On functions having coincident p-norms

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Abstract. In a measure space $(X, \mathcal{A}, \mu)$ we consider two measurable functions $f, g : E \to \mathbb{R}$ for some $E \in \mathcal{A}$. We characterize the property of having equal p-norms when p varies in an infinite set P in $[1, +\infty)$. In a first theorem we consider the case of bounded functions when P is unbounded with $\sum_{p\in P}(1/p) = +\infty$. The second theorem deals with the possibility of unbounded functions, when P has a finite accumulation point in $[1, +\infty)$.

1 Introduction

We consider a measure space $(X, \mathcal{A}, \mu)$ and two measurable functions $f, g : E \to \mathbb{R}$, for some $E \in \mathcal{A}$. The aim of this paper is to characterize the property of having equal p-norms when p varies in an infinite set $P \subseteq [1, +\infty)$.

Before stating our main results, let us recall the standard notation for the norms in $\mathcal{L}^p(E)$:

$$\|f\|_p = \left(\int_E |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \quad \text{if } 1 \le p < +\infty \,, \qquad \|f\|_\infty = \operatorname*{ess\,sup}_E |f| \,.$$

Moreover, for a measurable function $f : E \to \mathbb{R}$, we write

$$\{f > \alpha\} = \{x \in E : f(x) > \alpha\} \,, \qquad \mu(f > \alpha) = \mu(\{f > \alpha\}) \,.$$

Here is our first result.

Theorem 1. *Let $f, g \in \mathcal{L}^1(E) \cap \mathcal{L}^\infty(E)$. If P is an unbounded subset of $[1, +\infty)$, with*

$$\sum_{p\in P} \frac{1}{p} = +\infty \,, \tag{1}$$

then the following two conditions are equivalent:

i) $\|f\|_p = \|g\|_p$ *for all $p \in P$;*

ii) $\mu(|f| > \alpha) = \mu(|g| > \alpha)$ *for all $\alpha \ge 0$.*

Theorem 1 applies for example if $P = \mathbb{N} \setminus \{0\}$, or $P = \{n \ln n : n \in \mathbb{N} \setminus \{0\}\}$. On the contrary, the set $P = \{n^2 : n \in \mathbb{N} \setminus \{0\}\}$ is not admissible, since in this case the series in (1) converges.

In the case when f and g do not belong to $\mathcal{L}^\infty(E)$ the result is, in general, no longer true. We will give a counterexample in Section 4. On the other hand, we have the following second result.

Theorem 2. *Let $f, g \in \mathcal{L}^p(E)$ for all $p \ge 1$. If P has an accumulation point in $(1, +\infty)$, then the same conclusion of Theorem 1 holds.*

In this case condition (1) is trivially satisfied, and the existence of a finite accumulation point is necessary, otherwise the same counterexample developed in Section 4 applies. The case when 1 is the only accumulation point of P can be treated, provided that $f, g \in \mathcal{L}^\infty(E)$. As a direct consequence of the above two theorems, we have the following.

Corollary 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $E \in \mathcal{A}$, with $\mu(E) < +\infty$, and $f : E \to \mathbb{R}$ such that $f \in \mathcal{L}^p(E)$ for all $p \geq 1$. Let C be a non negative constant. Then, if P has an accumulation point in $(1, +\infty)$, the following two conditions are equivalent:*

$$i) \quad \left(\frac{1}{\mu(E)}\right)^{\frac{1}{p}} \|f\|_p = C \qquad \text{for all } p \in P\,;$$

$$ii) \quad |f(x)| = C \qquad \text{for a.e. } x \in E\,.$$

Else, if P is unbounded and (1) *holds, the same two conditions are equivalent.*

The paper is organized as follows. In Section 2 we recall some results in measure theory and a variant of the Müntz – Szász theorem. Section 3 is then devoted to the proof of Theorem 1. In Section 4 we construct a counterexample to the conclusion of Theorem 1 if the boundedness hypothesis on f and g is dropped. Moreover we show that this will remain a valid counterexample to Theorem 2 if the hypothesis of the existence a finite accumulation point is not fulfilled. In Section 5 we provide the proof of Theorem 2 by means of elementary complex analysis and the Mellin transform. In the last section we provide some complementary results and final remarks.

2 Some preliminary results

In this section we recall some results in measure theory that will be useful in the sequel.

Lemma 1. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$; suppose that $1 \leq p_1 \leq p \leq p_2 \leq +\infty$ and*

$$\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2} \qquad \text{for some } \alpha \in [0,1]\,. \tag{2}$$

If $f \in \mathcal{L}^{p_1}(E) \cap \mathcal{L}^{p_2}(E)$, then $f \in \mathcal{L}^p(E)$ and

$$\|f\|_p \leq \|f\|_{p_1}^{\alpha} \|f\|_{p_2}^{1-\alpha}. \tag{3}$$

Proof. If $p = +\infty$, then $p_2 = +\infty$ and either $\alpha = 0$ or $p_1 = +\infty$, so that in both cases (3) is fulfilled. If $p < +\infty$, multiplying (2) by p we have that $\frac{p_1}{\alpha p}$ and $\frac{p_2}{(1-\alpha)p}$ are conjugate exponents, and applying the Hölder inequality to $|f|^p = |f|^{p\alpha}|f|^{p(1-\alpha)}$ the lemma is proved. □

Lemma 2. *Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$. If $f \in \mathcal{L}^\infty(E) \cap \mathcal{L}^r(E)$ for some $r \in [1, +\infty)$, then*

$$\lim_{p \to +\infty} \|f\|_p = \|f\|_\infty\,.$$

Proof. If E has finite measure the proof can be found, for example, in [8]. Otherwise, by Lemma 1, for all $p \geq r$, choosing $p_1 = r$, $p_2 = +\infty$, $\alpha = \frac{r}{p} \leq 1$, we have that $f \in \mathcal{L}^r(E)$ and

$$\|f\|_p \leq \|f\|_r^{\frac{r}{p}} \|f\|_\infty^{1-\frac{r}{p}}\,. \tag{4}$$

Setting $C = \|f\|_r^r$ and passing to $\limsup$ in (4) gives us

$$\limsup_{p \to +\infty} \|f\|_p \leq \limsup_{p \to +\infty} C^{\frac{1}{p}} \|f\|_\infty^{1-\frac{r}{p}} = \|f\|_\infty\,. \tag{5}$$

Let us prove that

$$\liminf_{p\to+\infty}\|f\|_p \geq \|f\|_\infty\,. \tag{6}$$

Let $t < \|f\|_\infty$; then there exists a set $B \subseteq E$ of positive finite measure such that $|f| > t$ over B.

$$\|f\|_p^p = \int_E |f|^p \,\mathrm{d}\mu \geq \int_B |f|^p \,\mathrm{d}\mu \geq \mu(B)t^p\,,$$

hence

$$\|f\|_p \geq \mu(B)^{\frac{1}{p}}t\,.$$

Then,

$$\liminf_{p\to+\infty}\|f\|_p \geq t\,,$$

and being $t < \|f\|_\infty$ arbitrary, we arrive to (6). Putting together (5) and (6), the lemma is proved. □

Lemma 3. *Let $P \subseteq [1,+\infty)$ be such that* (1) *holds. If $\varphi \in \mathcal{L}^1[0,1]$ satisfies*

$$\int_0^1 \varphi(x)x^{p-1}\,\mathrm{d}x = 0 \qquad \textit{for all } p \in P\,, \tag{7}$$

then $\varphi(x) = 0$ for almost every $x \in [0,1]$.

Proof. By a generalization of Müntz–Szász theorem given in [1, 3] a necessary and sufficient condition for the powers x^s, with $s \in S \subseteq \mathbb{R}$, to span a dense subset of $\mathcal{L}^r[0,1]$ is that $s > -\frac{1}{r}$ for all $s \in S$, and

$$\sum_{s\in S}\frac{s+\frac{1}{r}}{\left(s+\frac{1}{r}\right)^2+1} = +\infty\,.$$

Choosing $r = 1$, $s = p-1$ and $S = \{p-1 : p \in P\}$, we have that if

$$\sum_{p\in P}\frac{p}{p^2+1} = +\infty\,,$$

then

$$\langle x^{p-1},\ p \in P\rangle \text{ is dense in } \mathcal{L}^1[0,1]\,.$$

Notice that, if $P \subseteq [1,+\infty)$, then

$$\sum_{p\in P}\frac{p}{p^2+1} \qquad \text{and} \qquad \sum_{p\in P}\frac{1}{p}$$

both diverge or both converge, as can be seen from the inequalities

$$\frac{1}{2p} \leq \frac{p}{p^2+1} \leq \frac{1}{p}\,.$$

Consequently, (7) holds if and only if $\varphi = 0$ almost everywhere. □

3 Proof of Theorem 1

If $\|f\|_\infty = 0$ or $\|g\|_\infty = 0$ then either $i)$ or $ii)$ imply that $f = g = 0$ almost everywhere, and the result is achieved. By hypothesis, we have that $f, g \in \mathcal{L}^1(E) \cap \mathcal{L}^\infty(E)$ so, by Lemma 1, $f, g \in \mathcal{L}^p(E)$ for every $p \in [1, +\infty]$. Moreover, by Lemma 2,

$$\lim_{p\to+\infty} \|f\|_p = \|f\|_\infty \,.$$

Hence, if condition $i)$ holds, then $\|f\|_\infty = \|g\|_\infty$. Without loss of generality, we can suppose $\|f\|_\infty = \|g\|_\infty = 1$. Indeed,

$$\begin{aligned}\|f\|_p = \|g\|_p &\Leftrightarrow \|f\|_p^p = \|g\|_p^p \\ &\Leftrightarrow \int_E \|f\|_\infty^p \left(\frac{|f|}{\|f\|_\infty}\right)^p \mathrm{d}\mu = \int_E \|g\|_\infty^p \left(\frac{|g|}{\|g\|_\infty}\right)^p \mathrm{d}\mu \\ &\Leftrightarrow \int_E \left(\frac{|f|}{\|f\|_\infty}\right)^p \mathrm{d}\mu = \int_E \left(\frac{|g|}{\|g\|_\infty}\right)^p \mathrm{d}\mu \,.\end{aligned}$$

Being f and g in $\mathcal{L}^p(E)$ for every $p \in [1, +\infty]$, the functions

$$t \to \mu\left(|f| > t^{\frac{1}{p}}\right) \qquad \text{and} \qquad t \to \mu\left(|g| > t^{\frac{1}{p}}\right)$$

are finite almost everywhere and their difference is defined almost everywhere. Therefore the function

$$\mathcal{I}(p) = \int_E [\, |f|^p - |g|^p \,] \, \mathrm{d}\mu \tag{8}$$

is well defined and finite for every $p \in [1, +\infty)$, and

$$\mathcal{I}(p) = \int_E \big(|f|^p - |g|^p\big)\mathrm{d}\mu = \int_0^1 \big[\mu(|f|^p > t) - \mu(|g|^p > t)\big]\mathrm{d}t = \int_0^1 \Big[\mu\left(|f| > t^{\frac{1}{p}}\right) - \mu\left(|g| > t^{\frac{1}{p}}\right)\Big]\mathrm{d}t \,.$$

Substituting $z = t^{\frac{1}{p}}$, the integral becomes

$$\mathcal{I}(p) = p\int_0^1 \big[\mu(|f| > z) - \mu(|g| > z)\big] z^{p-1}\,\mathrm{d}z = p\int_0^1 \varphi(z) z^{p-1}\,\mathrm{d}z \,,$$

where

$$\varphi(z) = \mu(|f| > z) - \mu(|g| > z) \,.$$

Notice that $\varphi : [0, 1] \to \mathbb{R}$ is defined almost everywhere and it is measurable, being the difference of two monotone functions. As a consequence,

$$\|f\|_p = \|g\|_p \;\Leftrightarrow\; \mathcal{I}(p) = 0 \,.$$

By Lemma 3, $\mathcal{I}(p) = 0$ for all $p \in P$ if and only if

$$\mu(|f| > \alpha) = \mu(|g| > \alpha) \qquad \text{for a.e. } \alpha \geq 0 \,.$$

We want to prove now that the level sets must coincide *for all* $\alpha \geq 0$. Let

$$U = \{\alpha \in [0, +\infty) : \mu(|f| > \alpha) = \mu(|g| > \alpha)\} \,.$$

By contradiction assume that there exists $\bar{\alpha} \geq 0$ not belonging to U. Being U dense in $[0, \|f\|_\infty]$, there exists a decreasing sequence $(\alpha_n)_n$ in U such that $\alpha_n \to \bar{\alpha}$, and

$$\{|f| > \bar{\alpha}\} = \bigcup_n \{|f| > \alpha_n\} .$$

Passing to measure,

$$\mu\left(|f| > \bar{\alpha}\right) = \mu\left(\bigcup_n \{|f| > \alpha_n\}\right) = \lim_{n\to\infty} \mu(|f| > \alpha_n) ,$$

and similarly

$$\mu(|g| > \bar{\alpha}) = \lim_{n\to\infty} \mu(|g| > \alpha_n) .$$

Then $\mu\left(|f| > \bar{\alpha}\right) = \mu\left(|g| > \bar{\alpha}\right)$, so $\bar{\alpha} \in U$, a contradiction. The proof is thus concluded. □

4 Construction of the counterexample

In this section we want to show that, in general, the boundedness hypothesis on f and g in Theorem 1 cannot be removed. In the first part we give some definitions to set the problem in a more general frame, then we develop the counterexample. Precisely, we will firstly build a continuous function φ defined on the positive real semiaxis and orthogonal to every monomial (and for linearity to every polynomial). Then, we will prove that this function is continuous and it is of bounded variation on $[0, +\infty)$. So, it can be written as the difference of two strictly decreasing functions; their inverses are the functions we are looking for. To conclude we show, as corollaries of independent interest, that modifying a bit this function φ firstly we can make it smooth, and secondly it could be orthogonal to every rational power of x, with fixed denominator. For an in-depth analysis of this argument see e.g. [5, 6].

Lemma 4. *There exists a continuous function $\varphi : [0, +\infty) \to \mathbb{R}$, not identically 0 , such that*

$$\int_0^{+\infty} x^n \varphi(x)\, \mathrm{d}x = 0 \qquad \textit{for all } n \in \mathbb{N} .$$

Proof. Set

$$I_n = \int_0^{+\infty} x^n e^{-(1-i)x} \mathrm{d}x .$$

Being $|x^n e^{-(1-i)x}| = x^n e^{-x}$, we have that the integral I_n is well defined for all n. Moreover, letting $z = 1 - i$, performing the change of variables $zx = y$ we obtain:

$$I_n = \int_0^{+\infty} x^n e^{-(1-i)x}\, \mathrm{d}x = \int_0^{+\infty} x^n e^{-zx}\, \mathrm{d}x = z^{-n-1} \int_\gamma y^n e^{-y}\, \mathrm{d}y ,$$

where γ is the half-line starting at the origin, containing the point $1 - i$. Consider the triangular path T_N in the complex plane joining the points 0, N, $N - iN$. Being $y^n e^{-y}$ analytic over the interior of T_N the integral along T_N is 0. Moreover:

$$\left| \int_N^{N-iN} y^n e^{-y}\, \mathrm{d}y \right| = \left| \int_0^N (N - it)^n e^{-N+it}\, \mathrm{d}t \right| \leq \int_0^N |N^2 + t^2|^{\frac{n}{2}} e^{-N}\, \mathrm{d}t \to 0$$

and so

$$\int_0^N y^n e^{-y}\,\mathrm{d}y + \int_N^{N-iN} y^n e^{-y}\,\mathrm{d}y + \int_{N-iN}^0 y^n e^{-y}\,\mathrm{d}y = 0.$$

Then, passing to the limit for N tending to $+\infty$, the first term tends to $\Gamma(n+1)$, the second term tends to 0, and the third tends to $-z^{n+1}I_n$ hence:

$$I_n = z^{-n-1}\Gamma(n+1) = z^{-n-1}n!\,.$$

Then,

$$\begin{aligned} I_n &= n!\cdot(1-i)^{-n-1} = n!\cdot(1+i)^{n+1}\cdot 2^{-n-1} = \\ &= n!\cdot\left[\frac{(1+i)}{\sqrt{2}}\right]^{n+1}\cdot 2^{-n-1}\cdot 2^{\frac{n+1}{2}} = n!\cdot e^{\frac{(n+1)i\pi}{4}}\cdot 2^{-\frac{n+1}{2}}\,. \end{aligned}$$

So,

$$I_{4p+3}\in\mathbb{R}\quad\text{for all } p\in\mathbb{N}\,,$$

and then

$$\Im\left(I_{4p+3}\right) = 0\quad\text{for all } p\in\mathbb{N}\,,$$

so that

$$0 = \Im(I_{4p+3}) = \int_0^{+\infty} x^{4p+3}e^{-x}\Im(e^{ix})\,\mathrm{d}x = \int_0^{+\infty} x^{4p+3}e^{-x}\sin(x)\,\mathrm{d}x\quad\text{for all }\ p\in\mathbb{N}\,.$$

Letting $x = u^{\frac{1}{4}}$, we arrive to

$$\int_0^{+\infty} u^p e^{-\sqrt[4]{u}}\sin(\sqrt[4]{u})\,\mathrm{d}u = 0\quad\text{for all }\ p\in\mathbb{N}\,.$$

The function

$$\varphi(x) = e^{-\sqrt[4]{x}}\sin(\sqrt[4]{x})$$

has the requested properties. □

Lemma 5. *The function φ defined in Lemma 4 belongs to $BV([0,+\infty))$.*

Proof. Observe preliminarily that $\varphi(0) = 0$ and φ tends to 0 at infinity; moreover,

$$\varphi'(x) = \frac{e^{-\sqrt[4]{x}}\cos\left(\sqrt[4]{x}\right)}{4x^{3/4}} - \frac{e^{-\sqrt[4]{x}}\sin\left(\sqrt[4]{x}\right)}{4x^{3/4}} = \frac{\sqrt{2}e^{-\sqrt[4]{x}}}{4x^{3/4}}\sin\left(\frac{\pi}{4}-\sqrt[4]{x}\right)\ ,$$

and so

$$\varphi'(x) = 0 \ \Leftrightarrow\ \frac{\sqrt{2}e^{-\sqrt[4]{x}}}{4x^{3/4}}\sin\left(\frac{\pi}{4}-\sqrt[4]{x}\right) = 0 \ \Leftrightarrow\ \sqrt[4]{x} = \frac{\pi}{4}+k\pi\quad\text{for } k\in\mathbb{N}\,.$$

The second derivative of φ is given by

$$\varphi''(x) = \frac{e^{-\sqrt[4]{x}}\left(3\sin\left(\sqrt[4]{x}\right) - \left(2\sqrt[4]{x}+3\right)\cos\left(\sqrt[4]{x}\right)\right)}{16\,x^{7/4}}\,.$$

Letting

$$x_n = \left(\frac{\pi}{4}+n\pi\right)^4,$$

we see that $(\varphi''(x_n))_n$ has alternating signs, since

$$\varphi''(x_n) = (-1)^{n+1} \frac{256\sqrt{2}e^{\pi\left(-\left(n+\frac{1}{4}\right)\right)}}{(4\pi n + \pi)^6} \, .$$

So, the total variation of φ is the series of variations between each stationary point. Writing $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, we have

$$\begin{aligned}
V_{\mathbb{R}^+}(\varphi) &= \sum_{n\geq 0} |\varphi(x_{n+1}) - \varphi(x_n)| \\
&= \sum_{n\in 2\mathbb{N}} |\varphi(x_{n+1}) - \varphi(x_n)| + \sum_{n\in 2\mathbb{N}+1} |\varphi(x_{n+1}) - \varphi(x_n)| \\
&= \sum_{n\in \mathbb{N}} |\varphi(x_{2n+1}) - \varphi(x_{2n})| + \sum_{n\in \mathbb{N}} |\varphi(x_{2n+2}) - \varphi(x_{2n+1})| \\
&= \sum_{n\in\mathbb{N}} \left| -\frac{e^{-\frac{1}{4}\pi(8n+5)}}{\sqrt{2}} - \frac{e^{-\frac{1}{4}\pi(8n+1)}}{\sqrt{2}} \right| + \sum_{n\in\mathbb{N}} \left| \frac{e^{-\frac{1}{4}\pi(8n+9)}}{\sqrt{2}} + \frac{e^{-\frac{1}{4}\pi(8n+5)}}{\sqrt{2}} \right| \\
&= \frac{e^{-\frac{\pi}{4}}\left(e^{\pi}+1\right)}{\sqrt{2}\left(e^{\pi}-1\right)} .
\end{aligned}$$

We are now ready to construct the counterexample. Define

$$\phi(t) = P(t, +\infty) + \frac{1}{(t+1)} \, , \qquad \psi(t) = N(t, +\infty) + \frac{1}{(t+1)} \, ,$$

where $P(t, +\infty)$ and $N(t, +\infty)$ are, respectively, the positive and the negative variation of φ on $(t, +\infty)$. The functions ϕ and ψ are positive, strictly decreasing, bounded, and achieve their maximum in 0. Moreover,

$$\lim_{t\to+\infty} \phi(t) = \lim_{t\to+\infty} \psi(t) = 0 \, ,$$

and

$$\phi(t) - \psi(t) = \varphi(t) \, .$$

Restricting the codomain of ϕ to $(0, \phi(0)]$ and that of ψ to $(0, \psi(0)]$, we obtain two invertible functions

$$\hat{\phi} : [0, +\infty) \to (0, \phi(0)] \, , \qquad \hat{\psi} : [0, +\infty) \to (0, \psi(0)] \, .$$

Moreover their inverses are also non negative decreasing functions. Define

$$f = \hat{\phi}^{-1} : (0, \phi(0)] \to [0, +\infty) \, , \qquad g = \hat{\psi}^{-1} : (0, \psi(0)] \to [0, +\infty) \, ,$$

and notice that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = +\infty \, .$$

Extend f and g to all $\mathbb{R}$ by setting them equal to 0 outside their domain, and call them $\tilde{f}$ and $\tilde{g}$. These are the functions we are looking for. Indeed, we will now prove that $\mu(|\tilde{f}| > \alpha)$ does not coincide with $\mu(|\tilde{g}| > \alpha)$ for a.e. $\alpha \geq 0$. By contradiction suppose that

$$\mu(|\tilde{f}| > \alpha) = \mu(|\tilde{g}| > \alpha) \qquad \text{for a.e. } \alpha \geq 0 \, .$$

Being $\tilde{f}$ and $\tilde{g}$ non-negative and being their level sets coincident with those of f and g, we have

$$\mu(f > \alpha) = \mu(g > \alpha) \qquad \text{for a.e. } \alpha \geq 0 \, .$$

But f and g are monotonically strictly decreasing, so $\{f > \alpha\} = [0, f^{-1}(\alpha))$ and $\{g > \alpha\} = [0, g^{-1}(\alpha))$, hence

$$f^{-1}(\alpha) = g^{-1}(\alpha) \qquad \text{for a.e. } \alpha \geq 0\,.$$

Being $f^{-1} = \phi$ and $g^{-1} = \psi$,

$$\phi(\alpha) = \psi(\alpha) \qquad \text{for a.e. } \alpha \geq 0\,.$$

Recall then the definition of ϕ and ψ to obtain

$$P(\alpha, +\infty) + \frac{1}{(\alpha+1)} = N(\alpha, +\infty) + \frac{1}{(\alpha+1)} \qquad \text{for a.e. } \alpha \geq 0\,,$$

so

$$P(\alpha, +\infty) = N(\alpha, +\infty) \qquad \text{for a.e. } \alpha \geq 0\,,$$

and then

$$\varphi(\alpha) = P(\alpha, +\infty) - N(\alpha, +\infty) = 0 \qquad \text{for a.e. } \alpha \geq 0$$

finding a contradiction. The proof is then completed. □

In the following corollary, we want to extend Lemma 4 to find a continuous function orthogonal to every fractional power of x with fixed denominator.

Corollary 2. *Fix $q \in \mathbb{N} \setminus \{0\}$. There exists a continuous function $\varphi_q : (0, +\infty) \to \mathbb{R}$, not identically 0, such that*

$$\int_0^{+\infty} x^{\frac{n}{q}} \varphi_q(x)\,\mathrm{d}x = 0 \qquad \textit{for all } n \in \mathbb{N}\,.$$

Proof. Define I_n as before. We have

$$\int_0^{+\infty} x^{4p+3} e^{-x} \sin(x)\,\mathrm{d}x = 0 \quad \text{for all } p \in \mathbb{N}\,.$$

Letting $x = u^{\frac{1}{4q}}$ we arrive to

$$\int_0^{+\infty} u^{\frac{p}{q}} e^{-\sqrt[4q]{u}} \sin(\sqrt[4q]{u})\, u^{\frac{1-q}{q}}\,\mathrm{d}u = 0 \quad \text{for all } p \in \mathbb{N}\,.$$

The function

$$\varphi(x) = e^{-\sqrt[4q]{x}} \sin(\sqrt[4q]{x})\, x^{\frac{1-q}{q}}$$

is the one we were looking for. □

The aim of the subsequent theorem is to show that, if we multiply the functions $\varphi(x)$ and $\varphi_q(x)$ found respectively in Lemma 4 and Corollary 2 by a suitable power of x, we obtain two new functions that maintain the same property of orthogonality but are arbitrarily regular. We achieve this result applying Faà di Bruno's formula.

Lemma 6. *Let $w \in \mathcal{C}^\infty(\mathbb{R})$ and $0 < \alpha < 1$. Then the function $g_n : [0, +\infty) \to \mathbb{R}$,*

$$g_n(x) = x^n w(x^\alpha)$$

is of class $\mathcal{C}^n$ su $[0, +\infty)$, with $g_n^{(j)}(0) = 0$ for all $j = 0, 1, 2, \ldots, n$.

Proof. A central tool of this proof will be Faà di Bruno's formula that we will recall briefly. Let w and u be $\mathcal{C}^m$ real valued functions such that the composition $w \circ u$ is defined; then $(w \circ u)(x)$ is of class $\mathcal{C}^m$ and for $x > 0$ we have

$$(w \circ u)^{(j)}(x) = j! \sum_{k=1}^{j} \left[\frac{w^{(k)}(u(x))}{k!} \sum_{h_1+\cdots+h_k=j} \frac{u^{(h_1)}(x)}{h_1!} \cdots \frac{u^{(h_k)}(x)}{h_k!} \right]$$

or

$$(w \circ u)^{(j)}(x) = \sum_{k_1+2k_2+\cdots+jk_j=j} \frac{j!}{k_1! \cdots k_j!} w^{(k_1+\cdots+k_j)}(u(x)) \cdot \left(\frac{u'(x)}{1!}\right)^{k_1} \cdot \left(\frac{u''(x)}{2!}\right)^{k_2} \cdots \left(\frac{u^{(j)}(x)}{j!}\right)^{k_j}.$$

For a proof of this formula look at [4]. We have that

$$g_n^{(j)}(x) = \sum_{h=0}^{j} \binom{j}{h} n(n-1)\cdots(n-h+1) x^{n-h} \left[w(x^\alpha)\right]^{(n-h)},$$

and each term of this sum is of the form

$$C\, x^{n-h} [w(x^\alpha)]^{(n-h)}, \tag{9}$$

where C is a real number depending on j, h and n. Now we use the Faà di Bruno's formula to express the derivatives of w. In our case $u(x) = x^\alpha$ and so

$$u^{(h)}(x) = (\alpha)_h\, x^{\alpha-h} \qquad \text{where} \qquad (\alpha)_h = \alpha(\alpha-1)\cdots(\alpha-h+1).$$

Consequently

$$u^{(h_1)}(x) \cdots u^{(h_k)}(x) = (\alpha)_{h_1} (\alpha)_{h_2} \cdots (\alpha)_{h_k} x^{\alpha-h_1} x^{\alpha-h_2} \cdots x^{\alpha-h_k},$$

and if $h_1 + \cdots + h_k = j$,

$$u^{(h_1)}(x) \cdots u^{(h_k)}(x) = C(h_1, \ldots h_k) x^{k\alpha-j}.$$

So, applying Faà di Bruno's formula to (9), each term has the form

$$x^{n-h} \sum_{k=1}^{n-h} C_k w^{(k)}(x^\alpha) \cdot x^{k\alpha-(n-h)} = \sum_{k=1}^{n-h} C_k w^{(k)}(x^\alpha) \cdot x^{k\alpha}.$$

To conclude observe that

$$g_n^{(j)}(x) = \sum_{k=1}^{j} C_k' w^{(k)}(x^\alpha)\, x^{k\alpha},$$

and apply the theorem on the limit of the derivative. □

As a consequence of Lemma 6, we have that the function $\varphi : [0, +\infty) \to \mathbb{R}$ in the statement of Lemma 4 can be chosen to be arbitrarily regular (but not $\mathcal{C}^\infty$). For example, taking

$$\varphi(x) = e^{-\sqrt[4]{x}} \sin(\sqrt[4]{x}) x^n \qquad n \in \mathbb{N},$$

by Lemma 6 choosing $w(x) = e^{-x}\sin(x)$ and $\alpha = \frac{1}{4}$, we see that $\varphi(x)$ is of class $\mathcal{C}^n$. The same reasoning choosing the same w and $\alpha = \frac{1}{4q}$ allows to conclude that also the function φ_q is of class $\mathcal{C}^n$ if multiplied by x^{n+1}.

5 Proof of Theorem 2

The Mellin transform of a function $\upsilon(t)$ is defined as

$$\{\mathcal{M}\upsilon\}(z) = F(z) = \int_0^\infty \upsilon(t)t^{z-1}\mathrm{d}t\,, \quad z \in \mathbb{C}\,,$$

whenever the integral exists for at least one value z_0 of z (cf. [7, 9, 10]).

Lemma 7. *Let $\upsilon : [0, +\infty) \to \mathbb{R}$ a function such that*

$$\upsilon(t)\, t^{z-1} \in \mathcal{L}^1([0, +\infty)) \qquad \textit{for all } \; z \geq 1.$$

Then $\mathcal{M}\upsilon$ is analytic in $S = \{w \in \mathbb{C} : \Re(w) > 1\}$.

Proof. Let $\gamma : [0, 1] \to \mathbb{C}$ be a triangle in S and let $\Phi(s, z) = \upsilon(s)s^{z-1}$. Then,

$$\int_\gamma F = \int_\gamma \left(\int_0^{+\infty} \Phi(s, z)\,\mathrm{d}s\right)\mathrm{d}z = \int_0^1 \left(\int_0^{+\infty} \Phi(s, \gamma(t))\gamma'(t)\,\mathrm{d}s\right)\mathrm{d}t\,,$$

with $\gamma'(t)$ defined for all but three points $t \in [0, 1]$. Observe that

$$\int_\gamma |F| = \int_0^1 \left(\int_0^{+\infty} |\Phi(s, \gamma(t))\gamma'(t)|\,\mathrm{d}s\right)\mathrm{d}t \leq \int_0^1 \left(\int_0^{+\infty} |\upsilon(s)||s^{\gamma(t)-1}|\,|\gamma'(t)|\,\mathrm{d}s\right)\mathrm{d}t\,.$$

Being γ a triangle, $|\gamma'(t)|$ is constant on every side and then there exists C_1 such that $|\gamma'(t)| < C_1$ for all $t \in [0, 1]$ where the tangent vector is defined. Let $R > 0$ be such that $\mathrm{Supp}(\gamma) \subseteq B(0, R)$. Then,

$$|s^{\gamma(t)-1}| = s^{\Re[\gamma(t)-1]} \leq s^{R+1},$$

and so

$$\int_0^1 \left(\int_0^{+\infty} |\upsilon(s)||s^{\gamma(t)-1}|\,|\gamma'(t)|\,\mathrm{d}s\right)\mathrm{d}t \leq C_1 \int_0^1 \left(\int_0^{+\infty} |\upsilon(s)s^{R+1}|\,\mathrm{d}s\right)\mathrm{d}t\,.$$

By hypothesis, $\upsilon(s)s^p$ is Lebesgue integrable for all $p \geq 0$, so

$$C_1 \int_0^1 \left(\int_0^{+\infty} |\upsilon(s)s^{R+1}|\,\mathrm{d}s\right)\mathrm{d}t \leq C_1 C_R < +\infty\,.$$

Then, by Fubini-Tonelli Theorem,

$$\int_\gamma F = \int_\gamma \left(\int_0^{+\infty} \Phi(s, z)\,\mathrm{d}s\right)\mathrm{d}z = \int_0^{+\infty}\left(\int_\gamma \Phi(s, z)\,\mathrm{d}z\right)\mathrm{d}s = \int_0^{+\infty}\left(\int_\gamma \upsilon(s)s^{z-1}\,\mathrm{d}z\right)\mathrm{d}s\,.$$

But now $\upsilon(s)s^{z-1}$ is a holomorphic function of z, and then by the Cauchy integral theorem

$$\int_\gamma \upsilon(s)s^{z-1}\,\mathrm{d}z = 0\,,$$

and then

$$\int_\gamma F = 0\,,$$

for every triangular path. Consequently, by Morera's theorem for triangles (see for example [2]), $F(s)$ is holomorphic on $\{w \in \mathbb{C} : \Re(w) > 1\}$. □

Now we are ready to prove Theorem 2.

Proof. Suppose that there exists an accumulation point in $(1, \infty)$. Define, as in Theorem 1,

$$\varphi(z) = \mu(|f| > z) - \mu(|g| > z). \tag{10}$$

Being $\varphi : [0, +\infty) \to \mathbb{R}$ the difference of two monotone functions, it is differentiable almost everywhere, and hence continuous almost everywhere. Moreover, it is of bounded variation on $[a, +\infty)$ for all $a > 0$ and then of bounded variation in a neighbourhood of each $y \in (0, +\infty)$. Recalling now the definition (8), in the case of unbounded functions f and g we have

$$\mathcal{I}(p) = p \int_0^\infty \varphi(z) z^{p-1} \, dz \,. \tag{11}$$

Notice that the integral in the right-hand side of (11) is the Mellin transform of φ , hence

$$\mathcal{I}(p) = 0 \iff \{\mathcal{M}\varphi\}(p) = 0 \,.$$

By [9, Chapter 6.9, Theorem 28], for every $c \in (1, +\infty)$,

$$\frac{1}{2} \left[\varphi(x + 0) + \varphi(x - 0)\right] = \frac{1}{2\pi i} \lim_{T \to +\infty} \int_{c-iT}^{c+iT} \{\mathcal{M}\varphi\}(p) x^{-p} \, \mathrm{d}p \,, \tag{12}$$

where

$$\varphi(x + 0) = \lim_{t \to x^+} \varphi(t) \qquad \text{and} \qquad \varphi(x - 0) = \lim_{t \to x^-} \varphi(t) \,.$$

By Lemma 7, $\mathcal{M}\varphi$ is holomorphic on $\{w \in \mathbb{C} : \Re(w) > 1\}$. But

$$\{\mathcal{M}\varphi\}(p) = 0 \qquad \text{for all } \; p \in P \,,$$

and P has an accumulation point in $\{w \in \mathbb{C} : \Re(w) > 1\}$. Then, by the identity theorem of complex analytic functions,

$$\mathcal{M}\varphi \equiv 0 \qquad \text{on } \{w \in \mathbb{C} : \Re(w) > 1\}.$$

The inversion formula (12) then becomes

$$\frac{1}{2} \left[\varphi(x + 0) + \varphi(x - 0)\right] = \frac{1}{2\pi i} \lim_{T \to +\infty} \int_{c-iT}^{c+iT} 0 \cdot \, x^{-p} \, \mathrm{d}p = 0 \,.$$

Being $\varphi(x)$ continuous almost everywhere, we have that $\varphi(x) = 0$ for almost every x. The conclusion easily follows recalling the definition of φ in (10) and the last part of the proof of Theorem 1. □

6 Final remarks

Theorem 2 remains valid supposing the existence of an accumulation point of P in $(0, 1]$ and supposing $f, g \in \mathcal{L}^p(E)$ for all $p > 0$. In this case, $\|f\|_p$ is formally defined as before, although this is not a norm anymore. To prove this, first notice that, without loss of generality, we can always assume that $f, g \geq 0$. Let $\delta \in (0, 1]$ be an accumulation point of P. For all $p \in P \cap \left(\frac{\delta}{2}\, , 2\right)$,

$$\int_E |f|^p \mathrm{d}\mu = \int_E |g|^p \mathrm{d}\mu \iff \left(\int_E |f^{\frac{\delta}{2}}|^{\frac{2p}{\delta}} \mathrm{d}\mu\right)^{\frac{\delta}{2p}} = \left(\int_E |g^{\frac{\delta}{2}}|^{\frac{2p}{\delta}} \mathrm{d}\mu\right)^{\frac{\delta}{2p}} \iff \|f^{\frac{\delta}{2}}\|_{\frac{2p}{\delta}} = \|g^{\frac{\delta}{2}}\|_{\frac{2p}{\delta}}$$

The set $\widetilde{P} = \{\frac{2p}{\delta} : p \in P \cap \left(\frac{\delta}{2}, 2\right)\}$, is contained in $(1, +\infty)$ and it has an accumulation point there. We can now apply Theorem 2 to find that

$$\mu(|f^{\frac{\delta}{2}}| > \alpha) = \mu(|g^{\frac{\delta}{2}}| > \alpha) \qquad \text{for all } \alpha \geq 0\,,$$

and so $\mu(|f| > \alpha) = \mu(|g| > \alpha)$ for all $\alpha \geq 0$.

If P has 0 as an accumulation point, the argument in the proof of Theorem 1 can be adapted assuming $f, g \in \mathcal{L}^p$ for every $p \in (0, +\infty]$, provided that

$$\sum_{p \in P} \frac{p}{p^2 + 1} = +\infty\,. \tag{13}$$

Indeed the Müntz–Szász theorem still applies in this case, providing the analogue of Lemma 3 which is needed to get the conclusion of Theorem 1.

It would be interesting to see whether condition (13) is also necessary for the conclusion of Theorem 1. We acknowledge the anonymous referee for raising this problem, which I have not been able to settle.

In the last part of this section we propose an application of Theorem 1 to ℓ^p spaces. We recall that, for a sequence $A = (a_n)_n$, we can define the ℓ^p norms as follows:

$$\|A\|_p = \left(\sum_{n=0}^{\infty} |a_n|^p\right)^{\frac{1}{p}}, \qquad \|A\|_\infty = \sup_n |a_n|\,.$$

The result is the following.

Theorem 3. *Let $A = (a_n)_n$ and $B = (b_n)_n$ be two sequences of real numbers in ℓ^1. If P is a subset of $[1, +\infty)$ satisfying* (1) *and*

$$\|A\|_p = \|B\|_p \quad \text{for all } p \in P\,,$$

then the sequences

$$|A| = (|a_n|)_n \qquad \text{and} \qquad |B| = (|b_n|)_n$$

can be obtained one from the other by permutation, appending or removing some zeroes.

Proof. Choosing $X = \mathbb{N}$, $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and μ the counting measure, by Theorem 1 we have that

$$\#\,(|A| > \alpha) = \#\,(|B| > \alpha) \qquad \text{for all } \alpha \geq 0\,.$$

Without loss of generality we can suppose that $a_n, b_n > 0$ for all $n \in \mathbb{N}$. Being $(a_n)_n$ and $(b_n)_n$ absolutely convergent we can rearrange them in such a way that A and B are non increasing without modifying the ℓ_p norms, thus obtaining $\hat{A} = (\hat{a}_n)_n$ and $\hat{B} = (\hat{b}_n)_n$, respectively. Clearly

$$\#\,(A > \alpha) = \#\left(\hat{A} > \alpha\right) \quad \text{and} \quad \#\,(B > \alpha) = \#\left(\hat{B} > \alpha\right)\,.$$

If $\hat{a}_n = \hat{b}_n$ for all n then the theorem is proved. Assume by contradiction that $\hat{A} \neq \hat{B}$ and let $\bar{n}$ be the smallest index such that $\hat{a}_{\bar{n}} \neq \hat{b}_{\bar{n}}$. Suppose for instance that $\hat{a}_{\bar{n}} > \hat{b}_{\bar{n}}$ and choose

$$\alpha = \frac{\hat{a}_{\bar{n}} + \hat{b}_{\bar{n}}}{2}\,.$$

With this choice we have

$$\#\left(\hat{A} > \alpha\right) \geq \bar{n} > \#\left(\hat{B} > \alpha\right)\,,$$

a contradiction. □

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