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CANONICAL AND PERTURBATIVE QUANTUM GRAVITY

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Abstract. After a review of Dirac's theory of constrained Hamiltonian systems and their quantization, canonical quantum gravity is studied relying on the Arnowitt-Deser-Misner formalism. First-class constraints of the theory are studied in some detail following De Witt's work, and geometrical and topological properties of Wheeler's superspace are discussed following the mathematical work of Fisher.

Perturbative quantum gravity is then formulated in terms of amplitudes of going from a three-metric and a matter-field configuration on a spacelike surface Σ to a three-metric and a field configuration on a spacelike surface Σ' . The Wick-rotated quantum amplitudes are here studied under the assumption that the analytic continuation to the real Riemannian section of the complexified space-time is possible, but this is not a generic property. Within the background-field method, one then expands both the four-metric g and matter fields ϕ about a configuration (g_0, ϕ_0) which is a solution of the classical equations of motion. If the one-loop approximation holds, the part of the action quadratic in the fluctuations about (g_0, ϕ_0) gives the dominant contribution to the quantum amplitudes. This leads to Gaussian integrals and to formally divergent amplitudes, since the one-loop result involves the determinant of second-order elliptic operators.

The corresponding divergences are regularized using the zeta-function method. For this purpose, following Hawking, one first defines a generalized zeta-function $\zeta(s)$ obtained from the eigenvalues of the elliptic operator $\mathcal{B}$ appearing in the calculation. Such $\zeta(s)$ can be analytically extended to a meromorphic function which only has poles at some finite values of s . The values of ζ and its first derivative at the origin enable one to express the one-loop quantum amplitudes, whose scaling properties only depend on $\zeta(0)$ under suitable

assumptions on the measure in the path integral. Although it frequently happens that the eigenvalues of $\mathcal{B}$ cannot be computed exactly, the regularized $\zeta(0)$ value can be obtained by studying the heat equation for the elliptic operator $\mathcal{B}$. The corresponding integrated heat kernel $G(\tau)$ has an asymptotic expansion as $\tau \rightarrow 0^+$ for those boundary conditions which ensure self-adjointness of $\mathcal{B}$. The $\zeta(0)$ value is then given by the constant term in the asymptotic form of $G(\tau)$, and it also determines the one-loop divergences of physical theories. The zeta-function technique has been recently applied to the study of one-loop properties of supersymmetric field theories in the presence of boundaries.

Some relevant examples of gravitational background fields are then studied. These gravitational instantons are complete, four-dimensional Riemannian manifolds whose metric solves the Einstein equations with cosmological constant: $R(X,Y) - \Lambda g(X,Y) = 0$. The possible boundary conditions are asymptotically Euclidean, asymptotically locally Euclidean, asymptotically flat, asymptotically locally flat, compact without boundary.

1. Introduction

Despite the lack of a mathematically consistent theory of quantum gravity, the quantization of general relativity or of alternative theories of gravitation is still receiving careful consideration in the current literature. The two main motivations for this analysis are the singularity problem in cosmology, and various attempts to understand perturbative or non-perturbative renormalization of quantum field theories [1]. The aim of this review paper is therefore to present, in a way accessible to a large number of physicists and mathematicians, the minimal set of well-established mathematical techniques that should be familiar to all those working on quantum gravity.

For this purpose, in sections 2, 3 and 4 Dirac's theory of constrained Hamiltonian systems is described, discussing in detail primary and secondary constraints, first-class and second-class constraints, Dirac brackets, effective Hamiltonian, total Hamiltonian and extended Hamiltonian. Dirac's prescription for quantizing first-class and second-class systems is then given, jointly with examples and discussion. The gravitational field, which is a first-class constrained system, is then studied in section 5, using the Arnowitt-Deser-Misner $3 + 1$ decomposition of space-time geometry, and applying Dirac's theory to derive constraints and write down the quantum version of these constraints. These can be regarded as functional differential equations on Wheeler's superspace, i.e. a quotient space whose points are equivalence classes of metrics related by the action of the diffeomorphism group of a compact spacelike three-surface. More precisely, the superspace $S(\mathcal{M})$ is defined as $S(\mathcal{M}) \equiv \text{Riem}(\mathcal{M})/\text{Diff}(\mathcal{M})$. With this notation, $\mathcal{M}$ is a compact, connected,

orientable, Hausdorff, C^∞ three-manifold without boundary. $Riem(\mathcal{M})$ is the space of C^∞ Riemannian metrics on $\mathcal{M}$, and $Diff(\mathcal{M})$ is the group of C^∞ orientation-preserving diffeomorphisms of $\mathcal{M}$. Geometrical and topological properties of Wheeler's superspace are thus studied in section 6.

In section 7, the problems of the path-integral approach to quantum gravity are first presented. Although the full theory via path integrals does not exist, and although quantized general relativity is perturbatively non-renormalizable [which has led to the study of non-perturbative renormalization via canonical quantum gravity], the one-loop approximation can be studied in detail in some cases. The corresponding Gaussian integrals are regularized using the generalized-zeta-function technique in curved backgrounds (section 8). This involves the fascinating study of the eigenvalues of self-adjoint elliptic operators. The background gravitational fields are studied in section 9 in the asymptotically locally Euclidean, asymptotically flat, and compact cases. Concluding remarks and open research problems are presented in section 10.

2. Hamiltonian Methods in Physics

It is quite often the case that theories of interest in modern physics are formulated as constrained systems. This happens whenever the Lagrangian L is singular (we begin by studying the case of a finite number of degrees of freedom for simplicity), so that there is no unique solution of Hamilton's equations of motion expressing the velocities in terms of

the canonical coordinates q^i and conjugate momenta p_i (i.e. $\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0$). One then finds that certain functions $\varphi_m^{(1)}(q, p)$ exist such that

$$\varphi_m^{(1)}(q, p) \approx 0 \quad . \quad (2.1)$$

Following [2] and [3], we say that the primary constraint $\varphi_m^{(1)}$ is weakly zero; in other words, in working out Poisson brackets on the phase space with other canonical variables, $\varphi_m^{(1)}$ can be set to zero only *after* these brackets have been computed, and some of these brackets do not vanish. Note that, on the constraint manifold Σ defined by (2.1), no Poisson bracket can be defined, since if some $F(q, p) \approx 0$ on Σ , its gradient does not necessarily vanish weakly on Σ . The problem thus arises to *extend* out of Σ the Legendre transform H_c of L by adding a linear combination of constraints. One is then led to define an effective Hamiltonian $\tilde{H}$ given by the sum

$$\tilde{H} = H_c + u_m(q, p)\varphi_m^{(1)}(q, p) \quad . \quad (2.2)$$

Note also that the coefficients u_m of linear combination are not constants, but depend on the canonical variables q, p . In light of (2.1), the new equations of motion generated by $\tilde{H}$ are

$$\dot{q}^i \equiv \left\{ q^i, \tilde{H} \right\} \approx \frac{\partial H_c}{\partial p_i} + u_m(q, p) \frac{\partial \varphi_m^{(1)}}{\partial p_i} \quad , \quad (2.3)$$

$$\dot{p}_i \equiv \left\{ p_i, \tilde{H} \right\} \approx -\frac{\partial H_c}{\partial q^i} - u_m(q, p) \frac{\partial \varphi_m^{(1)}}{\partial q^i} \quad , \quad (2.4)$$

where curly brackets denote Poisson brackets. We now have to make sure that the primary constraints $\varphi_m^{(1)}$ are preserved in time. This implies that

$$\dot{\varphi}_m^{(1)} \equiv \left\{ \varphi_m^{(1)}, \tilde{H} \right\} \approx \left\{ \varphi_m^{(1)}, H_c \right\} + u_j(q, p) \left\{ \varphi_m^{(1)}, \varphi_j^{(1)} \right\} \approx 0 \quad . \quad (2.5)$$

Essentially three possibilities occur:

- (1) Eq. (2.5) already holds by virtue of (2.1);
- (2) Eq. (2.5) can be solved for $u_j = u_j(q, p)$;
- (3) Eq. (2.5) leads to secondary constraints $\varphi_j^{(2)}(q, p)$ independent of $u_h(q, p)$.

The outlined procedure is then repeated until all secondary constraints, and finally all $u_j = u_j(q, p)$ have been found, so that $\tilde{H} = \tilde{H}(q, p)$. The set S_c of all constraints is then given by (we define secondary all constraints which are not primary)

$$S_c \equiv \left\{ \varphi_m^{(1)}, m = 1, \dots, L; \quad \varphi_j^{(2)}, j = 0, 1, \dots, M \right\} \quad . \quad (2.6)$$

In geometrical language, one can compute the secondary constraints $\varphi_j^{(2)}(q, p)$ by defining the vector field (cf. (2.3-4))

$$\Gamma \equiv \left(\frac{\partial H_c}{\partial p_i} + u_m(q, p) \frac{\partial \varphi_m^{(1)}}{\partial p_i} \right) \frac{\partial}{\partial q^i} - \left(\frac{\partial H_c}{\partial q^i} + u_m(q, p) \frac{\partial \varphi_m^{(1)}}{\partial q^i} \right) \frac{\partial}{\partial p_i} \quad ,$$

and then taking the Lie derivative $L_\Gamma \varphi_m^{(1)}$.

A more convenient (and fundamental) division of the set S_c can be, however, obtained. For this purpose, we note that the Poisson bracket of any two elements of S_c may or may not be a linear combination of constraints. This property can be made precise by giving the following definitions [2,3]:

Definition 2.1. The function $F(q, p)$ is a first-class quantity if

$$\{F, \varphi\} \approx 0 \quad , \quad \forall \varphi \in S_c \quad . \quad (2.7)$$

Definition 2.2. The function $F(q, p)$ is a second-class quantity if

$$\exists \varphi \in S_c : \{F, \varphi\} \not\approx 0 \quad . \quad (2.8)$$

Note that a number of second-class constraints might be brought into the first class by means of *independent* linear combinations. We call irreducible those second-class constraints which cannot be brought into the first class. In what follows, we always assume the second-class constraints we deal with are irreducible. First-class constraints are here denoted by $\varphi_m^{(I)}(q, p)$, and second-class constraints by $\varphi_j^{(II)}(q, p)$. An equivalent definition of the set S_c of all constraints is then given by

$$S_c \equiv \left\{ \varphi_m^{(I)}, m = 0, 1, \dots, L'; \quad \varphi_j^{(II)}, j = 0, 1, \dots, M' \right\} \quad , \quad (2.9)$$

where $L' + M' = L + M$. Moreover, with our notation, $\varphi_m^{(1,I)}(q, p)$ and $\varphi_m^{(1,II)}(q, p)$ denote primary first-class and primary second-class constraints, whereas $\varphi_m^{(2,I)}(q, p)$ and $\varphi_m^{(2,II)}(q, p)$ denote secondary first-class and secondary second-class constraints respectively.

Relevant examples of first-class Hamiltonian systems are electromagnetism and general relativity, whereas second-class constraints occur for example in relativistic theories of gravitation with nonvanishing torsion, or after imposing gauge constraints on a first-class system.

The definition of second-class constraints enables one to understand which quantities can be set to zero also *before* computing any bracket. For this purpose, one first proves [2] that the matrix $C_{lm} = \left\{ \varphi_l^{(II)}, \varphi_m^{(II)} \right\}$ of Poisson brackets of second-class constraints is non-singular. The second step [3] is to define, for any given $G(q, p)$, a new variable $\tilde{G}(q, p)$ given by

$$\tilde{G} \equiv G - \left\{ G, \varphi_l^{(II)} \right\} C_{lm}^{-1} \varphi_m^{(II)} \quad , \quad (2.10)$$

so that $\left\{ \tilde{G}, \varphi_r^{(II)} \right\} \approx 0, \forall r \in \{0, 1, \dots, M'\}$. In other words, $\tilde{G}$ has vanishing Poisson brackets with all second-class constraints, whereas $\left\{ \tilde{G}, \varphi_m^{(I)} \right\}$ may be $\neq 0$ for $m \in \{0, 1, \dots, L'\}$. Thus, defining the Dirac bracket of $F_1(q, p)$ and $F_2(q, p)$ as

$$\left\{ F_1, F_2 \right\}^* \equiv \left\{ F_1, F_2 \right\} - \left\{ F_1, \varphi_l^{(II)} \right\} C_{lm}^{-1} \left\{ \varphi_m^{(II)}, F_2 \right\} \quad , \quad (2.11)$$

one finds the fundamental results

$$\left\{ F_1, F_2 \right\}^* \approx \left\{ \tilde{F}_1, \tilde{F}_2 \right\} \approx \left\{ \tilde{F}_1, F_2 \right\} \approx \left\{ F_1, \tilde{F}_2 \right\} \quad , \quad (2.12)$$

$$\left\{ G, \varphi_n^{(II)} \right\}^* \approx \left\{ G, \varphi_n^{(II)} \right\} - \left\{ G, \varphi_l^{(II)} \right\} C_{lm}^{-1} C_{mn} = 0 \quad . \quad (2.13)$$

In other words, second-class constraints are now strongly vanishing, and Dirac brackets are the tool needed to achieve this and deal only with first-class constraints in the classical theory (see also section 4).

The Hamiltonians we are interested in are essentially of four kinds:

(1) The canonical Hamiltonian H_c , i.e. the Legendre transform of L . This is the relevant Hamiltonian if L is non-singular;

(2) The effective Hamiltonian $\tilde{H}$, i.e.

$$\tilde{H} \equiv H_c + u_m(q, p) \varphi_m^{(1)}(q, p) = H_c - \left\{ H_c, \varphi_l^{(1)} \right\} C_{lm}^{-1} \varphi_m^{(1)} \quad . \quad (2.14)$$

This is the relevant Hamiltonian for theories with second-class constraints (section 4).

(3) The total Hamiltonian H_T , i.e.

$$H_T \equiv \tilde{H} + r_m(q, p) \varphi_m^{(1,I)}(q, p) \quad . \quad (2.15)$$

(4) The extended Hamiltonian H_E , i.e.

$$\begin{aligned} H_E &\equiv \tilde{H} + r_m^{(A)}(q, p) \varphi_m^{(1,I)}(q, p) + r_j^{(B)}(q, p) \varphi_j^{(2,I)}(q, p) \\ &= H_T + r_j^{(B)}(q, p) \varphi_j^{(2,I)}(q, p) \quad . \end{aligned} \quad (2.16)$$

Thus H_T is given by the effective Hamiltonian $\tilde{H}$ plus a linear combination of primary first-class constraints, whereas H_E is given by H_T plus a linear combination of secondary first-class constraints. This means that it is possible to include secondary constraints in the Hamiltonian, provided they are first-class.

In fact, as explained in [2] (see below), there are certain changes in the q 's and p 's that do not correspond to a change of state, and which have as generators secondary first-class constraints. One is then led to generalize the equations of motion in order to allow as variation of a dynamical variable also any variation which does not correspond to a change of state. This is obtained using the extended Hamiltonian H_E . The Hamiltonians $\tilde{H}$, H_T and H_E are sharply distinguished ways of defining a Hamiltonian function on the whole

phase space, and they all reduce to H_c on the constraint manifold defined by (2.1) (and by (2.6) in the case of H_E). In particular, the H_E -formalism is reasonable from the physical point of view (see below), although it does not correspond to the original one deduced from a Lagrangian [2,4].

Note that, in the case of first-class constrained systems, (2.15) becomes

$$H_T \equiv H_c + s_m(q, p) \varphi_m^{(1, I)}(q, p) \quad . \quad (2.17)$$

For example, this is what happens for electromagnetism. An intermediate step is also possible for $U(1)$ gauge theory (and general relativity), following [2]. Namely, one only includes secondary first-class constraints in the Hamiltonian H_I defined as

$$H_I \equiv H_c + r_j^{(B)}(q, p) \varphi_j^{(2, I)}(q, p) \quad . \quad (2.18)$$

Whenever the extended Hamiltonian H_E is used, the equations of motion take the form

$$\begin{aligned} \dot{q}^i &\equiv \{q^i, H_E\} \approx \{q^i, \tilde{H}\} \\ &+ r_m^{(A)}(q, p) \{q^i, \varphi_m^{(1, I)}\} + r_j^{(B)}(q, p) \{q^i, \varphi_j^{(2, I)}\} \quad , \end{aligned} \quad (2.19)$$

$$\begin{aligned} \dot{p}_i &\equiv \{p_i, H_E\} \approx \{p_i, \tilde{H}\} \\ &+ r_m^{(A)}(q, p) \{p_i, \varphi_m^{(1, I)}\} + r_j^{(B)}(q, p) \{p_i, \varphi_j^{(2, I)}\} \quad . \end{aligned} \quad (2.20)$$

The physical state of the system is not affected by the infinitesimal contact transformations generated by the first-class constraints of the theory [2,3]. This is a crucial point, and Dirac's original argument is as follows [2].

Given at time t any function $g_t(q, p)$, we study its time evolution. At time $t + \epsilon$, $g_{t+\epsilon}$ is found by definition as

$$g_{t+\epsilon} = g_t + \epsilon \left\{ g, H_T \right\} . \quad (2.21)$$

Setting $r_m = 0$ in (2.15), this yields

$$g_{t+\epsilon} = g_t + \epsilon \left\{ g, \tilde{H} \right\} . \quad (2.22)$$

However, we may also take $r_m = c_m \neq 0$. This leads to

$$\hat{g}_{t+\epsilon} = g_t + \epsilon \left\{ g, \tilde{H} \right\} + \epsilon c_m \left\{ g, \varphi_m^{(1,I)} \right\} . \quad (2.23)$$

Both choices must correspond to the same physical state at time $t + \delta t$, since the physical state at $t + \epsilon$ is the one arising from the given initial physical state at time t . From (2.22-23) we find

$$\hat{g}_{t+\epsilon} - g_{t+\epsilon} = \epsilon c_m \left\{ g, \varphi_m^{(1,I)} \right\} . \quad (2.24)$$

Thus all primary first-class constraints $\varphi_m^{(1,I)}$, regarded as generators of a contact transformation, give rise to a transformation which does not change the physical state.

Dirac's next step is to consider two of these transformations. Denoting by $\chi_a^{(1,I)}$ and $\psi_a^{(1,I)}$ two primary first-class constraints, the first transformation changes g into $g + \epsilon \left\{ g, \chi_a^{(1,I)} \right\}$, and the second changes $g + \epsilon \left\{ g, \chi_a^{(1,I)} \right\}$ into

$$g_1 = g + \epsilon \left\{ g, \chi_a^{(1,I)} \right\} + \epsilon' \left\{ g + \epsilon \left\{ g, \chi_a^{(1,I)} \right\}, \psi_a^{(1,I)} \right\} . \quad (2.25)$$

He then neglects ϵ^2 and ϵ'^2 , but retains $\epsilon\epsilon'$ in the calculation. Even though ϵ^2 , ϵ'^2 and $\epsilon\epsilon'$ are of the same order, this approximation is valid, since otherwise, by retaining terms involving ϵ^2 and ϵ'^2 , one would obtain an equation holding for all values of ϵ and ϵ' ; one has thus to set to zero the coefficients of ϵ^2 , of ϵ'^2 and of $\epsilon\epsilon'$. This would lead to three equations, but the first two are trivial so that one is not interested in them. Now, by applying the two transformations in the reverse order, one obtains

$$g_2 = g + \epsilon \left\{ g, \chi_a^{(1,I)} \right\} + \epsilon' \left\{ g, \psi_a^{(1,I)} \right\} + \epsilon\epsilon' \left\{ \left\{ g, \psi_a^{(1,I)} \right\}, \chi_a^{(1,I)} \right\} . \quad (2.26)$$

In light of (2.25), this leads to

$$g_1 - g_2 = \epsilon\epsilon' \left\{ g, \left\{ \chi_a^{(1,I)}, \psi_a^{(1,I)} \right\} \right\} . \quad (2.27)$$

Thus, using the group property of all transformations which leave the physical state unchanged, Dirac finds there must be further transformations of this type which do not affect the physical state. He then points out that the only generalization of the argument is that the primary first-class constraints $\chi_a^{(1,I)}$ and $\psi_a^{(1,I)}$ might be replaced by secondary first-class constraints $\rho_a^{(2,I)}$ and $\sigma_a^{(2,I)}$ also leading to an equation formally identical to (2.27) and thus generating transformations which do not change the physical state.

However, Dirac made it clear he had not been able to obtain a rigorous mathematical proof that all first-class constraints, whether primary or secondary, do not change the physical state. This has been proved only much later in [5]. As discussed in detail in [5,6], if a function $f(q, p)$ is first-class, it must be gauge-invariant, and

$$\dot{f}(q, p) \approx \left\{ f, H_T \right\} \approx \left\{ f, H_E \right\} . \quad (2.28)$$

In other words, the total Hamiltonian H_T and the extended Hamiltonian H_E generate the same time evolution for the gauge-invariant functions $f(q, p)$, and are thus physically equivalent. By contrast, if $f(q, p)$ is gauge-dependent, H_T and H_E generate *different* equations of motion, and the two formalisms cannot be compared. The detailed calculations in [6] add evidence in favour of Dirac's interpretation of first-class constraints being correct.

3. Dirac's Quantization of First-Class Constrained Hamiltonian Systems

Suppose we deal with a theory where all constraints are first-class. The canonical coordinates q^i , with conjugate momenta p_i , are made into operators satisfying canonical commutation relations (hereafter referred to as CCR) corresponding to the Poisson brackets of the classical theory. The mathematically rigorous form of these CCR is the exponentiated Weyl form [7,8]

$$U(a_1)U(a_2) = U(a_1 + a_2) \quad , \quad (3.1)$$

$$V(b_1)V(b_2) = V(b_1 + b_2) \quad , \quad (3.2)$$

$$U(a)V(b) = e^{i\hbar ab}V(b)U(a) \quad , \quad (3.3)$$

where $U(a) = e^{-ia\hat{p}}$ and $V(b) = e^{-ib\hat{q}}$. By virtue of the Stone-von Neumann theorem, the unique (up to unitary equivalence) unitary representation of (3.1-3) is

$$(V(b)\psi)(q) = e^{-ibq}\psi(q) \quad , \quad (3.4)$$

$$(U(a)\psi)(q) = \psi(q - \hbar a) \quad . \quad (3.5)$$

The generators of $U(a)$ and $V(b)$ are $\hat{p}$ and $\hat{q}$ respectively, and satisfy the familiar relations

$$(\hat{q}\psi)(q) \equiv q\psi(q) \quad , \quad (3.6)$$

$$(\hat{p}\psi)(q) \equiv -i\hbar \frac{\partial \psi}{\partial q}(q) \quad , \quad (3.7)$$

together with the CCR

$$[\hat{q}, \hat{p}] = i\hbar \quad . \quad (3.8)$$

This holds on the dense domain of infinitely differentiable functions of compact support.

Of course, the more general form of (3.8) we are interested in is

$$[\hat{q}^i, \hat{p}_j] = i\hbar \delta_j^i \quad . \quad (3.9)$$

We then study a Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}_T \psi \quad , \quad (3.10)$$

where ψ is the wave function, and $\hat{H}_T$ the suitably defined operator corresponding to the first-class Hamiltonian (2.17), also denoted by $\hat{H}'$ in the literature [2,3]. The next step in the quantization program is to impose *all first-class constraints* as supplementary conditions on the wave function, so that

$$\hat{\varphi}_l^{(I)} \psi = 0 \quad . \quad (3.11)$$

The naturally-occurring question is whether the equations (3.11) are consistent with one another. Indeed, for $l' \neq l$, we know that

$$\hat{\varphi}_{l'}^{(I)} \psi = 0 \quad . \quad (3.12)$$

We now multiply (3.12) by $\hat{\varphi}_l^{(I)}$ and (3.11) by $\hat{\varphi}_{l'}^{(I)}$. The subtraction of the resulting equations leads to

$$\left[\hat{\varphi}_l^{(I)}, \hat{\varphi}_{l'}^{(I)} \right] \psi = 0 \quad . \quad (3.13)$$

Note that, in the classical theory, (3.13) would be obviously satisfied, since by definition the Poisson bracket of any two first-class constraints is again a linear combination of first-class constraints. In the quantum theory, however, it is not *a priori* obvious that

$$\left[\hat{\varphi}_l^{(I)}, \hat{\varphi}_{l'}^{(I)} \right] = c_{ll'm}(q, p) \hat{\varphi}_m^{(I)}(q, p) \quad . \quad (3.14)$$

In other words, for (3.13) to be a consequence of (3.11-12), the *additional* condition (3.14) should hold. Since $c_{ll'm}$ depends on all q 's and p 's, it does not commute with the $\hat{\varphi}_m^{(I)}$ in the quantum theory. The problem is thus to make sure that $c_{ll'm}$ appears on the left in (3.14), and no extra terms occur.

If (3.14) holds, the first-class constraints are consistently imposed as supplementary conditions on ψ . By contrast, if it is not possible to obtain (3.14) with the help of suitable factor-ordering prescriptions, the quantum theory we are looking for is ill-defined. In the favourable case, one should go on, and check whether the conditions (3.11) are also consistent with the Schrödinger equation (3.10). Since $\hat{H}_T$ is a first-class Hamiltonian, this means one should find

$$\left[\hat{\varphi}_l^{(I)}, \hat{H}_T \right] = 0 \quad , \quad (3.15)$$

which holds provided

$$\left[\hat{\varphi}_l^{(I)}, \hat{H}_T \right] = b_{lm}(q, p) \hat{\varphi}_m^{(I)}(q, p) \quad . \quad (3.16)$$

Again, we find a condition which is certainly satisfied in the classical theory, whereas in the quantum theory there may be serious problems, since b_{lm} depends on the q 's and p 's, and thus does not necessarily appear on the left when we compute the commutator $[\hat{\varphi}_l^{(I)}, \hat{H}_T]$.

4. Dirac's Quantization of Second-Class Constrained Hamiltonian Systems

In section 2 we remarked that irreducible second-class constraints can be eliminated in the classical theory, after they are set strongly to zero by using the Dirac brackets (2.11). Following [2], it can be instructive to see what happens in the simplest case, i.e. when two second-class constraints exist of the form

$$q^l \approx 0 \quad , \quad p_l \approx 0 \quad . \quad (4.1)$$

Of course, the corresponding quantum operators cannot be used to impose supplementary conditions on the wave function of the form

$$\hat{q}^l \psi = 0 \quad , \quad \hat{p}_l \psi = 0 \quad , \quad \Rightarrow [\hat{q}^l, \hat{p}_l] \psi = 0 \quad ,$$

since these would be inconsistent with the CCR (3.9)

$$[\hat{q}^l, \hat{p}_l] \psi = i\hbar \psi \quad .$$

Now, by virtue of (4.1), one might point out that q^l and p_l are not of interest. One is thus led to define a new bracket $\left\{ \quad , \quad \right\}^*$ in the classical theory, where the degree of freedom

corresponding to the index l has been discarded, so that

$$\{A, B\}^* \equiv \sum_{n \neq l} \left(\frac{\partial A}{\partial q^n} \frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n} \frac{\partial B}{\partial q^n} \right) . \quad (4.2)$$

One then tries to set up the quantum theory in terms of all degrees of freedom but for the value $n = l$. We are then looking for an operator representation of Dirac brackets, such that second-class constraints are realized strongly, i.e. as equations between operators. In the above example, since Dirac brackets are

$$\{q^l, p_l\}^* = 0 \quad , \quad (4.3)$$

$$\{q^i, p_j\}^* = \delta_j^i \quad , \quad \forall i, j \neq l \quad , \quad (4.4)$$

an irreducible representation of Dirac brackets is given by

$$\hat{Q}^l = \hat{p}_l = 0 \quad , \quad (4.5)$$

$$\hat{Q}^i \psi = q^i \psi \quad , \quad \forall i \neq l \quad , \quad (4.6)$$

$$\hat{p}_i \psi = -i\hbar \frac{\partial \psi}{\partial q^i} \quad , \quad \forall i \neq l \quad , \quad (4.7)$$

i.e. the usual Schrödinger representation for q^i and p_i , $\forall i \neq l$, described in section 3, and the zero operator for $\hat{Q}^l$ and $\hat{p}_l$. Dirac's quantization program when second-class constraints exist can be thus described as follows.

- (1) One picks out irreducible second-class constraints;
- (2) Using Dirac brackets (2.11), these constraints are set strongly to zero;

(3) Since, for any $g(q, p)$, $\{g, H_T\}^* \approx \{g, H_T\}$, the classical equations of motion take the form

$$\dot{g} \approx \{g, H_T\}^* \quad ;$$

(4) On quantization, commutation relations are taken to correspond to Dirac-bracket relations, and second-class constraints are realized as equations between operators;

(5) With the exception of some (simple) examples, one has to bear in mind that it may be not possible to find an irreducible representation of the Dirac-brackets algebra. This remains an open problem;

(6) Remaining first-class constraints are imposed as supplementary conditions on the wave function;

(7) For these first-class constraints, one has to check that (3.14) and (3.16) hold.

In light of points (5) and (7) as above, Dirac's quantization in the presence of second-class constraints is far from being straightforward. It may be thus very helpful to study a nontrivial example of second-class constrained systems. For this purpose, we here describe, following in part [9], a second-class system with no secondary constraints. The four primary second-class constraints may be reduced by one. In the corresponding theory there are three primary second-class constraints and one secondary second-class constraint [10]. The theory is then quantized using Dirac's method.

In other words, as done in [9], one begins by studying a theory described by the Lagrangian

$$L = (q_2 + q_3)\dot{q}_1 + q_4\dot{q}_3 + V(q_2, q_3, q_4) \quad , \quad (4.8)$$

where $V(q_2, q_3, q_4) \equiv \frac{1}{2}(q_4^2 - 2q_2q_3 - q_3^2)$. The study of (4.8) is motivated by a generalization of the work by Feynman recently described in [11], and we here use, for simplicity of notation, a convention for the indices of q 's and p 's different from what we have done so far. In light of (4.8) and of the definitions $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$, one finds the four primary constraints

$$\rho_1 \equiv (p_1 - q_2 - q_3) \approx 0 \quad , \quad (4.9)$$

$$\rho_2 \equiv p_2 \approx 0 \quad , \quad (4.10)$$

$$\rho_3 \equiv (p_3 - q_4) \approx 0 \quad , \quad (4.11)$$

$$\rho_4 \equiv p_4 \approx 0 \quad . \quad (4.12)$$

As usual, the weak-equality symbol means that the constraints only vanish identically on the constraint manifold Σ , but may have nonvanishing Poisson brackets with some canonical variables. The canonical Hamiltonian H_c , i.e. the Legendre transform of L , is then given by

$$\begin{aligned} H_c &\equiv \sum_{\alpha=1}^4 p_\alpha \dot{q}_\alpha - L \\ &= (p_1 - q_2 - q_3) \dot{q}_1 + p_2 \dot{q}_2 + (p_3 - q_4) \dot{q}_3 \\ &\quad + p_4 \dot{q}_4 - V(q_2, q_3, q_4) \quad . \end{aligned} \quad (4.13)$$

However, as we said in section 2, we want to extend H_c out of Σ . For this purpose we define the effective Hamiltonian $\tilde{H}$ (cf. (2.2)) on the whole phase space, which coincides with H_c on Σ . In our case, $\tilde{H}$ becomes

$$\begin{aligned} \tilde{H} &\equiv H_c + \Lambda_1 \rho_1 + \Lambda_2 \rho_2 + \Lambda_3 \rho_3 + \Lambda_4 \rho_4 \\ &= \lambda_1 \rho_1 + \lambda_2 \rho_2 + \lambda_3 \rho_3 + \lambda_4 \rho_4 - V(q_2, q_3, q_4) \quad , \end{aligned} \quad (4.14)$$

where $\lambda_i = \lambda_i(q, p) \equiv (\Lambda_i + \dot{q}_i)$. Interestingly, the preservation in time of the primary constraints (4.9-12) leads to no secondary constraints in the theory, since

$$\{\rho_1, \tilde{H}\} = -(\lambda_2 + \lambda_3) \quad , \quad \{\rho_2, \tilde{H}\} = \lambda_1 - q_3 \quad , \quad (4.15a)$$

$$\{\rho_3, \tilde{H}\} = \lambda_1 - \lambda_4 - q_2 - q_3 \quad , \quad \{\rho_4, \tilde{H}\} = -\lambda_3 + q_4 \quad , \quad (4.15b)$$

which implies

$$\lambda_1 = q_3 \quad , \quad \lambda_2 = -q_4 \quad , \quad \lambda_3 = q_4 \quad , \quad \lambda_4 = -q_2 \quad . \quad (4.16)$$

Moreover, the reader can easily check that the determinant of the 4×4 matrix of Poisson brackets is nonvanishing, so that this matrix is invertible and the constraints are second-class. Using the definition (2.11) of Dirac brackets, the nonvanishing Dirac brackets of the theory are found to be

$$\{q_1, q_2\}^* = \{q_3, q_4\}^* = -\{q_2, q_4\}^* = 1 \quad , \quad (4.17)$$

$$\{p_3, q_2\}^* = -\{p_k, q_k\}^* = 2 \quad , \quad \forall k = 1, 3 \quad . \quad (4.18)$$

The calculation expressed by (4.17-18) is very important, since nonvanishing Dirac brackets play the key role on quantization (cf. (4.4) and (4.6-7)). It may be now instructive to reduce the theory described by (4.8) to a three-dimensional one setting $q_4 = \text{constant} = k$. One thus deals with a model described by the Lagrangian

$$\tilde{L} = (q_2 + q_3)\dot{q}_1 + k\dot{q}_3 + W(q_2, q_3) \quad , \quad (4.19)$$

where $W(q_2, q_3) = V(q_2, q_3, k) = \frac{1}{2}(k^2 - 2q_2q_3 - q_3^2)$. Using again the definition of canonical momenta p_i , one now finds three primary constraints

$$\psi_1 \equiv (p_1 - q_2 - q_3) \approx 0 \quad , \quad (4.20)$$

$$\psi_2 \equiv p_2 \approx 0 \quad , \quad (4.21)$$

$$\psi_3 \equiv (p_3 - k) \approx 0 \quad . \quad (4.22)$$

This leads to a secondary constraint, i.e. $q_2 \approx 0$, and all constraints are second-class [10].

One thus finds that the only nonvanishing Dirac brackets are

$$\{p_1, q_1\}^* = -2 \quad , \quad (4.23)$$

$$\{q_1, q_3\}^* = 1 \quad . \quad (4.24)$$

The passage to quantum theory is then made replacing Dirac brackets by the commutators of the corresponding quantum operators, as described in the first part of this section.

However, we should say that progress has been recently made in developing new methods to study similar problems. As pointed out in [12], physicists have become interested in theories where the splitting into first- and second-class constraints is not so desirable. The main motivations for taking this point of view seem to be [12]:

- (1) In the case of the Green-Schwartz formulation of the super-string, one finds that the requirement of a manifestly supersymmetric description of the system is incompatible with the classical elimination of second-class constraints.

(2) It may happen that the first-class constraint algebra develops an anomaly upon quantization. In the quantum theory one is then dealing with what is essentially a second-class system, but there is no obvious way to classically remove these constraints, since they were originally first-class. Relevant examples are given by the quantization of the bosonic string, and by the anomaly in Gauss's law when gauge theories are coupled to chiral fermions.

In [12], second-class constraints are not eliminated at the classical level. The author is instead able to show that one can deal with second-class constraints in the quantum theory, and he shows what are the additional conditions to impose on a state so as to call it a physical state. Attention is there restricted to the case where second-class constraints $\xi_a^{(II)}$ can be decomposed into two first-class subsets $(\varphi_a^{(I)}, \chi_b^{(I)})$, where the $\varphi_a^{(I)}$ are linear in momentum and the $\chi_a^{(I)}$ are gauge-fixing terms with no momentum dependence. Interestingly, the example here studied in Eqs. (4.1-7) is there studied in a completely different way, finding the two physical states of the second-class system. The details can be found in section 4 of [12], but cannot be given here, since this would force us to become very technical.

From the point of view of gravitational physics, it appears both desirable and possible to apply the ideas in [12] to the quantization of theories of gravity with torsion, which are relativistic theories of gravitation with second-class constraints ([13] and references therein).

5. ADM Formalism and Constraints in Canonical Gravity

In section 3 we have studied Dirac's general method to quantize first-class constrained systems, and we are now aiming to apply this technique to general relativity. For this purpose, it may be useful to describe the main ideas of the Arnowitt-Deser-Misner (hereafter referred to as ADM) formalism. This is a canonical formalism for general relativity that enables one to re-write Einstein's field equations in first-order form and explicitly solved with respect to a time variable. For this purpose, one assumes that four-dimensional Lorentzian space-time (M, g) is topologically $\Sigma \times R$ and can be foliated by a family of $t = \text{constant}$ spacelike surfaces S_t all diffeomorphic to Σ , giving rise to a $3 + 1$ decomposition of the original four-geometry. The basic geometric data of this decomposition are as follows [14].

(1) The induced three-metric h of the three-dimensional spacelike surfaces S_t . This yields the intrinsic geometry of the three-space. h is also called the first fundamental form of S_t , and is positive-definite with our conventions.

(2) The way each S_t is embedded in (M, g) . This is known once we are able to compute the spatial part of the covariant derivative of the normal n to S_t . Denoting by ∇ the four-connection of (M, g) , one is thus led to define the tensor

$$K_{ij} \equiv -\nabla_j n_i \quad . \quad (5.1)$$

Note that K_{ij} is symmetric if and only if ∇ is symmetric [1]. The tensor K is called extrinsic-curvature tensor, or second fundamental form of S_t .

(3) How the coordinates are propagated off the surface S_t . For this purpose one defines the vector $(N, N^1, N^2, N^3)dt$ connecting the point (t, x^i) with the point $(t + dt, x^i)$.

Thus, given the surface $x^0 = t$ and the surface $x^0 = t + dt$, $Ndt \equiv d\tau$ specifies a proper displacement normal to the surface $x^0 = t$. Moreover, $N^i dt$ yields the displacement from the point (t, x^i) to the foot of the normal to $x^0 = t$ through $(t + dt, x^i)$. In other words, Ndt specifies the proper-time separation of the $t = \text{constant}$ surfaces, and the N^i arise since the $x^i = \text{constant}$ lines do not coincide in general with the normals to the $t = \text{constant}$ surfaces (cf. Fig. 4.2 in [14]). According to a well-established terminology, N is the lapse function, and the N^i are the shift functions. They are the tool needed to achieve the desired space-time foliation.

In light of points (1-3) as above, the four-metric g can be locally cast in the form

$$\begin{aligned} g_L &= h_{ij} (dx^i + N^i dt) \otimes (dx^j + N^j dt) - N^2 dt \otimes dt \\ &= (-N^2 + N_i N^i) dt \otimes dt + N_i (dt \otimes dx^i + dx^i \otimes dt) + h_{ij} dx^i \otimes dx^j \quad . \end{aligned} \quad (5.2)$$

This implies that

$$g_{00} = -(N^2 - N_i N^i) \quad , \quad (5.3)$$

$$g_{i0} = g_{0i} = N_i \quad , \quad (5.4)$$

$$g_{ij} = h_{ij} \quad , \quad (5.5)$$

whereas, using the property $g^{\lambda\nu} g_{\nu\mu} = \delta^\lambda_\mu$, one finds

$$g^{00} = -\frac{1}{N^2} \quad , \quad (5.6)$$

$$g^{i0} = g^{0i} = \frac{N^i}{N^2} \quad , \quad (5.7)$$

$$g^{ij} = h^{ij} - \frac{N^i N^j}{N^2} \quad . \quad (5.8)$$

Interestingly, the covariant g_{ij} and h_{ij} coincide, whereas the contravariant g^{ij} and h^{ij} differ as shown in (5.8). In terms of N, N^i and h , the extrinsic-curvature tensor defined in (5.1) takes the form

$$K_{ij} \equiv \frac{1}{2N} \left(-\frac{\partial h_{ij}}{\partial t} + N_{i|j} + N_{j|i} \right) \quad , \quad (5.9)$$

where the stroke $|$ denotes three-dimensional covariant differentiation on the spacelike three-surface S_t , and indices of K_{ij} are raised using h^{il} . Eq. (5.9) can be also written as

$$\frac{\partial h_{ij}}{\partial t} = N_{i|j} + N_{j|i} - 2NK_{ij} \quad . \quad (5.10)$$

Eq. (5.10) should be supplemented by another first-order equation expressing the time derivative of the momenta p^{ij} conjugate to the three-metric, or equivalently the time evolution of K_{ij} (since p^{ij} is related to K^{ij}). The details can be found for example in [14,15].

Using the ADM variables described so far, the form of the Lorentzian action integral I for pure gravity that is stationary under variations of the metric vanishing on the boundary is (in $c = 1$ units)

$$\begin{aligned} I &\equiv \frac{1}{16\pi G} \int_M {}^{(4)}R \sqrt{-\det g} \, d^4x + \frac{1}{8\pi G} \int_{\partial M} K^i_i \sqrt{\det h} \, d^3x \\ &= \frac{1}{16\pi G} \int_M \left[{}^{(3)}R + K_{ij} K^{ij} - \left(K^i_i \right)^2 \right] N \sqrt{\det h} \, d^3x \, dt \quad . \end{aligned} \quad (5.11)$$

The boundary term appearing in (5.11) is necessary since ${}^{(4)}R$ contains second derivatives of the metric, and integration by parts in the Einstein-Hilbert part

$$I_H \equiv \frac{1}{16\pi G} \int_M {}^{(4)}R \sqrt{-\det g} \, d^4x$$

of the action also leads to a boundary term equal to $-\frac{1}{8\pi G} \int_{\partial M} K^i_i \sqrt{\det h} \, d^3x$. Denoting by G_{ab} the Einstein tensor $G_{ab} \equiv {}^{(4)}R_{ab} - \frac{1}{2}g_{ab} {}^{(4)}R$, and defining

$$\delta\Gamma_{ab}^d \equiv \frac{1}{2}g^{dl} \left[\nabla_a (\delta g_{lb}) + \nabla_b (\delta g_{la}) - \nabla_l (\delta g_{ab}) \right] , \quad (5.12)$$

one then finds [16]

$$\begin{aligned} 16\pi G \delta I_H = & - \int_M \sqrt{-\det g} \, G^{ab} \delta g_{ab} \, d^4x \\ & + \int_{\partial M} \sqrt{-\det g} \left(g^{ab} \delta_d^c - g^{ac} \delta_d^b \right) \delta\Gamma_{ab}^d \left(d^3x \right)_c , \end{aligned} \quad (5.13)$$

which clearly shows that I_H is stationary if the Einstein equations hold, and the normal derivatives of the variations of the metric vanish on the boundary ∂M . In other words, I_H is not stationary under *arbitrary* variations of the metric, and stationarity is only achieved after adding to I_H the boundary term appearing in (5.11), if δg_{ab} is fixed on ∂M . Other useful forms of the boundary term can be found in [16,17]. Note also that, strictly, in writing down (5.11) one should also take into account a term arising from I_H and given by [18]

$$I_t \equiv \frac{1}{8\pi G} \int dt \int_{\partial M} d^3x \, \partial_i \left[\sqrt{\det h} \left(K^i_l N^l - h^{ij} N_{|j} \right) \right] .$$

However, we have not explicitly included I_t since it does not modify the results derived or described hereafter. With our conventions, and defining

$$\nabla^2 N \equiv h^{ij} N_{|ij} \quad , \quad (5.14)$$

the Lorentzian ADM formulae for the curvature are found to be [1]

$$\begin{aligned} {}^{(4)}R_{0i0j} &= \frac{1}{N} \frac{\partial K_{ij}}{\partial t} + K_i^l K_{lj} \\ &\quad - \frac{1}{N} \left(N^m K_{ij|m} + N_{|i}^m K_{jm} + N_{|j}^m K_{im} \right) + \frac{N_{|ij}}{N} \quad , \end{aligned} \quad (5.15)$$

$${}^{(4)}R_{ijl}^0 = - \left(K_{il|j} - K_{ij|l} \right) \quad , \quad (5.16)$$

$${}^{(4)}R_i^0 = - \left[\left(K_l^l \right)_{|i} - K_{i|l}^l \right] \quad , \quad (5.17)$$

$${}^{(4)}R_0^0 = - \left[\frac{1}{N} \frac{\partial K_l^l}{\partial t} - K_{ij} K^{ij} - \frac{N^l}{N} \left(K_m^m \right)_{|l} + \frac{\nabla^2 N}{N} \right] \quad , \quad (5.18)$$

$$\begin{aligned} {}^{(4)}R_{ij} &= {}^{(3)}R_{ij} + \left[K_{ij} K_l^l - \frac{1}{N} \frac{\partial K_{ij}}{\partial t} - 2 K_{im} K_j^m \right] \\ &\quad + \frac{1}{N} \left(N^m K_{ij|m} + N_{|i}^m K_{jm} + N_{|j}^m K_{im} \right) - \frac{N_{|ij}}{N} \quad , \end{aligned} \quad (5.19)$$

$${}^{(4)}R = {}^{(3)}R + K_{ij} K^{ij} + \left(K_l^l \right)^2 - \frac{2}{N} \frac{\partial K_l^l}{\partial t} - \frac{2}{N} \left[\nabla^2 N - N^p \left(K_l^l \right)_{|p} \right] \quad . \quad (5.20)$$

Indeed, in quantum gravity one is often interested in the real Riemannian section of a complex space-time, where the metric is positive-definite (see section 7). In that case, the ADM form of the four-metric is

$$g_R = \left(N^2 + N_i N^i \right) d\tau \otimes d\tau + N_i \left(d\tau \otimes dx^i + dx^i \otimes d\tau \right) + h_{ij} dx^i \otimes dx^j \quad , \quad (5.21)$$

where τ is the so-called Euclidean time. The Euclidean formula for the extrinsic-curvature tensor is obtained multiplying by i Eq. (5.9) when its right-hand side is evaluated at $t = -i\tau$. This leads to

$$K_{ij}^{(E)} = \frac{1}{2N} \left(\frac{\partial h_{ij}}{\partial \tau} - N_{i|j} - N_{j|i} \right) . \quad (5.22)$$

Thus, by virtue of Eqs. (5.21-22) one finds [1]

$$\begin{aligned} {}^{(4)}R_{0i0j} &= -\frac{1}{N} \frac{\partial K_{ij}}{\partial \tau} + K_i{}^l K_{lj} \\ &+ \frac{1}{N} \left(N^m K_{ij|m} + N_{|i}^m K_{jm} + N_{|j}^m K_{im} \right) + \frac{N_{|ij}}{N} , \end{aligned} \quad (5.23)$$

$${}^{(4)}R_0^0 = -\frac{1}{N} \frac{\partial K_l^l}{\partial \tau} - K_{ij} K^{ij} + \frac{N^l}{N} \left(K_m^m \right)_{|l} + \frac{\nabla^2 N}{N} , \quad (5.24)$$

$$\begin{aligned} {}^{(4)}R_{ij} &= {}^{(3)}R_{ij} - \left[K_{ij} K_l^l + \frac{1}{N} \frac{\partial K_{ij}}{\partial \tau} - 2K_{im} K_j{}^m \right] \\ &+ \frac{1}{N} \left(N^m K_{ij|m} + N_{|i}^m K_{jm} + N_{|j}^m K_{im} \right) + \frac{N_{|ij}}{N} , \end{aligned} \quad (5.25)$$

$${}^{(4)}R = {}^{(3)}R - K_{ij} K^{ij} - \left(K_l^l \right)^2 - \frac{2}{N} \frac{\partial K_l^l}{\partial \tau} + \frac{2}{N} \left[\nabla^2 N + N^p \left(K_l^l \right)_{|p} \right] . \quad (5.26)$$

We are now ready to apply Dirac's technique to general relativity, so as to derive the constraint equations. Since the action functional (5.11) is independent of the time derivatives of N and N^i , the corresponding conjugate momenta, denoted by $p(x)$ and $p^i(x)$ respectively, give rise to the primary constraints

$$p(x) \approx 0 , \quad (5.27)$$

$$p^i(x) \approx 0 \quad . \quad (5.28)$$

Requiring the preservation in time of (5.27-28) one finds the secondary constraints

$$\mathcal{H} \equiv (16\pi G)^{-1} \sqrt{\det h} \left[K_{ij} K^{ij} - \left(K^l_l \right)^2 - {}^{(3)}R \right] \approx 0 \quad , \quad (5.29)$$

$$\mathcal{H}^i \equiv -2p^{ij}{}_{|j} \approx 0 \quad . \quad (5.30)$$

Dirac's extended Hamiltonian H_E for the gravitational field is thus given by

$$H_E \equiv \int \left[N\mathcal{H} + N_i \mathcal{H}^i + \lambda p + \lambda_i p^i \right] d^3x \quad , \quad (5.31)$$

where lapse and shift, originally defined in a geometrical way, play the role of Lagrange multipliers that can be freely specified, and additional arbitrary multipliers λ and λ_i have been introduced for the primary constraints. Eq. (5.31) can be used since we are dealing with a first-class constrained Hamiltonian system. This result is proved as follows. As we know, consistency of the constraints is shown if one finds that their commutators lead to no new constraints. For this purpose, it may be useful to recall the commutation relations of the canonical variables

$$\left[N(x), p(x') \right] = i\delta(x, x') \quad , \quad (5.32)$$

$$\left[N_i(x), p^j(x') \right] = i\delta_i^{j'} \quad , \quad (5.33)$$

$$\left[h_{ij}, p^{k'l'} \right] = i\delta_{ij}{}^{k'l'} \quad . \quad (5.34)$$

Note that, following [18], primes have been used, either on indices or on the variables themselves, to distinguish different points of three-space. In other words, we define

$$\delta_i^{j'} \equiv \delta_i^j \delta(x, x') \quad , \quad (5.35)$$

$$\delta_{ij}{}^{k'l'} \equiv \delta_{ij}{}^{kl} \delta(x, x') \quad , \quad (5.36)$$

$$\delta_{ij}{}^{kl} \equiv \frac{1}{2} \left(\delta_i^k \delta_j^l + \delta_i^l \delta_j^k \right) \quad . \quad (5.37)$$

The reader can easily check that

$$\left[p(x), p^i(x') \right] = \left[p(x), \mathcal{H}^i(x') \right] = \left[p(x), \mathcal{H}(x') \right] = \left[p^i(x), \mathcal{H}^j(x') \right] = \left[p^i(x), \mathcal{H}(x') \right] = 0 \quad . \quad (5.38)$$

It now remains to compute the three commutators $\left[\mathcal{H}_i, \mathcal{H}_{j'} \right]$, $\left[\mathcal{H}_i, \mathcal{H}' \right]$, $\left[\mathcal{H}, \mathcal{H}' \right]$. The first two commutators are obtained using Eq. (5.30) and defining $\mathcal{H}_i \equiv h_{ij} \mathcal{H}^j$. Interestingly, $\mathcal{H}_i$ is homogeneous bilinear in the h_{ij} and p^{ij} , with the momenta always to the right. Since the correct version of the Hamiltonian and momentum constraints (5.29-30) is

$$\int \mathcal{H} \xi \, d^3 x \, \psi = 0 \quad \forall \xi \quad , \quad (5.39)$$

$$\int \mathcal{H}_i \xi^i \, d^3 x \, \psi = 0 \quad \forall \xi^i \quad , \quad (5.40)$$

we begin by computing [18]

$$\left[h_{ij}, i \int \mathcal{H}_{k'} \delta \xi^{k'} \, d^3 x' \right] = -h_{ij,k} \delta \xi^k - h_{kj} \delta \xi^k{}_{,i} - h_{ik} \delta \xi^k{}_{,j} \quad , \quad (5.41)$$

$$\left[p^{ij}, i \int \mathcal{H}_{k'} \delta \xi^{k'} d^3 x' \right] = - \left(p^{ij} \delta \xi^k \right)_{,k} + p^{kj} \delta \xi^i_{,k} + p^{ik} \delta \xi^j_{,k} . \quad (5.42)$$

This calculation shows that the $\mathcal{H}_i$ are indeed generators of three-dimensional coordinate transformations $\bar{x}^i = x^i + \delta \xi^i$. Thus, using the definition of structure constants of the general coordinate-transformation group [18]

$$c^{k''}_{ij'} \equiv \delta^{k''}_{i,l''} \delta^{l''}_{j'} - \delta^{k''}_{j',l''} \delta^{l''}_i , \quad (5.43)$$

the results (5.41-42) may be used to show that

$$\left[\mathcal{H}_i(x), \mathcal{H}_j(x') \right] = -i \int \mathcal{H}_{k''} c^{k''}_{ij'} d^3 x'' , \quad (5.44)$$

$$\left[\mathcal{H}_i(x), \mathcal{H}(x') \right] = i \mathcal{H} \delta_{,i}(x, x') . \quad (5.45)$$

Note that the only term of $\mathcal{H}$ which might lead to difficulties is the one quadratic in the momenta. However, all factors appearing in this term have homogeneous linear transformation laws under the three-dimensional coordinate-transformation group. They thus remain undisturbed in position when commuted with $\mathcal{H}_i$ [18].

Finally, we have to study the commutator $\left[\mathcal{H}(x), \mathcal{H}(x') \right]$. The following remarks are now in order:

(i) Terms quadratic in momenta contain no derivatives of h_{ij} or p^{ij} with respect to three-space coordinates. Hence they commute;

(ii) The terms $\sqrt{\det h(x)} \binom{(3)}{R}(x)$ and $\sqrt{\det h(x')} \binom{(3)}{R}(x')$ contain no momenta, so that they also commute;

(iii) The only commutators we are left with are the cross commutators, and they can be evaluated using the variational formula [18]

$$\begin{aligned} \delta\left(\sqrt{\det h} {}^{(3)}R\right) &= \sqrt{\det h} h^{ij} h^{kl} \left(\delta h_{ik,jl} - \delta h_{ij,kl}\right) \\ &\quad - \sqrt{\det h} \left[{}^{(3)}R^{ij} - \frac{1}{2} h^{ij} \left({}^{(3)}R\right) \right] \delta h_{ij} \quad , \end{aligned} \quad (5.46)$$

which leads to

$$\left[\int \mathcal{H}_{\xi_1} d^3x, \int \mathcal{H}_{\xi_2} d^3x \right] = i \int \mathcal{H}^i \left(\xi_1 \xi_{2,i} - \xi_{1,i} \xi_2 \right) d^3x \quad . \quad (5.47)$$

The commutators (5.44-45) and (5.47) clearly show that the constraint equations of canonical quantum gravity are first-class. As we said in section 1, the Wheeler-De Witt equation (5.39) is an equation on the superspace $S(\mathcal{M}) \equiv Riem(\mathcal{M})/Diff(\mathcal{M})$. In Wheeler's superspace-based hybrid scheme the spatial diffeomorphisms are factored, but the operator constraint (5.39) is retained [8].

Two very useful classical formulae frequently used in Lorentzian canonical gravity are

$$\mathcal{H} \equiv (16\pi G) G_{ijkl} p^{ij} p^{kl} - \frac{\sqrt{\det h}}{16\pi G} \left({}^{(3)}R\right) \quad , \quad (5.48)$$

$$\mathcal{H} \equiv (16\pi G)^{-1} \left[G^{ijkl} K_{ij} K_{kl} - \sqrt{\det h} \left({}^{(3)}R\right) \right] \quad , \quad (5.49)$$

where G_{ijkl} , called De Witt supermetric [19,20], is such that

$$G_{ijkl} \equiv \frac{1}{2\sqrt{\det h}} \left(h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl} \right) \quad , \quad (5.50)$$

$$G^{ijkl} \equiv \frac{\sqrt{\det h}}{2} \left(h^{ik} h^{jl} + h^{il} h^{jk} - 2h^{ij} h^{kl} \right) \quad , \quad (5.51)$$

and p^{ij} is here defined as $-\frac{\sqrt{\det h}}{16\pi G}(K^{ij} - h^{ij}K^l_l)$. Note that the factor -2 multiplying $h^{ij}h^{kl}$ in (5.51) is needed so as to obtain the identity

$$G_{ijmn}G^{mnkl} = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k) \quad . \quad (5.52)$$

Interestingly, since on the real Riemannian section of a complex space-time we can think of K_{ij} as being replaced by $-K_{ij}$ in the sense specified by Eqs. (5.9) and (5.22), whereas the coefficient of N^2 in the four-metric changes sign and the three-metric is unaffected by the Wick rotation, the Hamiltonian constraint takes the Euclidean-time form

$$\mathcal{H} \equiv (16\pi G)^{-1} \left[-G^{ijkl}K_{ij}K_{kl} - \sqrt{\det h}({}^{(3)}R) \right] \approx 0 \quad . \quad (5.53)$$

Eq. (5.48) clearly shows that $\mathcal{H}$ contains a part *quadratic* in the momenta and a part proportional to ${}^{(3)}R$. On quantization, it is then hard to give a well-defined meaning to the second functional derivative $\frac{\delta^2}{\delta h_{ij}\delta h_{kl}}$, whereas the occurrence of ${}^{(3)}R$ makes it even more difficult to solve exactly the Wheeler-De Witt equation. Nevertheless, since our paper deals with canonical quantum gravity within the geometrodynamical approach, our next aim is to describe the relevant geometrical and topological features of Wheeler's superspace.

6. Mathematical Theory of Wheeler's Superspace

Let $\mathcal{M}$ be a compact, connected, orientable, Hausdorff, C^∞ three-manifold without boundary. Following [21], we say $\mathcal{M}$ is a *superspatial*. If g is a Riemannian (i.e. positive-definite) C^∞ metric on a superspatial $\mathcal{M}$, the pair $(\mathcal{M}, g)$ is also called a superspatial. For a given superspatial $\mathcal{M}$, one denotes by $Riem(\mathcal{M})$ the space of Riemannian C^∞ metrics on $\mathcal{M}$, and by $Diff(\mathcal{M})$ the group of C^∞ orientation-preserving diffeomorphisms of $\mathcal{M}$. $Diff(\mathcal{M})$ acts as a transformation group on $Riem(\mathcal{M})$; its action maps (f, g) to f^*g , where $f \in Diff(\mathcal{M})$ and $g \in Riem(\mathcal{M})$. The space of all orbits of $Diff(\mathcal{M})$

$$S(\mathcal{M}) \equiv Riem(\mathcal{M})/Diff(\mathcal{M}) \quad , \quad (6.1)$$

is called *superspace*. In other words, $\forall g \in Riem(\mathcal{M})$, we consider all metrics obtained from g by the action of elements $f \in Diff(\mathcal{M})$. If two metrics g and $\bar{g}$ are on the same orbit, a diffeomorphism f of $\mathcal{M}$ onto itself exists such that

$$f^*g = \bar{g} \quad , \quad (6.2)$$

which implies that g and $\bar{g}$ are isometric. Two metrics are isometric if and only if they lie on the same orbit. $S(\mathcal{M})$ is thus the set of geometries of $\mathcal{M}$, which are equivalence classes of isometric Riemannian metrics.

An important topological property of superspace, which is necessary to prove theorems on its structure, is the following result [21]:

Theorem 6.1. Metrization Theorem for Superspace: The superspace $S(\mathcal{M})$ is a connected, second-countable, metrizable space (i.e. a countable basis of open sets exists

for its topology, and there also exists a metric on $S(\mathcal{M})$ inducing on $S(\mathcal{M})$ the given topology).

Note that, since there are symmetric geometries on $\mathcal{M}$, there are neighbourhoods of $S(\mathcal{M})$ not homeomorphic to neighbourhoods of non-symmetric geometries. This implies the superspace defined in (6.1) cannot be a manifold. However, all geometries with the same kind of symmetry have homeomorphic neighbourhoods and they are thus a manifold. Two theorems hold which enable one to understand how these manifolds can be put together to give rise to superspace. For us to be able to state these theorems, some further definitions are in order [21].

(i) Let U be a compact subgroup of $Diff(\mathcal{M})$, and let us denote by (U) all compact subgroups of $Diff(\mathcal{M})$ that are conjugate to U by an element in $Diff(\mathcal{M})$, i.e.

$$(U) \equiv \{fUf^{-1} \mid f \in Diff(\mathcal{M})\} \quad (6.3)$$

If some element of (H) is included in some element of (U) , we say that $(H) \leq (U)$. The relation $\leq$ is a partial ordering. The partially-ordered set of conjugacy classes of compact subgroups of $Diff(\mathcal{M})$ is used to index a partition of $S(\mathcal{M})$.

(ii) A partition of a second-countable Hausdorff space X , is a set of non-empty subspaces $\{X_\alpha\}$ such that:

$$X = \bigcup_{\alpha} X_{\alpha} \quad , \quad \alpha \in A \quad , \quad (6.4)$$

$$X_{\alpha} \cap X_{\beta} \neq \emptyset \quad \Rightarrow \quad \alpha = \beta \quad . \quad (6.5)$$

The subspace X_α is Hausdorff, second-countable, and its components

$$\left\{ \left\{ X_\alpha^i \right\} \mid i \in C_\alpha \right\}$$

yield a partition of X_α indexed by C_α . If $\{X_\alpha\}$ is a partition for X , $\{X_\alpha^i\}$ is the complete partition of X , indexed by

$$\left\{ (\alpha, i) \mid \alpha \in A, i \in C_\alpha \right\} = \prod_{\alpha} C_\alpha .$$

A partition is said to be a manifold partition if each X_α is a manifold.

(iii) A *stratification* (respectively, an *inverted stratification*) of a connected, second-countable, Hausdorff topological space X , is a countable, partially-ordered, manifold partition of X whose complete partition has the frontier property (respectively, inverted frontier property)

$$X_\alpha^i \cap \overline{X_\beta^j} \neq \emptyset \quad , \quad \alpha \neq \beta \quad \left(\Leftrightarrow (\alpha, i) \neq (\beta, j) \right)$$

$$\Rightarrow X_\alpha^i \subset \overline{X_\beta^j} \quad \text{and} \quad \alpha < \beta \quad \left(\Leftrightarrow (\alpha, i) < (\beta, j) \right)$$

(respectively : $\beta < \alpha \Leftrightarrow (\beta, j) < (\alpha, i)$)).

The manifolds X_α are said the *strata* of the stratification, and the manifolds $\{X_\alpha^i\}$ are the connected strata.

We are now in a position to state the following theorems [21]:

Theorem 6.2. Decomposition Theorem for Superspace: The decomposition of $S(\mathcal{M})$ by the subspaces $\{S_G(\mathcal{M})\}$ is a countable, partially-ordered, C^∞ -Fréchet manifold

partition [A locally convex space which is metrizable and complete is called a Fréchet space, and manifolds can be modeled on any linear space in which one has a theory of differential calculus. Fréchet manifolds are thus differentiable manifolds whose charts have values in a Fréchet space].

Theorem 6.3. Stratification Theorem for Superspace: The manifold partition $\{S_G(\mathcal{M})\}$ of $S(\mathcal{M})$ is an inverted stratification indexed by the symmetry type.

This means that geometries with a given symmetry are completely contained within the boundary of less symmetric geometries. For example, the so-called minisuperspace models studied in the literature only consider certain strata of superspace.

Since physicists are interested in writing down and studying differential equations, it would be of much help if superspace could be extended to a suitable manifold. This extended superspace is obtained from $Riem(\mathcal{M})$ by the action of a subgroup of $Diff(\mathcal{M})$. With the help of a suitable choice of such subgroup, the resulting orbit space is a manifold. One can then prove what follows [21].

Theorem 6.4. Extension Theorem for Superspace: For every n -dimensional superspatial $\mathcal{M}$, the superspace $S(\mathcal{M})$ can be extended to a suitable manifold $S^{ext}(\mathcal{M})$, such that

$$\dim \left(S^{ext}(\mathcal{M})/S(\mathcal{M}) \right) = n(n+1) \quad . \quad (6.6)$$

7. The One-Loop Approximation in Perturbative Quantum Gravity

In the path-integral approach to quantum gravity, the space-time topology is no longer bound to be $\Sigma \times R$ as in canonical gravity, but one deals with a (mathematically ill-defined) formalism which should enable one to consider, in principle, all possible topologies. The basic postulate is that the amplitude $\tilde{A}$ of going from a three-metric h and a matter-field configuration ϕ on a spacelike surface Σ to a three-metric h' and a field configuration ϕ' on a spacelike surface Σ' is formally given by [1]

$$\tilde{A}(h', \phi' | h, \phi) = \int_C D[g] D[\Phi] e^{-I_E[g, \Phi]} \quad , \quad (7.1)$$

where I_E is the action integral for gravitation and matter fields in the Euclidean-time regime, the four-metrics g inducing the three-metrics on Σ and Σ' and belonging to the class C are required to be positive-definite (i.e. we deal with Riemannian four-geometries), and the fields Φ match ϕ on Σ and ϕ' on Σ' . More precisely, we assume that real Lorentzian space-time (M, g) with contravariant metric g^{ab} is embedded as a real four-dimensional submanifold in a complex four-manifold M_c , called the complexification of M . We also assume that in M_c a complex contravariant tensor field $g_{(c)}^{ab}$ is given of rank 2, such that the restriction of $g_{(c)}^{ab}$ to the cotangent space of the submanifold is real. M is then said to be a real section of M_c . Local coordinates for M_c are (x, x^*) , and M is defined by the condition $x = x^*$ [1].

Unfortunately, the measure $D[g]D[\Phi]$ in (7.1) has not yet been given a rigorous mathematical meaning, we do not know how to evaluate path integrals over all four-geometries

and all possible topologies, and the Euclidean action is not positive-definite, so that the integrand in (7.1) may blow up exponentially after suitable conformal rescalings of the four-metric. Moreover, there are topological obstructions to a Wick rotation, and in general a real Lorentzian metric has not a section of the complexified space-time where the metric is real and positive-definite (and viceversa). We thus restrict our analysis to the semiclassical approximation of the (as yet lacking) full theory, since this approximation has shed new light on thermodynamical properties of black holes, gravitational instantons (section 9) and one-loop quantum gravity (section 10).

For this purpose, we study perturbative quantum gravity within the background-field method. In the one-loop approximation (also called stationary-phase or WKB method) one first expands *both* the metric g and matter fields ϕ about a metric g_0 and a field ϕ_0 which are solutions of the classical field equations

$$g = g_0 + \bar{g} \quad , \quad (7.2)$$

$$\phi = \phi_0 + \bar{\phi} \quad . \quad (7.3)$$

One then assumes that the fluctuations $\bar{g}$ and $\bar{\phi}$ are so small that the dominant contribution to the path integral comes from the quadratic order in the Taylor-series expansion of the action about the background fields g_0 and ϕ_0 [22]

$$I_E[g, \phi] = I_E[g_0, \phi_0] + I_2[\bar{g}, \bar{\phi}] + \text{higher - order terms} \quad , \quad (7.4)$$

so that the logarithm of the quantum-gravity amplitude $\tilde{A}$ can be expressed as

$$\log(\tilde{A}) \sim -I_E[g_0, \phi_0] + \log \int D[\bar{g}, \bar{\phi}] e^{-I_2[\bar{g}, \bar{\phi}]} \quad . \quad (7.5)$$

Various properties of background fields will be studied in section 9. For our present purposes we are instead interested in the second term appearing on the right-hand side of (7.5). An useful factorization is obtained if ϕ_0 can be set to zero. One then finds that $I_2[\bar{g}, \bar{\phi}] = I_2[\bar{g}] + I_2[\bar{\phi}]$, which implies [22]

$$\log(\tilde{A}) \sim -I_E[g_0] + \log \int D[\phi] e^{-I_2[\phi]} + \log \int D[\bar{g}] e^{-I_2[\bar{g}]} \quad . \quad (7.6)$$

We here recall some basic results, following again [22].

A familiar form of $I_2[\phi]$ is

$$I_2[\phi] = \frac{1}{2} \int \phi B \phi \sqrt{\det g_0} \, d^4x \quad , \quad (7.7)$$

where the elliptic differential operator B depends on the background metric g_0 . Note that B is a second-order operator for bosonic fields, whereas it is first-order for fermionic fields. In light of (7.7) it is clear we are interested in the eigenvalues $\{\lambda_n\}$ of B , with corresponding eigenfunctions $\{\phi_n\}$. If boundaries are absent, it is sometimes possible to know explicitly the eigenvalues with their degeneracies. This is what happens for example in de Sitter space [23]. If boundaries are present, however, very little is known about the detailed form of the eigenvalues, once boundary conditions have been imposed.

We here assume for simplicity to deal with bosonic fields subject to (homogeneous) Dirichlet conditions on the boundary surface: $\phi = 0$ on ∂M , and $\phi_n = 0$ on ∂M , $\forall n$. It is in fact well-known that the Laplace operator subject to Dirichlet conditions has a

positive-definite spectrum (page 9 of [24]). The field ϕ can then be expanded in terms of the eigenfunctions ϕ_n of $\mathcal{B}$ as

$$\phi = \sum_{n=n_0}^{\infty} y_n \phi_n \quad , \quad (7.8)$$

where the eigenfunctions ϕ_n are normalized so that

$$\int \phi_n \phi_m \sqrt{\det g_0} \, d^4x = \delta_{nm} \quad . \quad (7.9)$$

Another formula we need is the one expressing the measure on the space of all fields ϕ as

$$D[\phi] = \prod_{n=n_0}^{\infty} \mu \, dy_n \quad , \quad (7.10)$$

where the normalization parameter μ has dimensions of mass or (length)⁻¹. Note that, if gauge fields appear in the calculation, the choice of gauge-averaging and the form of the measure in the path integral are not a trivial problem. The reader should always bear in mind this remark, and he will find a detailed (although incomplete) study of these issues in [1]. Using well-known results about Gaussian integrals, the one-loop matter amplitudes $\tilde{A}_\phi^{(1)}$ can be now obtained as

$$\begin{aligned} \tilde{A}_\phi^{(1)} &\equiv \int D[\phi] e^{-I_2[\phi]} \\ &= \prod_{n=n_0}^{\infty} \int \mu \, dy_n \, e^{-\frac{\lambda_n}{2} y_n^2} \\ &= \prod_{n=n_0}^{\infty} \left(2\pi \mu^2 \lambda_n^{-1} \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{\det \left(\frac{1}{2} \pi^{-1} \mu^{-2} \mathcal{B} \right)}} \quad . \end{aligned} \quad (7.11)$$

In the particular (and relevant) case of a complex scalar field in a complex or real Riemannian space-time, the complex conjugate ϕ^* of ϕ can be replaced by its analytic continuation $\tilde{\phi}$, where $\tilde{\phi}$ is now completely independent of ϕ . The formula (7.7) for the one-loop term is then replaced by

$$I_2[\phi, \tilde{\phi}] = \frac{1}{2} \int \tilde{\phi} B \phi \sqrt{\det g_0} d^4x \quad . \quad (7.12)$$

The adjoint operator $B^\dagger$ has now eigenfunctions $\tilde{\phi}_n$, and the field $\tilde{\phi}$ can be expanded in terms of these $\tilde{\phi}_n$ as

$$\tilde{\phi} = \sum_{n=n_0}^{\infty} \tilde{y}_n \tilde{\phi}_n \quad , \quad (7.13)$$

whereas the measure $D[\phi, \tilde{\phi}]$ takes the form

$$D[\phi, \tilde{\phi}] = \prod_{n=n_0}^{\infty} \mu^2 dy_n d\tilde{y}_n \quad . \quad (7.14)$$

Since one has to integrate over y_n and $\tilde{y}_n$ independently, one now finds (cf. (7.11))

$$\tilde{A}_\phi^{(1)} = \frac{1}{\det\left(\frac{1}{2}\pi^{-1}\mu^{-2}B\right)} \quad . \quad (7.15)$$

When fermionic fields appear in the path integral, one deals with a first-order elliptic operator, the Dirac operator, acting on independent spinor fields ψ and $\tilde{\psi}$. These are anticommuting Grassmann variables obeying Berezin integration rules

$$\int dw = 0 \quad , \quad \int w dw = 1 \quad . \quad (7.16)$$

The formulae (7.16) are all what we need, since powers of w greater than or equal to 2 vanish in light of the anticommuting property. The reader can then check that the one-loop amplitude for fermionic fields is

$$\tilde{A}_\psi^{(1)} = \det \left(\frac{1}{2} \mu^{-2} B \right) . \quad (7.17)$$

The main difference with respect to bosonic fields is the direct proportionality to the determinant. The following comments can be useful in understanding the meaning of (7.17).

Let us denote by γ^μ the gamma matrices, and by λ_i the eigenvalues of the Dirac operator $\gamma^\mu D_\mu$, and suppose that no zero-modes exist. More precisely, the eigenvalues of $\gamma^\mu D_\mu$ occur in equal and opposite pairs : $\pm\lambda_1, \pm\lambda_2, \dots$, whereas the eigenvalues of the Laplace operator on spinors occur as $(\lambda_1)^2$ twice, $(\lambda_2)^2$ twice, and so on. For Dirac fermions (D) one thus finds

$$\det_D (\gamma^\mu D_\mu) = \left(\prod_{i=1}^{\infty} |\lambda_i| \right) \left(\prod_{i=1}^{\infty} |\lambda_i| \right) = \prod_{i=1}^{\infty} |\lambda_i|^2 , \quad (7.18)$$

whereas in the case of Majorana spinors (M), for which the number of degrees of freedom is halved, one finds

$$\det_M (\gamma^\mu D_\mu) = \prod_{i=1}^{\infty} |\lambda_i| = \sqrt{\det_D (\gamma^\mu D_\mu)} . \quad (7.19)$$

8. Zeta-Function Regularization of Path Integrals

The formal expression (7.11) for the one-loop quantum amplitude clearly diverges since the eigenvalues λ_n increase without bound, and a regularization is thus necessary. For this purpose, the following technique has been described and applied by many authors [22,25].

Bearing in mind that Riemann's zeta-function $\zeta_R(s)$ is defined as

$$\zeta_R(s) \equiv \sum_{n=1}^{\infty} n^{-s} \quad , \quad (8.1)$$

one first defines a generalized zeta-function $\zeta(s)$ obtained from the (positive) eigenvalues of the second-order, self-adjoint operator $\mathcal{B}$. Such $\zeta(s)$ can be defined as

$$\zeta(s) \equiv \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} d_m(n) \lambda_{n,m}^{-s} \quad . \quad (8.2)$$

This means that all the eigenvalues are completely characterized by two integer labels n and m , while their degeneracy d_m only depends on n . This is the case studied in [1]. Note that formal differentiation of (8.2) at the origin yields

$$\det(\mathcal{B}) = e^{-\zeta'(0)} \quad . \quad (8.3)$$

This result can be given a sensible meaning since, in four dimensions, $\zeta(s)$ converges for $\text{Re}(s) > 2$, and one can perform its analytic extension to a meromorphic function of s

which only has poles at $s = \frac{1}{2}, 1, \frac{3}{2}, 2$. Since $\det(\mu B) = \mu^{\zeta(0)} \det(B)$, one finds the useful formula

$$\log(\tilde{A}_\phi) = \frac{1}{2}\zeta'(0) + \frac{1}{2}\log(2\pi\mu^2)\zeta(0) \quad . \quad (8.4)$$

As we said following (7.7), it may happen quite often that the eigenvalues appearing in (8.2) are unknown, since the eigenvalue condition, i.e. the equation leading to the eigenvalues by virtue of the boundary conditions, is a complicated equation which cannot be solved exactly for the eigenvalues. However, since the scaling properties of the one-loop amplitude are still given by $\zeta(0)$ (and $\zeta'(0)$) as shown in (8.4), efforts have been made to compute the regularized $\zeta(0)$ also in this case. The various steps of this program are as follows [25].

(1) One first studies the heat equation for the operator B

$$\frac{\partial}{\partial\tau}F(x,y,\tau) + BF(x,y,\tau) = 0 \quad , \quad (8.5)$$

where the Green's function F satisfies the initial condition $F(x,y,0) = \delta(x,y)$.

(2) Assuming completeness of the set $\{\phi_n\}$ of eigenfunctions of B , the field ϕ can be expanded as

$$\phi = \sum_{n=n_0}^{\infty} a_n \phi_n \quad .$$

(3) The Green's function $F(x,y,\tau)$ is then given by

$$F(x,y,\tau) = \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} e^{-\lambda_{n,m}\tau} \phi_{n,m}(x) \otimes \phi_{n,m}(y) \quad . \quad (8.6)$$

(4) The corresponding (integrated) heat kernel is then

$$G(\tau) = \int_M \text{Tr} F(x, x, \tau) \sqrt{\det g} d^4x = \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} e^{-\lambda_{n,m}\tau} . \quad (8.7)$$

(5) In light of (8.2) and (8.7), the generalized zeta-function can be also obtained as an integral transform of the integrated heat kernel

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \tau^{s-1} G(\tau) d\tau . \quad (8.8)$$

(6) The hard part of the analysis is now to prove that $G(\tau)$ has an asymptotic expansion as $\tau \rightarrow 0^+$ [26]. This property has been proved for all boundary conditions such that the Laplace operator is self-adjoint. The corresponding asymptotic expansion of $G(\tau)$ can be written as

$$G(\tau) \sim b_1 \tau^{-2} + b_2 \tau^{-\frac{3}{2}} + b_3 \tau^{-1} + b_4 \tau^{-\frac{1}{2}} + b_5 + O(\sqrt{\tau}) , \quad (8.9)$$

which implies

$$\zeta(0) = b_5 . \quad (8.10)$$

The result (8.10) is proved splitting the integral in (8.8) into an integral from 0 to 1 and an integral from 1 to ∞ . The asymptotic expansion of $\int_0^1 \tau^{s-1} G(\tau) d\tau$ is then obtained using (8.9).

In other words, for a given second-order self-adjoint elliptic operator, we study the corresponding heat equation, and the integrated heat kernel $G(\tau)$. The regularized $\zeta(0)$ value is then given by the constant term appearing in the asymptotic expansion of $G(\tau)$

as $\tau \rightarrow 0^+$, and it yields the one-loop divergences of the theory for bosonic and fermionic fields (section 10).

9. Gravitational Instantons

This section is devoted to the study of the background gravitational fields appearing in Eqs. (7.2-6). These gravitational instantons are complete four-geometries solving the Einstein equations $R(X,Y) - \Lambda g(X,Y) = 0$ when the four-metric g has signature $+4$ (i.e. it is positive-definite, and thus called Riemannian). Following [27], essentially three cases can be studied.

9.1. Asymptotically locally Euclidean instantons

Even though it might seem natural to define first the asymptotically Euclidean instantons, it turns out there is not much choice in this case, since the only asymptotically Euclidean instanton is flat space. It is in fact well-known that the action of an asymptotically Euclidean metric with vanishing scalar curvature is ≥ 0 , and it vanishes if and only if the metric is flat. Suppose now such a metric is a solution of the Einstein equations $R(X,Y) = 0$. Its action should be thus stationary also under constant conformal rescalings $g \rightarrow k^2 g$ of the metric. However, the whole action rescales then as $I_E \rightarrow k^2 I_E$, so that it can only be stationary and finite if $I_E = 0$. By virtue of the theorem previously mentioned, the metric g must then be flat.

In the asymptotically locally Euclidean case, however, the boundary at infinity has topology S^3/Γ rather than S^3 , where Γ is a discrete subgroup of the local tetrad rotation

group $SO(4)$. Many examples can then be found. The simplest was discovered by Eguchi and Hanson, and corresponds to $\Gamma = Z_2$ and $\partial M = RP^3$. This instanton is conveniently described using three left-invariant one-forms $\{\omega_i\}$ on the three-sphere, satisfying the $SU(2)$ algebra: $d\omega_i = -\frac{1}{2}\epsilon_i^{jk}\omega_j \wedge \omega_k$, and parametrized by Euler angles as follows:

$$\omega_1 = (\cos \psi)d\theta + (\sin \psi)(\sin \theta)d\phi \quad , \quad (9.1)$$

$$\omega_2 = -(\sin \psi)d\theta + (\cos \psi)(\sin \theta)d\phi \quad , \quad (9.2)$$

$$\omega_3 = d\psi + (\cos \theta)d\phi \quad , \quad (9.3)$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$. The metric of the Eguchi-Hanson instanton may be thus written in the Bianchi-IX form [27]

$$g_1 = \left(1 - \frac{a^4}{r^4}\right) dr \otimes dr + \frac{r^2}{4} \left[(\omega_1)^2 + (\omega_2)^2 + \left(1 - \frac{a^4}{r^4}\right) (\omega_3)^2 \right] \quad , \quad (9.4)$$

where $r \in [a, \infty[$. The singularity of g_1 at $r = a$ is only a coordinate singularity. We may get rid of it defining $4\frac{\rho^2}{a^2} \equiv 1 - \frac{a^4}{r^4}$, so that, as $r \rightarrow a$, the metric g_1 is approximated by the metric

$$g_2 = d\rho \otimes d\rho + \rho^2 \left[d\psi + (\cos \theta)d\phi \right]^2 + \frac{a^2}{4} \left[d\theta \otimes d\theta + (\sin \theta)^2 d\phi \otimes d\phi \right] \quad . \quad (9.5)$$

Regularity of g_2 at $\rho = 0$ is then guaranteed provided one identifies ψ with period 2π . This implies in turn the local surfaces $r = \text{constant}$ have topology RP^3 rather than S^3 , as we claimed. Note that at $r = a \Rightarrow \rho = 0$ the metric becomes that of a two-sphere of radius $\frac{a}{2}$. Following [28], we say that $r = a$ is a bolt (i.e. a fixed point of the action which is a

two-surface), where the action of the Killing vector $\frac{\partial}{\partial \psi}$ has a two-dimensional fixed-point set [27].

A whole family of multi-instanton solutions is obtained taking the group $\Gamma = Z_k$. They all have a self-dual Riemann-curvature tensor [22], and their metric takes the form

$$g = V^{-1} \left(d\tau + \underline{\gamma} \cdot \underline{dx} \right)^2 + V \underline{dx} \cdot \underline{dx} \quad . \quad (9.6)$$

Following [27], $V = V(\underline{x})$ and $\underline{\gamma} = \underline{\gamma}(\underline{x})$ on an auxiliary flat three-space with metric $\underline{dx} \cdot \underline{dx}$. This metric g solves the Einstein vacuum equations provided $\text{grad } V = \text{rot } \underline{\gamma}$, which implies $\nabla^2 V = 0$. If we take

$$V = \sum_{i=1}^n \frac{1}{|\underline{x} - \underline{x}_i|} \quad , \quad (9.7)$$

we obtain the desired asymptotically locally Euclidean multi-instantons. In particular, if $n = 1$ in (9.7), g describes flat space, whereas $n = 2$ leads to the Eguchi-Hanson instanton. If $n > 2$, there are $3n - 6$ arbitrary parameters, related to the freedom to choose the positions $\underline{x}_i$ of the singularities in V . These singularities correspond actually to coordinate singularities in (9.5), and can be removed using suitable coordinate transformations [27].

9.2. Asymptotically flat instantons

This name is chosen since the underlying idea is to deal with metrics in the path integral which tend to the flat metric in three directions but are periodic in the Euclidean-time dimension. The basic example is provided by the Riemannian version $g_R^{(1)}$ (also called Euclidean) of the Schwarzschild solution

$$g_R^{(1)} = \left(1 - 2\frac{M}{r} \right) d\tau \otimes d\tau + \left(1 - 2\frac{M}{r} \right)^{-1} dr \otimes dr + r^2 \Omega \quad , \quad (9.8)$$

where $\Omega = d\theta \otimes d\theta + (\sin \theta)^2 d\phi \otimes d\phi$ is the metric on a unit two-sphere. It is indeed well-known that, in the Lorentzian case, the metric g_L is more conveniently written using Kruskal-Szekeres coordinates

$$g_L = 32M^3 r^{-1} e^{-\frac{r}{2M}} \left(-dz \otimes dz + dy \otimes dy \right) + r^2 \Omega \quad , \quad (9.9)$$

where z and y obey the relations

$$-z^2 + y^2 = \left(\frac{r}{2M} - 1 \right) e^{\frac{r}{2M}} \quad , \quad (9.10)$$

$$\frac{(y+z)}{(y-z)} = e^{\frac{t}{2M}} \quad . \quad (9.11)$$

In the Lorentzian case, the coordinate singularity at $r = 2M$ can be thus avoided, whereas the curvature singularity at $r = 0$ remains and is described by the surface $z^2 - y^2 = 1$. However, if we set $\zeta = iz$, the analytic continuation to the section of the complexified space-time where ζ is real yields the positive-definite (i.e. Riemannian) metric

$$g_R^{(2)} = 32M^3 r^{-1} e^{-\frac{r}{2M}} \left(d\zeta \otimes d\zeta + dy \otimes dy \right) + r^2 \Omega \quad , \quad (9.12)$$

where

$$\zeta^2 + y^2 = \left(\frac{r}{2M} - 1 \right) e^{\frac{r}{2M}} \quad . \quad (9.13)$$

It is now clear that also the curvature singularity at $r = 0$ has disappeared, since the left-hand side of (9.13) is ≥ 0 , whereas the right-hand side of (9.13) would be equal to -1

at $r = 0$. Note also that, setting $z = -i\zeta$ and $t = -i\tau$ in (9.11), and writing $\zeta^2 + y^2$ as $(y + i\zeta)(y - i\zeta)$ in (9.13), one finds

$$y + i\zeta = e^{\frac{i\tau}{4M}} \sqrt{\frac{r}{2M} - 1} e^{\frac{r}{4M}} , \quad (9.14)$$

$$y = \cos\left(\frac{\tau}{4M}\right) \sqrt{\frac{r}{2M} - 1} e^{\frac{r}{4M}} , \quad (9.15)$$

which imply that the Euclidean time τ is periodic with period $8\pi M$. This periodicity on the Euclidean section leads to the interpretation of the Riemannian Schwarzschild solution as describing a black hole in thermal equilibrium with gravitons at a temperature $(8\pi M)^{-1}$ [27]. Moreover, the fact that any matter-field Green's function on this Schwarzschild background is also periodic in imaginary time leads to some of the thermal-emission properties of black holes. This is one of the greatest conceptual revolutions in modern gravitational physics.

There is also a local version of the asymptotically flat boundary condition in which ∂M has the topology of a nontrivial S^1 -bundle over S^2 , i.e. S^3/Γ , where Γ is a discrete subgroup of $SO(4)$. However, unlike the asymptotically Euclidean boundary condition, the S^3 is distorted and expands with increasing radius in only two directions rather than three [27]. The simplest example of an asymptotically locally flat instanton is the self-dual Taub-NUT solution, which can be regarded as a special case of the two-parameter Taub-NUT metrics

$$g = \frac{(r + M)}{(r - M)} dr \otimes dr + 4M^2 \frac{(r - M)}{(r + M)} (\omega_3)^2 + (r^2 - M^2) [(\omega_1)^2 + (\omega_2)^2] , \quad (9.16)$$

where the $\{\omega_i\}$ have been defined in (9.1-3). The main properties of the metric (9.16) are:

- (I) $r \in [M, \infty[$, and $r = M$ is a removable coordinate singularity provided ψ appearing in (9.1-3) is identified modulo 4π ;
- (II) the $r = \text{constant}$ surfaces have S^3 topology;
- (III) $r = M$ is a point at which the isometry generated by the Killing vector $\frac{\partial}{\partial \psi}$ has a zero-dimensional fixed-point set.

In other words, $r = M$ is a nut, i.e. a fixed point of the action which is an isolated point [28].

There is also a family of asymptotically locally flat multi-Taub-NUT instantons. Their metric takes the form (9.6), but one should bear in mind that the formula (9.7) is replaced by

$$V = 1 + \sum_{i=1}^n \frac{2M}{|\underline{x} - \underline{x}_i|} \quad (9.17)$$

Again, the singularities at $\underline{x} = \underline{x}_i$ can be removed, and the instantons are all self-dual.

9.3. Compact instantons

Compact gravitational instantons occur in the course of studying the topological structure of the gravitational vacuum. This can be done by first of all normalizing all metrics in the functional integral to have a given four-volume V , and then evaluating the instanton contributions to the partition function as a function of their topological complexity. One then sends the volume V to infinity at the end of the calculation. If one wants to constrain the metrics in the path integral to have a volume V , this can be obtained by adding a

term $\frac{\Lambda}{8\pi}V$ to the action. The stationary points of the modified action are solutions of the Einstein equations with cosmological constant Λ : $R(X, Y) - \Lambda g(X, Y) = 0$.

The few compact instantons that are known can be described as follows [27].

(1) The four-sphere S^4 , i.e. the Riemannian version of de Sitter space obtained by analytic continuation to positive-definite metrics. Setting to 3 for convenience the cosmological constant, the metric on S^4 takes the conformally-flat form [27]

$$g_I = d\beta \otimes d\beta + \frac{1}{4}(\sin \beta)^2 \left[(\omega_1)^2 + (\omega_2)^2 + (\omega_3)^2 \right] , \quad (9.18)$$

where $\beta \in [0, \pi]$. The apparent singularities at $\beta = 0, \pi$ can be made into regular nuts, provided the Euler angle ψ appearing in (9.1-3) is identified modulo 4π . The $\beta = \text{constant}$ surfaces are topologically S^3 , and the isometry group of the metric (9.18) is $SO(5)$.

(2) If in C^3 we identify (z_1, z_2, z_3) and $(\lambda z_1, \lambda z_2, \lambda z_3)$, $\forall \lambda \in C - \{0\}$, we obtain, by definition, CP^2 . For this two-dimensional complex space one can find a real four-dimensional metric, which solves the Einstein equations with cosmological constant Λ . If we set Λ to 6 for convenience, the metric of CP^2 takes the form [27]

$$g_{II} = d\beta \otimes d\beta + \frac{1}{4}(\sin \beta)^2 \left[(\omega_1)^2 + (\omega_2)^2 + (\cos \beta)^2 (\omega_3)^2 \right] , \quad (9.19)$$

where $\beta \in [0, \frac{\pi}{2}]$. A bolt exists at $\beta = \frac{\pi}{2}$, where $\frac{\partial}{\partial \psi}$ has a two-dimensional fixed-point set. The isometry group of g_{II} is locally $SU(3)$, which has a $U(2)$ subgroup acting on the three-spheres $\beta = \text{constant}$.

(3) The Einstein metric on the product manifold $S^2 \times S^2$ is obtained as the direct sum of the metrics on 2 two-spheres

$$g = \frac{1}{\Lambda} \sum_{i=1}^2 \left(d\theta_i \otimes d\theta_i + (\sin \theta_i)^2 d\phi_i \otimes d\phi_i \right) . \quad (9.20)$$

The metric (9.20) is invariant under the $SO(3) \times SO(3)$ isometry group of $S^2 \times S^2$, but is not of Bianchi-IX type as (9.18-19). This can be achieved by a coordinate transformation leading to [27]

$$g_{III} = d\beta \otimes d\beta + (\cos \beta)^2 (\omega_1)^2 + (\sin \beta)^2 (\omega_2)^2 + (\omega_3)^2 , \quad (9.21)$$

where $\Lambda = 2$ and $\beta \in [0, \frac{\pi}{2}]$. Regularity at $\beta = 0, \frac{\pi}{2}$ is obtained provided that ψ is identified modulo 2π (cf. (9.18)). Remarkably, this is a regular Bianchi-IX Einstein metric in which the coefficients of ω_1, ω_2 and ω_3 are all different.

(4) The nontrivial S^2 -bundle over S^2 has a metric which, setting $\Lambda = 3$, may be cast in the form [27]

$$g_{IV} = (1 + \nu^2) \left[\frac{(1 - \nu^2 x^2)}{(3 - \nu^2 - \nu^2(1 + \nu^2)x^2)} \frac{dx \otimes dx}{(1 - x^2)} + \frac{(1 - \nu^2 x^2)}{(3 + 6\nu^2 - \nu^4)} ((\omega_1)^2 + (\omega_2)^2) \right. \\ \left. + \frac{(3 - \nu^2 - \nu^2(1 + \nu^2)x^2)}{(3 - \nu^2)(1 - \nu^2 x^2)} (1 - x^2)(\omega_3)^2 \right] , \quad (9.22)$$

where $x \in [0, 1]$, and ν is the positive root of

$$w^4 + 4w^3 - 6w^2 + 12w - 3 = 0 . \quad (9.23)$$

The isometry group corresponding to (9.22) may be shown to be $U(2)$.

(5) Another compact instanton of fundamental importance is the $K3$ surface, whose explicit metric has not yet been found. $K3$ is defined as the compact complex surface whose first Betti number and first Chern class are vanishing. The mathematically-oriented reader might now like to recall the following basic concepts.

C1. The p -th Betti number B_p can be seen as the number of independent closed p -surfaces that are not boundaries of some $(p + 1)$ -surface [22].

C2. A complex structure on a real manifold M is a tensor field J^μ_ν such that $J^\mu_\nu J^\nu_\sigma = -\delta^\mu_\sigma$, and satisfying an integrability condition that enables one to introduce local complex coordinates z^j on M so that transition functions between different coordinate patches are holomorphic.

C3. Given a complex structure J^μ_ν , a Hermitian metric is a Riemannian metric g such that $J^\mu_\rho J^\nu_\sigma g_{\mu\nu} = g_{\rho\sigma}$.

C4. Let $g_{j\bar{k}}$ be a Hermitian metric, and consider the real $(1,1)$ form $J \equiv ig_{j\bar{k}} dz^j \wedge d\bar{z}^{\bar{k}}$. A Kähler metric is by definition a Hermitian metric with $dJ = 0$. If $g_{j\bar{k}}$ is Kähler, the corresponding closed form $J = \omega$ is called the Kähler form. A Hermitian metric is Kähler if and only if J^μ_ν is covariantly constant with respect to the connection defined by g . This means that, for Kähler metrics, the Riemannian structure is compatible with the complex structure.

C5. The Ricci tensor of a Kähler metric [29] is a (real, symmetric) bilinear form of type $(1,1)$, and the associated two-form is the Ricci form ρ .

C6. The first real Chern class c_1 is represented by the two-form $\frac{\rho}{2\pi}$.

C7. A Kähler metric on a complex manifold is said to be Kähler-Einstein if the Ricci form ρ is proportional to the Kähler form ω : $\rho = \lambda\omega$.

C8. According to Calabi's conjecture, given a compact Kähler manifold M , its Kähler form ω , its real first Chern class $c_1(M)$, then any closed (real) two-form of type $(1,1)$ belonging to $2\pi c_1(M)$ is the Ricci form of one and only one Kähler metric in the Kähler class of ω .

So far twistor theory has given important contributions to the $K3$ -metric problem, since it provides a method so as to obtain explicit approximations to the $K3$ metric [30], and it leads to a proof of the existence of Kähler-Einstein metrics on $K3$ [31,32]. We now focus on the latter development, trying to explain its importance, its limits and what remains to be done.

Indeed, Yau's proof [33] of Calabi's conjecture [34] already leads to an existence theorem for Kähler-Einstein metrics on $K3$, but it does not lead to explicit calculations. Topiwala obtained the same result using the following idea [31,32]: if a Kähler-Einstein metric on $K3$ exists, then it gives rise to a canonical one-parameter deformation of the complex structure. Before going on, it is worth saying that a one-parameter deformation is a set of deformations indexed by t , where t can either be real or complex. The deformation is said to be canonical if it depends only on the Kähler-Einstein metric, and not on any

choice of coordinate used to express structures (e.g. a coordinate description of metrics). The advantage of Topiwala's method is that it shows under which conditions the twistor space for $K3$ is biholomorphic to the one for the Eguchi-Hanson metric (9.4). One thus gets an explicit result, proving Page's conjecture [35]. However, it should be emphasized that this method is a substitute for the existence theorem obtained using partial differential equations, and does not lead to the knowledge of the metric. Thus, it seems we are left with two main possibilities:

- (a) To go on with twistor theory. Remarkably, in order to know *a priori* the twistor space, one should know all complex structures on $K3$. A number of them has already been studied [30], but the task seems to be very hard;
- (b) To use deformation theory [36,37].

Two topological invariants exist which may be used to characterize the various gravitational instantons studied so far. These invariants are the Euler number χ and the Hirzebruch signature τ . The Euler number can be defined as an alternating sum of Betti numbers

$$\chi \equiv B_0 - B_1 + B_2 - B_3 + B_4 \quad . \quad (9.24)$$

The Hirzebruch signature can be defined as

$$\tau \equiv B_2^+ - B_2^- \quad , \quad (9.25)$$

where B_2^+ is the number of self-dual harmonic two-forms [38], and B_2^- is the number of anti-self-dual harmonic two-forms [in terms of the Hodge-star operator $*F_{ab} \equiv \frac{1}{2}\epsilon_{abcd}F^{cd}$, self-duality of a two-form F is expressed as $*F = F$, and anti-self-duality as $*F = -F$].

In the case of compact four-dimensional manifolds without boundary, χ and τ can be expressed as integrals of the curvature [22]

$$\chi = \frac{1}{128\pi^2} \int_M R_{abcd} R_{efgh} \epsilon^{abef} \epsilon^{cdgh} \sqrt{\det g} d^4x \quad , \quad (9.26)$$

$$\tau = \frac{1}{96\pi^2} \int_M R_{abcd} R^{ab}{}_{ef} \epsilon^{cdef} \sqrt{\det g} d^4x \quad . \quad (9.27)$$

For the instantons previously listed one finds [27]

Eguchi-Hanson: $\chi = 2, \tau = 1$.

Asymptotically locally Euclidean multi-instantons: $\chi = n, \tau = n - 1$.

Schwarzschild: $\chi = 2, \tau = 0$.

Taub-NUT: $\chi = 1, \tau = 0$.

Asymptotically locally flat multi-Taub-NUT instantons: $\chi = n, \tau = n - 1$.

S^4 : $\chi = 2, \tau = 0$.

CP^2 : $\chi = 3, \tau = 1$.

$S^2 \times S^2$: $\chi = 4, \tau = 0$.

S^2 -bundle over S^2 : $\chi = 4, \tau = 0$.

$K3$: $\chi = 24, \tau = 16$.

10. Concluding Remarks

Recent work on canonical quantum gravity has emphasized the role of connections [15], rather than Wheeler's geometrodynamics. In Ashtekar's formalism, the space-time metric is a secondary object, while the new configuration variable is the restriction to a three-manifold of a $SL(2, C)$ spin-connection. The momentum conjugate to this variable is a $SU(2)$ soldering form which turns internal $SU(2)$ indices into $SU(2)$ spinor indices. Remarkably, in terms of these new variables, the scalar curvature ${}^{(3)}R$ of the three-manifolds no longer appears, and the constraints are polynomial. Further progress has been made using the Rovelli-Smolín loop-variables approach. However, the Rovelli-Smolín transform remains a *formal* tool because it has not been given a rigorous mathematical meaning in $3 + 1$ space-time dimensions [39].

By contrast, since the author's research in quantum gravity has only used the ADM formalism and the corresponding form of the Wheeler-De Witt equation, we have described in this review paper those properties of Dirac's theory and of quantum geometrodynamics which are also relevant to perturbative quantum gravity via (Euclidean-time) path integrals [1,40]. A naturally-occurring application of the techniques described in sections 7 and 8 has emerged in the last few years, i.e. the study of perturbative properties of physical theories in the presence of boundaries within the framework of one-loop quantum cosmology [1]. The corresponding problems are as follows.

(i) *Choice of locally supersymmetric boundary conditions* [1]: they involve the normal to the boundary and the field for spin $\frac{1}{2}$, the normal to the boundary and the spin- $\frac{3}{2}$

potential for gravitinos, Dirichlet conditions for real scalar fields, magnetic or electric field for electromagnetism, mixed boundary conditions for the four-metric of the gravitational field (and in particular Dirichlet conditions on the perturbed three-metric).

(ii) *Quantization techniques*: one-loop amplitudes can be evaluated by first reducing the classical theory to the physical degrees of freedom (hereafter referred to as PDF) by choice of gauge and then quantizing, or by using the gauge-averaging method of Faddeev and Popov, or by applying the extended-phase-space Hamiltonian path integral of Batalin, Fradkin and Vilkovisky [1].

(iii) *Regularization techniques*: the generalized Riemann zeta-function and its regularized $\zeta(0)$ value (which yields both the scaling of the one-loop prefactor and the one-loop divergences of physical theories) can be obtained by studying the eigenvalue equations obeyed by perturbative modes, once the corresponding degeneracies and boundary conditions are known, or by using geometrical formulae for one-loop counterterms which generalize well-known results for scalar fields, but make no use of mode-by-mode eigenvalue conditions and degeneracies.

It turns out that one-loop quantum cosmology may add further evidence in favour of different approaches to quantizing gauge theories being inequivalent. Studying flat Euclidean backgrounds bounded by a three-sphere, for electromagnetism the PDF method yields $\zeta(0) = -\frac{77}{180}$ and $\zeta(0) = \frac{13}{180}$ in the magnetic and electric cases respectively [1], whereas the *indirect* Faddeev-Popov method (i.e. when one-loop amplitudes are expressed using the boundary-counterterms technique and evaluating the various coefficients as in [41]) is found to yield $\zeta(0) = -\frac{38}{45}$ in both cases [1,42]. For $N = 1$ supergravity, the PDF

method yields *partial* cancellations between spin 2 and spin $\frac{3}{2}$ [1], whereas the *indirect* Faddeev-Popov method yields a one-loop amplitude which is even more divergent than in the pure-gravity case [41]. Finally, for pure gravity, the PDF method yields $\zeta(0) = -\frac{278}{45}$ in the Dirichlet case, whereas the *indirect* Faddeev-Popov method yields $\zeta(0) = -\frac{803}{45}$ [1,41]. Moreover, within the PDF approach, it is possible to set to zero on S^3 the linearized magnetic curvature. This yields a well-defined one-loop calculation, and the corresponding $\zeta(0)$ value is $\frac{112}{45}$ [1]. By contrast, using the Faddeev-Popov formula, magnetic boundary conditions for pure gravity are ruled out [41].

It is therefore necessary to get a better understanding of the manifestly gauge-invariant formulae for one-loop amplitudes used so far in the literature, by performing a mode-by-mode analysis of the eigenvalue equations, rather than relying on general formulae which contain no explicit information about degeneracies and eigenvalue conditions. As shown in [1], this detailed analysis can be attempted for vacuum Maxwell theory studied at one-loop about flat Euclidean backgrounds bounded by a three-sphere, recently considered in perturbative quantum cosmology. Working within the Faddeev-Popov formalism and making a $3 + 1$ split of the vector potential, the full $\zeta(0)$ value takes into account the contribution of the physical degrees of freedom, i.e. the transverse part $A^{(T)}$ of the vector potential, the gauge modes, i.e. the A_0 component and the longitudinal part $A^{(L)}$ of the vector potential, and the ghost action. Interestingly, a gauge-averaging term can be found such that the contributions to $\zeta(0)$ of physical degrees of freedom and of decoupled mode for A_0 add up to $-\frac{61}{90}$ both in the electric and in the magnetic case. However, remaining modes for A_0 and $A^{(L)}$ are always found to obey a coupled system of second-order ordinary

differential equations. This system has been solved exactly [1], but unfortunately the power series appearing in its solution are not (obviously) related to well-known special functions. The corresponding asymptotic analysis (i.e. at large values of the eigenvalues) is therefore much harder, and remains a stimulating challenge for applied mathematicians and theoretical physicists. It also turns out that, in the presence of boundaries, gauge modes should remain coupled, for us to be able to find a linear, elliptic second-order operator corresponding to the ghost action. Since the difficulties concerning gauge modes and ghost fields are technical in nature and not completely unfamiliar (i.e. systems of ordinary differential equations, fourth-order algebraic equations, finite parts of diverging series), there is hope that the research initiated in [1] will shed new light on one-loop properties of physical theories in the presence of boundaries.

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References

- [1] G. ESPOSITO, Quantum Gravity, Quantum Cosmology and Lorentzian Geometries, Lecture Notes in Physics, New Series m: Monographs, Volume m12 (Springer-Verlag, Berlin, 1992).
- [2] P. A. M. DIRAC, Lectures on Quantum Mechanics, Belfer Graduate School of Science (Yeshiva University, New York, 1964).
- [3] A. HANSON, T. REGGE and C. TEITELBOIM, Constrained Hamiltonian Systems, Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche e loro Applicazioni, n. 22 (Roma, 1976).
- [4] G. MARMO, N. MUKUNDA and J. SAMUEL, Riv. Nuovo Cimento **6** (1983) 1.
- [5] M. E. V. COSTA, H. D. GIROTTI and T. J. M. SIMOES, Phys. Rev. **D 32** (1985) 405.
- [6] Q. ZHI, Int. J. Theor. Phys. **31** (1992) 115.
- [7] M. REED and B. SIMON, Methods of Modern Mathematical Physics, Volume 1 (Academic Press, New York, 1972).
- [8] C. J. ISHAM, in Relativity, Groups and Topology II, Les Houches 1983, Session XL, eds. B. S. De Witt and R. Stora (North Holland, Amsterdam, 1984) 1059.
- [9] U. KULSHRESHTHA, J. Math. Phys. **33** (1992) 633.
- [10] A. CORICHI, J. Math. Phys. **33** (1992) 4066.
- [11] F. J. DYSON, Am. J. Phys. **58** (1990) 209.
- [12] D. McMULLAN, Nucl. Phys. **B 363** (1991) 451.

- [13] G. ESPOSITO, *Nuovo Cimento B* 104 (1989) 199; *Nuovo Cimento B* 106 (1991) 1315.
- [14] M. A. H. MacCALLUM, in *Quantum Gravity: an Oxford Symposium*, eds. C. J. Isham, R. Penrose and D. W. Sciama (Clarendon Press, Oxford, 1975) 174.
- [15] A. ASHTEKAR, *New Perspectives in Canonical Gravity* (Bibliopolis, Napoli, 1988).
- [16] J. W. YORK, *Found. of Phys.* 16 (1986) 249.
- [17] G. W. GIBBONS and S. W. HAWKING, *Phys. Rev. D* 15 (1977) 2752.
- [18] B. S. DE WITT, *Phys. Rev.* 160 (1967) 1113.
- [19] S. W. HAWKING, in *Relativity, Groups and Topology II*, Les Houches 1983, Session XL, eds. B. S. De Witt and R. Stora (North Holland, Amsterdam, 1984) 333.
- [20] O. PEKONEN, *J. Geom. Phys.* 4 (1987) 493.
- [21] A. E. FISHER, in *Relativity*, eds. M. Carmeli, S. I. Fickler and L. Witten (Plenum Press, New York, 1970) 303.
- [22] S. W. HAWKING, in *General Relativity, an Einstein Centenary Survey*, eds. S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979) 746.
- [23] B. ALLEN, *Nucl. Phys. B* 226 (1983) 228.
- [24] I. CHAVEL, *Eigenvalues in Riemannian Geometry* (Academic Press, New York, 1984).
- [25] S. HAWKING, *Commun. Math. Phys.* 55 (1977) 133.
- [26] P. GREINER, *Archs. Ration. Mech. Analysis* 41 (1971) 163.
- [27] C. N. POPE, in *Quantum Gravity, a second Oxford Symposium*, eds. C. J. Isham, R. Penrose and D. W. Sciama (Clarendon Press, Oxford, 1981) 377.
- [28] G. W. GIBBONS and S. W. HAWKING, *Commun. Math. Phys.* 66 (1979) 291.

- [29] S. S. CHERN, Complex Manifolds without Potential Theory (Springer-Verlag, Berlin, 1979).
- [30] N. J. HITCHIN, in Global Riemannian Geometry, eds. T. J. Willmore and N. J. Hitchin (Ellis Horwood, Chichester, 1984) 115.
- [31] P. N. TOPIWALA, Invent. Math. **89** (1987) 425.
- [32] P. N. TOPIWALA, Invent. Math. **89** (1987) 449.
- [33] S. T. YAU, Commun. Pure Appl. Math. **31** (1978) 339.
- [34] E. CALABI, Proc. Int. Congr. Math. **2** (1954) 206.
- [35] P. N. TOPIWALA, A Twistor Approach to the Einstein Metric on K3, Ph.D. Thesis (University of Michigan, 1985).
- [36] W. BARTH, C. PETERS and A. van de VEN, Compact Complex Surfaces (Springer-Verlag, Berlin, 1984).
- [37] K. KODAIRA, Complex Manifolds and Deformation of Complex Structures (Springer-Verlag, Berlin, 1986).
- [38] R. S. WARD and R. O. WELLS, Twistor Geometry and Field Theory (Cambridge University Press, Cambridge, 1990).
- [39] A. ASHTEKAR and C. J. ISHAM, Class. Quantum Grav. **9** (1992) 1433.
- [40] J. J. HALLIWELL and J. B. HARTLE, Phys. Rev. D **43** (1991) 1170.
- [41] S. POLETTI, Phys. Lett. **249 B** (1990) 249.
- [42] I. G. MOSS and S. POLETTI, Phys. Lett. **245 B** (1990) 355.