NUMERICALLY FLAT HIGGS VECTOR BUNDLES

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ABSTRACT. After providing a suitable definition of numerical effectiveness for Higgs bundles, and a related notion of numerical flatness, in this paper we prove, together with some side results, that all Chern classes of a Higgs-numerically flat Higgs bundle vanish, and that a Higgs bundle is Higgs-numerically flat if and only if it is has a filtration whose quotients are flat stable Higgs bundles. We also study the relation between these numerical properties of Higgs bundles and (semi)stability.

1. INTRODUCTION.

A notion of ampleness for a vector bundle E on a scheme X was introduced in [9]: E is said to be ample if the tautological bundle $\mathcal{O}_{\mathbb{P}E}(1)$ on the projectivization $\mathbb{P}E$ is ample. A related notion is that of numerical effectiveness: E is numerical effective if $\mathcal{O}_{\mathbb{P}E}(1)$ is so (in some literature, the terms "pseudo-ample" or "semi-positive" are used instead of "numerically effective"). Moreover, E is said to be *numerically flat* if both E and the dual bundle E^* are numerically effective.

Properties of numerically effective and numerically flat vector bundles were studied in several papers, see e.g. [6, 4, 5]. In particular, all Chern classes of a numerically flat vector bundle vanish (the same result has been proved for principal bundles in [2]). In this paper we explore some properties of the Higgs bundles that satisfy numerical effectiveness or flatness conditions; these are a modification of those introduced in [3]. We give a recursive definition: a Higgs bundle $\mathfrak{E}$ of rank one is said to be *Higgs-numerically effective* (resp. *Higgs-ample*) if it is numerically effective (resp. ample) in the usual sense, while if its rank is greater than one we require that its determinant bundle is nef (resp. ample), and all universal Higgs quotient bundles on the Grassmannian varieties of Higgs quotients are Higgs-numerically effective (resp. Higgs-ample). Finally, a Higgs bundle $\mathfrak{E}$ is said to be *Higgs-numerically flat* if both $\mathfrak{E}$ and its dual are Higgs-numerically effective.

Among other things, we prove that the Chern classes of a numerically flat Higgs bundle vanish. One encounters here a similar situation as in the case of the Bogomolov inequality: semistable (ordinary) bundles satisfy the Bogomolov inequality, but this is also implied by the weaker property of Higgs-semistability. In the same line, the condition of Higgs-numerical flatness is weaker than ordinary numerical flatness, but it is nevertheless enough to ensure that all Chern classes vanish. We also prove that a Higgs bundle is Higgs-numerically flat if and only if admits a filtration whose quotients are flat stable Higgs bundles.

One of the main ingredients of our proofs is the numerical characterization of the semistability of Higgs-bundles that has been established in [3] (see Theorem 2.8 in this paper).

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2. AMPLE AND NUMERICALLY EFFECTIVE HIGGS BUNDLES

All varieties are projective varieties over the complex field. Let E be a vector bundle of rank r on X, and let s be a positive integer less than r. We shall denote by $\mathrm{Gr}_s(E)$

the Grassmann bundle of s-planes in E, with projection $p_s : \mathrm{Gr}_s(E) \to X$. When $s = 1$, the Grassmann bundle reduces to the projectivization of $\mathbb{P}E$ of E, defined as $\mathbb{P}E = \mathbf{Proj}(\mathcal{S}(E))$, where $\mathcal{S}(E)$ is the symmetric algebra of the sheaf of sections of E.

The Grassmann bundle $\mathrm{Gr}_s(E)$ is a parametrization of the rank s locally-free quotients of E. There is a universal exact sequence

$$(1) \qquad 0 \to S_{r-s,E} \xrightarrow{\psi} p_s^*(E) \xrightarrow{\eta} Q_{s,E} \to 0$$

of vector bundles on $\mathrm{Gr}_s(E)$, with $S_{r-s,E}$ the universal rank $r - s$ subbundle and $Q_{s,E}$ the universal rank s quotient bundle.

Definition 2.1. *A Higgs sheaf $\mathfrak{E}$ on X is a coherent sheaf E on X endowed with a morphism $\phi \colon E \to E \otimes \Omega_X$ of $\mathcal{O}_X$-modules such that $\phi \wedge \phi = 0$, where Ω_X is the cotangent sheaf to X. A Higgs subsheaf F of a Higgs sheaf $\mathfrak{E} = (E, \phi)$ is a subsheaf of E such that $\phi(F) \subset F \otimes \Omega_X$. A Higgs bundle is a Higgs sheaf $\mathfrak{E}$ such that E is a locally-free $\mathcal{O}_X$-module.*

There exists a stability condition for Higgs sheaves analogous to that for ordinary sheaves, which makes reference only to ϕ-invariant subsheaves.

Definition 2.2. *Let X be a smooth projective variety equipped with a polarization. A Higgs sheaf $\mathfrak{E} = (E, \phi)$ is semistable (resp. stable) if E is torsion-free, and $\mu(F) \leq \mu(E)$ (resp. $\mu(F) < \mu(E)$) for every proper nontrivial Higgs subsheaf F of $\mathfrak{E}$.*

Given a Higgs bundle $\mathfrak{E}$, we may construct closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subset \mathrm{Gr}_s(E)$ parameterizing rank s locally-free Higgs quotients, i.e., locally-free quotients of E whose corresponding kernels are ϕ-invariant. With reference to the exact sequence eq. (1), we define $\mathfrak{Gr}_s(\mathfrak{E})$ as the closed subvariety of $\mathrm{Gr}_s(E)$ where the composed morphism

$$(\eta \otimes 1) \circ p_s^*(\phi) \circ \psi \colon S_{r-s,E} \to Q_{s,E} \otimes p_s^* \Omega_X$$

vanishes. We denote by ρ_s the projections $\mathfrak{Gr}_s(\mathfrak{E}) \to X$. The restriction of (1) to the scheme $\mathfrak{Gr}_s(\mathfrak{E})$ provides the universal exact sequence

$$0 \to S_{r-s,\mathfrak{E}} \xrightarrow{\psi} \rho_s^*(E) \xrightarrow{\eta} Q_{s,\mathfrak{E}} \to 0$$

and $Q_{s,\mathfrak{E}}$ is a rank s universal Higgs quotient vector bundle, i.e., for every morphism $f \colon Y \to X$ and every rank s Higgs quotient F of f^*E there is a morphism $\psi_F \colon Y \to \mathfrak{Gr}_s(\mathfrak{E})$ such that $f = \rho_s \circ \psi_F$ and $F \simeq \psi_F^*(Q_{s,\mathfrak{E}})$. Note that the kernel $S_{r-s,\mathfrak{E}}$ of the morphism $\rho_s^*(E) \to Q_{s,\mathfrak{E}}$ is ϕ-invariant.

The scheme $\mathfrak{Gr}_s(\mathfrak{E})$ will be called the *Grassmannian of locally free rank s Higgs quotients* of $\mathfrak{E}$.

We give now our definition of ampleness and numerical effectiveness for Higgs bundles. This is inspired by, but is different from, the definition given in [3] (Def. 4.1): on the one hand it is less restrictive (though it is enough for proving the results we have in mind), while on the other hand we add a requirement on the determinant bundle to avoid the inclusion of somehow pathological situations, cf. Example 2.5 below.

Definition 2.3. *A Higgs bundle $\mathfrak{E}$ of rank one is said to be Higgs-ample if it is ample in the usual sense. If* $\mathrm{rk}\,\mathfrak{E} \geq 2$ *we require that:*

(i) *all bundles $Q_{s,\mathfrak{E}}$ are Higgs-ample;*
(ii) *the line bundle $\det(E)$ is ample.*

A Higgs bundle $\mathfrak{E}$ of rank one is said to be Higgs-numerically effective if it is numerically effective in the usual sense. If $\mathrm{rk}\,\mathfrak{E} \geq 2$ *we require that:*

(i) *all bundles $Q_{s,\mathfrak{E}}$ are Higgs-nef;*
(ii) *the line bundle $\det(E)$ is nef.*

If both $\mathfrak{E}$ and $\mathfrak{E}^*$ are Higgs-numerically effective, $\mathfrak{E}$ is said to be Higgs-numerically flat.

For short we shall use the abbreviations *H-ample* for Higgs-ample, *H-nef* for Higgs-numerically effective, and *H-nflat* for Higgs-numerically flat. Note that if $\mathfrak{E} = (E, \phi)$, with E ample in the usual sense (resp. nef) in the usual sense, than $\mathfrak{E}$ is H-ample (resp. H-nef). Moreover, if $\phi = 0$, the Higgs bundle $\mathfrak{E} = (E, 0)$ is H-ample (resp. H-nef) if and only if E is ample (resp. nef) in the usual sense.

Example 2.4. Examples of Higgs bundles that are H-nflat but not numerically flat as ordinary bundles may be constructed in terms of a Higgs bundle $\mathfrak{E} = (E, \phi)$ which is semistable as a Higgs bundle but not as an ordinary bundle. Let $\mathfrak{F} = \mathfrak{E} \otimes \mathfrak{E}^* = (E \otimes E^*, \psi)$. Since $\mathfrak{E}$ is semistable, and $c_1(E \otimes E^*) = 0$, by Proposition 3.3 below the Higgs bundle $\mathfrak{E}$ is H-nef, and hence H-nflat. On the other hand if $E \otimes E^*$ were numerically flat it would be semistable as a vector bundle (cf. Corollary 3.6), and then E would be semistable as well — a case we are excluding. $\triangle$

Example 2.5. Let us motivate the appearance of the condition that $\det(E)$ is ample, or nef, in Definition 2.3. Let X be a smooth projective curve, and let $\mathfrak{E} = (E, \phi)$ be a

rank 2 nilpotent Higgs bundle, i.e., $E = L_1 \oplus L_2$ (where L_1, L_2 are line bundles), and $\phi\colon L_1 \to L_2 \otimes \Omega_X$, $\phi(L_2) = 0$. It is shown in [3] that $\mathfrak{Gr}_1(\mathfrak{E}) = \mathbb{P}(L_1) \cup \mathbb{P}(Q)$, where

$$Q = \operatorname{coker}(\phi \otimes \operatorname{id})\colon E \otimes T_X \to E\,,$$

and T_X is the tangent bundle to X. This implies that $\mathfrak{E}$ has only two Higgs quotients, i.e., L_1 and $\bar{Q}$ which is Q modulo torsion. Note that $\deg(\bar{Q}) \geq \deg(L_1)$. If the genus of X is at least 2, one can for instance take $\deg(L_1) = 0$ and $\deg(L_2) = -2$. Without the condition that $\det(E)$ is nef this Higgs bundle, which has negative degree, would be H-nef. $\triangle$

We prove now some properties of H-nef Higgs bundles that will be useful in the sequel. These generalize properties given in [9, 4] for ordinary vector bundles.

Proposition 2.6. *Let X be a smooth projective variety.*

    (i) *A Higgs bundle $\mathfrak{E} = (E, \phi)$ on X is H-nef if and only if the Higgs bundle $\mathfrak{E} \otimes \mathcal{O}_X(D) = (E \otimes \mathcal{O}_X(D), \phi \otimes \operatorname{id})$ is H-ample for every ample $\mathbb{Q}$-divisor D in X.*

    (ii) *If $f\colon Y \to X$ is a finite surjective morphism of smooth projective varieties, and $\mathfrak{E}$ is a Higgs bundle on X, then $\mathfrak{E}$ is H-ample (resp. H-nef) if and only if $f^*\mathfrak{E}$ is H-ample (resp. H-nef).*

    (iii) *Every quotient Higgs bundle of a H-nef Higgs bundle $\mathfrak{E}$ on X is H-nef.*

Proof. (i) If $\operatorname{rk} \mathfrak{E} = 1$ this is a standard property of nef line bundles. Then we use induction on $\operatorname{rk} \mathfrak{E}$ using the fact that under the isomorphism $\mathfrak{Gr}_s(\mathfrak{E} \otimes \mathcal{O}_X(D)) \simeq \mathfrak{Gr}_s(\mathfrak{E})$ the universal quotients of $\mathfrak{E} \otimes \mathcal{O}_X(D)$ are identified with $Q_{s,\mathfrak{E}} \otimes \rho_s^* \mathcal{O}_X(D)$.

(ii) Again, this is standard in the rank one case [9]. In the higher rank case we first notice that $\det(f^*(E)) \simeq f^*(\det(E))$, so that the condition on the determinant is fulfilled. Moreover, by functoriality the morphism f induces a morphism $\bar{f}\colon \mathfrak{Gr}_s(f^*\mathfrak{E}) \to \mathfrak{Gr}_s(\mathfrak{E})$, and $Q_{s,f^*\mathfrak{E}} \simeq \bar{f}^*(Q_{s,\mathfrak{E}})$. One concludes by induction.

(iii) Let $\mathfrak{F} = (F, \phi_F)$ be a rank s Higgs quotient of $\mathfrak{E}$. This corresponds to a section $\sigma\colon X \to \mathfrak{Gr}_s(\mathfrak{E})$ such that $\mathfrak{F} \simeq \sigma^*(Q_{s,\mathfrak{E}})$. Since $Q_{s,\mathfrak{E}}$ is H-nef, $\mathfrak{F}$ is H-nef as well by the previous point. $\square$

In [10] Miyaoka introduced a numerical class λ in the projectivization $\mathbb{P}E$ which, when X is a curve, is nef if and only if E is semistable. In [3] some generalizations of the class λ were introduced. In the case of ordinary bundles one defines, for $s = 1, \ldots, r-1$,

$$\lambda_{s,E} = \left[c_1(\mathcal{O}_{\mathbb{P}Q_{s,E}}(1)) \right] - \frac{1}{r}\, q_s^*(c_1(E)) \in N^1(\mathbb{P}Q_{s,E})\,,$$

where $q_s \colon \mathbb{P}Q_{s,E} \to X$ is the natural epimorphism, and

$$\theta_{s,E} = [c_1(Q_{s,E})] - \frac{s}{r} p_s^*(c_1(E)) \in N^1(\mathfrak{Gr}_s(E)) \,.$$

Here, for every scheme Z, we denote by $N^1(Z)$ the vector space of divisors modulo numerical equivalence:

$$N^1(Z) = \frac{\mathrm{Pic}(Z)}{num.eq.} \otimes \mathbb{R}. \,.$$

It will be useful to have a formula relating the first Chern class of the universal bundles to the classes $\theta_{s,E}$, that we write in the following form.

Lemma 2.7. *One has*

$$c_1(S_{r-s,E}) = -\theta_{s,E} + \frac{r-s}{r} p_s^*(c_1(E)) \,.$$

Proof. The result is obtained by tensoring the dual of the exact sequence (1) by $Q_{s,E}$ and recalling that $T_{\mathrm{Gr}_s(E)/X} \simeq S_{r-s,E}^* \otimes Q_{s,E}$, cf. [7]. $\qquad\square$

In the case of a Higgs bundle $\mathfrak{E}$ on a smooth projective variety X one defines

$$\lambda_{s,\mathfrak{E}} = \left[c_1(\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1))\right] - \frac{1}{r}\pi_s^*(c_1(E)) \in N^1(\mathbb{P}Q_{s,\mathfrak{E}})$$

$$\theta_{s,\mathfrak{E}} = [c_1(Q_{s,\mathfrak{E}})] - \frac{s}{r} \rho_s^*(c_1(E)) \in N^1(\mathfrak{Gr}_s(\mathfrak{E})),$$

where $\pi_s \colon \mathbb{P}Q_{s,\mathfrak{E}} \to X$ and $\rho_s \colon \mathfrak{Gr}_s(\mathfrak{E}) \to X$ are again the natural epimorphisms.

Miyaoka's criterion for semistability has been generalized in [3] to Higgs bundles on smooth projective varieties of any dimension (the same criterion has been generalized to principal bundles in [1]). Let $\Delta(E)$ be the characteristic class

$$\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 = \frac{1}{2r} c_2(E \otimes E^*) \,.$$

Following theorem was partly proved in [3].

Theorem 2.8. *Let $\mathfrak{E}$ be a Higgs bundle on a smooth projective variety. The following conditions are equivalent.*

   (i) *All classes $\lambda_{s,\mathfrak{E}}$ are nef, for $0 < s < r$.*
   (ii) *$\mathfrak{E}$ is semistable and $\Delta(E) = 0$.*
   (iii) *All classes $\theta_{s,\mathfrak{E}}$ are nef, for $0 < s < r$.*
   (iv) *For any smooth projective curve C in X, the restriction $\mathfrak{E}_{|C}$ is semistable.*

Proof. The equivalence between (i) and (ii) was proved in [3].

(i) implies (iii): the class $\lambda_{s,\mathfrak{E}}$ may be regarded as the numerical class of the hyperplane bundle of the Higgs $\mathbb{Q}$-bundle $\mathfrak{F}_s = \mathfrak{Q}_{s,\mathfrak{E}} \otimes \rho_s^*(\det^{-1/r}(\mathfrak{E}))$, which therefore is nef. As a consequence, the class $\theta_{s,\mathfrak{E}} = [c_1(\mathfrak{F}_s)]$ is nef.

(iii) implies (iv): the restriction $\theta_{s,\mathfrak{E}_{|C}}$ of $\theta_{s,\mathfrak{E}}$ to $\mathfrak{Gr}_s(\mathfrak{E}_{|C})$ is nef. If $\mathfrak{E}'$ is a rank s Higgs quotient of $\mathfrak{E}_{|C}$, let $\sigma\colon C \to \mathfrak{Gr}_s(\mathfrak{E}_{|C})$ be the corresponding section. Then

$$\theta_{s,\mathfrak{E}_{|C}} \cdot [\sigma(C)] = s(\mu(E') - \mu(E_{|C})) \geq 0$$

so that $\mathfrak{E}_{|C}$ is semistable.

(iv) implies (i): Let C' be an irreducible curve in $\mathbb{P}Q_{s,\mathfrak{E}}$ and let $\overline{f}\colon C \to C'$ be it is a normalization. If $f = \pi_s \circ \overline{f}$, the pullback $f^*\mathfrak{E}$ is semistable, so that by Miyaoka's criterion the divisor $\lambda_{s,f^*\mathfrak{E}}$ is nef. Since

$$\lambda_{s,\mathfrak{E}} \cdot [C'] = \deg(\lambda_{s,f^*\mathfrak{E}}) \geq 0$$

the divisors $\lambda_{s,\mathfrak{E}}$ are nef. $\square$

This admits a simple corollary [3].

Corollary 2.9. *A semistable Higgs bundle $\mathfrak{E} = (E,\phi)$ on an n-dimensional polarized smooth projective variety (X,H) such that $c_1(E) \cdot H^{n-1} = \mathrm{ch}_2(E) \cdot H^{n-2} = 0$ is H-nflat.*

Theorem 2.8 makes use of Theorem 2 in [11], which will also be further needed in the present paper. We recall it here in a simplified form which is enough for our purposes.

Theorem 2.10. *Let $\mathfrak{E} = (E,\phi)$ be a semistable Higgs bundle on an n-dimensional polarized smooth projective variety (X,H), and assume $c_1(E) \cdot H^{n-1} = \mathrm{ch}_2(E) \cdot H^{n-2} = 0$. Then $\mathfrak{E}$ admits a filtration whose quotients are stable and have vanishing Chern classes.*

3. NUMERICALLY FLAT HIGGS BUNDLES AND STABILITY

In this section we explore some properties of Higgs bundles related to the notion of H-numerical effectiveness and H-numerical flatness. We start by showing that all Chern classes of an H-nflat Higgs bundle vanish.

Theorem 3.1. *Let $\mathfrak{E} = (E,\phi)$ be an H-nef Higgs bundle on a smooth polarized projective variety (X,H) whose first Chern class is numerically equivalent to zero, $c_1(E) \equiv 0$. Then all Chern classes $c_k(E)$ vanish.*

Proof. Since all bundles $Q_{s,\mathfrak{E}}$ are H-nef, and $c_1(E) \equiv 0$, the classes $\theta_{s,\mathfrak{E}}$ are nef, so that $\mathfrak{E}$ is semistable and $\Delta(E) = 0$ by Theorem 2.8. Since $c_1(E) \equiv 0$ then $\mathrm{ch}_2(E) \cdot H^{n-2} = 0$ (where $n = \dim X$), and Theorem 2.10 implies that all Chern classes $c_k(E)$ vanish. $\qquad\square$

Corollary 3.2. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle on a smooth polarized projective variety X. If $\mathfrak{E}$ is H-nflat, then all Chern classes $c_k(E)$ vanish.*

Proof. Since $\det(E)$ is numerically flat, the class $c_1(E)$ is numerically equivalent to zero. Moreover, $\mathfrak{E}$ is H-nef, so that Theorem 3.1 applies. $\qquad\square$

The next result generalizes Corollary 3.6 in [3] and Theorem 1.2 in [8]. The proof does not differ much from the one given in [8] but we include it here for the reader's convenience.

Proposition 3.3. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle on a smooth projective variety X such that all classes $\lambda_{s,\mathfrak{E}}$ are nef.*

   (i) *If the class $c_1(E)$ is nef, then all universal quotient bundles $Q_{s,\mathfrak{E}}$ are nef (so that $\mathfrak{E}$ is H-nef).*
   (ii) *If X is a curve and $c_1(E)$ is ample, then all universal quotient bundles $Q_{s,\mathfrak{E}}$ are ample (so that $\mathfrak{E}$ is H-ample).*
   (iii) *If $c_1(E)$ is positive (i.e., $c_1(E) \cdot [C] > 0$ for all irreducible curves $C \subset X$), then the class $c_1(Q_{s,\mathfrak{E}})$ is positive for all s.*

Proof. (i). If $Q_{s,\mathfrak{E}}$ is not nef there is an irreducible curve $C \subset \mathbb{P}Q_{s,\mathfrak{E}}$ such that $c_1(\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1)) \cdot [C] < 0$. Let $f: C' \to C$ be the normalization of C, and let $p: C' \to \mathfrak{Gr}_s(\mathfrak{E})$ be the induced map. If L is the pullback of $\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1)$ to C', then L is a Higgs quotient of $p^* \circ \rho_s^*(E)$, and

$$\deg(L) = [f(C')] \cdot c_1(\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1)) < 0\,.$$

On the other hand, one has

$$\deg(p^* \circ \rho_s^*(E)) = [p(C')] \cdot c_1(\rho_s^*(E)) \geq 0$$

since $c_1(E)$ is nef, so that

$$(2) \qquad\qquad\qquad \mu(L) < \mu(p^* \circ \rho_s^*(E)).$$

Now, in view of Theorem 2.8, the fact that all classes $\lambda_{s,\mathfrak{E}}$ are nef implies that $\mathfrak{E}$ is semistable, and also that the restriction of $\mathfrak{E}$ to any smooth projective curve in X is semistable. Combining this with Lemma 3.3 in [3], one shows that $p^* \circ \rho_s^*(E)$ is semistable. But then eq. (2) is a contradiction.

(ii). This proof is a slight variation of the previous one, due to the fact that Nakai's criterion for ampleness requires to check positive intersections with subvarieties of all dimensions. Let C be a smooth projective curve and $f\colon C \to X$ a morphism which is of degree larger than $r = \operatorname{rk} E$. Given a point $p \in C$ let F be the class of the fibre of $\mathbb{P}(f^*Q_{s,\mathfrak{E}})$ over p. The Higgs bundle $\mathfrak{E}' = f^*\mathfrak{E} \otimes \mathcal{O}_C(-p)$ is semistable by the same argument as in the previous proof. Moreover, $\deg(\mathfrak{E}') > 0$, so that $\mathfrak{E}'$ is H-nef by the previous point. If L is the pullback to C of the bundle $\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1)$, then $L(-F)$ is nef since it is the hyperplane bundle in $\mathbb{P}Q_{s,\mathfrak{E}'}$. If V is any subvariety of $\mathbb{P}(f^*Q_{s,\mathfrak{E}})$ of dimension k, then $c_1(L)^k \cdot [V] > 0$, so that L is ample. Thus the pullback of $\mathfrak{E}$ to C is H-ample, and hence $\mathfrak{E}$ is H-ample as well by Proposition 2.6.

Claim (iii) is proved as Claim (ii). $\qquad\qquad\square$

Corollary 3.4. *Given a Higgs bundle $\mathfrak{E} = (E,\phi)$, if all classes $\lambda_{s,\mathfrak{E}}$ are nef, and $c_1(E)$ is numerically equivalent to zero, then $\mathfrak{E}$ is H-nflat.*

We study now the relations between the conditions of H-numerical effectiveness and flatness and (semi)stability.

Proposition 3.5. *Let $\mathfrak{E} = (E,\phi)$ be a Higgs bundle on a smooth polarized projective variety X, such that all universal quotients $Q_{s,\mathfrak{E}}$ and $Q_{s,\mathfrak{E}^*}$ are nef. Then $\mathfrak{E}$ is semistable. If $\deg(E) \neq 0$, then $\mathfrak{E}$ is stable.*

Proof. Under the isomorphism $\mathfrak{Gr}_{r-s}(\mathfrak{E}^*) \simeq \mathfrak{Gr}_s(\mathfrak{E})$ the bundle $S_{r-s,\mathfrak{E}}$ is identified with $Q^*_{r-s,\mathfrak{E}^*}$. Therefore all the universal quotient bundles $Q_{s,\mathfrak{E}}$ and the bundles $S^*_{r-s,\mathfrak{E}}$ on $\mathfrak{Gr}_s(\mathfrak{E})$ are nef. From Lemma 2.7 we have, after restricting to $\mathfrak{Gr}_s\mathfrak{E}$,

$$(3) \qquad\qquad c_1(S^*_{r-s,\mathfrak{E}}) = \theta_{s,\mathfrak{E}} + \frac{s-r}{r}\rho_s{}^*(c_1(E))\,.$$

By [4, Prop. 1.2 (11)] this class is nef.

Assume for a while that X is a curve, and let us suppose that $\deg(E) \geq 0$. By a slight generalization of [5, Prop. 1.8(i)] or [6, Prop. 2.2], the class $p_s^*(c_1(E))$ is positive, and as $\mathfrak{Gr}_s(\mathfrak{E})$ is a closed subscheme of $\operatorname{Gr}_s(E)$, the class $\rho_s^*(c_1(E))$ is positive as well. But since $c_1(S^*_{r-s,\mathfrak{E}})$ is nef this implies that all classes $\theta_{s,\mathfrak{E}}$ are nef and so from Proposition 2.8 it follows that $\mathfrak{E}$ is semistable.

If $\deg(E) \leq 0$, the same argument shows that $\mathfrak{E}^*$ is semistable, and then $\mathfrak{E}$ is semistable as well.

We now show that if $\deg(E) \neq 0$ then $\mathfrak{E}$ is stable. Assume for instance that $\deg(E) > 0$. Proposition 3.3 proves that in this case $c_1(Q_{s,\mathfrak{E}}) > 0$ for all s. Without loss of generality we may assume that $\mathfrak{Gr}_s(\mathfrak{E})$ has a section $\sigma : X \to \mathfrak{Gr}_s(\mathfrak{E})$. Then the bundle $Q_\sigma = \sigma^*(Q_{s,\mathfrak{E}})$ is an ample Higgs quotient of $\mathfrak{E}$. So one has the exact sequence

$$0 \to K \to E \to Q_\sigma \to 0$$

and $-c_1(K) = c_1(Q_\sigma \otimes \det^{-1} E) = \sigma^*(c_1(S_{r-s,\mathfrak{E}}^*))$ is nef as well. Thus $c_1(K) \leq 0$ and $\mu(K) < \mu(E)$. Hence $\mathfrak{E}$ is stable. If $\deg(E) < 0$ by applying the same argument to the dual of $\mathfrak{E}$ we obtain that $\mathfrak{E}^*$ is stable, and hence $\mathfrak{E}$ is stable again.

These results are then extended to higher dimensional X with the usual induction on the dimension of X, by considering a smooth divisor in the linear system $|mH|$ for m big enough. $\qquad\square$

Corollary 3.6. *An H-nflat Higgs bundle is semistable.*

Proof. It is enough to check that if $\mathfrak{E}$ is H-nef and $c_1(E) \equiv 0$, then all universal quotient bundles $Q_{s,\mathfrak{E}}$ are nef. Indeed, in this case the classes $\theta_{s,\mathfrak{E}} = [c_1(Q_{s,\mathfrak{E}})]$ are nef, so that the classes $\lambda_{s,\mathfrak{E}}$ are nef by Theorem 2.8. But $\lambda_{s,\mathfrak{E}} = [c_1(\mathcal{O}_{\mathbb{P}Q_{s,\mathfrak{E}}}(1))]$, so that $Q_{s,\mathfrak{E}}$ is nef. $\qquad\square$

Corollary 3.7. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle on a smooth projective curve X, such that all universal quotients $Q_{s,\mathfrak{E}}$ and $Q_{s,\mathfrak{E}^*}$ are nef. Assume that $\mathfrak{E}$ is properly semistable, i.e., it is not stable. Then $\mathfrak{E}$ is H-nflat.*

Proposition 3.5 raises the question of the existence of stable H-nflat Higgs bundles. An example bundle is provided by a Higgs bundle $\mathfrak{E}$ as in Example 2.5 with $\deg(L_1) = 1$, $\deg(L_2) = -1$ and the genus of the curve X at least 2. Then $\mathfrak{E}$ is stable and H-nflat.

Proposition 3.5 has a simple consequence, which generalizes [5, Thm. 1.18]. This provides an important characterization of H-nflat Higgs bundles.

Theorem 3.8. *A Higgs bundle $\mathfrak{E}$ on X is H-nflat if and only if it admits a filtration whose quotients are flat stable Higgs bundles.*

Proof. If $\mathfrak{E}$ is H-nflat by Corollary 3.6 it is semistable. Since all Chern classes of $\mathfrak{E}$ vanish, by Theorem 2.10 $\mathfrak{E}$ has a filtration whose quotients are stable and have vanishing Chern classes. We may assume that $\mathfrak{E}$ is an extension

$$(4) \qquad\qquad\qquad 0 \to \mathfrak{F} \to \mathfrak{E} \to \mathfrak{G} \to 0$$

of stable Higgs bundles with vanishing Chern classes, otherwise one simply iterates the following argument. Let us consider the bundle $\mathfrak{F} = (F, \phi_F)$; the same will apply to $\mathfrak{G}$. By results given in [11], the bundle F admits a Hermitian-Yang-Mills metric. Let Ω be the curvature of the associated Chern connection. Since $c_1(F) = c_2(F) = 0$, we have

$$0 = \int_X \operatorname{tr}(\Omega \wedge \Omega) \cdot H^{n-2} = \gamma_1 \|\Omega\|^2 - \gamma_2 \|\Lambda \Omega_i\|^2 = \gamma_1 \|\Omega\|^2$$

for some positive constants γ_1, γ_2, so that the Chern connection of F is flat, i.e., F is flat.

Conversely, let assume that $\mathfrak{E}$ has a filtration as in the statement. Then $\mathfrak{E}$ is semistable with vanishing Chern classes, and by Corollary 2.9 it is numerically flat. $\qquad \square$

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