever–Novikov bases) are summarized in subsection 3.1. The rest of section 3 is devoted to the monodromy of the DS systems: it is particularly pertinent to analyze it in cohomological terms and, in fact, we show that it is an $SL(2, \mathbb{C})$-valued cocycle.

Next we turn to the study of the solutions of the Liouville equation which can be obtained via solutions of the DS systems. Our aim is to identify the solutions which are local and single–valued on a Riemann surface with two punctures. In section 4 we discuss single–valuedness, which imposes two set of constraints on $\mathfrak{p}$ and $\bar{\mathfrak{p}}$: the first set tells us that the zero modes of the chiral and antichiral DS systems around the two punctures and around the homology cycles should be the same (this is a generalization of the genus 0 constraint); the second set expresses the fact that the monodromies of the DS systems must be represented by coboundaries (such type of condition is not needed in genus 0). In section 5 we introduce a symplectic structure on the space of DS connections. We are then able to calculate the exchange algebra and to impose locality. It turns out that the first set of constraints required for univalence is first class, while the second set is second class. Finally we introduce the corresponding Dirac brackets.

In section 6 we pass to quantization. We find the quantum exchange algebra and the quantum analogs of the above two sets of constraints which ensure univalence and locality. Section 7 is devoted to the generalization of the previous results to Toda theories. We think the example of the sl_3 Toda theory is enough to convince the reader that everything works for these theories too, up to minor technical modifications.

We mentioned above the two physical contexts in which the Liouville equation appears. It may therefore be interesting to compare the two corresponding types of solutions. In section 8 we make a detailed comparison between the solutions obtained via the DS systems and the uniformizing solution for a compact Riemann surface. We conclude that the latter is not included among the former.

Finally Appendices A and B contain detailed developments omitted in the main text, Appendix C is devoted to analyzing a non–standard family of solutions of the Liouville equation and, finally, Appendix D contains a discussion of the conformal properties of the Bloch wave basis.

The main results of our paper are: 1) the identification of the cohomological properties of the monodromy of the DS system (section 3); 2) the identification of the constraints (41), (42), (43); 3) the exchange algebras and the Dirac brackets of section 5; 4) the quantum exchange algebra and quantum coboundary conditions (69) and (70) of section 6; 6) the generalization to the sl_3 Toda field theory in section 7.

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2 The Liouville equation on Riemann surfaces

The basic objects of our analysis will be the solutions of the Liouville equation. Let X be a fixed compact Riemann surface of genus $g \geq 2$[1]. Usually $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ will denote a complex atlas on X. Fixing the complex atlas is tantamount to giving the complex structure, which will be held fixed throughout.

The Liouville equation is

$$\partial \bar{\partial} \varphi = e^{2\varphi} \tag{1}$$

where $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$, z being any local coordinate. In (1) we drop the chart label due to its invariance under a change of local coordinates. For if $(U_\alpha, z_\alpha) \rightarrow (U_\beta, z_\beta)$ with holomorphic coordinate change $z_\alpha = f_{\alpha\beta}(z_\beta)$, eq.(1) will not change its form if

$$\varphi_\beta(z_\beta) = \varphi_\alpha(f_{\alpha\beta}(z_\beta)) + \frac{1}{2} \log |f'_{\alpha\beta}(z_\beta)|^2 \tag{2}$$

This implies in particular that $e^{2\varphi}$ transforms as a $(1,1)$–form. We can consider it as the Kähler form of a metric on X if $e^{2\varphi}$ is regular. This is true in particular if the solution happens to be the one coming from the uniformization of X.

2.1 The DS linear system

To find a large class of solutions of the Liouville equation we proceed as in genus zero [7],[8] and write the linear system associated to it

$$\partial \mathcal{Q} = (\mathfrak{p}H - E_+)\mathcal{Q}, \qquad \bar{\partial}\mathfrak{p} = 0 \tag{3}$$

$$\bar{\partial}\bar{\mathcal{Q}} = -\bar{\mathcal{Q}}(\bar{\mathfrak{p}}H - E_-), \qquad \partial\bar{\mathfrak{p}} = 0 \tag{4}$$

where once again we drop the chart label (since we will impose the form of these equations to hold in any coordinate patch), and

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The solution matrices $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ have the form

$$\mathcal{Q} = \begin{pmatrix} \sigma_1 & \sigma_2 \\ 0 & \sigma_1^{-1} \end{pmatrix}, \qquad \bar{\mathcal{Q}} = \begin{pmatrix} \bar{\sigma}_1 & 0 \\ \bar{\sigma}_2 & \bar{\sigma}_1^{-1} \end{pmatrix}$$

and locally

$$\sigma_1(z) = e^{\int^z \mathfrak{p}(w)dw}, \qquad \sigma_2(z) = -\sigma_1(z) \int^z \sigma_1(w)^{-2} dw \tag{5}$$

$$\bar{\sigma}_1(\bar{z}) = e^{-\int^{\bar{z}} \bar{\mathfrak{p}}(\bar{w})d\bar{w}}, \qquad \bar{\sigma}_2(\bar{z}) = \bar{\sigma}_1(\bar{z}) \int^{\bar{z}} \bar{\sigma}_1(\bar{w})^{-2} d\bar{w} \tag{6}$$

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[1] This limitation will be held throughout the paper in order to avoid lengthy specifications on genus 0 and 1, which could be done anyway. The genus 0 case, in particular, has been treated at length elsewhere.

Then, in a local patch,

$$e^{-\varphi} = \sigma M \bar{\sigma}, \qquad \sigma = (\sigma_1, \sigma_2), \qquad \bar{\sigma} = \begin{pmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \end{pmatrix} \qquad (7)$$

is a solution of the Liouville equation, for any constant matrix M.

What we have done so far is well–defined in any local chart, but now we have to define the DS system in a global way on X. Let us consider (3) first. We remark that in order for $e^{2\varphi}$ to be a $(1,1)$ form, σ_i and $\bar{\sigma}_i$ must be tensors of weight $-1/2$ for $i = 1, 2$. This implies that the rows of $\mathcal{Q}$ have weights $(-1/2, 0)$ and $(1/2, 0)$. Thus our arena will be the holomorphic vector bundle $V = K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}$, where $K^{\frac{1}{2}}$ is a square root of the canonical line bundle K. Since there are many possible choices, we fix one once for all.

The next step will be to define the DS linear system as an analytic connection on this vector bundle. This is, we believe, the correct and easiest way to put a differential equation in a global context. Generally speaking, an analytic connection in a holomorphic vector bundle E over X is a map

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$$

where $\mathcal{O}_X$ is the sheaf of holomorphic functions on X, $\mathcal{E}$ is the sheaf of holomorphic sections of E and Ω^1_X the sheaf of holomorphic differentials on X, i.e. the sheaf of holomorphic sections of K. Analytic connections do not always exist [11]. According to a theorem of Weil, their existence is equivalent to E being a direct sum of indecomposable analytically flat bundles [11, 12]. This is certainly not the case for V, since $c_1(K^{\frac{1}{2}}) = g - 1$ and the genus g is supposed to be ≥ 2. Thus in such a case an analytic connection must be more properly defined as a map [13]

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X(*Y)$$

where now $\Omega^1_X(*Y)$ is the sheaf of meromorphic differentials, holomorphic outside a subset Y of X.[2] Allowing for poles trivializes the cohomological obstructions to the existence of (analytic) connections. Therefore on these general grounds we expect the DS connection to be meromorphic.

After these general remarks let us go into more detail. We start from our DS connection in a local chart

$$\nabla^{DS} = \partial + \begin{pmatrix} -\mathfrak{p}(z) & 1 \\ 0 & \mathfrak{p}(z) \end{pmatrix} dz$$

and require this form to be maintained in any other local chart, that is, applying ∇ to a section of V should give a (meromorphic) section of $V \otimes K$. This will give conditions on the coefficients of the connection. First of all, the "1" into

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[2]One can consider also replacing the sheaf $\Omega^1_X(*Y)$ with $\mathcal{M}^1_X$, the sheaf of meromorphic 1-differentials on X.

the connection matrix has an invariant meaning if we notice that 1 is a section of $Hom(K^{\frac{1}{2}}, K^{-\frac{1}{2}} \otimes K)$. The condition on $\mathfrak{p}$ is the following transformation rule:

$$\frac{d}{dz_\beta} \log k_{\alpha\beta}^{1/2} = \mathfrak{p}_\beta(z_\beta) - \mathfrak{p}_\alpha(z_\alpha(z_\beta))\frac{dz_\alpha}{dz_\beta} \tag{8}$$

where $k_{\alpha\beta} = dz_\beta/dz_\alpha$ are the transition functions of K and $k_{\alpha\beta}^{1/2}$ is a suitably chosen collection of square roots defining $K^{\frac{1}{2}}$. The condition (8) has the form of a relation in Čech cohomology. Indeed, introducing $c_{\alpha\beta} = \frac{d}{dz_\beta} \log k_{\alpha\beta}^{1/2}$, we easily have $\{c_{\alpha\beta}\} \in Z^1(\mathfrak{U}, \Omega_X^1)$ and (8) takes the form $\{c_{\alpha\beta}\} = \delta\{\mathfrak{p}_\alpha\}$, where δ is the Čech coboundary map. Thus $\{\mathfrak{p}_\alpha\} \in C^0(\mathfrak{U}, \Omega_X^1)$ is the cochain whose coboundary is $\{c_{\alpha\beta}\}$. However, it is not difficult to see that the above relation cannot take place, since the cocycle $\{c_{\alpha\beta}\}$ is the one defining the Chern class of $K^{\frac{1}{2}}$ [14], so that (8) can be realized only if we take $\{\mathfrak{p}_\alpha\} \in C^0(\mathfrak{U}, \mathcal{M}_X^1)$. This confirms our statement that an analytic connection will in general be meromorphic. Meromorphic connections on K are treated in [24].

Analogous things can be repeated for the antiholomorphic DS (4). Once this is done our linear system is well–defined on X.

2.2 The problem

Therefore from now on our problem is:

1) to parametrize the space $\mathfrak{F}$ of solutions of the Liouville equation determined through the recipe (7);

2) to define a symplectic structure in $\mathfrak{F}$ and to identify the subspace $\mathfrak{F}_0 \subset \mathfrak{F}$ of solutions which are single–valued and local with respect to this structure;

3) to quantize $\mathfrak{F}_0$.

3 Explicit representation and properties of the DS system

We set out to solve the problem just formulated. To this end it is very convenient to rely on an explicit representation of the connections $\mathfrak{p}$ and $\bar{\mathfrak{p}}$. We saw that they must be meromorphic connections on a spin 1/2 bundle. In analogy with the genus zero treatment, we will choose the simplest possible arrangement, that is we will suppose that all their poles are concentrated only in two generic points P_+ and P_- of X. Next we have to parametrize the space of these connections. Using the property that it is an affine space, we can describe all of them writing

$$\mathfrak{p} = \Gamma_0 + p$$

where Γ_0 is a fixed reference connection and p is a meromorphic one–form. As for Γ_0 we may choose

$$\Gamma_0 = -\partial \log h \tag{9}$$

h being a meromorphic section of $K^{\frac{1}{2}}$. If $K^{\frac{1}{2}}$ represents an odd theta characteristic the section h can be chosen holomorphic [15]. On the other hand any p can be represented as

$$p = \sum_k p_k \, \omega^k \tag{10}$$

where $\{\omega^k\}$ is a complete basis of meromorphic differentials holomorphic outside the set $Y = P_+ \cup P_-$ (see below). The same we can do for $\bar{\mathfrak{p}}$. Therefore our two DS systems, and consequently also the solutions of the Liouville equation given by the reconstruction formula (7), are parametrized by the moments p_k and $\bar{p}_k$. This is the parametrization of the space $\mathfrak{F}$ we anticipated above.

Complete bases on X like $\{\omega^k\}$ do exists and were introduced some time ago by I.M.Krichever and S.P.Novikov. Due to their importance in our analysis, we devote the next subsection to recalling their definition and properties. For more detailed information see [16],[17],[18],[20].

Finally a remark concerning the modes p_k. In genus 0 they are called free bosonic oscillators. Here they are the closest thing one can define to free bosonic oscillators, but they are not truly free. The Poisson brackets (45) below reveal the complicated way these modes interact with the background geometry of the Riemann surface.

3.1   KN basis in X

In the following $Q, Q', Q_0, ..$ will denote points on X, but we will often stick to the habit of denoting these points with local coordinates $z, z', z_0, ...$. On X let us consider the two distinguished points P_+ and P_- and local coordinates z_+ and z_- around them, such that $z_\pm(P_\pm) = 0$. On X we can introduce complete bases of meromorphic tensors which are holomorphic in $X \setminus Y$. In particular we will need a basis of vector fields e_n, functions A_n, 1–differentials ω^n and quadratic differentials Ω^n. Here n is integer or half–integer according to whether g is even or odd. The behaviour near $P_\pm$ is given by

$$A_n(z_\pm) = a_n^\pm z_\pm^{\pm n - g/2}(1 + \mathcal{O}(z_\pm))$$
$$\omega^n(z_\pm) = b_n^\pm z_\pm^{\mp n + g/2 - 1}(1 + \mathcal{O}(z_\pm))dz_\pm$$
$$e_n(z_\pm) = c_n^\pm z_\pm^{\pm n - 3/2g + 1}(1 + \mathcal{O}(z_\pm))\frac{\partial}{\partial z_\pm}$$
$$\Omega^n(z_\pm) = d_n^\pm z_\pm^{\mp n + 3/2g - 2}(1 + \mathcal{O}(z_\pm))(dz_\pm)^2$$

For $|n| \leq g/2$ the definitions of A_n and ω^n must be modified, because of the Weiestrass theorem. We set $A_{g/2} = 1$, while for $n = g/2 - 1, ..., -g/2$ the power of z_- is lowered by one in the above definition of A_n. As for ω^n and $n = g/2 - 1, ..., -g/2$ the power of z_- must be raised by 1 in the above definition, while $\omega^{g/2}$ is set equal to the third kind differential

$$\omega^{g/2}(z_\pm) = \pm\frac{1}{z_\pm}(1 + \mathcal{O}(z_\pm))dz_\pm \tag{11}$$

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This differential is chosen to be normalized in such a way that the periods around any cycle are purely imaginary. This implies that the function

$$\tau(Q) = Re \oint_{Q_0}^{Q} \omega^{g/2}$$

is univalent, for a fixed $Q_0 \in X$. The level curves of this function will be denoted C_τ. They reduce to small circles around $P_\pm$ in the vicinity of these two points.

The above bases elements are uniquely determined up to numerical constants due to the Riemann–Roch theorem. So we can set for example $a_n^+ = 1$, the a_n^-'s being then completely determined. We can do the same for the $c_n^\pm$'s. As for the remaining constants they are fixed by the duality relations

$$\frac{1}{2\pi i} \oint_{C_\tau} dz \, A_n(z) \, \omega^m(z) = \delta_n^m \tag{12}$$

$$\frac{1}{2\pi i} \oint_{C_\tau} dz \, e_n(z) \, \Omega^m(z) = \delta_n^m \tag{13}$$

The Lie brackets of the basis elements e_n are

$$[e_n, e_m] = C_{nm}^k \, e_k \tag{14}$$

Here and throughout the paper summation over repeated upper and lower indices is understood, unless otherwise stated. One has

$$C_{nm}^k = \frac{1}{2\pi i} \oint_{C_\tau} [e_n, e_m] \Omega^k$$

Equation (14) define the KN algebra over X. Its central extension is defined by means of the cocycle

$$\chi(e_n, e_m) = \frac{1}{24\pi i} \oint_{C_\tau} \tilde{\chi}(e_n, e_m) \tag{15}$$

the integral is over any simple cycle surrounding P_+ in an anticlockwise way. For any two meromorphic vector fields $f = f(z)\frac{\partial}{\partial z}$ and $g = g(z)\frac{\partial}{\partial z}$, $\tilde{\chi}(f, g)$ is given by

$$\tilde{\chi}(f, g) = \left(\tfrac{1}{2}(f'''g - g'''f) - R(f'g - fg')\right) dz_+ \tag{16}$$

where $'$ denotes derivative with respect to z_+ and R is a Schwarzian connection. Then the extended KN algebra is defined by

$$[e_n, e_m] = C_{nm}^k \, e_k + t \, \chi(e_n, e_m), \qquad [e_n, t] = 0 \tag{17}$$

In the following we will also make use of the relations

$$dA_n = -\gamma_{nm}\omega^m, \qquad \gamma_{nm} = \frac{1}{2\pi i} \oint_{C_\tau} A_n dA_m$$

and of the definitions

$$N_i^n = \frac{1}{2\pi i} \oint_{a_i} \omega^n, \qquad M_i^n = \frac{1}{2\pi i} \oint_{b_i} \omega^n$$

where $\{a_i, b_i\}$, $i = 1, ..., g$ is a basis of homology cycles. It is easy to prove [19] the relations

$$N_i^n \gamma_{nm} = 0, \qquad M_i^n \gamma_{nm} = 0 \tag{18}$$

The structure constants C_{nm}^k and γ_{nm} vanish outside a finite band of values of $n + m$ around $n + m = 0$.

Finally we will need two remarkable relations proven in [19]:

$$N_i^n A_n(Q) =: \mathcal{A}_i = \frac{1}{2\pi i} \oint_{a_i} \omega^{g/2} \tag{19}$$

$$M_i^n A_n(Q) =: \mathcal{B}_i = \frac{1}{2\pi i} \oint_{b_i} \omega^{g/2} \tag{20}$$

In other words the LHS's are constants that can be explicitly calculated.

As bases for anti–holomorphic tensors we choose the complex conjugate of the above bases and will distinguish them from the above ones by means of a bar: $\bar{A}_n$, etc.. In particular, due to the choice of normalization for the third kind differential, we have

$$\bar{\mathcal{A}}_i = \mathcal{A}_i, \qquad \bar{\mathcal{B}}_i = \mathcal{B}_i \tag{21}$$

Using the explicit representation of $\mathfrak{p}$ in terms of (10) and (9), one can better appreciate what are the solutions of Liouville we are analyzing: they may in general be very singular at $P_\pm$, but this is no novelty with respect to genus 0. These are the solutions we need in order to construct a manageable phase space.

3.2   Properties of the DS solutions

This subsection is devoted to analyzing a few general properties of the solutions of the DS system (3). As we already explained, the DS connection is a map

$$\nabla^{DS} : \mathcal{V} \longrightarrow \mathcal{V} \otimes_{\mathcal{O}_X} \Omega_X^1(*Y)$$

where $\mathcal{V}$ is the sheaf of holomorphic section of the chosen vector bundle $V = K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}$. With respect to the covering $\mathfrak{U}$ it is a collection of meromorphic differential equations. Thus for any open set $U_\alpha \in \mathfrak{U}$ we can exhibit a fundamental solution $\mathcal{Q}_\alpha$ of the differential equation $\nabla^{DS} \mathcal{Q}_\alpha = 0$. This requires a choice of the integration constants in (5). Once this is done, $\mathcal{Q}_\alpha$ is a local frame for V on U_α, but when changing local chart, on $U_\alpha \cap U_\beta$ we will have the gluing law

$$\mathcal{Q}_\alpha T_{\alpha\beta}^\vee = \begin{pmatrix} k_{\alpha\beta}^{-1/2} & 0 \\ 0 & k_{\alpha\beta}^{1/2} \end{pmatrix} \cdot \mathcal{Q}_\beta$$

where $T_{\alpha\beta}^\vee$ is a constant matrix. This follows from the fact that $\mathcal{Q}_\alpha$ and the RHS of the above equation both solve the differential equation $\nabla^{DS} \mathcal{Q} = 0$ on $U_\alpha \cap U_\beta$ with respect to z_α. The duality symbol "$\vee$" will be explained later on.

Consistency on the triple intersections the matrices $\{T_{\alpha\beta}^\vee\}$ implies the cocycle condition:

$$T_{\alpha\beta}^\vee T_{\beta\gamma}^\vee = T_{\alpha\gamma}^\vee$$

Changing our choice of the local frames $\{Q_\alpha\}$ yields a new collection $\{Q'_\alpha\}$ such that

$$Q_\alpha = Q'_\alpha \, C_\alpha$$

for appropriate constant matrices $\{C_\alpha\}$. This in turn produces a new cocycle $T'^\vee_{\alpha\beta}$ such that

$$T'^\vee_{\alpha\beta} = C_\alpha^{-1} T^\vee_{\alpha\beta} C_\beta$$

and therefore describes the same cohomology class.

Thus we interpret the collection $\{T^\vee_{\alpha\beta}\}$ as defining a flat rank 2 vector bundle or, in other terms, a rank 2 local system. The local system, in turn, is a representation of the fundamental group into the relevant structure group. Due to the singularities of the connection, we must be cautious about what fundamental group we are talking about. The singularities of the connection come either from those of Γ_0 or from the poles of the KN basis. It is easy to see that the former can only produce poles, while the latter can produce in general essential singularities at $P_\pm$ in the solutions of the DS system. For this reason we had better remove the points P_+ and P_-. As a consequence the fundamental group we consider is the fundamental group of $X' = X \setminus Y$.

What we are going to do next is to build a collection of local fundamental solution matrices for the DS connection, and subsequently describe in detail the associated monodromy, i.e. the local system (this will be done in the next subsection).

Let us discuss first the solution σ_1 in (5). We consider the covering $\mathfrak{U} = (U_\alpha, z_\alpha)$ of X' and a collection $\{Q_\alpha\}$ of points of X, one for each open neighbourhood U_α. For every α and $Q \in Q_\alpha$ define

$$\mathcal{L}_\alpha(Q) := \int_{Q_\alpha}^{Q} \mathfrak{p}_\alpha(z_\alpha) \, dz_\alpha$$

where the integration goes along any path contained in U_α and joining Q_α and Q. Thus the local solution (5) is written as

$$\sigma_{1\alpha}(Q) = e^{\mathcal{L}_\alpha(Q)}, \quad \sigma_{2\alpha}(Q) = -e^{\mathcal{L}_\alpha(Q)} \int_{Q_\alpha}^{Q} e^{-2\mathcal{L}_\alpha(z_\alpha)} \, dz_\alpha \qquad (22)$$

Now, to glue two solutions, consider the following situation. Let Q belong to $U_\alpha \cap U_\beta$ and consider another point Q' still belonging to the intersection $U_\alpha \cap U_\beta$ and lying, say, on the path from Q_α to Q. Then we have

$$
\begin{aligned}
\mathcal{L}_\alpha(Q) &= \int_{Q_\alpha}^{Q} \mathfrak{p}_\alpha(z_\alpha) \, dz_\alpha = \left(\int_{Q_\alpha}^{Q'} + \int_{Q'}^{Q} \right) \mathfrak{p}_\alpha \, dz_\alpha \\
&= \int_{Q_\alpha}^{Q'} \mathfrak{p}_\alpha \, dz_\alpha + \int_{Q'}^{Q} \mathfrak{p}_\beta \, dz_\beta + \int_{Q'}^{Q} d\log\left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2} \\
&= \int_{Q_\alpha}^{Q'} \mathfrak{p}_\alpha \, dz_\alpha + \int_{Q'}^{Q_\beta} \mathfrak{p}_\beta \, dz_\beta + \int_{Q_\beta}^{Q} p_\beta \, dz_\beta + \log\left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}\Bigg|_{Q'}^{Q}
\end{aligned}
$$

$$= \mathcal{L}_\beta(Q) + \log \left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}(Q) + b_{\alpha\beta}$$

with

$$b_{\alpha\beta} = \int_{Q_\alpha}^{Q'} \mathfrak{p}_\alpha \, dz_\alpha - \int_{Q_\beta}^{Q'} \mathfrak{p}_\beta \, dz_\beta - \log \left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}(Q')$$

So we obtain the transformation rule:

$$\sigma_{1\alpha}(Q) = c_{\alpha\beta} \left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}(Q)\,\sigma_{1\beta}(Q) \tag{23}$$

with $c_{\alpha\beta} = e^{b_{\alpha\beta}}$. The transformation rule (23) is meaningful as $b_{\alpha\beta}$ (or $c_{\alpha\beta}$) is a number, i.e it does not depend on the point Q' used to calculate it. This is easily seen simply choosing another point Q''. If Q' and Q'' both lie in $U_\alpha \cap U_\beta$, by using (8) we obtain

$$\int_{Q''}^{Q'} \mathfrak{p}_\alpha \, dz_\alpha = \int_{Q''}^{Q'} \mathfrak{p}_\beta \, dz_\beta + \int_{Q''}^{Q'} d\log\left(\frac{dz_\alpha}{dz_\beta}\right)^{1/2}$$

which proves the assertion.

We have given explicitly the above derivation as a sample of the calculations we use. From now we will be much more succinct.

On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ it is easy to verify the cocycle condition:

$$c_{\alpha\beta} c_{\beta\gamma} = c_{\alpha\beta}$$

Thus we have constructed a collection $\{c_{\alpha\beta}\}$ with values in $\mathbb{C}^*$ satisfying the cocycle condition on our Riemann surface X'.

Finally we should discuss what happens if we change the reference collection from $\{Q_\alpha\}$ to $\{Q'_\alpha\}$. It does not present any difficulty to see that $\{c_{\alpha\beta}\}$ changes by a coboundary, that is

$$c_{\alpha\beta} \longrightarrow c'_{\alpha\beta} = a_\alpha\, c_{\alpha\beta} a_\beta^{-1}$$

where the non-zero complex number $a_\alpha = \exp(\int_{Q_\alpha}^{Q'_\alpha} p_\alpha)$.

In summary, the differential equation

$$\partial \sigma_1 = \mathfrak{p}\,\sigma_1$$

can be solved on the non compact Riemann surface $X' = X \setminus Y$. The solution involves the 1-cocycle $\{c_{\alpha\beta}\}$ with values in $\mathbb{C}^*$, and we denote by C the corresponding cohomology class in $H^1(X', \mathbb{C}^*)$. By the usual correspondence, C is a flat line-bundle over X'. Thus σ_1 is actually a meromorphic section of $C \otimes K^{-\frac{1}{2}}$ (because of the zeros of h).

Let us take up next the construction of σ_2. We use the same scheme as before, namely we consider the point Q lying in the intersection $U_\alpha \cap U_\beta$ together with the paths joining it to the reference points Q_α and Q_β. We find

$$\sigma_{2\alpha}(Q) = d_{\alpha\beta}\, k_{\alpha\beta}(Q)^{-1/2}\, \sigma_{1\beta}(Q) + c_{\alpha\beta}^{-1}\, k_{\alpha\beta}(Q)^{-1/2}\, \sigma_{2\beta}(Q) \tag{24}$$

with

$$d_{\alpha\beta} = -c_{\alpha\beta} \int_{Q_\alpha}^{Q'} e^{-2\mathcal{L}_\alpha(z_\alpha)} dz_\alpha + c_{\alpha\beta}^{-1} \int_{Q_\beta}^{Q'} e^{-2\mathcal{L}_\beta(z_\beta)} dz_\beta$$

where $Q \in U_\alpha \cap U_\beta$. Again, the number $d_{\alpha\beta}$ does not depend on the point used to calculate it. Therefore the transformation rule (24) is well–defined on X'. It is apparent that the field σ_2 does not simply transform as a spin $-1/2$ tensor. Although the conformal weight $-1/2$ is preserved, σ_1 and σ_2 get mixed upon changing the local chart:

$$\begin{pmatrix} \sigma_{1\alpha} \\ \sigma_{2\alpha} \end{pmatrix} = k_{\alpha\beta}^{-1/2} \begin{pmatrix} c_{\alpha\beta} & 0 \\ d_{\alpha\beta} & c_{\alpha\beta}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \sigma_{1\beta} \\ \sigma_{2\beta} \end{pmatrix}$$

or, passing to the collection $\{\mathcal{Q}_\alpha\}$ of fundamental matrices

$$\mathcal{Q}_\alpha = \begin{pmatrix} k_{\alpha\beta}^{-1/2} & 0 \\ 0 & k_{\alpha\beta}^{1/2} \end{pmatrix} \cdot \mathcal{Q}_\beta \cdot \begin{pmatrix} c_{\alpha\beta} & d_{\alpha\beta} \\ 0 & c_{\alpha\beta}^{-1} \end{pmatrix}$$

From now on we denote by $T =: \{T_{\alpha\beta}\}$ the collection of matrices

$$T_{\alpha\beta} = \begin{pmatrix} c_{\alpha\beta} & 0 \\ d_{\alpha\beta} & c_{\alpha\beta}^{-1} \end{pmatrix}$$

For T the cocycle condition $T_{\alpha\beta}T_{\beta\gamma} = T_{\alpha\gamma}$ holds. Indeed, written in terms of d, this means:

$$d_{\alpha\gamma} = c_{\beta\gamma}\, d_{\alpha\beta} + c_{\alpha\beta}^{-1}\, d_{\beta\gamma}$$

which can be checked straightforwardly in the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$.

It remains for us to examine what happens when we change the reference point collection $\{Q_\alpha\}$. Passing to the reference collection $\{Q'_\alpha\}$, $\{d_{\alpha\beta}\}$ transforms as

$$d_{\alpha\beta} = a_\alpha^{-1} a_\beta^{-1} d'_{\alpha\beta} + c'_{\alpha\beta} f_\alpha a_\beta^{-1} - c'_{\alpha\beta}{}^{-1} f_\beta a_\alpha^{-1}$$

and this relation can be recast in matrix form

$$T_{\alpha\beta} = \begin{pmatrix} a_\alpha & 0 \\ f_\alpha & a_\alpha^{-1} \end{pmatrix} \cdot T'_{\alpha\beta} \cdot \begin{pmatrix} a_\beta & 0 \\ f_\beta & a_\beta^{-1} \end{pmatrix}^{-1}$$

thus showing that the cocycle $\{T_{\alpha\beta}\}$ is replaced by a cohomologous one.

Thus the matrices $\{T_{\alpha\beta}\}$ do form an $SL(2,\mathbb{C})$-valued cocycle. We denote by T the corresponding cohomology class in $H^1(X', SL(2,\mathbb{C}))$. T is a rank-2 flat

$SL(2,\mathbb{C})$ *vector bundle on* X', *or, as we already said, a rank-2 local system* [14, 12].

The solution of the DS system we have just produced is to be properly interpreted as a section of $K^{-\frac{1}{2}} \otimes T$ on X'.

For the matrices $\{T^{\vee}_{\alpha\beta}\}$ introduced above we have

$$T^{\vee}_{\alpha\beta} = \begin{pmatrix} c^{-1}_{\alpha\beta} & -d_{\alpha\beta} \\ 0 & c_{\alpha\beta} \end{pmatrix} = {}^t T^{-1}_{\alpha\beta}$$

The notation has been chosen in such a way that $T^{\vee}$ is indeed the dual of T.

The next step is to describe in more detail the structure of the flat bundle T so constructed. Due to the fact that the cocycle T is associated to the multi–valuedness of the local determinations of the solution, it is natural to refer to T as the *monodromy* of the DS system.

3.3 A description of the monodromy of the DS system

In this subsection we want to express our monodromy T in a way as explicit as possible in terms of the variables p_k. We start with a few preliminary remarks.

The flat bundle T, i.e. the monodromy of the DS system, is the same thing as a representation of the fundamental group $\pi_1(X')$ into $SL(2,\mathbb{C})$ (up to conjugation). This follows from the general fact that for any connected manifold M and any (Lie) group G we have

$$H^1(M, G) \cong Hom(\pi_1(M), G)/G$$

where the quotient is taken with respect to the action of G on itself by conjugation [21, 14]. Given an element F of $H^1(M, G)$, which is a flat bundle on M we will denote by **F** the corresponding element in the other space and call it "the characteristic representation associated with F". We shall exploit the explicit form of the above isomorphism in order to produce representatives for the various cohomology classes directly in terms of suitable line integrals over closed paths.

Considered as a vector bundle our monodromy T has triangular transition functions, so from that point of view it is an extension

$$0 \longrightarrow C^{-1} \longrightarrow T \longrightarrow C \longrightarrow 0$$

where the flat line-bundle C appears as a quotient. In view of the isomorphism just described it is a representation taking place in the (lower) Borel subgroup of $SL(2,\mathbb{C})$. This fact allows us to separately analyze the components of the representation.

As for the flat line bundle C, in view of the isomorphism mentioned above, it is an element of

$$Hom(H_1(X'), \mathbb{C}^*)$$

This follows from the fact that being $\mathbb{C}^*$ commutative, the homomorphisms of the fundamental group to $\mathbb{C}^*$ factor through $\pi_1(X')/[\pi_1(X'), \pi_1(X')] \cong H_1(X')$.

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We recall that the projection of the fundamental group of the non compact surface X' onto $H_1(X')$ kills the commutator subgroup, so that the latter is freely generated by the symbols $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, c_0, where c_0 is a small circle surrounding P_+[3].

Thus the line bundle C is the same thing as a character of the first homology group and it is determined by its value on the generators $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, c_0 of the group. Below we will see that the parametrization provided by the KN basis gives a formula for the character $\mathbf{c}$ so determined. But before doing this, we treat, at the same level of generality, the off-diagonal term in the monodromy.

We recall that T is the extension of C by C^{-1}. This means that the extension class represented by T is an element of the cohomology group

$$H^1(X', \mathcal{C}^{-2})$$

which means the first cohomology group of X' with values in the sheaf $\mathcal{C}^{-2}$ of *locally constant* sections of the flat bundle $C^{-2} = C^{-1} \otimes C^\vee$ [12].

This element in $H^1(X', \mathcal{C}^{-2})$ is determined in the following way. Recall the identity :

$$d_{\alpha\gamma} = c_{\beta\gamma} \, d_{\alpha\beta} + c_{\alpha\beta}^{-1} \, d_{\beta\gamma}$$

satisfied by the quantities $\{d_{\alpha\beta}\}$ introduced in (24). Following [12], we rewrite it by introducing the 1-cochain $\{s_{\alpha\beta}\} = \{c_{\alpha\beta} \, d_{\alpha\beta}\}$, thereby obtaining the identity

$$s_{\alpha\gamma} = c_{\beta\gamma}^2 \, s_{\alpha\beta} + s_{\beta\gamma}$$

This identity is in fact the cocycle condition for the 1-cochain $s = \{s_{\alpha\beta}\}$ with values in the locally constant sections of the flat line bundle C^{-2}. However we should verify that our procedure does define an element of a cohomology group. In other words, a new cocycle $\{s'_{\alpha\beta}\}$ which differs from the previous one by a coboundary, should be associated essentially to the same data for our differential equation. The cochain relation $s' - s = \delta(f)$ has the explicit form

$$s'_{\alpha\beta} = s_{\alpha\beta} + f_\beta - c_{\alpha\beta}^2 f_\alpha$$

Now suppose such a collection $\{f_\alpha\}$ is given. Dividing by $c_{\alpha\beta}$ and taking into account the explicit form for $d_{\alpha\beta}$ previously found, we see that shifting by a coboundary precisely amounts to a change in the integration constants in the indefinite integrals defining $\{\sigma_{2\alpha}\}$. In other words, this is the same as changing the initial points in the integral defining $\{\sigma_{2\alpha}\}$ in (22).

Thus the off-diagonal element of the monodromy also has a cohomological interpretation.

Let us now give an explicit representation for the classes in $H^1(X', \mathbb{C}^*)$ and $H^1(X', \mathcal{C}^{-2})$ in terms of the KN parametrization. More precisely, for any element

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[3]We make the slight abuse of language of denoting with the same letter both an element of the fundamental group and its image in the first homology group.

γ of the fundamental group, we provide representatives $\mathbf{c}_\gamma$ and $\mathbf{s}_\gamma$ in terms of line integrals. Let

$$p = \Gamma_0 + p_n\,\omega^n$$

where Γ_0 is given by (9). We recall that the sum over repeated indices is understood. Now we plug this expansion in the local expression (22) for σ_1. We keep the usual definitions and notations for the covering $\mathfrak{U}$ and the reference points $\{Q_\alpha\}$. With an easy integration we find

$$\sigma_{1\alpha}(Q) = e^{\mathcal{L}_\alpha(Q)} = h_\alpha(Q)^{-1}h_\alpha(Q_\alpha)\exp\left(p_n\int_{Q_\alpha}^{Q}\omega^n\right)$$

h_α is the determination of h in the chart U_α. Consequently we get

$$c_{\alpha\beta} = h_\alpha(Q_\alpha)h_\beta(Q_\beta)^{-1}\exp\left(p_n\int_{Q_\alpha}^{Q_\beta}\omega^n\right)$$

Now we exploit the description of the fundamental group of X' by means of chains of open sets (see Appendix A). We fix a base point $Q_0 \in X'$ and an open set U_0 containing it. We consider a path γ and a covering chain $(U_{\alpha_0}, U_{\alpha_1}, \cdots, U_{\alpha_n}, U_{\alpha_0})$, $U_{\alpha_0} = U_0$. The character $\mathbf{c}$ associated with the cocycle $\{C_{\alpha\beta}\}$ is given by the formula (see Appendix A)

$$\mathbf{c}_\gamma = \prod_{i=0}^{n} c_{\alpha_i\alpha_{i+1}}$$

which, together with explicit form for the 1-cocycle quoted above yields

$$\mathbf{c}_\gamma = \exp\left(p_n\oint_\gamma\omega^n\right) \tag{25}$$

Being a character of the first homology group of X', $\mathbf{c}$ is defined by its values on the generators:

$$\begin{aligned}
\mathbf{c}_{a_i} &= \exp(2\pi i\,N_i^n\,p_n) \\
\mathbf{c}_{b_i} &= \exp(2\pi i\,M_i^n\,p_n)
\end{aligned}$$

where the numbers M_i^n and N_i^n where defined in subsection 3.1 and are in principle explicitly computable, using the concrete expression for the KN basis [18, 20] in terms of θ-functions and prime form of X, the compact completion of X'.

The value of $\mathbf{c}$ on the cycle $C_\tau \equiv c_0$ around the puncture P_+ has a very simple form:

$$\mathbf{c}_{c_0} \equiv \mathbf{c}_0 = \exp\left(p_n\oint_{c_0}\omega^n\right) = e^{2\pi i p_{g/2}}$$

Next we have to characterize the off-diagonal element of the monodromy in the same terms as we did for $\mathbf{c}$. This is a rather long calculation and it is postponed to the Appendix A. Here we record the result. The representative $\mathbf{s}_\gamma$

corresponding to the closed path γ is an algebraic cocycle for the group $\pi_1(X')$ with values in $\mathbb{C}$, that is a map $\mathbf{s} : \pi_1(X') \to \mathbb{C}$ satisfying the cocycle condition

$$\mathbf{s}_{\gamma_1 \cdot \gamma_2} = \mathbf{c}_{\gamma_2}^2 \mathbf{s}_{\gamma_1} + \mathbf{s}_{\gamma_2}$$

or, in technical terms, what is called a *crossed homomorphism* of $\pi_1(X')$ into $\mathbb{C}$ with respect to the character $\mathbf{c}^{-2}$. The set of all crossed homomorphisms, modulo the trivial ones, forms a group denoted $H^1(\pi_1(X'), \mathbf{c}^{-2})$ which is actually isomorphic to $H^1(X', \mathcal{C}^{-2})$ (see Appendix A and [12]). The actual element determined by the DS system is given by

$$\mathbf{s}_\gamma = -h_0(Q_0)^{-2} e^{2p_k \oint_\gamma \omega^k} \oint_\gamma e^{-2p_k \int_{Q_0}^z \omega^k} h(z)^2 \tag{26}$$

where γ is a closed path on X' based at Q_0 covered by an appropriate chain of open sets. The $\mathbf{s}_\gamma$ when $\gamma = c_0$ will be denoted for simplicity $\mathbf{s}_0$.

A few comments are in order. The integral on the RHS of (26), as it stands, should be properly defined on the universal cover of X'. This is due to the fact that the integrand is multivalued on the surface. Thus formula (26) can be read in two ways. Interpreting the integral on the RHS as an integral over the universal covering space, (26) becomes an instance of the fact that elements of the group $H^1(\pi_1(M), \mathbf{F})$ can be represented by means of differential 1-forms on the universal covering space of M "twisted" by the character $\mathbf{F}$ [12]. On the other hand, (26) says that the LHS can be used to *define* the integral on the RHS, thereby giving a full meaning to a way of naively continuing the integrals in (22), outside their domain of definition, along a complete path on the surface.

An immediate formal manipulation of the integral formula (26) so obtained yields very easily the cocycle condition for $\mathbf{s}_\gamma$. The latter will rigorously follow from the formulas in Appendix A.

Finally we can do the same for the anti–holomorphic DS system (4). In particular we find another representation of $\pi_1(X')$, with representatives of the generators which can be written in the following form:

$$\bar{\mathbf{c}}_\gamma = \exp\left(-\bar{p}_n \oint_\gamma \bar{\omega}^n\right) \tag{27}$$

and

$$\bar{\mathbf{s}}_\gamma = \bar{h}_0(Q_0)^2 e^{-2\bar{p}_k \oint_\gamma \bar{\omega}^k} \oint_\gamma e^{2\bar{p}_k \int_{Q_0}^{\bar{z}} \bar{\omega}^k} h(\bar{z})^{-2} \tag{28}$$

4 Single–valued solutions

In the previous section we clarified the geometrical meaning and found explicit expression for the cocycles $\mathbf{c}_\gamma$ and $\mathbf{s}_\gamma$ for any cycle γ. In this section we will select in the space $\mathfrak{F}$ the solutions of the Liouville equation which are single–valued on X'. As we will see this corresponds to putting constraints on $\mathfrak{F}$. We will proceed in two steps. First we find the conditions for single–valuedness around P_+, then around the homotopy generators of X.

4.1 Single–valuedness around P_+

We use the reconstruction formula (7) and determine the conditions for the solution to be single–valued around P_+. The problem is the same as in genus 0 and we will simply summarize the procedure. We start from the solutions

$$\sigma(Q) = (\sigma_1(Q), \sigma_2(Q)), \qquad \bar{\sigma}(Q) = \left(\begin{array}{c} \bar{\sigma}_1(Q) \\ \bar{\sigma}_2(Q) \end{array} \right) \tag{29}$$

of the DS systems (3) and (4), respectively.

$$\sigma_1(Q) = e^{\int_{Q_0}^{Q} \mathfrak{p}(z)dz}, \qquad \sigma_2(Q) = -\sigma_1(Q) \int_{Q_0}^{Q} \sigma_1(z)^{-2}dz \tag{30}$$

$$\bar{\sigma}_1(Q) = e^{-\int_{Q_0}^{Q} \bar{\mathfrak{p}}(\bar{z})d\bar{z}}, \qquad \bar{\sigma}_2(Q) = \bar{\sigma}_1(Q) \int_{Q_0}^{Q} \bar{\sigma}_1(\bar{z})^{-2}d\bar{z} \tag{31}$$

where the only difference with respect to (5) and (6) is that we have fixed the initial integration point Q_0 once for all: we fix a curve C_τ and we understand that Q_0 is a fixed point in it. This allows us to order the points in C_τ in an anticlockwise order with respect to P_+ starting from Q_0. Such a choice is very convenient and does not hinder the generality of our results since we showed in the previous section that nothing really depends on Q_0.[4]

The monodromy of these solutions is easily found to be

$$\sigma(Q + C_\tau) = \sigma(Q)\,{}^tT_0, \qquad T_0 = \left(\begin{array}{cc} \mathbf{c}_0 & 0 \\ \mathbf{c}_0^{-1}\mathbf{s}_0 & \mathbf{c}_0^{-1} \end{array} \right) \tag{32}$$

and

$$\bar{\sigma}(Q + C_\tau) = \bar{T}_0\bar{\sigma}(Q), \qquad \bar{T}_0 = \left(\begin{array}{cc} \bar{\mathbf{c}}_0 & 0 \\ \bar{\mathbf{c}}_0^{-1}\bar{\mathbf{s}}_0 & \bar{\mathbf{c}}_0^{-1} \end{array} \right) \tag{33}$$

where $\mathbf{c}_0, \mathbf{s}_0, \bar{\mathbf{c}}_0, \bar{\mathbf{s}}_0$ were defined in the previous section. The notation T_0 for the monodromy matrix agrees with the notation of subsection 3.2 of the matrix $T_{\alpha\beta}$. In T_0 the entries are representatives of the cocycles c and s along the cycle C_τ.

Now equation (7) gives a single–valued solution around P_+ if M is given by

$$M = g_0\rho\bar{\rho}\bar{g}_0^{-1} \tag{34}$$

where g_0 diagonalizes ${}^tT_0$

$${}^tT_0 = g_0 D_0 g_0^{-1}, \qquad D_0 = \left(\begin{array}{cc} \mathbf{c}_0 & 0 \\ 0 & \mathbf{c}_0^{-1} \end{array} \right), \qquad g_0 = \left(\begin{array}{cc} 1 & \frac{\mathbf{c}_0^{-1}\mathbf{s}_0}{\mathbf{c}_0^{-1}-\mathbf{c}_0} \\ 0 & 1 \end{array} \right) \tag{35}$$

and, similarly

$$\bar{T}_0 = \bar{g}_0\bar{D}_0\bar{g}_0^{-1}, \qquad \bar{D}_0 = \left(\begin{array}{cc} \bar{\mathbf{c}}_0 & 0 \\ 0 & \bar{\mathbf{c}}_0^{-1} \end{array} \right), \qquad \bar{g}_0 = \left(\begin{array}{cc} 1 & 0 \\ \frac{\bar{\mathbf{c}}_0^{-1}\bar{\mathbf{s}}_0}{\bar{\mathbf{c}}_0-\bar{\mathbf{c}}_0^{-1}} & 1 \end{array} \right) \tag{36}$$

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[4]Changing the base–point is an isomorphism of the fundamental group.

The matrices

$$\rho = e^{qH}, \qquad \bar\rho = e^{-\bar qH} \tag{37}$$

where q and $\bar q$ are constants, do not play any role here, and they are only introduced for later purposes.

The result of this construction is that if we introduce

$$\psi(Q) = \sigma(Q)g_0\rho, \qquad \bar\psi(Q) = \bar\rho\bar g_0^{-1}\bar\sigma(Q) \tag{38}$$

on this new basis, *the Bloch wave basis*, the monodromy is diagonal

$$\psi(Q + C_\tau) = \psi(Q)D_0, \qquad \bar\psi(Q + C_\tau) = \bar D_0\bar\psi(Q)$$

Thanks to this property the solution of the Liouville equation given by

$$e^{-\varphi} = \psi\bar\psi \tag{39}$$

is single valued around P_+ if $D_0\bar D_0$ is the identity matrix, i.e. if

$$\mathbf{c}_0^{-1} = \bar{\mathbf{c}}_0, \qquad \text{or} \qquad p_{\frac{g}{2}} = \bar p_{\frac{g}{2}} \tag{40}$$

This is the constraint on $\mathfrak{F}$ that guarantees single–valuedness around P_+.

4.2   Single–valuedness around the remaining loops

The phase space $\mathfrak{F}$ has to be further restricted if we want the solution (39) to be single–valued on the whole Riemann surface. A simple way to find the constraints is as follows. Let γ be any homotopically non–trivial loop for X, then

$$\psi(Q + \gamma) = \psi(Q)\rho^{-1}g_0^{-1}\,{}^tT_\gamma\,g_0\rho$$
$$\bar\psi(Q + \gamma) = \bar\rho\bar g_0^{-1}\bar T_\gamma\,\bar g_0\bar\rho^{-1}\bar\psi(Q)$$

Therefore univalence is guaranteed if

$$^tT_\gamma M\bar T_\gamma = M$$

where M is the same as in eq.(34). A simple calculation shows that this implies

$$\mathbf{c}_\gamma^{-1} = \bar{\mathbf{c}}_\gamma \tag{41}$$

$$\mathcal{F}_\gamma \equiv \mathbf{s}_\gamma - \frac{\mathbf{c}_0^{-1}\mathbf{s}_0}{\mathbf{c}_0^{-1} - \mathbf{c}_0}(1 - \mathbf{c}_\gamma^2) = 0 \tag{42}$$

$$\bar{\mathcal{F}}_\gamma \equiv \bar{\mathbf{s}}_\gamma - \frac{\bar{\mathbf{c}}_0^{-1}\bar{\mathbf{s}}_0}{\bar{\mathbf{c}}_0^{-1} - \bar{\mathbf{c}}_0}(1 - \bar{\mathbf{c}}_\gamma^2) = 0 \tag{43}$$

The conditions (42) and (43) tell us, from a cohomological point of view, that the cocycles $\mathbf{s}_\gamma$ and $\bar{\mathbf{s}}_\gamma$, respectively, are coboundaries (see subsection 3.3).

It is clear that, in order to guarantee univalence of the solutions (39), we have to impose $2g$ such sets of constraints, one for each generator in $\pi_1(X)$.

The constraints (41) can be written

$$M_i^n \, p_n = \bar{M}_i^n \bar{p}_n, \qquad N_i^n \, p_n = \bar{N}_i^n \bar{p}_n, \qquad i = 1, ..., g \qquad (44)$$

As for the remaining constraints, let us denote them, for practical reasons, $\mathcal{F}_r = 0$ and $\bar{\mathcal{F}}_r = 0$, $r = 1, ..., 2g$.

In conclusion, *the phase space of the single–valued solutions is $\mathfrak{F}$ restricted by the $2g + 1$ conditions (44) and by the $4g$ additional constraints $\mathcal{F}_r = 0$ and $\bar{\mathcal{F}}_r = 0$, $r = 1, ..., 2g$.*

To conclude this section, we remark that so far we have talked only about sufficient conditions for single–valuedness. However it is easy to convince oneself that the constraints given above are also necessary. Anyhow this will be clear from Appendix B, where a less simple–minded derivation of the results of this section is given. In Appendix C we present a family of solutions of the Liouville equation disconnected from the ones discussed so far. They will not be included in our phase space and are unexplored from the quantization point of view.

5   Exchange algebra and locality

In this section we define a symplectic structure on the phase space $\mathfrak{F}$, calculate the exchange algebra for the Bloch wave basis and discuss locality for the solutions of the Liouville equation in X'. Throughout the section we fix a curve C_τ and a reference point Q_0 on it.

5.1   The symplectic structure

A symplectic structure in $\mathfrak{F}$ can be defined by means of the Poisson bracket

$$\{p_n, p_m\} = -\gamma_{nm} \qquad (45)$$

If we remember that

$$p_n = \tfrac{1}{2\pi i} \oint_{C_\tau} p(z) A_n(z) dz, \qquad \mathfrak{p}(Q) = p(Q) + \Gamma_0(Q)$$

we find immediately

$$\{\mathfrak{p}(Q), \mathfrak{p}(Q')\} = -\partial \Delta(Q, Q')$$

where $\Delta(Q, Q')$ is the δ–function appropriate for 0– and 1–forms along C_τ.

The symplectic structure thus defined is degenerate, for we have

$$\{p_{\frac{g}{2}}, p_n\} = 0, \qquad \{N_i^m p_m, p_n\} = 0, \qquad \{M_i^m p_m, p_n\} = 0, \quad \forall n, i \qquad (46)$$

as a consequence of eq.(18). We can eliminate the degeneracy by enlarging the phase space with the addition of the a new variable q such that

$$\{q, p_n\} = -A_n(Q_0) \qquad (47)$$

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We have in particular

$$\{q, p_{\frac{g}{2}}\} = -1, \qquad \{q, N_i^m p_m\} = -\mathcal{A}_i, \qquad \{q, M_i^m p_m\} = -\mathcal{B}_i \qquad (48)$$

Due to (19) and (20) and the way we normalized the third kind differential, we see that the degeneracy is eliminated. The q introduced here is the one appearing in eq.(37).

We have to define the symplectic structure also for the anti–holomorphic degrees of freedom. So we set

$$\{\bar{p}_n, \bar{p}_m\} = -\bar{\gamma}_{nm}$$

Moreover we introduce the multi–conjugate variable

$$\{\bar{q}, \bar{p}_n\} = -\bar{A}_n(Q_0)$$

All the remaining Poisson brackets vanish

$$\{p_n, \bar{p}_m\} = 0, \qquad \{q, \bar{p}_m\} = 0, \qquad \{\bar{q}, p_n\} = 0, \qquad \forall n, m$$

In Appendix D we investigate compatibility between the symplectic structure introduced here and the tensorial properties of the various bases introduced in section 4. In fact one expects the tensor transformation properties to be generated by the energy–momentum tensor through the above Poisson brackets. *Only the ψ basis fulfills such a compatibility requirement.*

5.2   The classical exchange algebra

The exchange algebra consists of the Poisson brackets of the components of the σ basis or the ψ basis among themselves, evaluated at two different points Q and Q' of C_τ. In the following it is essential to keep in mind what we said in subsection 4.1 about the ordering of the points on C_τ with respect to Q_0. We will write $Q > Q'$ or $Q < Q'$ according to whether Q comes after or before Q', if we run C_τ starting from Q_0 in an anticlockwise way as seen from P_+.

The calculation of the exchange algebra is not very different from the genus 0 case (see, for example [7]). Therefore we will avoid a detailed exposition, but we cannot avoid a few specifications. For example a simple calculation gives

$$\{\sigma_1(Q), \sigma_1(Q')\} = \gamma_{nm} \int_{Q_0}^{Q} \omega^n(z) dz \int_{Q_0}^{Q'} \omega^m(w) dw$$

Using the properties of the KN bases one finds that the RHS of this equation can be written as

$$2\pi i \Big(\epsilon_{Q_0}(Q, Q') - \epsilon_{Q_0}(Q, Q_0) + \epsilon_{Q_0}(Q', Q_0) \Big)$$

where we have introduced the symbol

$$\epsilon_{Q_0}(Q, Q') = \theta_{Q_0}(Q, Q') - \theta_{Q_0}(Q', Q)$$

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and the θ function is defined as follows for any 1–form ϕ

$$\oint_{C_\tau} dQ'\phi(Q')\theta_{Q_0}(Q,Q') = \int_{Q_0}^{Q} dQ'\phi(Q')$$

To simplify our life we choose henceforth both Q and $Q' > Q_0$, so that

$$\gamma_{nm} \int_{Q_0}^{Q} \omega^n(z)dz \int_{Q_0}^{Q'} \omega^m(w)dw = 2\pi i\, \epsilon_{Q_0}(Q,Q') \tag{49}$$

From now on the calculation is straightforward and one finds – henceforth we adopt a simplified notation to avoid the awkward factor $\frac{1}{2\pi i}$ in the formulas: therefore, unless otherwise explicitly stated, $\{\ ,\ \}$ will stand for $\frac{1}{2\pi i}\{\ ,\ \}$ –

$$\begin{aligned}
\{\sigma_1(Q),\sigma_1(Q')\} &= \epsilon_{Q_0}(Q,Q')\sigma_1(Q)\sigma_1(Q') \\
\{\sigma_2(Q),\sigma_2(Q')\} &= \epsilon_{Q_0}(Q,Q')\sigma_2(Q)\sigma_2(Q') \\
\{\sigma_1(Q),\sigma_2(Q')\} &= -\epsilon_{Q_0}(Q,Q')\sigma_1(Q)\sigma_2(Q') - 4\theta_{Q_0}(Q',Q)\sigma_2(Q)\sigma_1(Q')
\end{aligned} \tag{50}$$

We eventually need the exchange algebra in the ψ basis. We have

$$\psi_1 = \sigma_1 e^q, \qquad \psi_2 = \sigma_2 e^{-q} + \sigma_1 \frac{\mathbf{c}_0^{-1}\mathbf{s}_0}{\mathbf{c}_0^{-1} - \mathbf{c}_0} e^{-q}$$

and the calculation is straightforward on the basis of the previous remarks and the rules of the previous subsection. One obtains

$$\begin{aligned}
\{\psi_1(Q),\psi_1(Q')\} &= \epsilon_{Q_0}(Q,Q')\psi_1(Q)\psi_1(Q') \\
\{\psi_2(Q),\psi_2(Q')\} &= \epsilon_{Q_0}(Q,Q')\psi_2(Q)\psi_2(Q') \\
\{\psi_1(Q),\psi_2(Q')\} &= -\epsilon_{Q_0}(Q,Q')\psi_1(Q)\psi_2(Q') - \\
&\quad - 4\Big(\frac{\mathbf{c}_0}{\mathbf{c}_0^{-1} - \mathbf{c}_0} + \theta_{Q_0}(Q',Q)\Big)\psi_2(Q)\psi_1(Q')
\end{aligned} \tag{51}$$

Similarly for the antichiral half one obtains [5]

$$\begin{aligned}
\{\bar\psi_1(Q),\bar\psi_1(Q')\} &= -\epsilon_{Q_0}(Q,Q')\bar\psi_1(Q)\bar\psi_1(Q') \\
\{\bar\psi_2(Q),\bar\psi_2(Q')\} &= -\epsilon_{Q_0}(Q,Q')\bar\psi_2(Q)\bar\psi_2(Q') \\
\{\bar\psi_1(Q),\bar\psi_2(Q')\} &= \epsilon_{Q_0}(Q,Q')\bar\psi_1(Q)\bar\psi_2(Q') + \\
&\quad + 4\Big(\frac{\bar{\mathbf{c}}_0}{\bar{\mathbf{c}}_0^{-1} - \bar{\mathbf{c}}_0} + \bar\theta_{Q_0}(Q',Q)\Big)\bar\psi_2(Q)\bar\psi_1(Q')
\end{aligned} \tag{52}$$

---

[5]One may wonder why in the antichiral exchange algebra we do not use the $\bar\epsilon$ and the $\bar\theta$ symbol, i.e. the ϵ and θ distributions expressed in terms of the antichiral basis. The reason is that, when limited to C_τ, ϵ (θ) and $\bar\epsilon$ ($\bar\theta$) are different representations of the same objects.

5.3 Locality

By local solutions we mean those for which

$$\{e^{-\varphi(Q)}, e^{-\varphi(Q')}\} = 0, \qquad Q, Q' \in C_\tau$$

With the above exchange algebras in the Bloch wave basis we can easily compute

$$\{\psi(Q)\bar{\psi}(Q), \psi(Q')\bar{\psi}(Q')\} =$$
$$= 4\frac{\mathbf{c}_0\bar{\mathbf{c}}_0 - \mathbf{c}_0^{-1}\bar{\mathbf{c}}_0^{-1}}{(\mathbf{c}_0^{-1} - \mathbf{c}_0)(\bar{\mathbf{c}}_0^{-1} - \bar{\mathbf{c}}_0)}\Big(\psi_1(Q')\bar{\psi}_1(Q)\psi_2(Q)\bar{\psi}_2(Q') - \psi_1(Q)\bar{\psi}_1(Q')\psi_2(Q')\bar{\psi}_2(Q)\Big)$$

Therefore locality is guaranteed if

$$\mathbf{c}_0^{-1} = \bar{\mathbf{c}}_0$$

i.e. the same as (40).

In conclusion, *the phase space $\mathfrak{F}_0$ of the single–valued and local solutions is $\mathfrak{F}$ restricted by the $2g + 1$ conditions (44) and by the $4g$ additional constraints $\mathcal{F}_r = 0$ and $\bar{\mathcal{F}}_r = 0$, $r = 1, ..., 2g$.*

5.4 Other remarkable Poisson brackets

One may be interested in the exchange algebra in the covering space of the Riemann surface, in particular in what happens when we Poisson commute our solutions after going around P_+ a certain number of times. The answer is particularly simple in the Bloch wave basis. Let us set

$$\psi^{(n)}(Q) = \psi(Q + nC_\tau)$$

Then it is easy to calculate the exchange algebra

$$\{\psi_1^{(n)}(Q), \psi_1^{(m)}(Q')\} = \Big(\epsilon_{Q_0}(Q, Q') + (n - m)\Big)\psi_1^{(n)}(Q)\psi_1^{(m)}(Q')$$
$$\{\psi_2^{(n)}(Q), \psi_2^{(m)}(Q')\} = \Big(\epsilon_{Q_0}(Q, Q') + (n - m)\Big)\psi_2^{(n)}(Q)\psi_2^{(m)}(Q') \qquad (53)$$
$$\{\psi_1^{(n)}(Q), \psi_2^{(m)}(Q')\} = -\Big(\epsilon_{Q_0}(Q, Q') + (n - m)\Big)\psi_1^{(n)}(Q)\psi_2^{(m)}(Q') -$$
$$- 4\Big(\frac{\mathbf{c}_0}{\mathbf{c}_0^{-1} - \mathbf{c}_0} + \theta_{Q_0}(Q', Q)\Big)\psi_2^{(m)}(Q)\psi_1^{(n)}(Q')$$

Another interesting question is the calculation of the Poisson brackets of the constraints $\mathcal{F}_r$ found above. They are essential in order to compute the Dirac brackets (see below). To this end we have to compute the Poisson brackets of the cocycles $\mathbf{s}_\gamma$ with one another. From the previous experience we know that we have to be able to define an ordering of the points on the curves over which the cocycles in question are defined. We proceed as follows. We consider the curve C_τ passing through a fixed point Q_0 and, keeping Q_0 fixed, we continuously deform C_τ so that eventually C_τ overlap the curves in question (except possibly

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for a set of points of measure zero). One can convince oneself that this is always possible, and can be done in a definite order, for example $a_1, ..., a_g, b_g, ..., b_1, c_0$. This establishes an ordering according to which we run one after the other the various curves, starting from Q_0 and following the sense determined by the initial curve C_τ. Therefore, given two different curves γ_1 and γ_2, we can evaluate

$$\{\mathbf{s}_{\gamma_1}, \mathbf{s}_{\gamma_2}\} = \mathbf{c}_{\gamma_1} \mathbf{c}_{\gamma_2} \{\oint_{\gamma_1} \sigma_1^{-2}(z) dz, \oint_{\gamma_2} \sigma_1^{-2}(w) dw\} = \tag{54}$$

$$= 4\mathbf{c}_{\gamma_1} \mathbf{c}_{\gamma_2} \oint_{\gamma_1} dz \oint_{\gamma_2} dw \sigma_1^{-2}(z) \sigma_1^{-2}(w) \epsilon_{Q_0}(z, w) = \pm 4 \mathbf{s}_{\gamma_1} \mathbf{s}_{\gamma_2}$$

the $+$ ($-$) sign depending on γ_1 coming after (before) γ_2 according to the above mentioned ordering.

Other useful Poisson brackets are

$$\{\rho, \mathbf{c}_\gamma\} = -\rho \mathbf{c}_\gamma \chi_\gamma H \tag{55}$$

$$\{\rho, \mathbf{s}_\gamma\} = 2(1 - \chi_\gamma) \rho \mathbf{s}_\gamma H \tag{56}$$

5.5   Dirac Brackets

In order to guarantee univalence of the solutions of the Liouville equation we had to impose, in section 4, the constraints (41), (42) and (43). The first set are first class constraints, for

$$\{e^{-\varphi(Q)}, \mathbf{c}_\gamma^{-1} - \bar{\mathbf{c}}_\gamma\} = e^{-\varphi(Q)}(\mathbf{c}_\gamma^{-1} \chi_\gamma - \bar{\mathbf{c}}_\gamma \bar{\chi}_\gamma) = e^{-\varphi(Q)}(\mathbf{c}_\gamma^{-1} - \bar{\mathbf{c}}_\gamma) \chi_\gamma \tag{57}$$

where χ_γ is either $\mathcal{A}_i$ if $\gamma = a_i$, or $\mathcal{B}_i$ if $\gamma = b_i$, or 1 if $\gamma = \mathbf{c}_0$, and similarly for the barred quantities. The last equality follows from eq.(21).

The $\mathcal{F}$ and $\bar{\mathcal{F}}$ constraints, on the contrary, are second class. It is not hard to compute the Poisson brackets

$$\{\mathcal{F}_r, \mathcal{F}_s\} = C_{rs},$$
$$\{\bar{\mathcal{F}}_r, \bar{\mathcal{F}}_s\} = \bar{C}_{rs},$$

while

$$\{\mathcal{F}_r, \bar{\mathcal{F}}_s\} = 0$$

for all $r, s = 1, ..., 2g$. The matrices C and $\bar{C}$ are nonsingular. Therefore, for any two functions F and G on the phase space, we can define the Dirac brackets

$$\{F, G\}_* = \{F, G\} - \sum_{a,b} \{F, \mathcal{F}_a\} \mathcal{C}_{ab}^{-1} \{\mathcal{F}_b, G\} \tag{58}$$

where we use the collective notation $\mathcal{F}_a = (\mathcal{F}_r, \bar{\mathcal{F}}_r)$, $a = 1, ...4g$ and $\mathcal{C}$ is the direct sum of C and $\bar{C}$.

We have to use these brackets in the restricted phase space $\mathfrak{F}_0$ whenever we require univalence. In particular we should be careful about the first class constraints (41). However it is easy to verify that the $\mathcal{F}_a$ constraints Poisson

commute with $\mathbf{c}_\gamma^{-1} - \bar{\mathbf{c}}_\gamma$, therefore eq.(57) continues to hold even if we replace the Poisson brackets with the Dirac ones. Let us also notice that passing to the Dirac brackets preserves the conformal properties of the Bloch wave basis, see Appendix D.

Finally, a remark concerning locality. Locality is defined with respect to a fixed contour C_τ. Therefore, the proof of locality we gave above is perfectly adequate: locality is referred to the initial Poisson brackets not to the Dirac brackets just introduced [6]. In other words the logical order should be the following: first secure locality with respect to the original Poisson brackets, then impose univalence.

6 Quantization

Quantizing the Liouville theory is the necessary step to the ultimate aim of calculating correlation functions. Here we do not arrive that far but limit ourselves to the preliminary step of exposing the operator structure of the quantum theory. This is not a difficult task, as this problem is quite analogous to the genus zero case and we can follow the procedure outlined in ref.[9]. There, quantization was considered on a lattice as lattice quantization is particularly adapted to reveal the operator structure. However the main result can be immediately translated into a continuum language.

A hint for quantization comes from the substitution

$$[\ , \ ] = i\hbar \{ \ , \ \}$$

which means in particular

$$[p_n, p_m] = -i\hbar \gamma_{nm}$$

for the bosonic oscillators.

One can construct a set of consistent rules that lead in particular to the following quantum exchange algebra in the σ–basis.

$$\sigma_1(Q)\sigma_2(Q') = \sigma_2(Q')\sigma_1(Q)R_{12}^\pm, \qquad \begin{array}{lll} + & \text{when} & Q > Q' \\ - & \text{when} & Q < Q' \end{array} \qquad (59)$$

In this section the labels 1 and 2 appended to σ do not represent the two components of σ as before, but

$$\sigma_1 = \sigma \otimes 1, \qquad \sigma_2 = 1 \otimes \sigma$$

---

[6]Imposing locality with respect to the Dirac brackets would completely distort the sense of locality as we have seen that the Dirac brackets understand an arbitrary deformation of the contour C_τ. Consequently this kind of locality would mean Poisson commutativity at two generic points, instead of commutativity with respect to two points on a fixed C_τ, i.e. *at fixed Euclidean time.*

and $R_{12}^+ = (R_{21}^-)^{-1}$ is the well–known sl_2 quantum R matrix in the defining representation

$$R_{12}^+ = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{60}$$

where $q = \exp(-i\hbar)$ and $x = q - q^{-1}$. An algebra like (59) is discussed in [22].

After diagonalizing the quantum monodromy matrix one can then define the Bloch wave basis in just the same way as we did in the classical case, and find the corresponding quantum exchange algebra

$$\psi_1(Q)\psi_2(Q') = \psi_2(Q')\psi_1(Q)\mathcal{R}_{12}^\pm(\mathbf{c}_0), \qquad \begin{array}{cc} + & \text{when} \quad Q > Q' \\ - & \text{when} \quad Q < Q' \end{array} \tag{61}$$

where $\mathcal{R}_{12}^+(\mathbf{c}_0)) = (\mathcal{R}_{21}^-(\mathbf{c}_0))^{-1}$ is the quantum R matrix appropriate for the ψ–basis

$$\mathcal{R}_{12}^+(\mathbf{c}_0) = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & -xb_0 & 0 \\ 0 & xa_0 & 1 - x^2 a_0 b_0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{62}$$

Here

$$a_0 = \frac{\mathbf{c}_0}{\mathbf{c}_0 - \mathbf{c}_0^{-1}}, \qquad b_0 = \frac{\mathbf{c}_0^{-1}}{\mathbf{c}_0 - \mathbf{c}_0^{-1}}$$

and $\mathbf{c}_0$ is the quantum version of the cocycle denoted with the same symbol in the previous sections.

6.1   Quantum locality and univalence

Quantum locality of

$$e^{-\varphi(Q)} = \psi(Q)\bar\psi(Q)$$

can be easily discussed along the lines of ref.[9]. It can be easily seen to depend on the condition

$$\mathbf{c}_0\bar{\mathbf{c}}_0 = 1 \tag{63}$$

If this condition can be imposed, locality is guaranteed. We will see that this is indeed so. But to understand this and the forthcoming points we have to know the quantum algebra of the $\mathbf{c}_\gamma$'s and $\mathbf{s}_\gamma$'s among themselves and with the remaining degrees of freedom, in particular we should define the quantum analogs of the Poisson brackets of subsection 5.4. Following the general formulas of [9], this is not difficult. For any $\gamma = a_1, ..., a_g, b_1, ..., b_g, c_0$, we have

$$\rho\mathbf{c}_\gamma = q^{\chi_\gamma H}\mathbf{c}_\gamma\rho \tag{64}$$

while $\mathbf{c}_\gamma$ commute with any other operator of the theory. We recall that χ_γ was defined in connection with eq.(57). As for $\mathbf{s}_\gamma$ we have

$$\rho \mathbf{s}_\gamma = q^{2(\chi_\gamma - 1)H} \mathbf{s}_\gamma \rho \tag{65}$$

$$\mathbf{s}_{\gamma_1} \mathbf{s}_{\gamma_2} = q^{\pm 4} \mathbf{s}_{\gamma_2} \mathbf{s}_{\gamma_1} \tag{66}$$

The $\pm$ sign in the last equation has the same meaning as in eq.(54). It is then easy to verify that

$$\mathbf{c}_\gamma \bar{\mathbf{c}}_\gamma e^{-\varphi(Q)} = e^{-\varphi(Q)} \mathbf{c}_\gamma \bar{\mathbf{c}}_\gamma \tag{67}$$

therefore we are allowed to impose the condition

$$\mathbf{c}_\gamma \, \bar{\mathbf{c}}_\gamma = 1 \tag{68}$$

which, in particular, guarantees locality when $\mathbf{c}_\gamma$ coincides with $\mathbf{c}_0$[7].

Next we study the condition of quantum univalence. We can impose for any γ the quantum condition

$${}^t T_\gamma M \bar{T}_\gamma = q^{-\chi_\gamma} M$$

where M is the quantum version of eq.(34). A simple calculation shows that this implies (68) together with

$$\mathbf{s}_\gamma = \frac{\mathbf{c}_0^{-1} \mathbf{s}_0}{\mathbf{c}_0^{-1} - \mathbf{c}_0} (1 - \mathbf{c}_\gamma^2) \tag{69}$$

$$\bar{\mathbf{s}}_\gamma = \frac{\bar{\mathbf{c}}_0^{-1} \bar{\mathbf{s}}_0}{\bar{\mathbf{c}}_0^{-1} - \bar{\mathbf{c}}_0} (1 - \bar{\mathbf{c}}_\gamma^2) \tag{70}$$

The conditions (42) and (43) are the quantum analogs for the cocycles $\mathbf{s}_\gamma$ and $\bar{\mathbf{s}}_\gamma$ to be coboundaries.

In conclusion, if we impose the conditions (68), (69) and (70), $\exp(-\varphi)$ is univalent around γ up to the numerical factor $q^{-\chi_\gamma}$.

In order to impose the above constraints on the states of the theory, an approach á la BRST should be convenient.

7   Higher rank Toda field theories

So far we have implemented a method for constructing solutions to the Liouville equation based on the DS construction, in much the same fashion as in [7]. There the construction was for Toda field theories (ToFT) based on an arbitrary simple Lie algebra $\mathfrak{g}$ in a representation independent way. Therefore one may wonder if the setting we are considering here extends to these more general cases. Although we have not yet worked out the problem in its full generality, we can show (through a simple example) that the $SL(n, \mathbb{C})$–ToFT show the same features described above for the Liouville case. In particular second-class constraints corresponding to the off-diagonal monodromies will show up.

---

[7]Actually in order to impose (68) one has to suitably normalize the quantum bases ψ and $\bar{\psi}$.

To begin with, let us write down the *rank n* chiral DS system in the form[8]

$$\partial \mathcal{Q} = (\mathfrak{P} - \mathcal{E}_+)\mathcal{Q} \tag{71}$$

where $\mathfrak{P}$ takes its values in the Cartan subalgebra $\mathfrak{h}$ of $sl(n,\mathbb{C})$ – i.e. it is a diagonal matrix with vanishing trace – and $\mathcal{E}_+$ is the constant matrix $\mathcal{E}_+ = \sum_{i=1}^{n-1} E_{i\,i+1}$, E_{ij} being the matrix with the (i,j)–th entry equal to 1 and zero elsewhere. For simplicity we do not explicitly write down here the corresponding formulas for the anti-chiral part.

Relying on [7] and our previous experience, we promote the collection of DS linear systems defined for each local chart to an analytic connection ∇^{DS} on the vector bundle

$$W = V^{\otimes^n_S} = K^{-\frac{n}{2}} \oplus \cdots \oplus K^{\frac{n}{2}}$$

where $\otimes^n_S$ denotes the n-th symmetric tensor power. Therefore all the remarks previously made concerning the poles of the connection conserve their validity here, since W is a direct sum of line bundles with non vanishing first Chern class. We remark also that, the set of all DS connections being obviously affine over the vector space $\mathfrak{h} \otimes \mathcal{M}^1_X$, we can still introduce the KN parametrization

$$\mathfrak{P} = \Gamma_0 + \sum_k p_k \omega^k$$

where now the $\{p_k\}$ are $\mathfrak{h}$–valued modes and $\{\omega^k\}$ is the usual KN basis of 1–forms. As in the Liouville case, from the first row of the fundamental solution $\mathcal{Q}$ we obtain the chiral multiplet $\{\sigma_\alpha(z_\alpha)\}$ behaving like a meromorphic section of $W \otimes T$ over $X' = X \setminus \{P_+, P_-\}$, and T is the appropriate monodromy to be calculated as in $SL(2,\mathbb{C})$ case.

Let us give some explicit formulas in the relatively simple case $G = SL(3,\mathbb{C})$. The DS connection has therefore the following form

$$\nabla^{DS} = \partial + \begin{pmatrix} -\mathfrak{p}_1 & 1 & 0 \\ 0 & -\mathfrak{p}_2 & 1 \\ 0 & 0 & -\mathfrak{p}_3 \end{pmatrix} dz$$

with $\mathfrak{p}_1 + \mathfrak{p}_2 + \mathfrak{p}_3 = 0$. Requiring ∇^{DS} to define a connection on $W = K^{-1} \oplus \mathbb{C} \oplus K$ implies that

- $-\mathfrak{p}_1$ is a (meromorphic) connection on $K^{-1} = T$;
- $-\mathfrak{p}_2$ is a (meromorphic) 1–form;
- $-\mathfrak{p}_3$ is a (meromorphic) connection on K.

At this point one could also easily write down explicit local formulas for $\sigma(z) = (\sigma_1(z), \sigma_2(z)\,\sigma_3(z))$ in a similar form to (5) and (6) (except that the level of nested integrals is augmented by one), but the resulting expressions are not

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[8]We shall stick to the case $G = SL(n,\mathbb{C})$ in the fundamental representation, so that this "n" is the same as the rank of the DS linear system.

very interesting here. Instead we shall be concerned with the resulting local monodromy matrices which we write in the form

$$T_{\alpha\beta} = \begin{pmatrix} c_{1\alpha\beta} & 0 & 0 \\ d_{1\alpha\beta} & c_{2\alpha\beta} & 0 \\ d_{3\alpha\beta} & d_{2\alpha\beta} & c_{3\alpha\beta} \end{pmatrix}, \qquad c_{1\alpha\beta}c_{2\alpha\beta}c_{3\alpha\beta} = 1 \qquad (72)$$

One can easily verify that the cocycle condition for $\{T_{\alpha\beta}\}$ indeed holds, so that it defines a flat $SL(3,\mathbb{C})$–bundle over X'. Accordingly, the extension it represents is given by the classes

$$[s_2] \in H^1(X', \mathcal{C}_3 \otimes \mathcal{C}_2^\vee), \qquad s_{2\alpha\beta} = c_{3\alpha\beta}^{-1} d_{2\alpha\beta}$$

$$[s_1] \in H^1(X', \mathcal{C}_2 \otimes \mathcal{C}_1^\vee), \qquad s_{1\alpha\beta} = c_{2\alpha\beta}^{-1} d_{1\alpha\beta}$$

whereas $\{d_{3\alpha\beta}\}$ is such that

$$[s_3, s_2] \in H^1(X', \mathcal{C}_3 \otimes \mathcal{F}_1^\vee), \qquad s_{3\alpha\beta} = c_{3\alpha\beta}^{-1} d_{3\alpha\beta}$$

if F_1 is the flat bundle whose representative cocycle is given by

$$F_{1\alpha\beta} = \begin{pmatrix} c_{1\alpha\beta} & 0 \\ d_{1\alpha\beta} & c_{2\alpha\beta} \end{pmatrix}$$

and $\mathcal{F}_1$ is the corresponding sheaf of locally constant sections, so that T is thought as the extension

$$0 \longrightarrow C_3 \longrightarrow T \longrightarrow F_1 \longrightarrow 0$$

On the other hand, if F_2 is the flat bundle defined by the cocycle

$$F_{2\alpha\beta} = \begin{pmatrix} c_{2\alpha\beta} & 0 \\ d_{2\alpha\beta} & c_{3\alpha\beta} \end{pmatrix}$$

then

$$\left[F_{2\alpha\beta}^{-1} \begin{pmatrix} d_{1\alpha\beta} \\ d_{3\alpha\beta} \end{pmatrix} \right] \in H^1(X', \mathcal{F}_2 \otimes \mathcal{C}_1^\vee)$$

and T appears as

$$0 \longrightarrow F_2 \longrightarrow T \longrightarrow C_1 \longrightarrow 0$$

Thus the flat structure coming out from the DS differential equation is an agreement with the general theory [12], as expected.

Now, using also the antichiral half of the theory, we require the bilinear form

$$\sigma_\alpha(z_\alpha) M_\alpha \bar\sigma_\alpha(\bar z_\alpha)$$

to be univalent (up to the tensor transformation properties, of course) upon changing local chart. Here $\{M_\alpha\}$ is a collection of constant $SL(3,\mathbb{C})$ matrices. Quite obviously there is also the representation space picture, so that the univalence condition translates into the familiar one

$$M \, \bar{\mathbf{T}}_\gamma = \mathbf{T}_\gamma^\vee M$$

for any $\gamma \in \pi_1(X')$. It is clear that identical formulas must hold in the more general $SL(n, \mathbb{C})$ case.

If $n = 3$ things are still sufficiently simple that one can work out a componentwise calculation. The result is the following set of necessary and (obviously) sufficient conditions

$$
\begin{aligned}
\bar{\mathbf{c}}_{i\gamma} &= \mathbf{c}_{i\gamma}{}^{-1}, \quad i = 1, 2, 3 \\
\mathbf{s}_{1\gamma} &= \frac{\Delta_{21}}{\Delta_{11}}(1 - \mathbf{c}_{1\gamma}\mathbf{c}_{2\gamma}^{-1}) \\
\bar{\mathbf{s}}_{1\gamma} &= \frac{\Delta_{12}}{\Delta_{11}}(1 - \bar{\mathbf{c}}_{1\gamma}\bar{\mathbf{c}}_{2\gamma}^{-1}) \\
\mathbf{s}_{2\gamma} &= \frac{m_{23}}{m_{33}}(1 - \mathbf{c}_{2\gamma}\mathbf{c}_{3\gamma}^{-1}) \\
\bar{\mathbf{s}}_{2\gamma} &= \frac{m_{32}}{m_{33}}(1 - \bar{\mathbf{c}}_{2\gamma}\bar{\mathbf{c}}_{3\gamma}^{-1}) \\
\mathbf{s}_{3\gamma} &= \frac{m_{13}}{m_{33}}(1 - \mathbf{c}_{1\gamma}\mathbf{c}_{3\gamma}^{-1}) - \frac{m_{23}}{m_{33}}\mathbf{c}_{2\gamma}\mathbf{c}_{3\gamma}^{-1}\mathbf{s}_{1\gamma} \\
\bar{\mathbf{s}}_{3\gamma} &= \frac{m_{31}}{m_{33}}(1 - \bar{\mathbf{c}}_{1\gamma}\bar{\mathbf{c}}_{3\gamma}^{-1}) - \frac{m_{32}}{m_{33}}\bar{\mathbf{c}}_{2\gamma}\bar{\mathbf{c}}_{3\gamma}^{-1}\bar{\mathbf{s}}_{1\gamma}
\end{aligned}
\tag{73}
$$

where m_{ij} are the elements of M and Δ_{ij} is the subdeterminant relative to m_{ij}. A little thought shows that these are precisely the conditions for T and $\bar{T}$ to be both equivalent diagonal bundles (i.e. direct sums).

Since from [7] it follows that the monodromies around P_+ are diagonalizable, this gives values to the ratios appearing in the conditions (73), which can in turn be taken as constraints with respect to the remaining generators of $\pi_1(X')$. As a further side–effect, the values of the ratios in (73) force M to be of the form

$$
M = N_+ \, D \, N_- \tag{74}
$$

where $N_\pm$ are completely determined upper and lower unipotent matrices. Thus the resulting indeterminacy is on the diagonal factor D.

We notice that this result relies ultimately on the assumption that M admits a Gauss factorization. Indeed the conditions (73) are meaningful only if $m_{33} \neq 0$ and $\Delta_{11} \neq 0$ and this is precisely the condition M must satisfy to admit a Gauss factorization in the form (74). As such, our result must hold in any rank n, for, if $M \in SL(n, \mathbb{C})$ exists and can be represented in the form (74), then the equation

$$
M \, \bar{\mathbf{T}}_\gamma = \mathbf{T}_\gamma^\vee \, M \,, \qquad \gamma \in \pi_1(X')
$$

can be recast into the form

$$
N_- \, \bar{\mathbf{T}}_\gamma \, N_-^{-1} = D^{-1} \, (N_+ \, \mathbf{T}_\gamma^\vee \, N_+^{-1}) \, D
$$

which clearly can hold only if both $N_- \, \bar{\mathbf{T}}_\gamma \, N_-^{-1}$ and $N_+ \, \mathbf{T}_\gamma^\vee \, N_+^{-1}$ are diagonal for any $\gamma \in \pi_1(X')$. Conversely, diagonalizing the monodromy around P_+ and imposing the constraints for $SL(n, \mathbb{C})$ clearly produces an intertwiner admitting

a Gauss factorization. Thus our result can be stated more precisely by saying that diagonalizability of the monodromy is a necessary and sufficient condition for a solution of the Toda Field Equations in terms of a Bloch wave basis to exist.

Finally we can carry out the quantization of this Toda theory in the same way and with the same limitations as for the Liouville theory in the previous section. However here we omit explicit formulas which can be easily inferred from those of ref.[23].

8 Liouville equation and uniformization

The emergence of the Liouville equation in the context of uniformization theory is such a celebrated result in mathematics that we can hardly avoid clarifying what is similar and what is different in the DS and the uniformization approaches. To such purpose we devote this last section.

Let us briefly review the uniformization theory of curves based on pseudogroups and differential equations [14, 25]. Although it does not yield such strong results as the one based on discontinuous actions of Fuchsian groups [26, 27], in recent times it has become popular among physicists because it is based on the idea of symmetry. Remember that we are sticking to the genus $g > 1$ case.

By uniformizing we mean finding a collection of functions $\{A_\alpha\}$, subordinate to the atlas $\{U_\alpha, z_\alpha\}$

$$A_\alpha \ : \ U_\alpha \longrightarrow \ V_\alpha \subset \mathbb{P}^1$$

which are required to be local homeomorphisms and such that:

$$A_\alpha(z_\alpha) = \frac{a_{\alpha\beta} A_\beta(z_\beta) + b_{\alpha\beta}}{c_{\alpha\beta} A_\beta(z_\beta) + d_{\alpha\beta}} \tag{75}$$

In this way, the new complex atlas $\{U_\alpha, A_\alpha \circ z_\alpha\}$ is such that all local charts are connected through projective transformations. The collection $\{A_\alpha\}$ can be thought of as a section of a flat $PSL(2, \mathbb{C})$ bundle on X. These remarks are used in [14] to explicitly construct the uniformizing atlas in terms of sections of adequate vector bundles. The result is as follows. A projective structure (subordinate to the complex one on X) is constructed taking a section $\{\xi_\alpha = (\xi_{1\alpha}, \xi_{2\alpha})\}$ of the bundle

$$T \otimes K^{-\frac{1}{2}}$$

where $K^{\frac{1}{2}}$ is a square root of the canonical bundle and T is a flat $SL(2, \mathbb{C})$ bundle, that is an element $T \in H^1(X, SL(2, \mathbb{C}))$. The coordinate sections $\{A_\alpha\}$ are

$$A_\alpha(z_\alpha) = \frac{\xi_{1\alpha}(z_\alpha)}{\xi_{2\alpha}(z_\alpha)} \tag{76}$$

and $\xi_{1\alpha}$, $\xi_{2\alpha}$ are two independent solutions of the differential equation

$$\frac{d^2}{dz_\alpha^2} \xi_\alpha + \frac{1}{2} u_\alpha(z_\alpha) \xi_\alpha(z_\alpha) = 0 \,, \qquad i = 1, 2 \tag{77}$$

with normalized Wronskian. The equation in (77) glues coherently on X as $\{u_\alpha(z_\alpha)\}$ is a projective connection [14, 28], that is a 1-coboundary for the 1-cocycle $\{f_{\alpha\beta}, z_\beta\}dz_\beta{}^2$, namely:

$$\{f_{\alpha\beta}, z_\beta\}dz_\beta{}^2 = u_\beta(z_\beta)dz_\beta{}^2 - u_\alpha(z_\alpha)dz_\alpha{}^2$$

where $\{f_{\alpha\beta}, z_\beta\}$ is the Schwarzian derivative and $z_\alpha = f_{\alpha\beta}(z_\beta)$. By Serre duality, $H^1(X, K^2) = 0$, so that projective connections certainly exist and are in one-to-one correspondence with projective structures [14].

Let us now define the collection of matrices

$$F_\alpha(z_\alpha) = \begin{pmatrix} \xi'_{1\alpha}(z_\alpha) & \xi_{1\alpha}(z_\alpha) \\ \xi'_{2\alpha}(z_\alpha) & \xi_{2\alpha}(z_\alpha) \end{pmatrix} \tag{78}$$

One can prove that there exists a flat vector bundle which supports a truly analytic connection

$$\nabla_\alpha = \partial_\alpha + \Lambda_\alpha, \qquad \Lambda_\alpha = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2}u_\alpha(z_\alpha) & 0 \end{pmatrix} \tag{79}$$

such that (77) can be rewritten as a linear system

$$\nabla_\alpha F_\alpha = 0$$

and T is realized as the holonomy of $\{\nabla_\alpha\}$.

Once a uniformization of X is found, one would write down a solution of the Liouville equation (1) by using the projective charts to pull back on X the standard Poincaré metric on the upper half plane $\mathbb{H}$, obtaining the very classical formula

$$e^{2\varphi_\alpha} = \frac{|\partial A_\alpha|^2}{(\mathrm{Im} A_\alpha(z_\alpha))^2} \tag{80}$$

One can verify that (80) defines a $(1,1)$–form by using the projective transformation (75). This requires the structure group to be reduced from $PSL(2, \mathbb{C})$ to $PSU(1,1)$ and the coordinate sections $\{A_\alpha\}$ to take their values in $\mathbb{H}$. We must notice that this is too optimistic, in general. What one *does* obtain, is the so–called "developing map" [29, 30]

$$f : \tilde{X} \longrightarrow \Omega \subset \mathbb{P}^1$$

where $\tilde{X}$ is the universal cover of X, which is equivariant with respect to the action of $\pi_1(X)$ on $\tilde{X}$ and of a certain group $\Gamma \subset PSL(2, \mathbb{C})$ on Ω. We shall not dwell on the definition of f any longer, except to mention that f is the global equivariant map corresponding to the section of the flat $PSL(2, \mathbb{C})$-bundle over X, therefore it is locally a projective chart [30]. The homomorphism $\rho : \pi_1(X) \to \Gamma$ with respect to which f is equivariant is the holonomy of the projective structure [30].

Since X has genus $g > 1$, f is a covering map, but Ω is not analytically equivalent to $\mathbb{H}$ in general [29]. Thus formula (80) is perhaps to be interpreted by

saying that one has to use local sections to f to transfer the standard hyperbolic metric to Ω (as in ref. [2] for the case of Schottky uniformization) and then locally pull–back this last one down to X.

Bearing in mind these warnings, let us briefly mention how various classical formulas [1] arise in this formalism. Consider the (improved) Liouville energy-momentum tensor:

$$\mathcal{T}_\alpha(z_\alpha) = (\partial\varphi_\alpha)^2 - \partial^2\varphi_\alpha$$

It is holomorphic on–shell. Plugging in (80), we find:

$$\mathcal{T}_\alpha(z_\alpha) = -\frac{1}{2}\left\{A_\alpha(z_\alpha), z_\alpha\right\}$$

so that we can identify it with the projective connection $-\frac{1}{2}u_\alpha(z_\alpha)$. Moreover, using the spin $-1/2$ realization of the projective atlas $A_\alpha = \xi_{1\alpha}/\xi_{2\alpha}$, we can write the Liouville field as:

$$e^{-\varphi_\alpha} = \xi_{1\alpha}\bar\xi_{2\alpha} - \xi_{2\alpha}\bar\xi_{1\alpha} \tag{81}$$

with

$$\xi_{1\alpha} = \frac{A_\alpha}{(\partial A_\alpha)^{1/2}} \quad, \quad \xi_{2\alpha} = \frac{1}{(\partial A_\alpha)^{1/2}}$$

8.1 A comparison between the two types of solutions

Thus far in this subsection we have presented the uniformization point of view concerning the Liouville equation. One might even superficially conclude that the reconstruction formula (81) is the same as the one in the previous subsection (7). Although the similarity between the two formulas are certainly not accidental, there are two important differences. The first one is that our approach in the previous subsection is inspired by conformal field theory. As is very profitably done in such a theory we split our problem into two *independent* holomorphic and anti–holomorphic parts. In the language of this section this amounts to considering the solutions of the Liouville equation of the form

$$e^{2\varphi_\alpha} = \frac{\partial A_\alpha(z_\alpha)\bar\partial B_\alpha(\bar z_\alpha)}{(A_\alpha(z_\alpha) - B_\alpha(\bar z_\alpha))^2}$$

where B is independent of A. It is obvious that in order to obtain a (1,1) form from this formula, one has to put constraints on the monodromy of $\{B_\alpha\}$. If one in addition requires reality of $e^{2\varphi_\alpha}dz_\alpha d\bar z_\alpha$, the flat cocycles are again reduced to $PSU(1,1)$ with the additional freedom of taking $B_\alpha(\bar z_\alpha)$ to be a projective transform of $\overline{A_\alpha(z_\alpha)}$.

The second important difference with the uniformization setting is that we actually consider only the solutions ensuing from the holomorphic and anti–holomorphic DS linear systems. This allows us to represent the solutions in terms of bosonic oscillators. We proved in [8] in genus 0 that there is a one–to–one correspondence between the space of free bosonic oscillators and hyperbolic

solutions of the Liouville equation. The same proof does not work in higher genus. A simple example will show the difficulties we run into (on this point, see [31]).

In order to extend the proof of [8], one should be able to 'transform' the connection Λ in (79) (for simplicity we drop the label α) into the upper triangular connection typical of the DS system. In this way one would end up with the bosonization formula

$$\xi_1 = e^{-\wp} \quad , \quad \bar{\partial}\wp = 0$$

and would find the relation:

$$-\partial^2\wp + (\partial\wp)^2 = -\frac{1}{2}\{A, z\} = \mathcal{T}(z) \tag{82}$$

which is a *Miura transformation*. It is matter of coordinate patching to prove that in order for (82) to be consistent, $\partial\wp$ should transform as a holomorphic connection on $K^{-\frac{1}{2}}$ and, as we proved above, such objects do not exist. Thus, paying attention to the residues, $\partial\wp$ can be taken to be meromorphic with simple poles. This is still not enough, since in a coordinate transformation ξ_1 and ξ_2 mix with each other and this is compatible with (82) if and only if the transition functions $\{T_{\alpha\beta}\}$ of the flat bundle T are triangular matrices. However this is hardly acceptable if T has to be a flat bundle arising from (or yielding) uniformization. For, in this case the projective structure should actually be an affine one, which cannot exist for a surface of genus $g > 1$ [14].

The last remark, based on a monodromy argument, points toward a clear–cut separation between the uniformization solution for a compact Riemann surface and the solutions based on the DS systems (which we rely on for quantizing the theory): we are not going to find the uniformization solution among the latter. In fact DS systems only define (branched) affine structures on X with some points removed (see below), rather than projective structures (as the uniformizing solutions do). In the next subsection we clarify this point.

8.2   The DS system and branched affine structures

An affine structure on a general surface X is a collection of local charts $\{A_\alpha\}$ related by the transformation rule

$$A_\alpha(z_\alpha) = a_{\alpha\beta}A_\beta(z_\beta) + b_{\alpha\beta} \tag{83}$$

The target space is assumed to be $\mathbb{C}$, although also $\mathbb{P}^1$ can be considered. An affine structure can be obviously considered as a special kind of projective structure. However, as proved in [29, 14], affine structures do not exist whenever the surface is compact (and of genus $g > 1$), as their existence is equivalent to the triviality of the canonical bundle K. Allowing for branch points, that is points where the local charts $\{A_\alpha\}$ may not be local homeomorphism, avoids that obstruction, so that these slightly generalized structures do exist even on compact surfaces [32].

From the analysis developed in section 2 and 3, it follows that the DS system is related to affine structures rather than to projective ones. Indeed, given a solution $(\sigma_{1\alpha}, \sigma_{2\alpha})$ of the DS system we can form the maps

$$f_\alpha = -\frac{\sigma_{2\alpha}}{\sigma_{1\alpha}} \tag{84}$$

which from the monodromy properties previously analyzed have exactly the transformation property (83). Now we have

$$\partial f_\alpha = \sigma_{1\alpha}^{-2}$$

and since

$$\sigma_{1\alpha}^{-2}(Q) = h_\alpha(Q)^2 h_\alpha(Q_\alpha)^{-2} \exp\left(-2p_n \int_{Q_\alpha}^{Q} \omega^n\right)$$

we see that branch points occur at the zeroes of h. Thus branch points unavoidably appear, even if we consider a non closed surface. The maps $\{f_\alpha\}$ will in general be badly behaved in the vicinity of the points P_+ and P_- where the fields have essential singularities. Note, though, that if we set to zero all the modes p_k except those corresponding to the holomorphic differentials in the KN basis, we can undelete the points P_+ and P_- and the only singular points left will be the zeroes of h. This is the standard case treated in the mathematical literature (see [32]).

The maps in (84) globalize to give a (branched) developing map [33]

$$f : \tilde{X}' \longrightarrow \mathbb{C}$$

and the holonomy homomorphism into the affine group:

$$\rho : \pi_1(X') \longrightarrow A(1, \mathbb{C})$$

Actually, from our previous results we are able to explicitly compute this homomorphism

$$\rho(\gamma)z = \mathbf{c}_{\gamma^{-1}}^2 z + \mathbf{s}_{\gamma^{-1}}$$

for any $z \in \mathbb{C}$.

Appendix A

Here we explicitly calculate the cocycle $\mathbf{s}_\gamma$. Before doing this, it is necessary to see in some detail how to realize the isomorphism

$$H^1(M, G) \cong Hom(\pi_1(M), G)/G$$

The main point is to describe the fundamental group from within a Čech setting. This is done in the following way. Fix a base-point on the manifold and

a suitable good-covering of which we fix a certain element U_0 containing the base point. Given a closed path γ, we cover it with a chain $(U_0, U_1, \ldots, U_n, U_0)$ of open sets starting and ending at U_0. Homotopic paths correspond to chains such that we can pass from one to the other by a finite sequence of simple operations consisting in replacing a pair (U_i, U_{i+1}) of consecutive elements by a triple (U_i, U_j, U_{i+1}) with non void intersection. The inverse path γ^{-1} correspond to the chain $(U_0, U_n, \ldots, U_1, U_0)$, and so on. This defines the group $\pi_1(\mathfrak{U}, U_0)$. The true fundamental group is obtained by taking the direct limit over the coverings $\mathfrak{U}$. Given the cocycle $g = \{g_{ij}\}$ the corresponding element in $Hom(\pi_1(M), G)$ is the one assigning to the chain $(U_0, U_1, \ldots, U_n, U_0)$, the group element $g_{01} \cdot g_{12} \cdots g_{n-1\,n} \cdot g_{n0}$. It is clear that this operation is well-defined up to conjugation and compatible with refinements of the covering (full details in [14]).

Next, in order to have the necessary formulas at hand, and also to put the significance of $\mathbf{s}_\gamma$ in the right context, we quote from [12] the following theorem: on any connected manifold M carrying a flat bundle F, there is an isomorphism

$$H^1(M, \mathcal{F}) \cong H^1(\pi_1(M), \mathbf{F})$$

where $\mathcal{F}$ is the sheaf of locally constant sections of F, $\mathbf{F}$ is the associated characteristic representation and the space at the RHS is the group of *crossed homomorphisms* of $\pi_1(M)$ into the representation space of $\mathbf{F}$ [9] modulo the trivial ones.

We recall that a crossed homomorphism u of a group Π into a Π-module V is a map $u : \Pi \to V$ satisfying

$$u(xy) = y^{-1} \cdot u(x) + u(y)$$

$x, y \in \Pi$, where the dot stands for the action of Π on V. The space of all crossed homomorphism is denoted by $Z^1(\Pi, V)$. The trivial crossed homomorphisms (i.e. the coboundaries) are those given by

$$u(x) = v - x^{-1} \cdot v$$

for $v \in V$. The space of the coboundaries is denoted by $B^1(\Pi, V)$. Thus $H^1(\Pi, V) = Z^1(\Pi, V)/B^1(\Pi, V)$.

It is now worth describing the explicit form of the isomorphism $H^1(M, \mathcal{F}) \cong H^1(\pi_1(M), \mathbf{F})$. We use the representation of $\pi_1(M)$ we are now familiar with, namely the one given by $\pi_1(\mathfrak{U}, U_0)$. For a cocycle $\{A_{\alpha\beta}\} \in Z^1(\mathfrak{U}, \mathcal{F})$, the corresponding element in $Z^1(\pi_1(\mathfrak{U}, U_0), \hat{F})$ is given by

$$
\begin{aligned}
\hat{A}_\gamma \;=\;& (\hat{F}_{\alpha_1\alpha_2} \cdots \hat{F}_{\alpha_{p-1}\alpha_p})^{-1} \cdot A_{\alpha_0\alpha_1} + (\hat{F}_{\alpha_2\alpha_3} \cdots \hat{F}_{\alpha_{p-1}\alpha_p})^{-1} \cdot A_{\alpha_1\alpha_2} + \cdots \\
& \cdots + \hat{F}_{\alpha_{p-1}\alpha_p}^{-1} \cdot A_{\alpha_{p-2}\alpha_{p-1}} + A_{\alpha_{p-1}\alpha_p}
\end{aligned}
$$

where γ is the chain $(U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_p})$ based at U_0. It is a standard matter of tracing all the relations to verify that this correspondence is well defined on the

---

[9] The representation space of $\mathbf{F}$ can be taken to be $\mathbb{C}^r$ where r is the rank of F.

classes and goes through when taking the direct limit on the coverings. It is also easy to verify the cocycle condition starting directly from the formula above. We refer to [12] for the details.

The point is now simply to insert the KN parametrization into the expressions for $c_{\alpha\beta}$ and $d_{\alpha\beta}$ and apply the general formulas we quoted above. To keep all the matter conceptually (if not practically) more manageable, it is better to recast $s = s_{\alpha\beta}$ in slightly different form, perhaps giving it a nicer geometrical significance. Indeed notice that the transformation rule (23) for σ_1 can be rewritten in the following form

$$e^{-2\mathcal{L}_\alpha(z_\alpha)}dz_\alpha = c_{\alpha\beta}^{-2}e^{-2\mathcal{L}_\beta(z_\beta)}dz_\beta$$

and can be interpreted as the definition for each α, β of a 1-form $\phi_{\alpha\beta}$:

$$\phi_{\alpha\beta} = \begin{cases} e^{-2\mathcal{L}_\alpha}dz_\alpha & \text{on } U_\alpha \\ c_{\alpha\beta}^{-2}e^{-2\mathcal{L}_\beta}dz_\beta & \text{on } U_\beta \end{cases}$$

It is easy to see that the two prolongations $\phi_{\alpha\beta}$, $\phi_{\alpha\gamma}$ from U_α agree on the intersection $U_\beta \cap U_\gamma$, so that we can drop the second index: $\phi_{\alpha\beta} \to \phi_\alpha$. Inserting the explicit expression for $d_{\alpha\beta}$ and the definition of $s_{\alpha\beta}$, we find

$$s_{\alpha\beta} = \int_{(Q_\beta,Q_\alpha)} \phi_\beta = -c_{\alpha\beta}^2 \int_{(Q_\alpha,Q_\beta)} \phi_\alpha$$

where (Q_α, Q_β) is the 1-simplex joining Q_α with Q_β.

At this point formula (26) is inferred plugging the equation above into the expression of $\mathbf{s}_\gamma$ as an element of $Z^1(\pi_1(\mathfrak{U}, U_0), \mathbf{c}^{-2})$. Here are the first few steps. Consider for instance three open sets $U_\alpha, U_\beta, U_\gamma$. According to the quoted prescription we must consider

$$c_{\beta\gamma}^2 s_{\alpha\beta} + s_{\beta\gamma}$$

which, using the previously introduced forms $\{\phi_\alpha\}$, reads

$$-c_{\beta\gamma}^2 \int_{(Q_\alpha,Q_\beta)} \phi_\beta - \int_{(Q_\beta,Q_\gamma)} \phi_\gamma = -c_{\beta\gamma}^2 \int_{(Q_\alpha,Q_\beta)} \phi_\beta - c_{\beta\gamma}^2 \int_{(Q_\beta,Q_\gamma)} \phi_\beta$$
$$= -c_{\beta\gamma}^2 \int_{(Q_\alpha,Q_\gamma)} \phi_\beta$$

so that, inserting the KN parametrization into ϕ_β, we find

$$h_\gamma(Q_\gamma)^{-2} e^{2\sum_n p_n \int_{Q_\alpha}^{Q_\gamma} \omega_n} \int_{(Q_\alpha,Q_\gamma)} e^{-2\sum_k p_k \int_{Q_\alpha}^z \omega_k} h(z)^2$$

Next add U_δ at the end of the chain. The relevant quantity now is

$$c_{\gamma\delta}^2 c_{\beta\gamma}^2 s_{\alpha\beta} + c_{\gamma\delta}^2 s_{\beta\gamma} + s_{\gamma\delta}$$

and, using the result for three sets, we reexpress it as

$$-c_{\gamma\delta}^2 c_{\beta\gamma}^2 \int_{(Q_\alpha,Q_\gamma)} \phi_\beta - c_{\gamma\delta}^2 \int_{(Q_\gamma,Q_\delta)} \phi_\gamma$$

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but the forms ϕ_γ and $c_{\beta\gamma}^2\phi_\beta$ are the same on $U_\beta \cap U_\gamma$, so that the last expression can be written in term of the integral of a unique form we keep calling ϕ_γ :

$$c_{\gamma\delta}^2 c_{\beta\gamma}^2 s_{\alpha\beta} + c_{\gamma\delta}^2 s_{\beta\gamma} + s_{\gamma\delta} = -c_{\gamma\delta}^2 \int_{(Q_\alpha, Q_\delta)} \phi_\gamma$$

Plugging in again the expression in terms of the KN basis we find the same formula as the one relative to the three sets, except for the shift of indices:

$$h_\delta(Q_\delta)^{-2} e^{2\sum_n p_n \int_{Q_\alpha}^{Q_\delta} \omega_n} \int_{(Q_\alpha, Q_\delta)} e^{-2\sum_k p_k \int_{Q_\alpha}^z \omega_k} h(z)^2$$

... and so on. It is clear that we obtain the formula (26).

Appendix B

Here we show that conditions (41), (42), (43) are *equivalent* to the univalence of the solution obtained from (7). Sufficiency being obvious, only necessity is to be proved. Thus assume that a collection $\{M_\alpha\}$ such that

$$e^{-\varphi_\alpha} = k_{\alpha\beta}^{-1/2} \bar{k}_{\alpha\beta}^{-1/2} e^{-\varphi_\beta}$$

indeed exists, where each $e^{-\varphi_\alpha}$ is given by (7). This means that the quadratic form defined by (7) must be an $SL(2, \mathbb{C})$–scalar, which in turn is true if and only if

$$M_\beta = {}^t T_{\alpha\beta} M_\alpha \bar{T}_{\alpha\beta}.$$

In the representation space $Hom(\pi_1(X'), SL(2, \mathbb{C}))/SL(2, \mathbb{C})$ this is reformulated by saying that the characteristic representations $\bar{\mathbf{T}}$ and $\mathbf{T}^\vee$ must be equivalent, that is we must have

$$M \bar{\mathbf{T}}_\gamma = \mathbf{T}_\gamma^\vee M$$

for a certain $M \in SL(2, \mathbb{C})$ and any $\gamma \in \pi_1(X')$. Writing down the components of M as

$$M = \begin{pmatrix} x & y \\ u & v \end{pmatrix}, \qquad det\, M = 1$$

and working out the matrix products, one easily derives the following equations

$$\begin{aligned} \bar{\mathbf{c}}_\gamma &= \mathbf{c}_\gamma^{-1} \\ \mathbf{s}_\gamma &= \frac{y}{v} - \frac{y}{v}\mathbf{c}_\gamma^2 \\ \bar{\mathbf{s}}_\gamma &= \frac{u}{v} - \frac{u}{v}\bar{\mathbf{c}}_\gamma^2 \end{aligned}$$

A glance at Appendix A will immediately convince the reader that these relations exactly mean that the cocycles $\mathbf{s}$ and $\bar{\mathbf{s}}$ are coboundaries. This proves necessity.

Now it is an immediate consequence that the characteristic representations $\mathbf{T}^\vee$ and $\bar{\mathbf{T}}$ must be diagonalizable. Indeed if $f = \frac{y}{v}$, and from the above coboundary relations, it follows at once that

$$\mathbf{T}_\gamma = \begin{pmatrix} \mathbf{c}_\gamma & 0 \\ \mathbf{c}_\gamma^{-1}\mathbf{s}_\gamma & \mathbf{c}_\gamma^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -f & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{c}_\gamma & 0 \\ 0 & \mathbf{c}_\gamma^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}$$

A similar relation holds for $\bar{\mathbf{T}}_\gamma$ with $\bar{f} = \frac{u}{v}$. Diagonalizability means that the extension classes represented by $\mathbf{T}^\vee$ and $\bar{\mathbf{T}}$ are trivial, and so are the flat bundles T and $\bar{T}$.

Appendix C

In our analysis of the univalence for the Liouville solutions in Appendix B and the Toda solutions in section 7, we understood the hypothesis that the intertwiner M is Gauss factorizable. We also remarked how this condition is essential for the Bloch–wave basis representation to exist. However, it is interesting to see what happens if we drop this requirement. Let us analyze the Liouville case in some detail.

Let M be the matrix introduced in Appendix B. The condition $v \neq 0$ is necessary and sufficient for M to admit a Gauss factorization in the form

$$M = N_+ \, D \, N_-$$

with D diagonal and $N_\pm$ upper and lower unipotent matrices.

Thus let us suppose that $v = 0$, so that

$$M = \begin{pmatrix} x & y \\ u & 0 \end{pmatrix}, \qquad u = -\frac{1}{y}$$

Then the familiar equation

$$M \, \bar{\mathbf{T}}_\gamma = \mathbf{T}_\gamma^\vee \, M, \qquad \gamma \in \pi_1(X')$$

gives the conditions:

$$\bar{\mathbf{c}}_\gamma = \mathbf{c}_\gamma \tag{85}$$

$$\bar{\mathbf{s}}_\gamma = \frac{1}{y^2}\mathbf{s}_\gamma + \frac{x}{y}(1 - \mathbf{c}_\gamma^2) \tag{86}$$

Thus we see that in this case the conditions so obtained are such that the antichiral monodromy must represent the same flat bundle as the chiral one. Indeed, condition (85) forces the diagonal elements to be the same, while condition (86) tells us that the representations $\mathbf{T}$, $\bar{\mathbf{T}}$ differ by a conjugation in the lower Borel subgroup of $SL(2,\mathbb{C})$. One has directly

$$\bar{\mathbf{T}}_\gamma = \begin{pmatrix} y & 0 \\ x & y^{-1} \end{pmatrix} \cdot \mathbf{T}_\gamma \cdot \begin{pmatrix} y^{-1} & 0 \\ -x & y \end{pmatrix}$$

as desired. As for the flat bundles, this means that the extension

$$0 \longrightarrow C^{-1} \longrightarrow T \longrightarrow \quad C \quad \longrightarrow 0$$
$$0 \longrightarrow C^{-1} \longrightarrow \bar{T} \longrightarrow \quad C \quad \longrightarrow 0$$

differ by an automorphism.

It is to be noticed that in this case the flat bundle T is allowed to be a non trivial extension class.

Finally, it is worth mentioning the following similarity with the uniformization formulas. In the special case $x = 0$, $y = -1$ we have $\bar{\mathbf{s}}_\gamma = \mathbf{s}_\gamma$ so that $\bar{T} = T$ and

$$e^{-\varphi_\alpha} = \sigma_{2\alpha}\bar{\sigma}_{1\alpha} - \sigma_{1\alpha}\bar{\sigma}_{2\alpha}$$

The class of solutions analyzed in this Appendix do *not* belong to the phase space $\mathcal{F}_0$. For this reason we call them non–standard. The relevance of these solutions for quantization is an open question.

Appendix D

It is interesting to examine the conformal properties of the ψ–basis in terms of the symplectic structure introduced in section 5. Let us consider the energy momentum tensor

$$\mathcal{T} = \mathfrak{p}^2 + \mathfrak{p}'$$

which naturally appears in the DS system through the equation

$$\partial^2 \sigma_i = \mathcal{T}\sigma_i, \qquad i = 1, 2$$

As it turns out, $2\mathcal{T}$ is a projective connection. According to the decomposition (3), we can rewrite

$$\mathcal{T} = p^2 + \nabla p + \frac{R_0}{2}, \qquad R_0 = 2(\Gamma'_0 + \Gamma_0^2) \tag{87}$$

R_0 is a fixed projective connection, and $\nabla = \partial + 2n\Gamma_0$ represents in this Appendix the covariant derivative applied to the weight n tensors. Since p^2 and ∇p are holomorphic two forms on X', we can expand them on the basis of quadratic differentials

$$\tfrac{1}{2}(p^2 + \nabla p) = \sum_k \ell_k \Omega^k$$

i.e.

$$\ell_k = \tfrac{1}{2}\sum_{n,m} l_k^{nm} p_n p_m + \tfrac{1}{2}\zeta_k^n p_n \tag{88}$$

where

$$l_k^{nm} = \tfrac{1}{2\pi i}\oint_{C_\tau} e_k \omega^n \omega^m, \qquad \zeta_k^n = \tfrac{1}{2\pi i}\oint_{C_\tau} e_k \nabla \omega^n$$

Using the basic Poisson brackets (45) we find – in this Appendix we abandon the simplified notation for the Poisson brackets adopted in section 5, therefore the following Poisson brackets are the usual ones –

$$\{\ell_n, \ell_m\} = -C^k_{nm}\ell_k - \frac{1}{8\pi i}\oint_{C_\tau} \tilde{\chi}_0(\ell_n, \ell_m) \tag{89}$$

where $\tilde{\chi}_0$ has the same form as in eq.(16) provided the role of projective connection be played by R_0, eq.(87). Therefore, up to an irrelevant – sign, eq.(89) represents a realization of the extended KN algebra (17).

We can now easily work out, for example, the Poisson bracket

$$\{\ell_n, \psi_1(Q)\} = e_n(Q)p(Q)\psi_1(Q) - \tfrac{1}{2}\nabla e_n(Q)\psi_1(Q) \tag{90}$$

The RHS of this equation is nothing but $\mathcal{L}_{e_n}$, i.e. the Lie derivative along the vector field e_n, applied to the weight $-\frac{1}{2}$ tensor $\psi_1(Q)$. This is exactly what we expect.

As for $\psi_2(Q)$ the calculation is longer, but once the formalism is established we can safely rely on the genus zero results [7] which guarantee that $\psi_2(Q)$ behaves like a weight $-\frac{1}{2}$ tensor as well. We stress that the σ–basis does not have good tensorial properties with respect to the symplectic structure introduced in section 5.

We remark that the above equations (89) and (90), remain unchanged if we replace the Poisson brackets with the Dirac brackets defined in subsection 5.5. Indeed the Poisson brackets of the ℓ_n's with $\mathcal{F}_r$ and $\bar{\mathcal{F}}_r$ turn out to be proportional to $\mathcal{F}_r$ and $\bar{\mathcal{F}}_r$, respectively. Thus the energy–momentum tensor is the generator of the conformal transformations also with the correct bracket on the constrained manifold in the phase space.

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