HOMOGENISATION OF VECTORIAL FREE-DISCONTINUITY FUNCTIONALS WITH COHESIVE TYPE SURFACE TERMS

GIANNI DAL MASO AND DAVIDE DONATI

Abstract. The results on Γ-limits of sequences of free-discontinuity functionals with bounded cohesive surface terms are extended to the case of vector-valued functions. In this framework, we prove an integral representation result for the Γ-limit, which is then used to study deterministic and stochastic homogenisation problems for this type of functionals.

MSC codes: 49J45, 49Q20, 60G60, 74Q05, 74S60.
Keywords: Γ-convergence, Free-discontinuity Problems, Homogenisation, Stochastic Homogenisation, Integral Representation, Functionals Depending on Vector-valued Functions

1. Introduction

Free-discontinuity problems are minimisation problems for functionals of the form

$$\int_A f(x,\nabla u)\,dx + \int_{J_u} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1}, \tag{1.1}$$

where

- A is a bounded open set in $\mathbb{R}^d$,
- f and g are two given scalar functions,
- the unknown u is a function defined in A with values in $\mathbb{R}^k$,
- J_u is the $(d-1)$-dimensional essential discontinuity set of u, whose location and size are unknown,
- ∇u is the gradient of u in $A\setminus J_u$,
- ν_u is the measure theoretical unit normal to J_u,
- $[u]=u^+-u^-$, where u^+ and u^- are the traces of u on both sides of J_u,
- $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure.

Since the function u exhibits an essential discontinuity set, a suitable framework to study these problems is the space $BV(A;\mathbb{R}^k)$ of functions of bounded variation. Problems of this type in $BV(A;\mathbb{R}^k)$ have been extensively studied (see for instance [7, Chapter 4.6] and [11]).

In his pioneering work [34], Griffith introduced the idea that the crack growth in an elastic material is determined by the competition between the stored elastic energy and the energy spent to open a new portion of the crack. Adopting this point of view, Francfort and Marigo proposed in [29] a variational model to study crack growth which includes the requirement that at each time $t>0$ the displacement $u(t)$ of the elastic body minimises a free-discontinuity functional of the form (1.1), where g also depends on the cracks present before t. In these models, the crack at time t is the union of $J_{u(s)}$ for $s\le t$, the volume integral represents the stored elastic energy, while the surface integral is related the energy spent to produce the crack. For an overview on this subject we refer to [10].

In cohesive models of fracture mechanics, it is natural to assume that the function g satisfies the following growth conditions

$$c(|\zeta|\wedge 1)\le g(x,\zeta,\nu)\le C(|\zeta|\wedge 1), \tag{1.2}$$

for suitable constants $0<c\le C$, where for $s,t\in\mathbb{R}$, $s\wedge t=\min\{s,t\}$. In particular, the Dugdale model [26] can be reformulated in the language of [10], using $g(\zeta):=a(|\zeta|\wedge b)$, for suitable constants $a,b\in[0,+\infty)$. Unfortunately, under hypotheses (1.2) the functional (1.1) is never lower

Preprint SISSA 18/2024/MATE.

semicontinuous in $BV(A;\mathbb{R}^k)$ with respect to the weak* convergence. Under suitable hypotheses (see for instance [8, Theorem 3.1] and [9]), its lower semicontinuous envelope has the form

$$\int_A f(x,\nabla u)\,dx + \int_A f^\infty\Big(x,\frac{dD^cu}{d|D^cu|}\Big)\,d|D^cu| + \int_{J_u\cap A} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1}, \tag{1.3}$$

for different functions f and g, where f satisfy the growth conditions

$$c(|\xi|-1)\le f(x,\xi)\le C(|\xi|+1) \quad \text{for every } x\in\mathbb{R}^d \text{ and } \xi\in\mathbb{R}^{k\times d}, \tag{1.4}$$

and f^∞ is its recession function with respect ξ. Here and in the rest of the paper D^cu is the Cantor part of the measure Du (see [7, Definition 3.91]), and $dD^cu/d|D^cu|$ is the Radon-Nikodým derivative of D^cu with respect to its total variation.

Since condition (1.2) does not allow us to control the total variation of $|Du|(A)$ along a minimising sequence, to gain coerciveness it is convenient to extend the functional to a space larger than $BV(A;\mathbb{R}^k)$, where all terms of (1.3) can still be defined. We choose to extend these functionals to the space $GBV_\star(A;\mathbb{R}^k)$, studied by the second author in [24], and defined using the scalar version $GBV_\star(A)$ introduced in [19]. This is a vector space and, although Du is not defined for every $u\in GBV_\star(A;\mathbb{R}^k)$, one can always define ∇u and D^cu in a convenient way. Moreover, the bounds for a minimising sequence obtained from (1.2) are enough to apply a suitable compactness result with respect to convergence $\mathcal{L}^d$-a.e. (see [19, Theorem 5.4], [21, Theorem 7.13], and [24, Theorem 4.8]).

In this paper, we study the Γ-convergence with respect to the convergence in measure of sequences of functionals of the form (1.3), where f and g satisfy (1.4) and (1.2). We first prove a compactness result (Theorem 4.1), which shows that under suitable hypotheses (see Definitions 3.1 and 3.3) on the integrands f_n and g_n corresponding to a sequence of functionals $(E_n)_n$, there always exists a subsequence Γ-converging to a functional E, whose volume and jump terms can be written as integral functionals associated to some functions f and g (see Theorem 5.16).

To obtain a complete integral representation of E as in (1.3) it remains to deal with the term depending on the Cantor part. This requires stronger hypotheses on f_n and g_n, studied in Section 6, concerning quantitative estimates on the behaviour of f_n and g_n near ∞ and near 0, respectively. Under these assumptions, we show that f and g can be obtained by taking suitable limits of the infima of some minimisation problems for $E_n(\cdot,Q)$ on suitable small cubes Q and that, if f is independent of x, we have the integral representation

$$E(u,A)=\int_A f(\nabla u)\,dx+\int_A f^\infty\Big(\frac{dD^cu}{d|D^cu|}\Big)\,d|D^cu|+\int_{J_u\cap A} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1}.$$

These results are then applied to the case of homogenisation, where the functionals E_n are given by

$$E_n(u,A)=\int_A f\Big(\frac{x}{\varepsilon_n},\nabla u\Big)\,dx+\int_A f^\infty\Big(\frac{x}{\varepsilon_n},\frac{D^cu}{|D^cu|}\Big)\,d|D^cu|+\int_{J_u\cap A} g\Big(\frac{x}{\varepsilon_n},[u],\nu_u\Big)\,d\mathcal{H}^{d-1}, \tag{1.5}$$

for a sequence $\varepsilon_n\to 0^+$. We determine general conditions on f and g which imply that the Γ-limit of these sequence of functionals exists and has the form

$$E_{\rm hom}(u,A)=\int_A f_{\rm hom}(\nabla u)\,dx+\int_A f^\infty_{\rm hom}\Big(\frac{dD^cu}{d|D^cu|}\Big)\,d|D^cu|+\int_{J_u\cap A} g_{\rm hom}([u],\nu_u)\,d\mathcal{H}^{d-1}, \tag{1.6}$$

for some functions $f_{\rm hom}$ and $g_{\rm hom}$ independent of x. These functions $f_{\rm hom}$ and $g_{\rm hom}$ are obtained by taking the limits of infima of certain minimisation problems for E_n, with $\varepsilon_n=1$, on cubes whose sides tend to infinity.

Thanks to these properties, we can apply the previous results to stochastic homogenisation problems, where f and g are random integrands satisfying suitable properties (see Definition 9.3). Under these assumptions, we show that the sequence E_n Γ-converges to E given by (1.6) almost surely.

These results were obtained in the scalar case in [21, 20]. In these papers, the truncation $u^{(m)}:=(u\wedge m)\vee(-m)$ and the corresponding estimates for the functionals are frequently used. The

main difficulty in the vector-valued setting is that these truncations have to be replaced by a sort of *smooth* truncations of the form $\psi_m^i \circ u$, $i \in \{1, ..., m\}$ for suitable functions $\psi_m^i \in C_c^\infty(\mathbb{R}^k; \mathbb{R}^k)$ satisfying

$$\psi_m^i(y) = y \quad \text{for } |y| \le R_m \qquad \text{and} \qquad \psi_m^i(y) = 0 \quad \text{for } |y| \ge \sigma^m R_m,$$

where the sequence $R_m \to +\infty$ and the constant $\sigma > 2$ are prescribed. These types of truncations were already considered in previous works (see, for instance, [12, 13, 16, 28, 31]). The main difficulty here is that an estimate for $E_n(\psi_m^i \circ u, A)$ in terms of $E_n(u, A)$ cannot be obtained for every $i \in \{1, ..., m\}$, but only for a suitable choice of i, depending on n, m, u, and A. This requires deep changes in many technical results of [21] and [20], which introduce new terms depending on m and which cannot be neglected (see, for instance, Lemma 5.9 and Proposition 6.14).

Our results can be considered as a preliminary step for the study of the asymptotic behaviour of the crack growth in a heterogeneous cohesive material, when the size of the grains tends to zero. Indeed, according to the approach of [40] (see also [36, 39, 38, 41]), in the variational theory of rate-independent problems the convergence of the quasi-static evolutions requires the Γ-convergence of the corresponding energy-dissipation functionals and the existence of a joint recovery sequence. Our results give a complete answer to the first part of this program in the case of cohesive models for fracture. In this paper we do not address the construction of a joint recovery sequence.

The problem of homogenisation of free-discontinuity functionals has already been addressed in a wide variety of cases. In the now classical work [12], Braides, Defranceschi, and Vitali first dealt with the deterministic periodic case for integrands f and g satisfying

$$c|\xi|^p \le f(x, \xi) \le C(1 + |\xi|^p) \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^{k \times d}, \tag{1.7}$$

$$c(1 + |\zeta|) \le g(x, \zeta, \nu) \le C(1 + |\zeta|) \quad \text{for every } x \in \mathbb{R}^d, \zeta \in \mathbb{R}^k, \nu \in \mathbb{S}^{d-1} \tag{1.8}$$

for some suitable constants $c, C > 0$ and for $p > 1$. Under these growth conditions one can use the compactness results on SBV by Ambrosio [5, Theorem 2.1] (see also [4, 6]) and the functionals to homogenise are of type (1.1). The main result of their paper is a periodic homogenisation theorem for such functionals. They also show that, in this case, the bulk integrand f_{hom} of the Γ-limit is determined only by f and the surface integrand g_{hom} is determined only by g.

Later Giacomini and Ponsiglione studied the scalar case $k = 1$ in [32], assuming that f satisfies (1.7) and that g does not depend on ζ and

$$c \le g(x, \nu) \le C \quad \text{for every } x \in \mathbb{R}^d \text{ and } \nu \in \mathbb{S}^{d-1}.$$

Note that in their result no periodicity assumption on f and g is made. As in the periodic case studied by Braides et al. [12], under these hypotheses they are able to show that f_{hom} dependes only on f and that g_{hom} depends only on g.

More recently, Cagnetti et al. considered the vectorial case $k \ge 1$ in [13] with non-periodic integrands f and g, where f satisfies (1.7) and g satisfies

$$c \le g(x, \zeta, \nu) \le C(1 + |\zeta|) \quad \text{for every } x \in \mathbb{R}^d, \ \zeta \in \mathbb{R}^k, \text{ and } \nu \in \mathbb{S}^{d-1}. \tag{1.9}$$

In a subsequent work [14], the results of [13] are then employed to deal with the stochastic homogenisation of free-discontinuity integral functionals satisfying growth conditions (1.7) and (1.9). Under the standard assumptions of stochastic homogenisation they prove an almost sure Γ-convergence result for functionals of type (1.1).

Building on the techniques devised in [13, 14], the same authors tackled in [15] the problem of deterministic and stochastic homogenisation for sequences of type (1.5) under the hypotheses that both f and g have linear growth. In this case, the underlying function space becomes $BV(A; \mathbb{R}^k)$ and the integral depending on the Cantor part of the derivative has to be considered in (1.5).

Recently, the problem of homogenisation of free-discontinuity functionals was also addressed in the context of functions of bounded deformation. In [30], Friedrich, Peruguini, and Solombrino tackle in dimension $d = 2$ the Γ-convergence with respect to the convergence in measure of

functionals E_n given by

$$E_n(u,A)=\int_A f\Big(\frac{x}{\varepsilon_n},\mathcal{E}u\Big)\,dx+\int_A g\Big(\frac{x}{\varepsilon_n},[u],\nu_u\Big)\,\mathcal{H}^{d-1}, \tag{1.10}$$

where $A\subset\mathbb{R}^2$ is open and bounded, $u\in GSBD^p(A;\mathbb{R}^2)$, and $\mathcal{E}u=(\nabla u+\nabla u^T)/2$ is the approximate symmetric gradient of u, assuming that the integrand $f\colon\mathbb{R}^2\times\mathbb{R}^{2\times2}_{\rm sym}\to[0,+\infty)$ has p-growth and that g is bounded from below by a positive constant.

In a recent work [25], the stochastic homogenisation of functionals of type (1.10) restricted to piecewise rigid functions, i.e., functions $u(x)=\sum_{i=1}^\infty(R_ix+b_i)\chi_{E_i}$ for $R_i\in\mathbb{R}^{d\times d}$ skew-symmetric, $b_i\in\mathbb{R}^d$, and E_i of finite perimeter, was tackled in dimensions $d\geq2$.

Our paper is organised as follows. In Section 2 we fix the notation and lay out the basic tools used throughout the paper. We then introduce in Section 3 the collections of volume integrands $\mathcal{F}$ and surface integrands $\mathcal{G}$ which will be the object of our study; we also introduce a class of abstract functionals $\mathfrak{E}$ which contains the integral functionals corresponding to integrands in $\mathcal{F}$ and $\mathcal{G}$.

Section 4 is devoted to proving a compactness result for sequences of integral functionals. In Section 5 we investigate the properties of the class $\mathfrak{E}$, by proving an integral representation result for $\mathfrak{E}$, showing that if $E\in\mathfrak{E}$ is lower-semicontinuous with respect to the convergence in measure then its "absolutely continuous part" E^a and its "jump part" E^j can be represented by integral functionals, with integrands $f\in\mathcal{F}$ and $g\in\mathcal{G}$, respectively.

In Section 6 we introduce two smaller collections of integrands $\mathcal{F}^\alpha$ and $\mathcal{G}^\vartheta$ and study the Γ-limits of sequences of integral functionals associated with them. We then prove in Section 7 that under some suitable hypotheses these Γ-limits can be fully represented as the sum of three integral functionals as in (1.3), including the term depending on the Cantor part.

In Section 8 we exploit the representation result of Section 7 to give a necessary sufficient condition for the Γ-convergence of sequences of functionals of the form (1.3) with $f\in\mathcal{F}^\alpha$ and $g\in\mathcal{G}^\vartheta$. Finally, Section 9 is devoted to the study of the Γ-limit of functionals of type (1.5). As an application, these results are then employed in the final part of Section 9 to deal with stochastic homogenisation.

2. Notation and preliminaries

In this Section we fix the notation and introduce the basic tools used in the rest of the paper.

(a) Let $n\in\mathbb{N}$. The scalar product in $\mathbb{R}^n$ is denoted by $\cdot$ and the Euclidean norm of $\mathbb{R}^n$ is denoted by $|\ |$. Given $x\in\mathbb{R}^n$, the i-th component of x is denoted by x_i. For every $\rho>0$ and $x\in\mathbb{R}^n$ the open ball of radius ρ and center x is denoted by $B_\rho(x)$.

(b) We fix once and for all two positive integer numbers $d\geq1$, $k\geq1$. the unit spere in $\mathbb{R}^d$ is denote by $\mathbb{S}^{d-1}:=\{\nu\in\mathbb{R}^d:|\nu|=1\}$, . We also set $\mathbb{S}^{d-1}_\pm:=\{\nu\in\mathbb{S}^{d-1}:\pm\nu_{i(\nu)}>0\}$, where $i(\nu)\in\{1,...,d\}$ is the largest index such that $\nu^{i(\nu)}\neq0$.

(c) Vectors in $\mathbb{R}^d$ are identified with $1\times d$ matrices, while $\mathbb{R}^{k\times d}$ is identified with the space of all $k\times d$ matrices. For $\xi\in\mathbb{R}^{k\times d}$ and $x\in\mathbb{R}^d$ $\xi x\in\mathbb{R}^k$ is defined by the usual rules of matrix multiplication and ξ^i is the i-th row of ξ. Given a matrix $\xi=(\xi_{ij})\in\mathbb{R}^{k\times d}$, its Frobenius norm is defined by

$$|\xi|:=\Big(\sum_{i=1}^k\sum_{j=1}^d\xi_{ij}^2\Big)^{1/2}.$$

(d) For $\rho>0$ we set $Q(\rho):=\{y\in\mathbb{R}^d:|y\cdot e_i|<\rho/2\}$, where $(e_i)_{i=1}^d$ is the canonical basis of $\mathbb{R}^d$. Given $x\in\mathbb{R}^d$, we set $Q(x,\rho):=x+Q(\rho)$.

(e) For $n\in\mathbb{N}$ the space of all $n\times n$ orthonormal matrices R with $\det(R)=1$ is denoted by $SO(n)$. For every $\nu\in\mathbb{S}^{d-1}$ we fix once and for all an element $R_\nu\in SO(d)$ such that $R_\nu(e_d)=\nu$. We suppose that $R_{e_d}=I$, the identity matrix, that the restrictions of

$\nu \to \mathbb{S}^{d-1}$ to $\mathbb{S}^{d-1}_{\pm}$ is continuous, and that $R_\nu(Q(\rho)) = R_{-\nu}(Q(\rho))$ for every $\nu \in \mathbb{S}^{d-1}$ (see [14, Example A.1] for the proof of the existence of such $\nu \mapsto R_\nu$).

(f) For $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, $\lambda \geq 1$, and $\rho > 0$, we consider the rectangle

$$Q^\lambda_\nu(x,\rho) := x + R_\nu\Big(\Big(-\frac{\lambda\rho}{2},\frac{\lambda\rho}{2}\Big)^{d-1} \times \Big(-\frac{\rho}{2},\frac{\rho}{2}\Big)\Big);$$

we omit the indication of λ when $\lambda = 1$.

(g) Given an open set $\Omega \subset \mathbb{R}^d$, $\mathcal{A}(\Omega)$ (resp. $\mathcal{B}(\Omega)$) is the collection of all open (resp. Borel) sets $A \subset \Omega$. Given $A, B \in \mathcal{A}(\Omega)$, $A \subset\subset B$ means that A is relatively compact in B. We set $\mathcal{A}_c(\Omega) := \{A \in \mathcal{A}(\Omega)\colon\ A \subset\subset \Omega\}$.

(h) For every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ the two functions $\ell_\xi\colon \mathbb{R}^d \to \mathbb{R}^k$ and $u_{x,\zeta,\nu}\colon \mathbb{R}^d \to \mathbb{R}^k$ are defined for every $y \in \mathbb{R}^d$ by

$$\ell_\xi(y) := \xi y,$$
$$u_{x,\zeta,\nu}(y) := \begin{cases} \zeta & \text{if } (y-x)\cdot\nu \geq 0,\\ 0 & \text{if } (y-x)\cdot\nu < 0.\end{cases}$$

Moreover, we set $\Pi^\nu_x := \{x \in \mathbb{R}^d\colon (y-x)\cdot\nu = 0\}$.

(i) Given $A \in \mathcal{A}(\mathbb{R}^d)$, the space of $\mathbb{R}^n$-valued bounded Radon measures on A is denoted by $\mathcal{M}_b(A;\mathbb{R}^n)$. If $n = 1$ we omit the indication of $\mathbb{R}^n$. If $\mu \in \mathcal{M}_b(A;\mathbb{R}^{k\times d})$ and $\lambda \in \mathcal{M}_b(A)$ is non-negative, $d\mu/d\lambda$ denotes the Radon-Nikodým derivative of μ with respect to λ. The Lebesgue measure is denoted by $\mathcal{L}^d$ and the $(d-1)$-dimensional Hausdorff measure is denoted by $\mathcal{H}^{d-1}$. For $\mu \in \mathcal{M}_b(A;\mathbb{R}^{k\times d})$ the total variation $|\mu|$ is computed with respect to the Frobenius norm.

(j) For every $A \in \mathcal{A}(\mathbb{R}^d)$, $L^0(A;\mathbb{R}^n)$ is the space of all $\mathcal{L}^d$-measurable functions $u\colon A \to \mathbb{R}^k$ with the topology induced by the convergence in measure. We recall that such topology is metrisable and separable. When $n = 1$ the indication $\mathbb{R}^n$ is omitted.

(k) Given an $\mathcal{L}^d$-measurable set $E \subset \Omega$, a point $x \in \Omega$ such that

$$\limsup_{\rho\to 0^+} \frac{\mathcal{L}^d(E \cap B_\rho(x))}{\rho^d} > 0,$$

and an $\mathcal{L}^d$-measurable function $u\colon E \to \mathbb{R}^k$, we say that $a \in \mathbb{R}^k$ is the approximate limit of u at x, in symbols

$$\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in E}} u(y) = a,$$

if for every $\varepsilon > 0$ we have

$$\lim_{\rho\to 0^+} \frac{\mathcal{L}^d(\{|u-a| > \varepsilon\} \cap B_\rho(x))}{\rho^d} = 0,$$

where $\{|u-a| > \varepsilon\} := \{y \in E\colon |u(y)-a| > \varepsilon\}$.

(l) Given $A \in \mathcal{A}(\mathbb{R}^d)$ and an $\mathcal{L}^d$-measurable function $u\colon A \to \mathbb{R}^k$, the jump set J_u is the set of all points $x \in A$ such that there exists a triple $(u^+(x), u^-(x), \nu_u(x)) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{S}^{d-1}$, with $u^+(x) \neq u^-(x)$, such that, setting

$$H^+ := \{y \in A\colon\ (y-x)\cdot\nu_u(x) > 0\} \quad\text{and}\quad H^- := \{y \in A\colon\ (y-x)\cdot\nu_u(x) < 0\},$$

we have

$$\operatorname*{ap\,lim}_{\substack{y\to x\\ y\in H^+}} u(y) = u^+(x) \quad\text{and}\quad \operatorname*{ap\,lim}_{\substack{y\to x\\ y\in H^-}} u(y) = u^-(x).$$

The triple $(u^+(x), u^-(x), \nu_u(x))$ is well-defined up to interchanging the roles of $u^+(x)$ and $u^-(x)$ and swapping the sign of $\nu_u(x)$.

(m) Given $A \in \mathcal{A}(\mathbb{R}^d)$ the symbol $BV(A;\mathbb{R}^k)$ denotes the space of $\mathbb{R}^k$-valued functions with bounded variation on A. We refer the reader to [7] (see also [11, 27, 33]) for an exhaustive introduction to this function space. We recall that if $u \in BV(A;\mathbb{R}^k)$ then for $\mathcal{H}^{d-1}$-a.e. $x \in A \setminus J_u$ there exists

$$\tilde{u}(x) := \operatorname*{ap\,lim}_{\substack{y \to x \\ y \in A}} u(y).$$

We recall also that if $u \in BV(A;\mathbb{R}^k)$, then J_u is a $(d-1)$-countably rectifiable set, and for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$ the vector $\nu_u(x)$ is a measure theoretical normal to J_u. For every $x \in J_u$ we set

$$[u](x) := u^+(x) - u^-(x).$$

A change of sign of $\nu_u(x)$ obviously implies a change of sign in $[u](x)$.

(n) Given $u \in BV(A;\mathbb{R}^k)$, its distributional derivative Du, which is by definition a bounded $\mathbb{R}^{k\times d}$-valued Radon measure, can be decomposed as

$$Du = \nabla u \mathcal{L}^d + D^c u + [u] \otimes \nu_u \mathcal{H}^{d-1} \llcorner J_u,$$

where

- $\nabla u \in L^1(A;\mathbb{R}^{k\times d})$ is the approximate gradient of u, that is, the only $\mathbb{R}^{k\times d}$-valued function such that for $\mathcal{L}^d$-a.e $x \in A$ we have
$$\operatorname*{ap\,lim}_{\substack{y \to x \\ y \in A}} \frac{u(y) - \tilde{u}(x) - \nabla u(x)(y-x)}{|y-x|} = 0, \tag{2.1}$$
- $D^c u$, called the Cantor part of Du, is a measure singular with respect to $\mathcal{L}^d$ and vanishing on all Borel sets $B \in \mathcal{B}(A)$ with $\mathcal{H}^{d-1}(B) < +\infty$,
- $\otimes$ denotes the tensor product defined by $(a \otimes b)_{ij} = a_i b_j$ for $a \in \mathbb{R}^k$, $b \in \mathbb{R}^d$, $\mathcal{H}^{d-1} \llcorner J_u$ is the Borel measure on A defined by $\mathcal{H}^{d-1}(B) := \mathcal{H}^{d-1}(B \cap J_u)$ for every $B \in \mathcal{B}(A)$, and $[u] \otimes \nu_u \mathcal{H}^{d-1} \llcorner J_u$ denotes the measure with density $[u] \otimes \nu_u$ with respect to $\mathcal{H}^{d-1} \llcorner J_u$.

We briefly recall the definition and the main properties of the space $GBV_\star(A;\mathbb{R}^k)$, introduced in the scalar setting by the first author and Toader in [19] and in the vectorial setting by the second author in [24]. For $s,t \in \mathbb{R}$ we set $s \wedge t = \min\{s,t\}$ and $s \vee t = \max\{s,t\}$. For every $t \in \mathbb{R}$, $a \in \mathbb{R}^k$ and $m > 0$, we set $t^{(m)} := (t \wedge m) \vee (-m)$ and $a^{(m)} := (a_1^{(m)}, ..., a_k^{(m)})$. In the rest of the section, $A \subset \mathbb{R}^d$ will always be a bounded open set.

Definition 2.1. Let $u \in L^0(A;\mathbb{R}^k)$. Then $u \in GBV_\star(A;\mathbb{R}^k)$ if and only $u^{(m)} \in BV(A;\mathbb{R}^k)$ for every $m > 0$ and there exists $M > 0$ such that

$$\sup_{m>0} \int_A |\nabla u^{(m)}|\, dx + |D^c u^{(m)}|(A) + \int_{J_{u^{(m)}}} |[u^{(m)}]| \wedge 1 \, d\mathcal{H}^{d-1} \le M. \tag{2.2}$$

$GBV_\star(A)$ is defined similarly for scalar functions.

Remark 2.2. It follows immediately from the definition that u belongs to $GBV_\star(A;\mathbb{R}^k)$ if and only if each component u_i belongs to $GBV_\star(A)$. By [21, Theorem 3.9] this implies that $GBV_\star(A;\mathbb{R}^k)$ is a vector space.

To characterise $GBV_\star(A;\mathbb{R}^k)$ by means of smooth truncations we introduce the following functions.

Given a positive constant $\sigma > 2$, we fix a smooth radial function $\psi \in C_c^\infty(\mathbb{R}^k;\mathbb{R}^k)$ satisfying

$$\begin{cases} \psi(y) = y & \text{if } |y| \le 1, \\ \psi(y) = 0 & \text{if } |y| \ge \sigma, \\ |\psi(y)| \le \sigma, \\ \mathrm{Lip}(\psi) = 1. \end{cases} \tag{2.3}$$

It is not difficult to construct such a function (see for instance [13, Section 4]).

Given $R>0$, we set

$$\psi_R(y) := R\psi\Big(\frac{y}{R}\Big) \quad \text{for every } y\in\mathbb{R}^k. \tag{2.4}$$

Note that ψ_R satisfies

$$\begin{cases} \psi_R(y)=y & \text{if } |y|\le R,\\ \psi_R(y)=0 & \text{if } |y|\ge \sigma R,\\ |\psi_R(y)|\le \sigma R,\\ \mathrm{Lip}(\psi_R)=1. \end{cases} \tag{2.5}$$

The following proposition characterises $GBV_\star(A;\mathbb{R}^k)$ in terms of smooth truncations.

Proposition 2.3 ([24, Proposition 3.8])**.** *For every $u\in GBV_\star(A;\mathbb{R}^k)$ there exists a constant $C_u>0$ such that for every Lipschitz function ϕ with compact support the function $v:=\phi\circ u$ belongs to $BV(A;\mathbb{R}^k)$ and satisfies the inequality*

$$\int_A |\nabla v|\,dx + |D^c v|(A) + \int_{J_v} |[v]|\wedge 1\,d\mathcal{H}^{d-1} \le C_u(\mathrm{Lip}(\phi)\vee 1). \tag{2.6}$$

Conversely, if $u\in L^0(A;\mathbb{R}^k)$ and there exists a constant $C_u>0$ such that for every integer $R>0$ (2.6) holds with $v=\psi_R\circ u$ and $\phi=\psi_R$ then $u\in GBV_\star(A;\mathbb{R}^k)$.

In the following proposition we recall the fine properties of functions in $GBV_\star(A;\mathbb{R}^k)$.

Proposition 2.4. *Let $u\in GBV_\star(A;\mathbb{R}^k)$. Then*

(a) *for $\mathcal{H}^{d-1}$-a.e $x\in A\setminus J_u$ there exists*

$$\tilde u(x) := \operatorname*{ap\,lim}_{y\to x} u(y);$$

(b) *there exists a Borel function $\nabla u\in L^1(A;\mathbb{R}^{k\times d})$ such that for $\mathcal{L}^d$-a.e. $x\in A$ formula (2.1) holds true; moreover, for every $R>0$ we have*

$$\nabla u(x) = \nabla(\psi_R\circ u)(x) \quad \text{for } \mathcal{L}^d\text{-a.e. } x\in\{|u|\le R\};$$

(c) *there exists a unique Radon measure $D^c u\in\mathcal{M}_b(A;\mathbb{R}^{k\times d})$ such that for every $R>0$ we have $D^c u(B)=0$ for every $B\in\mathcal{B}(A)$ with $\mathcal{H}^{d-1}(B\setminus J_u)=0$ and $D^c u(B) = D^c(\psi_R\circ u)(B)$ for every $B\subset\{x\in A\setminus J_u\colon \widetilde{u}(x)$ exists and $|\widetilde{u}(x)|\le R\}$; moreover, for every $B\in\mathcal{B}(A)$ we have*

$$D^c u(B) = \lim_{R\to+\infty} D^c(\psi_R\circ u)(B), \tag{2.7}$$

$$|D^c u|(B) = \lim_{R\to+\infty} |D^c(\psi_R\circ u)|(B); \tag{2.8}$$

(d) *for every $R>0$ we have $J_{\psi_R\circ u}\subset J_u$ up to an $\mathcal{H}^{d-1}$-negligible set and $|[\psi_R\circ u]|\le|[u]|$ on $J_{\psi_R\circ u}\cap J_u$. Moreover, for $\mathcal{H}^{d-1}$-a.e x in J_u and every $R>|u^+(x)|\vee|u^-(x)|$ we have $|[\psi_R\circ u](x)| = |[u](x)|$.*

For the proof of these facts we refer the reader to [24, Proposition 3.7].

We conclude this section recalling some useful facts related to the Cantor part of compositions with smooth function with compact support.

Proposition 2.5. *Let $A\in\mathcal{A}_c(\mathbb{R}^d)$, $u\in GBV_\star(A;\mathbb{R}^k)$ and $\phi\in C^1_c(\mathbb{R}^k;\mathbb{R}^k)$. Then*

(i) $\nabla(\phi\circ u) = \nabla\phi(\widetilde{u})\nabla u \quad \mathcal{L}^d$*-a.e. in* A;

(ii) $D^c(\phi\circ u) = \nabla\phi(\widetilde{u})D^c u$ *as Radon measures on* A;

(iii) *for every $R>0$ we have*

$$\frac{dD^c(\psi_R\circ u)}{d|D^c(\psi_R\circ u)|} = \frac{dD^c u}{d|D^c u|} \quad |D^c u|\text{-a.e. in } A^R_{u,0},$$

where $A^R_{u,0} := \{x \in A\colon \widetilde{u}(x)$ *exists and* $|\widetilde{u}(x)| \le R\}$. *As a consequence we have*

$$\lim_{R\to+\infty} \frac{dD^c(\psi_R\circ u)}{d|D^c(\psi_R\circ u)|} = \frac{dD^c u}{d|D^c u|} \qquad |D^c u|\text{-a.e. in } A.$$

For the proof we refer the reader to [24]. In accordance with the notation of [21, 20] and [24], we introduce two functionals on $L^0(\mathbb{R}^d;\mathbb{R}^k)$ closely related to the space $GBV_\star(A;\mathbb{R}^k)$.

Definition 2.6. Given $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$, with components $u_1,...,u_k$, for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ we define

$$V(u,A) := \sum_{i=1}^k \Big(\int_A |\nabla u_i|\,dx + |D^c u_i|(A) + \int_{J_{u_i}\cap A} |[u_i]| \wedge 1\, d\mathcal{H}^{d-1}\Big),$$

if $u|_A \in GBV_\star(A)$ and we set $V = +\infty$ otherwise. The definition is then extended to $A \in \mathcal{A}(\mathbb{R}^d)$ by setting

$$V(u,A) := \sup\{V(u,A')\colon\ A' \subset A,\ A' \in \mathcal{A}_c(\mathbb{R}^d)\}$$

and then to $B \in \mathcal{B}(\mathbb{R}^d)$ by setting

$$V(u,B) := \inf\{V(u,A)\colon\ B \subset A,\ A \in \mathcal{A}(\mathbb{R}^d)\}.$$

Definition 2.7. Given $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$, for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ we define

$$V_2(u,A) := \int_A |\nabla u|\,dx + |D^c u|(A) + \int_{J_u\cap A} |[u]| \wedge 1\, d\mathcal{H}^{d-1},$$

if $u \in GBV_\star(A;\mathbb{R}^k)$ and we set $V_2(u,A) = +\infty$ otherwise. The definition is then extended to every Borel set as in Definition 2.6.

Remark 2.8. Let $A \in \mathcal{A}_c(\mathbb{R}^d)$ and let $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$. It follows immediately from Remark 2.2 that

$$u|_A \in GBV_\star(A;\mathbb{R}^k) \text{ if and only if } V(u,A) < +\infty.$$

Remark 2.9. It follows from [8, Theorem 2.1] and [19, Theorem 3.11] that the functional V of Definition 2.6 is lower semicontinuous with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. We don't know whether this property holds for the functional V_2 of Definition 2.7.

3. Volume and Surface integrands

Throughout the rest of the paper we fix six constants $c_1,...,c_6 \ge 0$ and a bounded continuous function $\tau\colon [0,+\infty) \to [0,+\infty)$. We assume that

$$\begin{aligned} &0 < c_1 \le 1 \le c_3 \le c_5, \quad c_6 \ge (c_3/c_1)k^{3/2}, \\ &\tau(0) = 0 \quad \text{and} \quad \tau(t) \ge c_3(t\wedge 1) \quad \text{for every } t \ge 0. \end{aligned} \tag{3.1}$$

We now introduce the collection of volume integrands considered in this paper.

Definition 3.1. Let $\mathcal{F}$ be the space of functions $f\colon \mathbb{R}^d\times\mathbb{R}^{k\times d} \to [0,+\infty)$ satisfying the following conditions:

(f1) f is Borel measurable;
(f2) $c_1\sum_{i=1}^k |\xi_i| - c_2 \le f(x,\xi)$ for every $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{k\times d}$;
(f3) $f(x,\xi) \le c_3\sum_{i=1}^k |\xi_i| + c_4$ for every $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{k\times d}$;
(f4) $|f(x,\xi) - f(x,\xi)| \le c_5|\xi-\eta|$ for every $x \in \mathbb{R}^d$ and $\xi,\eta \in \mathbb{R}^{k\times d}$.

Remark 3.2. It follows from the inequalities $|\xi| \le \sum_{i=1}^k |\xi_i| \le k^{1/2}|\xi|$ that

(f2′) $c_1|\xi| - c_2 \le f(x,\xi)$ for every $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{k\times d}$,
(f3′) $f(x,\xi) \le c_3k^{1/2}|\xi| + c_4$ for every $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{k\times d}$.

The following definition introduces the collection of surface integrands considered in this paper.

Definition 3.3. Let $\mathcal{G}$ be the space of functions $g : \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0, +\infty)$ that satisfy the following conditions:

(g1) g is Borel measurable;
(g2) $c_1 \sum_{i=1}^k (|\zeta_i| \wedge 1) \le g(x, \zeta, \nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$;
(g3) $g(x, \zeta, \nu) \le c_3 \sum_{i=1}^k (|\zeta_i| \wedge 1)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$;
(g4) $|g(x, \zeta, \nu) - g(x, \theta, \nu)| \le \tau(|\zeta - \theta|)$ for every $x \in \mathbb{R}^d$, $\zeta, \theta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$;
(g5) $g(x, -\zeta, -\nu) = g(x, \zeta, \nu)$ for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$;
(g6) for every $\zeta, \theta \in \mathbb{R}^k$ with $c_6 k|\zeta| \le |\theta|$ we have $g(x, \zeta, \nu) \le g(x, \theta, \nu)$ for every $x \in \mathbb{R}^d$ and $\nu \in \mathbb{S}^{d-1}$.

Remark 3.4. A variant of property (g6) was already used in [13]. Combining (g2) and (g3), it is easy to show that $g(x, \zeta, \nu) \le c_3/c_1 g(x, \theta, \nu)$ whenever $\sum_{i=1}^k |\zeta_i| \le \sum_{i=1}^k |\theta_i|$. Arguing as in [13, Remark 3.2] we can show that this property and (g6) are weaker than monotonicity in $|\zeta|$ of g.

Definition 3.5. For every $f \colon \mathbb{R}^d \times \mathbb{R}^{k\times d} \to [0, +\infty)$, the recession function $f^\infty \colon \mathbb{R}^d \times \mathbb{R}^{k\times d} \to [0, +\infty]$ (with respect to ξ) is the function defined by

$$f^\infty(x, \xi) := \limsup_{t\to+\infty} \frac{f(x, t\xi)}{t} \tag{3.2}$$

for every $x \in \mathbb{R}^d$ and for every $\xi \in \mathbb{R}^{k\times d}$.

Remark 3.6. For every $x \in \mathbb{R}^d$ the function $\xi \mapsto f^\infty(x, \xi)$ is positively 1-homogeneous. If for every $x \in \mathbb{R}^d$ the function $\xi \mapsto f(x, \xi)$ is convex, the lim sup in (3.2) is a limit. If f satisfies (f2) and (f3), then f^∞ satisfies

$$c_1|\xi| \le c_1 \sum_{i=1}^k |\xi_i| \le f^\infty(x, \xi) \le c_3 k^{1/2} |\xi| \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^{k\times d}. \tag{3.3}$$

If f satisfies (f4), then f^∞ satisfies

$$|f^\infty(x, \xi) - f^\infty(x, \eta)| \le c_5|\xi - \eta| \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi, \eta \in \mathbb{R}^{k\times d}.$$

In Section 8 we will consider also integrands g^0 that do not belong to $\mathcal{G}$. For this reason, in the following definition we do not assume $g \in \mathcal{G}$.

Definition 3.7. Given $f \in \mathcal{F}$ and a Borel function $g \colon \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0, +\infty)$ satisfying (g5), the functional $E^{f,g} : L^0(\mathbb{R}^d; \mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^d) \to [0, +\infty]$ is the functional defined for $A \in \mathcal{A}_c(\mathbb{R}^d)$ by

$$E^{f,g}(u, A) := \int_A f(x, \nabla u)\, dx + \int_A f^\infty\Big(x, \frac{D^c u}{|D^c u|}\Big)\, d|D^c u| + \int_{J_u \cap A} g(x, [u], \nu_u)\, \mathcal{H}^{d-1},$$

if $u|_A \in GBV_\star(A; \mathbb{R}^k)$, and by $E^{f,g}(u, A) = +\infty$ otherwise. The definition is then extended to $A \in \mathcal{A}(\mathbb{R}^d)$ by setting

$$E^{f,g}(u, A) := \sup\{E^{f,g}(u, A') : \ A' \subset A, \ A' \in \mathcal{A}_c(\mathbb{R}^d)\},$$

and then to $B \in \mathcal{B}(\mathbb{R}^d)$ by setting

$$E^{f,g}(u, B) := \inf\{E^{f,g}(u, A) : \ B \subset A, \ A \in \mathcal{A}(\mathbb{R}^d)\}.$$

To study the integral representation of Γ-limits of sequences of functionals of the form $E^{f,g}$ with $f \in \mathcal{F}$ and $g \in \mathcal{G}$, it is convenient to study the properties of functionals $E^{f,g}$ that pass to the Γ-limit. This leads to define the abstract space $\mathfrak{E}$ of functionals defined on $L^0(\mathbb{R}^d; \mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^d)$. We shall prove that the Γ-limit of a sequence of functionals in $\mathfrak{E}$ belongs to $\mathfrak{E}$ (see Theorem 4.1) and that for every lower semicontinuous functional in $\mathfrak{E}$ the volume and surface part admit and integral representation of the form

$$\int_A f(x, \nabla u)\, dx \quad \text{and} \quad \int_{J_u} g(x, [u], \nu_u)\, \mathcal{H}^{d-1}$$

with integrands $f \in \mathcal{F}$ and $g \in \mathcal{G}$ (see Theorem 5.16).

For technical reasons, in the definition of the space $\mathfrak{E}$, we consider the behaviour of the functionals with respect to some smooth truncations. To this aim, for every $R>0$ and every integer $i\ge 1$ we consider the function ψ_R^i defined by (2.4) with R replaced by $\sigma^{i-1}R$ and $\sigma = c_6k+2$. Note that by (2.5) for every integer $i\ge 1$ we have

$$\begin{cases} \psi_R^i(y) = y & \text{if } |y|\le \sigma^{i-1}R,\\ \psi_R^i(y) = 0 & \text{if } |y|\ge \sigma^{i}R,\\ |\psi_R^i(y)|\le \sigma^i R,\\ \mathrm{Lip}(\psi_R) = 1. \end{cases} \tag{3.4}$$

We are now in a position to introduce the space of abstract functionals $\mathfrak{E}$.

Definition 3.8. Let $\mathfrak{E}$ be the space of functionals $E\colon L^0(\mathbb{R}^d;\mathbb{R}^k)\times\mathcal{B}(\mathbb{R}^d)\to[0,+\infty]$ satisfying the following conditions:

(a) for every $A\in\mathcal{A}(\mathbb{R}^d)$ and for every $u,v\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ with $u=v$ $\mathcal{L}^d$-a.e. in A, we have $E(u,A)=E(v,A)$;

(b) for every $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ the set function $B\mapsto E(u,B)$ is a non-negative Borel measure on $\mathbb{R}^d$ and for every $B\in\mathcal{B}(\mathbb{R}^d)$ we have
$$E(u,B)=\inf\{E(u,A)\colon A\in\mathcal{A}(\mathbb{R}^d) \text{ and } B\subset A\};$$

(c1) for every $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$, and $B\in\mathcal{B}(\mathbb{R}^d)$ we have
$$c_1V(u,B)-c_2\mathcal{L}^d(B)\le E(u,B);$$

(c2) for every $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $B\in\mathcal{B}(\mathbb{R}^d)$ we have
$$E(u,B)\le c_3V(u,B)+c_4\mathcal{L}^d(B);$$

(d) for every $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$, $B\in\mathcal{B}(\mathbb{R}^d)$, and $a\in\mathbb{R}^k$ we have
$$E(u+a,B)=E(u,B);$$

(e) for every $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$, $\xi\in\mathbb{R}^{k\times d}$, $B\in\mathcal{B}(\mathbb{R}^d)$ we have
$$E(u+\ell_\xi,B)\le E(u,B)+c_5|\xi|\mathcal{L}^d(B);$$

(f) for every $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$, $B\in\mathcal{B}(\mathbb{R}^d)$, $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$, and $\nu\in\mathbb{S}^{d-1}$ we have
$$E(u+u_{x,\zeta,\nu},B)\le E(u,B)+\tau(|\zeta|)\mathcal{H}^{d-1}(B\cap\Pi_x^\nu);$$

(g) for every $m\in\mathbb{N}$, $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$, $B\in\mathcal{B}(\mathbb{R}^d)$, $w\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$, and $R>0$ we have
$$\frac1m\sum_{i=1}^m E(w+\psi_R^i\circ(u-w),B)\le E(u,B)+c_3k^{1/2}\int_{B^R_{u,w}}|\nabla w|\,dx+c_4\mathcal{L}^d(B^R_{u,w})$$
$$+\frac{C}{m}\Big(E(u,B)+\mathcal{L}^d(B)+\int_B|\nabla w|\,dx\Big), \tag{3.5}$$
where $B^R_{u,w}:=\{x\in B\colon |u(x)-w(x)|\ge R\}$ and $C:=\max\{9c_3k/c_1,2c_3k^{1/2},c_4\}$;

(h) for every $0<\lambda\le 1/(c_6k)$, $R\in SO(k)$, $B\in\mathcal{B}(\mathbb{R}^d)$, and $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ we have
$$E(\lambda Ru,B)\le E(u,B)+(c_4+c_2)\mathcal{L}^d(B).$$

The subspace of all functionals $E\in\mathfrak{E}$ such that for every $A\in\mathcal{A}(\mathbb{R}^d)$ the functional $E(\cdot,A)$ is lower semicontinuous with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$ is denoted by $\mathfrak{E}_{\mathrm{sc}}$.

Remark 3.9. Let $E\in\mathfrak{E}$, $A\in\mathcal{A}(\mathbb{R}^d)$, and $u\in L^0(A;\mathbb{R}^k)$. We can define $E(u,B)$ for every $B\in\mathcal{B}(A)$ by extending the function u to a function $v\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and setting $E(u,B):=E(v,B)$. The value of $E(u,B)$ does not depend on the chosen extension thanks to the locality property (a) of Definition 3.8.

Remark 3.10. The inequalities (c1) and (c2) imply that

$$c_1 V_2(u,B) - c_2 \mathcal{L}^d(B) \le E(u,B) \tag{c1$'$}$$

$$E(u,B) \le c_3 k V_2(u,B) + c_4, \tag{c2$'$}$$

for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $B \in \mathcal{B}(\mathbb{R}^d)$. This follows from the elementary inequalities $|\xi| \le \sum_{i=1}^k |\xi_i| \le k^{1/2}|\xi|$ and $|\zeta| \wedge 1 \le \sum_{i=1}^k |[\zeta_i]| \wedge 1 \le k(|\zeta| \wedge 1)$, where ξ_i is the i-th row of the matrix ξ and ζ_i is the i-th component of ζ.

Remark 3.11. The technical condition (g) replaces the simpler condition (g) of [21, Definition 3.1], which in the particular case $w_1 = -m$ and $w_2 = m$ reads as

$$E(u^{(m)}, A) \le E(u,A) + \mathcal{L}^d(\{|u| > m\}). \tag{3.6}$$

Although this truncation procedure is still available in the vector-valued case (see [24, Definition 3.1, Proposition 3.8]), acting componentwise, an estimate of the form (3.6) does not hold for $k > 1$, even for the prototypical integral functional V_2. This is one of the main difficulties in generalising these results to the vector-valued case. The new condition (g) of Definition 3.8 is crucial to introduce a different type of truncation operators with good estimates. Similar smooth truncations were already considered in [12, 13, 16, 28, 31].

Remark 3.12. It follows directly from Remark 2.8 and from Definition 3.8 that if $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ then

$$E(u,A) < +\infty \quad \text{if and only if} \quad u|_A \in GBV_\star(A;\mathbb{R}^k).$$

The following proposition shows that functionals of type $E^{f,g}$ belong to $\mathfrak{E}$.

Proposition 3.13. *Let $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Then the functional $E^{f,g}$ belongs to $\mathfrak{E}$ and for every $A \in \mathcal{A}_c(\mathbb{R}^d)$. Moreover, $u \in GBV_\star(A;\mathbb{R}^k)$ we have*

$$E^{f,g}(u,B) = \int_B f(x,\nabla u)\,dx + \int_B f^\infty\Big(x, \frac{D^c u}{|D^c u|}\Big)\,d|D^c u| + \int_{J_u \cap B} g(x,[u],\nu_u)\,\mathcal{H}^{d-1} \tag{3.7}$$

for every $B \in \mathcal{B}(A)$.

Proof. The proof of the fact that $E^{f,g}$ satisfies (3.7) and (a)-(f) of Definition 3.8 may be deduced from the proof of [21, Proposition 3.11], with minor changes.

We are left with showing that $E^{f,g}$ enjoys properties (g) and (h) of Definition 3.8. We begin proving (g). Since $E^{f,g}$ satisfies properties (a) and (b), it is enough to prove (3.5) for every $A \in \mathcal{A}_c(\mathbb{R}^d)$. Without loss of generality, we may assume that $u|_A \in GBV_\star(A;\mathbb{R}^k)$. Let $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^k)$, let $R > 0$, let $m \in \mathbb{N}$, and let $i \in \{1,\dots,m\}$. We set $v_R^i := w + \psi_R^i \circ (u - w)$ and note that $v_R^i|_A \in GBV_\star(A;\mathbb{R}^k)$, thanks to Proposition 2.3 and to the fact that $GBV_\star(A;\mathbb{R}^k)$ is a vector space.

Consider the set $A_{\mathrm{reg}} := \{x \in A\colon \widetilde{u}(x),\ \widetilde{w}(x) \text{ exist}\}$. We observe that by Proposition 2.4 A_{reg} and J_u are Borel sets and that $\mathcal{H}^{d-1}(A \setminus (A_{\mathrm{reg}} \cup J_u)) = 0$. We set

$$\begin{aligned} A^i_{\mathrm{in}} &:= \{x \in A_{\mathrm{reg}}\colon |\widetilde{u}(x) - \widetilde{w}(x)| \le \sigma^{i-1}R\},\\ A^i &:= \{x \in A_{\mathrm{reg}}\colon \sigma^{i-1}R < |\widetilde{u}(x) - \widetilde{w}(x)| < \sigma^i R\},\\ A^i_{\mathrm{out}} &:= \{x \in A_{\mathrm{reg}}\colon |\widetilde{u}(x) - \widetilde{w}(x)| \ge \sigma^i R\}. \end{aligned}$$

Thanks to (i) of Proposition 2.5, for every $i \in \{1,\dots,m\}$ we have that

$$\begin{aligned} \nabla v_R^i &= \nabla u \quad \mathcal{L}^d\text{-a.e. in } A^i_{\mathrm{in}},\\ \nabla v_R^i &= \nabla w + \nabla\psi_R^i \circ (u-w)(\nabla u - \nabla w) \quad \mathcal{L}^d\text{-a.e. in } A^i,\\ \nabla v_R^i &= \nabla w \quad \mathcal{L}^d\text{- a.e. in } A^i_{\mathrm{out}}. \end{aligned} \tag{3.8}$$

Therefore,

$$\int_A f(x,\nabla v_R^i)\,dx = \int_{A^i_{\mathrm{in}}} f(x,\nabla u)\,dx + \int_{A^i_{\mathrm{str}}} f(x,\nabla v_R^i)\,dx + \int_{A^i_{\mathrm{out}}} f(x,\nabla w)\,dx. \tag{3.9}$$

Exploiting (f2′), (f3′), and (3.8), and recalling that $\mathrm{Lip}(\psi_R^i)=1$, we obtain

$$\begin{aligned}\int_{A^i} f(x,\nabla v_R^i)\,dx &\le 2c_3k^{1/2}\int_{A^i}|\nabla w|\,dx + c_3k^{1/2}\int_{A^i}|\nabla u|\,dx + c_4\mathcal{L}^d(A^i)\\ &\le 2c_3k^{1/2}\int_{A^i}|\nabla w|\,dx + \frac{c_3k^{1/2}}{c_1}\int_{A^i} f(x,\nabla u)\,dx + c_4\mathcal{L}^d(A^i),\\ \int_{A^i_{\rm out}} f(x,\nabla w)\,dx &\le c_3k^{1/2}\int_{A^R_{u,w}}|\nabla w|\,dx + c_4\mathcal{L}^d(A^R_{u,w}).\end{aligned}$$

These inequalites and (3.9) lead to

$$\begin{aligned}\sum_{i=1}^m\int_A f(x,\nabla v_R^i)\,dx &\le m\Big(\int_A f(x,\nabla u)\,dx + c_3k^{1/2}\int_{A^R_{u,w}}|\nabla w|\,dx + c_4\mathcal{L}^d(A^R_{u,w})\Big)\\ &\quad + \frac{c_3k^{1/2}}{c_1}\int_A f(x,\nabla u)\,dx + 2c_3\int_A|\nabla w|\,dx + c_4\mathcal{L}^d(A).\end{aligned}\tag{3.10}$$

To estimate the term depending on D^cu in $E^{f,g}$, we use Proposition 2.5(ii) to rewrite

$$\begin{aligned}\int_A f^\infty\Big(x,\frac{D^cv_R^i}{|D^cv_R^i|}\Big)\,d|D^cv_R^i| &= \int_{A^i_{\rm in}} f^\infty\Big(x,\frac{dD^cu}{d|D^cu|}\Big)\,d|D^cu|\\ &\quad+\int_{A^i} f^\infty\Big(x,\frac{\nabla\psi_R^i(\widetilde u-\widetilde w)}{|\nabla\psi_R^i(\widetilde u-\widetilde w)|}\frac{dD^cu}{d|D^cu|}\Big)\,|\nabla\psi_R^i(\widetilde u-\widetilde w)|d|D^cu| + \int_{A^i_{\rm out}} f^\infty(x,0)\,d|D^cu|\\ &\le \int_A f^\infty\Big(x,\frac{dD^cu}{d|D^cu|}\Big)\,d|D^cu| + \int_{A^i} f^\infty\Big(x,\frac{\nabla\psi_R^i(\widetilde u-\widetilde w)}{|\nabla\psi_R^i(\widetilde u-\widetilde w)|}\frac{dD^cu}{d|D^cu|}\Big)\,|\nabla\psi_R^i(\widetilde u-\widetilde w)|d|D^cu|,\end{aligned}$$

where we have used that $w\in W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$ and that $f^\infty(x,0)=0$. Taking advantage once again of (3.3) and of $\mathrm{Lip}(\psi_R^i)=1$, we infer that

$$\begin{aligned}&\int_{A^i} f^\infty\Big(x\frac{\nabla\psi_R^i(\widetilde u-\widetilde w)}{|\nabla\psi_R^i(\widetilde u-\widetilde w)|}\frac{dD^cu}{d|D^cu|}\Big)\,|\nabla\psi_R^i(\widetilde u-\widetilde w)|d|D^cu|\\ &\qquad\le c_3k^{1/2}\int_{A^i}|\nabla\psi_R^i(\widetilde u-\widetilde w)|d|D^cu| \le \frac{c_3k^{1/2}}{c_1}\int_{A^i} f^\infty\Big(x,\frac{dD^cu}{|D^cu|}\Big)\,d|D^cu|.\end{aligned}$$

Together with the previous inequalities this leads to

$$\begin{aligned}\sum_{i=1}^m\int_A f^\infty\Big(x,\frac{D^cv_R^i}{|D^cv_R^i|}\Big)\,d|D^cv_R^i| &\le m\int_A f^\infty\Big(x,\frac{dD^cu}{d|D^cu|}\Big)\,d|D^cu|\\ &\quad+\frac{c_3k^{1/2}}{c_1}\int_A f^\infty\Big(x,\frac{dD^cu}{|D^cu|}\Big)\,d|D^cu|.\end{aligned}\tag{3.11}$$

We now estimate the surface term in $E^{f,g}$. To this scope, we first remark that $J_{v_R^i}\subset A\setminus A_{\rm reg}$ and that for $\mathcal{H}^{d-1}$-a.e $x\in J_{v_R^i}$ we have $\nu_{v_R^i}(x)=\nu_u(x)$. Moreover, since $w\in W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$, for $\mathcal{H}^{d-1}$-a.e. $x\in A$ the approximate limit $\widetilde w(x)$ exists, so that for $\mathcal{H}^{d-1}$-a.e. every $x\in J_{v_R^i}$ we have

$$(v_R^i)^+(x)=\widetilde w(x)+\psi_R^i(u^+(x)-\widetilde w(x)),\tag{3.12}$$

$$(v_R^i)^-(x)=\widetilde w(x)+\psi_R^i(u^-(x)-\widetilde w(x)).\tag{3.13}$$

For $i \in \{1, ..., m\}$ we introduce the following partition of $\widetilde{J}_{v_R^i} := \{x \in J_{v_R^i} : \widetilde{w}(x) \text{ exists}\}$:

$$\begin{aligned}
S_1^i &:= \{x \in \widetilde{J}_{v_R^i} : |u^+(x) - \widetilde{w}(x)| < \sigma^{i-1}R,\ |u^-(x) - \widetilde{w}(x)| < \sigma^{i-1}R\},\\
S_2^i &:= \{x \in \widetilde{J}_{v_R^i} : |u^+(x) - \widetilde{w}(x))|, |u^-(x) - \widetilde{w}(x)| \in [\sigma^{i-1}R, \sigma^i R]\},\\
S_3^i &:= \{x \in \widetilde{J}_{v_R^i} : |u^+(x) - \widetilde{w}(x)| > \sigma^i R,\ |u^-(x) - \widetilde{w}(x)| > \sigma^i R\}\},\\
S_4^i &:= \{x \in \widetilde{J}_{v_R^i} : |u^+(x) - \widetilde{w}(x)| < \sigma^{i-1}R,\ |u^-(x) - \widetilde{w}(x)| > \sigma^i R\},\\
S_5^i &:= \{x \in \widetilde{J}_{v_R^i} : |u^+(x) - \widetilde{w}(x)| > \sigma^i R,\ |u^-(x) - \widetilde{w}(x)| < \sigma^{i-1}R\},\\
S_6^i &:= \{x \in \widetilde{J}_{v_R^i} : |u^+(x) - \widetilde{w}(x)| \in [\sigma^{i-1}R, \sigma^i R],\ |u^-(x) - \widetilde{w}(x)| \notin [\sigma^{i-1}R, \sigma^i R]\},\\
S_7^i &:= \{x \in \widetilde{J}_{v_R^i} :\ |u^+(x) - \widetilde{w}(x)| \notin [\sigma^{i-1}R, \sigma^i R],\ |u^-(x) - \widetilde{w}(x)| \in [\sigma^{i-1}R, \sigma^i R]\}.
\end{aligned}$$

Note that for $\ell \in \{2, 6, 7\}$

$$\text{if } S_\ell^i \cap S_\ell^j \neq \emptyset \ \text{ then } \ |i - j| \le 1,$$

so that

$$\sum_{i=1}^{m} (\chi_{S_2^i} + \chi_{S_6^i} + \chi_{S_7^i}) \le 9. \tag{3.14}$$

By definition of S_1^i and of S_3^i, recalling (3.12) and (3.13), we have

$$\int_{S_1^i} g(x, [v_R^i], \nu_{v_R^i})\, d\mathcal{H}^{d-1} = \int_{S_1^i} g(x, [u], \nu_u)\, d\mathcal{H}^{d-1}, \tag{3.15}$$

$$\int_{S_3^i} g(x, [v_R^i], \nu_{v_R^i})\, d\mathcal{H}^{d-1} = \int_{S_3^i} g(x, 0, \nu_u)\, d\mathcal{H}^{d-1} = 0. \tag{3.16}$$

Since $\mathrm{Lip}(\psi_R^i) = 1$, we have $|[\psi_R^i(u - w)]| \le |[u - w]| = |[u]|$ in $\widetilde{J}_{v_R^i}$. Hence, by (g2) and (g3) we deduce that

$$\begin{aligned}
\int_{S_2^i \cup S_6^i \cup S_7^i} g(x, [v_R^i], \nu_{v_R^i})\, d\mathcal{H}^{d-1} &\le c_3 k \int_{S_2^i \cup S_6^i \cup S_7^i} |[\psi_R^i(u - w)]| \wedge 1\, d\mathcal{H}^{d-1}\\
\le c_3 k \int_{S_2^i \cup S_6^i \cup S_7^i} |[u]| \wedge 1\, d\mathcal{H}^{d-1} &\le \frac{c_3 k}{c_1} \int_{S_2^i \cup S_6^i \cup S_7^i} g(x, [u], \nu_u)\, d\mathcal{H}^{d-1}.
\end{aligned} \tag{3.17}$$

We are left with estimating the surface integral over S_4^i and S_5^i. By definition, if $x \in S_4^i$ then $|u^+(x) - \widetilde{w}(x)| < \sigma^{i-1}R$ and $|u^-(x) - \widetilde{w}(x)| > \sigma^i R$. Therefore,

$$\begin{aligned}
|[u](x)| = |u^+(x) - u^-(x)| &\ge |u^-(x) - \widetilde{w}(x)| - |u^+(x) - \widetilde{w}(x)|\\
&\ge \sigma^i R - \sigma^{i-1}R = \sigma^{i-1}R(\sigma - 1) \ge c_6 k \sigma^{i-1} R,
\end{aligned} \tag{3.18}$$

where we have used that $\sigma > c_6 k + 1$. On the other hand, $[v_R^i](x) = u^+(x) - \widetilde{w}(x)$ so that

$$c_6 k |[v_R^i]| = c_6 k |u^+(x) - \widetilde{w}(x)| \le c_6 k \sigma^{i-1} R.$$

From these inequalities we deduce that $c_6 k |[v_R^i]| \le |[u]|$, which by (g6) implies

$$g(x, [v_R^i], \nu_{v_R^i}) \le g(x, [u], \nu_{v_R^i}) = g(x, [u], \nu_u) \qquad \mathcal{H}^{d-1}\text{-a.e. in } S_4^i.$$

The same argument shows that $\mathcal{H}^{d-1}$-a.e. in S_5^i it holds

$$g(x, [v_R^i], \nu_{v_R^i}) \le g(x, [u], \nu_u).$$

From these last two inequalities we get

$$\int_{S_4^i \cup S_5^i} g(x, [v_R^i], \nu_{v_R^i})\, d\mathcal{H}^{d-1} \le \int_{S_4^i \cup S_5^i} g(x, [u], \nu_u)\, d\mathcal{H}^{d-1}. \tag{3.19}$$

Combining (3.15)-(3.19), we obtain

$$\int_{J_{v_R^i}} g(x,[v_R^i],\nu_{v_R^i})\,d\mathcal{H}^{d-1} \le \int_{J_u} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1} + \frac{c_3k}{c_1}\int_{S_2^i\cup S_6^i\cup S_7^i} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1},$$

which, in light of (3.14), implies that

$$\sum_{i=1}^m \int_{J_{v_R^i}} g(x,[v_R^i],\nu_{v_R^i})\,d\mathcal{H}^{d-1} \le \Big(m+\frac{9c_3k}{c_1}\Big)\int_{J_u} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1}. \tag{3.20}$$

Finally, from (3.10), (3.11), and (3.20) we deduce that

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m E^{f,g}(v_R^i,A) \;&\le\; E^{f,g}(u,A) + c_3k^{1/2}\int_{A_{u,w}^R}|\nabla w|\,dx + c_4\mathcal{L}^d(A_{u,w}^R)\\ &\quad + \frac{C}{m}\left(E^{f,g}(u,A)+\mathcal{L}^d(A)+\int_A|\nabla w|\,dx\right),\end{aligned}$$

where $C:=\max\{9c_3k/c_1, 2c_3k^{1/2}, c_4\}$. This shows that $E^{f,g}$ satisfies (g).

To see that $E^{f,g}$ satisfies (h) it is enough to verify it for $A\in\mathcal{A}_c(\mathbb{R}^d)$ and for $u|_A\in GBV_\star(A;\mathbb{R}^k)$. Let $\lambda\le 1/c_6$ and let $R\in SO(k)$. Note that by (3.1) we also have $\lambda\le (c_1/c_3)k^{-1/2}$. We set $v:=\lambda Ru$. By (f2'), (f3), and (g6) we can estimate

$$\begin{aligned}E^{f,g}(v,A) \;&=\; \int_A f(x,\nabla v)\,dx + \int_A f^\infty\Big(x,\frac{dD^cv}{d|D^cv|},\Big)\,d|D^cv| + \int_{A\cap J_u} g(x,[v],\nu)\,d\mathcal{H}^{d-1}\\ &\le\; c_3k^{1/2}\lambda\Big(\int_A|\nabla u|\,dx + |D^cu|(A)\Big) + \int_A g(x,[u],\nu)\,d\mathcal{H}^{d-1} + c_4\mathcal{L}^d(A)\\ &\le\; E^{f,g}(u,A) + (c_4+c_2)\mathcal{L}^d(A),\end{aligned}$$

concluding the proof. □

Remark 3.14. It follows from Remark 2.9 and Proposition 3.13 that the functional V of Definition 2.6 belongs to $\mathfrak{E}$.

The following lemma provides an estimate for smooth truncations of the form $\psi_R\circ u$, which can be considered as an extension to the vector-valued case of the simpler inequality (3.6), valid in the scalar case. The proof heavily relies on property (g) of Definition 3.8. More refined versions of the following result will be presented in the forthcoming sections.

Lemma 3.15. *Let $A_1,A_2\in\mathcal{A}_c(\mathbb{R}^d)$, let $u_1\in L^0(A_1;\mathbb{R}^k)$ and $u_2\in L^0(A_2;\mathbb{R}^k)$, and for $j=1,2$ let $E(\cdot,A_j)\colon L^0(A_j;\mathbb{R}^k)\to[0,+\infty]$ be a a lower semicontinuous functional satisfying (c2') and (g) of Definition 3.8 with $B=A_1$ and $B=A_2$. Then there exists a strictly increasing sequence $R_m>0$, with $R_m\to+\infty$ as $m\to+\infty$, such that*

$$\lim_{m\to+\infty} E(\psi_{R_m}\circ u_j,A_j) = E(u_j,A_j) \tag{3.21}$$

for $j=1,2$.

Proof. For every sequence $R_m\to+\infty$ as $m\to+\infty$, we have that $\psi_{R_m}\circ u\to u_j$ in $L^0(A;\mathbb{R}^k)$ as $m\to+\infty$, so that the lower semicontinuity of $E(\cdot,A_j)$ with respect to the topology of $L^0(A_j;\mathbb{R}^k)$ implies that

$$E(u_j,A_j)\le\liminf_{m\to\infty} E(\psi_{R_m}\circ u_j,A_j) \tag{3.22}$$

for $j=1,2$.

Assume now that $E(u_1,A_1)$ and $E(u_2,A_2)$ are both finite. We can choose a sequence r_m converging to $+\infty$ as $m\to+\infty$ such that

$$c_4\mathcal{L}^d(A_{u_j,0}^{r_m})\le\frac{1}{m}\quad\text{for every } m\in\mathbb{N} \text{ and for } j=1,2.$$

From (g) of Definition 3.8 applied with $w = 0$, we obtain that there exists $i(m) \in \{1, ..., m\}$, such that

$$E(\psi_{r_m}^{i(m)} \circ u_1, A_1) + E(\psi_{r_m}^{i(m)} \circ u_2, A_2) \leq E(u_1, A_1) + E(u_2, A_2) + \frac{2 + C(E(u_1, A_1) + E(u_2, A_2)) + 2c_4 C\mathcal{L}^d(A))}{m}.$$

We now set $R_m := \sigma^{i(m)} r_m$, so that $\psi_{r_m}^{i(m)} = \psi_{R_m}$. Hence, the previous inequality gives

$$\limsup_{m\to\infty}(E(\psi_{R_m} \circ u_1, A_1) + E(\psi_{R_m} \circ u_2, A_2)) \leq E(u_1, A_1) + E(u_2, A_2).$$

Combining this last inequality with (3.22), we obtain (3.21) in the case where $E(u_1, A_1)$ and $E(u_2, A_2)$ are both finite. A simpler proof yields the result when in the other cases. □

4. A compactness result

The main result of this section is a compactness theorem for the class $\mathfrak{E}$. The strategy here adopted is the one of [21, Lemma 3.24]. As already mentioned, the main difference in the proof this compactness result in the vectorial case lies in the different type of truncation used, which is dealt with in Lemma 4.5 below.

In what follows, given a sequence $(E_n)_n \subset \mathfrak{E}$ and $A \in \mathcal{A}(\mathbb{R}^d)$, we set

$$E'(\cdot, A) := \Gamma\text{-}\liminf_{n\to\infty} E_n(\cdot, A) \quad \text{and} \quad E''(\cdot, A) := \Gamma\text{-}\limsup_{n\to\infty} E_n(\cdot, A), \tag{4.1}$$

$$E'_-(\cdot, A) := \sup_{A' \in \mathcal{A}_c(A)} E'(\cdot, A) \quad \text{and} \quad E''_-(\cdot, A) := \sup_{A' \in \mathcal{A}_c(A)} E''(\cdot, A'), \tag{4.2}$$

where the Γ-$\liminf$ and the Γ-$\limsup$ are computed with respect to the topology of $L^0(\mathbb{R}^d; \mathbb{R}^k)$. Given $u \in L^0(\mathbb{R}^d; \mathbb{R}^k)$, it is immediate to check that $E'(u, \cdot)$, $E''(u, \cdot)$, $E'_-(u, \cdot)$, and $E''_-(u, \cdot)$ are all increasing set functions.

Theorem 4.1. *Let $(E_n)_n \subset \mathfrak{E}$ be a sequence of functionals. Then there exists a subsequence of $(E_n)_n$, not relabelled, and a functional $E \in \mathfrak{E}_{\rm sc}$ such that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_n(\cdot, A)$ Γ-converges to $E(\cdot, A)$ in the topology of $L^0(\mathbb{R}^d; \mathbb{R}^k)$.*

Proof. By a compactness theorem for Γ-convergence of increasing functionals (see [17, Theorem 16.9]), there exists a subsequence of $(E_n)_n$, not relabelled, such that

$$E(u, A) := E'_-(u, A) = E''_-(u, A) \quad \text{for every } u \in L^0(\mathbb{R}^d; \mathbb{R}^k) \text{ and for } A \in \mathcal{A}(\mathbb{R}^d). \tag{4.3}$$

Thus, we can define $E\colon L^0(\mathbb{R}^d; \mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^d) \to [0 + \infty]$ as

$$E(u, B) = \inf\{E(u, A) : \ A \in \mathcal{A}(\mathbb{R}^d), \ B \subset A\} \quad \text{for every } B \in \mathcal{B}(\mathbb{R}^d). \tag{4.4}$$

We claim that $E \in \mathfrak{E}_{\rm sc}$ and that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ we have $E(\cdot, A) = E'(\cdot, A) = E''(\cdot, A)$.

From some general Γ-convergence results (see [17, Proposition 16.15]), it follows that E satisfies property (a) of Definition 3.8.

To show that E enjoys also property (b), we make use of the De Giorgi-Letta Criterion for measures [22] (see also [17, Theorem 14.23] for the particular version of the theorem here employed). As already mentioned, for every $u \in L^0(\mathbb{R}^d; \mathbb{R}^k)$ the set function $E(u, \cdot)$ is increasing. The inner regularity follows by definition of $E(u, \cdot)$, while superadditivity is a consequence of [17, Proposition 16.12] and of the fact that E_n satisfies (b) of Definition 3.8. We are left with proving that $E(u, \cdot)$ is subadditive. To prove this, we make use of a truncation argument and of the following estimate for which we refer to [21, Lemma 3.19], the proof in the vectorial case being the same as in the scalar case.

Lemma 4.2 ([21, Lemma 3.19])**.** *Let $(E_n) \subset \mathfrak{E}$ be a sequence of functionals and let $A', A'', A, U \in \mathcal{A}_c(\mathbb{R}^d)$, with $A' \subset\subset A'' \subset\subset A$. Let $u \in L^1_{\rm loc}(\mathbb{R}^d; \mathbb{R}^k)$ and let $(w_n)_n, (v_n)_n \subset L^1_{\rm loc}(\mathbb{R}^d; \mathbb{R}^k)$ be two sequence converging to u in $L^1_{\rm loc}(\mathbb{R}^d; \mathbb{R}^k)$ and such that $(w_n|_A) \subset BV(A; \mathbb{R}^k)$, $(v_n|_U) \subset$*

$BV(U;\mathbb{R}^k)$. *Then for every* $\delta>0$ *there exists a sequence* $(\varphi_n)_n\subset C_c^\infty(\mathbb{R}^d,[0,1])$, *with* $supp(\varphi_n)\subset A''$ *and* $\varphi_n=1$ *in a neighborhood of* $\overline{A}'$, *such that the functions* u_n *defined by*

$$u_n:=\varphi_n w_n+(1-\varphi_n)v_n$$

converge to u *in* $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$, $u_n|_{A'\cup U}\in BV(A'\cup U;\mathbb{R}^k)$, *and*

$$\liminf_{n\to+\infty}E_n(u_n,A'\cup U)\le(1+\delta)\liminf_{n\to+\infty}(E_n(w_n,A)+E_n(v_n,U))+\delta,$$
$$\limsup_{n\to+\infty}E_n(u_n,A'\cup U)\le(1+\delta)\limsup_{n\to+\infty}(E_n(w_n,A)+E_n(v_n,U))+\delta.$$

We now consider the truncated functions $\psi_R\circ u$ and prove that $E'(\psi_R\circ u,\cdot)$ and $E''(\psi_R\circ u,\cdot)$ satisfy a weak subadditivity inequality.

Lemma 4.3. *Let* $(E_n)_n\subset\mathfrak{E}$ *be a sequence of functionals, let* E' *and* E'' *be the functionals defined by* (4.1), *let* $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$, *and let* $A',A,U\in\mathcal{A}_c(\mathbb{R}^d)$, *with* $A'\subset\subset A$. *Then for every* $R>0$, $m\in\mathbb{N}$ *we have*

$$E'(\psi_R\circ u,A'\cup U)\le E''(\psi_R\circ u,A)+E'(\psi_R\circ u,U),\tag{4.5}$$
$$E''(\psi_R\circ u,A'\cup U)\le E''(\psi_R\circ u,A)+E''(\psi_R\circ u,U).\tag{4.6}$$

Proof. We prove (4.5). Without loss of generality, we assume that $E''(\psi_R\circ u,A)$ and $E'(\psi_R\circ u,U)$ are both finite. Let $w_n,v_n\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ be two sequence of functions converging in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ to $\psi_R\circ u$ and such that

$$\limsup_{n\to\infty}E_n(w_n,A)=E''(\psi_R\circ u,A)\quad\text{and}\quad\liminf_{n\to\infty}E_n(v_n,U)=E'(\psi_R\circ u,U).\tag{4.7}$$

We fix a subsequence $(E_{n_h})_h$ of $(E_n)_n$ such that

$$\lim_{h\to+\infty}E_{n_h}(v_{n_h},U)=\liminf_{n\to\infty}E_n(v_n,U)=E'(\psi_R\circ u,U).$$

Without loss of generality we may assume that there exists $M>0$ such that

$$E_{n_h}(w_{n_h},A)<M\quad\text{and}\quad E_{n_h}(v_{n_h},U)<M,$$

for every $h\in\mathbb{N}$. Remark 3.12 then implies $(w_{n_h}|_A)_h\subset GBV_\star(A;\mathbb{R}^k)$, $(v_{n_h}|_U)_h\subset GBV_\star(U;\mathbb{R}^k)$. By property (g) of Definition 3.8 for every $h,m\in\mathbb{N}$ we can find $i_{h,m},j_{h,m}\in\{1,...,m\}$ such that

$$E_{n_h}(\psi_{\sigma R}^{i_{h,m}}\circ w_{n_h},A)\le E_{n_h}(w_{n_h},A)+c_4\mathcal{L}^d(A^{\sigma R}_{w_{n_h},0})+C\frac{M+\mathcal{L}^d(A)}{m},\tag{4.8}$$
$$E_{n_h}(\psi_{\sigma R}^{j_{h,m}}\circ v_{n_h},U)\le E_{n_h}(v_{n_h},U)+c_4\mathcal{L}^d(U^{\sigma R}_{v_{n_h},0})+C\frac{M+\mathcal{L}^d(U)}{m}.\tag{4.9}$$

Since w_{n_h} and v_{n_h} converge in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ to $\psi_R\circ u$ as $n\to+\infty$, $\|\psi_{\sigma R}^{i_{h,m}}\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^k)}\le\sigma^{m+1}R$, $\|\psi_{\sigma R}^{j_{h,m}}\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^k)}\le\sigma^{m+1}R$, and $\|\psi_R\circ u\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^k)}\le\sigma R$, for every $m\in\mathbb{N}$ we infer that

$$\psi_{\sigma R}^{i_{h,m}}\circ w_{n_h}\to\psi_R\circ u\quad\text{in }L^1(\mathbb{R}^d;\mathbb{R}^k)\text{ as }h\to+\infty,$$
$$\psi_{\sigma R}^{j_{h,m}}\circ v_{n_h}\to\psi_R\circ u\quad\text{in }L^1(\mathbb{R}^d;\mathbb{R}^k)\text{ as }h\to+\infty,$$
$$\lim_{h\to\infty}\mathcal{L}^d(A^{\sigma R}_{w_{n_h},0})=\mathcal{L}^d(A^{\sigma R}_{\psi_R\circ u,0})=0,\tag{4.10}$$
$$\lim_{h\to\infty}\mathcal{L}^d(U^{\sigma R}_{v_{n_h},0})=\mathcal{L}^d(U^{\sigma R}_{\psi_R\circ u,0})=0.\tag{4.11}$$

For every $V\in\mathcal{A}(\mathbb{R}^d)$ we set

$$\widehat{E}'(\cdot,V):=\Gamma\text{-}\liminf_{h\to\infty}E_{n_h}(\cdot,V)\quad\text{and}\quad\widehat{E}''(\cdot,V):=\Gamma\text{-}\limsup_{h\to\infty}E_{n_h}(\cdot,V),\tag{4.12}$$

and observe that

$$E'(\cdot,V)\le\widehat{E}'(\cdot,V)\quad\text{and}\quad\widehat{E}''(\cdot,V)\le E''(\cdot,V).\tag{4.13}$$

By (4.7)-(4.11), we deduce that

$$\limsup_{h\to\infty} E_{n_h}(\psi_{\sigma R}^{i_{h,m}}\circ w_{n_h}, A) \le E''(\psi_R\circ u, A) + C\frac{M+\mathcal{L}^d(A)}{m} \tag{4.14}$$

$$\liminf_{h\to\infty} E_{n_h}(\psi_{\sigma R}^{j_{h,m}}\circ v_{n_h}, U) \le E'(\psi_R\circ u, U) + C\frac{M+\mathcal{L}^d(U)}{m}. \tag{4.15}$$

By Proposition 2.3 we have that $\psi_{\sigma R}^{i_{h,m}}\circ w_{n_h}|_A \in BV(A;\mathbb{R}^k)$ and that $\psi_{\sigma R}^{j_{h,m}}\circ v_{n_h}|U \in BV(U,\mathbb{R}^k)$. Thus, for $\delta>0$ and for every $m\in\mathbb{N}$ we may apply Lemma 4.2 to obtain a sequence of functions $(u_h^m)_h \subset BV(A'\cup U;\mathbb{R}^k)$ such that $u_h^m \to \psi_R\circ u$ in $L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$ as $h\to+\infty$ and such that

$$\begin{aligned}\liminf_{h\to+\infty} E_{n_h}(u_h^m, A'\cup U) &\le (1+\delta)\liminf_{h\to\infty}\Big(E_{n_h}(\psi_{\sigma R}^{i_{h,m}}\circ w_{n_h}, A) + E_{n_h}(\psi_{\sigma R}^{j_{h,m}}\circ v_{n_h}, U)\Big)+\delta\\ &\le (1+\delta)\limsup_{h\to\infty} E_{n_h}(\psi_{\sigma R}^{i_{h,m}}\circ w_{n_h}, A) + (1+\delta)\liminf_{h\to+\infty} E_{n_h}(\psi_{\sigma R}^{j_{h,m}}\circ v_{n_h}, U)+\delta\end{aligned}$$

This inequality, combined with (4.14), and (4.15), yields

$$\liminf_{h\to\infty} E_{n_h}(u_h^m, A'\cup U) \le (1+\delta)\Big(E''(\psi_R\circ u, A) + E'(\psi_R\circ u, U) + \frac{S}{m}\Big)+\delta,$$

where S is a positive constant independent of m. Recalling that $u_h^m\to\psi_R\circ u$ as $h\to+\infty$, by (4.12) and (4.13) we obtain that

$$E'(\psi_R\circ u, A'\cup U) \le \widehat{E}'(\psi_R\circ u, A'\cup U) \le (1+\delta)\Big(E''(\psi_R\circ u, A) + E'(\psi_R\circ u, U) + \frac{S}{m}\Big)+\delta.$$

Letting $m\to+\infty$ and $\delta\to 0^+$, we obtain (4.5).

A similar, but easier, argument shows that (4.6) holds true. □

To prove a weak subadditivity inequality for $E'(u,\cdot)$ we will approximate $E'(u,A)$ by $E'(\psi_R\circ u, A)$. For technical reasons, this approximation result is obtained using property (g) of Definition 3.8 for E', which is proved in the following lemma.

Lemma 4.4. *Let $(E_n)_n$ be a sequence in $\mathfrak{E}$, let E' be the functional defined by (4.1), let $A\in\mathcal{A}_c(\mathbb{R}^d)$, $u\in GBV_\star(A;\mathbb{R}^k)$, and let $w\in W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$. Then for every $R>0$ and $m\in\mathbb{N}$ we have*

$$\begin{aligned}\frac{1}{m}\sum_{i=1}^m E'(w+\psi_R^i\circ(u-w), A) \le E'(u,A) + c_3k^{1/2}\int_{A^R_{u,w}}|\nabla w| + c_4\mathcal{L}^d(A^R_{u,w})\\ + \frac{C}{m}\left(E'(u,A)+\mathcal{L}^d(A)+\int_A|\nabla w|\,dx\right),\end{aligned} \tag{4.16}$$

where $C=\{9c_3k/c_1, 2c_3k^{1/2}, c_4\}$.

Proof. Let us fix $u\in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $w\in W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$. Without loss of generality, we may assume that $E'(u,A)<+\infty$. Let $(u_n)_n\subset L^0(\mathbb{R}^d;\mathbb{R}^k)$ be a sequence of functions converging to u in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ and such that

$$\liminf_{n\to+\infty} E_n(u_n, A) = E'(u,A). \tag{4.17}$$

Fix $R>0$ and $m\in\mathbb{N}$. For every $i\in\{1,...,m\}$, we set $v_n^i := w+\psi_R^i\circ(u_n-w)$ and $v^i := w+\psi_R\circ(u-w)$. Note that for every $i\in\{1,...,m\}$ the sequence $(v_n^i)_n$ converges to v^i in $L^0(\mathbb{R}^d;\mathbb{R}^k)$, so that by Definition of E' we have

$$\frac{1}{m}\sum_{i=1}^m E'(v^i,A) \le \frac{1}{m}\sum_{i=1}^m \liminf_{n\to+\infty} E_n(v_n^i, A) \le \liminf_{n\to+\infty}\frac{1}{m}\sum_{i=1}^m E_n(v_n^i, A). \tag{4.18}$$

By property (g) and by (4.17) we get

$$\frac{1}{m}\sum_{i=1}^{m} E_n(v_n^i, A) \leq E_n(u_n, A) + \frac{C}{m}\Big(E_n(u_n, A) + \mathcal{L}^d(A) + \int_A |\nabla w|\Big) + c_3 k^{1/2} \int_{A^R_{u_n,w}} |\nabla w|\, dx + c_4 \mathcal{L}^d(A^R_{u_n,w}).$$

By (4.17), this inequality gives

$$\frac{1}{m}\sum_{i=1}^{m} E'(v^i, A) \leq E'(u, A) + \frac{C}{m}\Big(E'(u, A) + \mathcal{L}^d(A) + \int_A |\nabla w|\Big) + \limsup_{n\to+\infty}\Big(c_3 k^{1/2} \int_{A^R_{u_n,w}} |\nabla w|\, dx + c_4 \mathcal{L}^d(A^R_{u_n,w})\Big). \tag{4.19}$$

Since $u_n \to u$ in $L^0(\mathbb{R}^d;\mathbb{R}^k)$, we have that

$$\limsup_{n\to+\infty} \chi_{A^R_{u_n,w}}(x) \leq \chi_{A^R_{u,w}}(x) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in A.$$

Thus, from Fatou's Lemma, we deduce that

$$\limsup_{n\to+\infty}\Big(\int_{A^R_{u_n,w}} |\nabla w|\, dx + \mathcal{L}^d(A^R_{u_n,w})\Big) \leq \int_{A^R_{u,w}} |\nabla w|\, dx + \mathcal{L}^d(A^R_w), \tag{4.20}$$

which combined with (4.19) yields (4.16). □

To obtain the weak subadditivity of $E'(u,\cdot)$ we show that $E'(u, A)$ can be approximated by $E'(\psi_R \circ u, A)$ for a suitable choice of R.

Lemma 4.5. *Let $(E_n)_n$ be a sequence in $\mathfrak{E}$, let E' be the functional defined by (4.1), let $A_1, A_2 \in \mathcal{A}_c(\mathbb{R}^d)$, $u_1 \in L^0(A_1;\mathbb{R}^k)$ and $u_2 \in L^0(A_2;\mathbb{R}^k)$. Then there exists an increasing sequence $R_m > 0$, with $R_m \to +\infty$ as $m \to +\infty$, such that*

$$\lim_{m\to+\infty} E'(\psi_{R_m} \circ u_j, A_j) = E'(u_j, A_j) \tag{4.21}$$

for $j = 1, 2$.

Proof. It is immediate to check that E' satisfies property (c2), and by Lemma 4.4, we have that E' satisfies property (g) for $B = A_1$ and $B = A_2$. Since the functional $E'(\cdot, A_j)$ is lower semicontinuous with respect to the topology of $L^0(A_j;\mathbb{R}^k)$ for $j = 1, 2$ (see [17, Proposition 6.8]), an application of Lemma 3.15 then proves the claim. □

We now prove a weak subadditivity inequality for $E'(u,\cdot)$.

Lemma 4.6. *Let $(E_n)_n$ be a sequence of functionals in $\mathfrak{E}$ and let E', E'' be defined by (4.1). Assume that there exists a functional $E\colon L^0(\mathbb{R}^d;\mathbb{R}^k) \times \mathcal{A}(\mathbb{R}^d) \to [0, +\infty]$ such that (4.3) holds. Let $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$, and $A', A, U \in \mathcal{A}_c(\mathbb{R}^d)$ with $A' \subset\subset A$. Then we have*

$$E'(u, A' \cup U) \leq E'(u, A) + E'(u, U). \tag{4.22}$$

Proof. By [17, Proposition 15.15] we may choose $A'' \in \mathcal{A}_c(\mathbb{R}^d)$, with $A' \subset\subset A'' \subset\subset A$, such that

$$E''(v, A'') = E'(v, A'') = E(v, A'') \quad \text{for every } v \in L^0(\mathbb{R}^d;\mathbb{R}^k). \tag{4.23}$$

Lemma 4.5 implies that there exists $R_m \to +\infty$ as $m \to +\infty$ such that

$$\lim_{m\to+\infty} E'(\psi_{R_m} \circ u, A'') = E'(u, A''), \tag{4.24}$$

$$\lim_{m\to+\infty} E'(\psi_{R_m} \circ u, U) = E'(u, U). \tag{4.25}$$

Thanks, to Lemma 4.3, we obtain that

$$E'(\psi_{R_m} \circ u, A' \cup U) \leq E''(\psi_{R_m} \circ u, A'') + E'(\psi_{R_m} \circ u, U) \leq E'(\psi_{R_m} \circ u, A) + E'(\psi_{R_m} \circ u, U),$$

where in the equality we have used (4.23). Since $\psi_{R_m} \circ u \to u$ in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ as $m \to +\infty$, by the lower semicontinuity of E', (4.24), and (4.25) we obtain

$$E'(u, A' \cup U) \leq E'(u, A'') + E'(u, U) \leq E'(u, A) + E'(u, U),$$

concluding the proof. □

We are now ready to prove the subadditivity of $E(u,\cdot)$ on $\mathcal{A}(\mathbb{R}^d)$.

Lemma 4.7. *Let E_n be a sequence in $\mathfrak{E}$ for which (4.3) holds for some functional E. Then for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $A, U \in \mathcal{A}(\mathbb{R}^d)$ we have*

$$E(u, A \cup U) \leq E(u, A) + E(u, U).$$

Proof. The result follows from Lemma 4.6 and by standard arguments (see for instance the proof of [17, Lemma 18.4]). □

Finally, we show that $(E_n(\cdot, A))_n$ Γ-converges for every $A \in \mathcal{A}_c(\mathbb{R}^d)$.

Lemma 4.8. *Let $(E_n)_n$ be a sequence of functionals in $\mathfrak{E}$, let E', E'' be the functionals defined by (4.1). Assume that there exists a functional E such that (4.3) holds. Then $E(u, A) = E'(u, A) = E''(u, A)$ for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $A \in \mathcal{A}_c(\mathbb{R}^d)$.*

Proof. Fix $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $A \in \mathcal{A}_c(\mathbb{R}^d)$. We first show that $E(u, A) = E'(u, A)$. Since $E(u, A) \leq E'(u, A)$, it is enough to show the converse inequality. Without loss of generality we may suppose that $E(u, A) < +\infty$. It is immediate to check that E' and E satisfies properties (c2) Definition 3.8 on $\mathcal{A}(\mathbb{R}^d)$ and using Remark 2.9 we check that E' and E also satisfy (c1) on $\mathcal{A}(\mathbb{R}^d)$. Hence, $u \in GBV_\star(A;\mathbb{R}^k)$. Fix $\varepsilon > 0$ and consider a compact set $K \subset A$ such that

$$c_3 V(u, A \setminus K) + c_4 \mathcal{L}^d(A \setminus K) \leq \varepsilon. \tag{4.26}$$

Consider now $A', A'' \in \mathcal{A}_c(A)$ with $K \subset A' \subset\subset A''$ and let $U = A \setminus K$. Note that $A' \cup U = A$. Hence, by Lemma 4.6 to obtain

$$E'(u, A) \leq E'(u, A'') + E'(u, A \setminus K) \leq E(u, A) + \varepsilon, \tag{4.27}$$

where in the last inequality we have used (4.3), property (c2) for E' and (4.26). These arguments shows that $E(u, A) = E'(u, A)$ for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $A \in \mathcal{A}_c(\mathbb{R}^d)$.

We now prove that $E(u, A) = E''(u, A)$. Since $E(u, A) \leq E''(u, A)$ it is enough to show that $E''(u, A) \leq E(u, A)$. It is not restrictive to assume that $E(u, A) < +\infty$, so that $u \in GBV_\star(A;\mathbb{R}^k)$. Given $R > 0$, we can exploit the same argument used to obtain (4.27), replacing (4.22) of Lemma 4.6 with (4.6), to show that for every $\varepsilon > 0$

$$E''(\psi_R \circ u, A) \leq E(\psi_R \circ u, A) + \varepsilon = E'(\psi_R \circ u, A) + \varepsilon, \tag{4.28}$$

where we have used the equality $E = E'$. By Lemma 4.5 there exists a positive sequence $R_m \to +\infty$ such that

$$\lim_{m \to +\infty} E'(\psi_{R_m} \circ u, A) = E'(u, A) = E(u, A).$$

Since $\psi_{R_m} \circ u \to u$ in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ as $m \to +\infty$, exploiting the lower semicontinuity of E'' and this last equality, we deduce from (4.28) that

$$E''(u, A) \leq E(u, A) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $E''(u, A) \leq E(u, A)$, concluding the proof. □

Continuation of the Proof of Theorem 4.1. Thanks to Lemma 4.8, for every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $A \in \mathcal{A}_c(\mathbb{R}^d)$ we have that

$$E(u, A) = E'(u, A) = E''(u, A). \tag{4.29}$$

We are left with proving that $E \in \mathfrak{E}_{sc}$. We already noted that E satisfies (a) of Definition 3.8. From Lemma 4.7 and De Giorgi-Letta theorem, we deduce that E satisfies (b) as well. Properties (c2)-(f) can be derived arguing as in [21, Theorem 3.16].

By Remark 2.9 and (4.29), we infer that for ever $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$ and $A \in \mathcal{A}_c(\mathbb{R}^d)$ we have

$$c_1 V(u,A) - c_2 \mathcal{L}^d(A) \le E(u,A).$$

By inner regularity the inequality can be extended to $\mathcal{A}(\mathbb{R}^d)$ and, recalling (4.4), to $\mathcal{B}(\mathbb{R}^d)$. Property (g) for E on $\mathcal{A}_c(\mathbb{R}^d)$ is proved in Lemma 4.4, while (h) is trivial. The extension of (g) and (h) to all Borel sets can easily be obtained using (4.4).

Finally, by [17, Proposition 6.8] for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the functionals $E'(\cdot,A)$ and $E''(\cdot,A)$ are lower semicontinuous with respect to the convergence of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Since $E = E' = E''$, we conclude that $E \in \mathfrak{E}_{\rm sc}$. □

5. Partial Integral representation

In this Section we present and prove a (partial) representation result for functionals in $\mathfrak{E}_{\rm sc}$. We postpone the full representation to Section 7, where we will work with functionals in $\mathfrak{E}_{\rm sc}$ satisfying an additional property.

We introduce a splitting of E that mimics the structure of functionals in $E^{f,g}$.

Definition 5.1. Let $E\colon L^0(\mathbb{R}^d;\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^d) \to [0,+\infty]$ be a functional satisfying (b) and (c2) of Definition 3.8, and let $A \in \mathcal{A}_c(\mathbb{R}^d)$. For every $u \in L^0(\mathbb{R}^d;\mathbb{R}^k)$, we introduce $E^a(u,\cdot)$, $E^s(u,\cdot)$, $E^c(u,\cdot)$, and $E^j(u,A)$ as the Borel measures on $\mathcal{B}(A)$ defined as:

$$E^a(u,\cdot) \text{ is the absolutely continuous part of } E(u,\cdot) \text{ with respect to } \mathcal{L}^d,$$
$$E^s(u,\cdot) \text{ is the singular part of } E(u,\cdot) \text{ with respect to } \mathcal{L}^d,$$
$$E^j(u,B) = E^s(u, B \cap J_u) \quad \text{for every } B \in \mathcal{B}(A),$$
$$E^c(u,B) = E^s(u, B \setminus J_u) \quad \text{for every } B \in \mathcal{B}(A).$$

Note that by arguing as in [21, Remark 4.2], we see that when $u \in GBV_\star(A;\mathbb{R}^k)$, the measures $E^j(u,\cdot)$ and $E^c(u,\cdot)$ are the absolutely continuous parts with respect to $\mathcal{H}^{d-1} \llcorner J_u$ and $|D^c u|$, respectively. Additionally, for every $A \in \mathcal{A}_c(\mathbb{R})$ it holds

$$E(u,\cdot) = E^a(u,\cdot) + E^c(u,\cdot) + E^j(u,\cdot) \quad \text{on } \mathcal{B}(A).$$

The aim of this section is to show that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ and for every $u \in GBV_\star(A;\mathbb{R}^k)$ we may represent the two measures $E^a(u,\cdot)$ and $E^j(u,\cdot)$ as integrals.

As in [21], the idea is to take advantage of the representation results of [9] for functionals on $BV(A;\mathbb{R}^k)$ with linear growth. These results cannot be directly applied to $E^a(u,\cdot)$ and $E^j(u,\cdot)$, since functionals in $\mathfrak{E}$ are not bounded from below by the total variation measure $|Du|$. This difficulty is treated by restricting our attention first to functions $u \in BV(A;\mathbb{R}^k) \cap L^\infty(A,\mathbb{R}^k)$, and by considering the perturbed functionals $E^\delta(u,A) := E(u,A) + \delta|Du|(A)$. The absolutely continuous part and the jump part of E^δ can then be represented by means of an integral thanks to [9]. We then show that it is possible to pass to the limit as $\delta \to 0^+$ and to recover an integral representation of $E^a(u,\cdot)$ and of $E^j(u,\cdot)$ for every $u \in BV(A;\mathbb{R}^k)$.

The following result then allows us to obtain the result for a general $u \in GBV_\star(A;\mathbb{R}^k)$.

Lemma 5.2. *Let $E \in \mathfrak{E}_{\rm sc}$, let $A \in \mathcal{A}_c(\mathbb{R}^d)$, let $u \in GBV_\star(A;\mathbb{R}^k)$, and let $(R_m)_m$ be a sequence with $R_m > 0$ and $R_m \to +\infty$. Then*

$$\lim_{m\to+\infty}\Big(\frac{1}{m}\sum_{i=1}^{m} E(\psi^i_{R_m}\circ u, B)\Big) = E(u,B), \tag{5.1}$$

$$\lim_{m\to+\infty}\Big(\frac{1}{m}\sum_{i=1}^{m} E^a(\psi^i_{R_m}\circ u, B)\Big) = E^a(u,B), \tag{5.2}$$

$$\lim_{m\to+\infty}\Big(\frac{1}{m}\sum_{i=1}^{m} E^s(\psi^i_{R_m}\circ u, B)\Big) = E^s(u,B), \tag{5.3}$$

$$\lim_{m\to+\infty}\Big(\frac{1}{m}\sum_{i=1}^{m} E^j(\psi^i_{R_m}\circ u, B)\Big) = E^j(u,B), \tag{5.4}$$

$$\lim_{m\to+\infty}\Big(\frac{1}{m}\sum_{i=1}^{m} E^c(\psi^i_{R_m}\circ u, B)\Big) = E^c(u,B), \tag{5.5}$$

for every $B \in \mathcal{B}(A)$.

Proof. For every $m \in \mathbb{N}$ we set

$$\mu_m(B) := \frac{1}{m}\sum_{i=1}^{m} E(\psi^i_{R_m}\circ u, B) \quad \text{for every } B \in \mathcal{B}(A)$$

and note that this defines a finite Radon measure, being $u \in GBV_\star(A;\mathbb{R}^k)$.

Let $U \in \mathcal{A}(A)$. Thanks to property (g) of Definition 3.8, we have that

$$\mu_m(U) \le E(u,U) + c_4\mathcal{L}^d(U^{R_m}_{u,0}) + \frac{C}{m}\left(E(u,U) + \mathcal{L}^d(U)\right).$$

Taking the limsup for $m \to +\infty$ in the previous inequality, we get

$$\limsup_{m\to+\infty}\mu_m(U) \le E(u,U).$$

For every $m \in \mathbb{N}$, there exists $i(m) \in \{1,\dots,m\}$ such that

$$E(\psi^{i(m)}_{R_m}\circ u, U) \le \mu_m(U).$$

Since $\psi^{i(m)}_{R_m}\circ u \to u$ in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ when $m \to +\infty$ and $E(\cdot,U)$ is lower semicontinuous with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$, we have

$$E(u,U) \le \liminf_{m\to+\infty} E(\psi^{i(m)}_{R_m}\circ u, U) \le \limsup_{m\to+\infty}\mu_m(U) \le E(u,U),$$

so that (5.1) holds for every $U \in \mathcal{A}(A)$. From [21, Lemma 4.4] we then deduce that (5.1) holds for every $B \in \mathcal{B}(A)$.

To show that (5.2) holds true, consider a set $N \in \mathcal{B}(A)$, with $\mathcal{L}^d(N) = 0$, such that for every $B \in \mathcal{B}(A)$ we have $E^a(u,B) = E(u, B\setminus N)$ and $\mu^a_m(B) = \mu_m(B\setminus N)$ for every $m \in \mathbb{N}$, where μ^a is the absolutely continuous part of μ with respect to $\mathcal{L}^d$. By (5.1) we have that

$$\lim_{m\to+\infty}\mu^a_m(B) = \lim_{m\to+\infty}\mu_m(B\setminus N) = E(u,B\setminus N) = E^a(u,B),$$

which proves (5.2). Taking the difference of (5.1) and of (5.2) we obtain (5.3).

Finally, equalities (5.4) and (5.5) can be obtained arguing as in [21, Proposition 4.3], replacing Theorem 2.2(d) with our Proposition 2.4(d). □

The following result shows that the singular part E^s satisfies a simplified version of property (g) of Definition 3.8.

Lemma 5.3. *Let $E \in \mathfrak{E}$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, $u \in GBV_\star(A;\mathbb{R}^k)$, $R > 0$, $m \in \mathbb{N}$, $B \in \mathcal{B}(A)$, and $w \in W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$. Then*

$$\frac{1}{m}\sum_{i=1}^m E^s(w+\psi_R^i\circ(u-w),B) \le \Big(1+\frac{C}{m}\Big)E^s(u,B),$$

where $C = \max\{9c_3k/c_1, 2c_3k^{1/2}, c_4\}$.

Proof. Let $N \in \mathcal{B}(\mathbb{R}^d)$ be a Borel set, with $\mathcal{L}^d(N) = 0$, such that $E^s(u,B) = E(u, B\cap N)$ and $E^s(w+\psi_R^i\circ(u-w),B) = E(w+\psi_R^i\circ(u-w), B\cap N)$ for every $i \in \{1,...m\}$, and for every $B \in \mathcal{B}(A)$. Since (3.5) holds for every Borel set $B \in \mathcal{B}(A)$, we get

$$\begin{aligned}
&\frac{1}{m}\sum_{i=1}^m E^s(w+\psi_R^i\circ(u-w),B) = \frac{1}{m}\sum_{i=1}^m E(w+\psi_R^i\circ(u-w), B\cap N)\\
&\quad\le E(u,B\cap N) + \int_{B^R_{u,w}\cap N}(c_3k^{1/2}|\nabla w| + c_4)\,dx + \frac{C}{m}\left(E(u,B\cap N) + \int_{B\cap N}(|\nabla w|+1)\,dx\right)\\
&\quad= E(u,B\cap N) + \frac{C}{m}E(u,B\cap N) = \Big(1+\frac{C}{m}\Big)E^s(u,B),
\end{aligned}$$

which proves the claim. □

We now introduce the perturbed functionals E_δ which will play a fundamental role in the proof of our representation result for functionals in $\mathfrak{E}$.

Definition 5.4. Let $E \in \mathfrak{E}$ and let $A \in \mathcal{A}_c(\mathbb{R}^d)$ be given. For every $\delta > 0$ the functional $E_\delta\colon BV(A;\mathbb{R}^k)\times\mathcal{B}(A) \to [0,+\infty)$ is defined for every $u \in BV(A;\mathbb{R}^k)$ and $B \in \mathcal{B}(A)$ as

$$E_\delta(u,B) := E(u,B) + \delta|Du|(B). \tag{5.6}$$

Given $u \in BV(A;\mathbb{R}^k)$, the measures $E^a_\delta(u,\cdot)$, $E^s_\delta(u,\cdot)$, $E^j_\delta(u,\cdot)$, and $E^c_\delta(u,\cdot)$ are defined as in Definition 5.1 with E replaced by E_δ.

Remark 5.5. Let $E \in \mathfrak{E}$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, and $\delta > 0$. Thanks to (c1′) and (c2′) of Remark 3.10 we have that for every $u \in BV(A;\mathbb{R}^k)$ it holds

$$\delta|Du|(A) - c_2\mathcal{L}^d(A) \le E_\delta(u,A) \le (c_3+\delta)|Du|(A) + c_4\mathcal{L}^d(A).$$

Definition 5.6. Let $A \in \mathcal{A}_c(\mathbb{R}^d)$ and let $E\colon BV(A;\mathbb{R}^k)\times\mathcal{B}(A)\to[0,+\infty]$. For every $U \in \mathcal{A}(A)$ with Lipschitz boundary and $w \in BV(U;\mathbb{R}^k)$, we set

$$m^E(w,U) := \inf\{E(u,U) :\ u \in BV(U;\mathbb{R}^k),\ \mathrm{tr}_U u = \mathrm{tr}_U w\}, \tag{5.7}$$

where $\mathrm{tr}_U\colon BV(U;\mathbb{R}^k) \to L^1_{\mathcal{H}^{d-1}}(\partial U;\mathbb{R}^k)$ is the trace operator. Given $t > 0$, we set

$$m^E_t(w,U) := \inf\{E(u,U) :\ u \in BV(U;\mathbb{R}^k),\ \mathrm{tr}_U u = \mathrm{tr}_U w,\ \|u-w\|_{L^\infty(U;\mathbb{R}^k)} \le t\}. \tag{5.8}$$

We now introduce some functions which will play a crucial role in the integral representation of the bulk part E^a and the surface part E^j of a functional $E \in \mathfrak{E}_{\rm sc}$. We recall that the cubes $Q(x,\rho)$ and $Q_\nu(x,\rho)$ are defined in (f) of Section 2, while the functions ℓ_ξ and $u_{x,\zeta,\nu}$ are defined in (h) of Section 2.

Definition 5.7. Let $E \in \mathfrak{E}$ and $\delta > 0$. We define the functions $f, f_\delta \colon \mathbb{R}^d \times \mathbb{R}^{k\times d} \to [0,+\infty)$ and $g, g_\delta \colon \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{S}^{d-1} \to [0,+\infty)$ as

$$f(x,\xi) := \limsup_{\rho\to 0^+} \frac{m^E(\ell_\xi, Q(x,\rho))}{\rho^d} \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^{k\times d}, \tag{5.9}$$

$$f_\delta(x,\xi) := \limsup_{\rho\to 0^+} \frac{m^{E_\delta}(\ell_\xi, Q(x,\rho))}{\rho^d} \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^{k\times d}, \tag{5.10}$$

$$g(x,\zeta,\nu) := \limsup_{\rho\to 0^+} \frac{m^E(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} \quad \text{for every } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^k, \text{ and } \nu \in \mathbb{S}^{d-1}, \tag{5.11}$$

$$g_\delta(x,\zeta,\nu) := \limsup_{\rho\to 0^+} \frac{m^{E_\delta}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} \quad \text{for every } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^k, \text{ and } \in \mathbb{S}^{d-1}. \tag{5.12}$$

Remark 5.8. It is immediate to check that the functions $\delta \mapsto f_\delta(x,\xi)$ and $\delta \mapsto g_\delta(x,\zeta,\nu)$ are non-decreasing and that

$$f(x,\xi) \le f_\delta(x,\xi) \quad \text{and} \quad g(x,\zeta,\nu) \le g_\delta(x,\zeta,\nu)$$

for every $\delta > 0$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

When $E_{\rm sc}$, we shall see that the functions f_δ and g_δ will be used in the integral representation of E^a_δ and E^j_δ thanks to the results of [9]. This will lead to an integral representation of E^a and E^j by means of the functions (see the proof of Theorem 5.16)

$$\hat f(x,\xi) := \inf_{\delta>0} f_\delta(x,\xi) \quad \text{for every } x \in \mathbb{R}^d,\ \xi \in \mathbb{R}^{k\times d}, \tag{5.13}$$

$$\hat g(x,\zeta,\nu) := \inf_{\delta>0} g_\delta(x,\zeta,\nu) \quad \text{for every } x \in \mathbb{R}^d,\ \zeta \in \mathbb{R}^k,\ \nu \in \mathbb{S}^{d-1}.$$

For the applications to homogenisation it is important to prove that $\hat f = f$ and $\hat g = g$, so that by (5.9) and (5.11) the integrands used in the bulk and surface part can be obtained by solving some auxiliary minum problems on small cubes.

The proof of the equality $\hat f = f$ is not direct and requires a lot of technical tools, one of which being the following truncation lemma. For future use we prove the result also for the rectangles $Q^\lambda_\nu(x,\rho)$, defined in (f) of Section 2. Given $\xi \in \mathbb{R}^{k\times d}$ and $m \in \mathbb{N}$, we set

$$c_{\xi,m} := (\sigma^m + 1) d^{1/2} \Big(|\xi| + \frac{c_2 + c_4 + 1}{c_1} \Big) \tag{5.14}$$

Lemma 5.9. *Let $E \in \mathfrak{E}$. Assume that there exists a function $\hat f \in \mathcal{F}$ such that*

$$E^a(u,A) = \int_A \hat f(x, \nabla u)\, dx, \tag{5.15}$$

for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u \in BV(A;\mathbb{R}^k)$. Then there exists a set $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, satisfying the following property: for every $x \in \mathbb{R}^d \setminus N$, $m \in \mathbb{N}$, $\xi \in \mathbb{R}^{k\times d}$, $\nu \in \mathbb{S}^{d-1}$, and $\lambda \ge 1$ there exists $\rho^{\lambda,\nu}_{m,\xi}(x) > 0$ such that for every $\rho \in (0, \rho^{\lambda,\nu}_{m,\xi}(x))$ and $u \in BV(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)$, with $tr_{Q^\lambda_\nu(x,\rho)} u = tr_{Q^\lambda_\nu(x,\rho)} \ell_\xi$, there exists $v \in BV(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)$, with $tr_{Q^\lambda_\nu(x,\rho)} v = tr_{Q^\lambda_\nu(x,\rho)} \ell_\xi$ and $\|v - \ell_\xi\|_{L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)} \le c_{\xi,m}\lambda\rho$, satisfying the inequality

$$E(v, Q^\lambda_\nu(x,\rho)) \le \Big(1 + \frac{C}{m}\Big) E(u, Q^\lambda_\nu(x,\rho)) + \frac{C}{m} \lambda^{d-1} \rho^d, \tag{5.16}$$

where C is a constant depending only on the structural constants c_1, c_2, c_3, c_4, c_5, and k. Moreover, if f is continuous on $\mathbb{R}^d \times \mathbb{R}^{k\times d}$, then $N = \varnothing$. Finally, if $\hat f$ is independent of x, then $\rho^{\lambda,\nu}_{m,\xi}(x) = +\infty$.

Proof. Let us fix $m \in \mathbb{N}$. Arguing as in [21, Lemma 4.16], one can construct a Borel function $\omega_m : \mathbb{R}^d \to \mathbb{R}^{k\times d}$ such that

$$f(x, \omega_m(x)) \le f(x,\xi) + \frac{1}{m} \tag{5.17}$$

for every $x \in \mathbb{R}^d$ and every $\xi \in \mathbb{R}^{k\times d}$. Note that (f2′), (f3), and (5.17) imply that for every $x \in \mathbb{R}^d$ one has

$$|\omega_m(x)| \le \frac{c_2 + c_4 + 1}{c_1}. \tag{5.18}$$

Since $\omega_m \in L^\infty(\mathbb{R}^d;\mathbb{R}^{k\times d})$, by the Lebesgue Differentiation Theorem there exists $N_m \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N_m) = 0$, such that for every $x \in \mathbb{R}^d \setminus N_m$, $\nu \in \mathbb{S}^{d-1}$, and $\lambda \ge 1$ there exists $\rho^{\lambda,\nu}_{m,\xi}(x) > 0$ such that for every $\rho \in (0, \rho^{\lambda,\nu}_{m,\xi}(x))$ we have

$$\int_{Q^\lambda_\nu(x,\rho)} |\omega_m(x) - \omega_m(y)|\, dy \le \frac{1}{m}\rho^d\lambda^{d-1}.$$

We set $N := \bigcup_{m\in\mathbb{N}} N_m$ and note that $\mathcal{L}^d(N) = 0$. Using (f5) we get that for every $x \in \mathbb{R}^d \setminus N$, $m \in \mathbb{N}$, $\xi \in \mathbb{R}^{k\times d}$, $\nu \in \mathbb{S}^{d-1}$ and $\rho \in (0, \rho^{\lambda,\nu}_{m,\xi}(x))$ we have

$$\int_{Q^\lambda_\nu(x,\rho)} |\hat f(y,\omega_m(x)) - \hat f(y,\omega_m(y))|\, dy \le \frac{c_5}{m}\rho^d\lambda^{d-1}. \tag{5.19}$$

For every $y \in Q^\lambda_\nu(x,\rho)$ and $i \in \{1,...,m\}$ we set

$$\begin{aligned} w(y) &:= \omega_m(x)(y-x) + \ell_\xi(x),\\ R &:= d^{1/2}\Big(|\xi| + \frac{c_2 + c_4 + 1}{c_1}\Big)\lambda\rho,\\ v^i(y) &:= w(y) + \psi^i_R(u(y) - w(y)). \end{aligned} \tag{5.20}$$

Note that with this choice of R we have $|\mathrm{tr}_{Q^\lambda_\nu(x,\rho)}(u-w)| \le R$ $\mathcal{H}^{d-1}$-a.e. on $\partial Q^\lambda_\nu(x,\rho)$. Recalling (3.4), we obtain $\mathrm{tr}_{Q^\lambda_\nu(x,\rho)} v^i = \mathrm{tr}_{Q^\lambda_\nu(x,\rho)}\ell_\xi$ and

$$\|v^i - \ell_\xi\|_{L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)} \le (\sigma^m + 1)R = c_{\xi,m}\lambda\rho.$$

We claim that there exists $i \in \{1,...,m\}$ such that (5.16) holds with $v = v^i$. To this aim, for every $i \in \{1,...,m\}$ we consider the following partition of $Q^\lambda_\nu(x,\rho)$:

$$\begin{aligned} Q^i_{\rm in} &:= \{y \in Q^\lambda_\nu(x,\rho)\colon |u(y) - w(y)| \le \sigma^{i-1}R\},\\ Q^i &:= \{y \in Q^\lambda_\nu(x,\rho)\colon \sigma^{i-1}R < |u(y) - w(y)| < \sigma^i R\},\\ Q^i_{\rm out} &:= \{y \in Q^\lambda_\nu(x,\rho)\colon |u(y) - w(y)| \ge \sigma^i R\}. \end{aligned}$$

Using (b) of Definition 3.8 we may write

$$E(v^i, Q^\lambda_\nu(x,\rho)) = E(v^i, Q^i_{\rm in}) + E(v^i, Q^i) + E(v^i, Q^i_{\rm out}).$$

Recalling (3.4) and (5.20), from (5.15) we obtain

$$E^a(v^i, Q^\lambda_\nu(x,\rho)) = \int_{Q^i_{\rm in}} \hat f(y,\nabla u(y))\, dy + \int_{Q^i} \hat f(y,\nabla v^i(y))\, dy + \int_{Q^i_{\rm out}} \hat f(y,\omega_m(x))\, dy.$$

Taking advantage of (5.18)-(5.20), by (f2′) and (f3) we get

$$\begin{aligned} \int_{Q^i} \hat f(y,\nabla v^i(y))\, dy &\le c_3k^{1/2}\int_{Q^i} |\omega_m(x) + \nabla\psi^i_R(u(y) - w(y))(\nabla u(y) - \omega_m(x))|\, dy + c_4\mathcal{L}^d(Q^i)\\ &\le c_3k^{1/2}\int_{Q^i} |\nabla u(y)|\, dy + (2c_3k^{1/2}|\omega_m(x)| + c_4)\mathcal{L}^d(Q^i)\\ &\le \frac{c_3k^{1/2}}{c_1}\int_{Q^i} \hat f(y,\nabla u(y))\, dy + \Big(2c_3k^{1/2}\frac{c_2 + c_4 + 1}{c_1} + c_2 + c_4\Big)\mathcal{L}^d(Q^i). \end{aligned}$$

By (5.17) and (5.19) we have

$$\int_{Q^i_{\rm out}} \hat f(y,\omega_m(x))\, dy \le \int_{Q^i_{\rm out}} \hat f(y,\omega(y))\, dy + \frac{c_5}{m}\lambda^{d-1}\rho^d \le \int_{Q^i_{\rm out}} \hat f(y,\nabla u(y))\, dy + \frac{c_5 + 1}{m}\lambda^{d-1}\rho^d.$$

From the previous inequalities, we get

$$E^a(v^i,Q^\lambda_\nu(x,\rho)) \le \int_{Q^i_{\rm in}} \hat f(y,\nabla u(y))\,dy + \frac{c_3k^{1/2}}{c_1}\int_{Q^i} \hat f(y,\nabla u(y))\,dy$$
$$+C_1\mathcal{L}^d(Q^i) + \int_{Q^i_{\rm out}} \hat f(y,\nabla u(y))\,dy + \frac{c_5+1}{m}\lambda^{d-1}\rho^d,$$

for some constant C_1 depending only on c_1,c_2,c_3,c_4,c_5, and k. Summing over $i\in\{1,...,m\}$ and dividing by m, we obtain

$$\frac1m\sum_{i=1}^m E^a(v^i,Q^\lambda_\nu(x,\rho)) \le \Big(1+\frac{C}{m}\Big)E^a(u,Q^\lambda_\nu(x,\rho)) + \frac{C}{m}\lambda^{d-1}\rho^d, \tag{5.21}$$

for some constant C depending only on c_1,c_2,c_3,c_4,c_5, and k, which we may assume to be larger than the constant of Lemma 5.3. Finally, using that lemma, we get

$$\frac1m\sum_{i=1}^m E^s(v^i,Q^\lambda_\nu(x,\rho)) \le \Big(1+\frac{C}{m}\Big)E^s(u,Q^\lambda_\nu(x,\rho)), \tag{5.22}$$

so that combining (5.21) with (5.22) we obtain that there exists $i\in\{1,...,m\}$ such that (5.16) holds with $v=v^i$.

Suppose now that $\hat f$ is continuous in $\mathbb{R}^d\times\mathbb{R}^{k\times d}$. Thanks to (f2′) and to (f4), we have that for every $x\in\mathbb{R}^d$ there exists $\omega(x)$ which minimizes the continuous function $\xi\mapsto\hat f(x,\xi)$.

Since $\hat f\in\mathcal{F}$, we also have that

$$|\omega(x)| \le \frac{c_2+c_4}{c_1}.$$

This inequality, together with the uniform continuity of $\hat f$ on compact sets of $\mathbb{R}^d\times\mathbb{R}^{k\times d}$, implies that there exists a $\rho^{\lambda,\nu}_m(x)>0$ such that for every $\rho\in(0,\rho^{\lambda,\nu}_m(x))$ one has

$$|\hat f(y,\xi)-\hat f(x,\xi)| \le \frac1m \tag{5.23}$$

for every $y\in Q^\lambda_\nu(x,\rho)$ and $|\xi|\le(c_2+c_4)/c_1$. In particular, we have

$$|\hat f(y,\omega(x))-\hat f(x,\omega(x))| \le \frac1m. \tag{5.24}$$

Exploiting these inequalities and the minimality of $\omega(x)$, we obtain that

$$\hat f(y,\omega(x)) \le \hat f(y,\xi)+\frac2m \tag{5.25}$$

for every $y\in Q^\lambda_\nu(x,\rho)$ and $\xi\in\mathbb{R}^{k\times d}$. Indeed, if $|\xi|\le(c_2+c_4)/c_1$, we apply (5.23) twice to get

$$\hat f(y,\omega(x)) \le \hat f(x,\omega(x))+\frac1m \le \hat f(x,\xi)+\frac1m \le \hat f(y,\xi)+\frac2m.$$

If $|\xi|>(c_4+c_2)/c_1$, recalling the minimality of $\omega(x)$, by (f3) and (f2′) we have $\hat f(x,\omega(x))\le\hat f(x,0)\le c_4\le\hat f(y,\xi)$, which, togheter with (5.24), implies (5.25). As in the previous part of the proof, for every $y\in\mathbb{R}^d$ we denote $w(y):=\omega(x)(y-x)+\ell_\xi(x)$. We can replace (5.19) by (5.25) in the argument that we used in the case where $\hat f$ was not assumed to be continuous on $\mathbb{R}^d\times\mathbb{R}^{k\times d}$ and this leads to the existence of $i\in\{1,...,m\}$ such that

$$E(w+\psi^i_R\circ(u-w),Q^\lambda_\nu(x,\rho)) \le \Big(1+\frac{C}{m}\Big)E(u,Q^\lambda_\nu(x,\rho)) + \frac{C}{m}\lambda^{d-1}\rho^d.$$

In the case where $\hat f$ does not depend on x the same is true for ω and (5.25) holds for every $\rho>0$, since it is a direct consequence of the minimality. □

The next result follows immediately from Lemma 5.9. Given $\xi \in \mathbb{R}^{k\times d}$, we set

$$C_\xi := 2c_3k^{1/2}C|\xi| + 2C(c_4+1), \tag{5.26}$$

where C is the constant of Lemma 5.9. We also recall that the constant $c_{\xi,m}$ is given by (5.14).

Corollary 5.10. *Let $E \in \mathfrak{E}$ and assume that there exists a function $\hat f \in \mathcal{F}$ satisfying (5.15). Then there exists a set $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, such that for every $x \in \mathbb{R}^d \setminus N$, $m \in \mathbb{N}$, $\xi \in \mathbb{R}^{k\times d}$, $\nu \in \mathbb{S}^{d-1}$, and $\lambda \ge 1$ there exists $\rho^{\nu,\lambda}_{m,\xi}(x) > 0$ such that for every $\rho \in (0, \rho^{\nu,\lambda}_{m,\xi}(x))$ there exists $u \in BV(Q^\lambda_\nu(x,\rho);\mathbb{R}^k) \cap L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)$, with $tr_{Q^\lambda_\nu(x,\rho)}u = tr_{Q^\lambda_\nu(x,\rho)}\ell_\xi$ and $\|u - \ell_\xi\|_{L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)} \le c_{\xi,m}\lambda\rho$, such that*

$$E(u, Q^\lambda_\nu(x,\rho)) \le m^E(\ell_\xi, Q^\lambda_\nu(x,\rho)) + \frac{C_\xi\lambda^{d-1}\rho^d}{m}. \tag{5.27}$$

In particular for $t = c_{\xi,m}\lambda$, we have

$$m^E_{t\rho}(\ell_\xi, Q^\lambda_\nu(x,\rho)) \le m^E(\ell_\xi, Q^\lambda_\nu(x,\rho)) + \frac{C_\xi\lambda^{d-1}\rho^d}{m}. \tag{5.28}$$

Moreover, if $\hat f$ is continuous in $\mathbb{R}^d \times \mathbb{R}^{k\times d}$ then $N = \varnothing$. Finally, if $\hat f$ is independent on x then $\rho^{\eta,\lambda}_{\xi,\nu}(x) = +\infty$.

Proof. Let $N \in \mathcal{B}(\mathbb{R}^d)$ and $\rho^{\nu,\lambda}_{m,\xi}(x) > 0$ be as in Lemma 5.9. Consider a $v \in BV(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)$, with $\mathrm{tr}_{Q^\lambda_\nu(x,\rho)}v = \mathrm{tr}_{Q^\lambda_\nu(x,\rho)}\ell_\xi$, such that

$$E(v, Q^\lambda_\nu(x,\rho)) \le m^E(\ell_\xi, Q^\lambda_\nu(x,\rho)) + \frac{\lambda^{d-1}\rho^d}{m}.$$

Since the function ℓ_ξ is a competitor for the minimisation problem in the right-hand side of the previous inequality, we also get

$$E(v, Q^\lambda_\nu(x,\rho)) \le (c_3k^{1/2}|\xi| + c_4 + 1)\lambda^{d-1}\rho^d.$$

Thus, we may apply Lemma 5.9 to obtain a function $u \in BV(Q^\lambda_\nu(x,\rho);\mathbb{R}^k) \cap L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)$, with $\mathrm{tr}_{Q^\lambda_\nu(x,\rho)}u = \mathrm{tr}_{Q^\lambda_\nu(x,\rho)}\ell_\xi$ and $\|u - \ell_\xi\|_{L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)} \le c_{\xi,m}\lambda\rho$, such that (5.27) holds, concluding the proof. □

The following proposition shows that under the hypotheses of Lemma 5.9 the functions f_δ and f^∞_δ converge $\mathcal{L}^d$-a.e. as $\delta \to 0^+$ to f and f^∞, respectively.

Proposition 5.11. *Let $E \in \mathfrak{E}$. Assume that there exists $\hat f \in \mathcal{F}$ such that (5.15) holds. Then*

$$f(x,\xi) = \lim_{\delta\to0^+} f_\delta(x,\xi) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \text{ and for every } \xi \in \mathbb{R}^{k\times d}, \tag{5.29}$$

$$f^\infty(x,\xi) = \lim_{\delta\to0^+} f^\infty_\delta(x,\xi) \quad \text{for } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \text{ and for every } \xi \in \mathbb{R}^{k\times d}. \tag{5.30}$$

Proof. From Remark 5.8, we deduce that the limits in the right-hand side of (5.29) and of (5.30) exist and that

$$f(x,\xi) \le \inf_{\delta>0} f_\delta(x,\xi),$$
$$f^\infty(x,\xi) \le \inf_{\delta>0} f^\infty_\delta(x,\xi).$$

We are left with proving that the converse inequality also holds. Let $N \in \mathcal{B}(\mathbb{R}^d)$ the $\mathcal{L}^d$-negligible set of Corollary 5.10, let $x \in \mathbb{R}^d \setminus N$, $m \in \mathbb{N}$, $\xi \in \mathbb{R}^{k\times d}$, and $t \ge 1$. By (5.9), for $\rho > 0$ small enough we have

$$\frac{m^E(t\ell_\xi, Q(x,\rho))}{t\rho^d} \le \frac{f(x,t\xi)}{t} + \frac{1}{tm}. \tag{5.31}$$

We can now apply Corollary 5.10, with ξ replaced by $t\xi$, to obtain a function $u \in BV(Q(x,\rho);\mathbb{R}^k)$, with $\mathrm{tr}_{Q(x,\rho)}u = \mathrm{tr}_{Q(x,\rho)}t\ell_\xi$ and $\|u - t\ell_\xi\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \le c_{t\xi,m}\rho$, such that (5.27) holds with ξ

replaced by $t\xi$. By taking $\rho > 0$ small enough, we may also suppose that $\|u - t\ell_\xi\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \leq 1/2$. Inequality (5.31) then yields

$$\frac{E(u, Q(x,\rho))}{t\rho^d} \leq \frac{f(x,t\xi)}{t} + \frac{C_{t\xi}+1}{tm} \leq \frac{f(x,t\xi)}{t} + \frac{C_\xi+1}{m},$$

where $C_\xi > 0$ is the constant given by (5.26). Note that in the last inequality we have used the estimate $C_{t\xi} \leq tC_\xi$ for every $t \geq 1$.

We now compare $E_\delta(u, Q(x,\rho))$ and $E(u, Q(x,\rho))$. Since $\|u - t\ell_\xi\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \leq 1/2$, we have $|[u]| \leq 1$ $\mathcal{H}^{d-1}$-a.e. on J_u. Hence, by (c1′) we get

$$\begin{aligned} E_\delta(u, Q(x,\rho)) &= E(u, Q(x,\rho)) + \delta|Du|(Q(x,\rho)) \\ &\leq E(u, Q(x,\rho)) + \delta\int_{Q(x,\rho)} |\nabla u|\,dx + \delta|D^c u|(Q(x,\rho)) + \delta\int_{J_u} |[u]| \wedge 1\,d\mathcal{H}^{d-1} \\ &\leq \Big(1 + \frac{\delta}{c_1}\Big) E(u, Q(x,\rho)) + \frac{c_2\delta}{c_1}\rho^d. \end{aligned}$$

Letting δ be so small that $\delta/c_1 \leq 1/m$ and $\delta c_2/c_1 \leq 1/m$, we then obtain

$$\frac{E_\delta(u, Q(x,\rho))}{t\rho^d} \leq \Big(1 + \frac{1}{m}\Big)\frac{E(u, Q(x,\rho))}{t\rho^d} + \frac{1}{m} \leq \Big(1 + \frac{1}{m}\Big)\Big(\frac{f(x,t\xi)}{t} + \frac{C_\xi+1}{m}\Big)$$

and, recalling that $\mathrm{tr}_{Q(x,\rho)} u = \mathrm{tr}_{Q(x,\rho)} t\ell_\xi$, we get

$$\frac{m^{E_\delta}(\ell_{t\xi}, Q(x,\rho))}{t\rho^d} \leq \Big(1 + \frac{1}{m}\Big)\Big(\frac{f(x,t\xi)}{t} + \frac{C_\xi+1}{m}\Big).$$

Evaluating this last inequality at $t = 1$ and taking the lim sup as $\rho \to 0^+$, we deduce

$$f_\delta(x,\xi) \leq \Big(1 + \frac{1}{m}\Big)\Big(f(x,\xi) + \frac{C_\xi+1}{m}\Big),$$

while taking the lim sup as $\rho \to 0^+$ first and letting then $t \to +\infty$, we obtain

$$f_\delta^\infty(x,\xi) \leq \Big(1 + \frac{1}{m}\Big)\Big(f^\infty(x,\xi) + \frac{C_\xi+1}{m}\Big).$$

Finally, letting $\delta \to 0^+$ first and then taking the limit for $m \to +\infty$, we conclude the proof. □

We now pass to the study of the minimisation problems used to define the integrands for the surface terms.

Lemma 5.12. *Let $E \in \mathfrak{E}$, $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, $m \in \mathbb{N}$, $\zeta \in \mathbb{R}^k$, and $\rho > 0$. Then there exists $u \in BV(Q_\nu(x,\rho);\mathbb{R}^k) \cap L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)$, with $tr_{Q_\nu(x,\rho)} u = tr_{Q_\nu(x,\rho)} u_{x,\zeta,\nu}$ and $\|u\|_{L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)} \leq \sigma^m$, such that*

$$E(u, Q_\nu(x,\rho)) \leq m^E(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \frac{K(|\zeta| \wedge 1)}{m}\rho^{d-1} + K\rho^d, \tag{5.32}$$

where $K := \max\{c_3 kC, (C + c_4 + 1)\}$ and C is the constant of (g) of Definition 3.8.

Proof. By (5.7) there exists $v \in BV(Q_\nu(x,\rho);\mathbb{R}^k)$, with $\mathrm{tr}_{Q_\nu(x,\rho)} u = \mathrm{tr}_{Q_\nu(x,\rho)} u_{x,\zeta,\nu}$, such that

$$E(v, Q_\nu(x,\rho)) \leq m^E(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) + \rho^d \leq c_3 k(|\zeta| \wedge 1)\rho^{d-1} + \rho^d, \tag{5.33}$$

where the second inequality follows from (c2′) and the fact that $u_{x,\zeta,\nu}$ is a competitor for the minimisation problem.

Let us fix $R \geq |\zeta|$. Note that with this choice of R by (3.4) we have $\mathrm{tr}_{Q_\nu(x,\rho)} u = \mathrm{tr}_{Q_\nu(x,\rho)} u_{x,\zeta,\nu}$ and $\|u\|_{L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)} \leq \sigma^m$. By property (g) of Definition 3.8 and (5.33) there exists $i \in \{1, ..., m\}$ such that $u := \psi_R^i \circ v$ satisfies

$$\begin{aligned} E(u, Q_\nu(x,\rho)) &\leq E(v, Q_\nu(x,\rho))\,dx + c_4\rho^d + \frac{C}{m}\left(E(v, Q_\nu(x,\rho)) + \rho^d\right) \\ &\leq m^E(u_{x,\zeta,\nu}, Q_\nu(x,\rho)) + (C + c_4 + 1)\rho^d + \frac{c_3 kC(|\zeta| \wedge 1)}{m}\rho^{d-1}, \end{aligned}$$

concluding the proof. □

We are now ready to prove that g_δ converges to g.

Proposition 5.13. *Let $E \in \mathfrak{E}$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$. Then*

$$g(x,\zeta,\nu) = \lim_{\delta\to 0^+} g_\delta(x,\zeta,\nu). \tag{5.34}$$

Proof. Thanks to Remark 5.8, the limit in the right-hand side of (5.34) exists and

$$g(x,\zeta,\nu) \le \lim_{\delta\to 0^+} g_\delta(x,\zeta,\nu).$$

We now prove that the converse inequality holds true as well. Let us fix $m \in \mathbb{N}$. By (5.11) for $\rho > 0$ small enough we have that

$$\frac{m^E(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} \le g(x,\zeta,\nu) + \frac{1}{m}. \tag{5.35}$$

We can now apply Lemma 5.12 to obtain a function $u \in BV(Q_\nu(x,\rho);\mathbb{R}^k) \cap L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)$, with $\mathrm{tr}_{Q_\nu(x,\rho)} u = \mathrm{tr}_{Q_\nu(x,\rho)} u_{x,\zeta,\nu}$ and $\|u\|_{L^\infty(Q_\nu(x,\rho):\mathbb{R}^k)} \le \sigma^m$, such that

$$E(u, Q_\nu(x,\rho)) \le m^E(u_{x,\zeta,\nu}.Q_\nu(x,\rho)) + \frac{K}{m}\rho^{d-1} + K\rho^d.$$

From (5.35) we then deduce that

$$\frac{E(u, Q_\nu(x,\rho))}{\rho^{d-1}} \le g(x,\zeta,\nu) + \frac{K+1}{m} + K\rho.$$

To conclude, we compare $E_\delta(u, Q_\nu(x,\rho))$ with $E(u, Q_\nu(x,\rho))$. Since $\|u\|_{L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)} \le \sigma^m$, we have $|[u]| \le 2\sigma^m |[u]| \wedge 1$. Hence, we get

$$\begin{aligned} E_\delta(u, Q_\nu(x,\rho)) &= E(u, Q_\nu(x,\rho)) + \delta|Du|(Q_\nu(x,\rho)) \\ &\le E(u, Q_\nu(x,\rho)) + \delta\int_{Q_\nu(x,\rho)} |\nabla u|\,dx + \delta|D^c u|(Q_\nu(x,\rho)) + 2\delta\sigma^m \int_{J_u} |[u]| \wedge 1\, \mathcal{H}^{d-1} \\ &\le \Big(1 + \frac{2\delta\sigma^m}{c_1}\Big) E(u, Q_\nu(x,\rho)) + \frac{2c_2\delta\sigma^m}{c_1}\rho^d. \end{aligned}$$

Letting δ be so small that $2\delta\sigma^m/c_1 \le 1/m$ and $2c_2\delta\sigma^m/c_1 \le 1$, we get

$$\frac{E_\delta(u, Q_\nu(x,\rho))}{\rho^{d-1}} \le \Big(1 + \frac{1}{m}\Big) g(x,\zeta,\nu) + \frac{K+1}{m} + \Big(1+K\Big)\rho.$$

Recalling the definition of m^{E_δ}, from this last estimate we deduce that

$$\frac{m^{E_\delta}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} \le \Big(1 + \frac{1}{m}\Big) g(x,\zeta,\nu) + \frac{K+1}{m} + \Big(1+K\Big)\rho.$$

Taking the limsup as $\rho \to 0^+$, we obtain

$$g_\delta(x,\zeta,\nu) \le \Big(1 + \frac{1}{m}\Big) g(x,\zeta,\nu) + \frac{K+1}{m}.$$

Taking the lim sup for $\delta \to 0^+$ first, and the limit for $m \to +\infty$ then, we conclude the proof. □

As the following proposition shows, the functions f and g defined by (5.9) and (5.11), belong to $\mathcal{F}$ and to $\mathcal{G}$, respectively.

Proposition 5.14. *Let $E \in \mathfrak{E}_{\rm sc}$, f, $\hat f$, and g be the functions defined by (5.9), (5.13), and (5.11), respectively. Then $\hat f, f \in \mathcal{F}$, $g \in \mathcal{G}$.*

Proof. The proof of the inclusions $\hat{f}$, $f \in \mathcal{F}$ and of properties (g1)-(g4) for g, can be obtained by adapting the arguments of [21, Section 5], with minor changes.

We are left with proving that g satisfies (g5). To see this, let $\zeta_1, \zeta_2 \in \mathbb{R}^k \setminus \{0\}$ with $c_6 k|\zeta_1| \le |\zeta_2|$. We set $\lambda := |\zeta_1|/|\zeta_2|$ and note that $\lambda \le 1/(c_6 k)$. Let $R \in SO(k)$ be a rotation that maps $\lambda\zeta_2$ to ζ_1. Let $x \in \mathbb{R}^d$, $\nu \in \mathbb{S}^{d-1}$, and $m \in \mathbb{N}$. By (5.11) there exists $u \in BV(Q_\nu(x,\rho)); \mathbb{R}^k)$, with $\mathrm{tr}_{Q_\nu(x,\rho)} u = \mathrm{tr}_{Q_\nu(x,\rho)} u_{x,\zeta_2,\nu}$, such that

$$E(u, Q_\nu(x,\rho)) \le m^E(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + \rho^d. \tag{5.36}$$

We set $v := \lambda R u$ and note that $\mathrm{tr}_{Q_\nu(x,\rho)} v = \mathrm{tr}_{Q_\nu(x,\rho)} u_{x,\zeta_1,\nu}$. Then, by means of (h) of Definition 3.8, we have

$$E(v, Q_\nu(x,\rho)) \le E(u, Q_\nu(x,\rho)) + (c_4 + c_2)\rho^d,$$

so that, by (5.36), we infer

$$E(v, Q_\nu(x,\rho)) \le m^E(u_{x,\zeta_2,\nu}, Q_\nu(x,\rho)) + (c_4 + c_2 + 1)\rho^d. \tag{5.37}$$

Dividing this last inequality by ρ^{d-1} and taking the lim sup for $\rho \to 0^+$, we obtain

$$\limsup_{\rho\to 0^+} \frac{E(v, Q_\nu(x,\rho))}{\rho^{d-1}} \le g(x, \zeta_2, \nu),$$

which, in light of the fact that $\mathrm{tr}_{Q_\nu(x,\rho)} v = \mathrm{tr}_{Q_\nu(x,\rho)} u_{x,\zeta_1,\nu}$, implies

$$g(x, \zeta_1, \nu) \le g(x, \zeta_2, \nu),$$

which concludes the proof. □

Using the results of [9], we can now establish an integral representation both for the absolutely continuous part and the jump part of the perturbed functionals E_δ. This representation is achieved by means of the functions f_δ and g_δ defined above.

Proposition 5.15. *Let $E \in \mathfrak{E}_{\mathrm{sc}}$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, and $\delta > 0$. Let E^a_δ and E^j_δ be the functionals introduced in Definition 5.4 and let f_δ and g_δ be the functions defined by (5.10) and (5.12), respectively. Then*

$$E^a_\delta(u, B) = \int_B f_\delta(x, \nabla u)\, dx, \tag{5.38}$$

$$E^j_\delta(u, B) = \int_{B \cap J_u} g_\delta(x, [u], \nu)\, \mathcal{H}^{d-1}, \tag{5.39}$$

for every $u \in BV(A; \mathbb{R}^k)$ and for every $B \in \mathcal{B}(A)$.

Proof. The result follows from [9], using the same arguments of [21, Theorem 6.1]. □

With this proposition at hand, we are ready to prove the integral representation of E^a and of E^j.

Theorem 5.16. *Let $E \in \mathfrak{E}_{\mathrm{sc}}$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, and let f and g be the functions defined by (5.9) and (5.11), respectively. Then*

$$E^a(u, B) = \int_B f(x, \nabla u)\, dx, \tag{5.40}$$

$$E^j(u, B) = \int_{J_u \cap B} g(x, [u], \nu)\, \mathcal{H}^{d-1} \tag{5.41}$$

for every $u \in GBV_\star(A; \mathbb{R}^k)$ and for every $B \in \mathcal{B}(A)$.

Proof. Let $\hat{f}$ be the function defined by (5.13). We first show that

$$E^a(u, B) = \int_B \hat{f}(x, \nabla u)\, dx \quad \text{for every } u \in BV(A; \mathbb{R}^k) \text{ and } B \in \mathcal{B}(A). \tag{5.42}$$

To this aim, we begin noting that $E^a_\delta(u,B) = E^a(u,B) + \delta \int_B |\nabla u|$ for every $u \in BV(A;\mathbb{R}^k)$ and for every $B \in \mathcal{B}(A)$, so that

$$E^a(u,B) = \inf_{\delta>0} E^a_\delta(u,B) = \lim_{\delta>0} E^a_\delta(u,B).$$

By definition of f_δ, we have that

$$f_\delta(x,\xi) \le (c_3 k^{1/2} + \delta)|\xi| + c_4,$$

for every $x \in \mathbb{R}^d$ and for every $\xi \in \mathbb{R}^{k\times d}$, so that by invoking the Dominate Convergence Theorem, we obtain (5.42).

We now prove that

$$E^j(u,B) = \int_B g(x,[u],\nu_u)\, d\mathcal{H}^{d-1} \quad \text{for every } u \in BV(A;\mathbb{R}^k) \text{ and } B \in \mathcal{B}(A). \tag{5.43}$$

To show this, note that $E^j_\delta(u,B) = E^j(u,B) + \delta \int_{J_u\cap B} |[u]| d\mathcal{H}^{d-1}$ for every $u \in BV(A;\mathbb{R}^k)$ and $B \in \mathcal{B}(A)$; therefore,

$$E^j(u,B) = \inf_{\delta>0} E^j_\delta(u,B) = \lim_{\delta>0} E^j_\delta(u,B).$$

It is immediate to see that for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$ and $\nu \in \mathbb{S}^{d-1}$ we have

$$g_\delta(x,\zeta,\nu) \le (c_3 k + \delta)|\zeta| \wedge 1.$$

Hence, recalling Proposition 5.13 and using (5.39), by the Dominated Convergence Theorem we obtain (5.41).

Consider now $u \in GBV_\star(A;\mathbb{R}^k)$. Let $R_m > 0$ with $R_m \to +\infty$. . By Lemma 5.2, both (5.2) and (5.4) hold for every $B \in \mathcal{B}(A)$. To conclude, it is enough to show that

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{i=1}^m E^a(\psi^i_{R_m} \circ u, A) = \int_A f(x,\nabla u)\, dx, \tag{5.44}$$

$$\lim_{m\to+\infty} \frac{1}{m} \sum_{i=1}^m E^j(\psi^i_{R_m} \circ u, A) = \int_{J_u\cap A} g(x,[u],\nu_u)\, d\mathcal{H}^{d-1}. \tag{5.45}$$

Since for every $i \in \{1,...m\}$ we have that $\psi^i_{R_m} \circ u \in BV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$, from (5.40) and (5.43) we deduce that

$$E^a(\psi^i_{R_m} \circ u, A) = \int_A f(x,\nabla(\psi^i_{R_m} \circ u))\, dx,$$

$$E^j(\psi^i_{R_m} \circ u, A) = \int_{J_{\psi^i_{R_m}(u)}\cap A} g(x,[\psi^i_{R_m} \circ u],\nu_u)\, \mathcal{H}^{d-1}.$$

By (3.4) we have $|\nabla(\psi^i_{R_m} \circ u)| \le |\nabla u|$ for every $m \in \mathbb{N}$ and $i \in \{1,...,m\}$. Morover, $\nabla u \in L^1(\mathbb{R}^d;\mathbb{R}^{k\times d})$ by Proposition 2.4(a). For every $m \in \mathbb{N}$ there exists $i(m), j(m) \in \{1,...,m\}$ such that

$$\int_A f(x,\nabla(\psi^{i(m)}_{R_m} \circ u))\, dx \le \frac{1}{m} \sum_{i=1}^m E^a(\psi^i_{R_m} \circ u, A) \le \int_A f(x,\nabla(\psi^{j(m)}_{R_m} \circ u))\, dx.$$

We observe that both $\nabla(\psi^{i(m)}_{R_m} \circ u)$ and $\nabla(\psi^{j(m)}_{R_m} \circ u)$ converge to ∇u pointwise $\mathcal{L}^d$-a.e. in A as $m \to +\infty$. Hence, recalling that by Proposition 5.14 the function f belongs to $\mathcal{F}$, the Dominated Convergence Theorem implies (5.44).

As for (5.45), by Proposition 2.4(d), for every $i \in \{1,...,m\}$ we have that $|[\psi^i_{R_m} \circ u]| \wedge 1 \le |[u]| \wedge 1$ and that $[\psi^i_{R_m} \circ u](x) \to [u](x)$ for $\mathcal{H}^{d-1}$-a.e. $x \in J_u$. For every $m \in \mathbb{N}$ we choose $i(m), j(m) \in \{1,...,m\}$ such that

$$\int_{J_u\cap A} g(x,[\psi^{i(m)}_{R_m} \circ u],\nu_u)\, d\mathcal{H}^{d-1} \le \frac{1}{m} \sum_{i=1}^m E^j(\psi^i_{R_m} \circ u, A) \le \int_{J_u\cap A} g(x,[\psi^{j(m)}_{R_m} \circ u],\nu_u)\, d\mathcal{H}^{d-1}.$$

Since by Proposition 5.14 the function g belongs to $\mathcal{G}$, an application of the Dominated Convergence Theorem yields (5.45), concluding the proof. □

6. A smaller collection of integrands

As the scalar case studied in [21] and [20] suggests, to recover a full integral representation for functionals in $\mathfrak{E}_{\rm sc}$, it is convenient to consider a smaller collection of integrands, whose definition is closely related to those studied in [15]. In particular, we will show that the Cantor part E^c can be represented as an integral functional whenever $E \in \mathfrak{E}_{\rm sc}$ is the Γ-limit of a sequence of functionals associated to integrands in this in this smaller class and the volume integrand corresponding to E does not depend on x (see Theorem 7.1).

In the rest of the paper we fix two new constants $c_7 > 0$ and $\alpha \in (0,1)$. Moreover, we fix a continuous non-decreasing function $\vartheta\colon [0,+\infty) \to [0,+\infty)$ such that

$$\vartheta(0) = 0 \quad \text{and} \quad \vartheta(t) \geq \frac{c_1}{c_3} t - 1 \quad \text{for every } t \geq 0. \tag{6.1}$$

The smaller collection of volume integrands is introduced in the following definition.

Definition 6.1. Let $\mathcal{F}^\alpha$ be the collection of functions $f \in \mathcal{F}$ such that

$$\left| \frac{f(x,s\xi)}{s} - \frac{f(x,t\xi)}{t} \right| \leq \frac{c_7}{s} f(x,s\xi)^{1-\alpha} + \frac{c_7}{s} + \frac{c_7}{t} f(x,t\xi)^{1-\alpha} + \frac{c_7}{t} \tag{6.2}$$

for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and for every $s,t > 0$ and $\xi \in \mathbb{R}^{k\times d}$.

Remark 6.2. Arguing as in [20, Remark 4.3], one can show that $f \in \mathcal{F}^\alpha$ if and only if for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and for every $\xi \in \mathbb{R}^{k\times d}$ we have

$$f^\infty(x,\xi) = \lim_{s\to+\infty} \frac{1}{s} f(x,s\xi),$$

and

$$\left| \frac{1}{s} f(x,s\xi) - f^\infty(x,\xi) \right| \leq \frac{c_7}{s} + \frac{c_7}{s} f(x,s\xi)^{1-\alpha}$$

for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$, for every $\xi \in \mathbb{R}^{k\times d}$ and for every $s > 0$. This is closely related to condition (H4) of [9].

We now introduce the smaller collection of surface integrands.

Definition 6.3. Let $\mathcal{G}^\vartheta$ be the collection of functions $g \in \mathcal{G}$ such that

$$\left| \frac{g(x,s\zeta,\nu)}{s} - \frac{g(x,t\zeta,\nu)}{t} \right| \leq \vartheta(s|\zeta|) \frac{g(x,s\zeta,\nu)}{s} + \vartheta(t|\zeta|) \frac{g(x,t,\nu)}{t},$$

for every $s,t > 0$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$.

Remark 6.4. The arguments of [20, Remark 4.5] show that $g \in \mathcal{G}^\vartheta$ if and only if for $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$ and $\nu \in \mathbb{S}^{d-1}$ we have that the limit

$$g^0(x,\zeta,\nu) := \lim_{s\to 0^+} \frac{1}{s} g(x,s\zeta,\nu) \tag{6.3}$$

exists and

$$\left| \frac{1}{s} g(x,s\zeta,\nu) - g^0(x,\zeta,\nu) \right| \leq \vartheta(s|\zeta|) \frac{1}{s} g(x,s\zeta,\nu) \leq c_3 k \vartheta(s|\zeta|)|\zeta|$$

for every $s > 0$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$. By [15, Remark 3.5] this is closely related to condition (g5) of that paper.

Remark 6.5. The class $\mathcal{G}^\vartheta$ is non-empty, since the function $(x,\zeta,\nu) \mapsto \sum_{i=1}^k ((c_1|\zeta_i|) \wedge c_3)$ belongs to $\mathcal{G}^\vartheta$. Arguing as in [20, Remark 4.7], it is possible to show that $\mathcal{G}^\vartheta \neq \emptyset$ if and only if ϑ satisfies (6.1) .

As in Section 3, we introduce a space of functionals $\mathfrak{E}^{\alpha,\vartheta}$ containing $E^{f,g}$ for every $f \in \mathcal{F}^\alpha$ and $g \in \mathcal{G}^\vartheta$.

Definition 6.6. Let $\mathfrak{E}^{\alpha,\vartheta}$ be the space of functionals $E \in \mathfrak{E}$ satisfying the condition

$$\Big|\frac{E(su,A)}{s} - \frac{E(tu,A)}{t}\Big| \le \frac{c_7}{s}\mathcal{L}^d(A)^\alpha E(su,A)^{1-\alpha} + \vartheta(su^A)\frac{E(su,A)}{s} + \frac{c_7}{s}\mathcal{L}^d(A) + \frac{c_7}{t}\mathcal{L}^d(A)^\alpha E(tu,A)^{1-\alpha} + \vartheta(tu^A)\frac{E(tu,A)}{t} + \frac{c_7}{t}\mathcal{L}^d(A), \quad (6.4)$$

for every $t,s>0$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, and $u \in BV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$, where $u^A := \operatorname{osc}_A u := \operatorname{ess\,sup}_{x,y\in A} |u(x)-u(y)|$. We also set $\mathfrak{E}^{\alpha,\vartheta}_{\rm sc} := \mathfrak{E}^{\alpha,\vartheta} \cap \mathfrak{E}_{\rm sc}$.

Proposition 6.7. *Let $f \in \mathcal{F}^\alpha$ and $g \in \mathcal{G}^\vartheta$. Then the functional $E^{f,g}$ of Definition 3.7 belongs to $\mathfrak{E}^{\alpha,\vartheta}$.*

Proof. The result is proved as in [20, Proposition 4.9], replacing Remark 2.9 by our Proposition 3.13. □

To study the Γ-limit of sequences of functionals $(E_n)_n \subset \mathfrak{E}^{\alpha,\vartheta}$, given $A \in \mathcal{A}_c(\mathbb{R}^d)$ and a function $u \in BV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$, it is important to be able to find approximate recovery sequences for u which are bounded in $L^\infty(A;\mathbb{R}^k)$. This is taken care of in the next lemma.

Lemma 6.8. *Let $m \in \mathbb{N}$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, $u \in BV(A;\mathbb{R}^k) \cap L^\infty(A;\mathbb{R}^k)$, $E \in \mathfrak{E}$, and $(E_n)_n \subset \mathfrak{E}$, with $E_n(\cdot,A)$ Γ-converging to E in the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then there exist sequences $(u_n)_n \subset BV(A;\mathbb{R}^k)$ and $(v_n)_n \subset BV(A;\mathbb{R}^k)$ converging to u in $L^1(A;\mathbb{R}^k)$ such that*

$$u_n^A \le 4\sigma^m u^A \quad \text{and} \quad \|v_n\|_{L^\infty(A;\mathbb{R}^k)} \le 2\sigma^m \|u\|_{L^\infty(A;\mathbb{R}^k)} \quad \text{for every } n \in \mathbb{N}, \quad (6.5)$$

$$\limsup_{n\to+\infty} E_n(u_n,A) \vee \limsup_{n\to+\infty} E_n(v_n,A) \le E(u,A) + C\frac{E(u,A)+\mathcal{L}^d(A)}{m}, \quad (6.6)$$

where C is the constant in property (g) of Definition 3.8.

Proof. By property (d) of Definition 3.8, it is not restrictive to assume that $u^A = \|u\|_{L^\infty(A;\mathbb{R}^k)}$. By Γ-convergence there exists a sequence $(w_n) \subset L^0(\mathbb{R}^d;\mathbb{R}^k)$ converging to u in $L^0(\mathbb{R}^d;\mathbb{R}^k)$ such that

$$\lim_{n\to+\infty} E_n(w_n,A) = E(u,A). \quad (6.7)$$

Let us fix $R := 2\|u\|_{L^\infty(A;\mathbb{R}^k)}$. By property (g) of Definition 3.8 for every $n \in \mathbb{N}$ there exists an index $i(n) \in \{1,\dots,m\}$ such that

$$E_n(\psi_R^{i(n)} \circ w_n, A) \le E_n(w_n,A) + C\frac{E(u,A)+\mathcal{L}^d(A)}{m} + c_4\mathcal{L}^d(A^R_{v_n,0}).$$

Since $w_n \to u$ in $L^0(\mathbb{R}^d;\mathbb{R}^k)$, by our choice of R we have $\mathcal{L}^d(A^R_{w_n,0}) \to 0$. Setting $u_n := \psi_R^{i(n)} \circ w_n$, we deduce from (6.7) and the previous inequality that

$$\limsup_{n\to+\infty} E_n(u_n,A) \le E(u,A) + C\frac{E(u,A)+\mathcal{L}^d(A)}{m}.$$

We conclude noting that by (3.4) we have $u_n^A \le 2\|u_n\|_{L^\infty(A;\mathbb{R}^k)} \le 4\sigma^m\|u\|_{L^\infty(A;\mathbb{R}^k)} = 4\sigma^m u^A$ and $u_n \to u$ in $L^1(A;\mathbb{R}^k)$.

The construction of v_n is similar. □

We now want study Γ-limits of sequences of functionals in $\mathfrak{E}^{\alpha,\vartheta}$. To this aim, it is convenient to introduce a family of subspaces of $\mathfrak{E}$, in which (6.4) holds in a weaker form. Given $m \in \mathbb{N}$, we set

$$\vartheta_m(t) := \vartheta(4\sigma^m t) \quad (6.8)$$

for every $t \ge 0$.

Definition 6.9. Given $m \in \mathbb{N}$, we denote by $\mathfrak{E}_m^{\alpha,\vartheta_m}$ the space of functionals $E \in \mathfrak{E}$ satisfying the condition

$$\begin{aligned}\Big|\frac{E(su,A)}{s} - \frac{E(tu,A)}{t}\Big| &\le \frac{c_7}{s}\mathcal{L}^d(A)^\alpha E(su,A)^{1-\alpha} + \vartheta_m(su^A)\frac{E(su,A)}{s} + \frac{c_7}{s}\mathcal{L}^d(A)\\ &+ \frac{c_7}{t}\mathcal{L}^d(A)^\alpha E(tu,A)^{1-\alpha} + \vartheta_m(tu^A)\frac{E(tu,A)}{t} + \frac{c_7}{t}\mathcal{L}^d(A) + R_{m,s}(u,A) + R_{m,t}(u,A) \quad (6.9)\end{aligned}$$

for every $t,s>0$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, $u \in BV(A;\mathbb{R}^k)\cap L^\infty(A,\mathbb{R}^k)$, where ϑ_m is the function defined by (6.8) and for every $t>0$

$$R_{m,t}(u,A) := C\frac{1+\vartheta_m(tu^A)}{t}\frac{E(tu,A)+\mathcal{L}^d(A)}{m} + C^\alpha\frac{c_7}{t}\Big(\frac{E(tu,A)+\mathcal{L}^d(A)}{m}\Big)^{1-\alpha}\mathcal{L}^d(A)^\alpha, \quad (6.10)$$

C being the constant of property (g) of Definition 3.8. We also set $\mathfrak{E}_w^{\alpha,\vartheta} := \Big(\bigcap_{m=1}^\infty \mathfrak{E}_m^{\alpha,\vartheta_m}\Big)\cap\mathfrak{E}_{\rm sc}$.

We now show that Γ-limits of sequences in $\mathfrak{E}^{\alpha,\vartheta}$ belong to the larger space $\mathfrak{E}_w^{\alpha,\vartheta}$.

Proposition 6.10. *Let $(E_n)_n$ be a sequence of functionals in $\mathfrak{E}^{\alpha,\vartheta}$ and let $E\in\mathfrak{E}$. Assume that for every $A\in\mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_n(\cdot,A)$ Γ-converges to $E(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then $E\in\mathfrak{E}_m^{\alpha,\vartheta_m}\cap\mathfrak{E}_{\rm sc}$ for every $m\in\mathbb{N}$. In particular, $E\in\mathfrak{E}_w^{\alpha,\vartheta}$.*

Proof. By the semicontinuity of Γ-limits we have $E\in\mathfrak{E}_{\rm sc}$. Thus, given $m\in\mathbb{N}$, we only need to prove that $E\in\mathfrak{E}_m^{\alpha,\vartheta_m}$.

By the continuity of ϑ_m and exchanging the roles of s and t, to conclude it is enough to show that

$$\begin{aligned}(1-\vartheta_m(su^A))\frac{E(su,A)}{s} - \frac{c_7}{s}\mathcal{L}^d(A) - \frac{c_7}{s}\mathcal{L}^d(A)^\alpha E(su,A)^{1-\alpha}&\\ \le (1+\vartheta_m(tu^A))\frac{E(tu,A)}{t} + \frac{c_7}{t}\mathcal{L}^d(A) + \frac{c_7}{t}\mathcal{L}^d(A)^\alpha E(tu,A)^{1-\alpha} + R_{m,t}(u,A),& \quad (6.11)\end{aligned}$$

assuming that the left-hand side is stricty positive.

Let $A\in\mathcal{A}_c(\mathbb{R}^d)$, $u\in L^\infty(A;\mathbb{R}^k)$, and $t,s>0$. By Lemma 6.8, there exists a sequence $(u_n)_n\subset BV(A;\mathbb{R}^k)\cap L^\infty(A;\mathbb{R}^k)$ such that $u_n\to u$ in $L^1(A;\mathbb{R}^k)$, $u_n^A\le 4\sigma^m u^A$, and

$$\limsup_{n\to+\infty} E_n(tu_n,A) \le E(tu,A) + C\frac{E(tu,A)+\mathcal{L}^d(A)}{m}. \quad (6.12)$$

Since $u_n\in BV(A;\mathbb{R}^k)\cap L^\infty(A;\mathbb{R}^k)$, by (6.4) we get

$$\begin{aligned}(1-\vartheta(su_n^A))\frac{E_n(su_n,A)}{s} - \frac{c_7}{s}\mathcal{L}^d(A) - \frac{c_7}{s}\mathcal{L}^d(A)^\alpha E_n(su_n,A)^{1-\alpha}&\\ \le (1+\vartheta(tu_n^A))\frac{E_n(tu_n,A)}{t} + \frac{c_7}{t}\mathcal{L}^d(A) + \frac{c_7}{t}\mathcal{L}^d(A)^\alpha E_n(tu_n,A)^{1-\alpha}.&\end{aligned}$$

Taking the limsup as $n\to+\infty$ and using the monotonicity of ϑ and (6.12), we deduce that

$$\begin{aligned}&\limsup_{n\to+\infty}\Big((1-\vartheta_m(su^A))\frac{E_n(su_n,A)}{s} - \frac{c_7}{s}\mathcal{L}^d(A) - \frac{c_7}{s}\mathcal{L}^d(A)^\alpha E_n(su_n,A)^{1-\alpha}\Big)\\ &\quad\le \limsup_{n\to+\infty}\Big((1-\vartheta(su_n^A))\frac{E_n(su_n,A)}{s} - \frac{c_7}{s}\mathcal{L}^d(A) - \frac{c_7}{s}\mathcal{L}^d(A)^\alpha E_n(su_n,A)^{1-\alpha}\Big)\\ &\quad\le \limsup_{n\to+\infty}\Big(1+\vartheta(tu_n^A))\frac{E_n(tu_n,A)}{t} + \frac{c_7}{t}\mathcal{L}^d(A) + \frac{c_7}{t}\mathcal{L}^d(A)^\alpha E_n(tu_n,A)^{1-\alpha}\Big)\\ &\quad\le (1+\vartheta_m(tu^A))\frac{E(tu,A)}{t} + \frac{c_7}{t}\mathcal{L}^d(A) + \frac{c_7}{t}\mathcal{L}^d(A)^\alpha E(tu,A)^{1-\alpha} + R_{m,t}(u,A), \quad (6.13)\end{aligned}$$

where $R_{m,t}(u,A)$ is defined by (6.10). To deal with the first term in the previous chain of inequalities we introduce the function Φ defined for every $z\in[0,+\infty)$ as

$$\Phi(z):=(1-\vartheta_m(su^A))\frac{z}{s}-\frac{c_7}{s}\mathcal{L}^d(A)^{1-\alpha}z^{1-\alpha}-\frac{c_7}{s}\mathcal{L}^d(A).$$

Since the left-hand side of (6.11) is strictly positive, we have that $\Phi(E(su,A))>0$, which implies $(1-\vartheta_m(su^A))>0$. We set $z_0:=c_7^{1/\alpha}(1-\vartheta_m(su^A))^{-1/\alpha}\mathcal{L}^d(A)$ and observe that Φ is increasing on $(z_0,+\infty)$ and that if $\Phi(z)>0$ then $z>z_0$; in particular, $E(su,A)>z_0$. Finally, from the Γ-convergence of $E_n(\cdot,A)$ to $E(\cdot,A)$ and from the convergence of u_n to u in $L^0(\mathbb{R}^d;\mathbb{R}^k)$, we deduce that

$$E(su,A)\le\liminf_{n\to+\infty}E_n(su_n,A),$$

which, by the monotonicity of Φ, implies

$$\Phi(E(su,A))\le\liminf_{n\to+\infty}\Phi(E_n(su_n,A)).$$

Recalling the definition of Φ, from this inequality and (6.13) we obtain (6.11), concluding the proof. □

The next technical results will used in the proof of the representation theorem presented in Section 7. Given $\xi\in\mathbb{R}^{k\times d}$, and $m\in\mathbb{N}$, we set

$$\kappa_{\xi,m}:=2c_{\xi,m}+2d^{1/2}|\xi|, \tag{6.14}$$

where $c_{\xi,m}$ the constant defined by (5.14).

Lemma 6.11. *For every $\xi\in\mathbb{R}^{k\times d}$ there exists a constant with the following property: for every $m\in\mathbb{N}$ and $E\in\mathfrak{E}^{\alpha,\vartheta_m}\cap\mathfrak{E}_{\rm sc}$ there exists a set Let $m\in\mathbb{N}$ and $E\in\mathfrak{E}^{\alpha,\vartheta_m}_m\cap\mathfrak{E}_{\rm sc}$. Then there exists $N_m\in\mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N_m)=0$, satisfying the following property: for every $x\in\mathbb{R}^d\setminus N_m$, $\lambda\ge1$, $\nu\in\mathbb{S}^{d-1}$, $s,t>0$, and $\xi\in\mathbb{R}^{k\times d}$ there exist $\rho^{\nu,\lambda}_{m,\xi,t,s}(x)>0$, and M_ξ, the latter depending only on $|\xi|$ and on the structurual constants $c_1,...,c_7,k$ and α, such that for every $\rho\in(0,\rho^{\nu,\lambda}_{m,\xi,t,s}(x))$ we have*

$$\begin{aligned}
&\Big|\frac{m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))}{s}-\frac{m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))}{t}\Big|\\
&\le\frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{d\alpha}m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))^{1-\alpha}+\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{(s\wedge1)(t\wedge1)}\Big)\frac{m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))}{s}+\frac{c_7}{s}\lambda^{d-1}\rho^d\\
&\quad+\frac{c_7}{t}\lambda^{\alpha(d-1)}\rho^{d\alpha}m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))^{1-\alpha}+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\lambda\rho}{(s\wedge1)(t\wedge1)}\Big)\frac{m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))}{t}+\frac{c_7}{t}\lambda^{d-1}\rho^d\\
&\quad+\Big(1+\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{(s\wedge1)(t\wedge1)}\Big)\Big)\frac{M_\xi}{(s\wedge1)m}\lambda^{d-1}\rho^d+\Big(\frac{1}{s^\alpha}+\frac{1}{s}\Big)\frac{M_\xi}{m^{1-\alpha}}\lambda^{d-1}\rho^d\\
&\quad+\Big(1+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\lambda\rho}{(s\wedge1)(t\wedge1)}\Big)\Big)\frac{M_\xi}{(t\wedge1)m}\lambda^{d-1}\rho^d+\Big(\frac{1}{t^\alpha}+\frac{1}{t}\Big)\frac{M_\xi}{m^{1-\alpha}}\lambda^{d-1}\rho^d,
\end{aligned}\tag{6.15}$$

where $\kappa_{\xi,m}$ is the constant defined by (6.14). If, in addition the function f defined by (5.9) does not depend on x, the set $N_m=\emptyset$ and $\rho^{\nu,\lambda}_{m,\xi,t,s}(x)=+\infty$.

Proof. By hypothesis $E\in\mathfrak{E}_{\rm sc}$, so that by Theorem 5.16, the function f defined by (5.9) satisfies (5.15) for every $A\in\mathcal{A}_c(\mathbb{R}^d)$ and $u\in BV(A;\mathbb{R}^k)$. Hence, we may apply Corollary 5.10 to obtain a set $N_m\in\mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N_m)=0$, satisfying the following property: for every $x\in\mathbb{R}^d\setminus N_m$, $\xi\in\mathbb{R}^{k\times d}$, $t>0$, $\nu\in\mathbb{S}^{d-1}$, and $\lambda\ge1$ there exists $\rho^{\nu,\lambda}_{m,t\xi}(x)$ such that for every $\rho\in(0,\rho^{\nu,\lambda}_{m,t\xi}(x))$, there exists $u\in BV(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)\cap L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)$, with $\mathrm{tr}_{Q^\lambda_\nu(x,\rho)}u=\mathrm{tr}_{Q^\lambda_\nu(x,\rho)}\ell_\xi$ and $\|u-\ell_\xi\|_{L^\infty(Q^\lambda_\nu(x,\rho);\mathbb{R}^k)}\le\frac{1}{t}c_{t\xi,m}\lambda\rho\le\frac{1}{t\wedge1}c_{\xi,m}\lambda\rho$, such that

$$E(tu,Q^\lambda_\nu(x,\rho))\le m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))+\frac{C_{t\xi}}{m}\lambda^{d-1}\rho^d\le m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))+\frac{(t\vee1)C_\xi}{m}\lambda^{d-1}\rho^d,\tag{6.16}$$

where C_ξ is defined by (5.26). Note that the oscillation of u satisfies

$$u^{Q^\lambda_\nu(x,\rho)} \le (\tfrac{2}{t\wedge 1}c_{\xi,m} + \tfrac{2}{t\wedge 1}d^{1/2}|\xi|)\lambda\rho = \frac{\kappa_{\xi,m}}{t\wedge 1}\lambda\rho. \tag{6.17}$$

We can estimate $m^E(t\ell_\xi, Q^\lambda_\nu(x,\rho))$ by evaluating $E(\cdot, Q^\lambda_\nu(x,\rho))$ at $t\ell_\xi$. Recalling (6.16) and (c2′) of Definition 3.8, this leads to

$$E(tu, Q^\lambda_\nu(x,\rho)) \le \big(c_3k^{1/2}t|\xi| + c_4 + (t\vee 1)C_\xi\big)\lambda^{d-1}\rho^d. \tag{6.18}$$

Since $E \in \mathfrak{E}^{\alpha,\vartheta_m}_m$, by (6.9) for every $s>0$ we have that

$$\begin{aligned}
&(1-\vartheta_m(su^{Q^\lambda_\nu(x,\rho)}))\frac{E(su,Q^\lambda_\nu(x,\rho))}{s} - \frac{c_7}{s}(\lambda^{d-1}\rho^d)^\alpha E(su,Q^\lambda_\nu(x,\rho))^{1-\alpha}\\
&\qquad -\frac{c_7}{s}\lambda^{d-1}\rho^d - R_{m,s}(u,Q^\lambda_\nu(x,\rho))\\
&\le (1+\vartheta_m(tu^{Q^\lambda_\nu(x,\rho)}))\frac{E(tu,Q^\lambda_\nu(x,\rho))}{t} + \frac{c_7}{t}(\lambda^{d-1}\rho^d)^\alpha E(tu,Q^\lambda_\nu(x,\rho))^{1-\alpha}\\
&\qquad +\frac{c_7}{t}\lambda^{d-1}\rho^d + R_{m,t}(u,Q^\lambda_\nu(x,\rho)),
\end{aligned} \tag{6.19}$$

where R is defined by (6.10). Using the monotonicity of ϑ_m, (6.16), (6.17), (6.18), and the subadditivity of the function $z\mapsto z^{1-\alpha}$ on $[0,+\infty)$ we see that there exists a positive constant M_ξ, independent of m, such that the left-hand side can be bounded from below by

$$\begin{aligned}
&\Big(1-\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{E(su,Q^\lambda_\nu(x,\rho))}{s} - \frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{\alpha d}E(su,Q^\lambda_\nu(x,\rho))^{1-\alpha}\\
&-\frac{c_7}{s}\lambda^{d-1}\rho^d - C\Big(1+\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{E(su,Q^\lambda_\nu(x,\rho))+\lambda^{d-1}\rho^d}{sm}\\
&-C^\alpha\frac{c_7}{s}\frac{E(su,Q^\lambda_\nu(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}+\lambda^{d-1}\rho^d}{m^{1-\alpha}},
\end{aligned} \tag{6.20}$$

while the right-hand side of (6.19) can be bounded from above by

$$\begin{aligned}
&\Big(1+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))}{t} + \frac{c_7}{t}m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}\\
&+\frac{c_7}{t}\lambda^{d-1}\rho^d + \Big(1+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{C(c_3k^{1/2}|\xi|+c_4+C_\xi+1)+C_\xi}{(t\wedge 1)m}\lambda^{d-1}\rho^d\\
&+\Big(C^\alpha\frac{c_7}{t^\alpha}+\frac{c_7}{t}\Big)\frac{\big(c_3k^{1/2}|\xi|+c_4+C_\xi\big)^{1-\alpha}+1+C_\xi^{1-\alpha}}{m^{1-\alpha}}\lambda^{d-1}\rho^d.
\end{aligned} \tag{6.21}$$

Therefore, by (6.19)-(6.21) there exists a constant M_ξ, independent of m, such that

$$\begin{aligned}
&\Big(1-\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{E(su,Q^\lambda_\nu(x,\rho))}{s} - \frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{\alpha d}E(su,Q^\lambda_\nu(x,\rho))^{1-\alpha} - \frac{c_7}{s}\lambda^{d-1}\rho^d\\
&-C\Big(1+\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{E(su,Q^\lambda_\nu(x,\rho))+\lambda^{d-1}\rho^d}{sm} - C^\alpha\frac{c_7}{s}\frac{E(su,Q^\lambda_\nu(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}+\lambda^{d-1}\rho^d}{m^{1-\alpha}}\\
&\le \Big(1+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))}{t} + \frac{c_7}{t}m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}\\
&+\frac{c_7}{t}\lambda^{d-1}\rho^d + \Big(1+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{M_\xi}{(t\wedge 1)m}\lambda^{d-1}\rho^d + \Big(\frac{1}{t^\alpha}+\frac{1}{t}\Big)\frac{M_\xi}{m^{1-\alpha}}\lambda^{d-1}\rho^d.
\end{aligned}$$

We claim that the previous inequality still holds replacing $E(su,Q^\lambda_\nu(x,\rho))$ by $m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))$. To prove the claim, for given $s,t>0$ we introduce the function Φ defined for every $z\in[0,+\infty)$

as

$$\Phi(z) := \Big(1-\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{z}{s} - \frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{\alpha d}z^{1-\alpha} - \frac{c_7}{s}\lambda^{d-1}\rho^d$$
$$- C\Big(1+\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda}{t\wedge 1}\rho\Big)\Big)\frac{z+\lambda^{d-1}\rho^d}{sm} - C^\alpha\frac{c_7}{s}\frac{z^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}+\lambda^{d-1}\rho^d}{m^{1-\alpha}},$$

so that the left-hand side of (6.20) is equal to $\Phi(E(su,Q^\lambda_\nu(x,\rho))$. We now show that

$$\Phi(m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))) \le \Phi(E(su,Q^\lambda_\nu(x,\rho))). \tag{6.22}$$

Since the righ-hand side is clearly larger than zero, it is enough to prove this inequality when $\Phi(m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho)))>0$. Note that this positivity condition implies that $1-\vartheta_m(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1})>0$. Some straightforward computations, show that if $\Phi(z)>0$ then $z>z_0$, where

$$z_0 := c_7^{1/\alpha}\Big((1-\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big) - \Big(\Big(1+\vartheta_m(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{C}{m}\Big)^{-1/\alpha}\Big(1+\frac{C^\alpha}{m^{1-\alpha}}\Big)^{1/\alpha}\lambda^{d-1}\rho^d.$$

One can also see that Φ is increasing on $(z_0,+\infty)$. Since $\mathrm{tr}_{Q^\lambda_\nu(x,\rho)}su = \mathrm{tr}_{Q^\lambda_\nu(x,\rho)}s\ell_\xi$, we obtain that

$$m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))) \le E(su,Q^\lambda_\nu(x,\rho)). \tag{6.23}$$

Since we assumed that $\Phi(m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho)))>0$, we have that $m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))>z_0$. Hence, recalling that Φ is increasing on $(z_0,+\infty)$, from (6.23) we obtain that (6.22). Thus, in (6.20) we can substitute $E(su,Q^\lambda_\nu(x,\rho))$ by $m^E(s\ell_\xi,Q(x,\rho))$. This new inequality, together with

$$\frac{m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))}{s} \le \frac{c_3k^{1/2}|\xi|+c_4}{s\wedge 1}\lambda^{d-1}\rho^d,$$

implies that

$$\Big(1-\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))}{s} - \frac{c_7}{s}\lambda^{\alpha(d-1)}\rho^{\alpha d}m^E(s\ell_\xi,Q^\lambda_\nu(x,\rho))^{1-\alpha}$$
$$-\frac{c_7}{s}\lambda^{d-1}\rho^d - \Big(1+\vartheta_m\Big(\frac{s\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{M_\xi}{(s\wedge 1)m}\lambda^{d-1}\rho^d - \frac{1}{s}\frac{M_\xi}{m^{1-\alpha}}\lambda^{d-1}\rho^d$$
$$\le \Big(1+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))}{t} + \frac{c_7}{t}m^E(t\ell_\xi,Q^\lambda_\nu(x,\rho))^{1-\alpha}\lambda^{\alpha(d-1)}\rho^{\alpha d}$$
$$+\frac{c_7}{t}\lambda^{d-1}\rho^d + \Big(1+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\lambda\rho}{t\wedge 1}\Big)\Big)\frac{M_\xi}{(t\wedge 1)m}\lambda^{d-1}\rho^d + \Big(\frac{1}{t^\alpha}+\frac{1}{t}\Big)\frac{M_\xi}{m^{1-\alpha}}\lambda^{d-1}\rho^d.$$

Exchanging the roles of s and t, this gives (6.15).

If, in addition, the function f defined by (5.9) does not depend on x, then in Corollary 5.10 we have $N_m=\emptyset$ and $\rho^{\nu,\lambda}_{m,t\xi}(x)=\rho^{\nu,\lambda}_{m,s\xi}(x)=+\infty$, concluding the proof in this case. □

The following lemma deals with the case with boundary conditions related to the the jump functions $u_{x,\zeta,\nu}$.

Lemma 6.12. *There exists a positive constant $M>0$, depending only on the structural constants $c_1,...,c_7$, k, and on α, such that for every $m\in\mathbb{N}$, $E\in\mathfrak{E}^{\alpha,\vartheta_m}_m\cap\mathfrak{E}_{\rm sc}$, $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$, $\nu\in\mathbb{S}^{d-1}$,*

$s,t>0$, and $\rho>0$ we have

$$\begin{aligned}
&\Big|\frac{m^E(su_{x,\zeta,\nu},Q_\nu(x,\rho))}{s}-\frac{m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))}{t}\Big|\\
&\quad\le \frac{c_7}{s}\rho^{d\alpha}m^E(su_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}+\vartheta_{2m}(s|\zeta|)\frac{m^E(su_{x,\zeta,\nu},Q_\nu(x,\rho))}{s}+\frac{c_7}{s}\rho^d\\
&\qquad\frac{c_7}{t}\rho^{d\alpha}m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}+\vartheta_{2m}(t|\zeta|)\frac{m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))}{t}+\frac{c_7}{t}\rho^d\\
&\qquad+(1+\vartheta_{2m}(s|\zeta|))\Big(\frac{M|\zeta|}{m}\rho^{d-1}+\frac{M}{sm}\rho^d\Big)+\frac{1}{s^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha}+\frac{1}{s}\frac{M}{m^{1-\alpha}}\rho^d\\
&\qquad+(1+\vartheta_{2m}(t|\zeta|))\Big(\frac{M|\zeta|}{m}\rho^{d-1}+\frac{M}{tm}\rho^d\Big)+\frac{1}{t^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha}+\frac{1}{t}\frac{M}{m^{1-\alpha}}\rho^d. \qquad (6.24)
\end{aligned}$$

Proof. By Lemma 5.12 for every $x\in\mathbb{R}^d$, $\zeta\in\mathbb{R}^k$, $t>0$, $\nu\in\mathbb{S}^{d-1}$, and $\rho>0$ there exists a function $u\in BV(Q_\nu(x,\rho);\mathbb{R}^k)\cap L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)$ such that $\mathrm{tr}_{Q_\nu(x,\rho)}u=\mathrm{tr}_{Q_\nu(x,\rho)}u_{x,\zeta,\nu}$, $\|u\|_{L^\infty(Q_\nu(x,\rho);\mathbb{R}^k)}\le\sigma^m|\zeta|$, and

$$E(tu,Q_\nu(x,\rho))\le m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))+\frac{tK|\zeta|}{m}\rho^{d-1}+K\rho^d, \qquad (6.25)$$

where $K>0$ is the constant in Lemma 5.12. Note that the oscillation of u satisfies $u^{Q_\nu(x,\rho)}\le 2\sigma^m|\zeta|$. We now estimate $m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))$ by evaluating $E(\cdot,Q_\nu(x,\rho))$ at $tu_{x,\zeta,\nu}$ and by (c2) we get

$$m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))\le c_3kt|\zeta|\rho^{d-1}+c_4\rho^d,$$

so that from (6.25) we deduce

$$E(tu,Q_\nu(x,\rho))\le t|\zeta|\Big(c_3k+\frac{K}{m}\Big)\rho^{d-1}+(c_4+K)\rho^d. \qquad (6.26)$$

Since $E\in\mathfrak{E}_m^{\alpha,\vartheta_m}$, by (6.9) for every $s>0$ we have that

$$\begin{aligned}
&(1-\vartheta_m(su^{Q_\nu(x,\rho)}))\frac{E(su,Q_\nu(x,\rho))}{s}-\frac{c_7}{s}\rho^{\alpha d}E(su,Q_\nu(x,\rho))^{1-\alpha}-\frac{c_7}{s}\rho^d-R_s(u,Q_\nu(x,\rho))\\
&\quad\le(1+\vartheta_m(tu^{Q_\nu(x,\rho)}))\frac{E(tu,Q_\nu(x,\rho))}{t}+\frac{c_7}{t}\rho^{\alpha d}E(tu,Q_\nu(x,\rho))^{1-\alpha}+\frac{c_7}{t}\rho^d+R_t(u,Q_\nu(x,\rho)).
\end{aligned}$$

Using the inequality $u^{Q_\nu(x,\rho)}\le 2\sigma^m|\zeta|$, the monotonicity of ϑ_m, and the subadditivity of $z\mapsto z^{1-\alpha}$, we deduce that the left-hand side of the last inequality can be estimated from below by

$$\begin{aligned}
&(1-\vartheta_{2m}(s|\zeta|))\frac{E(su,Q_\nu^\lambda(x,\rho))}{s}-\frac{c_7}{s}\rho^{\alpha d}E(su,Q_\nu(x,\rho))^{1-\alpha}-\frac{c_7}{s}\rho^d\\
&-C(1+\vartheta_{2m}(s|\zeta|))\frac{E(su,Q_\nu(x,\rho))+\rho^d}{sm}-C^\alpha\frac{c_7}{s}\frac{E(su,Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d}+\rho^d}{m^{1-\alpha}}.
\end{aligned}$$

while the right-hand side can be bounded from above by

$$\begin{aligned}
&(1+\vartheta_{2m}(t|\zeta|))\frac{m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))}{t}+\frac{c_7}{t}m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d}+\frac{c_7}{t}\rho^d\\
&\quad+(1+\vartheta_{2m}(t|\zeta|))\frac{(C+1)|\zeta|(kc_3+2K)}{m}\rho^{d-1}+\frac{(1+\vartheta_{2m}(t|\zeta|))}{t}\frac{(C+1)(c_4+2K+1)}{m}\rho^d\\
&\quad+\frac{c_7}{t^\alpha}\frac{|\zeta|^{1-\alpha}(C^\alpha+1)\Big(k^{1-\alpha}c_3^{1-\alpha}+2K^{1-\alpha}\Big)}{m^{1-\alpha}}\rho^{d-1+\alpha}+\frac{c_7}{t}\frac{(C^\alpha+1)(c_4^{1-\alpha}+2K^{1-\alpha}+1)}{m^{1-\alpha}}\rho^d,
\end{aligned}$$

Therefore, there exists a constant $M>0$ such that

$$
\begin{aligned}
&(1-\vartheta_{2m}(s|\zeta|))\frac{E(su,Q_\nu(x,\rho))}{s}-\frac{c_7}{s}\rho^{\alpha d}E(su,Q_\nu(x,\rho))^{1-\alpha}-\frac{c_7}{s}\rho^d\\
&-C(1+\vartheta_{2m}(s|\zeta|))\frac{E(su,Q_\nu(x,\rho))+\rho^d}{sm}-C^\alpha\frac{c_7}{s}\frac{E(su,Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d}+\rho^d}{m^{1-\alpha}}\\
&\le(1+\vartheta_{2m}(t|\zeta|))\frac{m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))}{t}+\frac{c_7}{t}m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d}+\frac{c_7}{t}\rho^d\\
&+(1+\vartheta_{2m}(t|\zeta|))\frac{M|\zeta|}{m}\rho^{d-1}+\frac{(1+\vartheta_{2m}(t|\zeta|))}{t}\frac{M}{m}\rho^d+\frac{1}{t^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha}+\frac{1}{t}\frac{M}{m^{1-\alpha}}\rho^d,
\end{aligned}
$$

Arguing as in the proof of Lemma 6.11 we may substitute $E(su,Q_\nu(x,\rho))$ by $m^E(su_{x,\zeta,\nu},Q_\nu(x,\rho))$, so that, taking into account the estimate

$$
m^E(su_{x,\zeta,\nu},Q_\nu(x,\rho))\le c_3s|\zeta|\rho^{d-1}+c_4\rho^d,
$$

we obtain

$$
\begin{aligned}
&(1+\vartheta_{2m}(s|\zeta|))\frac{m^E(su_{x,\zeta,\nu},Q_\nu(x,\rho))}{s}+\frac{c_7}{s}m^E(su_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d}+\frac{c_7}{s}\rho^d\\
&+(1+\vartheta_{2m}(s|\zeta|))\frac{M|\zeta|}{m}\rho^{d-1}+\frac{(1+\vartheta_{2m}(s|\zeta|))}{s}\frac{M}{m}\rho^d+\frac{1}{s^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha}+\frac{1}{s}\frac{M}{m^{1-\alpha}}\rho^d\\
&\le(1+\vartheta_{2m}(t|\zeta|))\frac{m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))}{t}+\frac{c_7}{t}m^E(tu_{x,\zeta,\nu},Q_\nu(x,\rho))^{1-\alpha}\rho^{\alpha d}+\frac{c_7}{t}\rho^d\\
&+(1+\vartheta_{2m}(t|\zeta|))\frac{M|\zeta|}{m}\rho^{d-1}+\frac{(1+\vartheta_{2m}(t|\zeta|))}{t}\frac{M}{m}\rho^d+\frac{1}{t^\alpha}\frac{M|\zeta|^{1-\alpha}}{m^{1-\alpha}}\rho^{d-1+\alpha}+\frac{1}{t}\frac{M}{m^{1-\alpha}}\rho^d.
\end{aligned}
$$

Exchanging the roles of s,t, we obtain (6.24). □

We now investigate the properties of the integrands f and g associated with functionals in $\mathfrak{E}^{\alpha,\vartheta}_w$.

Proposition 6.13. *Let $E\in\mathfrak{E}^{\alpha,\vartheta}_w$. Then the function f defined by (5.9) belongs to $\mathcal{F}^\alpha$.*

Proof. For every $m\in\mathbb{N}$ let $N_m\in\mathcal{B}(\mathbb{R}^d)$ be the $\mathcal{L}^d$-negligible set of Lemma 6.11 and let $N:=\bigcup_{m\in\mathbb{N}}N_m$. By the same lemma for every $x\in\mathbb{R}^d\setminus N$, $\xi\in\mathbb{R}^{k\times d}$, $m\in\mathbb{N}$, $s,t>0$ and every ρ small enough we have

$$
\begin{aligned}
&\Big|\frac{m^E(s\ell_\xi,Q(x,\rho))}{s}-\frac{m^E(t\ell_\xi,Q(x,\rho))}{t}\Big|\\
&\le\frac{c_7}{s}\rho^{d\alpha}m^E(s\ell_\xi,Q(x,\rho))^{1-\alpha}+\vartheta_m\Big(\frac{s\kappa_{\xi,m}\rho}{(s\wedge1)(t\wedge1)}\Big)\frac{m^E(s\ell_\xi,Q(x,\rho))}{s}+\frac{c_7}{s}\rho^d\\
&\quad+\frac{c_7}{t}\rho^{d\alpha}m^E(t\ell_\xi,Q(x,\rho))^{1-\alpha}+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\rho}{(s\wedge1)(t\wedge1)}\Big)\frac{m^E(t\ell_\xi,Q(x,\rho))}{t}+\frac{c_7}{t}\rho^d\\
&\quad+\Big(1+\vartheta_m\Big(\frac{s\kappa_{\xi,m}\rho}{(s\wedge1)(t\wedge1)}\Big)\Big)\frac{M_\xi}{(s\wedge1)m}\rho^d+\Big(\frac{1}{s^\alpha}+\frac{1}{s}\Big)\frac{M_\xi}{m^{1-\alpha}}\rho^d\\
&\quad+\Big(1+\vartheta_m\Big(\frac{t\kappa_{\xi,m}\rho}{(s\wedge1)(t\wedge1)}\Big)\Big)\frac{M_\xi}{(t\wedge1)m}\rho^d+\Big(\frac{1}{t^\alpha}+\frac{1}{t}\Big)\frac{M_\xi}{m^{1-\alpha}}\rho^d,
\end{aligned}
\tag{6.27}
$$

where $\kappa_{\xi,m}$ is the constant defined by (6.14). Since ϑ_m is continuous and $\vartheta_m(0)=0$, we have that

$$
\begin{aligned}
&\limsup_{\rho\to0^+}\Big(\vartheta_m\Big(\frac{s\kappa\rho}{(s\wedge1)(t\wedge1)}\Big)\frac{m^E(s\ell_\xi,Q(x,\rho))}{s\rho^d}\Big)=0,\\
&\limsup_{\rho\to0^+}\Big(\vartheta_m\Big(\frac{t\kappa\rho}{(s\wedge1)(t\wedge1)}\Big)\frac{m^E(t\ell_\xi,Q(x,\rho))}{t\rho^d}\Big)=0.
\end{aligned}
$$

Therefore, dividing (6.27) by ρ^d and taking the limsup as $\rho \to 0^+$, we obtain

$$\begin{aligned}\frac{f(x,s\xi)}{s} \;\leq\; & \frac{f(x,t\xi)}{t} + \frac{c_7}{s} f(x,s\xi)^{1-\alpha} + \frac{c_7}{s} + f(x,t\xi)^{1-\alpha} + \frac{c_7}{t} \\ & + \frac{M_\xi}{(s\wedge 1)m} + \Big(\frac{1}{s^\alpha} + \frac{1}{s}\Big)\frac{M_\xi}{m^{1-\alpha}} + \frac{M_\xi}{(t\wedge 1)m} + \Big(\frac{1}{t^\alpha} + \frac{1}{t}\Big)\frac{M_\xi}{m^{1-\alpha}}.\end{aligned}$$

Letting $m \to +\infty$ and exchanging the roles of s and t, we recover (6.2), which concludes the proof. □

Proposition 6.14. *Let $E \in \mathfrak{E}^{\alpha,\vartheta}_w$, $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$, and $s,t > 0$. Then the function g defined by (5.11) satisfies*

$$\begin{aligned}\Big|\frac{g(x,s\zeta,\nu)}{s} - \frac{g(x,t\zeta,\nu)}{t}\Big| \;\leq\; & \vartheta_{2m}(s|\zeta|)\frac{g(x,s\zeta,\nu)}{s} + \vartheta_{2m}(t|\zeta|)\frac{g(x,t\zeta,\nu)}{t} \\ & + (2 + \vartheta_{2m}(s|\zeta|) + \vartheta_{2m}(t|\zeta|))\frac{M|\zeta|}{m},\end{aligned}$$

where $M > 0$ is the constant of Lemma 6.12.

Proof. Thanks to Lemma 6.12, inequality (6.24) holds. Dividing this inequality by ρ^{d-1} and taking the limsup as $\rho \to 0^+$, we immediately get

$$\begin{aligned}\frac{g(x,s\zeta,\nu)}{s} \leq \frac{g(x,t\zeta,\nu)}{t} + \vartheta_{2m}(s|\zeta|)\frac{g(x,\zeta,\nu)}{s} + \vartheta_{2m}(t|\zeta|)\frac{g(x,\zeta,\nu)}{t} \\ + (1 + \vartheta_{2m}(s|\zeta|))\frac{M|\zeta|}{m} + (1 + \vartheta_{2m}(t|\zeta|))\frac{M|\zeta|}{m}.\end{aligned}$$

Exchanging the roles of s and t we conclude the proof. □

Remark 6.15. In the scalar case, due to the different nature of the required vertical truncations, it is possible to prove a stronger version of Propositions 6.10 and 6.14. First, the space $\mathfrak{E}^{\alpha,\vartheta}$ turns out to be closed under Γ-convergence with respect to the topology of $L^0(\mathbb{R}^d)$. Then, exploiting this stronger version of Proposition 6.10, it is shown that the function g associated to E by (5.11) belongs to $\mathcal{G}^\vartheta$.

However, due to the presence of error terms in the truncation procedure, which depend on ϑ_m, in the case $k > 1$ one cannot recover the closedness of $\mathfrak{E}^{\alpha,\vartheta}$ and subsequently conclude that $g \in \mathcal{G}^\vartheta$.

7. Full Integral representation

We are finally ready to state and prove the full integral representation result for functionals in the class $\mathfrak{E}^{\alpha,\vartheta}_w$. More precisely, we will show that if $E \in \mathfrak{E}^{\alpha,\vartheta}_w$ and if the bulk integrand f defined by (5.9) does not depend on x, then the Cantor part E^c of the functional E can be represented by means of f^∞. Since this result will be employed to obtain an integral representation for functionals arising from homogenisation of functionals in $\mathfrak{E}^{\alpha,\vartheta}$, the hypothesis that f does not depend on x is not restrictive for our purposes.

We now state the main result of this section.

Theorem 7.1. *Let $E \in \mathfrak{E}^{\alpha,\vartheta}_w$ and f and g be defined by (5.9) and by (5.11), respectively. Assume that there exists a function $\hat{f}\colon \mathbb{R}^{k\times d} \to [0,+\infty)$ such that*

$$f(x,\xi) = \hat{f}(\xi) \quad \text{for every } x \in \mathbb{R}^d \text{ and for every } \xi \in \mathbb{R}^{k\times d}. \tag{7.1}$$

Then $E = E^{f,g}$, where $E^{f,g}$ is the functional introduced in Definition 3.7.

To prove this result, we try to characterise the Radon-Nikodým derivative of the measures $E^c(u,\cdot)$ with respect to $|D^c u|$ for any $u \in GBV_\star(A;\mathbb{R}^k)$ with $A \in \mathcal{A}_c(\mathbb{R}^d)$. This is done via a careful truncation procedure and taking advantage of the results of [9]. We recall that given $A \in \mathcal{A}_c(\mathbb{R}^d)$ and $u \in BV(A;\mathbb{R}^k)$, Alberti's Rank-One Theorem (see [2, Corollary 4.6] and [3, 23, 37])

ensures that for $|D^c u|$-a.e. $x \in A$ the matrix $dD^c u/d|D^c u|(x)$ has rank one, i.e., there exists two Borel functions $a_u\colon A \to \mathbb{S}^{k-1}$ and $\nu_u\colon A \to \mathbb{S}^{d-1}$ such that for $|D^c u|$-a.e. $x \in A$ we have

$$\frac{dD^c u}{d|D^c u|}(x) = a_u(x) \otimes \nu_u(x). \tag{7.2}$$

Lemma 7.2. *Let $E \in \mathfrak{E}_{\rm sc}$, $A \in \mathcal{A}_c(\mathbb{R}^d)$, and $u \in BV(A;\mathbb{R}^k)$. Assume that there exists a function $\hat f\colon \mathbb{R}^{k\times d} \to [0,+\infty)$ satisfying (7.1). For $|D^c u|$-a.e. $x \in \mathbb{R}^d$, for every $\lambda \ge 1$, and $\rho > 0$ we set*

$$s^\lambda_\rho(x) := \frac{|D^c u|(Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d} \quad \text{and} \quad \xi^\lambda_\rho(x) := s^\lambda_\rho(x)(a_u(x)\otimes\nu_u(x)), \tag{7.3}$$

where $(a_u(x),\nu_u(x)) \in \mathbb{S}^{k-1}\times\mathbb{S}^{d-1}$ is given by (7.2). Then

$$\lim_{\rho\to+\infty} s^\lambda_\rho(x) = +\infty \quad \text{and} \quad \lim_{\rho\to+\infty} \rho s^\lambda_\rho(x) = 0 \quad \text{for every } \lambda \ge 1, \tag{7.4}$$

$$\frac{dE^c(u,\cdot)}{d|D^c u|}(x) = \lim_{\lambda\to+\infty} \limsup_{\rho\to0^+} \frac{m^E(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}, \tag{7.5}$$

for $|D^c u|$-a.e. $x \in A$.

Proof. Let us fix $\delta > 0$ and consider the function E_δ defined by (5.6). It is shown in [9, Lemma 3.7] that (7.4) holds true for $|D^c u|$-a.e. $x \in A$ and that for $\delta > 0$ one has

$$\frac{dE^c(u,\cdot)}{d|D^c u|}(x) + \delta = \frac{dE^c_\delta(u,\cdot)}{d|D^c u|}(x) = \lim_{\lambda\to+\infty} \limsup_{\rho\to0^+} \frac{m^{E_\delta}(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}.$$

Since $E \le E_\delta$, we deduce that

$$\frac{dE^c(u,\cdot)}{d|D^c u|}(x) + \delta \ge \limsup_{\lambda\to+\infty} \limsup_{\rho\to0^+} \frac{m^{E}(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}.$$

Thus, it is enough to show that there exists a constant $K > 0$, depending only on $c_1,\dots,c_4$ and on k, such that for every $\lambda \ge 1$, for every $\delta > 0$, and for $|D^c u|$-a.e. $x \in A$ we have

$$\limsup_{\rho\to0^+} \frac{m^{E_\delta}(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} \le \Big(1 + K\delta\Big) \limsup_{\rho\to0^+} \frac{m^{E}(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}. \tag{7.6}$$

We fix $m \in \mathbb{N}$, $\lambda \ge 1$, and consider a point $x \in A$ such that (7.2) and (7.4) hold. By Theorem 5.16 hypothesis (5.15) of Corollary 5.10 is satisfied. Thus, there exists a function $u^x_\rho \in BV(Q^\lambda_{\nu_u(x)}(x,\rho);\mathbb{R}^k)$, with $\operatorname{tr}_{Q^\lambda_{\nu_u(x)}(x,\rho)} u^x_\rho = \operatorname{tr}_{Q^\lambda_{\nu_u(x)}(x,\rho)} \ell_{\xi^\lambda_\rho(x)}$ and $\|u^x_\rho - \ell_{\xi^\lambda_\rho(x)}\|_{L^\infty(Q^\lambda_{\nu_u(x)}(x,\rho);\mathbb{R}^k)} \le c_{\xi^\lambda_\rho(x),m}\lambda\rho$, such that

$$E(u^x_\rho, Q^\lambda_{\nu_u(x)}(x,\rho)) \le m^E(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho)) + \frac{C_{\xi^\lambda_\rho(x)}}{m}\lambda^{d-1}\rho^d, \tag{7.7}$$

where $c_{\xi^\lambda_\rho(x),m}$ and $C_{\xi^\lambda_\rho(x)}$ are the positive constants defined by (5.14) and (5.26), respectively. Recalling that $\|u^x_\rho - \ell_{\xi^\lambda_\rho(x)}\| \le c_{\xi^\lambda_\rho(x),m}\lambda\rho$ and observing that $|[u^x_\rho]| = |[u^x_\rho - \ell_{\xi^\lambda_\rho(x)}]|$, we deduce that $|[u^x_\rho]| \le 2c_{\xi^\lambda_\rho(x),m}\lambda\rho$. Therefore, we may estimate

$$\begin{aligned}\int_{J_{u^x_\rho}} |[u^x_\rho]|\,d\mathcal{H}^{d-1} &\le \int_{(J_{u^x_\rho}\setminus J^1_{u^x_\rho})} |[u^x_\rho]|\,d\mathcal{H}^{d-1} + 2c_{\xi^\lambda_\rho(x),m}\lambda\rho\mathcal{H}^{d-1}(J^1_{u^x_\rho}) \\ &\le (1 + 2c_{\xi^\lambda_\rho(x),m}\lambda\rho)\int_{J_{u^x_\rho}} |[u^x_\rho]| \wedge 1\,d\mathcal{H}^{d-1}.\end{aligned}$$

From this inequality, by (c1) and (c2) of Definition 3.8 we see that

$$\begin{aligned}E_\delta(u^x_\rho, Q^\lambda_{\nu_u(x)}(x,\rho)) &= E(u^x_\rho, Q^\lambda_{\nu_u(x)}(x,\rho)) + \delta|D^c u^x_\rho|(Q^\lambda_{\nu_u(x)}(x,\rho))\\ &\le E(u^x_\rho, Q^\lambda_{\nu_u(x)}(x,\rho)) + \delta\int_{Q^\lambda_{\nu_u(x)}(x,\rho)} |\nabla u^x_\rho|\,dx + \delta|D^c u|(Q^\lambda_{\nu_u(x)}(x,\rho))\\ &\quad +\delta(1+2c_{\xi^\lambda_\rho(x),m}\lambda\rho)\int_{J_{u^x_\rho}} |[u^x_\rho]|\wedge 1\,d\mathcal{H}^{d-1}\\ &\le \Big(1+\delta\frac{1+2c_{\xi^\lambda_\rho(x),m}\lambda\rho}{c_1}\Big)E(u^x_\rho, Q^\lambda_{\nu_u(x)}(x,\rho)) + \delta\frac{c_2}{c_1}(1+2c_{\xi^\lambda_\rho(x),m}\lambda\rho)\lambda^{d-1}\rho^d.\end{aligned}$$

Since $\mathrm{tr}_{Q^\lambda_{\nu_u(x)}(x,\rho)}u^x_\rho = \mathrm{tr}_{Q^\lambda_{\nu_u(x)}(x,\rho)}\ell_{\xi^\lambda_\rho(x)}$, combining the previous inequality with (7.7), we get

$$\begin{aligned}\frac{m^{E_\delta}(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} &\le \Big(1+\delta\frac{1+2c_{\xi^\lambda_\rho(x),m}\lambda\rho}{c_1}\Big)\Big(\frac{m^E(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} + \frac{C_{\xi^\lambda_\rho(x)}}{s^\lambda_\rho(x)m}\Big)\\ &\quad + \delta\frac{c_2}{c_1}(1+2c_{\xi^\lambda_\rho(x),m}\lambda\rho)\frac{1}{s^\lambda_\rho(x)}.\end{aligned}\tag{7.8}$$

It is immediate to check that when $s^\lambda_\rho(x)\ge 1$ we have

$$\frac{m^E(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} \le c_3 k^{1/2} + c_4.$$

Hence, from this inequality, (5.14), (5.26), and (7.4), we see that there exist two cosntants C_1 and C_2 depending only on the structural constants $c_1,\dots,c_4$, k, and d such that

$$\begin{aligned}&\limsup_{\rho\to 0^+}\frac{\delta(1+2c_{\xi^\lambda_\rho(x),m}\lambda\rho)}{c_1}\Big(\frac{m^E(\ell_{\xi^\lambda_\rho(x)}, Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)} + \frac{C_{\xi^\lambda_\rho(x)}}{s^\lambda_\rho(x)m}\Big)\\ &\le \limsup_{\rho\to 0^+}\frac{\delta}{c_1}\Big(1+2(\sigma^m+1)\lambda\rho\big(s^\lambda_\rho(x)d^{1/2}+C_1\big)\Big)\Big((c_3k^{1/2}+c_4) + \frac{C_{\xi^{\lambda\rho}(x)}}{s^\lambda_\rho(x)m}\Big)\\ &\le \frac{\delta}{c_1}\Big(c_3k^{1/2}+c_4+\frac{C_2}{m}\Big),\end{aligned}\tag{7.9}$$

As for the remaining terms of (7.8), by (5.26) we see that

$$\limsup_{\rho\to 0^+}\frac{C_{\xi^\lambda_\rho(x)}}{s^\lambda_\rho(x)m} = \limsup_{\rho\to 0^+}\frac{2c_3Cs^\lambda_\rho(x)+2C(c_4+1)}{s^\lambda_\rho(x)m} = \frac{2c_3C}{m},\tag{7.10}$$

while by (5.14) and (7.4) we get that

$$\limsup_{\rho\to 0^+}\frac{c_2}{c_1}\delta\big(1+2c_{\xi^\lambda_\rho(x),m}\lambda\rho\big)\frac{1}{s^\lambda_\rho(x)} = 0.\tag{7.11}$$

Finally, using (7.9)-(7.11) we can take the limsup in (7.8) as $\rho\to 0^+$, obtaining an estimate that depends on m and δ. Then, taking the limit as $m\to+\infty$ we get (7.6), concluding the proof. □

The next result ensures that estimates on the volume integrand f translate into estimates on the right-hand side of (7.5). For the proof of this result we refer the reader to [20, Lemma 5.3].

Lemma 7.3. *Let $E\in\mathfrak{E}_{\mathrm{sc}}$, $\xi\in\mathbb{R}^{k\times d}$, $\lambda\ge 1$, $\nu\in\mathbb{S}^{d-1}$, $t>0$, and $\mu\in[0,+\infty)$. Assume that for every $x\in\mathbb{R}^d$ and $\rho>0$ we have*

$$m^E_{t\rho}(\ell_\xi, Q(x,\rho)) \le \mu\rho^d.$$

Then

$$m^E_{t\lambda\rho}(\ell_\xi, Q^\lambda_\nu(x,\rho)) \le \mu\lambda^{d-1}\rho^d$$

for every $x \in \mathbb{R}^d$ *and* $\rho > 0$. *If, in addition, there exists some* $x_0 \in \mathbb{R}^d$ *such that*

$$\limsup_{\rho\to 0^+} \frac{m^E_{t\rho}(\ell_\xi, Q(x_0,\rho))}{\rho^d} = \mu,$$

then

$$\limsup_{\rho\to 0^+} \frac{m^E_{t\lambda\rho}(\ell_\xi, Q^\lambda_\nu(x_0,\rho))}{\lambda^{d-1}\rho^d} = \mu.$$

With these two lemmas at hand, we are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. Thanks to Theorem 5.16 and to (7.1), for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ we have

$$E^a(u,B) = \int_B \hat f(\nabla u)\,dx \quad \text{and} \quad E^s(u,B) = \int_{J_u\cap B} g(x,[u],\nu_u)\,d\mathcal{H}^{d-1} \tag{7.12}$$

for every $u \in GBV_\star(A;\mathbb{R}^k)$ and $B \in \mathcal{B}(A)$.

We are left with proving that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ we can represent the Cantor part E^c as

$$E^c(u,B) = \int_B \hat f^\infty\Big(\frac{dD^cu}{d|D^cu|}\Big)\,d|D^cu| \tag{7.13}$$

for every $u \in GBV_\star(A;\mathbb{R}^k)$ and $B \in \mathcal{B}(A)$.

Since ℓ_ξ is an admissible function for the minimisation problem $m^E(\ell_\xi, Q(x,\rho))$, by (7.12) we get that

$$m^E(\ell_\xi, Q(x,\rho)) \le \hat f(\xi)\rho^d$$

for every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{k\times d}$, and $\rho > 0$. Recalling (5.9) and (7.1), applying Lemma 7.3 we obtain

$$\limsup_{\rho\to 0^+} \frac{m^E(\ell_\xi, Q^\lambda_\nu(x,\rho)))}{\lambda^{d-1}\rho^d} = \hat f(\xi)$$

for every $\lambda \ge 1$. In particular, for every $t > 0$ we have

$$\limsup_{\rho\to 0^+} \frac{m^E(t\ell_\xi, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t} = \frac{\hat f(t\xi)}{t} \tag{7.14}$$

so that, taking the lim sup for $t \to +\infty$, we get

$$\limsup_{t\to+\infty}\limsup_{\rho\to 0^+} \frac{m^E(t\ell_\xi, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t} = \hat f^\infty(\xi). \tag{7.15}$$

We claim that for every $\xi \in \mathbb{R}^{k\times d}$, $\lambda \ge 1$, and $x \in \mathbb{R}^d$ we have

$$\limsup_{\rho\to 0^+} \frac{m^E(s_\rho\ell_\xi, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho} = \hat f^\infty(\xi) \tag{7.16}$$

whenever $s_\rho \to +\infty$ and $\rho s_\rho \to 0^+$ as $\rho \to 0^+$. To see this, first note that by (5.9) and (7.1) for every $\rho > 0$ we have $m^E(s_\rho\ell_\xi, Q^\lambda_\nu(x,\rho)) \le \hat f(s_\rho\xi)\lambda^{d-1}\rho^d$, whence

$$\limsup_{\rho\to 0^+} \frac{m^E(s_\rho\ell_\xi, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho} \le \hat f^\infty(\xi).$$

Thus, to prove (7.16) it is enough to show that

$$\hat f^\infty(\xi) \le \limsup_{\rho\to 0^+} \frac{m^E(s_\rho\ell_\xi, Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}. \tag{7.17}$$

Let $\eta > 0$. By definition of $\hat f^\infty$, there exists $t_\eta > 1/\eta$ such that

$$\hat f^\infty(\xi) - \eta < \frac{\hat f(t_\eta\xi)}{t_\eta}. \tag{7.18}$$

Hence, recalling (7.14) we have

$$\hat f^\infty(\xi)-\eta\le\limsup_{\rho\to0^+}\frac{m^E(t_\eta\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta}. \tag{7.19}$$

Let us fix $m\in\mathbb{N}$. By Lemma 6.11 we obtain

$$\begin{aligned}\limsup_{\rho\to0^+}\frac{m^E(t_\eta\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho t_\eta}&\le\limsup_{\rho\to+\infty}\frac{m^E(s_\rho\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\\&+\limsup_{\rho\to0^+}\Big(\vartheta_m(s_\rho\kappa_{\xi,m}\lambda\rho)\frac{m^E(s_\rho\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}+\frac{c_7}{s_\rho^\alpha}\Big(\frac{m^E(s_\rho\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\Big)^{1-\alpha}+\frac{c_7}{s_\rho}\\&\quad+(1+\vartheta_m(s_\rho\kappa_{\xi,m}\lambda\rho))\frac{M_\xi}{m}+\Big(\frac{1}{s_\rho^\alpha}+\frac{1}{s_\rho}\Big)\frac{M_\xi}{m^{1-\alpha}}\Big)\\&+\limsup_{\rho\to0^+}\Big(\vartheta_m(t_\eta\kappa_{\xi,m}\lambda\rho)\frac{m^E(t_\eta\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta}+\frac{c_7}{t_\eta^\alpha}\Big(\frac{m^E(t_\eta\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta}\Big)^{1-\alpha}+\frac{c_7}{t_\eta}\\&\quad+(1+\vartheta_m(t_\eta\kappa_{\xi,m}\lambda\rho))\frac{M_\xi}{m}+\Big(\frac{1}{t_\eta^\alpha}+\frac{1}{t_\eta}\Big)\frac{M_\xi}{m^{1-\alpha}}\Big),\end{aligned} \tag{7.20}$$

where $M_\xi>0$ is the constant of Lemma 6.11, which we recall is independent of ρ, s_ρ, t, and m.

We separately study the summands of this last expression. Since ϑ_m is continuous and $\vartheta_m(0)=0$, while $s_\rho\rho\to0$ and $s_\rho\to+\infty$ as $\rho\to0^+$, recalling that $\frac{m^E(s_\rho\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}$ is bounded uniformly with respect to ρ by (c2) of Definition 3.8, we obtain

$$\begin{aligned}\limsup_{\rho\to0^+}\Big(\vartheta_m(s_\rho\kappa_{\xi,m}\lambda\rho)\frac{m^E(s_\rho\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}+\frac{c_7}{s_\rho^\alpha}\Big(\frac{m^E(s_\rho\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}\Big)^{1-\alpha}\\+\frac{c_7}{s_\rho}+(1+\vartheta_m(s_\rho\kappa_{\xi,m}\lambda\rho))\frac{M_\xi}{m}+\Big(\frac{1}{s_\rho^\alpha}+\frac{1}{s_\rho}\Big)\frac{M_\xi}{m^{1-\alpha}}\Big)=\frac{M_\xi}{m}.\end{aligned} \tag{7.21}$$

By (c2) of Definition 3.8 there exists $N_\xi>0$, independent of ρ and t_η, such that

$$\frac{m^E(t_\eta\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta}\le N_\xi.$$

Therefore, arguing as in the proof of (7.21), by (7.18) and (7.19), we obtain

$$\begin{aligned}&\limsup_{\rho\to0^+}\Big(\vartheta_m(t_\eta\kappa_{\xi,m}\lambda\rho)\frac{m^E(t_\eta\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta}+\frac{c_7}{t_\eta^\alpha}\Big(\frac{m^E(t_\eta\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d t_\eta}\Big)^{1-\alpha}+\frac{c_7}{t_\eta}\\&\qquad+(1+\vartheta_m(t_\eta\kappa_{\xi,m}\lambda\rho))\frac{M_\xi}{m}+\Big(\frac{1}{t_\eta^\alpha}+\frac{1}{t_\eta}\Big)\frac{M_\xi}{m^{1-\alpha}}\Big)\\&\le\frac{N_\xi^{1-\alpha}}{t_\eta^\alpha}+\frac{c_7}{t_\eta}+\Big(\frac{1}{t_\eta^\alpha}+\frac{1}{t_\eta}\Big)\frac{M_\xi}{m^{1-\alpha}}<N_\xi^{1-\alpha}\eta^\alpha+c_7\eta+(\eta+\eta^\alpha)\frac{M_\xi}{m^{1-\alpha}},\end{aligned} \tag{7.22}$$

where in the last inequality we have used that $t_\eta>1/\eta$. Finally, combining (7.19) and (7.22), we get

$$\hat f^\infty(\xi)-\eta-\frac{M_\xi}{m}-N_\xi^{1-\alpha}\eta^\alpha+c_7\eta+(\eta+\eta^\alpha)\frac{M_\xi}{m^{1-\alpha}}\le\limsup_{\rho\to0^+}\frac{m^E(s_\rho\ell_\xi,Q^\lambda_\nu(x,\rho))}{\lambda^{d-1}\rho^d s_\rho}.$$

Letting $m\to+\infty$ and $\eta\to0^+$, we obtain (7.17).

Consider now a function $u\in BV(A;\mathbb{R}^k)$. Let $a_u(x)$, $\nu_u(x)$, and $s^\lambda_\rho(x)$ be as in (7.2) and (7.3), and set $\xi(x):=a_u(x)\otimes\nu_u(x)$. By Lemma 7.2 for $|D^cu|$-a.e. $x\in A$ we have

$$\frac{dE^c(u,\cdot)}{d|D^cu|}(x)=\lim_{\lambda\to+\infty}\limsup_{\rho\to0^+}\frac{m^E(s^\lambda_\rho(x)\ell_{\xi(x)},Q^\lambda_{\nu_u(x)}(x,\rho))}{\lambda^{d-1}\rho^d s^\lambda_\rho(x)}. \tag{7.23}$$

Recalling (7.4), in light of (7.16) and of (7.23), we then infer that

$$\frac{dE^c(u,\cdot)}{d|D^cu|}(x) = \hat f^\infty(a_u(x)\otimes\nu_u(x)),$$

which by (7.2) gives (7.13) for every $u\in BV(A;\mathbb{R}^k)$ and $B\in\mathcal{B}(A)$.

Let us assume now that $u\in GBV_\star(A;\mathbb{R}^k)$ and for every $R>0$ we consider the set A^R_u introduced in Proposition 2.5. We claim that it is enough to prove (7.13) for every $B\in\mathcal{B}(A)$ for which there exists $R>0$ such that $B\subset A^R_u$. Indeed, by Proposition 2.4 we have $A^R_{u,0}\nearrow A_{\rm reg}:=\{x\in A\colon \widetilde u(x) \text{ exists}\}$ and that $|D^cu|(A\setminus A_{\rm reg})=0$. Hence, every $B\in\mathcal{B}(A)$ can be written, up to a $|D^cu|$-negligble set, as the increasing union of Borel sets each contained in $A^R_{u,0}$, for some R. This proves the claim.

Let us fix $R>0$ and a Borel set $B\subset A^R_u$. Let $R_m>R$ be a sequence with $R_m\to+\infty$. Thanks to Lemma 5.2, we have

$$\lim_{m\to+\infty}\frac1m\sum_{i=1}^m E^c(\psi^i_{R_m}\circ u,B)=E^c(u,B). \tag{7.24}$$

Since each function $\psi^i_{R_m}\circ u$ belongs to $BV(A;\mathbb{R}^k)$ and the integral representation holds in $BV(A;\mathbb{R}^k)$, we have

$$E^c(\psi^i_{R_m}\circ u,B)=\int_B\hat f^\infty\Big(\frac{dD^c(\psi^i_{R_m}\circ u)}{d|D^c(\psi^i_{R_m}\circ u)|}\Big)\,d|D^c(\psi^i_{R_m}\circ u)| \tag{7.25}$$

for every $i\in\{1,...,m\}$. Recalling that $B\subset A^R_u\subset A^{R_m}_u$, from Proposition 2.5 and from (3.4) we obtain that

$$\frac{dD^c(\psi^i_{R_m}\circ u)}{d|D^c(\psi^i_{R_m}\circ u)|}=\frac{dD^cu}{d|D^cu|}\quad |D^cu|\text{-a.e. in } A^R_u,$$
$$|D^c(\psi^i_{R_m}\circ u)|=|D^cu|\quad \text{as measures in } A^R_u,$$

for every $i\in\{1,...,m\}$. These equalities, together with (7.24) and (7.25), give (7.13), concluding the proof. □

8. Integrands of the Γ-limits

In this section we consider a sequence $(E_n)_n$ of functionals in $\mathfrak{E}$. In the first part we assume that $(E_n)_n$ Γ-converges to some functional $E\in\mathfrak{E}_{\rm sc}$ and we characterise the bulk and surface integrands f and g of E at a point $x\in\mathbb{R}^d$ by taking first the limit as $n\to+\infty$ of the infima of suitable minimisation problems for E_n on small cubes and taking then the limit as these cubes shrink to x.

In the last part we assume that each functional E_n belongs to $\mathfrak{E}^{\alpha,\vartheta}$ and prove the converse of the previous results: if the limits mentioned above exist and are independent of x, then they define two integrands f and g such that the sequence $(E_n)_n$ Γ-converges to the functional corresponding to f and g.

We begin this analysis by showing that it is possible to obtain the function f introduced in (5.9) by means of limits of constrained minimisation problems.

Lemma 8.1. *Let $E\in\mathfrak{E}_{\rm sc}$ and let f be the function defined by (5.9). Then there exists a $\mathcal{L}^d$-negligible set $N\in\mathcal{B}(\mathbb{R}^d)$, such that for every $x\in\mathbb{R}^d\setminus N$ and for every $\xi\in\mathbb{R}^{k\times d}$ we have*

$$f(x,\xi)=\lim_{m\to+\infty}\limsup_{\rho\to0^+}\frac{m^E_{\rho c_{\xi,m}}(\ell_\xi,Q(x,\rho))}{\rho^d}, \tag{8.1}$$

where $c_{\xi,m}$ is the constant defined by (5.14) and $m^E_t(\ell_\xi,Q(x,\rho))$ is given by (5.8) with $t=c_{\xi,m}\rho$. If, in addition, there exists a function $\hat f\colon\mathbb{R}^{k\times d}\to[0,+\infty)$ such that $f(x,\xi)=\hat f(\xi)$ for $\mathcal{L}^d$-a.e. $x\in\mathbb{R}^d$ and every $\xi\in\mathbb{R}^{k\times d}$, then (8.1) holds for every $x\in\mathbb{R}^d$ and $\xi\in\mathbb{R}^{k\times d}$.

Proof. It follows immediately from (5.7) and (5.8) that for every $m \in \mathbb{N}$ and $\xi \in \mathbb{R}^{k\times d}$ we have $m^E \le m^E_{c_{\xi,m}\rho}$. Thus, we only have to prove that

$$\lim_{m\to+\infty} \limsup_{\rho\to 0^+} \frac{m^E_{\rho c_{\xi,m}}(\ell_\xi, Q(x,\rho))}{\rho^d} \le \limsup_{\rho\to 0^+} \frac{m^E(\ell_\xi, Q(x,\rho))}{\rho^d} \tag{8.2}$$

for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and for every $\xi \in \mathbb{R}^{k\times d}$. Thanks to Theorem 5.16, the equality (5.15) is satisfied. Hence, by Corollary 5.10 there exists a set $N \in \mathcal{B}(\mathbb{R}^d)$, with $\mathcal{L}^d(N) = 0$, satisfying the following property: for every $x \in \mathbb{R}^d \setminus N$, $\xi \in \mathbb{R}^{k\times d}$, $m \in \mathbb{N}$ and $\rho > 0$ small enough there exists a function $u \in BV(Q(x,\rho);\mathbb{R}^k)$ such that with $\operatorname{tr}_{Q(x,\rho)} u = \operatorname{tr}_{Q(x,\rho)} \ell_\xi$, $\|u - \ell_\xi\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \le c_{\xi,m}\rho$, and such that

$$m^E_{\rho c_{\xi,m}}(\ell_\xi, Q(x,\rho)) \le m^E(\ell_\xi, Q(x,\rho)) + \frac{C_\xi}{m}\rho^d,$$

where $C_\xi > 0$ is the constant defined by (5.26). Dividing this inequality by ρ^d and letting $\rho \to 0^+$, we conclude that

$$\limsup_{\rho\to 0^+} \frac{m^E_{\rho c_{\xi,m}}(u, Q(x,\rho))}{\rho^d} \le \limsup_{\rho\to 0^+} \frac{m^E(u, Q(x,\rho))}{\rho^d} + \frac{C_\xi}{m}.$$

Taking the limit for $m \to +\infty$, we obtain (8.2).

To conclude the proof, we note that under the additional hypothesis we have $N = \emptyset$ in Corollary 5.10. □

The next result is useful to understand the relation between the minima of problems associated with a sequence $(E_n)_n$ on an open set A' and the minimisation of the problem associated with their Γ-limit E, computed on a larger open set A''.

Lemma 8.2. *Let $(E_n)_n \subset \mathfrak{E}$, $E \in \mathfrak{E}_{\rm sc}$, and let $A', A'' \in \mathcal{A}_c(\mathbb{R}^d)$ with Lipschitz boundary and such that $A' \subset\subset A''$. Assume that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$, the sequence $E_n(\cdot, A)$ Γ-converges to $E(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then for every $w \in W^{1,1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^k)$, we have*

$$m^E(w, A'') \le \liminf_{n\to+\infty} m^{E_n}(w, A') + c_3 k^{1/2} \int_{A\setminus A'} |\nabla w|\, dx + c_4 \mathcal{L}^d(A'' \setminus A').$$

Proof. The proof can be obtained by adapting the arguments of [20, Proposition 3.1], replacing [21, Theorem 7.13] by [24, Theorem 3.22]. □

We now prove a result that allows to compare the limit of the minima of problems associated with a sequence $(E_n)_n$ on a cube $Q(x,\rho)$ with the minimum of the problem associated with the Γ-limit E, computed on the same cube.

Lemma 8.3. *Let $(E_n)_n \subset \mathfrak{E}$, $E \in \mathfrak{E}$, $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{k\times d}$, $m \in \mathbb{N}$, $\rho > 0$, and $s > d^{1/2}|\xi|\rho$. Assume that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_n(\cdot, A)$ Γ-converges to $E(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then*

$$\limsup_{n\to+\infty} m^{E_n}_t(\ell_\xi, Q(x,\rho)) \le m^E_s(\ell_\xi, Q(x,\rho)) + \frac{K_\xi}{m}\rho^d \tag{8.3}$$

where $t := 2\sigma^m(s + d^{1/2}|\xi|\rho) + d^{1/2}|\xi|\rho$ and $K_\xi > 0$ is a constant depending on ξ, but not on $(f_n)_n$, $(g_n)_n$, m, s, and ρ.

Proof. Let us fix $0 < \eta < 1$. Consider a function $u \in BV(Q(x,\rho);\mathbb{R}^k)$, with $\|u - \ell_\xi\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)} \le s$ and $\operatorname{tr}_{Q(x,\rho)} u = \operatorname{tr}_{Q(x,\rho)} \ell_\xi$, such that

$$E(u, Q(x,\rho)) \le m^E_s(\ell_\xi, Q(x,\rho)) + \eta \le (c_3 k^{1/2}|\xi| + c_4)\rho^d + \eta. \tag{8.4}$$

By Lemma 6.8 there exists a sequence of functions $(v_n)_n \subset BV(Q(x,\rho);\mathbb{R}^k)$, with $\|v_n\|_{L^\infty(Q(x,\rho))} \le 2\sigma^m(s + d^{1/2}|\xi|\rho)$, such that $v_n \to u$ in $L^1(\mathbb{R}^d;\mathbb{R}^k)$

$$\limsup_{n\to+\infty} E_n(v_n, Q(x,\rho)) \le E(u, Q(x,\rho)) + C\frac{E(u, Q(x,\rho)) + \rho^d}{m}. \tag{8.5}$$

Let us fix $0<\eta<1$ and $0<r<\rho$. For every $0<\delta\le\eta$ we apply Lemma 4.2 to the open sets $A=Q(x,r)$ and $U=Q(x,\rho)\setminus\overline{Q(x,r)}$ to obtain a sequence $(u_n)_n\subset BV(Q(x,\rho);\mathbb{R}^k)$ converging to u in $L^1(\mathbb{R}^d;\mathbb{R}^k)$, with $\mathrm{tr}_{Q(x,\rho)}u_n=\mathrm{tr}_{Q(x,\rho)}\ell_\xi$ and $\|u_n-\ell_\xi\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)}\le 2\sigma^m(s+d^{1/2}|\xi|\rho)+d^{1/2}|\xi|\rho=t$, such that

$$\limsup_{n\to+\infty}E_n(u_n,Q(x,\rho))\le(1+\delta)\limsup_{n\to+\infty}\Big(E_n(v_n,Q(x,\rho))+E_n(\ell_\xi,Q(x,\rho)\setminus\overline{Q(x,r)})\Big)+\eta$$

Exploiting (8.4) and (8.5), from this last inequality we deduce that

$$\begin{aligned}\limsup_{n\to+\infty}E_n(u_n,Q(x,\rho))&\le m_s^E(u,Q(x,\rho))+\eta+\delta\big((c_3k^{1/2}|\xi|+c_4)\rho^d+\eta\big)\\&\quad+(1+\delta)C\frac{(c_3k^{1/2}|\xi|+c_4+1)\rho^d+\eta}{m}+(1+\delta)(c_3k^{1/2}|\xi|+c_4)(\rho^d-r^d)+\eta.\end{aligned}\tag{8.6}$$

Choosing r so that $2(c_3k^{1/2}|\xi|+c_4)(\rho^d-r^d)\le\eta$ and δ such that $\delta((c_3k^{1/2}|\xi|+c_4)\rho^d+\eta)\le\eta$, recalling that $\mathrm{tr}_{Q(x,\rho)}u_n=\mathrm{tr}_{Q(x,\rho)}\ell_\xi$ and $\|u_n-\ell_\xi\|_{L^\infty(Q(x,\rho);\mathbb{R}^k)}\le t$, from (8.6) we obtain

$$\limsup_{n\to+\infty}m_t^{E_n}(\ell_\xi,Q(x,\rho))\le m_s^E(u,Q(x,\rho))+2C\frac{(c_3k^{1/2}|\xi|+c_4+1)\rho^d+\eta}{m}+4\eta.$$

We conclude the proof by letting $\eta\to0^+$. □

The next result shows that when E is the Γ-limit of a sequence of functionals $(E_n)_n$, the value of its bulk integrand f at (x,ξ) can be obtained by taking first the limit of $m^{E_n}(\ell_\xi,Q(x,\rho))/\rho^d$ as $n\to+\infty$ and then the limit as $\rho\to0^+$. For technical reasons, we need also a similar result where we replace $m^{E_n}(\ell_\xi,Q(x,\rho))$ by its constrained version $m_t^{E_n}(\ell_\xi,Q(x,\rho))$, for a suitable choice of $t>0$.

Given $m\in\mathbb{N}$ and $\xi\in\mathbb{R}^{k\times d}$, the constraint t will be given by $\rho\beta_{\xi,m}$, with

$$\beta_{\xi,m}:=\sigma^m(c_{\xi,m}+d^{1/2}|\xi|)+d^{1/2}|\xi|,\tag{8.7}$$

where $c_{\xi,m}>0$ is the constant defined by (5.14).

Proposition 8.4. *Let $(E_n)_n\subset\mathfrak{E}$, $E\in\mathfrak{E}_{\rm sc}$, and let f be the function defined by (5.9). Assume that for every $A\in\mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_n(\cdot,A)$ Γ-converges to $E(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$. Then there exist an $\mathcal{L}^d$-negligible set $N\in\mathcal{B}(\mathbb{R}^d)$ such that for every $x\in\mathbb{R}^d\setminus N$ and $\xi\in\mathbb{R}^{k\times d}$ we have*

$$f(x,\xi)=\limsup_{\rho\to0^+}\liminf_{n\to\infty}\frac{m^{E_n}(\ell_\xi,Q(x,\rho))}{\rho^d}=\limsup_{\rho\to0^+}\limsup_{n\to\infty}\frac{m^{E_n}(\ell_\xi,Q(x,\rho))}{\rho^d},\tag{8.8}$$

$$f(x,\xi)=\lim_{m\to+\infty}\limsup_{\rho\to0^+}\liminf_{n\to\infty}\frac{m^{E_n}_{\rho\beta_{\xi,m}}(\ell_\xi,Q(x,\rho))}{\rho^d}=\lim_{m\to+\infty}\limsup_{\rho\to0^+}\limsup_{n\to\infty}\frac{m^{E_n}_{\rho\beta_{\xi,m}}(\ell_\xi,Q(x,\rho))}{\rho^d},\tag{8.9}$$

where $\beta_{\xi,m}$ is defined by (8.7).

If in addition there exists a function $\hat f\colon\mathbb{R}^{k\times d}\to[0,+\infty)$ such that $f(x,\xi)=\hat f(\xi)$ for $\mathcal{L}^d$-a.e. $x\in\mathbb{R}^d$ and for every $\xi\in\mathbb{R}^{k\times d}$, then (8.8) and (8.9) hold for every $x\in\mathbb{R}^d$ and for every $\xi\in\mathbb{R}^{k\times d}$.

Proof. Let $N\in\mathcal{B}(\mathbb{R}^d)$ be the union of the $\mathcal{L}^d$-negligible sets of Corollary 5.10 and Lemma 8.1. We fix $m\in\mathbb{N}$, let $\rho>0$, and set $r:=\rho+\rho^2$. Using first Lemma 8.2, then Lemma 8.3, and finally

Corollary 5.10, for every $x \in \mathbb{R}^d \setminus N$ and $\rho > 0$ small enough we have

$$\begin{aligned} m^E(\ell_\xi; Q(x,r)) &\le \liminf_{n\to+\infty} m^{E_n}(\ell_\xi, Q(x,\rho)) + (c_3 k^{1/2}|\xi| + c_4)(r^d - \rho^d) \\ &\le \liminf_{n\to+\infty} m^{E_n}_{\rho\beta_{\xi,m}}(\ell_\xi, Q(x,\rho)) + (c_3 k^{1/2}|\xi| + c_4)(r^d - \rho^d) \\ &\le \limsup_{n\to+\infty} m^{E_n}_{\rho\beta_{\xi,m}}(\ell_\xi, Q(x,\rho)) + (c_3 k^{1/2}|\xi| + c_4)(r^d - \rho^d) \\ &\le m^E_{\rho c_{\xi,m}}(\ell_\xi, Q(x,\rho)) + (c_3 k^{1/2}|\xi| + c_4)(r^d - \rho^d) + \frac{K_\xi}{m}\rho^d \\ &\le m^E(\ell_\xi, Q(x,\rho)) + (c_3 k^{1/2}|\xi| + c_4)(r^d - \rho^d) + \frac{K_\xi + C_\xi}{m}\rho^d, \end{aligned}$$

where K_ξ is the constant of Lemma 8.3 and C_ξ is given by (5.26). We divide all terms of the previous chain of inequalities by ρ^d, and take first the limsup for $\rho \to 0^+$ and then the limit for $m \to +\infty$. By (5.9) in this way we obtain (8.9), since $(r^d - \rho^d)/\rho^d \to 0$ as $\rho \to 0^+$.

To prove (8.8), one can simply replace the expression in the second line of the previous chain of inequalities by

$$\limsup_{n\to+\infty} m^{E_n}(\ell_\xi, Q(x,\rho)) + (c_3 k^{1/2}|\xi| + c_4)(r^d - \rho^d).$$

If the additional hypothesis is satisfied, the last lines of Corollary 5.10 and of Lemma 8.8 ensure that $N = \emptyset$, which concludes the proof. □

In the next proposition we show that an equality similar to (8.8) holds also for the surface integrand g.

Proposition 8.5. *Let $(E_n)_n \subset \mathfrak{E}$, let $E \in \mathfrak{E}_{\rm sc}$, and let g be given by (5.11). Assume that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_n(\cdot, A)$ Γ-converges to $E(\cdot, A)$ with respect to the topology of $L^0(\mathbb{R}^d)$. Then for every $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ we have*

$$g(x,\zeta,\nu) = \limsup_{\rho\to0^+}\limsup_{n\to+\infty} \frac{m^{E_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} = \limsup_{\rho\to0^+}\limsup_{n\to+\infty} \frac{m^{E_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}}. \quad (8.10)$$

Proof. The proof can be obtained with the same arguments of [20, Proposition 3.3]. □

We conclude this section by stating a fundamental result for the proof of the homogenisation theorem, which will be the object of the next section. We show that for sequences of functions $(f_n)_n \subset \mathcal{F}^\alpha$ and $(g_n)_n \subset \mathcal{G}^\vartheta$, a sufficient conditions for E^{f_n,g_n} to Γ-converge on every bounded open set is that (8.9) and (8.10) hold and that the function f is independent of x.

Theorem 8.6. *Let $(E_n)_n \subset \mathfrak{E}^{\alpha,\vartheta}$. Assume that there exist $\hat f\colon \mathbb{R}^{k\times d} \to [0,+\infty)$ and $\hat g\colon \mathbb{R}^d \times \mathbb{S}^{d-1} \to [0,+\infty)$ such that*

$$\hat f(\xi) = \lim_{m\to+\infty}\limsup_{\rho\to0^+}\liminf_{n\to+\infty} \frac{m^{E_n}_{\rho\beta_{\xi,m}}(\ell_\xi, Q(x,\rho))}{\rho^d} = \lim_{m\to+\infty}\limsup_{\rho\to0^+}\limsup_{n\to+\infty} \frac{m^{E_n}_{\rho\beta_{\xi,m}}(\ell_\xi, Q(x,\rho))}{\rho^d},$$

$$\hat g(x,\zeta,\nu) = \limsup_{\rho\to0^+}\limsup_{n\to+\infty} \frac{m^{E_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}} = \limsup_{\rho\to0^+}\liminf_{n\to+\infty} \frac{m^{E_n}(u_{x,\zeta,\nu}, Q_\nu(x,\rho))}{\rho^{d-1}},$$

for every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$, where $\beta_{\xi,m}$ is given by (8.7). Then $\hat f \in \mathcal{F}^\alpha$, $\hat g \in \mathcal{G}$ and for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_n(\cdot, A)$ Γ-converges to $E^{\hat f,\hat g}$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$, where $E^{\hat f,\hat g}$ is as in Definition 3.7.

Proof. The proof follows closely the lines of [20, Theorem 5.4]. By Theorem 4.1 there exists a subsequence, not relabelled, and a functional $E \in \mathfrak{E}$ such that for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_n(\cdot, A)$ Γ-converges to E with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$ and by Proposition 6.10 $E \in \mathfrak{E}^{\alpha,\vartheta}_w$. Thanks to Theorem 5.16, the functions f and g defined by (5.9) and (5.11), respectively, satisfy (5.40) and (5.41), and $f \in \mathcal{F}^\alpha$ and $g \in \mathcal{G}$. Additionally, by Proposition 8.4 and Proposition

8.5, $f = \hat f$ and $g = \hat g$, so that (7.1) holds. Hence, by Theorem 7.1 we have that $E = E^{\hat f,\hat g}$. Since the functions $\hat f$ and $\hat g$ are independent of the chosen subsequence, for every $A \in \mathcal{A}_c(\mathbb{R}^d)$ by the Urysohn property of Γ-convergence (see [17, Proposition 8.3]) the original sequence $E_n(\cdot, A)$ Γ-converges to $E^{\hat f,\hat g}(\cdot, A)$, concluding the proof. □

9. Homogenisation of free-discontinuity functionals

We are now ready to deal with the homogenisation of functionals in $\mathfrak{E}^{\alpha,\vartheta}$. The arguments that we will make use of are based on the method devised by in [20, Section 6]. The main difference with respect to [20], is the dependence on m of the constant $\beta_{\xi,m}$ appearing in (8.6), which, in the problem of stochastic homogenisation, forces us to use the Subadditive Ergodic Theorem at m fixed and only then to pass to the limit for $m \to +\infty$.

Before introducing the stochastic setting, we state a result which shows that, in the case of homogenisation, the sufficient conditions for Γ-convergence presented in Theorem 8.6 can be rewritten in terms of cubes whose side length r tends to $+\infty$. This formulation will be more suitable for the stochastic setting.

Given $f \in \mathcal{F}^\alpha$ and $g \in \mathcal{G}^\vartheta$ for every $\varepsilon > 0$ we set $f_\varepsilon(x,\xi) := f(x/\varepsilon,\xi)$ and $g_\varepsilon(x,\zeta,\nu) := g(x/\varepsilon,\zeta,\nu)$ for every $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{k\times d}$, $\zeta \in \mathbb{R}^k$, $\nu \in \mathbb{S}^{d-1}$. We observe that $f_\varepsilon \in \mathcal{F}^\alpha$ and $g_\varepsilon \in \mathcal{G}^\vartheta$. We set $E_\varepsilon := E^{f_\varepsilon,g_\varepsilon}$ according to Definition 3.7. We recall that f^∞ and g^0 are the functions given by (3.2) and (6.3), that $f^\infty \in \mathcal{F}^\alpha$, while $g^0 \notin \mathcal{G}$, since it does not satisfy (g3).

The following theorem provides a general condition that guarantees the Γ-convergence of E_ε towards an integral functional whose integrands do not depend on x. We shall see in Remark 9.10 that the hypotheses are satisfied in the case where f and g are periodic with respect to x. The advantage of these formulation is that these hypotheses are satisfied almost surely under the standard hypotheses of stochastic homogenisation.

Theorem 9.1. *Assume that there exists a function $g_{\rm hom}\colon \mathbb{R}^k\times\mathbb{S}^{d-1}\to[0,+\infty)$ and that for every $m\in\mathbb{N}$ there exists a function $f^m_{\rm hom}\colon \mathbb{R}^{k\times d}\to[0,+\infty)$ such that*

$$f^m_{\rm hom}(\xi) = \lim_{r\to+\infty} \frac{m^{E^{f,g^0}}_{r\beta_{\xi,m}}(\ell_\xi, Q(rx,r))}{r^d} \qquad \text{for every } x\in\mathbb{R}^d,\ \xi\in\mathbb{R}^{k\times d}, \tag{9.1}$$

$$g_{\rm hom}(\zeta,\nu) = \lim_{r\to+\infty} \frac{m^{E^{f^\infty,g}}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} \qquad \text{for every } x\in\mathbb{R}^d,\ \zeta\in\mathbb{R}^k,\ \text{and } \nu\in\mathbb{S}^{d-1}. \tag{9.2}$$

Let $f_{\rm hom}\colon\mathbb{R}^{k\times d}\to[0,+\infty)$ be the function defined by

$$f_{\rm hom}(\xi) := \lim_{m\to+\infty} f^m_{\rm hom}(\xi) = \inf_{m\in\mathbb{N}} f^m_{\rm hom}(\xi) \tag{9.3}$$

for every $\xi\in\mathbb{R}^{k\times d}$. Then $f_{\rm hom}\in\mathcal{F}^\alpha$, $g_{\rm hom}\in\mathcal{G}$, $E^{f_{\rm hom},g_{\rm hom}}\in\mathfrak{E}^{\alpha,\vartheta}_w$, and for every $\varepsilon_n\to0^+$ and for every $A\in\mathcal{A}_c(\mathbb{R}^d)$ the sequence $E_{\varepsilon_n}(\cdot,A)$ Γ-converges to $E^{f_{\rm hom},g_{\rm hom}}(\cdot,A)$ with respect to the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$.

Proof. We first observe that the limit in (9.3) exists, since $m\mapsto m^{E^{f,g^0}}_{r\beta_{\xi,m}}$ is non-increasing, hence, the same property holds for $m\mapsto f^m_{\rm hom}$.

Let us fix a sequence $\varepsilon_n\to0^+$ as $n\to+\infty$ and set $E_n := E^{f_{\varepsilon_n},g_{\varepsilon_n}}$. To prove the result, it is enough to show that the hypotheses of Theorem 8.6 are satisfied by $f_{\rm hom}$ and $g_{\rm hom}$. Since the hypothesis concerning $g_{\rm hom}$ can checked by repeating verbatim the arguments of [20, Theorem 6.3], we only prove that the hypothesis concerning $f_{\rm hom}$ are satisfied.

To this aim, we fix $m\in\mathbb{N}$, $\rho>0$ and set $r_n := \rho/\varepsilon_n$. The same computations performed in [20, Lemma 6.1] show that for every $\varepsilon\in(0,1)$, $x\in\mathbb{R}^d$, $\xi\in\mathbb{R}^{k\times d}$ we have

$$\left|m^{E_\varepsilon}_{\rho\beta_{\xi,m}}(\ell_\xi,Q(x,\rho)) - \varepsilon^d m^{E^{f,g^0}}_{(\rho/\varepsilon)\beta_{\xi,m}}(\ell_\xi,Q(x/\varepsilon,\rho/\varepsilon))\right| \le K_\xi\vartheta(2\rho\beta_{\xi,m})\rho^d, \tag{9.4}$$

where $K_\xi > 0$ is a constant depending on ξ, but not on m, ρ, and x. Let us fix $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^{k\times d}$. Using twice the previous inequality with ε replaced by ε_n, by (9.1) we get that

$$\begin{aligned} f^m_{\rm hom}(\xi)\rho^d - K_\xi(2\beta_{\xi,m}\rho)\rho^d &= \lim_{n\to+\infty}(\rho/r_n)^d m^{E^{f,g^0}}_{r_n\beta_{\xi,m}}(\ell_\xi, Q(r_nx/\rho, r_n)) - K_\xi\vartheta(2\beta_{\xi,m}\rho)\rho^d \\ &\le \liminf_{n\to+\infty} m^{E_{\varepsilon_n}}_{\rho\beta_{\xi,m}}(\ell_\xi, Q(x,\rho)) \le \limsup_{n\to+\infty} m^{E_{\varepsilon_n}}_{\rho\beta_{\xi,m}}(\ell_\xi, Q(x,\rho)) \\ &\le \lim_{n\to+\infty}(\rho/r_n)^d m^{E^{f,g^0}}_{r_n\beta_{\xi,m}}(\ell_\xi, Q(r_nx/\rho, r_n)) + K_\xi\vartheta(2\beta_{\xi,m}\rho)\rho^d = f^m_{\rm hom}(\xi)\rho^d + K_\xi\vartheta(2\beta_{\xi,m}\rho)\rho^d. \end{aligned}$$

Since ϑ is continuous and $\vartheta(0) = 0$, dividing by ρ^d and taking the limsup for $\rho \to 0^+$ we obtain

$$f^m_{\rm hom}(\xi) = \lim_{\rho\to0^+}\liminf_{n\to+\infty}\frac{m^{E_n}_{\rho\beta_{\xi,m}}(\ell_\xi, Q(x,\rho))}{\rho^d} = \lim_{\rho\to0^+}\limsup_{n\to+\infty}\frac{m^{E_n}_{\rho\beta_{\xi,m}}(\ell_\xi, Q(x,\rho))}{\rho^d}.$$

Taking the limit as $m \to +\infty$ we obtain that the hypothesis for $f_{\rm hom}$ in Theorem 8.6 is satisfied, so that the proof is concluded. □

The following result shows that hypothesis (9.1) can be slightly weakend

Lemma 9.2. *Assume that for every $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{Q}^{k\times d}$, the space of $k \times d$ matrices with rational entries, the limit*

$$f^m_{\rm hom}(\xi) := \lim_{r\to+\infty}\frac{m^{E^{f,g^0}}_{r\beta_{\xi,m}}(\ell_\xi, Q(rx,r))}{r^d}$$

exists and is independent of x. Then the function $f^m_{\rm hom}$ can be extended to a continuous function, still denoted by $f^m_{\rm hom}$, defined on the whole $\mathbb{R}^{k\times d}$ and such that (9.1) is satisfied.

Proof. It is enough to repeat for every $m \in \mathbb{N}$ the arguments of [20, Lemma 6.4]. □

We now introduce the stochastic setting in which we are going to deal with the homogenisation problem.

We fix a probability space $(\Omega, \mathcal{T}, P)$ and a group $(\tau_z)_{z\in\mathbb{Z}^d}$ of P-preserving transformations on $(\Omega, \mathcal{T}, P)$; that is, a family $(\tau_z)_{z\in\mathbb{Z}^d}$ of $\mathcal{T}$-measurable bijective maps $\tau_z\colon \Omega \to \Omega$ such that

(a) for every $E \in \mathcal{T}$ and every $z \in \mathbb{Z}^d$ we have $P(\tau_z^{-1}(E)) = P(E)$;
(b) $\tau_0 = \mathrm{id}$, the identity map on Ω and for every $z, z' \in \mathbb{Z}^d$ one has $\tau_z \circ \tau'_z = \tau_{z+z}$.

A group $(\tau_z)_{z\in\mathbb{Z}^d}$ of P-preserving transformations is said to be ergodic if for every set $E \in \mathcal{T}$ with the property that $\tau_z(E) = E$ for every $z \in \mathbb{Z}^d$, has probability either 0 or 1. In analogy with [20], we introduce two classes of stochastic integrands.

Definition 9.3. $\mathcal{SF}^\alpha$ is the collection of all $\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^d\times\mathbb{R}^{k\times d})$-measurable functions $f\colon \Omega\times\mathbb{R}^d\times\mathbb{R}^{k\times d} \to [0,+\infty)$ such that for every $\omega \in \Omega$ the function $f(\omega) := f(\omega,\cdot,\cdot)$ belongs to $\mathcal{F}^\alpha$ the following stochastic periodicity holds: for every $\omega \in \Omega$, $z \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^{k\times d}$ we have

$$f(\omega, x+z, \xi) = f(\tau_z(\omega), x, \xi).$$

$\mathcal{SG}^\vartheta$ is the collection of all $\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{S}^{d-1})$-measurable functions $g\colon \Omega\times\mathbb{R}^d\times\mathbb{R}^k\times\mathbb{S}^{d-1} \to [0,+\infty)$ such that for every $\omega \in \Omega$ the function $g(\omega) := g(\omega,\cdot,\cdot,\cdot)$ belongs to $\mathcal{G}^\vartheta$, and the following stochastic periodicity holds: for every $\omega \in \Omega$, $z \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$ we have

$$g(\omega, x+z, \zeta, \nu) = g(\tau_z(\omega), x, \zeta, \nu).$$

We now give the definition of subadditive process. Before doing this, we introduce $\mathcal{R}$ the collection of rectangles defined by

$$\mathcal{R} := \{R \in \mathbb{R}^d\colon R = [a_1, b_1)\times\ldots\times[a_d, b_d), \text{ for some } a, b \in \mathbb{R}^d \text{ with } a_i < b_i \text{ for } i \in \{1,\ldots,d\}\}.$$

We also introduce $(\Omega, \widehat{\mathcal{T}}, \widehat{P})$ the completion of $(\Omega, \mathcal{T}, P)$. It is immediate to see that $(\tau_z)_z$ is a group of P-preserving transformation on $(\Omega, \widehat{\mathcal{T}}, \widehat{P})$.

Definition 9.4. A function $\mu\colon \Omega\times\mathcal{R}\to\mathbb{R}$ is said to be a covariant subadditive process with respect to $(\tau_z)_{z\in\mathbb{Z}^d}$ if the following properties are satisfied

(a) for every $R\in\mathcal{R}$ the function $\mu(\cdot,R)$ is $\widehat{\mathcal{T}}$-measurable;
(b) for every $\omega\in\Omega$, $R\in\mathcal{R}$, and $z\in\mathbb{Z}^d$ we have $\mu(\omega,R+z)=\mu(\tau_z(\omega),R)$;
(c) given $R\in\mathcal{R}$ and a finite partition $(R_i)_{i=1}^n\subset\mathcal{R}$ of R, we have

$$\mu(\omega,R)\le\sum_{i=1}^n\mu(\omega,R_i)$$

for every $\omega\in\Omega$;
(d) there exists $C>0$ such that $0\le\mu(\omega,R)\le C\mathcal{L}^d(R)$ for every $\omega\in\Omega$ and $R\in\mathcal{R}$.

We recall the Subadditive Ergodic Theorem of Ackoglu and Krengel [1, Theorem 2.7]. For the particular version here used we refer the reader to [18, Proposition 1] (see also [35]).

Theorem 9.5. *Let μ be a subadditive process with respect to the group $(\tau_z)_{z\in\mathbb{Z}^d}$. Then there exist a $\mathcal{T}$-measurable set Ω', with $P(\Omega')=1$, and a function $\varphi\colon\Omega\to[0,+\infty)$ such that*

$$\lim_{r\to+\infty}\frac{\mu(\omega,Q(rx,r))}{r^d}=\varphi(\omega)$$

for every $x\in\mathbb{R}^d$ and every $\omega\in\Omega'$. If the group $(\tau_z)_{z\in\mathbb{Z}^d}$ is also ergodic, then φ is constant P-a.e.

Let $f\in\mathcal{SF}^\alpha$ and $g\in\mathcal{SG}^\vartheta$. For every $\omega\in\Omega$, we set $f^\infty(\omega):=f(\omega)^\infty$ and $g^0(\omega):=g(\omega)^0$.

The following lemma shows that it is possible to define a subadditive process closely related to condition (9.1).

Lemma 9.6. *Let $f\in\mathcal{SF}^\alpha$, let $g\in\mathcal{SG}^\vartheta$, let $\xi\in\mathbb{R}^{k\times d}$, and let $m\in\mathbb{N}$. For every $R\in\mathcal{R}$ let $\rho(R)$ be the length of the longest of its sides. Then the function $\Phi_{\xi,m}\colon\Omega\times\mathcal{R}\to[0,+\infty)$ defined by*

$$\Phi_{\xi,m}(\omega,R):=m^{E^{f(\omega),g^0(\omega)}}_{\rho(R)\beta_{\xi,m}}(\ell_\xi,R^\circ)\tag{9.5}$$

is a covariant subadditive process.

Proof. The proof can be obtained by arguing exactly as in [20, Lemma 6.9]. □

With this lemma at hand, we are ready to show that condition (9.1) with f replaced by $f(\omega)$ and g^0 replaced by $g^0(\omega)$ is satisfied for P-a.e $\omega\in\Omega$ and for every $m\in\mathbb{N}$.

Proposition 9.7. *Let $f\in\mathcal{SF}^\alpha$ and $g\in\mathcal{SG}^\vartheta$. Then there exist a $\mathcal{T}$-measurable set Ω', with $P(\Omega')=1$, such that for every $m\in\mathbb{N}$ there exists a function $f^m_{\rm hom}\colon\Omega\times\mathbb{R}^{k\times d}\to[0,+\infty)$, with $f^m_{\rm hom}(\cdot,\xi)$ $\mathcal{T}$-measurable for every $\xi\in\mathbb{R}^{k\times d}$, such that*

$$\lim_{r\to+\infty}\frac{m^{E^{f(\omega),g^0(\omega)}}_{r\beta_{\xi,m}}(\ell_\xi,Q(rx,r))}{r^d}=f^m_{\rm hom}(\omega,\xi)$$

for every $\omega\in\Omega'$, $x\in\mathbb{R}^d$, and $\xi\in\mathbb{R}^{k\times d}$. Moreover, the function $f_{\rm hom}\colon\Omega\times\mathbb{R}^{k\times d}\to[0,+\infty)$ defined by

$$f_{\rm hom}(\omega,\xi):=\lim_{m\to+\infty}f^m_{\rm hom}(\omega,\xi)=\inf_{m\in\mathbb{N}}f^m_{\rm hom}(\omega,\xi)\tag{9.6}$$

belongs to $\mathcal{SF}^\alpha$. If, in addition, $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Ω' appropriately, we have that $f^m_{\rm hom}$ and $f_{\rm hom}$ are independent of ω.

Proof. By Lemma 9.6 for every $m\in\mathbb{N}$ and for every $\xi\in\mathbb{Q}^{k\times d}$ the function $\Phi_{\xi,m}$ defined by (9.5) is a covariant subadditive process. Hence, by the Subadditive Ergodic Theorem 9.5, there exists a $\mathcal{T}$-measurable set Ω', with $P(\Omega')=1$, and for every $m\in\mathbb{N}$ a function $f^m_{\rm hom}\colon\Omega\times\mathbb{Q}^{k\times d}\to[0,+\infty)$, with $f^m_{\rm hom}(\cdot,\xi)$ $\mathcal{T}$-measurable for every $\xi\in\mathbb{Q}^{k\times d}$, such that

$$\lim_{r\to+\infty}\frac{m^{E^{f(\omega),f^0(\omega)}}_{r\beta_{\xi,m}}(\ell_\xi,Q(rx,r))}{r^d}=f^m_{\rm hom}(\omega,\xi)\tag{9.7}$$

for every $\omega \in \Omega'$, $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{Q}^{k\times d}$. Thanks to Lemma 9.2, the function $f^m_{\rm hom}$ can be extended to a function $f^m_{\rm hom}\colon \Omega\times\mathbb{R}^{k\times d} \to [0,+\infty)$, $\mathcal{T}$-measurable with respect to ω and continuous with respect to ξ, such that (9.7) holds for every $\omega \in \Omega'$, $m \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $\xi \in \mathbb{R}^{k\times d}$. We now fix $\omega_0 \in \Omega$ and we redefine $f^m_{\rm hom}$ on $\Omega \setminus \Omega' \times \mathbb{R}^{k\times d}$, by setting $f^m_{\rm hom}(\omega,\xi) = f^m_{\rm hom}(\omega_0,\xi)$ for every $\omega \in \Omega$ and $\xi \in \mathbb{R}^{k\times d}$. Note that for every $\omega \in \Omega$ and $\xi \in \mathbb{R}^{k\times d}$, the sequence $f^m(\omega,\xi)$ is non-increasing with respect to m, which justifies (9.6) and that $f_{\rm hom}(\cdot,\xi)$ is $\mathcal{T}$-measurable. By Theorem 9.1 for every $\omega \in \Omega$ the function $f_{\rm hom}(\omega) \in \mathcal{F}^\alpha$, hence, $f_{\rm hom} \in \mathcal{SF}^\alpha$.

If, in addition, $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, then $f^m_{\rm hom}(\cdot,\xi)$ is constant P-a.e. for every m. This leads to the last sentence of the statement, concluding the proof. □

The following proposition shows that P-a.e. in Ω condition (9.2) is also satisfied.

Proposition 9.8. *Let $f \in \mathcal{SF}^\alpha$, let $g \in \mathcal{SG}^\vartheta$. Then there exists a $\mathcal{T}$-measurable set Ω', with $P(\Omega') = 1$, and a $\mathcal{T}\otimes\mathcal{B}(\mathbb{R}^k\times\mathbb{S}^{d-1})$-measurable function $\hat g\colon \Omega\times\mathbb{R}^k\times\mathbb{S}^{d-1}\to[0,+\infty)$ such that*

$$\lim_{r\to+\infty} \frac{m^{E^{f^\infty(\omega),g(\omega)}}(u_{rx,\zeta,\nu}, Q_\nu(rx,r))}{r^{d-1}} = \hat g(\omega,\zeta,\nu) \tag{9.8}$$

for every $\omega \in \Omega'$, $x \in \mathbb{R}^d$, $\zeta \in \mathbb{R}^k$, and $\nu \in \mathbb{S}^{d-1}$. If, in addition, the group $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Ω' appropriately, we have that the function g is independent of ω.

Proof. The result is proved by adapting the same arguments used in [15, Proposition 9.3, Proposition 9.4, Proposition 9.5]. □

Combining Theorem 8.6 with Propositions 9.7 and Proposition 9.8, we are finally able to obtain the desired stochastic homogenisation theorem.

Theorem 9.9. *Let $f \in \mathcal{SF}^\alpha$, let $g \in \mathcal{SG}^\vartheta$, and for every $\varepsilon > 0$ and $\omega \in \Omega$ let $E^\omega_\varepsilon := E^{f_\varepsilon(\omega),g_\varepsilon(\omega)}$, according to Definition 3.7. Let $f_{\rm hom}\colon \Omega\times\mathbb{R}^{k\times d}\to[0,+\infty)$ and $g_{\rm hom}\colon \Omega\times\mathbb{R}^k\times\mathbb{S}^{d-1}\to[0,+\infty)$ be the functions given in Propositions 9.7 and 9.8. Then for the function $f_{\rm hom}(\omega,\cdot) \in \mathcal{F}^\alpha$ and $g_{\rm hom}(\omega,\cdot,\cdot) \in \mathcal{G}$ for every $\omega \in \Omega$ and there exists $\mathcal{T}$-measurable set Ω', with $P(\Omega') = 1$, such that for every sequence $\varepsilon_n \to 0^+$, $\omega \in \Omega'$, and $A \in \mathcal{A}_c(\mathbb{R}^d)$, the sequence $E^\omega_{\varepsilon_n}(\cdot,A)$ Γ-converges to $E^{f_{\rm hom}(\omega),g_{\rm hom}(\omega)}(\cdot,A)$ in the topology of $L^0(\mathbb{R}^d;\mathbb{R}^k)$.*

If, in addition, the group $(\tau_z)_{z\in\mathbb{Z}^d}$ is ergodic, by choosing Ω' appropriately, the functions $f_{\rm hom}$ and $g_{\rm hom}$ are independent of ω.

Remark 9.10. The periodic homogenisation in the deterministic case follows immediately from Theorem 9.9 once we note that in the case where Ω consists of a single point and $\tau_z = Id$ for every $z \in \mathbb{Z}^d$, the stochastic periodicity reduces to the 1-periodicity in each variable.

Acknowledgments. This article is based on work supported by the National Research Project PRIN 2022J4FYNJ "Variational methods for stationary and evolution problems with singularities and interfaces" funded by the Italian Ministry of University and Research. The authors are members of GNAMPA of INdAM.

References

[1] M. A. Akcoglu and U. Krengel. Ergodic theorems for superadditive processes. *J. Reine Angew. Math.*, 323:53–67, 1981.

[2] G. Alberti. Rank one property for derivatives of functions with bounded variation. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(2):239–274, 1993.

[3] G. Alberti, M. Csörnyei, and D. Preiss. Structure of null sets in the plane and applications. In *European Congress of Mathematics*, pages 3–22. Eur. Math. Soc., Zürich, 2005.

[4] L. Ambrosio. A compactness theorem for a new class of functions of bounded variation. *Boll. Un. Mat. Ital. B (7)*, 3(4):857–881, 1989.

[5] L. Ambrosio. Existence theory for a new class of variational problems. *Arch. Ration. Mech. Anal.*, 111(4):291–322, 1990.

[6] L. Ambrosio. A new proof of the SBV compactness theorem. *Calc. Var. Partial Differential Equations*, 3(1):127–137, 1995.

[7] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.
[8] G. Bouchitté, A. Braides, and G. Buttazzo. Relaxation results for some free discontinuity problems. *J. Reine Angew. Math.*, 458:1–18, 1995.
[9] G. Bouchitté, I. Fonseca, and L. Mascarenhas. A global method for relaxation. *Arch. Ration. Mech. Anal.*, 145(1):51–98, 1998.
[10] B. Bourdin, G. A. Francfort, and J.-J. Marigo. *The variational approach to fracture*. Springer, New York, 2008. [reprinted from. J. Elasticity 91, 5–148 (2008)].
[11] A. Braides. *Approximation of free-discontinuity problems*, volume 1694 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998.
[12] A. Braides, A. Defranceschi, and E. Vitali. Homogenization of free discontinuity problems. *Arch. Ration. Mech. Anal.*, 135(4):297–356, 1996.
[13] F. Cagnetti, G. Dal Maso, L. Scardia, and C. I. Zeppieri. Γ-convergence of free-discontinuity problems. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 36(4):1035–1079, 2019.
[14] F. Cagnetti, G. Dal Maso, L. Scardia, and C. I. Zeppieri. Stochastic homogenisation of free-discontinuity problems. *Arch. Ration. Mech. Anal.*, 233(2):935–974, 2019.
[15] F. Cagnetti, G. Dal Maso, L. Scardia, and C. I. Zeppieri. A global method for deterministic and stochastic homogenisation in BV. *Ann. PDE*, 8(1):Paper No. 8, 89, 2022.
[16] P. Celada and G. Dal Maso. Further remarks on the lower semicontinuity of polyconvex integrals. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 11(6):661–691, 1994.
[17] G. Dal Maso. *An Introduction to Γ-convergence*. Birkhäuser, Basel, 1990.
[18] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.*, 368:28–42, 1986.
[19] G. Dal Maso and R. Toader. A new space of generalised functions with bounded variation motivated by fracture mechanics. *NoDEA Nonlinear Differential Equations Appl.*, 29(Paper No. 63.), 2022.
[20] G. Dal Maso and R. Toader. Homogenisation problems for free discontinuity functionals with bounded cohesive surface term. http://cvgmt.sns.it/paper/6125/, 2023. cvgmt preprint.
[21] G. Dal Maso and R. Toader. Gamma-convergence and integral representation for a class of free discontinuity functionals. *J. Convex Anal.*, 31:411–476, 2024.
[22] E. De Giorgi and G. Letta. Une notion générale de convergence faible pour des fonctions croissantes d'ensemble. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 4(1):61–99, 1977.
[23] G. De Philippis and F. Rindler. On the structure of $\mathcal{A}$-free measures and applications. *Ann. of Math. (2)*, 184(3):1017–1039, 2016.
[24] D. Donati. A new space of generalised vector-valued functions of bounded variation. https://doi.org/10.48550/arXiv.2310.11538, 2023.
[25] A. F. Donnarumma and M. Friedrich. Stochastic homogenisation for functionals defined on asymptotically piecewise rigid functions. https://arxiv.org/abs/2312.12082, 2023.
[26] D. S. Dugdale. Yielding of steel sheets containing slits. *Journal of The Mechanics and Physics of Solids*, 8:100–104, 1960.
[27] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions, revised edition*. CRC Press, 2015.
[28] I. Fonseca and S. Müller. Quasi-convex integrands and lower semicontinuity in L^1. *SIAM J. Math. Anal.*, 23(5):1081–1098, 1992.
[29] G. A. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids*, 46(8):1319–1342, 1998.
[30] M. Friedrich, M. Perugini, and F. Solombrino. Γ-convergence for free-discontinuity problems in linear elasticity: homogenization and relaxation. *Indiana Univ. Math. J.*, 72(5):1949–2023, 2023.
[31] N. Fusco and J. E. Hutchinson. A direct proof for lower semicontinuity of polyconvex functionals. *Manuscripta Math.*, 87(1):35–50, 1995.
[32] A. Giacomini and M. Ponsiglione. A Γ-convergence approach to stability of unilateral minimality properties in fracture mechanics and applications. *Arch. Ration. Mech. Anal.*, 180(3):399–447, 2006.
[33] E. Giusti and G. H. Williams. *Minimal surfaces and functions of bounded variation*, volume 80. Springer, 1984.
[34] A. A. Griffith. The phenomena of rupture and flow in solids. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 221:163–198, 1921.
[35] C. Licht and G. Michaille. Global-local subadditive ergodic theorems and application to homogenization in elasticity. *Ann. Math. Blaise Pascal*, 9(1):21–62, 2002.
[36] A. Mainik and A. Mielke. Global existence for rate-independent gradient plasticity at finite strain. *J. Nonlinear Sci.*, 19(3):221–248, 2009.
[37] A. Massaccesi and D. Vittone. An elementary proof of the rank-one theorem for BV functions. *J. Eur. Math. Soc. (JEMS)*, 21(10):3255–3258, 2019.
[38] A. Mielke and T. Roubíček. Rate-independent systems. *Applied Mathematical Sciences*, 193, 2015.

[39] A. Mielke and T. s. Roubíček. Numerical approaches to rate-independent processes and applications in inelasticity. *M2AN Math. Model. Numer. Anal.*, 43(3):399–428, 2009.

[40] A. Mielke, T. s. Roubíček, and U. Stefanelli. Γ-limits and relaxations for rate-independent evolutionary problems. *Calc. Var. Partial Differential Equations*, 31(3):387–416, 2008.

[41] A. Mielke and A. M. Timofte. Two-scale homogenization for evolutionary variational inequalities via the energetic formulation. *SIAM J. Math. Anal.*, 39(2):642–668, 2007.

SISSA, Via Bonomea 265, 34136 Trieste