Reduced basis approximation and a-posteriori error estimation for the coupled Stokes-Darcy system

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Abstract The coupling of a free flow with a flow through porous media has many potential applications in several fields related with computational science and engineering, such as blood flows, environmental problems or food technologies. We present a reduced basis method for such coupled problems. The reduced basis method is a model order reduction method applied in the context of parametrized systems. Our approach is based on a heterogeneous domain decomposition formulation, namely the Stokes-Darcy problem. Thanks to an offline/online-decomposition, computational times can be drastically reduced. At the same time the induced error can be bounded by fast evaluable a-posteriori error bounds. In the offline-phase the proposed algorithms make use of the decomposed problem structure. Rigorous a-posteriori error bounds are developed, indicating the accuracy of certain lifting operators used in the offline-phase as well as the accuracy of the reduced coupled system. Also, a strategy separately bounding pressure and velocity errors is extended. Numerical experiments dealing with groundwater flow scenarios demonstrate the efficiency of the approach as well as the limitations regarding a-posteriori error estimation.

Keywords Reduced basis method · Stokes flow · Porous medium equation · Domain decomposition · Non-coercive problem · Error estimation

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1 Introduction

The fundamental idea of reduced basis (RB) methods [1, 24–26, 30] is approximating the solution manifold of a parametric partial differential equation (PDE) by a low-dimensional linear space, commonly spanned by solution instantiations (snapshots) for carefully selected parameter values. This RB space is built up on a high-dimensional approximation space, from which the solutions are taken, for example a finite element (FE) space. The efficiency of the method highly relies on the affine decomposition property of the weak form. It ensures an efficient decomposition of the computations in an offline- and online-phase. In the offline-phase the RB space is built up once for all RB simulations. This is a costly process, but independent of the ultimate simulation requests. In the online-phase RB solutions can be computed for every new parameter value of the parameter domain, but the computational load is independent of the high dimension. The quality of the approximation can be certified by efficiently computable a-posteriori error estimates.

The advantages of the method are worth while for several settings, where many parameter requests are posed (many-query context) or where fast solution evaluations are needed (real-time context). For example RB methods were applied to optimization problems [6, 23, 28]. The range of RB methods includes time dependent problems [11, 12] as well as noncoercive and nonlinear problems [26, 36]. More specifically, much investigation was performed in solving saddle-point problems evolving from fluid simulations [10, 15, 16, 18, 28, 29]. The idea to include domain decomposition methods [27, 34] in the computational framework of the RB method was introduced in [19, 20], following a Mortar-like approach for a coercive and linear PDE and extended in [15, 16, 18] to the Stokes equations. Other approaches are based on the Schur-complement in domain decomposition [9, 13, 35] and on the Dirichlet-Neumann solution scheme [21]. A method suited for the simulation of fluidic networks was developed in [15]. The a-posteriori error estimation for elliptic PDEs on decomposed domains is an issue that has been addressed from many different points of view [13, 18, 21]. Very recently a framework has been developed, that combines many of the previous ideas [33]. To our knowledge, there is no work for the noncoercive case.

In this paper a RB method for the heterogeneous coupling of the Stokes equations and the porous medium equation [7, 8, 17] is presented. The known procedures for homogeneous couplings of noncoercive PDEs [15] are extended to a more general setting. The algorithm takes account of the fact that the complexity of the problem with respect to the parameter is expected to be smaller on the interface between the subdomains than on the whole domain. With the interface values of a small number of global snapshots, we construct bases on the subdomains via a subdomain-lifting framework, that is integrated into the Greedy-algorithm [36] for the basis generation. The resulting Subdomain-Greedy adaptively builds the basis on the subdomains in parallel. Regarding a-posteriori error estimation of the global error, the standard non-coercive error estimate [26] can be very pessimistic due to a great difference of the stability constants of the subproblems. We present a modified error bound, which is more suited in that case and refines the error bounds of [10].

The outline of the paper is as follows. We introduce the coupled system of PDEs and the weak formulation in Section 2. In Section 3 the considered parameter

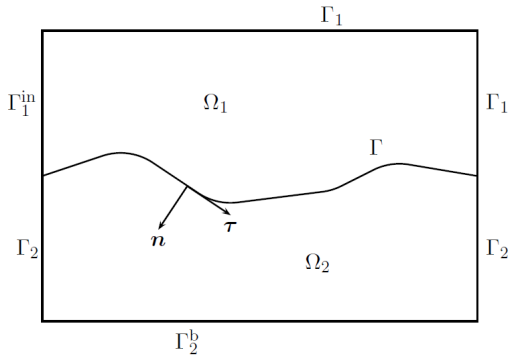

Fig. 1 Illustration of the computational domains and boundaries.

dependency is explained and the resulting parametrized problem is detailed. The analysis of the problem is summarized in Section 4. As we use the FE method as foundation, Section 5 serves to present the FE approximation of the coupled system and in addition the concept of lifting interface modes. Then we introduce the RB formulation together with the basis generation procedure in Section 6 and give the definitions and proofs of the error bounds in Section 7. An investigation by numerical tests dealing with groundwater flows is portrayed in Section 8. We conclude with some remarks on the results and future work.

2 Problem formulation

We consider the coupled Stokes-Darcy system [7,8,17] in a two-dimensional domain $\Omega \subset \mathbb{R}^2$, decomposed into two subdomains Ω_i, $i = 1, 2$ with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$. We denote by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ the common boundary or interface, depicted in Figure 1, with associated normal unit vector $\mathbf{n}$ pointing from Ω_1 to Ω_2 and tangent unit vector $\boldsymbol{\tau}$. All boundaries are assumed to be *Lipschitz-boundaries*.

The differential multidomain formulation is as follows. On Ω_1 we consider a *Stokes* flow: Find velocity $\mathbf{u}$ and pressure p such that

$$\left.\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega_1\,, \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega_1\,, \\ \mathbf{u} &= \mathbf{u}_{\text{in}} && \text{on } \Gamma_1^{\text{in}}\,, \\ \mathbf{u} &= 0 && \text{on } \Gamma_1\,, \end{aligned}\right\} \tag{1}$$

and on Ω_2 the porous medium equation: Find the pressure head φ such that

$$\left.\begin{aligned} -\nabla\cdot(\mathsf{K}\nabla\varphi) &= 0 && \text{in } \Omega_2\,, \\ -\mathsf{K}\nabla\varphi\cdot\mathbf{n}_2 &= 0 && \text{on } \Gamma_2\,, \\ \varphi &= \varphi_{\text{b}} && \text{on } \Gamma_2^{\text{b}}\,. \end{aligned}\right\} \tag{2}$$

Here ν is the kinematic viscosity, K is the hydraulic conductivity tensor, $\mathbf{f}$ a source term, $\mathbf{u}_{\text{in}}$ the inflow data, φ_{b} the *Darcy* pressure on the bottom and $\mathbf{n}_i$ the outward normal unit vector on $\partial\Omega_i$, $i = 1, 2$. The Darcy velocity is given by

$$\mathbf{u}_D = -\frac{1}{n}\mathsf{K}\nabla\varphi\,,$$

where n is the volumetric porosity constant. The systems are coupled through the following *interface* conditions on Γ:

$$\left.\begin{aligned} \mathbf{u}\cdot\mathbf{n} &= -\frac{1}{n}\mathsf{K}\nabla\varphi\cdot\mathbf{n} && \text{on } \Gamma\,,\\ -\nu\mathbf{n}\cdot\frac{\partial\mathbf{u}}{\partial\mathbf{n}} + p &= g\varphi && \text{on } \Gamma\,,\\ \mathbf{u}\cdot\boldsymbol{\tau} &= -\frac{\sqrt{k}}{\alpha_{\text{BJ}}}\boldsymbol{\tau}\cdot\frac{\partial\mathbf{u}}{\partial\mathbf{n}} && \text{on } \Gamma\,. \end{aligned}\right\} \tag{3}$$

The first two conditions represent the balance of mass and forces. The last one was proposed based on experiments and is a boundary condition rather than a transmission condition. Here g denotes the gravity acceleration, α_{BJ} the *Beavers-Joseph* constant depending on the permeable material and $k = \frac{1}{\nu}\boldsymbol{\tau}\cdot\mathsf{K}\cdot\boldsymbol{\tau}$. We introduce the weak formulation with the definition of the functional spaces:

$$\left.\begin{aligned} X_{1,\mathbf{u}} &= \{\mathbf{v}\in(H^1(\Omega_1))^2 : (\mathbf{v}\cdot\mathbf{n}_1)|_{\Gamma_1} = 0 \wedge (\mathbf{v}\cdot\mathbf{n}_1)|_{\Gamma_1^{\text{in}}} = 0\}\,,\\ X_{1,\mathbf{u}}^{\text{in}} &= \{\mathbf{v}\in(H^1(\Omega_1))^2 : (\mathbf{v}\cdot\mathbf{n}_1)|_{\Gamma_1} = 0 \wedge \mathbf{v}|_{\Gamma} = 0\}\,,\\ X_{1,p} &= L^2(\Omega_1)\,,\\ X_2 &= \{\psi\in H^1(\Omega_2) : \psi|_{\Gamma_2^{\text{b}}} = 0\}\,,\\ X_2^{\text{b}} &= \{\psi\in H^1(\Omega_2) : \psi|_{\Gamma} = 0\}\,. \end{aligned}\right\} \tag{4}$$

The inflow data is assumed to be in $(H_{00}^{1/2}(\Gamma_1^{\text{in}}))^2$ and lifted into the interior of Ω_1 by means of a continuous extension operator $\mathcal{E}_{\text{in}} : (H_{00}^{1/2}(\Gamma_1^{\text{in}}))^2 \to X_{1,\mathbf{u}}^{\text{in}}$ with $(\mathcal{E}_{\text{in}}\mathbf{v})|_{\Gamma_1^{\text{in}}} = \mathbf{v}$ for $\mathbf{v}\in(H_{00}^{1/2}(\Gamma_1^{\text{in}}))^2$. Here, $H_{00}^{1/2}(\Gamma_1^{\text{in}})$ is defined as the subspace of functions in $H^{1/2}(\Gamma_1^{\text{in}})$ whose extension by zero to $\partial\Omega_1$ lives in $H^{1/2}(\partial\Omega_1)$. We also introduce a continuous extension operator $\mathcal{E}_{\text{b}} : H^{1/2}(\Gamma_2^{\text{b}}) \to X_2^{\text{b}}$, such that $(\mathcal{E}_{\text{b}}\psi)|_{\Gamma_2^{\text{b}}} = \psi$ for $\psi\in H^{1/2}(\Gamma_2^{\text{b}})$. The weak solution of (1)–(3) is given by $(\mathbf{u}+\mathcal{E}_{\text{in}}\mathbf{u}_{\text{in}}, p, \varphi+\mathcal{E}_{\text{b}}\varphi_{\text{b}})$ with $(\mathbf{u}, p, \varphi)\in X_{1,\mathbf{u}}\times X_{1,p}\times X_2$ such that

$$\left.\begin{aligned} \mathcal{A}_1(\mathbf{u},\mathbf{v}) + \mathcal{B}_1(\mathbf{v},p) + \mathcal{C}_1(\varphi,\mathbf{v}) &= \mathcal{F}_{1,\mathbf{u}}(\mathbf{v}), && \forall\mathbf{v}\in X_{1,\mathbf{u}},\\ \mathcal{B}_1(\mathbf{u},q) &= \mathcal{F}_{1,p}(q), && \forall q\in X_{1,p},\\ \mathcal{A}_2(\varphi,\psi) + \mathcal{C}_2(\mathbf{u},\psi) &= \mathcal{F}_2(\psi), && \forall\psi\in X_2, \end{aligned}\right\} \tag{5}$$

where

$$\begin{aligned} \mathcal{A}_1(\mathbf{v},\mathbf{w}) &= \int_{\Omega_1}\nu\nabla\mathbf{v}:\nabla\mathbf{w}\,dx + \int_{\Gamma}\nu\frac{\alpha_{\text{BJ}}}{\sqrt{k}}(\mathbf{v}\cdot\boldsymbol{\tau})(\mathbf{w}\cdot\boldsymbol{\tau})\,ds,\\ \mathcal{B}_1(\mathbf{v},q) &= -\int_{\Omega_1} q\,\nabla\cdot\mathbf{v}\,dx, \qquad \mathcal{C}_1(\psi,\mathbf{v}) = \int_{\Gamma} g\psi\;\mathbf{v}\cdot\mathbf{n}\,ds,\\ \mathcal{F}_{1,\mathbf{u}}(\mathbf{v}) &= \int_{\Omega_1}\mathbf{f}\cdot\mathbf{v}\,dx - \mathcal{A}_1(\mathcal{E}_{\text{in}}\mathbf{u}_{\text{in}},\mathbf{v}), \qquad \mathcal{F}_{1,p}(q) = -\mathcal{B}_1(\mathcal{E}_{\text{in}}\mathbf{u}_{\text{in}},q) \end{aligned}$$

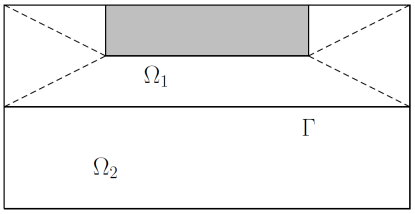

Fig. 2 Computational domain with parametrized properties: The height of the shaded area is parametrized. This is achieved with $R_1 = 5$ and $R_2 = 1$ affine transformations.

and

$$\mathcal{A}_2(\psi,\chi) = \int_{\Omega_2} \nabla\psi \cdot \mathsf{K}\nabla\chi \, dx,$$
$$\mathcal{C}_2(\mathbf{v},\psi) = -\int_{\Gamma} n\psi \; \mathbf{v}\cdot\mathbf{n} \, ds, \qquad \mathcal{F}_2(\psi) = -\mathcal{A}_2(\mathcal{E}_{\mathrm{b}}\varphi_{\mathrm{b}},\psi)\,.$$

3 The parametrized problem

We introduce a parameter vector $\boldsymbol{\mu} \in \mathcal{P}$, $\mathcal{P} \subset \mathbb{R}^P$, on which the material properties and also geometric quantities depend. Here we follow the methodology for RB approximation in parametrized geometries developed earlier [28,29]. We suppose that $\Omega = \Omega(\boldsymbol{\mu})$ is obtained from a reference domain Ω_{ref} through a piecewise affine geometry transformation. More precisely, let reference subdomains $\overline{\Omega}_{\mathrm{ref}} = \overline{\Omega}_{1,\mathrm{ref}} \cup \overline{\Omega}_{2,\mathrm{ref}}$, $\Omega_{1,\mathrm{ref}} \cap \Omega_{2,\mathrm{ref}} = \emptyset$ with decompositions $\overline{\Omega}_{i,\mathrm{ref}} = \bigcup_{r=1}^{R_i} \overline{\Omega}^r_{i,\mathrm{ref}}$ for $i = 1,2$ be given. Such decompositions originate from coarse triangulations, as is depicted in Figure 2. On each subdomain $\Omega^r_{i,\mathrm{ref}}$, $r = 1,\ldots,R_i$, $i = 1,2$ acts a bijective affine transformation $T^r_i(\cdot;\boldsymbol{\mu}) : \Omega^r_{i,\mathrm{ref}} \to \Omega(\boldsymbol{\mu})$ given through

$$T^r_i(\mathbf{x};\boldsymbol{\mu}) = \mathsf{G}^r_i(\boldsymbol{\mu})\mathbf{x} + \mathbf{t}^r_i(\boldsymbol{\mu}),$$

with a transformation matrix $\mathsf{G}^r_i : \mathcal{P} \to \mathbb{R}^{2\times 2}$ and a translation vector $\mathbf{t}^r_i : \mathcal{P} \to \mathbb{R}^2$. The transformation mappings have to fullfill the following continuity condition:

$$T^r_i(\mathbf{x};\boldsymbol{\mu}) = T^{r'}_{i'}(\mathbf{x};\boldsymbol{\mu}), \qquad \forall \mathbf{x} \in \overline{\Omega}^r_{i,\mathrm{ref}} \cap \overline{\Omega}^{r'}_{i',\mathrm{ref}},$$

for all $r = 1,\ldots,R_i, r' = 1,\ldots,R_{i'}$ and $i,i' = 1,2$. Then our assumption reads

$$\overline{\Omega}_i = \overline{\Omega_i(\boldsymbol{\mu})} = \bigcup_{r=1}^{R_i} \overline{\Omega^r_i(\boldsymbol{\mu})}, \qquad i = 1,2$$

with $\Omega^r_i(\boldsymbol{\mu}) = T^r_i(\Omega^r_{i,\mathrm{ref}};\boldsymbol{\mu})$ for all $r = 1,\ldots,R_i$, $i = 1,2$. Subsequently we define $\Gamma_{\mathrm{ref}} = \partial\Omega_{1,\mathrm{ref}} \cap \partial\Omega_{2,\mathrm{ref}}$ and $\Gamma^r_{\mathrm{ref}} = \Gamma_{\mathrm{ref}} \cap \Omega^r_{1,\mathrm{ref}}$ and have

$$\overline{\Gamma} = \overline{\Gamma(\boldsymbol{\mu})} = \bigcup_{r=1}^{R_1} \overline{T^r_1(\Gamma^r_{\mathrm{ref}};\boldsymbol{\mu})}.$$

Regarding the material constants, we assume $\nu = \nu(\boldsymbol{\mu})$, $n = n(\boldsymbol{\mu})$, $k = k(\boldsymbol{\mu})$ and $\alpha_{\mathrm{BJ}} = \alpha_{\mathrm{BJ}}(\boldsymbol{\mu})$ to be constant in space. Let $\boldsymbol{\mu} = (\boldsymbol{\mu}^{\mathrm{geo}}, \boldsymbol{\mu}^{\mathrm{mat}})$, where $\boldsymbol{\mu}^{\mathrm{geo}}$

bundles the parameters shaping the geometrical domain and $\boldsymbol{\mu}^{\mathrm{mat}}$ bundles the parameters representing material properties. Then $\nu(\boldsymbol{\mu}) = \nu(\boldsymbol{\mu}^{\mathrm{mat}})$, $n(\boldsymbol{\mu}) = n(\boldsymbol{\mu}^{\mathrm{mat}})$, $k(\boldsymbol{\mu}) = k(\boldsymbol{\mu}^{\mathrm{mat}})$ and $\alpha_{\mathrm{BJ}}(\boldsymbol{\mu}) = \alpha_{\mathrm{BJ}}(\boldsymbol{\mu}^{\mathrm{mat}})$. Conversely, we allow heterogeneity in $\mathsf{K} = \mathsf{K}(\boldsymbol{\mu})$ away from the interface $\Gamma(\boldsymbol{\mu})$ (thus $k(\boldsymbol{\mu})$ remains constant) and so

$$\mathsf{K}(\mathbf{x};\boldsymbol{\mu}) = \mathsf{K}_{\mathrm{ref}}\left(\left(\left(T_i^r(\cdot;\boldsymbol{\mu}^{\mathrm{geo}})\right)^{-1}(\mathbf{x});\boldsymbol{\mu}^{\mathrm{mat}}\right)\right), \qquad \mathbf{x} \in \Omega_i^r(\boldsymbol{\mu}), \tag{6}$$

for all $r = 1, \dots, R_i$, $i = 1, 2$.

Problem (5) is solved by tracing it back to Ω_{ref}. As a consequence, in addition to the dependency on the material parameters, we get dependency of (5) on the geometry parameters due to transformation formulas.

Remark 1 For the sake of readability we do not introduce new notations for the function spaces on Ω_{ref}, but understand the definitions (4) to be done on Ω_{ref}.

The resulting problem on Ω_{ref} then reads: For $\boldsymbol{\mu} \in \mathcal{P}$ find $(\mathbf{u}(\boldsymbol{\mu}), p(\boldsymbol{\mu}), \varphi(\boldsymbol{\mu})) \in X_{1,\mathbf{u}} \times X_{1,p} \times X_2$ such that

$$\mathcal{A}_1(\mathbf{u}(\boldsymbol{\mu}), \mathbf{v};\boldsymbol{\mu}) + \mathcal{B}_1(\mathbf{v}, p(\boldsymbol{\mu});\boldsymbol{\mu}) + \mathcal{C}_1(\varphi(\boldsymbol{\mu}), \mathbf{v};\boldsymbol{\mu}) = \mathcal{F}_{1,\mathbf{u}}(\mathbf{v};\boldsymbol{\mu}), \quad \forall \mathbf{v} \in X_{1,\mathbf{u}}, \tag{7a}$$

$$\mathcal{B}_1(\mathbf{u}(\boldsymbol{\mu}), q;\boldsymbol{\mu}) = \mathcal{F}_{1,p}(q;\boldsymbol{\mu}), \quad \forall q \in X_{1,p}, \tag{7b}$$

$$\mathcal{A}_2(\varphi(\boldsymbol{\mu}), \psi;\boldsymbol{\mu}) + \mathcal{C}_2(\mathbf{u}(\boldsymbol{\mu}), \psi;\boldsymbol{\mu}) = \mathcal{F}_2(\psi;\boldsymbol{\mu}), \quad \forall \psi \in X_2, \tag{7c}$$

with

$$\mathcal{A}_1(\mathbf{v}, \mathbf{w};\boldsymbol{\mu}) = \sum_{r=1}^{R_1} \int_{\Omega_{1,\mathrm{ref}}^r} (\nabla v_j)^T J_{\mathcal{A}_1}^r(\boldsymbol{\mu}) \nabla w_j \, dx + \sum_{r=1}^{R_1} \int_{\Gamma_{\mathrm{ref}}^r} K_{\mathcal{A}_1}^r(\boldsymbol{\mu})(\mathbf{v}\cdot\boldsymbol{\tau})(\mathbf{w}\cdot\boldsymbol{\tau}) \, ds,$$

$$\mathcal{B}_1(\mathbf{v}, q;\boldsymbol{\mu}) = \sum_{r=1}^{R_1} \int_{\Omega_{1,\mathrm{ref}}^r} q(J_{\mathcal{B}_1}^r(\boldsymbol{\mu})\nabla)\cdot\mathbf{v} \, dx,$$

$$\mathcal{C}_1(\psi, \mathbf{v};\boldsymbol{\mu}) = \sum_{r=1}^{R_1} \int_{\Gamma_{\mathrm{ref}}^r} K_{\mathcal{C}_1}^r(\boldsymbol{\mu})\psi\mathbf{v}\cdot\mathbf{n} \, ds,$$

$$\mathcal{F}_{1,\mathbf{u}}(\mathbf{v};\boldsymbol{\mu}) = \sum_{r=1}^{R_1} \int_{\Omega_{1,\mathrm{ref}}^r} K_{\mathcal{F}_1}^r(\boldsymbol{\mu})\mathbf{f}\cdot\mathbf{v} \, dx - \mathcal{A}_1(\mathcal{E}_{\mathrm{in}}\mathbf{u}_{\mathrm{in}}, \mathbf{v};\boldsymbol{\mu}),$$

$$\mathcal{F}_{1,p}(q;\boldsymbol{\mu}) = -\mathcal{B}_1(\mathcal{E}_{\mathrm{in}}\mathbf{u}_{\mathrm{in}}, q;\boldsymbol{\mu}),$$

where we omitted summation over j ($j = 1, 2$) and

$$J_{\mathcal{A}_1}^r(\boldsymbol{\mu}) = \nu(\boldsymbol{\mu})|\det(\mathsf{G}_1^r(\boldsymbol{\mu}))|\mathsf{G}_1^r(\boldsymbol{\mu})^{-1}\mathsf{G}_1^r(\boldsymbol{\mu})^{-T},$$

$$K_{\mathcal{A}_1}^r(\boldsymbol{\mu}) = \nu(\boldsymbol{\mu})\frac{\alpha_{\mathrm{BJ}}(\boldsymbol{\mu})}{\sqrt{k(\boldsymbol{\mu})}}|\mathsf{G}_1^r(\boldsymbol{\mu})\boldsymbol{\tau}|, \qquad J_{\mathcal{B}_1}^r(\boldsymbol{\mu}) = |\det(\mathsf{G}_1^r(\boldsymbol{\mu}))|\mathsf{G}_1^r(\boldsymbol{\mu})^{-T},$$

$$K_{\mathcal{C}_1}^r(\boldsymbol{\mu}) = g|\mathsf{G}_1^r(\boldsymbol{\mu})\boldsymbol{\tau}|, \qquad K_{\mathcal{F}_1}^r(\boldsymbol{\mu}) = |\det(\mathsf{G}_1^r(\boldsymbol{\mu}))|$$

and on the second subdomain

$$\mathcal{A}_2(\psi, \chi;\boldsymbol{\mu}) = \sum_{r=1}^{R_2} \int_{\Omega_{2,\mathrm{ref}}^r} (\nabla\psi)^T J_{\mathcal{A}_2}^r(\boldsymbol{\mu}) \nabla\chi \, dx,$$

$$\mathcal{C}_2(\mathbf{v}, \psi;\boldsymbol{\mu}) = -\sum_{r=1}^{R_2} \int_{\Gamma_{\mathrm{ref}}^r} K_{\mathcal{C}_2}^r(\boldsymbol{\mu})\psi \; \mathbf{v}\cdot\mathbf{n} \, ds, \qquad \mathcal{F}_2(\psi;\boldsymbol{\mu}) = -\mathcal{A}_2(\mathcal{E}_{\mathrm{b}}\varphi_{\mathrm{b}}, \psi;\boldsymbol{\mu}),$$

where

$$J^r_{\mathcal{A}_2}(\boldsymbol{\mu}) = |\det(\mathsf{G}^r_2(\boldsymbol{\mu}))|\mathsf{G}^r_2(\boldsymbol{\mu})^{-1}\mathsf{K}_{\mathrm{ref}}(\boldsymbol{\mu})\mathsf{G}^r_2(\boldsymbol{\mu})^{-T}, \qquad K^r_{\mathcal{C}_2}(\boldsymbol{\mu}) = n(\boldsymbol{\mu})|\mathsf{G}^r_2(\boldsymbol{\mu})\boldsymbol{\tau}|.$$

Here, $\boldsymbol{\tau}$ is the tangential unit vector on Γ_{ref}. Alternatively, one could use the Piola transformation [4,5] to treat the boundary terms for general geometrical transformations. However, this is avoided by our procedure. We only require that the geometry treatment yields an affinely parametrized system [24,26,30]. This means that decompositions of the following type can be applied:

$$\mathcal{A}_1(\mathbf{v},\mathbf{w};\boldsymbol{\mu}) = \sum_{q=1}^{Q_{\mathcal{A}_1}} \Theta^q_{\mathcal{A}_1}(\boldsymbol{\mu})\mathcal{A}^q_1(\mathbf{v},\mathbf{w}), \qquad \forall \mathbf{v} \in X_{1,\mathbf{u}}, \mathbf{w} \in X_{1,\mathbf{u}}, \boldsymbol{\mu} \in \mathcal{P}, \tag{8}$$

with rapidly evaluable parameter-dependent coefficients $\Theta^q_{\mathcal{A}_1}$ and parameter-independent bilinear forms $\mathcal{A}^q_1$. The size $Q_{\mathcal{A}_1}$ of the decomposition is preferably small. Analogous representations must hold for $\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{C}_i$, $i = 1, 2$ and for the linear forms $\mathcal{F}_{1,\mathbf{u}}$, $\mathcal{F}_{1,p}$ and $\mathcal{F}_2$.

If all parameter-dependent constants allow an affine decomposition, writing down those decompositions for the linear and bilinear forms is straightforward, except for the terms involving $|\mathsf{G}^r_i(\boldsymbol{\mu})\boldsymbol{\tau}|$, $i = 1, 2$, $r = 1, \ldots, R_i$ due to the square root. To get small-sized decompositions for those terms, the *Empirical Interpolation Method* (EIM) [2] could be considered. For our purposes it is sufficient to assume that the tangential vector $\boldsymbol{\tau}$ is constant on Γ^r_{ref}, for all $r = 1, \ldots, R_1$, so that $|\mathsf{G}^r_i(\boldsymbol{\mu})\boldsymbol{\tau}|$, $i = 1, 2$ only depends on $\boldsymbol{\mu}$ for all $r = 1, \ldots, R_i$.

4 Analysis of the problem

The wellposedness of system (7) has been studied in [7]. The proof makes use of the *Brezzi* stability theory for saddle-point problems [3,4]. We introduce here the properties of the involved linear and bilinear forms of the parametrized system, which ensure wellposedness. The function spaces are equipped with the following norms: $\|\cdot\|_{X_{1,\mathbf{u}}} = \|\cdot\|_{(H^1_0(\Omega_{1,\mathrm{ref}}))^2}$, $\|\cdot\|_{X_{1,p}} = \|\cdot\|_{L^2(\Omega_{1,\mathrm{ref}})}$ and $\|\cdot\|_{X_2} = \|\cdot\|_{H^1_0(\Omega_{2,\mathrm{ref}})}$. For all $\boldsymbol{\mu} \in \mathcal{P}$ the symmetric bilinear form $\mathcal{A}_1(\cdot,\cdot;\boldsymbol{\mu})$ is continuous and coercive on $X_{1,\mathbf{u}}$, that is

$$\begin{aligned} \gamma_{\mathcal{A}_1}(\boldsymbol{\mu}) &= \sup_{\mathbf{v}\in X_{1,\mathbf{u}}} \sup_{\mathbf{w}\in X_{1,\mathbf{u}}} \frac{\mathcal{A}_1(\mathbf{v},\mathbf{w};\boldsymbol{\mu})}{\|\mathbf{v}\|_{X_{1,\mathbf{u}}}\|\mathbf{w}\|_{X_{1,\mathbf{u}}}} < \infty, && \forall \boldsymbol{\mu} \in \mathcal{P}, \\ \alpha_{\mathcal{A}_1}(\boldsymbol{\mu}) &= \inf_{\mathbf{v}\in X_{1,\mathbf{u}}} \frac{\mathcal{A}_1(\mathbf{v},\mathbf{v};\boldsymbol{\mu})}{\|\mathbf{v}\|^2_{X_{1,\mathbf{u}}}} > 0, && \forall \boldsymbol{\mu} \in \mathcal{P}. \end{aligned} \tag{9}$$

Furthermore, $\sup_{\boldsymbol{\mu}\in\mathcal{P}} \gamma_{\mathcal{A}_1}(\boldsymbol{\mu}) < \infty$ and $\inf_{\boldsymbol{\mu}\in\mathcal{P}} \alpha_{\mathcal{A}_1}(\boldsymbol{\mu}) > 0$ is supposed. Here, $\gamma_{\mathcal{A}_1}(\boldsymbol{\mu})$ is called *continuity* constant and $\alpha_{\mathcal{A}_1}(\boldsymbol{\mu})$ *coercivity* constant. The same properties are assumed to be valid for $\mathcal{A}_2(\cdot,\cdot;\boldsymbol{\mu})$ on $X_2 \times X_2$ with corresponding constants $\gamma_{\mathcal{A}_2}(\boldsymbol{\mu})$ and $\alpha_{\mathcal{A}_2}(\boldsymbol{\mu})$. The bilinear form $\mathcal{B}_1(\cdot,\cdot;\boldsymbol{\mu})$ is continuous and inf-sup stable on $X_{1,\mathbf{u}} \times X_{1,p}$, that is

$$\beta(\boldsymbol{\mu}) = \inf_{q\in X_{1,p}} \sup_{\mathbf{v}\in X_{1,\mathbf{u}}} \frac{\mathcal{B}_1(\mathbf{v},q;\boldsymbol{\mu})}{\|\mathbf{v}\|_{X_{1,\mathbf{u}}}\|q\|_{X_{1,p}}} > 0, \qquad \forall \boldsymbol{\mu} \in \mathcal{P} \tag{10}$$

and we suppose $\inf_{\boldsymbol{\mu}\in\mathcal{P}} \beta(\boldsymbol{\mu}) > 0$. Here, $\beta(\boldsymbol{\mu})$ is called *inf-sup* constant. The bilinear forms $\mathcal{C}_i(\cdot,\cdot;\boldsymbol{\mu})$, $i = 1, 2$ are continuous on $X_{1,\mathbf{u}} \times X_2$ respectively $X_2 \times X_{1,\mathbf{u}}$ with corresponding constants $\gamma_{\mathcal{C}_i}(\boldsymbol{\mu})$ and $\sup_{\boldsymbol{\mu}\in\mathcal{P}} \gamma_{\mathcal{C}_i}(\boldsymbol{\mu}) < \infty$ for $i = 1, 2$. Last but not least the linear forms $\mathcal{F}_{1,\mathbf{u}}(\cdot;\boldsymbol{\mu})$, $\mathcal{F}_{1,p}(\cdot;\boldsymbol{\mu})$ and $\mathcal{F}_2(\cdot;\boldsymbol{\mu})$ are continuous on $X_{1,\mathbf{u}}$, $X_{1,p}$ and X_2, respectively.

We point out that, beyond those standard assumptions, the following condition on the coupling terms holds by definition: For all $\boldsymbol{\mu} \in \mathcal{P}$ there exist constants $\alpha_{\mathcal{C}_i}(\boldsymbol{\mu}) > 0$, $i = 1, 2$, such that

$$\alpha_{\mathcal{C}_1}(\boldsymbol{\mu})\mathcal{C}_1(\psi, \mathbf{v}; \boldsymbol{\mu}) + \alpha_{\mathcal{C}_2}(\boldsymbol{\mu})\mathcal{C}_2(\mathbf{v}, \psi; \boldsymbol{\mu}) = 0, \qquad \forall \mathbf{v} \in X_{1,\mathbf{u}}, \psi \in X_2. \tag{11}$$

We report now the a-priori result [7] adapted to our notation.

Proposition 1 *For every $\boldsymbol{\mu} \in \mathcal{P}$ there exists a unique solution $(\mathbf{u}(\boldsymbol{\mu}), p(\boldsymbol{\mu}), \varphi(\boldsymbol{\mu})) \in X_{1,\mathbf{u}} \times X_{1,p} \times X_2$ of the system (7). The following a-priori estimates are valid:*

$$\begin{aligned}
\|(\mathbf{u}(\boldsymbol{\mu}), \varphi(\boldsymbol{\mu}))\|_{X_{1,\mathbf{u}}\times X_2} &\leq \frac{1}{\tilde{\alpha}_{\mathbf{U}}(\boldsymbol{\mu})} \left(\sqrt{2} C_{\mathcal{F},\mathbf{U}}(\boldsymbol{\mu}) + \frac{\tilde{\gamma}_{\mathbf{U}}(\boldsymbol{\mu}) + \tilde{\alpha}_{\mathbf{U}}(\boldsymbol{\mu})}{\beta(\boldsymbol{\mu})} C_{\mathcal{F},P}(\boldsymbol{\mu}) \right), \\
\|p(\boldsymbol{\mu})\|_{X_{1,p}} &\leq \frac{1}{\alpha_{\mathcal{C}_1}(\boldsymbol{\mu})\beta(\boldsymbol{\mu})} \left[\sqrt{2} \left(1 + \frac{\tilde{\gamma}_{\mathbf{U}}(\boldsymbol{\mu})}{\tilde{\alpha}_{\mathbf{U}}(\boldsymbol{\mu})} \right) C_{\mathcal{F},\mathbf{U}}(\boldsymbol{\mu}) \right. \\
&\quad \left. + \frac{\tilde{\gamma}_{\mathbf{U}}(\boldsymbol{\mu})(\tilde{\gamma}_{\mathbf{U}}(\boldsymbol{\mu}) + \tilde{\alpha}_{\mathbf{U}}(\boldsymbol{\mu}))}{\tilde{\alpha}_{\mathbf{U}}(\boldsymbol{\mu})\beta(\boldsymbol{\mu})} C_{\mathcal{F},P}(\boldsymbol{\mu}) \right],
\end{aligned}$$

with

$$\tilde{\alpha}_{\mathbf{U}}(\boldsymbol{\mu}) = \min_{i=1,2} \alpha_{\mathcal{C}_i}(\boldsymbol{\mu})\alpha_{\mathcal{A}_i}(\boldsymbol{\mu}), \tag{12}$$

$$\tilde{\gamma}_{\mathbf{U}}(\boldsymbol{\mu}) = 2 \max \left\{ \max_{i=1,2} \alpha_{\mathcal{C}_i}(\boldsymbol{\mu})\gamma_{\mathcal{A}_i}(\boldsymbol{\mu}), \max_{i=1,2} \alpha_{\mathcal{C}_i}(\boldsymbol{\mu})\gamma_{\mathcal{C}_i}(\boldsymbol{\mu}) \right\}, \tag{13}$$

$$\begin{aligned}
C_{\mathcal{F},\mathbf{U}}(\boldsymbol{\mu}) &= \max \left\{ \alpha_{\mathcal{C}_1}(\boldsymbol{\mu}) \|\mathcal{F}_{1,\mathbf{u}}(\cdot;\boldsymbol{\mu})\|_{X'_{1,\mathbf{u}}}, \alpha_{\mathcal{C}_2}(\boldsymbol{\mu}) \|\mathcal{F}_2(\cdot;\boldsymbol{\mu})\|_{X'_2} \right\}, \\
C_{\mathcal{F},P}(\boldsymbol{\mu}) &= \|\mathcal{F}_{1,p}(\cdot;\boldsymbol{\mu})\|_{X'_{1,p}}.
\end{aligned}$$

The results presented in Proposition 1 are obtained through the reformulation of the coupled problem (7) as a saddle-point problem. We introduce the respective notations here, since we will use them for error estimation in Section 7. With the product spaces $X_{\mathbf{U}} = X_{1,\mathbf{u}} \times X_2$ and $X_P = X_{1,p}$ we define the bilinear form $\mathcal{A}_{\mathbf{U}} : X_{\mathbf{U}} \times X_{\mathbf{U}} \times \mathcal{P} \to \mathbb{R}$ through

$$\begin{aligned}
\mathcal{A}_{\mathbf{U}}(\mathbf{V}, \mathbf{W}, \boldsymbol{\mu}) = \alpha_{\mathcal{C}_1}(\boldsymbol{\mu}) &\left(\mathcal{A}_1(\mathbf{v}, \mathbf{w}; \boldsymbol{\mu}) + \mathcal{C}_1(\psi, \mathbf{w}; \boldsymbol{\mu}) \right) \\
&+ \alpha_{\mathcal{C}_2}(\boldsymbol{\mu}) \left(\mathcal{A}_2(\psi, \chi; \boldsymbol{\mu}) + \mathcal{C}_2(\mathbf{v}, \chi; \boldsymbol{\mu}) \right),
\end{aligned}$$

and the linear form $\mathcal{F}_{\mathbf{U}} : X_{\mathbf{U}} \times \mathcal{P} \to \mathbb{R}$ through

$$\mathcal{F}_{\mathbf{U}}(\mathbf{V}, \boldsymbol{\mu}) = \alpha_{\mathcal{C}_1}(\boldsymbol{\mu})\mathcal{F}_{1,\mathbf{u}}(\mathbf{v}; \boldsymbol{\mu}) + \alpha_{\mathcal{C}_2}(\boldsymbol{\mu})\mathcal{F}_2(\psi; \boldsymbol{\mu}),$$

where $\mathbf{V} = (\mathbf{v}, \psi)$ and $\mathbf{W} = (\mathbf{w}, \chi)$. Additionally we define $\mathcal{B} : X_{\mathbf{U}} \times X_P \times \mathcal{P} \to \mathbb{R}$ through

$$\mathcal{B}(\mathbf{V}, Q; \boldsymbol{\mu}) = \alpha_{\mathcal{C}_1}(\boldsymbol{\mu})\mathcal{B}_1(\mathbf{v}, q; \boldsymbol{\mu})$$

and $\mathcal{F}_P : X_P \times \mathcal{P} \to \mathbb{R}$ through

$$\mathcal{F}_P(Q;\boldsymbol{\mu}) = \alpha_{\mathcal{C}_1}(\boldsymbol{\mu})\mathcal{F}_{1,p}(q;\boldsymbol{\mu}),$$

where $Q = q$. Then problem (7) can be written as

$$\begin{aligned} \mathcal{A}_{\mathbf{U}}(\mathbf{U}(\boldsymbol{\mu}),\mathbf{V};\boldsymbol{\mu}) + \mathcal{B}(\mathbf{V},P(\boldsymbol{\mu});\boldsymbol{\mu}) &= \mathcal{F}_{\mathbf{U}}(\mathbf{V};\boldsymbol{\mu}), \qquad \forall \mathbf{V} \in X_{\mathbf{U}}, \qquad (14a)\\ \mathcal{B}(\mathbf{U}(\boldsymbol{\mu}),Q;\boldsymbol{\mu}) &= \mathcal{F}_P(Q;\boldsymbol{\mu}), \qquad \forall Q \in X_P, \qquad (14b)\end{aligned}$$

where $\mathbf{U}(\boldsymbol{\mu}) = (\mathbf{u}(\boldsymbol{\mu}),\varphi(\boldsymbol{\mu})) \in X_{\mathbf{U}}$ and $P(\boldsymbol{\mu}) = p(\boldsymbol{\mu}) \in X_P$ for $\boldsymbol{\mu} \in \mathcal{P}$. Let $\gamma_{\mathbf{U}}(\boldsymbol{\mu})$ be the continuity and $\alpha_{\mathbf{U}}(\boldsymbol{\mu})$ the coercivity constant of $\mathcal{A}_{\mathbf{U}}(\cdot,\cdot;\boldsymbol{\mu})$. In fact, they are well-defined due to $\alpha_{\mathbf{U}}(\boldsymbol{\mu}) >= \tilde{\alpha}_{\mathbf{U}}(\boldsymbol{\mu})$ and $\gamma_{\mathbf{U}}(\boldsymbol{\mu}) <= \tilde{\gamma}_{\mathbf{U}}(\boldsymbol{\mu})$.

We continue with condensing the notation by introducing the product space $X = X_{1,\mathbf{u}} \times X_{1,p} \times X_2$ and define the bilinear form $\mathcal{A} : X \times X \times \mathcal{P} \to \mathbb{R}$ through

$$\begin{aligned}\mathcal{A}(\mathbf{V},\mathbf{W};\boldsymbol{\mu}) := \alpha_{\mathcal{C}_1}(\boldsymbol{\mu})(\mathcal{A}_1(\mathbf{v},\mathbf{w};\boldsymbol{\mu}) + \mathcal{B}_1(\mathbf{w},q;\boldsymbol{\mu}) + \mathcal{B}_1(\mathbf{v},r;\boldsymbol{\mu}) + \mathcal{C}_1(\psi,\mathbf{w};\boldsymbol{\mu}))\\ + \alpha_{\mathcal{C}_2}(\boldsymbol{\mu})(\mathcal{A}_2(\psi,\chi;\boldsymbol{\mu}) + \mathcal{C}_2(\mathbf{v},\chi;\boldsymbol{\mu}))\end{aligned}$$

and the linear form $\mathcal{F} : X \times \mathcal{P} \to \mathbb{R}$ through

$$\mathcal{F}(\mathbf{V};\boldsymbol{\mu}) := \alpha_{\mathcal{C}_1}(\boldsymbol{\mu})(\mathcal{F}_{1,\mathbf{u}}(\mathbf{v};\boldsymbol{\mu}) + \mathcal{F}_{1,p}(q;\boldsymbol{\mu})) + \alpha_{\mathcal{C}_2}(\boldsymbol{\mu})\mathcal{F}_2(\psi;\boldsymbol{\mu}),$$

where $\mathbf{V} = (\mathbf{v},q,\psi)$ and $\mathbf{W} = (\mathbf{w},r,\chi)$. Then problem (7) can be written in its most general form of a non-coercive problem as

$$\mathcal{A}(\hat{\mathbf{U}}(\boldsymbol{\mu}),\mathbf{V};\boldsymbol{\mu}) = \mathcal{F}(\mathbf{V};\boldsymbol{\mu}), \qquad \forall \mathbf{V} \in X, \qquad (15)$$

where $\hat{\mathbf{U}}(\boldsymbol{\mu}) = (\mathbf{u}(\boldsymbol{\mu}),p(\boldsymbol{\mu}),\varphi(\boldsymbol{\mu})) \in X$ for $\boldsymbol{\mu} \in \mathcal{P}$. The bilinear form $\mathcal{A}$ is known to fulfill the following (Babuška) inf-sup condition under our assumptions [37]:

$$\beta_{\mathcal{A}}(\boldsymbol{\mu}) := \inf_{\mathbf{V}\in X} \sup_{\mathbf{W}\in X} \frac{\mathcal{A}(\mathbf{V},\mathbf{W};\boldsymbol{\mu})}{\|\mathbf{V}\|_X \|\mathbf{W}\|_X} \geq \beta_{\mathcal{A}}^0 > 0, \qquad \forall \boldsymbol{\mu} \in \mathcal{P}. \qquad (16)$$

5 Finite Element approximation

We use the *Galerkin* FE method to approximate the solution to problem (7), the resulting FE solution acts as a reference for our RB approximation. Given a fine triangulation of Ω_{ref} which is consistent with the coarse triangulation introduced in Section 3, we use *Taylor-Hood* Elements on $\Omega_{1,\text{ref}}$ and second order *Lagrangian* Elements on $\Omega_{2,\text{ref}}$ to obtain FE approximation spaces [7] denoted by $X_{1,\mathbf{u}}^{\mathcal{N}} \subset X_{1,\mathbf{u}}$, $X_{1,\mathbf{u}}^{\text{in},\mathcal{N}} \subset X_{1,\mathbf{u}}^{\text{in}}$, $X_{1,p}^{\mathcal{N}} \subset X_{1,p}$, $X_2^{\mathcal{N}} \subset X_2$ and $X_2^{\text{b},\mathcal{N}} \subset X_2^{\text{b}}$ with finite dimensions $\mathcal{N}_{1,\mathbf{u}}$, $\mathcal{N}_{1,\mathbf{u}}^{\text{in}}$, $\mathcal{N}_{1,p}$, $\mathcal{N}_2$ and $\mathcal{N}_2^{\text{b}}$, respectively. The discrete problem on Ω_{ref} then reads: For $\boldsymbol{\mu} \in \mathcal{P}$ find $(\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu}),p^{\mathcal{N}}(\boldsymbol{\mu}),\varphi^{\mathcal{N}}(\boldsymbol{\mu})) \in X_{1,\mathbf{u}}^{\mathcal{N}} \times X_{1,p}^{\mathcal{N}} \times X_2^{\mathcal{N}}$ such that

$$\left.\begin{aligned}\mathcal{A}_1(\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu}),\mathbf{v}^{\mathcal{N}};\boldsymbol{\mu}) + \mathcal{B}_1(\mathbf{v}^{\mathcal{N}},p^{\mathcal{N}}(\boldsymbol{\mu});\boldsymbol{\mu})&\\ + \mathcal{C}_1(\varphi^{\mathcal{N}}(\boldsymbol{\mu}),\mathbf{v}^{\mathcal{N}};\boldsymbol{\mu}) = \mathcal{F}_{1,\mathbf{u}}(\mathbf{v}^{\mathcal{N}};\boldsymbol{\mu}), &\quad \forall \mathbf{v}^{\mathcal{N}} \in X_{1,\mathbf{u}}^{\mathcal{N}},\\ \mathcal{B}_1(\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu}),q^{\mathcal{N}};\boldsymbol{\mu}) = \mathcal{F}_{1,p}(q^{\mathcal{N}};\boldsymbol{\mu}), &\quad \forall q^{\mathcal{N}} \in X_{1,p}^{\mathcal{N}},\\ \mathcal{A}_2(\varphi^{\mathcal{N}}(\boldsymbol{\mu}),\psi^{\mathcal{N}};\boldsymbol{\mu}) + \mathcal{C}_2(\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu}),\psi^{\mathcal{N}};\boldsymbol{\mu}) = \mathcal{F}_2(\psi^{\mathcal{N}};\boldsymbol{\mu}), &\quad \forall \psi^{\mathcal{N}} \in X_2^{\mathcal{N}},\end{aligned}\right\} \qquad (17)$$

According to Section 4 we also define the product spaces $X_{\mathbf{U}}^{\mathcal{N}} = X_{1,\mathbf{u}}^{\mathcal{N}} \times X_2^{\mathcal{N}}$, $X_P^{\mathcal{N}} = X_{1,p}^{\mathcal{N}}$ and $X^{\mathcal{N}} = X_{1,\mathbf{u}}^{\mathcal{N}} \times X_{1,p}^{\mathcal{N}} \times X_2^{\mathcal{N}}$ and corresponding functions $\mathbf{U}^{\mathcal{N}}(\boldsymbol{\mu}) = (\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu}), \varphi^{\mathcal{N}}(\boldsymbol{\mu}))$, $P^{\mathcal{N}}(\boldsymbol{\mu}) = p^{\mathcal{N}}(\boldsymbol{\mu})$ and $\hat{\mathbf{U}}^{\mathcal{N}}(\boldsymbol{\mu}) = (\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu}), \varphi^{\mathcal{N}}(\boldsymbol{\mu}))$, which are solutions to respective condensed discrete problems.

We note that the properties stated in Section 4 hold true in the discrete setting. We adopt the notation of the continuity and coercivity constants $\gamma_{\mathcal{A}_i}^{\mathcal{N}}(\boldsymbol{\mu})$, $\alpha_{\mathcal{A}_i}^{\mathcal{N}}(\boldsymbol{\mu})$, $\gamma_{\mathcal{C}_i}^{\mathcal{N}}(\boldsymbol{\mu})$ for $i = 1, 2$ and

$$\beta^{\mathcal{N}}(\boldsymbol{\mu}) = \inf_{q^{\mathcal{N}} \in X_{1,p}^{\mathcal{N}}} \sup_{\mathbf{v}^{\mathcal{N}} \in X_{1,\mathbf{u}}^{\mathcal{N}}} \frac{\mathcal{B}_1(\mathbf{v}^{\mathcal{N}}, q^{\mathcal{N}}; \boldsymbol{\mu})}{\|\mathbf{v}^{\mathcal{N}}\|_{X_{1,\mathbf{u}}} \|q^{\mathcal{N}}\|_{X_{1,p}}} > 0, \qquad \forall \boldsymbol{\mu} \in \mathcal{P}. \tag{18}$$

Here, $\beta^{\mathcal{N}}(\boldsymbol{\mu}) > 0$ is ensured by using Taylor-Hood Elements. The inf-sup stability of the RB systems, that will be introduced in Section 6, is obtained by using pressure supremizers. For $\boldsymbol{\mu} \in \mathcal{P}$ the (pressure) supremizing operator is given through

$$\begin{aligned} &\mathcal{T}(\boldsymbol{\mu}) : X_{1,p}^{\mathcal{N}} \to X_{1,\mathbf{u}}^{\mathcal{N}} : q^{\mathcal{N}} \mapsto \mathcal{T}(\boldsymbol{\mu}) q^{\mathcal{N}} : \\ &\qquad (\mathcal{T}(\boldsymbol{\mu}) q^{\mathcal{N}}, \mathbf{v}^{\mathcal{N}}) = \mathcal{B}_1(\mathbf{v}^{\mathcal{N}}, q^{\mathcal{N}}; \boldsymbol{\mu}), \qquad \forall \mathbf{v}^{\mathcal{N}} \in X_{1,\mathbf{u}}^{\mathcal{N}}. \end{aligned} \tag{19}$$

Last but not least we introduce the notation $\gamma_{\mathbf{U}}^{\mathcal{N}}(\boldsymbol{\mu})$, $\alpha_{\mathbf{U}}^{\mathcal{N}}(\boldsymbol{\mu})$ and $\beta_{\mathcal{A}}^{\mathcal{N}}(\boldsymbol{\mu})$ in accordance to Section 4.

The RB procedure we present in the next section deals with so called interface modes, which are obtained by restricting discrete FE functions to the degrees of freedom on the interface. Accordingly, we define the trace operators

$$\begin{aligned} \gamma_1 : X_{1,\mathbf{u}}^{\mathcal{N}} \to H_{00}^{1/2}(\Gamma_{\mathrm{ref}}) : \mathbf{v}^{\mathcal{N}} \mapsto \mathbf{v}^{\mathcal{N}}|_{\Gamma_{\mathrm{ref}}} \cdot \mathbf{n}, \\ \gamma_2 : X_2^{\mathcal{N}} \to H^{1/2}(\Gamma_{\mathrm{ref}}) : \psi^{\mathcal{N}} \mapsto \psi^{\mathcal{N}}|_{\Gamma_{\mathrm{ref}}} \end{aligned}$$

and the resulting finite-dimensional function spaces

$$X_{1,\Gamma}^{\mathcal{N}} = \{\gamma_1 \mathbf{v}^{\mathcal{N}} : \mathbf{v}^{\mathcal{N}} \in X_{1,\mathbf{u}}^{\mathcal{N}}\}, \qquad X_{2,\Gamma}^{\mathcal{N}} = \{\gamma_2 \varphi^{\mathcal{N}} : \varphi^{\mathcal{N}} \in X_2^{\mathcal{N}}\}.$$

From the definition of the coupling functions $\mathcal{C}_i(\cdot, \cdot; \boldsymbol{\mu})$ it is clear that they are determined by above defined traces of their arguments. Thus $\mathcal{C}_i(\cdot, \cdot; \boldsymbol{\mu})$ can be evaluated at every $\nu^{\mathcal{N}} \in X_{1,\Gamma}^{\mathcal{N}}$ and $\eta^{\mathcal{N}} \in X_{2,\Gamma}^{\mathcal{N}}$ and we use the abbrevation $\mathcal{C}_1(\eta^{\mathcal{N}}, \cdot; \boldsymbol{\mu})$ for $\mathcal{C}_1(\psi^{\mathcal{N}}, \cdot; \boldsymbol{\mu})$, where $\psi^{\mathcal{N}} \in X_2^{\mathcal{N}}$ is an arbitrary function with $\gamma_2 \psi^{\mathcal{N}} = \eta^{\mathcal{N}}$ and $\mathcal{C}_2(\nu^{\mathcal{N}}, \cdot; \boldsymbol{\mu})$ for $\mathcal{C}_2(\mathbf{v}^{\mathcal{N}}, \cdot; \boldsymbol{\mu})$, where $\mathbf{v}^{\mathcal{N}} \in X_{1,\mathbf{u}}^{\mathcal{N}}$, $\gamma_1 \mathbf{v}^{\mathcal{N}} = \nu^{\mathcal{N}}$, respectively.

If the solution to (17) is known at the interface, it can be reconstructed by the following problem specific *lifting* operators:

$\mathcal{E}_1(\boldsymbol{\mu}) : X_{2,\Gamma}^{\mathcal{N}} \to X_{1,\mathbf{u}}^{\mathcal{N}} \times X_{1,p}^{\mathcal{N}} : \eta^{\mathcal{N}} \mapsto (\mathcal{E}_{1,\mathbf{u}}(\boldsymbol{\mu})\eta^{\mathcal{N}}, \mathcal{E}_{1,p}(\boldsymbol{\mu})\eta^{\mathcal{N}}) :$

$$\left.\begin{aligned} &\mathcal{A}_1(\mathcal{E}_{1,\mathbf{u}}(\boldsymbol{\mu})\eta^{\mathcal{N}}, \mathbf{v}^{\mathcal{N}}; \boldsymbol{\mu}) + \mathcal{B}_1(\mathbf{v}^{\mathcal{N}}, \mathcal{E}_{1,p}(\boldsymbol{\mu})\eta^{\mathcal{N}}; \boldsymbol{\mu}) \\ &\qquad = \mathcal{F}_{1,\mathbf{u}}(\mathbf{v}^{\mathcal{N}}; \boldsymbol{\mu}) - \mathcal{C}_1(\eta^{\mathcal{N}}, \mathbf{v}^{\mathcal{N}}; \boldsymbol{\mu}), \quad \forall \mathbf{v}^{\mathcal{N}} \in X_{1,\mathbf{u}}^{\mathcal{N}}, \\ &\mathcal{B}_1(\mathcal{E}_{1,\mathbf{u}}(\boldsymbol{\mu})\eta^{\mathcal{N}}, q^{\mathcal{N}}; \boldsymbol{\mu}) = \mathcal{F}_{1,p}(q^{\mathcal{N}}; \boldsymbol{\mu}), \quad \forall q^{\mathcal{N}} \in X_{1,p}^{\mathcal{N}}. \end{aligned}\right\} \tag{20}$$

Problem (20) is a (discrete) Stokes-Problem in Ω_1 with modified right-hand side.

$\mathcal{E}_2(\boldsymbol{\mu}) : X_{1,\Gamma}^{\mathcal{N}} \to X_2^{\mathcal{N}} : \nu^{\mathcal{N}} \mapsto \mathcal{E}_2(\boldsymbol{\mu})\nu^{\mathcal{N}} :$

$$\mathcal{A}_2(\mathcal{E}_2(\boldsymbol{\mu})\nu^{\mathcal{N}}, \psi^{\mathcal{N}}; \boldsymbol{\mu}) = \mathcal{F}_2(\psi^{\mathcal{N}}; \boldsymbol{\mu}) - \mathcal{C}_2(\nu^{\mathcal{N}}, \psi^{\mathcal{N}}; \boldsymbol{\mu}), \qquad \forall \psi^{\mathcal{N}} \in X_2^{\mathcal{N}}. \tag{21}$$

Problem (21) is a (discrete) porous-medium problem on Ω_2 with modified right-hand side. The solution to (17) is given through $\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu}) = \mathcal{E}_{1,\mathbf{u}}(\boldsymbol{\mu})\gamma_2\varphi^{\mathcal{N}}(\boldsymbol{\mu})$, $p^{\mathcal{N}}(\boldsymbol{\mu}) = \mathcal{E}_{1,p}(\boldsymbol{\mu})\gamma_2\varphi^{\mathcal{N}}(\boldsymbol{\mu})$ and $\varphi^{\mathcal{N}}(\boldsymbol{\mu}) = \mathcal{E}_2(\boldsymbol{\mu})\gamma_1\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu})$.

6 Reduced basis approximation

We seek to compute a global approximation of $\hat{\mathbf{U}}^{\mathcal{N}}(\boldsymbol{\mu}) \in X^{\mathcal{N}}$. The procedure we establish consists of three major steps:

6.1 Construct sets of interface modes $\Xi_{N,i,\Gamma} \subset X_\Gamma^{\mathcal{N}}$ for $i = 1, 2$.
6.2 Construct RB spaces $X_{N,1,\mathbf{u}} \subset X_{1,\mathbf{u}}^{\mathcal{N}}$, $X_{N,1,p} \subset X_{1,p}^{\mathcal{N}}$ and $X_{N,2} \subset X_2^{\mathcal{N}}$ on the subdomains for each variable from the interface modes.
6.3 For an incoming parameter value $\boldsymbol{\mu} \in \mathcal{P}$, compute the Galerkin projection of $\hat{\mathbf{U}}^{\mathcal{N}}(\boldsymbol{\mu})$ on $X_N = X_{N,1,\mathbf{u}} \times X_{N,1,p} \times X_{N,2}$.

The first two steps are part of the offline-phase. The third step comprises both offline and online computations. In the following subsections we explain those steps in detail. Further comments on the offline/online-decomposition are given in Subsection 6.4.

6.1 Interface modes

The interface modes are gained from global snapshots. To be more precise, the trace of the solution variable on Ω_2 is used as an interface mode for Ω_1 and vice versa: $\Xi_{N,1,\Gamma} = \{\gamma_2\varphi^{\mathcal{N}}(\boldsymbol{\mu}) : \boldsymbol{\mu} \in \mathcal{P}_{1,\mathrm{c}}\}$ and $\Xi_{N,2,\Gamma} = \{\gamma_1\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu}) : \boldsymbol{\mu} \in \mathcal{P}_{2,\mathrm{c}}\}$ where $\mathcal{P}_{i,\mathrm{c}} \subset \mathcal{P}$, $|\mathcal{P}_{i,\mathrm{c}}| = N_{i,\Gamma}$ for $i = 1, 2$ stem from a very coarse discretization of the parameter domain. Given those sets we introduce the notation $\xi_{N,i,\Gamma}^k, k = 1, \ldots, N_{i,\Gamma}$, $i = 1, 2$ for the interface modes (we omit the superscript $\mathcal{N}$ for the sake of readability), such that

$$\Xi_{N,i,\Gamma} = \left\{\xi_{N,i,\Gamma}^k, k = 1, \ldots, N_{i,\Gamma}\right\}, \qquad i = 1, 2.$$

The motivation of this construction is that the variation of the solutions to (17) on the interface Γ_{ref} is potentially smaller than on the whole domain Ω_{ref}. While using the classical RB approach the snapshots have to represent the solution on the whole domain accurately enough, here we are only aiming at the representation of the solutions traces. So the amount of global snapshots is expected to be reduced. In the next step (the generation of the RB spaces on the subdomains) only subproblems have to be solved, in contrast to global coupled problems in a "straightforward" RB procedure. This procedure also yields a certain flexibility in choosing between different interface mode options. One can think of solutions to a generalized eigenproblem on Γ_{ref} (see [9,13]) or *Fourier* interface functions (see [15]).

6.2 Subdomain RB spaces

Once the interface modes are given, the basis functions on the subdomains are obtained via the lifting operators introduced in Section 5. To simplify the reading we define the lifting of an interface mode via $\mathcal{E}_i^k(\boldsymbol{\mu}) := \mathcal{E}_i(\boldsymbol{\mu})\xi_{N,i,\Gamma}^k$ for $k = 1, \dots, N_{i,\Gamma}, i = 1, 2$. The parameter values for the liftings are chosen in an adaptive algorithm. For this, we use auxiliary RB problems on the subdomains. Suppose that on Ω_1 RB spaces $X_{N,1,\mathbf{u}} \subset X_{1,\mathbf{u}}^{\mathcal{N}}$ and $X_{N,1,p} \subset X_{1,p}^{\mathcal{N}}$ with dimensions $N_{1,\mathbf{u}} \ll \mathcal{N}_{1,\mathbf{u}}$ and $N_{1,p} \ll \mathcal{N}_{1,p}$ are given. We want to ensure that

$$\beta_N(\boldsymbol{\mu}) = \inf_{q \in X_{N,1,p}} \sup_{v \in X_{N,1,\mathbf{u}}} \frac{\mathcal{B}_1(\mathbf{v}, q; \boldsymbol{\mu})}{\|\mathbf{v}\|_{X_{1,\mathbf{u}}} \|q\|_{X_{1,p}}} > 0, \qquad \forall \boldsymbol{\mu} \in \mathcal{P} \tag{22}$$

plus $\inf_{\boldsymbol{\mu} \in \mathcal{P}} \beta_N(\boldsymbol{\mu}) > 0$. This inf-sup stability can be achieved by the so-called supremizer enrichment [10,28,29,31]. We use a supremizer option, for which stability in the sense of (22) was not demonstrated rigorously, but has been observed in many numerical test cases, including our test case. The procedure will be explained clearly in Algorithm 1. For $\boldsymbol{\mu} \in \mathcal{P}$ the RB approximation to $\mathcal{E}_1^k(\boldsymbol{\mu})$, $k = 1, \dots, N_{1,\Gamma}$ is defined through:

$\mathcal{E}_{N,1}^k(\boldsymbol{\mu}) = (\mathcal{E}_{N,1,\mathbf{u}}^k(\boldsymbol{\mu}), \mathcal{E}_{N,1,p}^k(\boldsymbol{\mu})) \in X_{N,1,\mathbf{u}} \times X_{N,1,p}$:

$$\left.\begin{aligned} \mathcal{A}_1(\mathcal{E}_{N,1,\mathbf{u}}^k(\boldsymbol{\mu}), \mathbf{v}; \boldsymbol{\mu}) + \mathcal{B}_1(\mathbf{v}, \mathcal{E}_{N,1,p}^k(\boldsymbol{\mu}); \boldsymbol{\mu}) & \\ = \mathcal{F}_{1,\mathbf{u}}(\mathbf{v}; \boldsymbol{\mu}) - \mathcal{C}_1(\xi_{N,1,\Gamma}^k, \mathbf{v}; \boldsymbol{\mu}), & \quad \forall \mathbf{v} \in X_{N,1,\mathbf{u}}, \\ \mathcal{B}_1(\mathcal{E}_{N,1,\mathbf{u}}^k(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = \mathcal{F}_{1,p}(q; \boldsymbol{\mu}), & \quad \forall q \in X_{N,1,p}. \end{aligned}\right\} \tag{23}$$

Suppose that on Ω_2 a RB space $X_{N,2} \subset X_2^{\mathcal{N}}$ with dimension $N_2 \ll \mathcal{N}_2$ is given. For $\boldsymbol{\mu} \in \mathcal{P}$ the RB approximation to $\mathcal{E}_2^k(\boldsymbol{\mu})$, $k = 1, \dots, N_{2,\Gamma}$ is defined through:

$\mathcal{E}_{N,2}^k(\boldsymbol{\mu}) \in X_{N,2}$:

$$\mathcal{A}_2(\mathcal{E}_{N,2}^k(\boldsymbol{\mu}), \psi; \boldsymbol{\mu}) = \mathcal{F}_2(\psi; \boldsymbol{\mu}) - \mathcal{C}_2(\xi_{N,2,\Gamma}^k, \psi; \boldsymbol{\mu}), \qquad \forall \psi \in X_{N,2}. \tag{24}$$

We now can formulate the algorithm in which parameter values are chosen to gradually extend the subdomain RB spaces. It is based on the Greedy algorithm for RB approximations [36] and especially on the Greedy procedures realized in [15] in the context of parametric boundary conditions.

Algorithm 1 (Subdomain Greedy) Let $\Xi_{N,i,\Gamma}$, $i = 1, 2$ and a finite training set $\mathcal{P}_{\mathrm{f}} \subset \mathcal{P}$ be given. In addition, let initial RB spaces $X_{N,1,\mathbf{u}}$, $X_{N,1,p}$ and $X_{N,2}$ and initial extended parameter value sets $\mathcal{S}_i \subset \mathcal{P} \times \{1, \dots, N_{i,\Gamma}\}$ for $i = 1, 2$ be given. We further assume that local error indicators $\mathcal{I}_i^k(\boldsymbol{\mu})$, $k = 1, \dots, N_{i,\Gamma}$, $i = 1, 2$ and a global error indicator $\mathcal{I}(\boldsymbol{\mu})$ are given. With tolerances $\epsilon_i > 0$ and $\epsilon > 0$ we perform

do
  $m = \max_{\boldsymbol{\mu}\in\mathcal{P}_\mathrm{f}} \mathcal{I}(\boldsymbol{\mu})$
  for $i = 1, 2$
    $m_i = \max_{(\boldsymbol{\mu},k)\in\mathcal{P}_\mathrm{f}\times\{1,\dots,N_{i,\Gamma}\}} \mathcal{I}_i^k(\boldsymbol{\mu})$
    if $m_i > \epsilon_i$ and $m > \epsilon$
      $(\boldsymbol{\mu}_i^*, k_i^*) = \arg\max_{(\boldsymbol{\mu},k)\in\mathcal{P}_\mathrm{f}\times\{1,\dots,N_{i,\Gamma}\}} \mathcal{I}_i^k(\boldsymbol{\mu})$
      $\mathcal{S}_i \leftarrow \mathcal{S}_i \cup \{(\boldsymbol{\mu}_i^*, k_i^*)\}$
      $(i = 1)$: $X_{N,1,\mathbf{u}} \leftarrow X_{N,1,\mathbf{u}} \oplus \operatorname{span}\{\mathcal{E}_{1,\mathbf{u}}^{k_1^*}(\boldsymbol{\mu}_1^*), \mathcal{T}(\boldsymbol{\mu}_1^*)\mathcal{E}_{1,p}^{k_1^*}(\boldsymbol{\mu}_1^*)\}$
        $X_{N,1,p} \leftarrow X_{N,1,p} \oplus \operatorname{span}\{\mathcal{E}_{1,p}^{k_1^*}(\boldsymbol{\mu}_1^*)\}$
      $(i = 2)$: $X_{N,2} \leftarrow X_{N,2} \oplus \operatorname{span}\{\mathcal{E}_2^{k_2^*}(\boldsymbol{\mu}_2^*)\}$
    end
  end
while $(m_1 > \epsilon_1$ or $m_2 > \epsilon_2)$ and $(m > \epsilon)$.

Here, $\mathcal{T}$ denotes the supremizer operator defined in (19). The error indicators $\mathcal{I}_i^k$ and $\mathcal{I}$ will be introduced in Section 7.

Remark 2 The initialization of the algorithm requires generating $\Xi_{N,i,\Gamma}$, $i = 1, 2$ by computing global solutions to (17). The subdomain RB spaces could be initialized with $\emptyset$ or some random snapshots. Note that the *for* loop can be processed in parallel since the subdomain computations are "decoupled". There is no guarantee that $m \leq \epsilon$ when the algorithm terminates. Obviously it will not be the case if the ϵ_i's are chosen too large. But it may also be caused by a low number of interface modes.

6.3 Global RB approximation

For $\boldsymbol{\mu} \in \mathcal{P}$ the solution $\hat{\mathbf{U}}^{\mathcal{N}}(\boldsymbol{\mu})$ of (7) is approximated by the solution $\hat{\mathbf{U}}_N(\boldsymbol{\mu}) = (\mathbf{u}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu}), \varphi_N(\boldsymbol{\mu})) \in X_{N,1,\mathbf{u}} \times X_{N,1,p} \times X_{N,2}$ of the following problem:

$$\left.\begin{aligned} \mathcal{A}_1(\mathbf{u}_N(\boldsymbol{\mu}), \mathbf{v}; \boldsymbol{\mu}) + \mathcal{B}_1(\mathbf{v}, p_N(\boldsymbol{\mu}); \boldsymbol{\mu}) & \\ + \mathcal{C}_1(\varphi_N(\boldsymbol{\mu}), \mathbf{v}; \boldsymbol{\mu}) = \mathcal{F}_{1,\mathbf{u}}(\mathbf{v}; \boldsymbol{\mu}), & \quad \forall \mathbf{v} \in X_{N,1,\mathbf{u}}, \\ \mathcal{B}_1(\mathbf{u}_N(\boldsymbol{\mu}), q; \boldsymbol{\mu}) = \mathcal{F}_{1,p}(q; \boldsymbol{\mu}), & \quad \forall q \in X_{N,1,p}, \\ \mathcal{A}_2(\varphi_N(\boldsymbol{\mu}), \psi; \boldsymbol{\mu}) + \mathcal{C}_2(\mathbf{u}_N(\boldsymbol{\mu}), \psi; \boldsymbol{\mu}) = \mathcal{F}_2(\psi; \boldsymbol{\mu}), & \quad \forall \psi \in X_{N,2}. \end{aligned}\right\} \tag{25}$$

The wellposedness of problem (25) is ensured by the assumptions of Section 4 and (22). The resulting linear system has reduced size and will be dense in general. In order to improve the stability we apply the *Gram-Schmidt* orthonormalization procedure to each basis. More details on the computational procedure and costs are given in the next section.

6.4 Offline/online-procedure

In Section 3 we stated that the system (7) and consequently (25) is an affinely parametrized system. Without that, assembling the corresponding linear system would involve computations of high complexity ($\mathcal{O}(\mathcal{N})$, $\mathcal{N} := \mathcal{N}_{1,\mathbf{u}} + \mathcal{N}_{1,p} + \mathcal{N}_2$).

But, given for example the decomposition (8) and a basis $\{\xi^l_{N,1,\mathbf{u}} \mid l = 1, \dots, N_{1,\mathbf{u}}\}$ of $X_{N,1,\mathbf{u}}$, the assembly of the matrix block $\underline{A}_{N,1}(\boldsymbol{\mu}) \in \mathbb{R}^{N_{1,\mathbf{u}} \times N_{1,\mathbf{u}}}$,

$$\big(\underline{A}_{N,1}(\boldsymbol{\mu})\big)_{ij} = \mathcal{A}_1(\xi^j_{N,1,\mathbf{u}}, \xi^i_{N,1,\mathbf{u}}; \boldsymbol{\mu}),$$

can be done in an offline/online-procedure. In the offline-phase matrix components $\underline{A}^q_{N,1} \in \mathbb{R}^{N_{1,\mathbf{u}} \times N_{1,\mathbf{u}}}$ for $q = 1, \dots, Q_{\mathcal{A}_1}$ are computed:

$$\Big(\underline{A}^q_{N,1}\Big)_{ij} = \mathcal{A}^q_1(\xi^j_{N,1,\mathbf{u}}, \xi^i_{N,1,\mathbf{u}}),$$

and in the online-phase the affine decomposition $\underline{A}_{N,1}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_{\mathcal{A}_1}} \Theta^q_{\mathcal{A}_1}(\boldsymbol{\mu}) \underline{A}^q_{N,1}$ is evaluated (for every parameter value $\boldsymbol{\mu}$). The online cost for this example is $\mathcal{O}(Q_{\mathcal{A}_1} N^2_{1,\mathbf{u}})$ and we recall that $N_{1,\mathbf{u}} \ll \mathcal{N}_{1,\mathbf{u}}$. The number of components $Q_{\mathcal{A}_1}$ is also expected to be quite low ($Q_{\mathcal{A}_1} \lesssim N_{1,\mathbf{u}}$). Analogous procedures are done for all matrix and vector blocks of system (25), for the assembly of systems (23), (24) and also for the computation of the residual norms as parts of the error estimators introduced in Section 7. The right-hand side of (23) and (24) depends on the interface mode index k. Given $\mathcal{C}_1(\psi, \mathbf{v}; \boldsymbol{\mu}) = \sum_{q=1}^{Q_{\mathcal{C}_1}} \Theta^q_{\mathcal{C}_1}(\boldsymbol{\mu}) \mathcal{C}^q_1(\psi, \mathbf{v})$ we compute offline the matrix components $\underline{C}^q_{N,1} \in \mathbb{R}^{N_{1,\mathbf{u}} \times N_{1,\Gamma}}$ for $q = 1, \dots, Q_{\mathcal{C}_1}$ via

$$\Big(\underline{C}^q_{N,1}\Big)_{ij} = \mathcal{C}^q_1(\xi^j_{N,1,\Gamma}, \xi^i_{N,1,\mathbf{u}}),$$

compute online $\underline{C}_{N,1}(\boldsymbol{\mu}) = \sum_{q=1}^{Q_{\mathcal{C}_1}} \Theta^q_{\mathcal{C}_1}(\boldsymbol{\mu}) \underline{C}^q_{N,1}$ and get the requested part of the right hand side by extracting a column of the matrix. In this way the computation of all liftings for $k = 1, \dots, N_{1,\Gamma}$ and fixed $\boldsymbol{\mu} \in \mathcal{P}$ (which is required in Algorithm 1) is very efficient.

The sizes of the linear systems that are solved during the online-phase are $N_{1,\mathbf{u}} + N_{1,p}$ for (23), N_2 for (24) and $N := N_{1,\mathbf{u}} + N_{1,p} + N_2$ for (25). Note that if on each subdomain $\tilde{N}$ parameter values have been chosen to build the bases, we have $N_{1,\mathbf{u}} = 2\tilde{N}$, $N_{1,p} = \tilde{N}$ and $N_2 = \tilde{N}$ due to the supremizer enrichment. So the size of the global RB system is $4\tilde{N}$.

7 A-posteriori error estimation

For convenience we first introduce some notations that will occur often in this section. We denote by z^{LB} a computable lower bound and by z^{UB} a computable upper bound of any respective constant quantity $z^{\mathcal{N}}$. For any vector space Z, we denote by Z' the dual space of Z.

As error indicators $\mathcal{I}^k_i(\boldsymbol{\mu})$ and $\mathcal{I}(\boldsymbol{\mu})$ for Algorithm 1 we use error estimates for the following errors:

$$e^k_{N,i}(\boldsymbol{\mu}) := \|\mathcal{E}^k_i(\boldsymbol{\mu}) - \mathcal{E}^k_{N,i}(\boldsymbol{\mu})\|_{X_i} \quad \text{and} \qquad \mathbf{e}_N(\boldsymbol{\mu}) := \|\hat{\mathbf{U}}^{\mathcal{N}}(\boldsymbol{\mu}) - \hat{\mathbf{U}}_N(\boldsymbol{\mu})\|_X.$$

Here and in the following we denote $X_1 = X_{1,\mathbf{u}} \times X_{1,p}$, $X^{\mathcal{N}}_1 = X^{\mathcal{N}}_{1,\mathbf{u}} \times X^{\mathcal{N}}_{1,p}$ and $X_{N,1} = X_{N,1,\mathbf{u}} \times X_{N,1,p}$. The usage of the errors itself would make an exhaustive search for the maximal error unrealistic, since $\mathcal{P}_{\mathrm{f}}$ is a fine discretization of $\mathcal{P}$. Thus we seek for fastly evaluable a-posteriori error estimates. In the case of the local

errors we rely on the well-known theory for coercive [30] and non-coercive PDE's [36]. To this end, let $(\mathbf{v}, q) \in X_1$ and $\psi \in X_2$. The residual $\mathcal{R}_1^k : X_1 \times \mathcal{P} \to \mathbb{R}$ is defined through

$$\begin{aligned}\mathcal{R}_1^k((\mathbf{v}, q); \boldsymbol{\mu}) = \mathcal{F}_{1,\mathbf{u}}(\mathbf{v}; \boldsymbol{\mu}) - \mathcal{A}_1(\mathcal{E}_{N,1,\mathbf{u}}^k(\boldsymbol{\mu}), \mathbf{v}; \boldsymbol{\mu}) - \mathcal{B}_1(\mathbf{v}, \mathcal{E}_{N,1,p}^k(\boldsymbol{\mu}); \boldsymbol{\mu}) \\ - \mathcal{C}_1(\xi_{N,1,\Gamma}^k, \mathbf{v}; \boldsymbol{\mu}) + \mathcal{F}_{1,p}(q; \boldsymbol{\mu}) - \mathcal{B}_1(\mathcal{E}_{N,1,\mathbf{u}}^k(\boldsymbol{\mu}), q; \boldsymbol{\mu}),\end{aligned}$$

and the residual $\mathcal{R}_2^k : X_2 \times \mathcal{P} \to \mathbb{R}$ is defined through

$$\mathcal{R}_2^k(\psi; \boldsymbol{\mu}) = \mathcal{F}_2(\psi; \boldsymbol{\mu}) - \mathcal{A}_2(\mathcal{E}_{N,2}^k(\boldsymbol{\mu}), \psi; \boldsymbol{\mu}) - \mathcal{C}_2(\xi_{N,2,\Gamma}^k, \psi; \boldsymbol{\mu}).$$

We also require the inf-sup constant of the Stokes system:

$$\begin{aligned}\tilde{\beta}^{\mathcal{N}}(\boldsymbol{\mu}) := \inf_{(\mathbf{v}^{\mathcal{N}}, q^{\mathcal{N}}) \in X_1^{\mathcal{N}}} \sup_{(\mathbf{w}^{\mathcal{N}}, r^{\mathcal{N}}) \in X_1^{\mathcal{N}}} \Bigg(\frac{\mathcal{A}_1(\mathbf{v}^{\mathcal{N}}, \mathbf{w}^{\mathcal{N}}; \boldsymbol{\mu}) + \mathcal{B}_1(\mathbf{w}^{\mathcal{N}}, q^{\mathcal{N}}; \boldsymbol{\mu})}{\|(\mathbf{v}^{\mathcal{N}}, q^{\mathcal{N}})\|_{X_1} \|(\mathbf{w}^{\mathcal{N}}, r^{\mathcal{N}})\|_{X_1}} \\ + \frac{\mathcal{B}_1(\mathbf{v}^{\mathcal{N}}, r^{\mathcal{N}}; \boldsymbol{\mu})}{\|(\mathbf{v}^{\mathcal{N}}, q^{\mathcal{N}})\|_{X_1} \|(\mathbf{w}^{\mathcal{N}}, r^{\mathcal{N}})\|_{X_1}} \Bigg).\end{aligned}$$

Proposition 2 [30,36] *For $\boldsymbol{\mu} \in \mathcal{P}$ and $i = 1, 2$ the errors $e_{N,i}^k(\boldsymbol{\mu})$ can be bounded in the X_i-norms by the a-posteriori error estimates*

$$\begin{aligned}\|\mathbf{e}_{N,1}^k(\boldsymbol{\mu})\|_{X_1} \leq \Delta_{N,1}^k(\boldsymbol{\mu}) := \frac{\|\mathcal{R}_1^k(\cdot; \boldsymbol{\mu})\|_{X_1'}}{\tilde{\beta}^{\mathrm{LB}}(\boldsymbol{\mu})}, \\ \|e_{N,2}^k(\boldsymbol{\mu})\|_{X_2} \leq \Delta_{N,2}^k(\boldsymbol{\mu}) := \frac{\|\mathcal{R}_2^k(\cdot; \boldsymbol{\mu})\|_{X_2'}}{\alpha_{\mathcal{A}_2}^{\mathrm{LB}}(\boldsymbol{\mu})}.\end{aligned}$$

Having these lifting error bounds, we continue with the global error analysis. A first ansatz to estimate the global error $\mathbf{e}_N(\boldsymbol{\mu}) = \|\hat{\mathbf{U}}^{\mathcal{N}}(\boldsymbol{\mu}) - \hat{\mathbf{U}}_N(\boldsymbol{\mu})\|_X$ is based on the representation (15) of the global problem (or rather its discrete form) and the corresponding stability factor $\beta_{\mathcal{A}}^{\mathcal{N}}(\boldsymbol{\mu})$. We define the global residual $\mathcal{R} : X \times \mathcal{P} \to \mathbb{R}$ through

$$\mathcal{R}(\mathbf{V}, \boldsymbol{\mu}) := \mathcal{F}(\mathbf{V}; \boldsymbol{\mu}) - \mathcal{A}(\hat{\mathbf{U}}_N(\boldsymbol{\mu}), \mathbf{V}; \boldsymbol{\mu}).$$

Then the following result holds:

Proposition 3 [26,29] *The error $\mathbf{e}_N(\boldsymbol{\mu})$ can be bounded in the X-norm by the a-posteriori error estimate*

$$\|\mathbf{e}_N(\boldsymbol{\mu})\|_X \leq \Delta_N(\boldsymbol{\mu}) := \frac{\|\mathcal{R}(\cdot; \boldsymbol{\mu})\|_{X'}}{\beta_{\mathcal{A}}^{\mathrm{LB}}(\boldsymbol{\mu})}.$$

The drawback of the estimator $\Delta_N(\boldsymbol{\mu})$ is that it does not take account of the possibility that the stability constants of the subproblems have different magnitudes. A critical issue is that $\beta_{\mathcal{A}}^{\mathrm{LB}}$ can be very small. Better results are obtained by a weighting of residual norms corresponding to the errors on the subdomains with constants related to the subproblems. In the following an error estimate is developed that partly solves this problem.

The second ansatz builds on the formulation (14) as a saddle-point problem. We define the residual $\mathcal{R}_{\mathbf{U}} : X_{\mathbf{U}} \times \mathcal{P} \to \mathbb{R}$ through

$$\mathcal{R}_{\mathbf{U}}(\mathbf{V};\boldsymbol{\mu}) = \mathcal{F}_{\mathbf{U}}(\mathbf{V};\boldsymbol{\mu}) - \mathcal{A}_{\mathbf{U}}(\mathbf{U}_N(\boldsymbol{\mu}),\mathbf{V};\boldsymbol{\mu}) - \mathcal{B}(\mathbf{V},P_N(\boldsymbol{\mu});\boldsymbol{\mu}),$$

where $\mathbf{U}_N(\boldsymbol{\mu}) = (\mathbf{u}_N(\boldsymbol{\mu}),\varphi_N(\boldsymbol{\mu}))$ and $P_N(\boldsymbol{\mu}) = p_N(\boldsymbol{\mu})$ is the reduced basis solution of problem (25). Additionally we define residuals $\mathcal{R}_{1,\mathbf{u}} : X_{1,\mathbf{u}} \times \mathcal{P} \to \mathbb{R}$, $\mathcal{R}_{1,p} : X_{1,p} \times \mathcal{P} \to \mathbb{R}$ and $\mathcal{R}_2 : X_2 \times \mathcal{P} \to \mathbb{R}$ through

$$\begin{aligned}
\mathcal{R}_{1,\mathbf{u}}(\mathbf{v};\boldsymbol{\mu}) &= \mathcal{F}_{1,\mathbf{u}}(\mathbf{v};\boldsymbol{\mu}) - \mathcal{A}_1(\mathbf{u}_N(\boldsymbol{\mu}),\mathbf{v};\boldsymbol{\mu}) - \mathcal{B}_1(\mathbf{v},p_N(\boldsymbol{\mu});\boldsymbol{\mu}) - \mathcal{C}_1(\varphi_N,\mathbf{v};\boldsymbol{\mu}),\\
\mathcal{R}_{1,p}(q;\boldsymbol{\mu}) &= \mathcal{F}_{1,p}(q;\boldsymbol{\mu}) - \mathcal{B}_1(\mathbf{u}_N(\boldsymbol{\mu}),q;\boldsymbol{\mu}),\\
\mathcal{R}_2(\psi;\boldsymbol{\mu}) &= \mathcal{F}_2(\psi;\boldsymbol{\mu}) - \mathcal{A}_2(\varphi_N(\boldsymbol{\mu}),\psi;\boldsymbol{\mu}) - \mathcal{C}_2(\mathbf{u}_N(\boldsymbol{\mu}),\psi;\boldsymbol{\mu}).
\end{aligned}$$

It holds $\mathcal{R}_{\mathbf{U}}(\mathbf{V};\boldsymbol{\mu}) = \alpha_{\mathcal{C}_1}(\boldsymbol{\mu})\mathcal{R}_{1,\mathbf{u}}(\mathbf{v};\boldsymbol{\mu}) + \alpha_{\mathcal{C}_2}(\boldsymbol{\mu})\mathcal{R}_2(\psi;\boldsymbol{\mu})$ for $\mathbf{V} = (\mathbf{v},\psi) \in X_{\mathbf{U}}$. With all the definitions we now introduce new error bounds, which extend the theory of error approximation for parametrized saddle-point problems of [10]. This means that the structure of the bounds is maintained, but the single components are refined due to our special subproblem structure.

Lemma 1 *The error $\mathbf{e}_{N,\mathbf{U}}(\boldsymbol{\mu}) = \mathbf{U}^{\mathcal{N}}(\boldsymbol{\mu}) - \mathbf{U}_N(\boldsymbol{\mu})$ can be bounded in the $X_{\mathbf{U}}$-norm by the a-posteriori error estimate $\|\mathbf{e}_{N,\mathbf{U}}(\boldsymbol{\mu})\|_{X_{\mathbf{U}}} \leq \Delta_{N,\mathbf{U}}(\boldsymbol{\mu})$ with*

$$\Delta_{N,\mathbf{U}}(\boldsymbol{\mu}) := \sqrt{\frac{2}{\alpha_{\mathbf{U}}^{\mathrm{LB}}(\boldsymbol{\mu})}} \max_{i=1,2} \Delta_{N,\mathbf{U},i}(\boldsymbol{\mu}) + \left(1 + \frac{\sqrt{2}\tilde{c}_1(\boldsymbol{\mu})}{\alpha_{\mathbf{U}}^{\mathrm{LB}}(\boldsymbol{\mu})}\right) \frac{\|\mathcal{R}_{1,p}(\cdot;\boldsymbol{\mu})\|_{X_{1,p}'}}{\beta^{\mathrm{LB}}(\boldsymbol{\mu})}, \tag{26}$$

with

$$\Delta_{N,\mathbf{U},1}(\boldsymbol{\mu}) := \sqrt{\frac{\alpha_{\mathcal{C}_1}(\boldsymbol{\mu})}{\alpha_{\mathcal{A}_1}^{\mathrm{LB}}(\boldsymbol{\mu})}}\|R_{1,\mathbf{u}}(\cdot;\boldsymbol{\mu})\|_{X_{1,\mathbf{u}}'}, \quad \Delta_{N,\mathbf{U},2}(\boldsymbol{\mu}) := \sqrt{\frac{\alpha_{\mathcal{C}_2}(\boldsymbol{\mu})}{\alpha_{\mathcal{A}_2}^{\mathrm{LB}}(\boldsymbol{\mu})}}\|R_2(\cdot;\boldsymbol{\mu})\|_{X_2'}$$

and

$$\tilde{c}_1(\boldsymbol{\mu}) = \max\left\{\alpha_{\mathcal{C}_1}(\boldsymbol{\mu})\gamma_{\mathcal{A}_1}^{\mathrm{UB}}(\boldsymbol{\mu}), \alpha_{\mathcal{C}_2}(\boldsymbol{\mu})\gamma_{\mathcal{C}_2}^{\mathrm{UB}}(\boldsymbol{\mu})\right\}.$$

The error $e_{N,P}(\boldsymbol{\mu}) = P^{\mathcal{N}}(\boldsymbol{\mu}) - P_N(\boldsymbol{\mu})$ can be bounded in the X_P-norm by the a-posteriori error estimate

$$\|e_{N,P}(\boldsymbol{\mu})\|_{X_P} \leq \Delta_{N,P}(\boldsymbol{\mu}) := \frac{\|\mathcal{R}_{1,\mathbf{u}}(\cdot;\boldsymbol{\mu})\|_{X_{1,\mathbf{u}}'}}{\beta^{\mathrm{LB}}(\boldsymbol{\mu})} + \sqrt{2}\tilde{c}_2(\boldsymbol{\mu})\frac{\Delta_{N,U}(\boldsymbol{\mu})}{\beta^{\mathrm{LB}}(\boldsymbol{\mu})}, \tag{27}$$

with

$$\tilde{c}_2(\boldsymbol{\mu}) = \max\left\{\gamma_{\mathcal{A}_1}^{\mathrm{UB}}(\boldsymbol{\mu}), \gamma_{\mathcal{C}_1}^{\mathrm{UB}}(\boldsymbol{\mu})\right\}.$$

Proof We define the linear operator $B(\boldsymbol{\mu}) : X_{1,\mathbf{u}}^{\mathcal{N}} \to X_{1,p}^{\mathcal{N}}$ through

$$(B(\boldsymbol{\mu})\mathbf{v}^{\mathcal{N}}, q^{\mathcal{N}}) = \mathcal{B}_1(\mathbf{v}^{\mathcal{N}}, q^{\mathcal{N}};\boldsymbol{\mu}), \qquad \forall \mathbf{v}^{\mathcal{N}} \in X_{1,\mathbf{u}}^{\mathcal{N}}, q \in X_{1,p}^{\mathcal{N}}.$$

The error $\mathbf{e}_{N,\mathbf{U}}(\boldsymbol{\mu}) \in X_{\mathbf{U}}^{\mathcal{N}}$ can be written as $\mathbf{e}_{N,\mathbf{U}}(\boldsymbol{\mu}) = (\mathbf{e}_{N,\mathbf{u}}(\boldsymbol{\mu}), e_{N,\varphi}(\boldsymbol{\mu}))$. Further, we exploit the unique decomposition $\mathbf{e}_{N,\mathbf{u}}(\boldsymbol{\mu}) = \mathbf{e}_{N,\mathbf{u}}^{0}(\boldsymbol{\mu}) + \mathbf{e}_{N,\mathbf{u}}^{\perp}(\boldsymbol{\mu})$,

where $\mathbf{e}^0_{N,\mathbf{u}}(\boldsymbol{\mu}) \in \ker(B(\boldsymbol{\mu}))$ and $\mathbf{e}^\perp_{N,\mathbf{u}}(\boldsymbol{\mu}) \in \ker(B(\boldsymbol{\mu}))^\perp$ and define $\mathbf{e}^0_{N,\mathbf{U}}(\boldsymbol{\mu}) = (\mathbf{e}^0_{N,\mathbf{u}}(\boldsymbol{\mu}), \mathbf{e}_{N,\varphi}(\boldsymbol{\mu}))$ and $\mathbf{e}^\perp_{N,\mathbf{U}}(\boldsymbol{\mu}) = (\mathbf{e}^\perp_{N,\mathbf{u}}(\boldsymbol{\mu}), 0)$. For the sake of readability we omit the $\boldsymbol{\mu}$-dependence in the following. It holds

$$\begin{aligned}\mathcal{A}_\mathbf{U}(\mathbf{e}^0_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}}) &= \mathcal{A}_\mathbf{U}(\mathbf{e}_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}}) - \mathcal{A}_\mathbf{U}(\mathbf{e}^\perp_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}}) \\ &= \mathcal{F}_\mathbf{U}(\mathbf{e}^0_{N,\mathbf{U}}) - \mathcal{A}_\mathbf{U}(\mathbf{U}_N, \mathbf{e}^0_{N,\mathbf{U}}) \\ &\quad - \mathcal{B}(\mathbf{e}^0_{N,\mathbf{U}}, P^\mathcal{N}) - \mathcal{A}_\mathbf{U}(\mathbf{e}^\perp_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}}).\end{aligned}$$

Here we use (14a). Now we add $0 = \mathcal{B}(\mathbf{e}^0_{N,\mathbf{U}}, P^\mathcal{N} - P_N)$ to the left hand side and get

$$\begin{aligned}\mathcal{A}_\mathbf{U}(\mathbf{e}^0_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}}) &= \mathcal{R}_\mathbf{U}(\mathbf{e}^0_{N,\mathbf{U}}) - \mathcal{A}_\mathbf{U}(\mathbf{e}^\perp_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}}) \\ &= \alpha_{\mathcal{C}_1}\mathcal{R}_{1,\mathbf{u}}(\mathbf{e}^0_{N,\mathbf{u}}) + \alpha_{\mathcal{C}_2}\mathcal{R}_2(e_{N,\varphi}) - \mathcal{A}_\mathbf{U}(\mathbf{e}^\perp_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}}). \qquad (28)\end{aligned}$$

The residual terms can be estimated by

$$\begin{aligned}&\alpha_{\mathcal{C}_1}\mathcal{R}_{1,\mathbf{u}}(\mathbf{e}^0_{N,\mathbf{u}}) + \alpha_{\mathcal{C}_2}\mathcal{R}_2(e_{N,\varphi}) \\ &\quad\le \alpha_{\mathcal{C}_1}\|\mathcal{R}_{1,\mathbf{u}}(\cdot)\|_{X'_{1,\mathbf{u}}}\|\mathbf{e}^0_{N,\mathbf{u}}\|_{X_{1,\mathbf{u}}} + \alpha_{\mathcal{C}_2}\|\mathcal{R}_2(\cdot)\|_{X'_2}\|e_{N,\varphi}\|_{X_2} \\ &\quad\le \sqrt{\frac{\alpha_{\mathcal{C}_1}}{\alpha^{\mathrm{LB}}_{\mathcal{A}_1}}}\|\mathcal{R}_{1,\mathbf{u}}(\cdot)\|_{X'_{1,\mathbf{u}}}\sqrt{\alpha_{\mathcal{C}_1}\mathcal{A}_1(\mathbf{e}^0_{N,\mathbf{u}}, \mathbf{e}^0_{N,\mathbf{u}})} \\ &\qquad + \sqrt{\frac{\alpha_{\mathcal{C}_2}}{\alpha^{\mathrm{LB}}_{\mathcal{A}_2}}}\|\mathcal{R}_2(\cdot)\|_{X'_2}\sqrt{\alpha_{\mathcal{C}_2}\mathcal{A}_2(e_{N,\varphi}, e_{N,\varphi})} \\ &\quad\le \max_{i=1,2}\Delta_{N,\mathbf{U},i}\left(\sqrt{\alpha_{\mathcal{C}_1}\mathcal{A}_1(\mathbf{e}^0_{N,\mathbf{u}}, \mathbf{e}^0_{N,\mathbf{u}})} + \sqrt{\alpha_{\mathcal{C}_2}\mathcal{A}_2(e_{N,\varphi}, e_{N,\varphi})}\right) \\ &\quad\le \sqrt{2}\max_{i=1,2}\Delta_{N,\mathbf{U},i}\sqrt{\mathcal{A}_\mathbf{U}(\mathbf{e}^0_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}})}.\end{aligned}$$

The last line follows from $a + b \le \sqrt{2}\sqrt{a^2 + b^2}$ for $a, b > 0$, the definitions of $\mathcal{A}_\mathbf{U}$, $\mathbf{e}^0_{N,\mathbf{U}}$ and (11). The last term in (28) can be estimated via

$$\begin{aligned}-\mathcal{A}_\mathbf{U}(\mathbf{e}^\perp_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}}) &= -\alpha_{\mathcal{C}_1}\mathcal{A}_1(\mathbf{e}^\perp_{N,\mathbf{u}}, \mathbf{e}^0_{N,\mathbf{u}}) - \alpha_{\mathcal{C}_2}\mathcal{C}_2(\mathbf{e}^\perp_{N,\mathbf{u}}, e_{N,\varphi}) \\ &\le \alpha_{\mathcal{C}_1}\gamma^{\mathrm{UB}}_{\mathcal{A}_1}\|\mathbf{e}^\perp_{N,\mathbf{u}}\|_{X_{1,\mathbf{u}}}\|\mathbf{e}^0_{N,\mathbf{u}}\|_{X_{1,\mathbf{u}}} \\ &\quad + \alpha_{\mathcal{C}_2}\gamma^{\mathrm{UB}}_{\mathcal{C}_2}\|\mathbf{e}^\perp_{N,\mathbf{u}}\|_{X_{1,\mathbf{u}}}\|e_{N,\varphi}\|_{X_2} \\ &\le \sqrt{2}\tilde{c}_1\|\mathbf{e}^\perp_{N,\mathbf{U}}\|_{X_\mathbf{U}}\|\mathbf{e}^0_{N,\mathbf{U}}\|_{X_\mathbf{U}} \\ &\le \frac{\sqrt{2}\tilde{c}_1}{\sqrt{\alpha^{\mathrm{LB}}_\mathbf{U}}}\|\mathbf{e}^\perp_{N,\mathbf{U}}\|_{X_\mathbf{U}}\sqrt{\mathcal{A}_\mathbf{U}(\mathbf{e}^0_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}})}.\end{aligned}$$

Combining those two results and dividing by $\sqrt{\mathcal{A}_\mathbf{U}(\mathbf{e}^0_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}})}$ yields

$$\sqrt{\mathcal{A}_\mathbf{U}(\mathbf{e}^0_{N,\mathbf{U}}, \mathbf{e}^0_{N,\mathbf{U}})} \le \sqrt{2}\max_{i=1,2}\Delta_{N,\mathbf{U},i} + \frac{\sqrt{2}\tilde{c}_1}{\sqrt{\alpha^{\mathrm{LB}}_\mathbf{U}}}\|\mathbf{e}^\perp_{N,\mathbf{U}}\|_{X_\mathbf{U}}.$$

With the coercivity of $\mathcal{A}_\mathbf{U}$ we have

$$\|\mathbf{e}^0_{N,\mathbf{U}}\|_{X_\mathbf{U}} \le \sqrt{\frac{2}{\alpha^{\mathrm{LB}}_\mathbf{U}}}\max_{i=1,2}\Delta_{N,\mathbf{U},i} + \frac{\sqrt{2}\tilde{c}_1}{\alpha^{\mathrm{LB}}_\mathbf{U}}\|\mathbf{e}^\perp_{N,\mathbf{U}}\|_{X_\mathbf{U}}. \qquad (29)$$

Now we estimate the error $\mathbf{e}_{N,\mathbf{U}}^{\perp}$. Equation (18) implies [3,4]

$$\beta^{\mathcal{N}}(\boldsymbol{\mu}) = \inf_{\mathbf{v}^{\mathcal{N}} \in \ker(B(\boldsymbol{\mu}))^{\perp}} \sup_{q^{\mathcal{N}} \in X_{1,p}^{\mathcal{N}}} \frac{\mathcal{B}_1(\mathbf{v}^{\mathcal{N}}, q^{\mathcal{N}}; \boldsymbol{\mu})}{\|\mathbf{v}^{\mathcal{N}}\|_{X_{1,\mathbf{u}}} \|q^{\mathcal{N}}\|_{X_{1,p}}}.$$

With $\|\mathbf{e}_{N,\mathbf{U}}^{\perp}\|_{X_{\mathbf{U}}} = \|\mathbf{e}_{N,\mathbf{u}}^{\perp}\|_{X_{1,\mathbf{u}}}$ this yields

$$\begin{aligned}
\|\mathbf{e}_{N,\mathbf{U}}^{\perp}\|_{X_{\mathbf{U}}} &\leq \frac{1}{\beta^{\mathrm{LB}}} \sup_{q^{\mathcal{N}} \in X_{1,p}^{\mathcal{N}}} \frac{\mathcal{B}_1(\mathbf{e}_{N,\mathbf{u}}^{\perp}, q^{\mathcal{N}})}{\|q^{\mathcal{N}}\|_{X_{1,p}}} = \frac{1}{\beta^{\mathrm{LB}}} \sup_{q^{\mathcal{N}} \in X_{1,p}^{\mathcal{N}}} \frac{\mathcal{B}_1(\mathbf{e}_{N,\mathbf{u}}, q^{\mathcal{N}})}{\|q^{\mathcal{N}}\|_{X_{1,p}}} \\
&= \frac{1}{\beta^{\mathrm{LB}}} \sup_{q^{\mathcal{N}} \in X_{1,p}^{\mathcal{N}}} \frac{\mathcal{F}_{1,p}(q^{\mathcal{N}}) - \mathcal{B}_1(\mathbf{u}_N, q^{\mathcal{N}})}{\|q^{\mathcal{N}}\|_{X_{1,p}}} \\
&\leq \frac{1}{\beta^{\mathrm{LB}}} \|\mathcal{R}_{1,p}(\cdot)\|_{X_{1,p}'},
\end{aligned}$$

using $\mathbf{e}_{N,\mathbf{u}}^{0} \in \ker B(\boldsymbol{\mu})$ and (7b). Combining the last estimate with (29) via the triangle-inequality $\|\mathbf{e}_{N,\mathbf{U}}\|_{X_{\mathbf{U}}} \leq \|\mathbf{e}_{N,\mathbf{U}}^{0}\|_{X_{\mathbf{U}}} + \|\mathbf{e}_{N,\mathbf{U}}^{\perp}\|_{X_{\mathbf{U}}}$ yields the desired result (26).

Secondly, we define $e_{N,p} = p^{\mathcal{N}} - p_N$ and get from (10):

$$\|e_{N,P}\|_{X_P} = \|e_{N,p}\|_{X_{1,p}} \leq \frac{1}{\beta^{\mathrm{LB}}} \sup_{\mathbf{v}^{\mathcal{N}} \in X_{1,\mathbf{u}}^{\mathcal{N}}} \frac{\mathcal{B}_1(\mathbf{v}^{\mathcal{N}}, e_{N,p})}{\|\mathbf{v}^{\mathcal{N}}\|_{X_{1,\mathbf{u}}}} \tag{30}$$

Using (7a) we have

$$\begin{aligned}
\mathcal{B}_1(\mathbf{v}^{\mathcal{N}}, e_{N,p}) &= \mathcal{F}_{1,\mathbf{u}}(\mathbf{v}^{\mathcal{N}}) - \mathcal{A}_1(\mathbf{u}^{\mathcal{N}}, \mathbf{v}^{\mathcal{N}}) - \mathcal{C}_1(\varphi^{\mathcal{N}}, \mathbf{v}^{\mathcal{N}}) - \mathcal{B}_1(\mathbf{v}^{\mathcal{N}}, p_N) \\
&= \mathcal{R}_{1,\mathbf{u}}(\mathbf{v}^{\mathcal{N}}) - \mathcal{A}_1(\mathbf{u}^{\mathcal{N}} - \mathbf{u}_N, \mathbf{v}^{\mathcal{N}}) - \mathcal{C}_1(\varphi^{\mathcal{N}} - \varphi_N, \mathbf{v}^{\mathcal{N}}) \\
&\leq \Big(\|\mathcal{R}_{1,\mathbf{u}}(\cdot)\|_{X_{1,\mathbf{u}}'} + \gamma_{\mathcal{A}_1}^{\mathrm{UB}} \|\mathbf{u}^{\mathcal{N}} - \mathbf{u}_N\|_{X_{1,\mathbf{u}}} \\
&\quad + \gamma_{\mathcal{C}_1}^{\mathrm{UB}} \|\varphi^{\mathcal{N}} - \varphi_N\|_{X_2} \Big) \|\mathbf{v}^{\mathcal{N}}\|_{X_{1,\mathbf{u}}} \\
&\leq \Big(\|\mathcal{R}_{1,\mathbf{u}}(\cdot)\|_{X_{1,\mathbf{u}}'} + \sqrt{2} \max\{\gamma_{\mathcal{A}_1}^{\mathrm{UB}}, \gamma_{\mathcal{C}_1}^{\mathrm{UB}}\} \|\mathbf{e}_{N,\mathbf{U}}\|_{X_{\mathbf{U}}} \Big) \|\mathbf{v}^{\mathcal{N}}\|_{X_{1,\mathbf{u}}}.
\end{aligned}$$

Inserting this estimate into (30) together with (26) yields the desired estimate (27). □

Remark 3 The quantities $\beta^{\mathrm{LB}}(\boldsymbol{\mu})$ and $\alpha_{\mathcal{A}_i}^{\mathrm{LB}}(\boldsymbol{\mu})$, $i = 1, 2$, respectively $\beta_{\mathcal{A}}^{\mathrm{LB}}(\boldsymbol{\mu})$ can be computed with the *Successive Constraint Method* (SCM) [14]. A recipe for the computation of $\beta_{\mathcal{A}}^{\mathrm{LB}}(\boldsymbol{\mu})$ is worked out in [22], in particular a lower bound for the *surrogate* inf-sup constant $\beta_{\mathcal{A}}^{\boldsymbol{\mu}^*}(\boldsymbol{\mu})$, defined through

$$\beta_{\mathcal{A}}^{\boldsymbol{\mu}^*}(\boldsymbol{\mu}) = \inf_{\mathbf{V}^{\mathcal{N}} \in X} \sup_{\mathbf{W}^{\mathcal{N}} \in X} \frac{\mathcal{A}(\mathbf{V}^{\mathcal{N}}, \mathbf{W}^{\mathcal{N}}; \boldsymbol{\mu})}{\|\hat{\mathcal{T}}(\boldsymbol{\mu}^*)\mathbf{V}^{\mathcal{N}}\|_X \|\mathbf{W}^{\mathcal{N}}\|_X},$$

is computed, where $\boldsymbol{\mu}^*$ is a fixed parameter value and $\hat{\mathcal{T}}(\boldsymbol{\mu}^*) : X^{\mathcal{N}} \to X^{\mathcal{N}}$ is the global supremizer operator defined through

$$(\hat{\mathcal{T}}(\boldsymbol{\mu}^*)\mathbf{V}^{\mathcal{N}}, \mathbf{W}^{\mathcal{N}})_X = \mathcal{A}(\mathbf{V}^{\mathcal{N}}, \mathbf{W}^{\mathcal{N}}, \boldsymbol{\mu}^*), \qquad \forall \mathbf{W}^{\mathcal{N}} \in X^{\mathcal{N}}.$$

This so-called *natural norm* approach overcomes the problem of quadratic complexity in the number of "$\mathcal{A}$-components" during the online-phase [22,32].

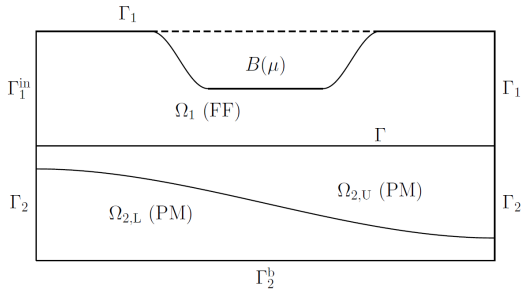

Fig. 3 Illustration of geometry for test model (FF = Free Flow, PM = Porous Medium). The obstacle $B(\boldsymbol{\mu})$ has a parametrized height.

The global error $\mathbf{e}_N(\boldsymbol{\mu})$ itself can be bounded with the above estimates via $\mathbf{e}_N(\boldsymbol{\mu})^2 = \mathbf{e}_{N,\mathbf{U}}(\boldsymbol{\mu})^2 + e_{N,P}(\boldsymbol{\mu})^2$. Since $e_{N,P}(\boldsymbol{\mu})$ is traced back to $\mathbf{e}_{N,\mathbf{U}}(\boldsymbol{\mu})$ to get the bound, the error bound for the primal variable $\mathbf{U}^{\mathcal{N}}(\boldsymbol{\mu})$ will be much sharper than the global error bound.

8 Numerical results

We consider a groundwater flow model as an instance of (1) - (3) with 1 geometry parameter and 2 material parameters. The model configurations, depicted in Figure 3, are $\Omega = \Omega(\boldsymbol{\mu}) = [0,4] \times [0,2] \setminus B(\boldsymbol{\mu})$, $\Omega_1 = \Omega_1(\boldsymbol{\mu}) = [0,4] \times [1,2] \setminus B(\boldsymbol{\mu})$, $\Omega_2 = [0,4] \times [0,1]$, $\Gamma_1^{\text{in}} = \{0\} \times [1,2]$ and $\Gamma_2^{\text{b}} = [0,4] \times \{0\}$. We consider a heterogeneous porous medium and, therefore, introduce a decomposition $\overline{\Omega}_2 = \overline{\Omega}_{2,\text{U}} \cup \overline{\Omega}_{2,\text{L}}$. The parameter vector lives in a three dimensional space: $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) \in \mathcal{P} \subset \mathbb{R}^3$ with $\mathcal{P} = [1,10]^2 \times [0.25, 0.75]$. The parameter μ_1 parametrizes the kinematic viscosity, the parameter μ_2 the hydraulic conductivity and μ_3 represents the height of the obstacle $B(\boldsymbol{\mu})$, so that Ω_2 is fixed and $\mathsf{K} = \mathsf{K}_{\text{ref}}$ (see (6)). For $\boldsymbol{\mu} \in \mathcal{P}$, we set $\nu(\boldsymbol{\mu}) = \mu_1 \cdot 10^{-2}$, $n(\boldsymbol{\mu}) = n = 0.4$ and $g = 9.807$. For $\boldsymbol{\mu} \in \mathcal{P}$, $\mathbf{x} \in \Omega_1$, we set $\mathbf{f}(\mathbf{x}) = \mathbf{f} = 0$ and for $\mathbf{x} \in \Omega_2$

$$\mathsf{K}(\mathbf{x};\boldsymbol{\mu}) = \frac{\tilde{\mathsf{K}}(\mathbf{x};\boldsymbol{\mu})}{\nu(\mathbf{x};\boldsymbol{\mu})}, \qquad \tilde{\mathsf{K}}(\mathbf{x};\boldsymbol{\mu}) = 1 \cdot 10^{-4} \left(\chi_{\Omega_{2,\text{U}}}(\mathbf{x}) + \frac{5}{\mu_2} \chi_{\Omega_{2,\text{L}}}(\mathbf{x}) \right).$$

Here $\chi_{\Omega_{2,*}}$ $(* = \text{L}, \text{U})$ denotes the characteristic function defined through $\chi_{\Omega_{2,*}}(\mathbf{x}) = 1$, if $\mathbf{x} \in \Omega_{2,*}$ and $\chi_{\Omega_{2,*}}(\mathbf{x}) = 0$, else. On the interface, we set $\alpha_{\text{BJ}}(\boldsymbol{\mu}) = \alpha_{\text{BJ}} = 0.05$ and on the outer boundaries, for $\mathbf{x} = (x_1, x_2)^T \in \Gamma_1^{\text{in}}$, we set

$$\mathbf{u}_{\text{in}}(\mathbf{x};\boldsymbol{\mu}) = \left(\left(\frac{\mu_1}{20} - 1 \right) x_2^2 + \left(2 - 3\frac{\mu_1}{20} \right) x_2 + \frac{\mu_1}{10}, 0 \right)^T$$

and $\varphi_{\text{b}}(\mathbf{x}) = \varphi_{\text{b}} = 100$ for $\mathbf{x} \in \Gamma_{\text{b}}$. By the parametrization of the inflow condition we achieve that the slip condition at $\mathbf{x} = (0,1)^T$ is fulfilled for every $\boldsymbol{\mu}$. We note that for the lifting operators $\mathcal{E}_1(\boldsymbol{\mu}) = \mathcal{E}_1((\mu_1, \mu_3))$ and $\mathcal{E}_2(\boldsymbol{\mu}) = \mathcal{E}_2((\mu_1, \mu_2))$, that is, the parameter domain reduces by 1 dimension for both subproblems. Two representative solutions to (17) are shown in Figures 4 and 5. Note that the plots are separated due to different value ranges.

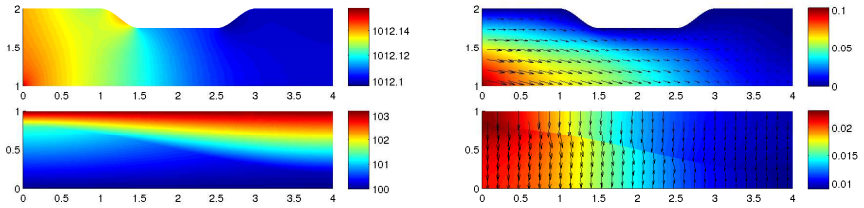

Fig. 4 Sample solutions of test model: $\mu = (10, 1, 0.25)$, left: pressures, right: velocities.

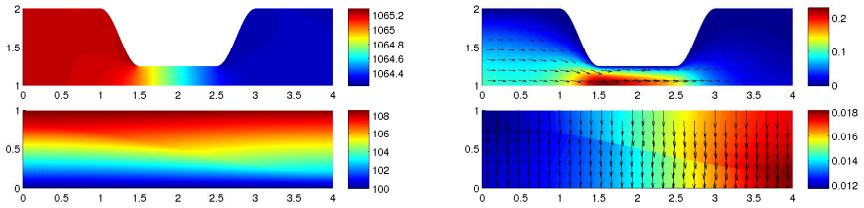

Fig. 5 Sample solutions of test model: $\mu = (10, 10, 0.75)$, left: pressures, right: velocities.

8.1 Basis generation process

The following quantities will be fixed throughout this section: The overall dimension of the FE approximation $\mathcal{N} = 92168$, the training set $\mathcal{P}_{\mathrm{f}}$ of 125 parameter values being the vertices of a regular grid on $\mathcal{P}$ and a test set $\mathcal{P}_{\mathrm{t}}$ of 100 uniformly distributed random parameter values. In addition, we will always consider $\mathcal{P}_{1,\mathrm{c}} = \mathcal{P}_{2,\mathrm{c}} \subset \mathcal{P}$, $N_{1,\Gamma} = N_{2,\Gamma}$ for the computation of the interface modes (see Subsection 6.1). To examine the basis generation with Algorithm 1 we choose tolerances $\epsilon_i = \epsilon = 1 \cdot 10^{-6}$ for $i = 1, 2$ and error indicators $\mathcal{I}_i^k(\boldsymbol{\mu}) = \Delta_i^k(\boldsymbol{\mu})$, $i = 1, 2$, $k = 1, \ldots, N_{i,\Gamma}$ and $\mathcal{I}(\boldsymbol{\mu}) = \Delta_{N,\mathbf{U}}(\boldsymbol{\mu})$. To simplify the notation we introduce

$$\Delta_{\mathbf{U}} := \max_{\boldsymbol{\mu} \in \mathcal{P}_{\mathrm{f}}} \Delta_{N,\mathbf{U}}(\boldsymbol{\mu}), \qquad \Delta_i := \max_{(\boldsymbol{\mu},k) \in \mathcal{P}_{\mathrm{f}} \times \{1,\ldots,N_{i,\Gamma}\}} \Delta_{N,i}^k(\boldsymbol{\mu}). \tag{31}$$

Given those settings, we compute interface modes with $N_{i,\Gamma} = 8$, $i = 1, 2$ as described in Subsection 6.1 and perform the algorithm. Figure 6 shows on the left-hand side the progress of the error indicators. We observe that the error bound for the global error stagnates, whereas the algorithm continues because the lifting approximations still improve. The actual termination is explained by the following considerations. In the offline/online procedure of the error bounds the squared residual norms are computed. An inaccuracy of the solver at $1 \cdot 10^{-12}$ leads to an inaccuracy at $1 \cdot 10^{-6}$ in the residual norms. Then one has to consider also the multiplication with the constants. As a consequence the newly chosen lifting snapshots are no longer orthogonal to the current bases, which is detected by our orthonormalization procedure. By computing more interface modes, we reach a higher global accuracy. This behaviour is shown in the right plot of Figure 6. In the following we use bases from intermediate steps, where the stagnation occurs.

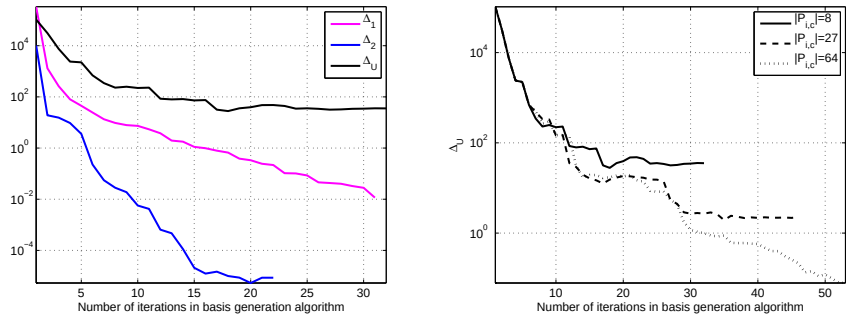

Fig. 6 Progress of the maximal error indicator values Δ_1, Δ_2 and $\Delta_{\mathbf{U}}$ (31) in the Subdomain-Greedy, left: global and local indicators for $N_{i,\Gamma} = 8$, right: global indicator for $N_{i,\Gamma} = 8, 27$ and 64.

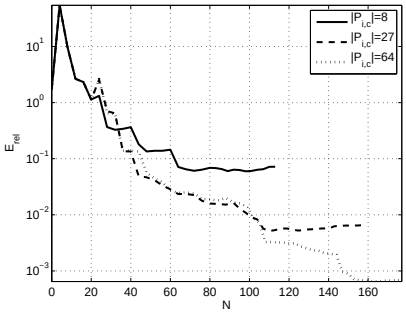

Fig. 7 Maximal relative global test error (100 parameter values) over basis size $N = N_{1,\mathbf{u}} + N_{1,p} + N_2$ for different numbers of interface modes $N_{i,\Gamma} = 8, 27$ and 64.

Table 1 Final basis size, maximal relative global test error, offline computational time, maximal online computational time and minimal speed-up factor for bases generated from different sets of interface modes.

$N_{i,\Gamma}$	N	E_{rel}	t_{offline}	$\max_{\boldsymbol{\mu}\in\mathcal{P}_{\mathrm{t}}} t_{\mathrm{online}}(\boldsymbol{\mu})$	$\min_{\boldsymbol{\mu}\in\mathcal{P}_{\mathrm{t}}} s(\boldsymbol{\mu})$
8	76	$6.37 \cdot 10^{-2}$	1h 2min	0.021 s	1232
27	110	$5.14 \cdot 10^{-3}$	1h 49min	0.024 s	1106
64	171	$6.72 \cdot 10^{-4}$	2h 55min	0.035 s	877

We investigate the approximation quality of the resulting reduced bases by computing the maximal relative global error, which is defined through

$$E_{\mathrm{rel}} = \max_{\boldsymbol{\mu}\in\mathcal{P}_{\mathrm{t}}} \left(\frac{\|\mathbf{e}_{N,\mathbf{u}}(\boldsymbol{\mu})\|^2_{X_{1,\mathbf{u}}}}{\|\mathbf{u}^{\mathcal{N}}(\boldsymbol{\mu})\|^2_{X_{1,\mathbf{u}}}} + \frac{\|e_{N,p}(\boldsymbol{\mu})\|^2_{X_{1,p}}}{\|p^{\mathcal{N}}(\boldsymbol{\mu})\|^2_{X_{1,p}}} + \frac{\|e_{N,\varphi}(\boldsymbol{\mu})\|^2_{X_2}}{\|\varphi^{\mathcal{N}}(\boldsymbol{\mu})\|^2_{X_2}} \right)^{1/2},$$

on the given test set $\mathcal{P}_{\mathrm{t}}$ of 100 parameters values. In Figure 7 the errors are plotted over the total size of the reduced basis $N = N_{1,\mathbf{u}} + N_{1,p} + N_2$.

We summarize the properties of the bases in Table 1. For a parameter value $\boldsymbol{\mu} \in \mathcal{P}$ we denote $t_{\mathrm{detail}}(\boldsymbol{\mu})$ the computational time for the detailed simulation (assembly and solution of (17)), $t_{\mathrm{online}}(\boldsymbol{\mu})$ the online computational time and $s(\boldsymbol{\mu}) = t_{\mathrm{detail}}(\boldsymbol{\mu})/t_{\mathrm{online}}(\boldsymbol{\mu})$ the speed-up factor. The offline computations (t_{offline}) include

Table 2 Final basis sizes and maximal relative test errors using Algorithm 1 and the global approach. In the latter case $N_{i,\Gamma}$ is the number of global snapshots.

$N_{i,\Gamma}$	N	$\tilde{N}$	E_{rel}	$\tilde{E}_{\mathrm{rel}}$
8	76	32	$6.37 \cdot 10^{-2}$	$1.02 \cdot 10^{-0}$
27	110	106	$5.14 \cdot 10^{-3}$	$9.98 \cdot 10^{-3}$
64	171	207	$6.72 \cdot 10^{-4}$	$2.18 \cdot 10^{-3}$

constructing the sets of interface modes, performing the Subdomain-Greedy and computing the components for system (25) and the error bound (26). The online computations comprise not only the global approximation (25) but also the error bound. More significance has the break even quantity, which also takes account of the offline computational time. Using the RB method pays off after 365, 492 and 634 solution evaluations in the three performed cases. This value is also important for a comparison with a "straightforward" RB procedure. It will further decrease, if the offline computational time decreases.

8.2 Global strategies

The global FE solutions computed for the interface mode sets $\Xi_{N,i,\Gamma}$ could as well serve as basis functions via restriction to the subdomains. We use the snapshots computed before in the cases $N_{i,\Gamma} = 8$, 27 and 64 for $i = 1, 2$ and construct RB spaces $X_{N,1,\mathbf{u}}$, $X_{N,1,p}$ and $X_{N,2}$ by restricting the snapshots to the subdomains, computing the supremizers and performing an orthonormalization step. We denote by $\tilde{N}$ the basis size obtained by this procedure and by $\tilde{E}_{\mathrm{rel}}$ the maximal relative test error on $\mathcal{P}_{\mathrm{t}}$ of the resulting RB approximation. A comparison of the global approach to the Subdomain Greedy procedure is shown in Table 2. One can clearly see that, using the Subdomain Greedy procedure, smaller errors can be reached using the same amount of global solutions whilst the magnitudes of the bases are comparable.

Also, a "straightforward" global Greedy procedure could be performed on the same training set $\mathcal{P}_f$. As already stated, the advantage of our method lies in a more efficient offline-phase due to the avoiding of many global solves.

8.3 Comparison of error estimators

We now investigate the error bound $\Delta_N(\boldsymbol{\mu})$ and also the new bounds introduced in Lemma 1, Section 7. We want to regard the effecctivities of the bounds with respect to the basis size N and use for this the bases generated with $N_{i,\Gamma} = 64$, $\mathcal{I}_i^k(\boldsymbol{\mu}) = \Delta_i^k(\boldsymbol{\mu})$, $i = 1, 2$, $k = 1, \ldots, N_{i,\Gamma}$ and $\mathcal{I}(\boldsymbol{\mu}) = \Delta_{N,\mathbf{U}}(\boldsymbol{\mu})$ and the same set of test parameter values as before. To simplify the notation we further introduce

$$\Delta := \max_{\boldsymbol{\mu} \in \mathcal{P}_{\mathrm{f}}} \Delta_N(\boldsymbol{\mu}), \qquad E := \max_{\boldsymbol{\mu} \in \mathcal{P}_{\mathrm{f}}} \|e_N(\boldsymbol{\mu})\|_X, \qquad \Delta_P := \max_{\boldsymbol{\mu} \in \mathcal{P}_{\mathrm{f}}} \Delta_{N,P}(\boldsymbol{\mu}),$$

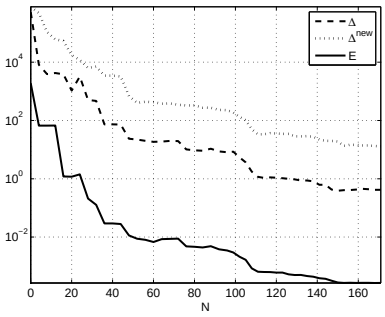

Fig. 8 Illustration of effectivities: Maximal global errors and error bounds on the test set $\mathcal{P}_\mathrm{t}$ over basis size N. The maximal effectivity value $\Delta_N(\boldsymbol{\mu})/\|e_N(\boldsymbol{\mu})\|_X$ is 10185, the mean value 940. The maximal effectivity value $\Delta_N^{\mathrm{new}}(\boldsymbol{\mu})/\|e_N(\boldsymbol{\mu})\|_X$ is 120010, the mean value 16036.

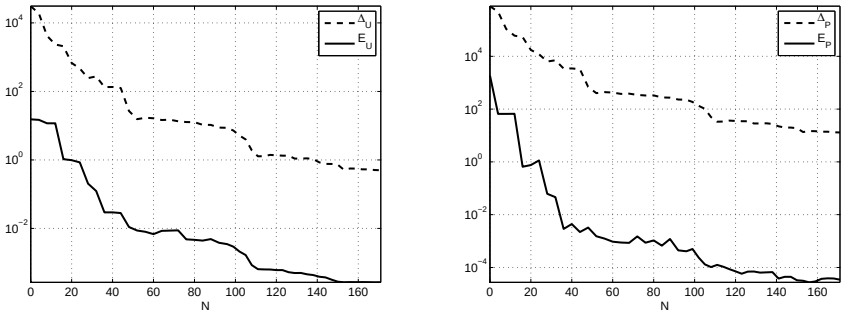

Fig. 9 Illustration of effectivities: Maximal errors and error bounds on the test set $\mathcal{P}_\mathrm{t}$ over basis size N, left: primal variable, right: dual variable.

$$E_{\mathbf{U}} := \max_{\boldsymbol{\mu}\in\mathcal{P}_\mathrm{f}} \|e_{N,\mathbf{U}}(\boldsymbol{\mu})\|_{X_\mathbf{U}}, \qquad E_P := \max_{\boldsymbol{\mu}\in\mathcal{P}_\mathrm{f}} \|e_{N,P}(\boldsymbol{\mu})\|_{X_P},$$
$$\Delta^{\mathrm{new}} := \max_{\boldsymbol{\mu}\in\mathcal{P}_\mathrm{f}} \Delta_N^{\mathrm{new}}(\boldsymbol{\mu}), \qquad \Delta_N^{\mathrm{new}} := \left(\Delta_{N,\mathbf{U}}(\boldsymbol{\mu})^2 + \Delta_{N,P}(\boldsymbol{\mu})^2\right)^{1/2}.$$

In Section 7 we already noted that a very small stability factor could be a problem with the bound $\Delta_N(\boldsymbol{\mu})$. This is confirmed by the results for the effectivity in Figure 8. Also, a comparison shows that Δ_N^{new} is an even less sharp bound. We expect to get an improvement, if we estimate only $\|\mathbf{e}_{N,\mathbf{U}}(\boldsymbol{\mu})\|_{X_\mathbf{U}}$ using $\Delta_{N,\mathbf{U}}(\boldsymbol{\mu})$ as error indicator, which was defined in (26). To do so in Algorithm 1 is reasonable, since $\mathbf{U}(\boldsymbol{\mu})$ is the primal variable of a saddle-point problem. Note, that this does not imply that we ascribe less importance to the pressure variable p in our applications. Figure 9 shows on the left-hand side the results for this bound. The maximal effectivity value $\Delta_{N,\mathbf{U}}(\boldsymbol{\mu})/\|e_{N,U}(\boldsymbol{\mu})\|_{X_{1,\mathbf{U}}}$ is 18257, the mean value 767. The results are comparable to those for Δ_N. The impracticality of Δ_N^{new} is due to the bad effectivity of the bound for the pressure variable. The gain of no longer dividing by a full global stability constant, but the square root, is neutralized by the fact, that the splitting into primal and dual variable leads to a rougher estimate. Nevertheless, the bound $\Delta_{N,\mathbf{U}}$ is competitive. For the sake of completeness, we show the results for the bound $\Delta_{N,P}(\boldsymbol{\mu})$, defined in (27), on the right-hand side of Figure 9. This is where the limitation of separately estimating pressure and velocity errors clearly sticks out, as the pressure error is highly overestimated.

9 Conclusions and perspectives

We presented a framework for RB approximation of the parametrized Stokes-Darcy system, picking up recent developments in the field of RB methods for homogeneous domain decomposition problems. Our algorithm allows for a heterogeneous problem structure and different ways of choosing interface modes on the inner boundary. The a-posteriori error estimation for inf-sup stable saddle point problems has been extended to the setting and a problem using the Babuška framework was revealed, which mirrors a general problem of a-posteriori error estimation in RB methods.

The strength of the method is that we need only few global snapshots. In most realistic cases the subproblems will depend on fewer parameters than the global problem. So high dimensions in the parameter space can presumably be better treated. Especially, large parts of the basis generation process can be performed in parallel, leading to important advantages and improvements.

Furthermore, there is a wide range of possible applications, including biological systems, environmental problems and food technologies. An extension of the method to fluid-structure interaction (FSI) problems is foreseen. Regarding future improvements, the conjunction of snapshot based basis generation and other basis enrichment schemes will be investigated more extensively.

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References

1. Almroth, B.O., Stern, P., Brogan, F.A.: Automatic choice of global shape functions in structural analysis. AIAA Journal **16**, 525–528 (1978)
2. Barrault, M., Maday, Y., Nguyen, N., Patera, A.: An "empirical interpolation" method: application to efficient reduced-basis discretization of partial differential equations. Comptes Rendus de l'Académie des Sciences, Series I **339**, 667–672 (2004)
3. Brezzi, F.: On the existence, uniqueness and approximation of saddle point problems arising from Lagrangian multipliers. R.A.I.R.O. **8**, 129–151 (1974)
4. Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods. Springer Verlag Berlin Heidelberg New York (1991)
5. Deparis, S., Løvgren, E.: Stabilized reduced basis approximation of incompressible three-dimensional Navier-Stokes equations in parametrized deformed domains. Journal of Scientific Computing **50(1)**, 198–212 (2012)
6. Dihlmann, M.A., Haasdonk, B.: Certified PDE-constrained parameter optimization using reduced basis surrogate models for evolution problems. Tech. rep., University of Stuttgart (2013). Submitted to *Computational Optimization and Applications*
7. Discacciati, M.: Domain decomposition methods for the coupling of surface and groundwater flows. Ph.D. thesis, EPFL, Lausanne (2004)
8. Discacciati, M., Miglio, E., Quarteroni, A.: Mathematical and numerical models for coupling surface and groundwater flows. Applied Numerical Mathematics **43**, 57–74 (2002)
9. Eftang, J.L., Patera, A.T.: Port reduction in parametrized component static condensation: approximation and a posteriori error estimation. International Journal for Numerical Methods in Engineering. (2013). DOI 10.1002/nme.4543
10. Gerner, A.L., Veroy, K.: Certified reduced basis methods for parametrized saddle point problems. SIAM Journal on Scientific Computing **34(5)**, A2812–A2836 (2012)

11. Grepl, M.A., Patera, A.T.: A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. Mathematical Modelling and Numerical Analysis **39**, 157–181 (2005)
12. Haasdonk, B., Ohlberger, M.: Reduced basis method for finite volume approximations of parametrized linear evolution equations. Mathematical Modelling and Numerical Analysis **42**, 277–302 (2008)
13. Huynh, D.B.P., Knezevic, D.J., Patera, A.T.: A static condensation reduced basis element method: Approximation and a posteriori error estimation. Mathematical Modelling and Numerical Analysis **47(1)**, 213–251 (2013)
14. Huynh, D.B.P., Rozza, G., Sen, S., Patera, A.T.: A successive constraint linear optimization method for lower bounds of parametric coercivity and inf-sup stability constants. Comptes Rendus de l'Académie des Sciences, Series I **345**, 473–478 (2007)
15. Iapichino, L.: Reduced basis method for the solution of parametrized PDEs in repetitive and complex networks with application to CFD. Ph.D. thesis, EPFL, Lausanne (2012)
16. Iapichino, L., Quarteroni, A., Rozza, G.: A reduced basis hybrid method for the coupling of parametrized domains represented by fluidic networks. Computer Methods in Applied Mechanics and Engineering **221–222**, 62–82 (2012)
17. Layton, W.J., Schieweck, F., Yotov, I.: Coupling fluid flow with porous media flow. SIAM Journal on Numerical Analysis **40(6)**, 2195–2218 (2003)
18. Løvgren, A.E., Maday, Y., Rønquist, E.M.: A reduced basis element method for the steady Stokes problem. Mathematical Modelling and Numerical Analysis **40**, 529–552 (2006)
19. Maday, Y., Rønquist, E.M.: A reduced-basis element method. Journal of Scientific Computing **17**, 447–459 (2002)
20. Maday, Y., Rønquist, E.M.: The reduced basis element method: Application to a thermal fin problem. Journal of Scientific Computing **26**, 240–258 (2004)
21. Maier, I., Haasdonk, B.: A Dirichlet-Neumann reduced basis method for homogeneous domain decomposition problems. Applied Numerical Mathematics **78**, 31–48 (2013)
22. Manzoni, A.: Reduced models for optimal control, shape optimization and inverse problems in haemodynamics. Ph.D. thesis, EPFL, Lausanne (2012)
23. Oliveira, I.B., Patera, A.T.: Reduced-basis techniques for rapid reliable optimization of systems described by affinely parametrized coercive elliptic partial differential equations. Optimization and Engineering **8**, 43–65 (2007)
24. Patera, A.T., Rozza, G.: Reduced Basis Approximation and A Posteriori Error Estimation for Parametrized Partial Differential Equations. MIT Pappalardo Graduate Monographs in Mechanical Engineering (2007). http://augustine.mit.edu
25. Prud'homme, C., Rovas, D.V., Veroy, K., Machiels, L., Maday, Y., Patera, A.T., Turinici, G.: Reliable real-time solution of parametrized partial differential equations: Reduced-basis output bound methods. Journal of Fluids Engineering **124**, 70–80 (2002)
26. Quarteroni, A., Rozza, G., Manzoni, A.: Certified reduced basis approximation for parametrized partial differential equations and applications. Journal of Mathematics in Industry **1:3** (2011). DOI 10.1186/2190-5983-1-3
27. Quarteroni, A., Valli, A.: Domain Decomposition Methods for Partial Differential Equations. Oxford University Press (1999)
28. Rozza, G.: Shape design by optimal flow control and reduced basis techniques: applications to bypass configurations in haemodynamics. Ph.D. thesis, EPFL, Lausanne (2005)
29. Rozza, G., Huynh, D.B.P., Manzoni, A.: Reduced basis approximation and a posteriori error estimation for Stokes flow in parametrized geometries: roles of the inf-sup stability constants. Numerische Mathematik **125**, 115–152 (2013)
30. Rozza, G., Huynh, D.B.P., Patera, A.T.: Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations. Archives of Computational Methods in Engineering **15**, 229–275 (2008)
31. Rozza, G., Veroy, K.: On the stability of the reduced basis method for Stokes equations in parametrized domains. Computer methods in applied mechanics and engineering **196**, 1244–1260 (2007)
32. Sen, S., Veroy, K., Huynh, D.B.P., Deparis, S., Nguyen, N.C., Patera, A.T.: "Natural norm" a posteriori error estimators for reduced basis approximations. Journal of Computational Physics **217**, 37–62 (2006)
33. Smetana, K.: A new certification framework for the port reduced static condensation reduced basis element method. Computer Methods in Applied Mechanics and Engineering Submitted March 2014

34. Toselli, A., Widlund, O.: Domain Decomposition Methods - Algorithms and Theory. Springer-Verlag Berlin Heidelberg New York (2005)
35. Vallaghé, S., Patera, A.T.: The static condensation reduced basis element method for a mixed-mean conjugate heat exchanger model. SIAM Journal on Scientific Computing **36(3)**, B294–B320 (2013)
36. Veroy, K., Prud'homme, C., Rovas, D.V., Patera, A.T.: A posteriori error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations. In: Proceedings of 16th AIAA computational fluid dynamics conference (2003). Paper 2003-3847
37. Xu, J., Zikatanov, L.: Some observations on Babuška and Brezzi theories. Numerische Mathematik **94**, 195–202 (2003)