ON THE PURE JUMP NATURE OF CRACK GROWTH FOR A CLASS OF PRESSURE-SENSITIVE ELASTO-PLASTIC MATERIALS

GIANNI DAL MASO AND RODICA TOADER

Abstract. In the framework of a model for the quasistatic crack growth in pressure-sensitive elasto-plastic materials in the planar case, we study the properties of the length $\ell(t)$ of the crack as a function of time. We prove that, under suitable technical assumptions on the crack path, the monotone function ℓ is a pure jump function.

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1. Introduction

After the results obtained in our previous paper [6] in a simplified situation, our aim in this paper is to give a further contribution to the mathematical derivation of the properties of the quasistatic crack growth in elastoplastic materials under more general hypotheses.

The investigation on this topic has a long history (see, e.g., [14, 15, 11]). In the quasistatic regime it can be studied within the framework of the variational approach to rate-independent systems, for which we refer to [12] and [13]. Using this approach, a model in the case of pressure-sensitive plastic materials in the small strain regime with no hardening (perfect plasticity) was developed in [5] in the two-dimensional case.

In this paper we consider the same problem with an additional constraint on the crack path and prove the pure jump nature of the crack growth in this case. More precisely, if $\Gamma(t)$ denotes the crack at time t in the reference configuration, we assume that $\Gamma(t)$ is connected and contained in a prescribed compact set K, composed of the union of a finite number of C^1 curves. We prove that, if $\ell(t)$ is the one-dimensional measure of $\Gamma(t)$, then $t \mapsto \ell(t)$ is a pure jump function, with a possibly infinite number of jumps.

This kind of behaviour has been noticed in several experiments, mainly in a dynamical regime, see, e.g., [7]. Its main consequence is the zig-zag shape of the crack lips at a microscopic scale. Moreover, recent numerical simulations (see [2]) for similar quasistatic problems show the intermittent character of crack growth also in this case.

In [6] we studied a simplified version of the crack growth problem in linearly elastic-perfectly plastic materials, assuming that both the crack and the plastic slips are concentrated on a prescribed straight line. Our analysis was limited to the antiplane case and the results were obtained using the maximum principle in the complement of this straight line. In this simplified situation we were also able to prove that the number of jumps of $t \mapsto \ell(t)$ is finite. We refer to [4] for a numerical simulation based on this model which highlights the dependence of the number of jumps on the parameters of the model.

For the problem considered in the present paper it is not possible to use the maximum principle, since the displacement function is vector valued and the plastic strain is not confined to a prescribed lower dimensional set. Instead we use completely different tools. An important one is an estimate of the jump of the displacement across a sufficiently regular curve in terms of the linearised strain on the complement of this curve, see Theorem 3.3. Another one is a suitable time rescaling of the plastic part of the strain obtained in Section 4. This produces a time dependent matrix valued Radon measure which is Lipschitz continuous

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in time and this property is crucial to prove the smallness of some terms depending on the plastic strains on small balls around the time dependent crack tips (see (5.15) and (5.17) in a special case, and (7.37) and (7.41) in the general case).

All these new difficulties are already present when the crack $\Gamma(t)$ can grow only on a prescribed segment, while the plastic part has no constraint. The proof of this special case is given in Section 5. The general case introduces additional difficulties, due to the fact that K has several branches and $\Gamma(t)$ can grow simultaneously on each of them.

2. The model and the main result

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary and let $T > 0$. The reference configuration of our problem in the time interval $[0, T]$ is the closure $\overline{\Omega}$ of Ω.

At time $t \in [0, T]$ the crack (in the reference configuration) is a compact connected set $\Gamma(t) \subset \overline{\Omega}$. Since the crack is irreversible we assume that the function $t \mapsto \Gamma(t)$ is increasing, i.e.,

$$\Gamma(t_1) \subset \Gamma(t_2) \quad \text{for every } t_1, t_2 \in [0, T] \text{ with } t_1 < t_2 . \tag{2.1}$$

According to Griffith's theory, the energy dissipated by the process of crack growth in the time interval $[t_1, t_2]$ is given by

$$\beta \mathcal{H}^1(\Gamma(t_2) \setminus \Gamma(t_1)) , \tag{2.2}$$

where $\beta > 0$ is the toughness of the material and $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure.

The elasto-plastic part of the body is given by

$$\hat{\Omega}_t := \overline{\Omega} \setminus \Gamma(t) , \tag{2.3}$$

whose interior is

$$\Omega_t := \Omega \setminus \Gamma(t) . \tag{2.4}$$

We shall prescribe a Dirichlet boundary condition on

$$\partial_D \Omega_t := \partial\Omega \setminus \Gamma(t) . \tag{2.5}$$

The displacement $u(t)$ is a function defined on Ω_t with values in $\mathbb{R}^2$, the elastic part $e(t)$ of the strain is a function defined in Ω_t with values in the space $\mathbb{R}^{2\times 2}_{sym}$ of symmetric 2×2 matrices, while the plastic part $p(t)$ of the strain is a measure on $\hat{\Omega}_t$ with values in the space $\mathbb{R}^{2\times 2}_{sym}$. The possible singular part of the measure $p(t)$ accounts for concentrated strains, which may occur in Ω_t and also on $\partial_D \Omega_t$.

To describe the functional setting of our problem we recall that for every open set $U \subset \mathbb{R}^2$ the space $BD(U)$ of functions of bounded deformation is defined as the space of functions $u \in L^1(U; \mathbb{R}^2)$ such that the symmetric part of the gradient $Eu := \frac{1}{2}(Du + (Du)^T)$ is a matrix-valued bounded Radon measure (see [17]). For every locally compact subset X of $\mathbb{R}^2$ and for every finite-dimensional Hilbert space Ξ, the space of Ξ-valued bounded Radon measures on X is denoted by $\mathcal{M}_b(X; \Xi)$.

For every $t \in [0, T]$ we assume that the displacement $u(t)$, the elastic strain $e(t)$, and the plastic strain $p(t)$ satisfy $u(t) \in BD(\Omega_t)$, $e(t) \in L^2(\Omega_t; \mathbb{R}^{2\times 2}_{sym})$, and $p(t) \in \mathcal{M}_b(\hat{\Omega}_t; \mathbb{R}^{2\times 2}_{sym})$. We also assume that the strain $Eu(t)$ satisfies the *additive decomposition* in Ω_t:

$$Eu(t) = e(t) + p(t) \quad \text{as measures in } \Omega_t . \tag{2.6}$$

To prescribe the Dirichlet boundary condition, for every $t \in [0, T]$ we fix a function $w(t) \in H^1(\Omega_t; \mathbb{R}^2)$. We would like to impose the equality $u(t) = w(t)$ in the sense of traces on $\partial_D \Omega_t$, but in general this cannot be achieved because of the presence of possible plastic slips at the boundary. For this reason the Dirichlet condition has to be relaxed requiring only that

$$p(t) = (w(t) - u(t)) \odot \nu_\Omega \mathcal{H}^1 \quad \text{as measures in } \partial_D \Omega_t , \tag{2.7}$$

where ν_Ω is the outer unit normal to $\partial\Omega$ and $a \odot b$ is the symmetrized tensor product between two vectors $a, b \in \mathbb{R}^2$, i.e., the symmetric matrix with entries $(a_i b_j + a_j b_i)/2$.

We now introduce a *constraint* on the possible cracks $\Gamma(t)$. We fix two (possibly empty) compact sets K_0 and K, with $K_0 \subset K \subset \overline{\Omega}$ and we impose that

$$K_0 \subset \Gamma(t) \subset K \quad \text{for every } t \in [0,T]. \tag{2.8}$$

We assume that K can be written as

$$K = \hat{K} \cup K_1 \cup \dots \cup K_k, \tag{2.9}$$

where $\hat{K}, K_1, \dots, K_k$ are compact sets, $\mathcal{H}^1(\hat{K}) = 0$, and, for $h = 1, \dots, k$, K_h is a simple C^1-curve with endpoints $x_h^1, x_h^2 \in \hat{K}$. We set

$$K_h^0 := K_h \setminus \{x_h^1, x_h^2\} \quad \text{and} \quad \check{K}_h := \hat{K} \cup \bigcup_{\substack{h'=1 \\ h' \neq h}}^{k} K_{h'}, \tag{2.10}$$

and we assume that for every $h = 1, \dots, k$ we have

$$K_h^0 \cap \check{K}_h = \emptyset, \tag{2.11}$$

$$\text{either } K_h^0 \subset \Omega \text{ or } K_k^0 \subset \partial\Omega. \tag{2.12}$$

Moreover, we assume that

$$\Omega \setminus K \text{ is the union of a finite number of open sets with Lipschitz boundary.} \tag{2.13}$$

The collection of all compact connected subsets of K containing K_0 is denoted by $\mathcal{K}$. We recall that (2.8) and the topological properties of $\Gamma(t)$ imply that $\Gamma(t) \in \mathcal{K}$ for every $t \in [0,T]$.

The stress $\sigma(t)$ at time $t \in [0,T]$ belongs to $L^2(\Omega_t; \mathbb{R}^{2\times2}_{sym})$ and depends on the elastic strain $e(t)$ through the linear relation

$$\sigma(t) := \mathbb{C}e(t),$$

where $\mathbb{C}$ is the *elasticity tensor*. As usual we assume that $\mathbb{C} \colon \mathbb{R}^{2\times2}_{sym} \to \mathbb{R}^{2\times2}_{sym}$ is a symmetric linear operator satisfying

$$\lambda_{\mathbb{C}} |A|^2 \le \mathbb{C}A : A \le \Lambda_{\mathbb{C}} |A|^2 \quad \text{for every } A \in \mathbb{R}^{2\times2}_{sym}, \tag{2.14}$$

for some constants $0 < \lambda_{\mathbb{C}} \le \Lambda_{\mathbb{C}}$. Here and henceforth $|A|$ denotes the Euclidean norm of A, while the colon denotes the corresponding scalar product between matrices, i.e., $A : B := \mathrm{tr}(AB^T) = \sum_{ij} A_{ij} B_{ij}$. These inequalities imply that

$$|\mathbb{C}A| \le \Lambda_{\mathbb{C}} |A| \quad \text{for every } A \in \mathbb{R}^{2\times2}_{sym}. \tag{2.15}$$

The *stored elastic energy* at time t depends only on the elastic strain $e(t)$ and is given by

$$\frac{1}{2} \int_{\Omega_t} \sigma(t) : e(t)\, dx = \int_{\Omega_t} Q(e(t))\, dx,$$

where

$$Q(A) := \tfrac{1}{2} \mathbb{C}A : A \quad \text{for every } A \in \mathbb{R}^{2\times2}_{sym}.$$

Since for every $t \in [0,T]$ the behaviour of the material in Ω_t is linearly elastic - perfectly plastic, we have a constraint on the stress of the form

$$\sigma(t,x) \in \mathbb{K} \quad \text{for a.e. } x \in \Omega_t, \tag{2.16}$$

where $\mathbb{K}$ is a prescribed closed and convex set in $\mathbb{R}^{2\times2}_{sym}$, depending on the material, whose boundary plays the role of the *yield surface*. We assume that there exist two constants $c_{\mathbb{K}}$ and $C_{\mathbb{K}}$, with $0 < c_{\mathbb{K}} \le C_{\mathbb{K}} < +\infty$, such that

$$\{A \in \mathbb{R}^{2\times2}_{sym} : |A| \le c_{\mathbb{K}}\} \subset \mathbb{K} \subset \{A \in \mathbb{R}^{2\times2}_{sym} : |A| \le C_{\mathbb{K}}\}. \tag{2.17}$$

Note that the second inclusion in (2.17) excludes the standard case where the constraint is imposed only on the deviatoric (i.e., stress free) part of the stress. Indeed, in our model the yield surface depends also the pressure component of the stress. For the study of the quasistatic evolution in the linearized theory of pressure-sensitive plastic materials without cracks we refer to [3] and the references therein.

The *support function* $H\colon \mathbb{R}^{2\times 2}_{sym} \to [0, +\infty[$ of $\mathbb{K}$ is defined by

$$H(A) := \sup_{B \in \mathbb{K}} B : A \tag{2.18}$$

and it turns out to be convex and positively homogeneous of degree one. By the previous bounds on $\mathbb{K}$ it follows that

$$c_{\mathbb{K}}|A| \le H(A) \le C_{\mathbb{K}}|A| \quad \text{for every } A \in \mathbb{R}^{2\times 2}_{sym}\,. \tag{2.19}$$

To obtain a convenient expression for the plastic dissipation, for every locally compact set $X \subset \mathbb{R}^2$ and every $\mu \in \mathcal{M}_b(X; \mathbb{R}^{2\times 2}_{sym})$ we introduce the measure $H(\mu) \in \mathcal{M}_b(X; \mathbb{R})$ defined by

$$H(\mu)(B) := \int_B H\big(\tfrac{d\mu}{d|\mu|}\big)d|\mu| \quad \text{for every Borel set } B \subset X\,, \tag{2.20}$$

where $|\mu|$ is the variation of μ and $\frac{d\mu}{d|\mu|}$ is the Radon-Nikodym derivative of μ with respect to $|\mu|$. Clearly for every Borel set $B \subset X$ the function $\mu \mapsto H(\mu)(B)$ is convex and positively homogeneous of degree one, therefore it satisfies the triangle inequality. Moreover, the map $\mu \mapsto H(\mu)$ is continuous for the strong topologies. Finally, by [9], for every open set $U \subset \mathbb{R}^2$ the function $\mu \mapsto H(\mu)(U \cap X)$ is lower semicontinuous with respect to the weak*-convergence in $\mathcal{M}_b(X; \mathbb{R}^{2\times 2}_{sym})$, which is defined by identifying $\mathcal{M}_b(X; \mathbb{R}^{2\times 2}_{sym})$ with the dual of the Banach space

$$C^0_0(X; \mathbb{R}^{2\times 2}_{sym}) := \{\varphi \in C^0(X; \mathbb{R}^{2\times 2}_{sym}) : \{x \in X : |\varphi(x)| \ge \varepsilon\} \text{ is compact for every } \varepsilon > 0\}\,,$$

endowed with the supremum norm. By (2.19) we obtain

$$c_{\mathbb{K}}|\mu|(B) \le H(\mu)(B) \le C_{\mathbb{K}}|\mu|(B) \quad \text{for every Borel set } B \subset X\,. \tag{2.21}$$

In our model the energy dissipated in a time interval depends on the evolution of the pair $(p(t), \Gamma(t))$ composed of the plastic strain and the crack. According to the terminology used in the theory of rate-independent systems (see [12, Section 7]), the *dissipation distance* between two pairs (p_2, Γ_2) and (p_1, Γ_1), with Γ_i compact connected subsets of $\overline{\Omega}$ and $p_i \in \mathcal{M}_b(\overline{\Omega} \setminus \Gamma_i; \mathbb{R}^{2\times 2}_{sym})$, is given by

$$d((p_2, \Gamma_2), (p_1, \Gamma_1)) := \begin{cases} H(p_2 - p_1)(\overline{\Omega} \setminus \Gamma_2) + \beta\mathcal{H}^1(\Gamma_2 \setminus \Gamma_1) & \text{if } \Gamma_1 \subset \Gamma_2\,, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.22}$$

where $H(p_2 - p_1)(\overline{\Omega} \setminus \Gamma_2)$ accounts for the plastic dissipation distance (see [13]) and $\beta\mathcal{H}^1(\Gamma_2 \setminus \Gamma_1)$ is the energy dissipated to produce the crack increment $\Gamma_2 \setminus \Gamma_1$ (see (2.2)).

Given a pair of functions (p, Γ), with $\Gamma\colon [0, T] \to \mathcal{K}$ increasing and $p(t) \in \mathcal{M}_b(\hat{\Omega}_t; \mathbb{R}^{2\times 2}_{sym})$ for every $t \in [0, T]$, the corresponding *total dissipation* on the time interval $[\tau_1, \tau_2] \subset [0, T]$ is given by

$$\mathcal{D}(p, \Gamma; \tau_1, \tau_2) := \sup \sum_{i=1}^{m} d((p(t_i), \Gamma(t_i)), (p(t_{i-1}), \Gamma(t_{i-1}))\,, \tag{2.23}$$

where the supremum is taken over all finite partitions $\tau_1 = t_0 \le t_1 \le \dots \le t_{m-1} \le t_m = \tau_2$. It is clear that, if $\tau_1 < \tau_2$, then nothing changes if the supremum is taken over all finite partitions $\tau_1 = t_0 < t_1 < \dots < t_{m-1} < t_m = \tau_2$.

It is convenient to write

$$\mathcal{D}(p, \Gamma; \tau_1, \tau_2) = \mathcal{D}_\Gamma(p; \tau_1, \tau_2) + \beta\mathcal{H}^1(\Gamma(\tau_2) \setminus \Gamma(\tau_1))\,, \tag{2.24}$$

where $\mathcal{D}_\Gamma(p;\tau_1,\tau_2)$ is the *plastic dissipation* (for a given Γ) on the time interval $[\tau_1,\tau_2]$, whose definition must take into account the dependence on t of the domain $\hat\Omega_t$ where $p(t)$ is defined, which, in turn, depends on $\Gamma(t)$. Clearly (2.23) and (2.24) imply that $\mathcal{D}_\Gamma(p;\tau_1,\tau_2)$ is given by

$$\mathcal{D}_\Gamma(p;\tau_1,\tau_2) := \sup\sum_{i=1}^m H(p(t_i)-p(t_{i-1}))(\hat\Omega_{t_i})\,, \tag{2.25}$$

where the supremum is taken over all finite partitions $\tau_1 = t_0 \le t_1 \le \cdots \le t_{m-1} \le t_m = \tau_2$.

For a concise presentation of the *global stability condition* of rate-independent systems it is convenient to introduce the following definition of *admissible triples*: given a compact connected set $\hat\Gamma \subset \overline\Omega$ and a function $\hat w \in H^1(\Omega\setminus\hat\Gamma;\mathbb{R}^2)$, the set $\mathcal{A}(\hat w,\hat\Gamma)$ of admissible triples is composed of all $(\hat u,\hat e,\hat p)$, with $\hat u \in BD(\Omega\setminus\hat\Gamma)$, $\hat e \in L^2(\Omega\setminus\hat\Gamma;\mathbb{R}^{2\times2}_{sym})$, and $\hat p \in \mathcal{M}_b(\overline\Omega\setminus\hat\Gamma;\mathbb{R}^{2\times2}_{sym})$, which satisfy the *weak kinematic admissibility conditions*

$$E\hat u = \hat e + \hat p \quad \text{as measures in } \Omega\setminus\hat\Gamma\,, \tag{2.26}$$

$$\hat p = (\hat w - \hat u)\odot\nu_\Omega\mathcal{H}^1 \quad \text{as measures in } \partial\Omega\setminus\hat\Gamma\,. \tag{2.27}$$

As for the function w used to prescribe the relaxed Dirichlet boundary condition (2.7), we assume that

$$w \in AC([0,T];H^1(\Omega\setminus K_0;\mathbb{R}^2))\,, \tag{2.28}$$

so that the time derivative $t\mapsto\dot w(t)$ belongs to $L^1([0,T];H^1(\Omega\setminus K_0;\mathbb{R}^2))$ and its spatial symmetric gradient $t\mapsto E\dot w(t)$ belongs to $L^1([0,T];L^2(\Omega\setminus K_0;\mathbb{R}^{2\times2}_{sym}))$.) d According to the energetic formulation of rate-independent processes the function $t\mapsto(u(t),e(t),p(t),\Gamma(t))$, satisfies the following properties

(a) *crack irreversibility*: the function $\Gamma\colon[0,T]\to\mathcal{K}$ is increasing, i.e., (2.1) holds;

(b) *global stability condition*: for every $t\in[0,T]$ we have $(u(t),e(t),p(t))\in\mathcal{A}(w(t),\Gamma(t))$ and

$$\int_{\Omega\setminus\Gamma(t)} Q(e(t))\,dx \le \int_{\Omega\setminus\hat\Gamma} Q(\hat e)\,dx + H(\hat p - p(t))(\overline\Omega\setminus\hat\Gamma) + \beta\mathcal{H}^1(\hat\Gamma\setminus\Gamma(t)) \tag{2.29}$$

for every $\hat\Gamma\in\mathcal{K}$, with $\hat\Gamma\supset\Gamma(t)$, and every $(\hat u,\hat e,\hat p)\in\mathcal{A}(w(t),\hat\Gamma)$;

(c) *energy-dissipation balance*: the function $t\mapsto e(t)$ belongs to $L^\infty([0,T];L^2(\Omega;\mathbb{R}^{2\times2}_{sym}))$ and for every $t\in[0,T]$ we have

$$\begin{aligned}&\int_{\Omega_t} Q(e(t))\,dx + \mathcal{D}_\Gamma(p;0,t) + \beta\mathcal{H}^1(\Gamma(t)\setminus\Gamma(0))\\ &= \int_{\Omega_0} Q(e(0))\,dx + \int_0^t\Big(\int_{\Omega_\tau}\sigma(\tau):E\dot w(\tau)\,dx\Big)\,d\tau\,.\end{aligned} \tag{2.30}$$

The last term in (2.30) represents the energy put into the system by the prescribed boundary displacement in the time interval $[0,t]$. Note that by Lemma 3.1 condition (b) implies (2.16).

Therefore conditions (a), (b), and (c) include all properties of our model. This leads to the following definition.

Definition 2.1. Let $w\in AC([0,T];H^1(\Omega\setminus K_0;\mathbb{R}^2))$. A quasistatic evolution for the crack growth problem in elasto-plastic materials (briefly *quasistatic evolution*) with boundary value w is a function $t\mapsto(u(t),e(t),p(t),\Gamma(t))$, with $u(t)\in BD(\Omega_t)$, $e(t)\in L^2(\Omega_t;\mathbb{R}^{2\times2}_{sym})$, $p(t)\in\mathcal{M}_b(\hat\Omega_t;\mathbb{R}^{2\times2}_{sym})$, and $\Gamma(t)\in\mathcal{K}$ for every $t\in[0,T]$, such that (a), (b), and (c) hold.

We are interested in the study of quasistatic evolutions with prescribed *initial data* (u_0,e_0,p_0,Γ_0), with $\Gamma_0\in\mathcal{K}$. By condition (b) (global stability) we must assume that $(u_0,e_0,p_0)\in\mathcal{A}(w(0),\Gamma_0)$ and

$$\int_{\Omega\setminus\Gamma_0} Q(e_0)\,dx \le \int_{\Omega\setminus\hat\Gamma} Q(\hat e)\,dx + H(\hat p - p_0)(\overline\Omega\setminus\hat\Gamma) + \beta\mathcal{H}^1(\hat\Gamma\setminus\Gamma_0)\,, \tag{2.31}$$

for every $\hat{\Gamma} \in \mathcal{K}$, with $\hat{\Gamma} \supset \Gamma_0$, and for every $(\hat{u}, \hat{e}, \hat{p}) \in \mathcal{A}(w(0), \hat{\Gamma})$.

The following existence theorem can be proved as in [5, Theorem 2.5], with $\mathcal{M}_b(\bar{\Omega}; \mathbb{R}^{2\times 2}_{sym})$ replaced by $\mathcal{M}_b(\bar{\Omega}; \mathbb{R}^{2\times 2}_{sym})$. Under our hypotheses there are obvious simplifications due to the constraint (2.8). In particular the approximation result [5, Lemma 4.4] can be obtained in a straightforward way, without using the crack transfer result [5, Theorem 4.2].

Theorem 2.2. *Let $w \in AC([0,T]; H^1(\Omega \setminus K_0; \mathbb{R}^2))$ and $(u_0, e_0, p_0) \in \mathcal{A}(w(0), \Gamma_0)$, with $\Gamma_0 \in \mathcal{K}$. Assume that* (2.31) *holds. Then there exists a quasistatic evolution with boundary value w and initial conditions $(u(0), e(0), p(0), \Gamma(0)) = (u_0, e_0, p_0, \Gamma_0)$.*

For every nondecreasing function $\alpha\colon [0,T] \to \mathbb{R}$ we set

$$\alpha(t-) := \lim_{\tau \to t-} \alpha(\tau) \quad \text{for every } t \in (0,T]\,, \tag{2.32}$$

$$\alpha(t+) := \lim_{\tau \to t+} \alpha(\tau) \quad \text{for every } t \in [0,T)\,. \tag{2.33}$$

We also set $\alpha(0-) := \alpha(0)$ and $\alpha(T+) := \alpha(T)$. Let J_α be the (at most countable) set of jump points of α. We can write $\alpha = \alpha^{cont} + \alpha^{jump}$, where α^{cont} is continuous and α^{jump} is the pure jump component of α, defined by

$$\alpha^{jump}(t) = \alpha(t) - \alpha(t-) + \sum_{\tau \in J_\alpha, \tau < t} [\alpha](\tau)\,, \quad \text{for every } t \in [0,T]\,.$$

The following theorem is the main result of the paper.

Theorem 2.3. *Let $w \in AC([0,T]; H^1(\Omega \setminus K_0))$, let $t \mapsto (u(t), e(t), p(t), \Gamma(t))$ be a quasistatic evolution with boundary value w, according to Definition 2.1, and let*

$$\ell(t) := \mathcal{H}^1(\Gamma(t))\,.$$

Then ℓ^{cont} is constant on the interval $[0,T]$.

The complete proof of this theorem will be given in Section 7. A simplified proof, which illustrates the main ideas, is presented in Section 5 when K is a segment and K_0 contains one of the endpoints of K.

3. The Euler condition and some useful estimates

In this section we establish some properties of quasistatic evolutions which follow from the global stability property (2.29). Moreover, we prove some estimates that will be crucial in the proof of our main result in Sections 5 and 7.

By the structural assumption on K there exists a continuous function $\nu\colon K \setminus \hat{K} \to \mathbb{R}^2$ such that for every $x \in K \setminus \hat{K}$ the vector $\nu(x)$ is normal to K and has norm 1. It is not restrictive to assume that $\nu(x) = \nu_\Omega(x)$ for every $x \in (K \setminus \hat{K}) \cap \partial\Omega$. Moreover, if $t \in [0,T]$ and $u \in BD(\Omega_t)$, then the trace of u on $\partial\Omega \setminus \hat{K}$, still denoted by u, is well-defined and belongs to $L^1(\partial\Omega \setminus \hat{K}; \mathbb{R}^2)$, while the traces u^+ and u^- of u from both sides of $\Gamma(t) \cap \Omega \setminus \hat{K}$, given at each point by the limits on half-balls determined by ν, are also well-defined and belong to $L^1(\Gamma(t) \cap \Omega; \mathbb{R}^2)$. The jump $[u]$ of u is defined by $[u] := u^+ - u^-$ $\mathcal{H}^1$-a.e. on $\Gamma(t) \cap \Omega$.

The Euler condition associated to the global stability property (2.29) is stated in the following lemma. Given a sufficiently regular function φ defined on a subset of $\mathbb{R}^2$, with values in $\mathbb{R}^{2\times 2}$, the divergence $\mathrm{div}\varphi$ of φ is the vector whose components are the divergence of the rows of φ. Moreover, for $0 \le t_1 \le t_2 \le T$ we set $\Gamma(t_1, t_2) := \Gamma(t_2) \setminus \Gamma(t_1)$.

Lemma 3.1. *Let $\Gamma\colon [0,T] \to \mathcal{K}$ be an increasing function, let $t_1 \in [0,T]$, let $w_1 \in H^1(\Omega_{t_1}; \mathbb{R}^2)$, let $(u_1, e_1, p_1) \in \mathcal{A}(w_1, \Gamma(t_1))$, and let $\sigma_1 := \mathbb{C}e_1$. Assume that*

$$\int_{\Omega_{t_1}} Q(e_1)\,dx \le \int_{\Omega_{t_1}} Q(e)\,dx + H(p - p_1)(\hat{\Omega}_{t_1}) \tag{3.1}$$

for every $(u,e,p)\in\mathcal{A}(w_1,\Gamma(t_1))$. *Then* $\sigma_1(x)\in\mathbb{K}$ *for a.e.* $x\in\Omega_{t_1}$ *and* $\mathrm{div}\sigma_1=0$ *in* Ω_{t_1}. *Moreover, for every* $t_2\in[t_1,T]$ *we have*

$$-\int_{\Omega_{t_1}}\sigma_1:\eta\,dx\le H(q)(\hat\Omega_{t_2})+\int_{\Gamma(t_1,t_2)\cap\Omega}H([v]\odot\nu)\,d\mathcal{H}^1+\int_{\Gamma(t_1,t_2)\cap\partial\Omega}H(-v\odot\nu_\Omega)\,d\mathcal{H}^1 \tag{3.2}$$

for every $(v,\eta,q)\in\mathcal{A}(0,\Gamma(t_2))$.

Proof. Let us fix $t_2\in[t_1,T]$ and $(v,\eta,q)\in\mathcal{A}(0,\Gamma(t_2))$. For every $\varphi\in C_c^\infty(\Omega_{t_1};\mathbb{R}^{2\times2}_{sym})$, integrating by parts on Ω_{t_2} (see, for instance, [1, formula (3.4)]), and taking into account the traces on $\Gamma(t_1,t_2)$ we obtain

$$-\int_{\Omega_{t_2}}v\,\mathrm{div}\varphi\,dx=\int_{\Omega_{t_2}}\varphi\,dEv+\int_{\Gamma(t_1,t_2)\cap\Omega}([v]\odot\nu):\varphi\,d\mathcal{H}^1-\int_{\Gamma(t_1,t_2)\cap\partial\Omega}(v\odot\nu_\Omega):\varphi d\mathcal{H}^1\,.$$

This implies that v, regarded as a function defined a.e. in Ω_{t_1}, belongs to $BD(\Omega_{t_1})$ and that its symmetrised gradient coincides with $[v]\odot\nu\mathcal{H}^1\llcorner\Gamma(t_1,t_2)$ on $\Gamma(t_1,t_2)\cap\Omega$. Let $\hat q\in\mathcal{M}_b(\hat\Omega_{t_1};\mathbb{R}^{2\times2}_{sym})$ be the measure defined by

$$\hat q(B):=q(B\cap\hat\Omega_{t_2})+\int_{B\cap\Gamma(t_1,t_2)\cap\Omega}[v]\odot\nu\,d\mathcal{H}^1-\int_{B\cap\Gamma(t_1,t_2)\cap\partial\Omega}v\odot\nu_\Omega\,d\mathcal{H}^1$$

for every Borel set $B\subset\hat\Omega_{t_1}$. By the previous remark we have $Ev=\eta+\hat q$ on Ω_{t_1} and $\hat q=-v\odot\nu_\Omega\mathcal{H}^1$ as measures on $\partial_D\Omega_{t_1}$, hence $(v,\eta,\hat q)\in\mathcal{A}(0,\Gamma(t_1))$, according to (2.26) and (2.27).

Given $\varepsilon>0$ we set $u=u_1+\varepsilon v$, $e=e_1+\varepsilon\eta$, and $p=p_1+\varepsilon\hat q$. Since $(u,e,p)\in\mathcal{A}(w_1,\Gamma(t_1))$, by (3.1) we obtain

$$\int_{\Omega_{t_1}}Q(e_1)\,dx-\int_{\Omega_{t_1}}Q(e_1+\varepsilon\eta)\,dx\le\varepsilon H(\hat q)(\hat\Omega_{t_1})\,.$$

Dividing by ε and taking the limit as $\varepsilon\to0+$ we obtain

$$-\int_{\Omega_{t_1}}\sigma_1:\eta\,dx\le H(\hat q)(\hat\Omega_{t_1})=H(q)(\hat\Omega_{t_2})+\int_{\Gamma(t_1,t_2)\cap\Omega}H([v]\odot\nu)\,d\mathcal{H}^1+\int_{\Gamma(t_1,t_2)\cap\partial\Omega}H(-v\odot\nu_\Omega)\,d\mathcal{H}^1\,,$$

which concludes the proof of (3.2).

In particular, if $v\in H^1_0(\Omega_{t_1};\mathbb{R}^2)$, $\eta=Ev$, and $q=0$, we obtain

$$\int_{\Omega_{t_1}}\sigma_1:Ev\,dx\ge0\,,$$

and since the same inequality holds for $-v$ we deduce that $\mathrm{div}\sigma_1=0$ in Ω_{t_1}. Moreover, taking $t_2=t_1$, $v=0$, $\eta=-\xi1_B$, and $q=\xi1_B$, with $\xi\in\mathbb{R}^{2\times2}_{sym}$ and B a Borel subset of Ω_{t_1}, we obtain

$$\int_B\sigma_1:\xi\,dx\le\int_B H(\xi)\,dx\,,$$

which, by the arbitrariness of ξ and B, gives $\sigma_1(x)\in\partial H(0)$ for a.e. $x\in\Omega_{t_1}$, where ∂H denotes the subdifferential of H. Since $\partial H(0)=\mathbb{K}$ (see, e.g., [16, Corollary 23.5.3]), we obtain that $\sigma_1(x)\in\mathbb{K}$ for a.e. $x\in\Omega_{t_1}$. The proof is thus concluded. □

We now prove an estimate for quasistatic evolutions which is a consequence of the Euler condition and of the energy-dissipation balance. It will be the starting point in the proof of our main result.

Proposition 3.2. *Under the hypotheses of Theorem 2.3 we have*

$$\frac{\lambda_{\mathbb{C}}}{2}\int_{\Omega_{t_2}}|e(t_2)-e(t_1)|^2dx+\beta(\ell(t_2)-\ell(t_1))\le\int_{t_1}^{t_2}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_1)):E\dot w(\tau))\,dx\Big)d\tau$$
$$+C_{\mathbb{K}}\int_{\Gamma(t_1,t_2)\cap\Omega}|[u(t_2)-u(t_1)]|\,d\mathcal{H}^1+C_{\mathbb{K}}\int_{\Gamma(t_1,t_2)\cap\partial\Omega}|u(t_2)-u(t_1)-w(t_2)+w(t_1)|\,d\mathcal{H}^1 \tag{3.3}$$

for every $t_1, t_2 \in [0,T]$, *with* $t_1 < t_2$.

Proof. Let us fix t_1 and t_2 as required. By the energy-dissipation balance (2.30) we have

$$\int_{\Omega_{t_2}} Q(e(t_2))\,dx + \mathcal{D}_\Gamma(p;t_1,t_2) + \beta(\ell(t_2)-\ell(t_1))$$
$$= \int_{\Omega_{t_1}} Q(e(t_1))\,dx + \int_{t_1}^{t_2}\Big(\int_{\Omega_\tau}\sigma(\tau):E\dot w(\tau)\,dx\Big)d\tau\,,$$

hence, using the definition of σ and the symmetry of $\mathbb{C}$, we get

$$\frac12\int_{\Omega_{t_2}}(\sigma(t_2)+\sigma(t_1)):(e(t_2)-e(t_1))\,dx + H(p(t_2)-p(t_1))(\hat\Omega_{t_2}) + \beta(\ell(t_2)-\ell(t_1))$$
$$\le \int_{t_1}^{t_2}\Big(\int_{\Omega_\tau}\sigma(\tau):E\dot w(\tau)\,dx\Big)d\tau\,. \tag{3.4}$$

We now apply Lemma 3.1 with $w_1 = w(t_1)$, $u_1 = u(t_1)$, $e_1 = e(t_1)$, $p_1 = p(t_1)$, $v = u(t_2)-u(t_1)-w(t_2)+w(t_1)$, $\eta = e(t_2)-e(t_1)-Ew(t_2)+Ew(t_1)$, and $q = p(t_2)-p(t_1)$, and we obtain

$$-\int_{\Omega_{t_1}}\sigma(t_1):(e(t_2)-e(t_1))\,dx + \int_{\Omega_{t_1}}\sigma(t_1):(Ew(t_2)-Ew(t_1))\,dx$$
$$\le H(p(t_2)-p(t_1))(\hat\Omega_{t_2}) + C_{\mathbb{K}}\int_{\Gamma(t_1,t_2)\cap\Omega}|[u(t_2)-u(t_1)]|\,d\mathcal{H}^1$$
$$+C_{\mathbb{K}}\int_{\Gamma(t_1,t_2)\cap\partial\Omega}|u(t_2)-u(t_1)-w(t_2)+w(t_1)|\,d\mathcal{H}^1\,.$$

Summing this inequality and (3.4) and using (2.14) we obtain (3.3). □

In the following theorem we obtain an estimate of the jump of u across a sufficiently regular curve G in terms of $|Eu|$ on the complement of G. This will play a crucial role in the proof of our main result.

Theorem 3.3. *There exists a constant* $c_0 > 0$ *with the following property: for every* $R > 0$ *and every* C^1 *function* $g\colon[-R,0]\to\mathbb{R}$, *with* $g(0) = g'(0) = 0$ *and* $|g'| \le \frac14$ *on* $[-R,0]$, *the estimate*

$$\int_{G_g\cap B_R(0)}|[u]|\,d\mathcal{H}^1 \le c_0|Eu|(B_R(0)\setminus G_g) \tag{3.5}$$

holds for every $u \in BD(B_R(0)\setminus G_g)$, *where* G_g *is the graph of* g.

Proof. Let us fix $R > 0$ and a function g satisfying the hypotheses of the theorem. We set $B := B_R(0)$ and

$$B^\pm := \{(x_1,x_2)\in B : x_1 \le 0 \text{ and } \pm x_2 > g(x_1)\}\cup\{(x_1,x_2)\in B : x_1 > 0 \text{ and } \pm x_2 > 0\}\,.$$

Finally, let $A := (0,\frac12 R)\times(-\frac34 R,\frac34 R)$ and $A^\pm := \{(x_1,x_2)\in A : \pm x_2 > 0\}$ (see Fig. 1).

The first step in the proof is to show that for every $u \in BD(B^+)$ we have the estimate

$$\int_{G_g\cap B}|u^+|\,d\mathcal{H}^1 \le \frac{c_1}{R}\int_{A^+}|u|\,dx + c_2|Eu|(B^+)\,, \tag{3.6}$$

for suitable constants c_1 and c_2 independent of R and g. By standard approximation properties of BD functions (see [17, Theorems 3.1 and 3.2]), it is enough to prove the result when $u \in C^\infty(\overline{B}^+;\mathbb{R}^2)$. For every $x_1 \in [-R,0]$ such that $(x_1,g(x_1)) \in \overline{B}$ we consider the line $L_{x_1} : \{(y_1,y_2)\in\mathbb{R}^2 : y_2 = g(x_1)+\frac13(y_1-x_1)\}$. Note that for every $(y_1,y_2)\in L_{x_1}$ we have

$$x_1 < y_1 \le 0 \implies -(R^2-y_1^2)^{1/2} < g(y_1) < y_2 < (R^2-y_1^2)^{1/2}, \tag{3.7}$$
$$0 \le y_1 \le \frac12 R \implies 0 \le y_2 \le \frac34 R. \tag{3.8}$$

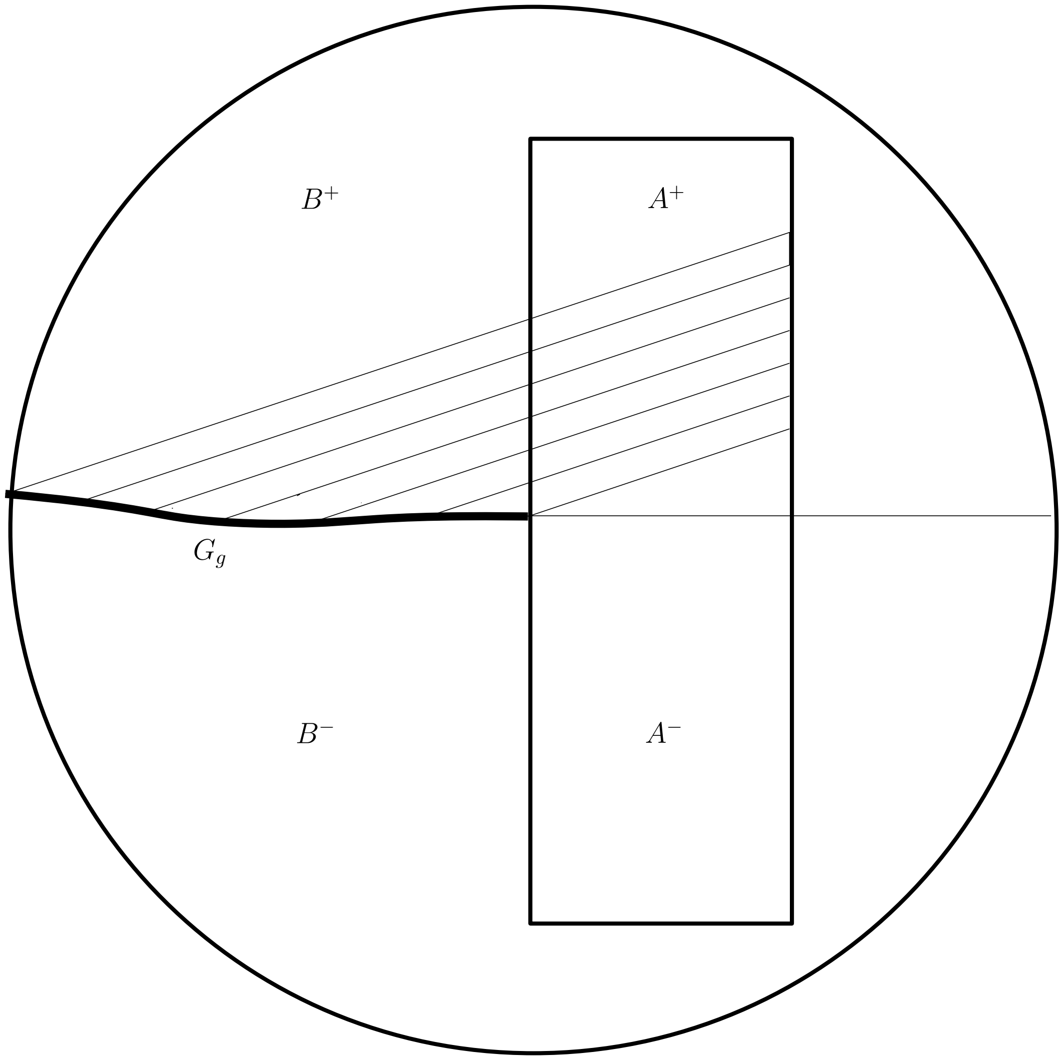

FIGURE 1. The geometry used in the proof of (3.6).

Let us fix $u \in C^\infty(\overline{B}^+;\mathbb{R}^2)$ and consider $v(y_1) := u(y_1, g(x_1) + \frac{1}{3}(y_1 - x_1)) \cdot (1, \frac{1}{3})$, where the dot denotes the scalar product. Then

$$v'(y_1) = Eu(y_1, g(x_1) + \tfrac{1}{3}(y_1 - x_1))(1, \tfrac{1}{3}) \cdot (1, \tfrac{1}{3})\,.$$

We now fix $z_1 \in (0, \frac{1}{2}R)$ and integrating v' between x_1 and z_1 we obtain

$$\begin{aligned} |u(x_1, g(x_1)) \cdot (1, \tfrac{1}{3})| &\le |u(z_1, g(x_1) + \tfrac{1}{3}(z_1 - x_1)) \cdot (1, \tfrac{1}{3})| \\ &+ \int_{x_1}^{z_1} |Eu(y_1, g(x_1) + \tfrac{1}{3}(y_1 - x_1))(1, \tfrac{1}{3}) \cdot (1, \tfrac{1}{3})| dy_1\,, \end{aligned}$$

hence

$$\begin{aligned} |u(x_1, g(x_1)) \cdot (1, \tfrac{1}{3})| &\le \tfrac{\sqrt{10}}{3} |u(z_1, g(x_1) + \tfrac{1}{3}(z_1 - x_1))| \\ &+ \tfrac{10}{9} \int_{x_1}^{z_1} |Eu(y_1, g(x_1) + \tfrac{1}{3}(y_1 - x_1))|\, dy_1\,. \end{aligned}$$

Integrating between 0 and $\frac{1}{2}R$ with respect to z_1 we obtain

$$|u(x_1,g(x_1))\cdot(1,\tfrac{1}{3})| \le \tfrac{2\sqrt{10}}{3R}\int_0^{R/2}|u(z_1,g(x_1)+\tfrac{1}{3}(z_1-x_1))|\,dz_1 + \tfrac{10}{9}\int_{x_1}^{R/2}|Eu(y_1,g(x_1)+\tfrac{1}{3}(y_1-x_1))|\,dy_1\,.$$

Let $a<0$ be such that $(a,g(a))\in\partial B$, so that

$$G_g\cap B=\{(x_1,g(x_1)) : a<x_1\le 0\}\,.$$

We now integrate the previous inequality with respect to x_1 between a and 0 and obtain

$$\int_a^0|u(x_1,g(x_1))\cdot(1,\tfrac{1}{3})|\,dx_1 \le \tfrac{2\sqrt{10}}{3R}\int_a^0\Big(\int_0^{R/2}|u(y_1,g(x_1)+\tfrac{1}{3}(y_1-x_1))|\,dy_1\Big)dx_1 + \tfrac{10}{9}\Big(\int_a^0\int_{x_1}^{R/2}|Eu(y_1,g(x_1)+\tfrac{1}{3}(y_1-x_1))|\,dy_1\Big)dx_1\,. \tag{3.9}$$

Since $|g'|\le\frac{1}{4}$ we have

$$\int_{G_g\cap B}|u^+\cdot(1,\tfrac{1}{3})|\,d\mathcal{H}^1 \le \tfrac{\sqrt{17}}{4}\int_a^0|u(x_1,g(x_1))\cdot(1,\tfrac{1}{3})|\,dx_1\,. \tag{3.10}$$

To evaluate the other terms in (3.9) we consider the function Φ defined by $\Phi(y_1,x_1):=(y_1,g(x_1)+\frac{1}{3}(y_1-x_1))$, for which we have the estimate $|\det\nabla\Phi(y_1,x_1)|=|g'(x_1)-\frac{1}{3}|\ge\frac{1}{12}$, where we used the inequality $|g'|\le\frac{1}{4}$. Therefore

$$\int_a^0\Big(\int_0^{R/2}|u(y_1,g(x_1)+\tfrac{1}{3}(y_1-x_1))|\,dy_1\Big)dx_1 \le 12\int_a^0\Big(\int_0^{R/2}|u(\Phi(y_1,x_1))|\,|\det\nabla\Phi(y_1,x_1)|\,dy_1\Big)dx_1 \le 12\int_{A^+}|u|\,dx\,, \tag{3.11}$$

where in the last inequality we used (3.8). Similarly, using also (3.7), we obtain that

$$\int_a^0\Big(\int_{x_1}^{R/2}|Eu(y_1,g(x_1)+\tfrac{1}{3}(y_1-x_1))|\,dy_1\Big)dx_1 \le 12\int_{B^+}|Eu|\,dx\,. \tag{3.12}$$

By (3.9)-(3.12) it follows that there exist two constants $\hat c_1,\hat c_2>0$ such that

$$\int_{G_g\cap B}|u^+\cdot(1,\tfrac{1}{3})|\,d\mathcal{H}^1 \le \frac{\hat c_1}{R}\int_{A^+}|u|\,dx+\hat c_2|Eu|(B^+)\,.$$

Repeating the same arguments with $v(y_1):=u(y_1,g(x_1)+\frac{7}{24}(y_1-x_1))\cdot(1,\frac{7}{24})$ we obtain two constants $\check c_1,\check c_2>0$ such that

$$\int_{G_g\cap B}|u^+\cdot(1,\tfrac{7}{24})|\,d\mathcal{H}^1 \le \frac{\check c_1}{R}\int_{A^+}|u|\,dx+\check c_2|Eu|(B^+)\,.$$

From the last two inequalities it follows that

$$\frac{1}{24}\int_{G_g\cap B}|u_2^+|\,d\mathcal{H}^1 \le \int_{G_g\cap B}|u_1^+ +\tfrac{1}{3}u_2^+|\,d\mathcal{H}^1+\int_{G_g\cap B}|u_1^+ +\tfrac{7}{24}u_2^+|\,d\mathcal{H}^1 \le \frac{\hat c_1+\check c_1}{R}\int_{A^+}|u|\,dx+(\hat c_2+\check c_2)|Eu|(B^+)\,. \tag{3.13}$$

Similarly we obtain that

$$\frac{1}{7}\int_{G_g\cap B}|u_1^+|\,d\mathcal{H}^1 \le \int_{G_g\cap B}|u_1^+ +\tfrac{1}{3}u_2^+|\,d\mathcal{H}^1+\frac{8}{7}\int_{G_g\cap B}|u_1^+ +\tfrac{7}{24}u_2^+|\,d\mathcal{H}^1 \le \frac{\hat c_1+\frac{8}{7}\check c_1}{R}\int_{A^+}|u|\,dx+(\hat c_2+\frac{8}{7}\check c_2)|Eu|(B^+)\,. \tag{3.14}$$

These chains of inequalities imply (3.6).

In the same way we prove that

$$\int_{G_g\cap B} |u^-|\, d\mathcal{H}^1 \le \frac{c_1}{R}\int_{A^-} |u|\, dx + c_2|Eu|(B^-)\,,$$

which, together with (3.6) gives

$$\int_{G_g\cap B} |[u]|\, d\mathcal{H}^1 \le \frac{c_1}{R}\int_{A} |u|\, dx + c_2|Eu|(B\setminus G_g)\,. \tag{3.15}$$

By the Poincaré-Wirtinger inequality in BD (see [17, Proposition 2.2 and Remark 1.1 of Chapter II]) there exists a constant $c_3>0$ independent of R such that for every $u\in BD(A)$ there exist a constant $\omega\in\mathbb{R}$ and a vector $b\in\mathbb{R}^2$ such that

$$\int_A |u-\omega J-b|\, dx \le c_3R|Eu|(A)\,,$$

where $J(x_1,x_2)=(x_2,-x_1)$. By applying (3.15) to the function $u(x)-\omega Jx-b$ we obtain

$$\int_{G_g\cap B} |[u]|\, d\mathcal{H}^1 \le c_1c_3|Eu|(A)+c_2|Eu|(B\setminus G_g)\le (c_1c_3+c_2)|Eu|(B\setminus G_g)\,,$$

where in the last inequality we used the inclusion $A\subset B\setminus G_g$. This concludes the proof of (3.5). ☐

To apply Theorem 3.3 we need a purely geometric lemma. For every $h=1,\dots,k$ and $x^1,x^2\in K_h$ let $K_h(x^1,x^2)$ be the arc of K_h between x^1 and x^2, including its endpoints x^1 and x^2. Recall that x^1_h and x^2_h are the endpoints of K_h.

Lemma 3.4. *There exists $\delta_0>0$ such that for every $h=1,\dots,k$ the following property holds: if $\delta\in(0,\delta_0)$ and $x^1,x^2\in K_h$, with $|x^1-x^2|=\delta$, then either $K_h(x^1,x^2)=K_h(x^1_h,x^2)\cap\overline{B_\delta(x^2)}$ or $K_h(x^1,x^2)=K_h(x^2_h,x^2)\cap\overline{B_\delta(x^2)}$. Moreover, there exists a suitable orthonormal coordinate system with origin at x^2, such that $K_h(x^1,x^2)$ can be represented as the intersection between the closed ball $\overline{B_\delta(0)}$ and the graph of a C^1 function $g:[-\delta,0]\to\mathbb{R}$ such that $g(0)=g'(0)=0$ and $|g'|\le\frac14$.*

Proof. All properties considered in the lemma are elementary consequences of the fact that K_h is a simple C^1 curve for every $h=1,\dots,k$. ☐

Corollary 3.5. *Let $\delta_0>0$ be the constant introduced in Lemma 3.4 and let $h=1,\dots,k$. If $\delta\in(0,\delta_0)$, $x^1,x^2\in K_h$, with $|x^1-x^2|=\delta$, then*

$$\int_{K_h(x^1,x^2)} |[u]|\, d\mathcal{H}^1 \le c_0|Eu|(B_\delta(x^2)\setminus K_h(x^1,x^2))\,, \tag{3.16}$$

for every $u\in BD(B_\delta(x^2)\setminus K_h(x^1,x^2))$, where c_0 is the constant in Theorem 3.3.

Proof. By Lemma 3.4 we can introduce an orthonormal coordinate system in which we can apply Theorem 3.3 to the ball $B_\delta(x^2)$, which becomes $B_\delta(0)$ in the new coordinate system. This gives (3.16). ☐

Corollary 3.6. *Let $\delta_0>0$ be the constant introduced in Lemma 3.4 and let $h=1,\dots,k$ be such that $K^0_h\subset\partial\Omega$. If $\delta\in(0,\delta_0)$, $x^1,x^2\in K^0_h$, with $|x^1-x^2|=\delta$, and $B_\delta(x^2)\cap\partial\Omega\subset K^0_h$ then*

$$\int_{K_h(x^1,x^2)} |u-w|\, d\mathcal{H}^1 \le c_0|E(u-w)|(B_\delta(x^2)\cap\Omega)+c_0\int_{B_\delta(x^2)\cap(\partial\Omega\setminus K_h(x^1,x^2))} |u-w|\, d\mathcal{H}^1\,, \tag{3.17}$$

for every $u\in BD(B_\delta(x^2)\cap\Omega)$ and every $w\in H^1(B_\delta(x^2)\cap\Omega;\mathbb{R}^2)$, where c_0 is the constant in Theorem 3.3.

Proof. Let us fix δ, x^1, x^2 as in the statement of the corollary. Let $u \in BD(B_\delta(x^2) \cap \Omega)$ and let $v : B_\delta(x^2) \to \mathbb{R}^2$ be defined by $v := u - w$ in $B_\delta(x^2) \cap \Omega$ and $v := 0$ in $B_\delta(x^2) \setminus \Omega$. It is easy to see that $v \in BD(B_\delta(x^2))$ and $[v] = w - u$ on $B_\delta(x^2) \cap \partial\Omega$. Hence $Ev = Eu - Ew$ in $B_\delta(x^2) \cap \Omega$, $Ev = 0$ in $B_\delta(x^2) \setminus \overline{\Omega}$, and $Ev = (w-u) \odot \nu_\Omega \mathcal{H}^1$ on $B_\delta(x^2) \cap \partial\Omega$. By Lemma 3.4 we can introduce an orthonormal coordinate system in which we can apply Theorem 3.3 to the function v on the ball $B_\delta(x^2)$, which becomes $B_\delta(0)$ in the new coordinate system. This gives

$$\int_{K_h(x^1,x^2)} |[v]| \, d\mathcal{H}^1 \le c_0 |Ev|(B_\delta(x^2) \setminus K_h(x^1, x^2)),$$

which implies (3.17). □

4. Functions with bounded dissipation

For every nondecreasing function $\Gamma \colon [0,T] \to \mathcal{K}$ we set

$$\Gamma(t-) := \mathrm{cl}\big(\bigcup_{\tau < t} \Gamma(\tau) \big) \quad \text{for } 0 < t \le T, \tag{4.1}$$

$$\Gamma(t+) := \bigcap_{\tau > t} \Gamma(\tau) \quad \text{for } 0 \le t < T, \tag{4.2}$$

where cl denotes the closure. We also set $\Gamma(0-) := \Gamma(0)$ and $\Gamma(T+) := \Gamma(T)$. For every $t \in [0,T]$ we consider $\hat{\Omega}_{t-} := \overline{\Omega} \setminus \Gamma(t-)$ and $\hat{\Omega}_{t+} := \overline{\Omega} \setminus \Gamma(t+)$.

Proposition 4.1. *Let $\Gamma \colon [0,T] \to \mathcal{K}$ be a nondecreasing function and for every $t \in [0,T]$ let $p(t) \in \mathcal{M}_b(\hat{\Omega}_t; \mathbb{R}^{2\times 2}_{sym})$. Assume that $\mathcal{D}_\Gamma(p; 0, T) < +\infty$, where $\mathcal{D}_\Gamma(p; 0,T)$ is defined in* (2.25)*. Then*

(a) *for every $t_0 \in (0,T]$ there exist $p(t_0-) \in \mathcal{M}_b(\hat{\Omega}_{t_0-}; \mathbb{R}^{2\times 2}_{sym})$ such that*

$$\lim_{t \to t_0-} |p(t) - p(t_0-)|(\hat{\Omega}_{t_0}) = \lim_{t \to t_0-} |p(t) - p(t_0-)|(\hat{\Omega}_{t_0-}) = 0, \tag{4.3}$$

(b) *for every $t_0 \in [0,T)$ there exist $p(t_0+) \in \mathcal{M}_b(\hat{\Omega}_{t_0+}; \mathbb{R}^{2\times 2}_{sym})$ such that*

$$\lim_{t \to t_0+} |p(t) - p(t_0+)|(\hat{\Omega}_t) = 0. \tag{4.4}$$

Proof. Let $V : [0,T] \to \mathbb{R}$ be the nondecreasing function defined by

$$V(t) := \mathcal{D}_\Gamma(p; 0, t) \qquad \text{for every } t \in [0,T]. \tag{4.5}$$

Using the triangle inequality for H it is easy to prove that

$$\mathcal{D}_\Gamma(p; t_1, t_2) = V(t_2) - V(t_1) \quad \text{for every } 0 \le t_1 \le t_2 \le T. \tag{4.6}$$

Let us fix $t_0 \in (0,T]$. To prove (4.3) we begin by observing that for $0 < t_1 \le t_2 < t_0$ we have

$$c_{\mathbb{K}} |p(t_2) - p(t_1)|(\hat{\Omega}_{t_0-}) \le c_{\mathbb{K}} |p(t_2) - p(t_1)|(\hat{\Omega}_{t_2}) \le H(p(t_2) - p(t_1))(\hat{\Omega}_{t_2}) \le V(t_2) - V(t_1),$$

where in the last inequality we used (4.6). Since $V(t)$ has a finite limit as $t \to t_0-$, the previous inequality implies that $t \mapsto p(t)$ satisfies the Cauchy condition in $\mathcal{M}_b(\hat{\Omega}_{t_0-}; \mathbb{R}^{2\times 2}_{sym})$ as $t \to t_0-$. Since this space is complete we obtain the existence of $p(t_0-) \in \mathcal{M}_b(\hat{\Omega}_{t_0-}; \mathbb{R}^{2\times 2}_{sym})$ such that the second equality in (4.3) holds. The first equality in (4.3) follows from the inclusion $\hat{\Omega}_{t_0} \subset \hat{\Omega}_{t_0-}$, which is a consequence of the fact that $\Gamma(t_0-) \subset \Gamma(t_0)$.

Let us fix now $t_0 \in [0,T)$. The proof of the existence of $p(t_0+)$ such that (4.4) holds is more delicate. Let us fix $\tau \in (t_0, T)$. For $t_0 < t_1 \le t_2 \le \tau$ we have

$$c_{\mathbb{K}} |p(t_2) - p(t_1)|(\hat{\Omega}_\tau) \le c_{\mathbb{K}} |p(t_2) - p(t_1)|(\hat{\Omega}_{t_2}) \le H(p(t_2) - p(t_1))(\hat{\Omega}_{t_2}) \le V(t_2) - V(t_1), \tag{4.7}$$

where in the last inequality we used (4.6) again. Since $V(t)$ has a finite limit as $t \to t_0+$, the previous inequality implies that the function $t \mapsto p(t)$ satisfies the Cauchy condition in

$\mathcal{M}_b(\hat\Omega_\tau;\mathbb{R}^{2\times 2}_{sym})$ as $t\to t_0+$. Therefore, for every $\tau\in(t_0,T)$ there exists $q_\tau\in\mathcal{M}_b(\hat\Omega_\tau;\mathbb{R}^{2\times 2}_{sym})$ such that $p(t)\to q_\tau$ strongly in $\mathcal{M}_b(\hat\Omega_\tau;\mathbb{R}^{2\times 2}_{sym})$ as $t\to t_0+$. By (4.7) we have

$$|p(t_2)-q_\tau|(\hat\Omega_\tau)\le\frac{1}{c_{\mathbb{K}}}(V(t_2)-V(t_0+))\,. \tag{4.8}$$

It is easy to see that

$$q_{\tau_1}=q_{\tau_2}\quad\text{as measures in }\hat\Omega_{\tau_2}\quad\text{for every }t_0<\tau_1\le\tau_2\,. \tag{4.9}$$

We now want to prove that for every Borel set $B\subset\hat\Omega_{t_0+}$ the limit

$$\lim_{\tau\to t_0+}q_\tau(B\cap\hat\Omega_\tau) \tag{4.10}$$

exists in $\mathbb{R}^{2\times 2}_{sym}$. We first show that the limit

$$L:=\lim_{t\to t_0+}|p(t)|(\hat\Omega_t) \tag{4.11}$$

exists and is finite. To prove this fact we observe that for $t_0<t_1\le t_2$ we have $|p(t_2)|(\hat\Omega_{t_2})\le|p(t_1)|(\hat\Omega_{t_2})+|p(t_2)-p(t_1)|(\hat\Omega_{t_2})\le|p(t_1)|(\hat\Omega_{t_1})+\frac{1}{c_{\mathbb{K}}}(V(t_2)-V(t_1))$, which shows that the function $t\mapsto|p(t)|(\hat\Omega_t)-\frac{1}{c_{\mathbb{K}}}V(t)$ is nonincreasing. Since V has a finite limit as $t\to t_0+$ we obtain the existence of a finite limit in (4.11).

For $t_0<t\le\tau_1\le\tau_2$, using the triangle inequality and the definition of V, we get $|p(t)|(\hat\Omega_{\tau_1}\setminus\hat\Omega_{\tau_2})=|p(t)|(\hat\Omega_{\tau_1})-|p(t)|(\hat\Omega_{\tau_2})\le|p(t)-p(\tau_1)|(\hat\Omega_{\tau_1})+|p(\tau_1)|(\hat\Omega_{\tau_1})-|p(\tau_2)|(\hat\Omega_{\tau_2})+|p(t)-p(\tau_2)|(\hat\Omega_{\tau_2})\le\frac{1}{c_{\mathbb{K}}}(V(\tau_1)-V(t))+|p(\tau_1)|(\hat\Omega_{\tau_1})-|p(\tau_2)|(\hat\Omega_{\tau_2})+\frac{1}{c_{\mathbb{K}}}(V(\tau_2)-V(t))$, where in the last inequality we used (4.6). Let us fix $\varepsilon>0$. By (4.11) and by the finiteness of $V(t_0+)$ there exists $\delta>0$ such that if $t_0<t\le\tau_1\le\tau_2<t_0+\delta$ we have $V(\tau_1)-V(t)<c_{\mathbb{K}}\varepsilon$, $V(\tau_2)-V(t)<c_{\mathbb{K}}\varepsilon$, and $||p(\tau_1)|(\hat\Omega_{\tau_1})-|p(\tau_2)|(\hat\Omega_{\tau_2})|<\varepsilon$. Therefore the estimate of $|p(t)|(\hat\Omega_{\tau_1}\setminus\hat\Omega_{\tau_2})$ gives

$$|p(t)|(\hat\Omega_{\tau_1}\setminus\hat\Omega_{\tau_2})<3\varepsilon\quad\text{for }t_0<t\le\tau_1\le\tau_2<t_0+\delta\,.$$

Passing to the limit as $t\to t_0+$ in the previous inequality we obtain

$$|q_\tau|(\hat\Omega_{\tau_1}\setminus\hat\Omega_{\tau_2})\le 3\varepsilon\quad\text{for }t_0<\tau\le\tau_1\le\tau_2<t_0+\delta\,. \tag{4.12}$$

We are now ready to prove the existence of the limit in (4.10). Let us fix a Borel set $B\subset\hat\Omega_{t_0+}$. For $t_0<\tau\le\tau_1\le\tau_2<t_0+\delta$, by (4.9), we have $|q_{\tau_2}(B\cap\hat\Omega_{\tau_2})-q_{\tau_1}(B\cap\hat\Omega_{\tau_1})|=|q_\tau(B\cap\hat\Omega_{\tau_2})-q_\tau(B\cap\hat\Omega_{\tau_1})|\le|q_\tau|(\hat\Omega_{\tau_1}\setminus\hat\Omega_{\tau_2})<3\varepsilon$, where in the last inequality we used (4.12). This shows that $\tau\mapsto q_\tau(B\cap\hat\Omega_\tau)$ satisfies the Cauchy condition as $\tau\to t_0+$ and concludes the proof of the existence of the limit (4.10).

For every Borel set $B\subset\hat\Omega_{t_0+}$ we define $p(t_0+)(B)$ as the limit in (4.10). By the Vitali-Hahn-Saks Theorem (see [8, III.7.2-4]) the set function $p(t_0+)$ belongs to $\mathcal{M}_b(\hat\Omega_{t_0+};\mathbb{R}^{2\times 2}_{sym})$. It remains to prove (4.4). To this aim we fix $t>t_0$ and show that

$$p(t_0+)(B)=q_t(B)\quad\text{for every Borel set }B\subset\hat\Omega_t\,. \tag{4.13}$$

Indeed, $p(t_0+)(B)$ is the limit of $q_\tau(B\cap\hat\Omega_\tau)$ as $\tau\to t_0+$. If $B\subset\hat\Omega_t$ we have $B\cap\hat\Omega_\tau=B$ for $t_0<\tau<t$, hence $q_\tau(B\cap\hat\Omega_\tau)=q_\tau(B)=q_t(B)$, where in the last equality we used (4.9). This proves (4.13), which implies that $|p(t)-p(t_0+)|(\hat\Omega_t)=|p(t)-q_t|(\hat\Omega_t)$. By (4.8) applied with $t_2=\tau=t$, the previous equality gives $|p(t)-p(t_0+)|(\hat\Omega_t)\le\frac{1}{c_{\mathbb{K}}}(V(t)-V(t_0+))$, which yields (4.4), taking into account the definition of $V(t_0+)$. □

The following Lemma shows the relation between $p(t)-p(t\pm)$ and $V(t)-V(t\pm)$.

Lemma 4.2. *Let $\Gamma\colon [0,T] \to \mathcal{K}$ be an increasing function and for every $t \in [0,T]$ let $p(t) \in \mathcal{M}_b(\hat\Omega_t;\mathbb{R}^{2\times 2}_{sym})$. Assume that $\mathcal{D}_\Gamma(p;0,T) < +\infty$. Let $V(t\pm)$ be defined as in* (2.32) *and* (2.33)*, with $V(t)$ given by* (4.5)*, and let $p(t\pm)$ be defined as in Proposition 4.1. Then*

$$V(t) - V(t-) = H(p(t) - p(t-))(\hat\Omega_t) \quad \text{for every } t \in (0,T]\,, \tag{4.14}$$

$$V(t+) - V(t) = H(p(t+) - p(t))(\hat\Omega_{t+}) \quad \text{for every } t \in [0,T)\,. \tag{4.15}$$

Proof. We begin by proving (4.14). Let us fix $t_0 \in (0,T]$. For every $\varepsilon > 0$ there exist $t_1 < t_0$ such that for every $t_1 \le t \le t_0$ we have $V(t_0-) - \varepsilon < V(t) \le V(t_0)$, hence

$$V(t_0) - V(t_0-) + \varepsilon > V(t_0) - V(t) \ge H(p(t_0) - p(t))(\hat\Omega_{t_0})\,,$$

where in the last inequality we used (4.6). By (4.3), passing to the limit as $t \to t_0-$ we obtain $V(t_0) - V(t_0-) + \varepsilon \ge H(p(t_0) - p(t_0-))(\hat\Omega_{t_0})$, and by the arbitrariness of ε we conclude that

$$V(t_0) - V(t_0-) \ge H(p(t_0) - p(t_0-))(\hat\Omega_{t_0})\,. \tag{4.16}$$

We now want to prove that

$$V(t_0) - V(t_0-) \le H(p(t_0) - p(t_0-))(\hat\Omega_{t_0})\,. \tag{4.17}$$

Let us fix $\varepsilon > 0$. By (4.3) and by the definition of $V(t_0-)$ there exists $t_1 < t_0$ such that, if $t_1 \le t < t_0$, then

$$V(t_0-) - \varepsilon < V(t) \le V(t_0-) \quad \text{and} \quad |p(t) - p(t_0-)|(\hat\Omega_{t_0}) < \frac{1}{C_{\mathbb{K}}}\varepsilon\,. \tag{4.18}$$

Hence $V(t_0) - V(t_0-) \le V(t_0) - V(t) = \mathcal{D}_\Gamma(p;t,t_0)$ for every $t_1 \le t < t_0$, where in the equality we used (4.6). From the definition of $\mathcal{D}_\Gamma(p;t,t_0)$ it follows that there exists a partition $t = \tau_0 < \tau_1 < \cdots < \tau_m = t_0$ such that

$$\begin{aligned} V(t_0) - V(t_0-) - \varepsilon \le \mathcal{D}_\Gamma(p;t,t_0) - \varepsilon &< \sum_{i=1}^{m} H(p(\tau_i) - p(\tau_{i-1}))(\hat\Omega_{\tau_i}) \\ &= \sum_{i=1}^{m-1} H(p(\tau_i) - p(\tau_{i-1}))(\hat\Omega_{\tau_i}) + H(p(t_0) - p(\tau_{m-1}))(\hat\Omega_{t_0}) \\ &\le V(\tau_{m-1}) - V(t_1) + H(p(t_0) - p(t_0-))(\hat\Omega_{t_0}) + H(p(t_0-) - p(\tau_{m-1}))(\hat\Omega_{t_0})\,. \end{aligned}$$

Since by (2.21) and (4.18) we have $V(\tau_{m-1}) - V(t_1) < \varepsilon$ and $H(p(\tau_{m-1}) - p(t_0-))(\hat\Omega_{t_0}) < \varepsilon$, the previous inequalities imply that

$$V(t_0) - V(t_0-) \le H(p(t_0) - p(t_0-))(\hat\Omega_{t_0}) + 3\varepsilon\,.$$

By the arbitrariness of ε we obtain (4.17), which together with (4.16) gives (4.14).

To prove (4.15) we proceed analogously. Let us fix $t_0 \in [0,T)$. First of all for every $\varepsilon > 0$ there exist $t_2 > t_0$ such that for every $t_0 \le t \le t_2$ we have $V(t_0+) + \varepsilon > V(t) \ge V(t_0)$, hence

$$\begin{aligned} V(t_0+) - V(t_0) + \varepsilon > V(t) - V(t_0) &\ge H(p(t) - p(t_0))(\hat\Omega_t) \\ &\ge -H(p(t_0+) - p(t))(\hat\Omega_t) + H(p(t_0+) - p(t_0))(\hat\Omega_t)\,. \end{aligned}$$

By (4.4), passing to the limit as $t \to t_0+$ we obtain $V(t_0+) - V(t_0) + \varepsilon \ge H(p(t_0+) - p(t_0))(\hat\Omega_{t_0+})$, and by the arbitrariness of ε we conclude that

$$V(t_0+) - V(t_0) \ge H(p(t_0+) - p(t_0))(\hat\Omega_{t_0+})\,. \tag{4.19}$$

It remains to prove that

$$V(t_0+) - V(t_0) \le H(p(t_0+) - p(t_0))(\hat\Omega_{t_0+})\,. \tag{4.20}$$

Let us fix $\varepsilon > 0$. By (4.4) and by the definition of $V(t_0+)$ there exists $t_2 > t_0$ such that, if $t_0 < t \leq t_2$, then

$$V(t_0+) \leq V(t) < V(t_0+) + \varepsilon \quad \text{and} \quad |p(t) - p(t_0+)|(\hat{\Omega}_t) < \frac{1}{C_{\mathbb{K}}}\varepsilon\,. \tag{4.21}$$

Hence $V(t_0+) - V(t_0) \leq V(t) - V(t_0) = \mathcal{D}_\Gamma(p; t_0, t)$ for every $t_0 < t \leq t_2$, where in the equality we used (4.6). From the definition of $\mathcal{D}_\Gamma(p; t_0, t)$ it follows that there exists a partition $t_0 = \tau_0 < \tau_1 < \cdots < \tau_m = t$ such that

$$\begin{aligned} V(t_0+) - V(t_0) - \varepsilon \leq \mathcal{D}_\Gamma(p; t_0, t) - \varepsilon &< \sum_{i=1}^{m} H(p(\tau_i) - p(\tau_{i-1}))(\hat{\Omega}_{\tau_i}) \\ &= H(p(\tau_1) - p(t_0))(\hat{\Omega}_{\tau_1}) + \sum_{i=2}^{m} H(p(\tau_i) - p(\tau_{i-1}))(\hat{\Omega}_{\tau_i}) \\ &\leq H(p(t_0+) - p(t_0))(\hat{\Omega}_{\tau_1}) + H(p(\tau_1) - p(t_0+))(\hat{\Omega}_{\tau_1}) + V(t_2) - V(\tau_1)\,. \end{aligned}$$

By (2.21) and (4.21) we have $V(t_2) - V(\tau_1) < \varepsilon$ and $H(p(\tau_1) - p(t_0+))(\hat{\Omega}_{\tau_1}) < \varepsilon$. Since by monotonicity we have also $H(p(t_0+) - p(t_0))(\hat{\Omega}_{\tau_1}) \leq H(p(t_0+) - p(t_0))(\hat{\Omega}_{t_0+})$, the previous chain of inequalities implies that

$$V(t_0+) - V(t_0) \leq H(p(t_0+) - p(t_0))(\hat{\Omega}_{t_0+}) + 3\varepsilon\,.$$

By the arbitrariness of ε we obtain (4.20), which together with (4.19) gives (4.15). □

In the remaining part of this section we construct a time-rescaled version of the plastic strain $p(t)$ and of the crack $\Gamma(t)$. We shall prove a Lipschitz property for the rescaled plastic strain which will provide some estimates involving time derivatives. As mentioned in the Introduction, these estimates will be crucial in the proof of the main result.

Under the assumptions of Lemma 4.2 let $V : [0, T] \to [0, +\infty)$ be the nondecreasing function defined by (4.5) and let $\rho : [0, T] \to [0, +\infty)$ be the increasing function defined by

$$\rho(t) := V(t) + t\,. \tag{4.22}$$

Since $\rho(t\pm) = V(t\pm) + t$, from Lemma 4.2 we obtain

$$\rho(t) - \rho(t-) = V(t) - V(t-) = H(p(t) - p(t-))(\hat{\Omega}_t), \tag{4.23}$$

$$\rho(t+) - \rho(t) = V(t+) - V(t) = H(p(t+) - p(t))(\hat{\Omega}_{t+}), \tag{4.24}$$

for every $t \in [0, T]$. Let $J_V \subset [0, T]$ be the finite or countable set of discontinuity points of $t \mapsto V(t)$, hence of $t \mapsto \rho(t)$. Since ρ is increasing we can write $[0, \rho(T)]$ as the disjoint union

$$[0, \rho(T)] = \rho([0, T]) \cup \bigcup_{t \in J_V} [\rho(t-), \rho(t)) \cup \bigcup_{t \in J_V} (\rho(t), \rho(t+)]\,. \tag{4.25}$$

For every $r \in \rho([0, T])$ there exists a unique $t \in [0, T]$ such that $\rho(t) = r$. We define

$$\Gamma^\circ(r) = \Gamma(t)\,, \quad \hat{\Omega}^\circ_r := \overline{\Omega} \setminus \Gamma^\circ(r) = \hat{\Omega}_t\,, \quad \text{and} \quad p^\circ(r) := p(t)\,. \tag{4.26}$$

It is clear that we have $p^\circ(r) \in \mathcal{M}_b(\hat{\Omega}^\circ_r; \mathbb{R}^{2\times 2}_{sym})$.

For every $r \in [\rho(t-), \rho(t))$, with $t \in J_V$, we set

$$\Gamma^\circ(r) := \Gamma(t) \quad \text{and} \quad \hat{\Omega}^\circ_r := \overline{\Omega} \setminus \Gamma^\circ(r) = \hat{\Omega}_t\,, \tag{4.27}$$

and we define $p^\circ(r) \in \mathcal{M}_b(\hat{\Omega}^\circ_r; \mathbb{R}^{2\times 2}_{sym})$ by

$$p^\circ(r) := p(t-) + \frac{r - \rho(t-)}{\rho(t) - \rho(t-)}(p(t) - p(t-))\,. \tag{4.28}$$

For every $r \in (\rho(t), \rho(t+)]$, with $t \in J_V$, we set

$$\Gamma^\circ(r) := \Gamma(t+) \quad \text{and} \quad \hat{\Omega}^\circ_r := \overline{\Omega} \setminus \Gamma^\circ(r) = \hat{\Omega}_{t+}\,, \tag{4.29}$$

and we define $p^\circ(r) \in \mathcal{M}_b(\hat{\Omega}^\circ_r; \mathbb{R}^{2\times 2}_{sym})$ by

$$p^\circ(r) := p(t) + \frac{r - \rho(t)}{\rho(t+) - \rho(t)}(p(t+) - p(t))\,. \tag{4.30}$$

The following lemma proves the Lipschitz continuity of p°, which will be used in the proof of our main result.

Lemma 4.3. *Let $\Gamma\colon [0,T] \to \mathcal{K}$ be an increasing function and for every $t \in [0,T]$ let $p(t) \in \mathcal{M}_b(\hat{\Omega}_t; \mathbb{R}^{2\times 2}_{sym})$. Assume that $\mathcal{D}_\Gamma(p; 0, T) < +\infty$ and let $\rho, \Gamma^\circ, p^\circ$ be defined by* (4.22) *and* (4.26)-(4.30)*. Then for every $t \in [0,T]$ we have*

$$\Gamma(t) = \Gamma^\circ(\rho(t)) \quad \textit{and} \quad p(t) = p^\circ(\rho(t))\,. \tag{4.31}$$

Moreover

$$c_{\mathbb{K}} |p^\circ(r_2) - p^\circ(r_1)|(\hat{\Omega}^\circ_{r_2}) \leq H(p^\circ(r_2) - p^\circ(r_1))(\hat{\Omega}^\circ_{r_2}) \leq r_2 - r_1\,, \tag{4.32}$$

for every $r_1, r_2 \in [0, \rho(T)]$ with $r_1 < r_2$.

Proof. Formula (4.31) follows from (4.26).

To prove (4.32) let us fix $r_1, r_2 \in [0, \rho(T)]$, with $r_1 < r_2$. We consider the following cases:

(a) $r_1, r_2 \in \rho([0,T])$,
(b) $r_1, r_2 \in [\rho(t-), \rho(t+)]$, with $t \in J_V$,
(c) $r_1 \in \rho([0,T])$ and $r_2 \in [\rho(t-), \rho(t+)]$, with $t \in J_V$,
(d) $r_1 \in [\rho(t-), \rho(t+)]$, with $t \in J_V$, and $r_2 \in \rho([0,T])$,
(e) $r_1 \in [\rho(t_1-), \rho(t_1+)]$ and $r_2 \in [\rho(t_2-), \rho(t_2+)]$, with $t_1, t_2 \in J_V$ and $t_1 < t_2$.

Case (a). Let $t_1, t_2 \in [0,T]$ be such that $\rho(t_1) = r_1$ and $\rho(t_2) = r_2$. Since ρ is increasing we have $t_1 < t_2$. By (4.26) we have $H(p^\circ(r_2) - p^\circ(r_1))(\hat{\Omega}^\circ_{r_2}) = H(p(t_2) - p(t_1))(\hat{\Omega}_{t_2}) \leq \mathcal{D}_\Gamma(p; t_1, t_2) = V(t_2) - V(t_1) \leq \rho(t_2) - \rho(t_1) = r_2 - r_1$. This concludes the proof of (4.32) in this case.

Case (b). Assume first that $r_1, r_2 \in [\rho(t-), \rho(t)]$. We claim that in this case

$$H(p^\circ(r_2) - p^\circ(r_1))(\hat{\Omega}^\circ_{r_2}) = r_2 - r_1\,. \tag{4.33}$$

First of all we observe that, since $r_1 < r_2$, it has to be $\rho(t-) < \rho(t)$. Therefore (4.27) and (4.28) give

$$p^\circ(r_2) - p^\circ(r_1) = \frac{r_2 - r_1}{\rho(t) - \rho(t-)}(p(t) - p(t-)) \quad \text{as measures in } \hat{\Omega}_t\,,$$

where we used also (4.26) for the case $r_2 = \rho(t)$. Since H is positively homogeneous of degree one, we have

$$\begin{aligned} H(p^\circ(r_2) - p^\circ(r_1))(\hat{\Omega}^\circ_{r_2}) &= \frac{r_2 - r_1}{\rho(t) - \rho(t-)} H(p(t) - p(t-))(\hat{\Omega}_t) \\ &= \frac{r_2 - r_1}{\rho(t) - \rho(t-)}(V(t) - V(t-)) = r_2 - r_1\,, \end{aligned}$$

where we used also (4.23).

Assume now that $r_1, r_2 \in [\rho(t), \rho(t+)]$. We claim that (4.33) holds also in this case. Arguing as in the previous step, by (4.29) and (4.30) we obtain

$$\begin{aligned} H(p^\circ(r_2) - p^\circ(r_1))(\hat{\Omega}^\circ_{r_2}) &= \frac{r_2 - r_1}{\rho(t+) - \rho(t)} H(p(t+) - p(t))(\hat{\Omega}_{t+}) \\ &= \frac{r_2 - r_1}{\rho(t+) - \rho(t)}(V(t+) - V(t)) = r_2 - r_1\,, \end{aligned}$$

where we used also (4.24).

Assume finally that $\rho(t-) \le r_1 < \rho(t) < r_2 \le \rho(t+)$ and let $r := \rho(t)$. Then (4.33) holds for r_1, r and for r, r_2. Therefore, using the triangle inequality, we obtain $H(p^\circ(r_2) - p^\circ(r_1))(\hat\Omega^\circ_{r_2}) \le H(p^\circ(r_2) - p^\circ(r))(\hat\Omega^\circ_{r_2}) + H(p^\circ(r) - p^\circ(r_1))(\hat\Omega^\circ_{r_2}) \le H(p^\circ(r_2) - p^\circ(r))(\hat\Omega^\circ_{r_2}) + H(p^\circ(r) - p^\circ(r_1))(\hat\Omega^\circ_{r}) = r_2 - r + r - r_1 = r_2 - r_1$. This concludes the proof of (4.32) in case (b).

Case (c). Since $r_1 \in \rho([0,T])$ there exists $t_1 \le t$ such that $r_1 = \rho(t_1) < r_2$. If $t_1 = t$ we have $r_1 = \rho(t) < r_2 \le \rho(t+)$ and we are in case (b), for which (4.32) is already proved. Therefore it is enough to consider the case $t_1 < t$.

Assume first that $r_2 = \rho(t-)$. If $\rho(t-) = \rho(t)$, then we are in case (a) for which (4.32) is already proved. If, instead, $\rho(t-) < \rho(t)$, by (4.27) and (4.28) we have $\hat\Omega^\circ_{r_2} = \hat\Omega_t$ and $p^\circ(r_2) = p(t-)$. Therefore

$$H(p^\circ(r_2) - p^\circ(r_1))(\hat\Omega^\circ_{r_2}) = H(p(t-) - p(t_1))(\hat\Omega_t)\,. \tag{4.34}$$

Using (4.3) we obtain

$$H(p(t-) - p(t_1))(\hat\Omega_t) = \lim_{\tau\to t-} H(p(\tau) - p(t_1))(\hat\Omega_t) \le V(t-) - V(t_1) \le \rho(t-) - \rho(t_1) = r_2 - r_1\,,$$

and by (4.34) we conclude that (4.32) holds in the special case $r_2 = \rho(t-)$.

Assume now that $r_2 \in (\rho(t-), \rho(t+)]$. We set $r := \rho(t-)$ and observe that (4.32) holds for r_1, r by the previous step and for r, r_2 by case (b). Therefore, using the triangle inequality, we obtain $H(p^\circ(r_2) - p^\circ(r_1))(\hat\Omega^\circ_{r_2}) \le H(p^\circ(r_2) - p^\circ(r))(\hat\Omega^\circ_{r_2}) + H(p^\circ(r) - p^\circ(r_1))(\hat\Omega^\circ_{r_2}) \le H(p^\circ(r_2) - p^\circ(r))(\hat\Omega^\circ_{r_2}) + H(p^\circ(r) - p^\circ(r_1))(\hat\Omega^\circ_{r}) \le r_2 - r + r - r_1 = r_2 - r_1$, which concludes the proof of (4.32) in case (c).

The proof in case (d) is similar to the proof in case (c), replacing $\rho(t-)$ by $\rho(t+)$. Finally, case (e) can be treated by interposing $r \in \rho([0,T])$ and using the triangle inequality and the estimates obtained in cases (c) and (d). □

We now show that p° has a time derivative a.e. in $[0, \rho(T)]$ in the sense of weak* convergence and that the increment of p° can be controlled using the integral of this time derivative.

Lemma 4.4. *Under the assumptions of Lemma 4.3 for a.e.* $r \in [0, \rho(T)]$ *there exists* $\dot p^\circ(r) \in \mathcal{M}_b(\hat\Omega^\circ_r; \mathbb{R}^{2\times2}_{sym})$ *with*

$$|\dot p^\circ(r)|(\hat\Omega^\circ_r) \le \frac{1}{c_{\mathbb{K}}}\,, \tag{4.35}$$

such that

$$\frac{p^\circ(r) - p^\circ(r-h)}{h} \rightharpoonup \dot p^\circ(r) \quad \text{weakly}^* \text{ in } \mathcal{M}_b(\hat\Omega^\circ_r; \mathbb{R}^{2\times2}_{sym}) \text{ as } h \to 0+\,. \tag{4.36}$$

Moreover for every open set $U \subset \mathbb{R}^2$, *the function* $r \mapsto |\dot p^\circ(r)|(\hat\Omega^\circ_r \cap U)$ *is measurable and*

$$|p^\circ(r_2) - p^\circ(r_1)|(\hat\Omega^\circ_{r_2} \cap U) \le \int_{r_1}^{r_2} |\dot p^\circ(r)|(\hat\Omega^\circ_r \cap U)\,dr\,. \tag{4.37}$$

for every $r_1, r_2 \in [0, \rho(T)]$, *with* $r_1 < r_2$.

Proof. The existence of $\dot p^\circ(r) \in \mathcal{M}_b(\hat\Omega^\circ_r; \mathbb{R}^{2\times2}_{sym})$ which satisfies (4.36) is proved in [5, Theorem 7.1]. Inequality (4.35) follows from (4.32).

Let us fix an open set $U \subset \mathbb{R}^2$. To prove the measurability of the function $r \mapsto |\dot p^\circ(r)|(\hat\Omega^\circ_r \cap U)$, for every $r \in [0, \rho(T)]$, $\varphi \in C^0_c(\hat\Omega^\circ_r; \mathbb{R}^{2\times2}_{sym})$, and $\mu \in \mathcal{M}_b(\hat\Omega^\circ_r; \mathbb{R}^{2\times2}_{sym})$ we define

$$\langle \varphi, \mu \rangle_{\hat\Omega^\circ_r} := \int_{\hat\Omega^\circ_r} \varphi : d\mu\,.$$

For every $\varphi \in C^0(\overline{\Omega}; \mathbb{R}^{2\times2}_{sym})$ we set

$$r_\varphi := \sup\{r \in [0, \rho(T)] : supp(\varphi) \cap \Gamma^\circ(r) = \emptyset\}\,,$$

and we observe that $supp(\varphi) \cap \Gamma^\circ(r) = \emptyset$, and hence $\varphi \in C^0_c(\hat\Omega^\circ_r; \mathbb{R}^{2\times 2}_{sym})$, whenever $r \in [0, r_\varphi)$. Since $\mathcal{D}_\Gamma(p; 0, T) < +\infty$, for every $\varphi \in C^0(\overline{\Omega}; \mathbb{R}^{2\times 2}_{sym})$ the functions $r \mapsto \langle \varphi, p^\circ(r)\rangle_{\hat\Omega^\circ_r}$ and $r \mapsto \langle \varphi, p^\circ(r-h)\rangle_{\hat\Omega^\circ_r}$ have bounded variation on $[h, r_\varphi - h]$ for every $h \in (0, r_\varphi/2)$. Therefore, using (4.36) we obtain that the function

$$r \mapsto \langle \varphi, \dot p^\circ(r)\rangle_{\hat\Omega^\circ_r} \quad \text{is measurable on } [0, r_\varphi). \tag{4.38}$$

It is well-known that for every $r \in [0, \rho(T)]$ we have

$$|\dot p^\circ(r)|(\hat\Omega^\circ_r \cap U) = \sup_{\substack{\varphi \in C^0(\overline{\Omega}; \mathbb{R}^{2\times 2}_{sym}) \\ \|\varphi\|_\infty \le 1,\ r_\varphi > r,\ \mathrm{supp}\varphi \subset U}} \langle \varphi, \dot p^\circ(r)\rangle_{\hat\Omega^\circ_r},$$

and that there exists a countable set $D \subset \{\varphi \in C^0(\overline{\Omega}; \mathbb{R}^{2\times 2}_{sym}) : \|\varphi\|_\infty \le 1,\ \mathrm{supp}\varphi \subset U\}$ with the following property: for every $\varphi \in C^0(\overline{\Omega}; \mathbb{R}^{2\times 2}_{sym})$, with $\|\varphi\|_\infty \le 1$, there exists a sequence of functions $\varphi_n \in D$ such that $\varphi_n \to \varphi$ uniformly on $\overline{\Omega}$ and $\mathrm{supp}\varphi_n \subset \mathrm{supp}\varphi$, which gives $r_{\varphi_n} \ge r_\varphi$. Therefore

$$|\dot p^\circ(r)|(\hat\Omega^\circ_r \cap U) = \sup_{\varphi \in D,\ r_\varphi > r} \langle \varphi, \dot p^\circ(r)\rangle_{\hat\Omega^\circ_r}.$$

The measurability of $r \mapsto |\dot p^\circ(r)|(\hat\Omega^\circ_r \cap U)$ follows from this equality and from (4.38).

To prove (4.37) we fix $r_1, r_2 \in [0, \rho(T)]$, with $r_1 < r_2$, and $\varphi \in C^0_c(\hat\Omega^\circ_{r_2}; \mathbb{R}^{2\times 2}_{sym})$ with $\|\varphi\|_\infty \le 1$ and $\mathrm{supp}\varphi \subset U$. Since the function $r \mapsto \langle \varphi, p^\circ(r)\rangle_{\hat\Omega^\circ_{r_2}}$ is Lipschitz continuous on $[0, r_2]$ by (4.32) and its derivative equals $\langle \varphi, \dot p^\circ(r)\rangle_{\hat\Omega^\circ_{r_2}}$ for a.e. $r \in [0, r_2]$ by (4.36), using the Fundamental Theorem of Calculus for absolutely continuous functions we obtain

$$\langle \varphi, p^\circ(r_2)\rangle_{\hat\Omega^\circ_{r_2}} - \langle \varphi, p^\circ(r_1)\rangle_{\hat\Omega^\circ_{r_2}} = \int_{r_1}^{r_2} \langle \varphi, \dot p^\circ(r)\rangle_{\hat\Omega^\circ_{r_2}} dr,$$

hence

$$|\langle \varphi, p^\circ(r_2) - p^\circ(r_1)\rangle_{\hat\Omega^\circ_{r_2}}| \le \int_{r_1}^{r_2} |\dot p^\circ(r)|(\hat\Omega^\circ_{r_2} \cap U)\, dr \le \int_{r_1}^{r_2} |\dot p^\circ(r)|(\hat\Omega^\circ_r \cap U)\, dr.$$

Taking the supremum over all $\varphi \in C^0_c(\hat\Omega^\circ_{r_2}; \mathbb{R}^{2\times 2}_{sym})$ with $\|\varphi\|_\infty \le 1$ and $\mathrm{supp}\varphi \subset U$ we obtain (4.37). ☐

5. The case of rectilinear cracks

This section is devoted to the proof of Theorem 2.3 in the special case in which the sets K_0 and K in (2.8) are segments. More precisely, we assume that

$$K \text{ is a closed segment of endpoints } x^1, x^2 \in \partial\Omega, \tag{5.1}$$

$$K_0 \text{ is a closed segment containing } x^1 \text{ and contained in } K. \tag{5.2}$$

Moreover, we assume that

$$\text{the interior points of } K \text{ are contained in } \Omega, \tag{5.3}$$

$$\Omega \setminus K \text{ is the union of two open sets with Lipschitz boundary.} \tag{5.4}$$

By (2.8) and (5.2) we have

$$x^1 \in \Gamma(t) \text{ for every } t \in [0, T]. \tag{5.5}$$

Since $\Gamma(t)$ is connected and contained in K, this implies that for every t there exists a unique point $x(t) \in K$ such that

$$\Gamma(t) \text{ is the segment of endpoints } x^1 \text{ and } x(t). \tag{5.6}$$

Let

$$a := \sup\{t \in [0,T] : x(t) = x^1\} = \sup\{t \in [0,T] : \Gamma(t) = \{x^1\}\}\,,$$
$$b := \inf\{t \in [0,T] : x(t) = x^2\} = \inf\{t \in [0,T] : \Gamma(t) = K\}\,,$$

with the usual convention $\sup\emptyset = 0$ and $\inf\emptyset = T$. It follows from the definition that $a \le b$ and that $\ell(t) = 0$ for $t \in [0,a)$, $0 < \ell(t) < L$ for $t \in (a,b)$, and $\ell(t) = L$ for $t \in (b,T]$.

To prove Theorem 2.3 it is enough to show that ℓ^{cont} is constant on (a,b), assuming $a < b$. To this aim we want to estimate the length increment for all pairs of consecutive points of a sufficiently fine subdivision of (a,b). By Proposition 3.2 for every $t_1, t_2 \in [0,T]$, with $t_1 < t_2$, we have

$$\frac{\lambda_{\mathbb{C}}}{2}\int_{\Omega_{t_2}} |e(t_2)-e(t_1)|^2 dx + \beta(\ell(t_2)-\ell(t_1)) \le \int_{t_1}^{t_2}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_1)) : E\dot w(\tau))\,dx\Big)d\tau$$
$$+C_{\mathbb{K}}\int_{\Gamma(t_1,t_2)} |[u(t_2)-u(t_1)]|\,d\mathcal{H}^1, \tag{5.7}$$

where now $\Gamma(t_1,t_2)$ is the segment with endpoints $x(t_1)$ (not included) and $x(t_2)$ (included).

In the next proposition we will improve now this estimate by replacing its last term with terms depending on the increment of the plastic strain p. To this aim it is convenient to introduce $B(t_1,t_2)$, the set obtained by removing $\Gamma(t_1,t_2)$ from the open ball centred at $x(t_2)$ and whose boundary contains $x(t_1)$. In other words, setting $\ell(t_1,t_2) := \ell(t_2)-\ell(t_1)$, we have that

$$B(t_1,t_2) := B_{\ell(t_1,t_2)}(x(t_2)) \setminus \Gamma(t_1,t_2)\,,$$

if $\ell(t_1,t_2) > 0$, and $B(t_1,t_2) := \emptyset$ if $\ell(t_1,t_2) = 0$.

Proposition 5.1. *Let $w \in AC([0,T]; H^1(\Omega\setminus K_0;\mathbb{R}^2))$, let $t \mapsto (u(t),e(t),p(t),\Gamma(t))$ be a quasistatic evolution with boundary value w, according to Definition 2.1, and let $\eta > 0$ be such that*

$$\frac{\pi c_0^2 C_{\mathbb{K}}^2 \eta}{\lambda_{\mathbb{C}}} < \frac{\beta}{2}\,, \tag{5.8}$$

where $c_0 > 0$ is the constant in Theorem 3.3. Assume (5.1)-(5.4). *Then*

$$\frac{\lambda_{\mathbb{C}}}{4}\int_{\Omega_{t_2}} |e(t_2)-e(t_1)|^2 + \frac{\beta}{2}(\ell(t_2)-\ell(t_1))$$
$$\le \int_{t_1}^{t_2}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_1)) : E\dot w(\tau)\,dx\Big)d\tau + c_0C_{\mathbb{K}}|p(t_2)-p(t_1)|(B(t_1,t_2)) \tag{5.9}$$

for every $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$, $\ell(t_2)-\ell(t_1) < \eta$, and $B(t_1,t_2) \subset \Omega_{t_2}$.

Proof. Let $t_1, t_2 \in [0,T]$ be as in the statement of the proposition. Since K is a segment, by Theorem 3.3, which is much easier in this case, we obtain

$$\int_{\Gamma(t_1,t_2)} |[u(t_2)-u(t_1)]|\,d\mathcal{H}^1 \le c_0|Eu(t_2)-Eu(t_1)|(B(t_1,t_2))\,.$$

Therefore (2.6) and Young's inequality give

$$\int_{\Gamma(t_1,t_2)} |[u(t_2)-u(t_1)]|\,d\mathcal{H}^1 \le c_0\int_{B(t_1,t_2)}|e(t_2)-e(t_1)|\,dx + c_0|p(t_2)-p(t_1)|(B(t_1,t_2))$$
$$\le \frac{\pi c_0^2 C_{\mathbb{K}}}{\lambda_{\mathbb{C}}}(\ell(t_2)-\ell(t_1))^2 + \frac{\lambda_{\mathbb{C}}}{4C_{\mathbb{K}}}\int_{B(t_1,t_2)}|e(t_2)-e(t_1)|^2 dx + c_0|p(t_2)-p(t_1)|(B(t_1,t_2))\,.$$

Hence (5.7) can be written as

$$\begin{aligned}&\frac{\lambda_{\mathbb{C}}}{2}\int_{\Omega_{t_2}}|e(t_2)-e(t_1)|^2+\beta(\ell(t_2)-\ell(t_1))\\ &\leq\int_{t_1}^{t_2}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_1)):E\dot w(\tau)\,dx\Big)d\tau+\frac{\pi c_0^2C_{\mathbb{K}}^2}{\lambda_{\mathbb{C}}}(\ell(t_2)-\ell(t_1))^2\\ &+\frac{\lambda_{\mathbb{C}}}{4}\int_{B(t_1,t_2)}|e(t_2)-e(t_1)|^2dx+c_0C_{\mathbb{K}}|p(t_2)-p(t_1)|(B(t_1,t_2))\,,\end{aligned}$$

and taking (5.8) into account we obtain (5.9). □

Proof of Theorem 2.3 under hypotheses (5.1)-(5.5). Let $\eta>0$ be such that (5.8) holds. Estimate (5.9) implies that

$$\begin{aligned}\frac{\beta}{2}(\ell(t_2)-\ell(t_1))\leq&\int_{t_1}^{t_2}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_1)):E\dot w(\tau)\,dx\Big)d\tau\\ &+c_0C_{\mathbb{K}}|p(t_2)-p(t_1)|(B(t_1,t_2))\end{aligned}\tag{5.10}$$

whenever $\ell(t_2)-\ell(t_1)<\eta$ and $B(t_1,t_2)\subset\Omega_{t_2}$. Since the inclusion $B(t_1,t_2)\subset\Omega$ does not necessarily hold for any pair $t_1,t_2\in(a,b)$ with $\ell(t_2)-\ell(t_1)<\eta$, in order to apply (5.10) we replace a and b by sequences a^δ and b^δ defined by

$$\begin{aligned}a^\delta&:=\inf\{t\in[a,b]:x(t)\notin B_\delta(x^1)\}\,,\\ b^\delta&:=\sup\{t\in[a,b]:x(t)\notin B_\delta(x^2)\}\,,\end{aligned}$$

with the convention that $\inf\emptyset=b$ and $\sup\emptyset=a$. Since

$$a^\delta\to a\quad\text{and}\quad b^\delta\to b\quad\text{as }\delta\to0+\,,$$

in order to prove the theorem it is enough to show that ℓ^{cont} is constant on (a^δ,b^δ) for $\delta>0$ small enough. Since the interior points of the segment K belong to Ω, for every $\delta>0$ there exists $\rho_\delta>0$ such that

$$\operatorname{dist}(x(t),\partial\Omega)>\rho_\delta\quad\text{for every }t\in[a^\delta,b^\delta]\,.$$

Let us fix $\delta>0$ such that $a\leq a^\delta<b^\delta\leq b$, and $\eta\in(0,\frac{\rho_\delta}{2})$ such that (5.8) is satisfied. Then (5.10) holds for every $t_1,t_2\in[a^\delta,b^\delta]$ with $t_1<t_2$ and $\ell(t_2)-\ell(t_1)<\eta$.

Given $\varepsilon\in(0,\eta)$, let $a^\delta=\tau_0<\tau_1<\dots<\tau_n=b^\delta$ be such that $[\ell](t)<\varepsilon$ for every $t\in[a^\delta,b^\delta]\setminus\{\tau_0,\tau_1,\dots,\tau_n\}$. Let us fix $\theta>0$ such that $5\theta<\tau_j-\tau_{j-1}$ for $j=1,\dots,n$. By [6, Lemma 4.8], applied to the intervals $[\tau_{j-1}+\theta,\tau_j-\theta]$, there exists $\zeta\in(0,\theta)$ such that if t_1, $t_2\in[\tau_{j-1}+\theta,\tau_j-\theta]$ for some $j=1,\dots,n$ and $0<t_2-t_1<\zeta$, then $\ell(t_2)-\ell(t_1)<\varepsilon<\eta$ and therefore

$$B(t_1,t_2)\subset B_\varepsilon(x(t_2))\cap\Omega_{t_2}\,.\tag{5.11}$$

Since a Lebesgue integral can be approximated by suitable Riemann sums (see, for instance, [6, Proposition 4.10]) there exists a subdivision $a^\delta=t_0<t_1<\dots<t_m=b^\delta$, with $t_i-t_{i-1}<\zeta$ for every $i=1,\dots,m$, such that

$$\sum_{i=1}^m\int_{t_{i-1}}^{t_i}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_{i-1})):E\dot w(\tau)\,dx\Big)d\tau<\varepsilon\,.\tag{5.12}$$

For $j=0,\dots,n-1$ there exists $m_j\in\{1,\dots,m\}$ such that $\tau_j+2\theta<t_{m_j}<\tau_j+3\theta$, and for $j=1,\dots,n$ there exists $n_j\in\{1,\dots,m\}$ such that $\tau_j-2\theta<t_{n_j}<\tau_j-\theta$. Note that, since $5\theta<\tau_j-\tau_{j-1}$ for $j=1,\dots,n$, we have $m_0<n_1<m_1<\dots<n_{n-1}<m_{n-1}<n_n$.

By (5.10) it follows that

$$\ell^{cont}(\tau_j-2\theta)-\ell^{cont}(\tau_{j-1}+3\theta)\le \ell^{cont}(t_{n_j})-\ell^{cont}(t_{m_{j-1}})\le \ell(t_{n_j})-\ell(t_{m_{j-1}})$$

$$\le \sum_{i=m_{j-1}}^{n_j}(\ell(t_i)-\ell(t_{i-1}))\le \frac{2}{\beta}\sum_{i=m_{j-1}}^{n_j}\int_{t_{i-1}}^{t_i}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_{i-1})):E\dot w(\tau)\,dx\Big)d\tau$$

$$+\frac{2c_0C_{\mathbb{K}}}{\beta}\sum_{i=m_{j-1}}^{n_j}|p(t_i)-p(t_{i-1})|(B(t_{i-1},t_i))\,. \tag{5.13}$$

The last term in (5.13) can be estimated using the change of variables introduced in (4.26)-(4.30). By (4.26) and (5.6) for every $r\in[0,\rho(T)]$ there exists $x^\circ(r)\in K$ such that $\Gamma^\circ(r)$ is the segment of endpoints x^1 and $x^\circ(r)$. Moreover $x(t)=x^\circ(\rho(t))$ for every $t\in[0,T]$. For every $r\in[0,\rho(T)]$ let $\ell^\circ(r):=\mathcal{H}^1(\Gamma^\circ(r))$, so that $\ell(t)=\ell^\circ(\rho(t))$ for every $t\in[0,T]$. By Lemma 4.4, taking into account (4.31) and (5.11), we obtain

$$|p(t_i)-p(t_{i-1})|(B(t_{i-1},t_i))\le |p(t_i)-p(t_{i-1})|(B_\varepsilon(x(t_i))\cap\Omega_{t_i})$$

$$=|p^\circ(\rho(t_i))-p^\circ(\rho(t_{i-1}))|(B_\varepsilon(x(t_i))\cap\hat\Omega^\circ_{\rho(t_i)})\le\int_{\rho(t_{i-1})}^{\rho(t_i)}|\dot p^\circ(r)|(B_\varepsilon(x(t_i))\cap\hat\Omega^\circ_r)dr\,. \tag{5.14}$$

If $m_{j-1}\le i\le n_j$ and $\rho(t_{i-1})\le r\le\rho(t_i)$ then we have $\ell(t_{i-1})=\ell^\circ(\rho(t_{i-1}))\le\ell^\circ(r)\le\ell^\circ(\rho(t_i))=\ell(t_i)$. Since $\ell(t_i)-\ell(t_{i-1})<\varepsilon$ we obtain $|x(t_i)-x^\circ(r)|=\ell(t_i)-\ell^\circ(r)<\varepsilon$, which implies that $B_\varepsilon(x(t_i))\subset B_{2\varepsilon}(x^\circ(r))$. Hence (5.14) gives

$$\sum_{i=m_{j-1}}^{n_j}|p(t_i)-p(t_{i-1})|(B(t_{i-1},t_i))\le\int_{\rho(\tau_{j-1})}^{\rho(\tau_j)}|\dot p^\circ(r)|(B_{2\varepsilon}(x^\circ(r))\cap\hat\Omega^\circ_r)\,dr\,, \tag{5.15}$$

which, together with (5.13), implies

$$\ell^{cont}(\tau_j-2\theta)-\ell^{cont}(\tau_{j-1}+3\theta)\le\frac{2}{\beta}\sum_{i=m_{j-1}}^{n_j}\int_{t_{i-1}}^{t_i}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_{i-1})):E\dot w(\tau)\,dx\Big)d\tau$$

$$+\frac{2c_0C_{\mathbb{K}}}{\beta}\int_{\rho(\tau_{j-1})}^{\rho(\tau_j)}|\dot p^\circ(r)|(B_{2\varepsilon}(x^\circ(r))\cap\hat\Omega^\circ_r)\,dr\,. \tag{5.16}$$

We note that $B_{2\varepsilon}(x^\circ(r))\cap\hat\Omega^\circ_r\to\emptyset$ as $\varepsilon\to0+$, since $x^\circ(r)\in\Gamma^\circ(r)$. Therefore by Lemma 4.4 we can apply the Dominated Convergence Theorem and we obtain

$$\int_{\rho(a^\delta)}^{\rho(b^\delta)}|\dot p^\circ(r)|(B_\varepsilon(x^\circ(r))\cap\hat\Omega^\circ_r)\,dr\to0\quad\text{as }\varepsilon\to0+\ .$$

Therefore there exists a continuous function $\omega\colon[0,+\infty)\to[0,+\infty)$ with $\omega(0)=0$ such that

$$\int_{\rho(a^\delta)}^{\rho(b^\delta)}|\dot p^\circ(r)|(B_\varepsilon(x^\circ(r))\cap\hat\Omega^\circ_r)\,dr<\omega(\varepsilon)\,. \tag{5.17}$$

Summing (5.16) over j we get

$$\sum_{j=1}^{n}(\ell^{cont}(\tau_j-2\theta)-\ell^{cont}(\tau_{j-1}+3\theta))$$

$$\le\frac{2}{\beta}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_{i-1})):E\dot w(\tau)\,dx\Big)d\tau$$

$$+\frac{2c_0C_{\mathbb{K}}}{\beta}\int_{\rho(a^\delta)}^{\rho(b^\delta)}|\dot p^\circ(r)|(B_{2\varepsilon}(x^\circ(r))\cap\hat\Omega^\circ_r)\,dr<\frac{2}{\beta}\varepsilon+\frac{2c_0C_{\mathbb{K}}}{\beta}\omega(2\varepsilon),$$

where in the last inequality we used (5.12) and (5.17). Taking the limit as $\theta \to 0+$, from the continuity of ℓ^{cont} we obtain $\ell^{cont}(b^\delta) - \ell^{cont}(a^\delta) \le \frac{2}{\beta}\varepsilon + \frac{2c_0 C_{\mathbb{K}}}{\beta}\omega(2\varepsilon)$, and as $\varepsilon \to 0+$ we obtain $\ell^{cont}(b^\delta) \le \ell^{cont}(a^\delta)$. Since ℓ^{cont} is nondecreasing, this inequality implies $\ell^{cont}(t) = \ell^{cont}(a^\delta)$ for every $t \in [a^\delta, b^\delta]$. This concludes the proof of Theorem 2.3 under assumption (5.1)-(5.5). □

6. Preparation of the proof in the general case

In this section we prepare the proof of Theorem 2.3 in the general case where the set K has several branches. To this aim we introduce some technical tools that will allow us to deal with this more complex structure. In particular, for every $h \in \{1, \dots, k\}$ we have to consider separately the time intervals in which $\mathcal{H}^1(\Gamma(t) \cap K_h) = 0$, $0 < \mathcal{H}^1(\Gamma(t) \cap K_h) < \mathcal{H}^1(K_h)$, or $\mathcal{H}^1(\Gamma(t) \cap K_h) = \mathcal{H}^1(K_h)$. The main result obtained at the end of this section is an estimate which adapts Proposition 5.1 to the case of a general K.

Let $\Gamma\colon [0,T] \to \mathcal{K}$ be an increasing function. To simplify the exposition we may assume that for every $t \in [0,T]$, with $\Gamma(t) \ne \emptyset$,

$$\text{there exists no } h \in \{1, \dots, k\} \text{ such that } \Gamma(t) \subset K_h^0, \tag{6.1}$$

where K_h^0 is defined in (2.10). Indeed, if there exists $h_0 \in \{1, \dots, k\}$ such that $\emptyset \ne \Gamma(t_0) \subset K_{h_0}^0$ for some $t_0 \in [0,T]$, we consider the intersection $\hat{\Gamma}$ of all nonempty sets $\Gamma(t)$ with $t \in [0,T]$, we select a point $\hat{x} \in \hat{\Gamma}$, and we replace $\hat{K}$ by $\hat{K} \cup \{\hat{x}\}$ and K_{h_0} by the two connected components of $K_{h_0} \setminus \{\hat{x}\}$. In this way we obtain a new decomposition (2.9) that satisfies (6.1) and we can prove Theorem 2.3 using this decomposition.

For every $t \in [0,T]$ and every $h = 1, \dots, k$ we define

$$\ell_h(t) := \mathcal{H}^1(\Gamma(t) \cap K_h). \tag{6.2}$$

Moreover, we set $L_h := \mathcal{H}^1(K_h)$ and

$$a_h := \sup\{t \in [0,T] : \ell_h(t) = 0\}, \tag{6.3}$$

$$b_h := \inf\{t \in [0,T] : \ell_h(t) = L_h\}, \tag{6.4}$$

with the usual convention $\sup \emptyset = 0$ and $\inf \emptyset = T$. It follows from the definition that $a_h \le b_h$ and that $\ell_h(t) = 0$ for $t \in [0, a_h)$, $0 < \ell_h(t) < L_h$ for $t \in (a_h, b_h)$, and $\ell_h(t) = L_h$ for $t \in (b_h, T]$. This justifies the following terminology: the branch K_h^0 at time t is void if $t \in [0, a_h)$, active if $t \in [a_h, b_h]$, and full if $t \in (b_h, T]$.

Let $\gamma_h\colon [0, L_h] \to \mathbb{R}^2$ be the arc-length parametrization of the curve K_h, with $\gamma_h(0) = x_h^1$ and $\gamma_h(L_h) = x_h^2$. Since $\Gamma(t)$ is connected and satisfies (6.1), if $a_h < b_h$ there exist two functions s_h^1, $s_h^2\colon [a_h, b_h] \to [0, L_h]$, with s_h^1 nondecreasing and s_h^2 nonincreasing, such that

$$\Gamma(t) \cap K_h^0 = \gamma_h((0, s_h^1(t)]) \cup \gamma_h([s_h^2(t), L_h)) \tag{6.5}$$

for every $t \in [a_h, b_h]$. Then $0 < \ell_h(t) = s_h^1(t) + L_h - s_h^2(t) < L_h$ for every $t \in (a_h, b_h)$.

To prove Theorem 2.3 we need a proposition similar to Proposition 5.1 to estimate the jump term and the boundary term in (3.3). Since now the set K has many branches, we introduce a notation to indicate which branches are void, active, or full at time t. For every $t \in [0,T]$ we consider the sets of indices $\check{I}(t)$, $I(t)$, and $\hat{I}(t)$ defined by

$$\check{I}(t) := \{h \in \{1, \dots, k\} : t \in [0, a_h)\}, \tag{6.6}$$

$$I(t) := \{h \in \{1, \dots, k\} : t \in [a_h, b_h]\}, \tag{6.7}$$

$$\hat{I}(t) := \{h \in \{1, \dots, k\} : t \in (b_h, T]\}. \tag{6.8}$$

To estimate the integrals on $\Gamma(t_1, t_2)$ in (3.3) we fix $t_1, t_2 \in [0,T]$, with $t_1 < t_2$, and we prove the estimate assuming in addition that

$$\check{I}(t_1) = \check{I}(t_2), \ I(t_1) = I(t_2), \text{ and } \hat{I}(t_1) = \hat{I}(t_2). \tag{6.9}$$

In particular, this implies that $\Gamma(t_1,t_2)\cap K_h^0=\emptyset$ unless $h\in I(t_1)=I(t_2)$. We also observe that the assumption $h\in I(t_1)=I(t_2)$ implies the inequalities $a_h\le t_1<t_2\le b_h$, hence s_h^1 and s_h^2 are well-defined.

Moreover, we assume that there exists

$$\eta\in(0,\delta_0)\,, \tag{6.10}$$

where δ_0 is the constant introduced in Lemma 3.4, with the following properties:
if $h\in I(t_1)=I(t_2)$ and $s_h^1(t_2)>0$, then

$$s_h^1(t_2)-s_h^1(t_1)<\eta\,, \tag{6.11}$$

$$B_{2\eta}(\gamma_h(s_h^1(t_2)))\cap(\partial\Omega\setminus K_h^0)=\emptyset\,, \tag{6.12}$$

$$B_{2\eta}(\gamma_h(s_h^1(t_2)))\cap\check K_h=\emptyset\,, \tag{6.13}$$

$$B_{2\eta}(\gamma_h(s_h^1(t_2)))\cap\Gamma(t_2)=B_{2\eta}(\gamma_h(s_h^1(t_2)))\cap\gamma_h([0,s_h^1(t_2)])\,; \tag{6.14}$$

if $h\in I(t_1)=I(t_2)$ and $s_h^2(t_2)<L_h$, then

$$s_h^2(t_1)-s_h^2(t_2)<\eta\,, \tag{6.15}$$

$$B_{2\eta}(\gamma_h(s_h^2(t_2)))\cap(\partial\Omega\setminus K_h^0)=\emptyset\,, \tag{6.16}$$

$$B_{2\eta}(\gamma_h(s_h^2(t_2)))\cap\check K_h=\emptyset\,, \tag{6.17}$$

$$B_{2\eta}(\gamma_h(s_h^2(t_2)))\cap\Gamma(t_2)=B_{2\eta}(\gamma_h(s_h^2(t_2)))\cap\gamma_h([s_h^2(t_2),L_h])\,. \tag{6.18}$$

Note that, if $s_h^1(t_1)=0$, we would have $\gamma_h(s_h^1(t_1))=x_h^1\in\hat K\subset\check K_h$, hence the inequality $s_h^1(t_2)>0$, together with (6.11) and (6.13), implies that

$$s_h^1(t_1)>0\,. \tag{6.19}$$

Similarly the inequality $s_h^2(t_2)<L_h$, together with(6.15) and (6.17), implies that

$$s_h^2(t_1)<L_h\,. \tag{6.20}$$

For future use we observe that, if (6.12)-(6.14) and (6.16)-(6.18) hold for some $\eta>0$, then they also hold for all smaller values of η.

For every $h\in I(t_1)=I(t_2)$ we define

$$K_h^1(t_1,t_2):=\gamma_h([s_h^1(t_1),s_h^1(t_2)])\,, \tag{6.21}$$

$$K_h^2(t_1,t_2):=\gamma_h([s_h^2(t_2),s_h^2(t_1)])\,. \tag{6.22}$$

By (6.9), we have

$$\begin{aligned}\Gamma(t_1,t_2)&\simeq\bigcup_{h=1}^k\Gamma(t_1,t_2)\cap K_h\simeq\bigcup_{h\in I(t_2)}\Gamma(t_1,t_2)\cap K_h\\&\simeq\bigcup_{h\in I(t_2)}K_h^1(t_1,t_2)\cup\bigcup_{h\in I(t_2)}K_h^2(t_1,t_2)\,,\end{aligned} \tag{6.23}$$

where $\simeq$ means equality up to an $\mathcal H^1$-negligible set.

Therefore, if $u(t_1)\in BD(\Omega_{t_1})$ and $u(t_2)\in BD(\Omega_{t_2})$ we have

$$\begin{aligned}\int_{\Gamma(t_1,t_2)\cap\Omega}|[u(t_2)-u(t_1)]|\,d\mathcal H^1&=\sum_{\substack{h\in I(t_2)\\ s_h^1(t_2)>0}}\int_{K_h^1(t_1,t_2)\cap\Omega}|[u(t_2)-u(t_1)]|\,d\mathcal H^1\\&+\sum_{\substack{h\in I(t_2)\\ s_h^2(t_2)<L_h}}\int_{K_h^2(t_1,t_2)\cap\Omega}|[u(t_2)-u(t_1)]|\,d\mathcal H^1\,.\end{aligned} \tag{6.24}$$

If in addition $w(t_1), w(t_2) \in H^1(\Omega \setminus K_0; \mathbb{R}^2)$ we also have

$$\begin{aligned}&\int_{\Gamma(t_1,t_2)\cap\partial\Omega} |u(t_2) - u(t_1) - w(t_2) + w(t_1)|\, d\mathcal{H}^1 \\ &= \sum_{\substack{h\in I(t_2)\\ s_h^1(t_2)>0}} \int_{K_h^1(t_1,t_2)\cap\partial\Omega} |u(t_2) - u(t_1) - w(t_2) + w(t_1)|\, d\mathcal{H}^1 \\ &+ \sum_{\substack{h\in I(t_2)\\ s_h^2(t_2)<L_h}} \int_{K_h^2(t_1,t_2)\cap\partial\Omega} |u(t_2) - u(t_1) - w(t_2) + w(t_1)|\, d\mathcal{H}^1 \,.\end{aligned} \tag{6.25}$$

If $h \in I(t_1) = I(t_2)$, let $B^{h,1}(t_1, t_2)$ be the intersection with $\overline{\Omega}$ of the set obtained by removing $K_h^1(t_1, t_2)$ from the open ball centred at $\gamma_h(s_h^1(t_2))$ and whose boundary contains $\gamma_h(s_h^1(t_1))$. In symbols, setting $r_h^1(t_1, t_2) := |\gamma_h(s_h^1(t_2)) - \gamma_h(s_h^1(t_1))| \le s_h^1(t_2) - s_h^1(t_1)$, we have that

$$B^{h,1}(t_1, t_2) := B_{r_h^1(t_1,t_2)}(\gamma_h(s_h^1(t_2))) \cap \overline{\Omega} \setminus K_h^1(t_1, t_2)\,, \tag{6.26}$$

and $B^{h,1}(t_1, t_2) := \emptyset$ if $r_h^1(t_1, t_2) = 0$. Similarly, we define

$$B^{h,2}(t_1, t_2) := B_{r_h^2(t_1,t_2)}(\gamma_h(s_h^2(t_2))) \cap \overline{\Omega} \setminus K_h^2(t_1, t_2)\,, \tag{6.27}$$

where $r_h^2(t_1, t_2) := |\gamma_h(s_h^2(t_2)) - \gamma_h(s_h^2(t_1))| \le s_h^2(t_1) - s_h^2(t_2)$, and $B^{h,2}(t_1, t_2) := \emptyset$ if $r_h^2(t_1, t_2) = 0$.

The following proposition improves the estimate obtained in Proposition 3.2. More precisely, it replaces the last two terms in (3.3) by terms depending on the increment of the plastic strain p.

Proposition 6.1. *Let $w \in AC([0, T]; H^1(\Omega\setminus K_0; \mathbb{R}^2))$ and let $t \mapsto (u(t), e(t), p(t), \Gamma(t))$ be a quasistatic evolution with boundary value w, according to Definition 2.1. Assume that* (6.1) *holds for every $t \in [0, T]$, with $\Gamma(t) \neq \emptyset$. Let $t_1, t_2 \in [0, T]$, with $t_1 < t_2$. Assume that* (6.9)-(6.18) *are satisfied and that*

$$\frac{\pi c_0^2 C_{\mathbb{K}}^2 \eta}{\lambda_{\mathbb{C}}} < \frac{\beta}{2}\,, \tag{6.28}$$

where $c_0 > 0$ is the constant in Theorem 3.3. Then

$$\begin{aligned}\frac{\lambda_{\mathbb{C}}}{4} \int_{\Omega_{t_2}} |e(t_2) - e(t_1)|^2 + \frac{\beta}{2}(\ell(t_2) - \ell(t_1)) &\le \int_{t_1}^{t_2} \Big(\int_{\Omega_\tau} (\sigma(\tau) - \sigma(t_1)) : E\dot w(\tau)\, dx\Big) d\tau \\ &+ c_0 C_{\mathbb{K}} \sum_{h\in I(t_2)} \int_{B^{h,1}(t_1,t_2)\cup B^{h,2}(t_1,t_2)} |Ew(t_2) - Ew(t_1)|\, dx \\ &+ \sqrt{2} c_0 C_{\mathbb{K}} \sum_{h\in I(t_2)} |p(t_2) - p(t_1)|(B^{h,1}(t_1, t_2) \cup B^{h,2}(t_1, t_2))\,.\end{aligned} \tag{6.29}$$

Proof. If $I(t_1) = I(t_2) = \emptyset$, then the remark after (6.9) implies that $\Gamma(t_1, t_2) \simeq \emptyset$ and the conclusion follows from Proposition 3.2.

If $I(t_1) = I(t_2) \neq \emptyset$ we fix $h \in I(t_1) = I(t_2)$. We observe that $r_h^\lambda(t_1, t_2) < \eta < \delta_0$ for $\lambda = 1, 2$, by (6.11), (6.15), and (6.10). If $K_h^0 \subset \Omega$ we can apply Corollary 3.5 with $x_1 = \gamma_h(s_h^\lambda(t_1))$ and $x_2 = \gamma_h(s_h^\lambda(t_2))$, and we obtain

$$\begin{aligned}&\int_{K_h^\lambda(t_1,t_2)} |[u(t_2) - u(t_1)]|\, d\mathcal{H}^1 \le c_0 |Eu(t_2) - Eu(t_1)|(B^{h,\lambda}(t_1, t_2)) \\ &\le c_0 \int_{B^{h,\lambda}(t_1,t_2)} |e(t_2) - e(t_1)|\, dx + c_0 |p(t_2) - p(t_1)|(B^{h,\lambda}(t_1, t_2))\,,\end{aligned}$$

where in the last inequality we used (2.6). Note that we also used the inclusion $B^{h,\lambda}(t_1,t_2) \subset \Omega_{t_2}$, which follows from Lemma 3.4 and (6.12)-(6.14) and (6.16)-(6.18).

If $K_h^0 \subset \partial\Omega$ we can apply Corollary 3.6 with $x_1 = \gamma_h(s_h^\lambda(t_1))$ and $x_2 = \gamma_h(s_h^\lambda(t_2))$, and, using again (2.6) we obtain

$$\begin{aligned}
&\int_{K_h^\lambda(x_1,x_2)} |u(t_2) - u(t_1) - w(t_2) + w(t_1)|\, d\mathcal{H}^1 \\
&\leq c_0 |Eu(t_2) - Eu(t_1)|(B^{h,\lambda}(t_1,t_2) \cap \Omega) + c_0 \int_{B^{h,\lambda}(t_1,t_2)} |Ew(t_2) - Ew(t_1)|\, dx \\
&\quad + c_0 \int_{B^{h,\lambda}(t_1,t_2)\cap\partial\Omega} |u(t_2) - u(t_1) - w(t_2) + w(t_1)|\, d\mathcal{H}^1 \\
&\leq c_0 \int_{B^{h,\lambda}(t_1,t_2)} |e(t_2) - e(t_1)|\, dx + \sqrt{2} c_0 |p(t_2) - p(t_1)|(B^{h,\lambda}(t_1,t_2)) \\
&\quad + c_0 \int_{B^{h,\lambda}(t_1,t_2)} |Ew(t_2) - Ew(t_1)|\, dx\,,
\end{aligned}$$

where in the last inequality we used (2.7) and the estimate $|a|\,|b| \leq \sqrt{2}|a \odot b|$. Therefore the previous inequalities give

$$\begin{aligned}
&\int_{K_h^\lambda(t_1,t_2)\cap\Omega} |[u(t_2) - u(t_1)]|\, d\mathcal{H}^1 + \int_{K_h^\lambda(x_1,x_2)\cap\partial\Omega} |u(t_2) - u(t_1) - w(t_2) + w(t_1)|\, d\mathcal{H}^1 \\
&\leq c_0 \int_{B^{h,\lambda}(t_1,t_2)} |e(t_2) - e(t_1)|\, dx + \sqrt{2} c_0 |p(t_2) - p(t_1)|(B^{h,\lambda}(t_1,t_2)) \\
&\quad + c_0 \int_{B^{h,\lambda}(t_1,t_2)} |Ew(t_2) - Ew(t_1)|\, dx \qquad (6.30)\\
&\leq \frac{\pi c_0^2 C_{\mathbb{K}}}{\lambda_{\mathbb{C}}} (s_h^\lambda(t_2) - s_h^\lambda(t_1))^2 + \frac{\lambda_{\mathbb{C}}}{4C_{\mathbb{K}}} \int_{B^{h,\lambda}(t_1,t_2)} |e(t_2) - e(t_1)|^2 dx \\
&\quad + c_0\sqrt{2} |p(t_2) - p(t_1)|(B^{h,\lambda}(t_1,t_2)) + c_0 \int_{B^{h,\lambda}(t_1,t_2)} |Ew(t_2) - Ew(t_1)|\, dx\,,
\end{aligned}$$

for $\lambda = 1, 2$, where in the last step we used Young's inequality.

By (3.3), (6.24), and (6.25) we have

$$\begin{aligned}
&\frac{\lambda_{\mathbb{C}}}{2} \int_{\Omega_{t_2}} |e(t_2) - e(t_1)|^2 + \beta(\ell(t_2) - \ell(t_1)) \leq \int_{t_1}^{t_2} \Big(\int_{\Omega_\tau} (\sigma(\tau) - \sigma(t_1)) : E\dot{w}(\tau)\, dx \Big) d\tau \\
&+ C_{\mathbb{K}} \sum_{h\in I(t_2)} \int_{K_h^1(t_1,t_2)} |[u(t_2) - u(t_1)]|\, d\mathcal{H}^1 + C_{\mathbb{K}} \sum_{h\in I(t_2)} \int_{K_h^2(t_1,t_2)} |[u(t_2) - u(t_1)]|\, d\mathcal{H}^1 \\
&\quad + C_{\mathbb{K}} \sum_{h\in I(t_2)} \int_{(K_h^1(x_1,x_2)\cup K_h^2(x_1,x_2))\cap\partial\Omega} |zu(t_2) - u(t_1) - w(t_2) + w(t_1)|\, d\mathcal{H}^1\,,
\end{aligned}$$

hence, using (6.11), (6.15), and (6.30), we obtain

$$\begin{aligned}\frac{\lambda_{\mathbb{C}}}{2}\int_{\Omega_{t_2}}|e(t_2)-e(t_1)|^2+\beta(\ell(t_2)-\ell(t_1))\leq\int_{t_1}^{t_2}\Big(\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_1)):E\dot{w}(\tau)\,dx\Big)d\tau\\+\frac{\pi c_0^2C_{\mathbb{K}}^2\eta}{\lambda_{\mathbb{C}}}\sum_{h\in I(t_2)}[|s_h^1(t_2)-s_h^1(t_1)|+|s_h^2(t_2)-s_h^2(t_1)|]\\+\frac{\lambda_{\mathbb{C}}}{4}\sum_{h\in I(t_2)}\int_{B^{h,1}(t_1,t_2)\cup B^{h,2}(t_1,t_2)}|e(t_2)-e(t_1)|^2dx\\+c_0C_{\mathbb{K}}\sum_{h\in I(t_2)}\int_{B^{h,1}(t_1,t_2)\cup B^{h,2}(t_1,t_2)}|Ew(t_2)-Ew(t_1)|\,dx\\+\sqrt{2}c_0C_{\mathbb{K}}\sum_{h\in I(t_2)}|p(t_2)-p(t_1)|(B^{h,1}(t_1,t_2)\cup B^{h,2}(t_1,t_2))\,.\end{aligned}$$

Since by (6.11)-(6.18) the sets $B^{h,1}(t_1,t_2)$ and $B^{h,2}(t_1,t_2)$ are pairwise disjoint, using (6.23), and (6.28), from the previous inequality we obtain

$$\begin{aligned}\frac{\lambda_{\mathbb{C}}}{2}\int_{\Omega_{t_2}}|e(t_2)-e(t_1)|^2+\beta(\ell(t_2)-\ell(t_1))\\\leq\int_{t_1}^{t_2}\int_{\Omega_\tau}(\sigma(\tau)-\sigma(t_1)):E\dot{w}(\tau)\,dx\,d\tau+\frac{\beta}{2}(\ell(t_2)-\ell(t_1))+\frac{\lambda_{\mathbb{C}}}{4}\int_{\Omega_{t_2}}|e(t_2)-e(t_1)|^2\\+c_0C_{\mathbb{K}}\sum_{h\in I(t_2)}\int_{B^{h,1}(t_1,t_2)\cup B^{h,2}(t_1,t_2)}|Ew(t_2)-Ew(t_1)|\,dx\\+\sqrt{2}c_0C_{\mathbb{K}}\sum_{h\in I(t_2)}|p(t_2)-p(t_1)|(B^{h,1}(t_1,t_2)\cup B^{h,2}(t_1,t_2))\,.\end{aligned}$$

This implies (6.29). □

7. Proof of the main result

This section is devoted to the proof of the pure jump nature of every quasistatic evolution according to Definition 2.1, under the general hypotheses (2.8)-(2.13) on the crack path.

Proof of Theorem 2.3. As we observed at the beginning of the previous section it is not restrictive to assume (6.1). For every $h=1,\dots,k$ let ℓ_h be the function defined in (6.2). Since $\ell^{cont}=\ell_1^{cont}+\dots+\ell_k^{cont}$, to prove Theorem 2.3 it is enough to show that ℓ_h^{cont} is constant.

Let us fix $h_0\in\{1,\dots,k\}$, and for every $h=1,\dots,k$ let a_h and b_h be defined by (6.3) and (6.4), respectively. Since $\ell_{h_0}(t)=0$ for every $t\in[0,a_{h_0})$ and $\ell_{h_0}(t)=\mathcal{H}^1(K_{h_0})$ for every $t>b_{h_0}$, by the continuity of $\ell_{h_0}^{cont}$ to prove the theorem it is enough to show that $\ell_{h_0}^{cont}$ is constant on (a_{h_0},b_{h_0}), assuming also that $a_{h_0}<b_{h_0}$. This will not be done directly. Instead, we shall subdivide the interval (a_{h_0},b_{h_0}) using suitable points $a_{h_0}=\alpha_0<\alpha_1<\dots<\alpha_\kappa=b_{h_0}$ and we shall prove that $\ell_{h_0}^{cont}$ is constant on each subinterval. Since $\ell_{h_0}^{cont}$ is continuous, this is enough to prove the theorem.

To construct these subintervals, for every $h=1,\dots,k$ we define

$$a_h^1:=\inf\{t\in[a_h,b_h]:s_h^1(t)>0\}\,,\tag{7.1}$$

$$a_h^2:=\inf\{t\in[a_h,b_h]:s_h^2(t)<L_h\}\,,\tag{7.2}$$

where we adopt now the convention $\inf\emptyset=b_h$. It is easy to see that $\min\{a_h^1,a_h^2\}=a_h$. Then we consider the set $[a_{h_0},b_{h_0}]\cap\bigcup_{h=1}^k\{a_h^1,a_h^2,b_h\}$, which will be written in the form

$$[a_{h_0}, b_{h_0}] \cap \bigcup_{h=1}^{k} \{a_h^1, a_h^2, b_h\} = \{\alpha_\iota : 0 \le \iota \le \kappa\}\,, \tag{7.3}$$

with $a_{h_0} = \alpha_0 < \alpha_1 < \cdots < \alpha_\kappa = b_{h_0}$.

We note that the sets of indices $\check I(t)$, $I(t)$, and $\hat I(t)$ introduced in (6.6)-(6.8), do not change if t varies in $(\alpha_{\iota-1}, \alpha_\iota)$ for some $1 \le \iota \le \kappa$. Indeed, they change only if t crosses one of the points of the set $\bigcup_{h=1}^k \{a_h, b_h\}$, which is contained in $\bigcup_{h=1}^k \{a_h^1, a_h^2, b_h\}$. Therefore, for every $\iota = 1, \dots, \kappa$ we can define $\check I_\iota := \check I(t)$, $I_\iota := I(t)$, and $\hat I_\iota := \hat I(t)$, where t is an arbitrary element of $(\alpha_{\iota-1}, \alpha_\iota)$. It follows from the definitions that, if $h \in I_\iota$, then

$$a_h \le \alpha_{\iota-1} < \alpha_\iota \le b_h\,. \tag{7.4}$$

Let us fix ι with $1 \le \iota \le \kappa$. To prove the constancy of $\ell_{h_0}^{cont}$ in the interval $(\alpha_{\iota-1}, \alpha_\iota)$, we want to apply the estimate of Proposition 6.1 to all pairs of consecutive times of a sufficiently fine subdivision of this interval. Unfortunately, this can not be done directly, since the hypotheses (6.9)-(6.18) are not satisfied by any pair of times $t_1, t_2 \in (\alpha_{\iota-1}, \alpha_\iota)$ with $t_1 < t_2$. To overcome this problem we shall construct two sequences $\hat\alpha_{\iota-1}^\delta$ and $\check\alpha_\iota^\delta$ with

$$\hat\alpha_{\iota-1}^\delta \to \alpha_{\iota-1} \quad \text{and} \quad \check\alpha_\iota^\delta \to \alpha_\iota \quad \text{as } \delta \to 0+\,, \tag{7.5}$$

such that for every $\delta > 0$ small enough there exists $\eta > 0$ with the following property: for every $t_1, t_2 \in [\hat\alpha_{\iota-1}^\delta, \check\alpha_\iota^\delta]$, with $t_1 < t_2$,

$$\ell(t_2) - \ell(t_1) < \eta \quad \Longrightarrow \quad \text{conditions (6.9)-(6.18) are satisfied.} \tag{7.6}$$

Therefore, if η satisfies also (6.28), the estimate of Proposition 6.1 holds for any pair of times $t_1, t_2 \in [\hat\alpha_{\iota-1}^\delta, \check\alpha_\iota^\delta]$ with $t_1 < t_2$ and $\ell(t_2) - \ell(t_1) < \eta$. As in Section 5, this will be used to prove that

$$\ell_{h_0}^{cont} \text{ is constant on } (\hat\alpha_{\iota-1}^\delta, \check\alpha_\iota^\delta)\,, \tag{7.7}$$

at least for $\delta > 0$ small enough. By (7.5) it is clear that this is enough to conclude the proof of the theorem.

Construction of $\hat\alpha_{\iota-1}^\delta$ *and* $\check\alpha_\iota^\delta$. For every $\delta > 0$ and every $h \in I_\iota$, we first define

$$a_h^{1,\delta} := \inf\{t \in [a_h, b_h] : \gamma_h(s_h^1(t)) \notin B_\delta(x_h^1)\}\,, \tag{7.8}$$

$$a_h^{2,\delta} := \inf\{t \in [a_h, b_h] : \gamma_h(s_h^2(t)) \notin B_\delta(x_h^2)\}\,, \tag{7.9}$$

$$b_h^\delta := \sup\{t \in [a_h, b_h] : (\gamma_h([0, s_h^1(t)]))_\delta \cap (\gamma_h([s_h^2(t), L_h]))_\delta = \emptyset\}\,, \tag{7.10}$$

where, for every set $E \subset \mathbb{R}^2$, we define $E_\delta := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, A) < \delta\}$. As before we use the convention that $\inf \emptyset = b_h$ and $\sup \emptyset = a_h$.

We consider the sets of indices

$$A_{\iota-1}^1 := \{h \in \{1, \dots, k\} : \alpha_{\iota-1} = a_h^1\}, \quad A_{\iota-1}^2 := \{h \in \{1, \dots, k\} : \alpha_{\iota-1} = a_h^2\}, \tag{7.11}$$

$$B_\iota := \{h \in \{1, \dots, k\} : \alpha_\iota = b_h\}\,, \tag{7.12}$$

and we define

$$\hat\alpha_{\iota-1}^{\lambda,\delta} := \begin{cases} \max\limits_{h \in A_{\iota-1}^\lambda} a_h^{\lambda,\delta} & \text{if } A_{\iota-1}^\lambda \ne \emptyset\,, \\ \alpha_{\iota-1} & \text{if } A_{\iota-1}^\lambda = \emptyset\,, \end{cases} \quad \text{for } \lambda = 1, 2\,, \tag{7.13}$$

$$\hat\alpha_{\iota-1}^\delta := \max\{\hat\alpha_{\iota-1}^{1,\delta}, \hat\alpha_{\iota-1}^{2,\delta}\}\,, \tag{7.14}$$

$$\check\alpha_\iota^\delta := \begin{cases} \min\limits_{h \in B_\iota} b_h^\delta & \text{if } B_\iota \ne \emptyset\,, \\ \alpha_\iota & \text{if } B_\iota = \emptyset\,. \end{cases} \tag{7.15}$$

It follows from the definition that

$$\hat\alpha_{\iota-1}^{\lambda,\delta} \ge \alpha_{\iota-1}\,, \quad \hat\alpha_{\iota-1}^\delta \ge \alpha_{\iota-1}\,, \quad \check\alpha_\iota^\delta \le \alpha_\iota\,. \tag{7.16}$$

Proof of (7.5). It is enough to show that

$$a_h^{1,\delta} \to a_h^1\,, \qquad a_h^{2,\delta} \to a_h^2\,, \qquad b_h^\delta \to b_h \tag{7.17}$$

as $\delta \to 0+$. To prove the first formula in (7.17) we observe that $a_h^{1,\delta} \geq a_h^1$. Therefore, if $a_h^1 = b_h$ we have $a_h^{1,\delta} = b_h$ and the conclusion is trivial. On the other hand, if $a_h^1 < b_h$, we fix $t \in (a_h^1, b_h)$. By (7.1) we have $s_h^1(t) > 0$, hence $\gamma_h(s_h^1(t)) \neq x_h^1$. Therefore there exists $\hat\delta > 0$ such that for every $0 < \delta < \hat\delta$ we have $\gamma_h(s_h^1(t)) \notin B_\delta(x_h^1)$, hence, $a_h^{1,\delta} \leq t$. This concludes the proof of the first part in (7.17). In a similar way we prove the second part.

To prove the last part in (7.17) we observe that the inequality $b_h^\delta \leq b_h$ follows immediately from the definition. Given $t \in (a_h, b_h)$, since by (6.4) we have $\mathcal{H}^1(\Gamma(t) \cap K_h) < L_h$, the sets $\gamma_h([0, s_h^1(t)])$ and $\gamma_h([s_h^2(t), L_h])$ are disjoint by (6.5). This implies that there exists $\hat\delta > 0$ such that for every $0 < \delta < \hat\delta$ we have $(\gamma_h([0, s_h^1(t)]))_\delta \cap (\gamma_h([s_h^2(t), L_h]))_\delta = \emptyset$, hence by the definition of b_h^δ we have $t < b_h^\delta$. This concludes the proof of (7.17), hence of (7.5).

By (7.5) and (7.16) there exists $\delta_1 > 0$ such that for every $\delta \in (0, \delta_1)$ we have

$$\alpha_{\iota-1} \leq \hat\alpha_{\iota-1}^\delta < \check\alpha_\iota^\delta \leq \alpha_\iota\,. \tag{7.18}$$

Proof of (7.6). Taking into account (2.11) it is easy to prove that there exists $\rho_\delta > 0$ such that for $\lambda = 1, 2$ and $t \in [a_h^{\lambda,\delta}, b_h^\delta]$ we have

$$\mathrm{dist}(\gamma_h(s_h^\lambda(t)), \check K_h) > \rho_\delta\,. \tag{7.19}$$

By (2.12) we may reduce the value of $\rho_\delta > 0$ so that we have also

$$\mathrm{dist}(\gamma_h(s_h^\lambda(t)), \partial\Omega \setminus K_h^0) > \rho_\delta \tag{7.20}$$

for $\lambda = 1, 2$ and $t \in [a_h^{\lambda,\delta}, b_h^\delta]$.

Let $\delta_0 > 0$ be the constant introduced in Lemma 3.4 and let us fix $\eta_0 \in (0, \delta_0)$ such that (6.28) holds for every $\eta \in (0, \eta_0)$. For every $\delta \in (0, \delta_1)$, with $\delta_1 > 0$ introduced before (7.18), let $\eta_\delta \in (0, \delta_0)$ be defined by

$$\eta_\delta := \min\{\eta_0, \rho_\delta/2, \delta/2\}\,, \tag{7.21}$$

where ρ_δ is the constant used in (7.19) and (7.20). We now prove that for every $\delta \in (0, \delta_1)$ condition (7.6) holds for every $\eta \in (0, \eta_\delta)$.

Let us fix δ and η as required and $t_1, t_2 \in [\hat\alpha_{\iota-1}^\delta, \check\alpha_\iota^\delta]$, with $t_1 < t_2$ and $\ell(t_2) - \ell(t_1) < \eta$. We now prove that conditions (6.9)-(6.18) are satisfied. Equalities (6.9) can be deduced from the remark after (7.3), while (6.10) follows from the inequalities $\eta < \eta_\delta \leq \eta_0 < \delta_0$. As for (6.11) and (6.15), they are consequences of the inequality $|s_h^\lambda(t_2) - s_h^\lambda(t_1)| \leq \ell(t_2) - \ell(t_1)$, for $\lambda = 1, 2$. Conditions (6.12), (6.13), (6.16), and (6.17) follow from (7.19) and (7.20), since $2\eta < \rho_\delta$.

To prove (6.14), we fix $h \in I_\iota = I(t_1) = I(t_2)$ with $s_h^1(t_2) > 0$. By (6.19) we have also $s_h^1(t_1) > 0$. From the definitions of $I(t_1)$, $I(t_2)$ and a_h^1 we have $a_h^1 \leq t_1 < t_2 \leq b_h$. Let us prove that we have also

$$a_h^1 \leq \alpha_{\iota-1} \leq t_1 < t_2 \leq \alpha_\iota \leq b_h\,. \tag{7.22}$$

The second, third, and fourth inequalities follow from (7.18) and the choice of t_1 and t_2, while the last one follows from (7.4). To prove the first one, assume by contradiction that $a_h^1 > \alpha_{\iota-1}$. Then $a_{h_0} \leq \alpha_{\iota-1} < a_h^1 < t_2 \leq \alpha_\iota \leq b_{h_0}$, which by (7.3) implies that there exists an index $\hat\iota > \iota - 1$ such that $\alpha_{\hat\iota} = a_h^1 < t_2$, in contradiction with $t_2 \leq \check\alpha_\iota^\delta \leq \alpha_\iota \leq \alpha_{\hat\iota}$. This shows that $a_h^1 \leq \alpha_{\iota-1}$.

Estimates (7.22) can be improved to show that

$$a_h^{1,\delta} \leq \hat\alpha_{\iota-1}^\delta \leq t_1 < t_2 \leq \check\alpha_\iota^\delta \leq b_h^\delta\,. \tag{7.23}$$

Let us prove first that $a_h^{1,\delta} \leq \hat\alpha_{\iota-1}^\delta$. Since $\hat\alpha_{\iota-1}^{1,\delta} \leq \hat\alpha_{\iota-1}^\delta$ it is enough to check that

$$a_h^{1,\delta} \leq \hat\alpha_{\iota-1}^{1,\delta}\,. \tag{7.24}$$

By (7.22) we have $a^1_h \le \alpha_{\iota-1}$. We shall consider three cases: $a^1_h = \alpha_{\iota-1}$, $a_{h_0} \le a^1_h < \alpha_{\iota-1}$, and $a^1_h < a_{h_0}$. In the first case $h \in A^1_{\iota-1}$ and (7.24) follows from (7.13). In the third case inequality (7.24) is trivial since $a^1_h < a_{h_0} \le \alpha_{\iota-1} \le \hat\alpha^{1,\delta}_{\iota-1}$ by (7.16). In the second case there exists $\iota^* < \iota$ such that $a^1_h = \alpha_{\iota^*-1}$, hence h belongs to the set $A^1_{\iota^*-1}$ defined as in (7.11) with ι replaced by ι^*. Let $\hat\alpha^{1,\delta}_{\iota^*-1}$ be defined by (7.13), with ι replaced by ι^*. Using also (7.16) with ι and (7.18) with ι^* we obtain $a^{1,\delta}_h \le \hat\alpha^{1,\delta}_{\iota^*-1} \le \alpha_{\iota^*} \le \alpha_{\iota-1} \le \hat\alpha^{1,\delta}_{\iota-1}$. This concludes the proof of (7.24). In a similar way we can show that $\check\alpha^\delta_\iota \le b^\delta_h$, which together with (7.24), gives (7.23), taking into account our choice of t_1, t_2.

Since $2\eta < \rho_\delta$, using (7.19) and (7.23), we obtain that $B_{2\eta}(\gamma_h(s^1_h(t_2))) \cap \check K_h = \emptyset$. Using (2.8)-(2.11), this equality implies that

$$
\begin{aligned}
&B_{2\eta}(\gamma_h(s^1_h(t_2))) \cap \Gamma(t_2) = B_{2\eta}(\gamma_h(s^1_h(t_2))) \cap (\Gamma(t_2) \cap K^0_h) \\
&= B_{2\eta}(\gamma_h(s^1_h(t_2))) \cap \big(\gamma_h((0, s^1_h(t_2)]) \cup \gamma_h([s^2_h(t_2), L_h))\big)\,,
\end{aligned}
$$

where in the last equality we used (6.5). Moreover, since $2\eta < \delta$, by (7.10) and (7.23) we have

$$
B_{2\eta}(\gamma_h(s^1_h(t_2))) \cap \gamma_h([s^2_h(t_2), L_h]) \subset (\gamma_h([0, s^1_h(t_2)]))_\delta \cap (\gamma_h([s^2_h(t_2), L_h]))_\delta = \emptyset\,.
$$

Together with the previous equality this gives (6.14). Similarly we can show that (6.18) holds. Summarizing, we have proved that (7.6) holds for $\delta \in (0, \delta_1)$ and $\eta \in (0, \eta_\delta)$, where δ_1 is the constant introduced before (7.18) and η_δ is defined by (7.21).

Proof of (7.7). We fix $\delta \in (0, \delta_1)$ and we want to prove that $\ell^{cont}_{h_0}$ is constant on $(\hat\alpha^\delta_{\iota-1}, \check\alpha^\delta_\iota)$. As in the proof for the case of a rectilinear crack given in Section 5 this will be done in several steps. First of all we fix $\eta \in (0, \eta_\delta)$, $\varepsilon \in (0, \eta)$, and a subdivision $\hat\alpha^\delta_{\iota-1} = \tau_0 < \tau_1 < \dots < \tau_n = \check\alpha^\delta_\iota$ such that $[\ell](t) < \varepsilon$ for every $t \in [\hat\alpha^\delta_{\iota-1}, \check\alpha^\delta_\iota] \backslash \{\tau_0, \tau_1, \dots, \tau_n\}$. Moreover we fix $\theta \in (0, \eta)$ such that $5\theta < \tau_j - \tau_{j-1}$ for $j = 1, \dots, n$.

By [6, Lemma 4.8], applied to the intervals $[\tau_{j-1} + \theta, \tau_j - \theta]$, there exists $\zeta \in (0, \theta)$ such that for every $j = 1, \dots, n$ we have that

$$
t_1, t_2 \in [\tau_{j-1} + \theta, \tau_j - \theta] \quad \text{and} \quad 0 < t_2 - t_1 < \zeta \quad \Longrightarrow \quad \ell(t_2) - \ell(t_1) < \varepsilon < \eta\,. \tag{7.25}
$$

Using (7.6) we get

$$
t_1, t_2 \in [\tau_{j-1} + \theta, \tau_j - \theta] \quad \text{and} \quad 0 < t_2 - t_1 < \zeta \quad \Longrightarrow \quad \text{(6.9)-(6.18) hold.} \tag{7.26}
$$

Since $\eta < \eta_\delta$, using (7.21) we obtain that η satisfies (6.28). Therefore we can apply Proposition 6.1 which gives the implication

$$
t_1, t_2 \in [\tau_{j-1} + \theta, \tau_j - \theta] \quad \text{and} \quad 0 < t_2 - t_1 < \zeta \quad \Longrightarrow \quad \text{(6.29) holds.} \tag{7.27}
$$

To conclude the proof of (7.7) we fix a subdivision $\hat\alpha^\delta_{\iota-1} = t_0 < t_1 < \dots < t_m = \check\alpha^\delta_\iota$, with $t_i - t_{i-1} < \zeta$ for every $i = 1, \dots, m$, such that

$$
\sum_{i=1}^m \int_{t_{i-1}}^{t_i} \int_{\Omega_\tau} (\sigma(\tau) - \sigma(t_{i-1})) : E\dot w(\tau)\, dx\, d\tau < \varepsilon\,. \tag{7.28}
$$

This can be done since a Lebesgue integral can be approximated by suitable Riemann sums (see, for instance, [6, Proposition 4.10]). By (7.27), if $\tau_{j-1} + \theta < t_{i-1} < t_i < \tau_j - \theta$, we have that (6.29) holds for the pair t_{i-1}, t_i. In the next steps we estimate the terms involved in (6.29).

Estimate between τ_{j-1} *and* τ_j. For $j = 0, \dots, n-1$ there exists $m_j \in \{1, \dots, m\}$ such that $\tau_j + 2\theta < t_{m_j} < \tau_j + 3\theta$, and for $j = 1, \dots, n$ there exists $n_j \in \{1, \dots, m\}$ such that $\tau_j - 2\theta < t_{n_j} < \tau_j - \theta$. Since $5\theta < \tau_j - \tau_{j-1}$ for $j = 1, \dots, n$, we have $m_0 < n_1 < m_1 < \dots < n_{n-1} < m_{n-1} < n_n$. If $m_{j-1} \le i \le n_j$, then $\tau_{j-1} + 2\theta < t_i < \tau_j - \theta$, and since $t_i - t_{i-1} < \zeta < \theta$ we have $\tau_{j-1} + \theta < t_{i-1} < t_i < \tau_j - \theta$. By (7.27) we conclude that (6.29) holds for the pair t_{i-1}, t_i. Therefore, using the obvious inequalities $\ell^{cont}_{h_0}(\tau_j - 2\theta) - \ell^{cont}_{h_0}(\tau_{j-1} + 3\theta) \leq \ell^{cont}_{h_0}(t_{n_j}) - \ell^{cont}_{h_0}(t_{m_{j-1}}) \leq \ell_{h_0}(t_{n_j}) - \ell_{h_0}(t_{m_{j-1}}) \leq \sum_{i=m_{j-1}}^{n_j} (\ell_{h_0}(t_i) - \ell_{h_0}(t_{i-1}))$, we obtain

$$\begin{aligned}\ell^{cont}_{h_0}(\tau_j - 2\theta) - \ell^{cont}_{h_0}(\tau_{j-1} + 3\theta) \leq \frac{2}{\beta} \sum_{i=m_{j-1}}^{n_j} \int_{t_{i-1}}^{t_i} \int_{\Omega_\tau} (\sigma(\tau) - \sigma(t_{i-1})) : E\dot{w}(\tau)\, dx\, d\tau \\ + \frac{2c_0 C_{\mathbb{K}}}{\beta} \sum_{i=m_{j-1}}^{n_j} \sum_{h\in I_\iota} \sum_{\lambda=1}^{2} \int_{B^{h,\lambda}(t_{i-1},t_i)} |Ew(t_i) - Ew(t_{i-1})|\, dx \\ + \frac{2\sqrt{2}c_0 C_{\mathbb{K}}}{\beta} \sum_{i=m_{j-1}}^{n_j} \sum_{h\in I_\iota} \sum_{\lambda=1}^{2} |p(t_i) - p(t_{i-1})|(B^{h,\lambda}(t_{i-1}, t_i))\,.\end{aligned} \tag{7.29}$$

The first term in the right-hand side can be estimated by (7.28), and we now have to estimate the other terms.

Estimate of the second term in the right-hand side of (7.29). By (6.5), (6.26), and (6.27) we have $|B^{h,\lambda}(t_{i-1}, t_i)| \leq \pi |s^\lambda_h(t_i) - s^\lambda_h(t_{i-1})|^2 \leq \pi(\ell(t_i) - \ell(t_{i-1}))^2 < \pi\varepsilon^2$, where the last inequality follows from (7.25). Hence by Hölder's inequality, for every $h \in I_\iota$, $i = m_{j-1}, ..., n_j$, and $\lambda = 1, 2$ we obtain

$$\int_{B^{h,\lambda}(t_{i-1},t_i)} |Ew(t_i) - Ew(t_{i-1})|\, dx \leq \varepsilon\sqrt{\pi} \|Ew(t_i) - Ew(t_{i-1})\|_L \leq \varepsilon\sqrt{\pi} \int_{t_{i-1}}^{t_i} \|E\dot{w}(\tau)\|_L d\tau\,,$$

where $L := L^2(\Omega \setminus K_0; \mathbb{R}^{2\times 2}_{sym})$. This gives

$$\frac{2c_0 C_{\mathbb{K}}}{\beta} \sum_{i=m_{j-1}}^{n_j} \sum_{h\in I_\iota} \sum_{\lambda=1}^{2} \int_{B^{h,\lambda}(t_{i-1},t_i)} |Ew(t_i) - Ew(t_{i-1})|\, dx \leq M_1 \varepsilon \int_{\tau_{j-1}}^{\tau_j} \|E\dot{w}(t)\|_L dt\,, \tag{7.30}$$

where $M_1 := \frac{4kc_0 C_{\mathbb{K}}\sqrt{\pi}}{\beta}$.

To estimate the last term in (7.29) we can not use the same argument since the function $t \mapsto p(t)$ is not absolutely continuous. However a similar argument can be used for the rescaled versions p° and Γ° of p and Γ, introduced in Section 4 and for a corresponding rescaled version of the functions s^1_h and s^2_h

Definition and properties of the rescaled functions $s^{1,\circ}_h$ *and* $s^{2,\circ}_h$. For $h \in I_\iota$ and $\lambda = 1, 2$ we define $s^{\lambda,\circ}_h \colon (\rho(a_h), \rho(b_h)) \to [0, +\infty)$ by

$$s^{\lambda,\circ}_h(r) = \begin{cases} s^\lambda_h(t) & \text{if } r = \rho(t) \text{ with } t \in (a_h, b_h)\,, \\ s^\lambda_h(t) & \text{if } r \in [\rho(t-), \rho(t)) \text{ with } t \in (a_h, b_h) \cap J_V\,, \\ s^\lambda_h(t+) & \text{if } r \in (\rho(t), \rho(t+)] \text{ with } t \in (a_h, b_h) \cap J_V\,. \end{cases} \tag{7.31}$$

Using (6.5) and the definition of $\Gamma^\circ(r)$ given in (4.26), (4.27), and (4.29), for $r \in (\rho(a_h), \rho(b_h))$ we have

$$\Gamma^\circ(r) \cap K^0_h = \gamma_h((0, s^{1,\circ}_h(r)]) \cup \gamma_h([s^{2,\circ}_h(r), L_h))\,. \tag{7.32}$$

Note that

$$s^\lambda_h(t) = s^{\lambda,\circ}_h(\rho(t)) \quad \text{for every } t \in (a_h, b_h)\,, \tag{7.33}$$

and that $s^{1,\circ}_h$ is nondecreasing, while $s^{2,\circ}_h$ is nonincreasing.

We claim that

$$\gamma_h(s^{\lambda,\circ}_h(r)) \in \Gamma^\circ(r) \tag{7.34}$$

for every $\lambda = 1, 2$, every $h \in I_\iota$, and every $r \in (\rho(a^\lambda_h), \rho(b_h))$. We prove the claim only in the case $\lambda = 1$, the other one being similar.

Let us fix h and r as required. Suppose that there exists $t \in (a_h, b_h)$ such that $r = \rho(t)$. Since ρ is increasing and $r > \rho(a^1_h)$, we have $t > a^1_h$. By definition $s^{1,\circ}_h(r) = s^1_h(t)$, hence $s^{1,\circ}_h(r) > 0$ by the definition of a^1_h. Then (7.34) is a consequence of (7.32).

Suppose now that $r \in [\rho(t-), \rho(t))$ for some $t \in (a_h, b_h) \cap J_V$. Since $r > \rho(a_h^1)$ and ρ is increasing we must have $t > a_h^1$ as we conclude as before.

Finally, suppose that $r \in (\rho(t), \rho(t+)]$ for some $t \in (a_h, b_h) \cap J_V$. Since ρ is increasing and $r > \rho(a_h^1)$, we have $t \geq a_h^1$. If $s_h^{1,\circ}(r) > 0$ we conclude as before. If $s_h^{1,\circ}(r) = 0$, then $\gamma_h(s_h^{1,\circ}(r)) = \gamma_h(0)$ and by definition $\Gamma^\circ(r) = \Gamma(t+)$. For every $\tau > t$ we have $\tau > a_h^1$, hence $s_h^1(\tau) > 0$. By (6.5) we have $\gamma_h((0, s_h^1(\tau)]) \subset \Gamma(\tau)$. Since $s_h^1(\tau) > 0$, the closure of $\gamma_h((0, s_h^1(\tau)])$ is $\gamma_h([0, s_h^1(\tau)])$, and since $\Gamma(\tau)$ is closed we have that $\gamma_h(0) \in \gamma_h([0, s_h^1(\tau)]) \subset \Gamma(\tau)$. By (4.2) we deduce that $\gamma_h(s_h^{1,\circ}(r)) = \gamma_h(0) \in \Gamma(t+) = \Gamma^\circ(r)$, which concludes the proof of (7.34).

Estimate of the last term in the right-hand side of (7.29). We first show that, if $t_{i-1}, t_i \in [\tau_{j-1} + \theta, \tau_j - \theta]$ for some $j = 1, \dots, n$ and $0 < t_i - t_{i-1} < \zeta$, then we have

$$B^{h,\lambda}(t_{i-1}, t_i) \subset B_\varepsilon(\gamma_h(s_h^\lambda(t_i))) \cap \hat{\Omega}_{t_i} \quad \text{for } \lambda = 1,2 \text{ and for every } h \in I_\iota\,, \tag{7.35}$$

where $B^{h,\lambda}(t_{i-1}, t_i)$ are defined in (6.26) and (6.27). Let us fix t_{i-1}, t_i, and h as required. Since $|s^1(t_i) - s^1(t_{i-1})| \leq \ell(t_i) - \ell(t_{i-1}) < \varepsilon$, we have $B^{h,1}(t_{i-1}, t_i) \subset B_{r_h^1(t_{i-1},t_i)}(\gamma_h(s_h^1(t_i))) \cap \overline{\Omega} \setminus \gamma_h([0, s_h^1(t_i)]) \subset B_\varepsilon(\gamma_h(s_h^1(t_i))) \cap \overline{\Omega} \setminus \gamma_h([0, s_h^1(t_i)]) = B_\varepsilon(\gamma_h(s_h^1(t_i))) \cap \hat{\Omega}_{t_i}$, where the last equality follows from (6.14) applied with η replaced by $\varepsilon/2$ (see the remark after (6.20)). Arguing in the same way for $B^{h,2}(t_{i-1}, t_i)$ we conclude the proof of (7.35).

For every $h \in I_\iota$, $\sigma > 0$, $r \in (\rho(a_h), \rho(b_h))$, and $\lambda = 1, 2$ we set

$$B_{h,\sigma}^{\lambda,\circ}(r) := B_\sigma(\gamma_h(s_h^{\lambda,\circ}(r)))\,. \tag{7.36}$$

By (7.34) for every $r \in (\rho(a_h^\lambda), \rho(b_h))$ we have $B_{h,\sigma}^{\lambda,\circ}(r) \cap \hat{\Omega}_r^\circ \to \emptyset$ as $\sigma \to 0+$. By Lemma 4.4, we can apply the Dominated Convergence Theorem and we obtain

$$\int_{\rho(a_h^\lambda)}^{\rho(b_h)} |\dot{p}^\circ(r)|(B_{h,\sigma}^{\lambda,\circ}(r) \cap \hat{\Omega}_r^\circ) dr \to 0 \quad \text{as } \sigma \to 0+\ .$$

Therefore there exists a continuous function $\omega\colon [0, +\infty) \to [0, +\infty)$, with $\omega(0) = 0$, such that

$$\int_{\rho(a_h^\lambda)}^{\rho(b_h)} |\dot{p}^\circ(r)|(B_{h,\sigma}^{\lambda,\circ}(r) \cap \hat{\Omega}_r^\circ) dr < \omega(\sigma) \tag{7.37}$$

for every $h \in I_\iota$, $\sigma \in (0, +\infty)$, and $\lambda = 1, 2$.

To estimate the last term in (7.29) we fix $h \in I_\iota$ and $m_{j-1} \leq i \leq n_j$. If $t_i < a_h^1$, then $s_h^1(t_i) = 0$. This implies $s_h^1(t_{i-1}) = 0$, which gives $B^{h,1}(t_{i-1}, t_i) = \emptyset$, hence

$$|p(t_i) - p(t_{i-1})|(B^{h,1}(t_{i-1}, t_i)) = 0\,.$$

A similar property can be proved if $t_i < a_h^2$. Therefore, for every $\lambda = 1, 2$ we have that

$$t_i < a_h^\lambda \quad \Rightarrow \quad |p(t_i) - p(t_{i-1})|(B^{h,\lambda}(t_{i-1}, t_i)) = 0\,. \tag{7.38}$$

If instead $t_i \geq a_h^\lambda$, then by (4.31), (7.33), (7.35), and Lemma 4.4 we obtain

$$\begin{aligned} |p(t_i) - p(t_{i-1})|(B^{h,\lambda}(t_{i-1}, t_i)) &\leq |p^\circ(\rho(t_i)) - p^\circ(\rho(t_{i-1}))|(B_{h,\varepsilon}^{\lambda,\circ}(\rho(t_i)) \cap \hat{\Omega}_{\rho(t_i)}^\circ) \\ &\leq \int_{\rho(t_{i-1})}^{\rho(t_i)} |\dot{p}^\circ(r)|(B_{h,\varepsilon}^{\lambda,\circ}(\rho(t_i)) \cap \hat{\Omega}_r^\circ)\, dr\,. \end{aligned} \tag{7.39}$$

If $\rho(t_{i-1}) \leq r \leq \rho(t_i)$, we have $s_h^1(t_{i-1}) = s_h^{1,\circ}(\rho(t_{i-1})) \leq s_h^{1,\circ}(r) \leq s_h^{1,\circ}(\rho(t_i)) = s_h^1(t_i)$. Since $s_h^1(t_i) - s_h^1(t_i) \leq \ell(t_i) - \ell(t_{i-1}) < \varepsilon$, we obtain $0 \leq s_h^{1,\circ}(\rho(t_i)) - s_h^{1,\circ}(r) < \varepsilon$. Similarly we can prove that $0 \leq s_h^{2,\circ}(r) - s_h^{2,\circ}(\rho(t_i)) < \varepsilon$. This implies that $B_{h,\varepsilon}^{\lambda,\circ}(\rho(t_i)) \subset B_{h,2\varepsilon}^{\lambda,\circ}(r)$. Therefore (7.39) gives that

$$|p(t_i) - p(t_{i-1})|(B^{h,\lambda}(t_{i-1}, t_i)) \leq \int_{\rho(t_{i-1})}^{\rho(t_i)} |\dot{p}^\circ(r)|(B_{h,2\varepsilon}^{\lambda,\circ}(r) \cap \hat{\Omega}_r^\circ)\, dr \tag{7.40}$$

for $m_{j-1} \le i \le n_j$ and $t_i \ge a_h^\lambda$.

By (7.26) conditions (6.9)-(6.18) are satisfied by t_{i-1}, t_i and, using (6.19) and (6.20), from the inequality $t_i \ge a_h^\lambda$ we deduce that $t_{i-1} \ge a_h^\lambda$. This implies that the intervals used in (7.40) satisfy $(\rho(t_{i-1}), \rho(t_i)) \subset (\rho(a_h^\lambda), \rho(b_h))$. Therefore, from (7.38) and (7.40) for every $h \in I_\iota$ we deduce that

$$\sum_{i=m_{j-1}}^{n_j} |p(t_i) - p(t_{i-1})|(B^{h,\lambda}(t_{i-1}, t_i)) \le \int_{\rho(\tau_{j-1} \vee a_h^\lambda)}^{\rho(\tau_j)} |\dot p^\circ(r)|(B^{\lambda,\circ}_{h,2\varepsilon}(r) \cap \hat\Omega^\circ_r)\, dr\,, \tag{7.41}$$

where $a \vee b := \max\{a, b\}$ for every a, $b \in \mathbb{R}$.

Conclusion of the proof of (7.7). The last inequality, together with (7.29) and (7.30), gives

$$\ell^{cont}_{h_0}(\tau_j - 2\theta) - \ell^{cont}_{h_0}(\tau_{j-1} + 3\theta) \le \frac{2}{\beta} \sum_{i=m_{j-1}}^{n_j} \int_{t_{i-1}}^{t_i} \Big(\int_{\Omega_t} (\sigma(t) - \sigma(t_{i-1})) : E\dot w(t)\, dx \Big) dt$$

$$+ M_1 \varepsilon \int_{\tau_{j-1}}^{\tau_j} \|E\dot w(t)\|_L dt + M_2 \sum_{\lambda=1}^{2} \sum_{h \in I_\iota} \int_{\rho(\tau_{j-1} \vee a_h^\lambda)}^{\rho(\tau_j)} |\dot p^\circ(r)|(B^{\lambda,\circ}_{h,2\varepsilon}(r) \cap \hat\Omega^\circ_r)\, dr \tag{7.42}$$

where $M_2 := \frac{2\sqrt{2} c_0 C_{\mathbb{K}}}{\beta}$. Recalling that, by (7.4), $a_h \le \tau_{j-1} < \tau_j \le b_h$ for every $h \in I_\iota$, for every nonnegative measurable function ψ we have

$$\sum_{j=1}^{n} \sum_{h \in I_\iota} \int_{\rho(\tau_{j-1} \vee a_h^\lambda)}^{\rho(\tau_j)} \psi(r)\, dr = \sum_{h \in I_\iota} \sum_{j=1}^{n} \int_{\rho(\tau_{j-1} \vee a_h^\lambda)}^{\rho(\tau_j)} \psi(r)\, dr \le \sum_{h \in I_\iota} \int_{\rho(a_h^\lambda)}^{\rho(b_h)} \psi(r)\, dr\,.$$

Therefore, summing (7.42) over j and using (7.28) and (7.37) we get

$$\sum_{j=1}^{n} (\ell^{cont}_{h_0}(\tau_j - 2\theta) - \ell^{cont}_{h_0}(\tau_{j-1} + 3\theta)) \le \frac{2}{\beta}\varepsilon + M_1 \varepsilon \int_0^T \|E\dot w(t)\|_L dt$$

$$+ M_2 \sum_{\lambda=1}^{2} \sum_{h \in I_\iota} \int_{\rho(a_h^\lambda)}^{\rho(b_h)} |\dot p^\circ(r)|(B^{\lambda,\circ}_{h,2\varepsilon}(r) \cap \hat\Omega^\circ_r)\, dr < M_3 \varepsilon + 2k M_2\, \omega(2\varepsilon)\,,$$

where $M_3 := 2/\beta + M_1 \int_0^T \|E\dot w(t)\|_L dt$. Taking the limit as $\theta \to 0+$, from the continuity of $\ell^{cont}_{h_0}$ we obtain $\ell^{cont}_{h_0}(\check\alpha^\delta_\iota) - \ell^{cont}_{h_0}(\hat\alpha^\delta_{\iota-1}) \le M_3 \varepsilon + 2k M_2\, \omega(2\varepsilon)$. As $\varepsilon \to 0+$, using (7.37) it follows that $\ell^{cont}_{h_0}(\check\alpha^\delta_\iota) \le \ell^{cont}_{h_0}(\hat\alpha^\delta_{\iota-1})$. Since $\ell^{cont}_{h_0}$ is nondecreasing, this inequality implies $\ell^{cont}_{h_0}(t) = \ell^{cont}_{h_0}(\hat\alpha^\delta_{\iota-1})$ for every $t \in [\hat\alpha^\delta_{\iota-1}, \check\alpha^\delta_\iota]$. This proves (7.7) and concludes the proof of the theorem. □

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(Gianni Dal Maso) SISSA, Via Bonomea 265, 34136 Trieste, Italy
Email address, Gianni Dal Maso: dalmaso@sissa.it

(Rodica Toader) DMIF, Università di Udine, Via delle Scienze 206, 33100, Udine, Italy
Email address, Rodica Toader: rodica.toader@uniud.it