ANALYSIS FOR NON-LOCAL PHASE TRANSITIONS CLOSE TO THE CRITICAL EXPONENT $s = \frac{1}{2}$

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Abstract

We analyze the behaviour of double-well energies perturbed by fractional Gagliardo squared seminorms in H^s close to the critical exponent $s = \frac{1}{2}$. This is done by computing a scaling factor $\lambda(\varepsilon, s)$, continuous in both variables, such that

$$\mathcal{F}_\varepsilon^{s_\varepsilon}(u) = \frac{\lambda(\varepsilon, s_\varepsilon)}{\varepsilon} \int W(u)dt + \lambda(\varepsilon, s_\varepsilon)\varepsilon^{(2s_\varepsilon - 1)^+}[u]^2_{H^{s_\varepsilon}}$$

Γ-converge, for any choice of $s_\varepsilon \to \frac{1}{2}$ as $\varepsilon \to 0$, to the sharp-interface functional found by Alberti, Bouchitté and Seppecher in [1] with the scaling $|\log \varepsilon|^{-1}$. Moreover, we prove that all the values $s \in [\frac{1}{2}, 1)$ are regular points for the functional $\mathcal{F}_\varepsilon^s$ in the sense of equivalence by Γ-convergence (see [3]), and that the Γ-limits as $\varepsilon \to 0$ are continuous with respect to s. In particular, the corresponding surface tensions, given by suitable non-local optimal-profile problems, are continuous on $[\frac{1}{2}, 1)$.

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1. Introduction and outline

Singular perturbations are often used to select solutions of non-convex problems with multiplicity of minimizers. In the case of theories of phase transitions, usually the non-convex problem at hand is an integral depending on a scalar variable u through a "double-well potential" W (that is, a function with two minimizers α and β). In the classical Cahn–Hilliard theory of phase transitions [5] such an energy is singularly perturbed by a term with the gradient of u depending on a small parameter ε as

$$\int_\Omega W(u)dt + \varepsilon^2 \int_\Omega |\nabla u|^2 dx$$

As $\varepsilon \to 0$ minimizers under a volume constraint converge to a function taking only the values α and β, and such that the interface between these two phases is minimal. This minimal-interface criterion had been conjectured by Gurtin [10] and proven by Modica [11] using the Γ-convergence (cf [6], [2]) results of a previous seminal paper by Modica and Mortola [12], obtaining that the functionals above, scaled by $\frac{1}{\varepsilon}$, Γ-converge to a functional defined on the space $BV(\Omega; \{\alpha, \beta\})$ of functions with bounded variation taking only the values α and β by

$$m_W \operatorname{Per}(\{u = \alpha\}; \Omega),$$

where $\mathrm{Per}(A;\Omega)$ denotes the perimeter of A in Ω and m_W (the *surface tension* between the phases) is a constant determined by W only. Since functions in $BV(\Omega;\{\alpha,\beta\})$ can be identified with sets of finite perimeter $\{u=\alpha\}$, this result provides a proof of the minimal-interface criterion. It is interesting that this result is one-dimensional in that m_W is characterized by the problems

$$m_W=\min\Big\{\int_{-\infty}^{+\infty}(W(v)+|\nabla v|^2)dt : v(-\infty)=\alpha,\ v(+\infty)=\beta\Big\}$$

(see [8]). The result in [11] has been extended in many ways, and in particular using higher-order gradients as in [7] and [4], with analogous formulas for the corresponding surface tension, depending on the order of the perturbation.

Motivated by an interest in non-local problems, perturbations have also been taken in fractional Sobolev spaces, with functionals of the form

$$\int_\Omega W(u)dx+\varepsilon^{2s}[u]^2_{H^s(\Omega)} \tag{1.1}$$

where $[u]_{H^s(\Omega)}$ denotes the Gagliardo seminorm in the fractional Sobolev space $H^s(\Omega)$ with $s\in(0,1)$ (see the works of Alberti, Bouchitté, and Seppecher [1] [9], Savin and Valdinoci [14] and Palatucci-Vincini [13]). If $s>\frac12$ then the scaling by $\frac1\varepsilon$ as in the Modica–Mortola case gives functionals

$$\frac1\varepsilon\int_\Omega W(u)dx+\varepsilon^{2s-1}[u]^2_{H^s(\Omega)},$$

which still provide a Γ-limit of perimeter type as above, with a surface tension m_s depending on s. Recently, the result has been proved to hold also for higher-order fractional perturbations by Solci [15], who also shows that the functionals can be slightly modified in such a way that the corresponding surface tension m_s is a continuous function for $s\in(\frac12,+\infty)$. A modification is necessary by the critical behaviour of the fractional norms at integer points.

The case $s=\frac12$ is *critical*, in the sense that the scaling by $\frac1\varepsilon$ makes the coefficient in front of the $H^{\frac12}$-seminorm equal to 1, so that, in order to obtain a phase-transition energy of perimeter type, in this case it is necessary to scale by a further logarithmic factor; that is, to consider functionals

$$\frac{1}{\varepsilon|\log\varepsilon|}\int_\Omega W(u)dx+\frac{1}{|\log\varepsilon|}[u]^2_{H^{\frac12}(\Omega)}.$$

With this scaling, the Γ-limit is still an energy of perimeter type with the explicit surface tension $m_{\frac12}=8$.

In the case $s<\frac12$, finally, Savin and Valdinoci have shown that the functionals scaled by ε^{-2s} have a non-local phase-transition limit in which the domain are H^2 functions taking only the values α and β.

In this paper we aim at describing more in detail the behaviour of functionals (1.1) for s *close* to $\frac12$. To that end, we introduce functionals

$$F^s_\varepsilon(u)=\frac1\varepsilon\int_{(0,1)}W(u)dt+\varepsilon^{(2s-1)^+}\iint_{(0,1)\times(0,1)}\frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}dxdy. \tag{1.2}$$

Here, W is a double-well potential satisfying the following assumptions:

- W is continuous;
- $W\geq0$ and $W(z)=0$ if and only if $z=\pm1$;

- $\liminf_{z\to\pm\infty} W(z) > 0$

Using the approach by [3], in order to provide a description as $s \to \frac{1}{2}$ and $\varepsilon \to 0$, we investigate the behaviour of the functionals F^s_ε under the assumption that $s = s_\varepsilon \to \frac{1}{2}$ as $\varepsilon \to 0$. The question then translates in finding a scaling factor $\lambda(\varepsilon, s)$ depending continuously on its variables such that

$$\lambda(\varepsilon, s_\varepsilon) F^{s_\varepsilon}_\varepsilon \xrightarrow{\Gamma} F^{\frac{1}{2}}_0 .$$

for all choices of $s_\varepsilon \to \frac{1}{2}$.

Note that a particular case is $s_\varepsilon = \frac{1}{2}$ for all ε, for which, in this one-dimensional case, the asymptotic result is that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \frac{1}{|\log(\varepsilon)|} F^{\frac{1}{2}}_\varepsilon = F^{\frac{1}{2}}_0 = \begin{cases} m_{\frac{1}{2}} \# S(u) & \text{if } u \in BV((0,1), \{-1,1\}), \\ +\infty & \text{otherwise.} \end{cases}$$

with $m_{\frac{1}{2}} = 8$. This remark implies that we may take $\lambda(\varepsilon, \frac{1}{2}) = \frac{1}{|\log\varepsilon|}$. Note that if the wells of W are in two points $\alpha < \beta$, the surface tension $m_{\frac{1}{2}}$ is equal to $2(\beta-\alpha)^2$. All the results we prove in this paper also hold in such generality.

In Sections 2 and 3 we prove that such a scaling is

$$\lambda(\varepsilon, s) = \begin{cases} \frac{2s-1}{1-\varepsilon^{2s-1}} & \text{if } s > \frac{1}{2} \\ \frac{1}{|\log\varepsilon|} & \text{if } s = \frac{1}{2}. \\ \frac{2s-1}{\varepsilon^{\frac{1-2s}{2s}}-1} & \text{if } s < \frac{1}{2} \end{cases}$$

In terms of asymptotic behaviour, in particular this implies that in the regime

$$|2s_\varepsilon - 1| << \frac{1}{|\log\varepsilon|},$$

we have a *separation of scales* effect; that is, the Γ-limit of $\lambda(\varepsilon, s)F^s_\varepsilon$ is the same as the one obtained first letting $s \to \frac{1}{2}$ with $\varepsilon > 0$ fixed, which gives $\frac{1}{|\log\varepsilon|} F^{1/2}_\varepsilon$, and then letting $\varepsilon \to 0$.

Furthermore, in Section 4 we show that this analysis extends to $s_\varepsilon \to s \in (\frac{1}{2}, 1)$ by studying the behaviour of the surface tensions

(1.3)
$$m_s = \inf \left\{ \int_{\mathbb{R}} W(u)dt + \iint_{\mathbb{R}\times\mathbb{R}} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s}} dxdy \mid u \in H^s(\mathbb{R}), u(\pm\infty) = \pm 1 \right\},$$

as $s \to \frac{1}{2}$ and proving that

$$\lim_{s\to\frac{1}{2}^+} (2s-1)m_s = m_{\frac{1}{2}}$$

This condition gives a continuity of the description by the scaled functionals. Indeed, since for $s > \frac{1}{2}$ and $s_\varepsilon \to s$ we have

$$\lambda(\varepsilon, s_\varepsilon) = \frac{1 - 2s_\varepsilon}{\varepsilon^{2s_\varepsilon - 1} - 1} \to 2s - 1$$

as $\varepsilon \to 0$, we obtain that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \lambda(\varepsilon, s_\varepsilon) F^{s_\varepsilon}_\varepsilon = \begin{cases} (2s-1)m_s \# S(u) & \text{if } u \in BV((0,1), \{-1,1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

which tends to $F_0^{1/2}$ as $s \to \frac{1}{2}^+$. Again, we highlight a separation of scales effect if $s_\varepsilon \to \frac{1}{2}^+$ and

$$2s_\varepsilon - 1 >> \frac{1}{|\log \varepsilon|}.$$

In this case, the Γ-limit of $\lambda(\varepsilon, s)F_\varepsilon^s$ is the same as the one obtained first letting $\varepsilon \to 0$ with $s > \frac{1}{2}$ fixed, which gives $(2s-1)\frac{m_s}{m_{1/2}}F_0^{1/2}$, and then letting $s \to \frac{1}{2}$.

These results can be expressed in the terminology of Γ-expansions [3] as the equivalence of the functionals F_ε^s and the functionals

$$G_\varepsilon^s(u) = \begin{cases} (1-\varepsilon^{2s-1})m_s F_0(u) & \text{if } \varepsilon > \frac{1}{2} \\ |\log \varepsilon| m_{\frac{1}{2}} F_0(u) & \text{if } \varepsilon = \frac{1}{2}, \end{cases}$$

where

$$F_0 = \begin{cases} \#S(u) & \text{if } u \in BV((0,1), \{-1,1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

for s varying uniformly on compact sets of $[\frac{1}{2}, 1)$. We note that this result can be extended to all compact subsets of $[\frac{1}{2}, +\infty)$ upon taking the correct extension to higher-order fractional perturbations as in [15].

2. Finding the correct scaling factor

In this section, we consider a sequence $(s_\varepsilon)_\varepsilon$ such that

$$\lim_{\varepsilon \to 0} s_\varepsilon = \frac{1}{2}.$$

Our aim is to find a scaling factor $\lambda(\varepsilon)$ such that the functionals $\lambda(\varepsilon)F_\varepsilon^{s_\varepsilon}$, where $F_\varepsilon^{s_\varepsilon}$ are given by (1.2), Γ-converge to $F_0^{\frac{1}{2}}$ as $\varepsilon \to 0$. If $s = \frac{1}{2}$ then the result by Alberti, Bouchitté and Seppecher gives $\lambda(\varepsilon) = \frac{1}{|\log \varepsilon|}$, so we can suppose $s_\varepsilon \neq \frac{1}{2}$.

We start by dealing with the case $s_\varepsilon \to \frac{1}{2}^+$. Let G_ε denote the unscaled functionals

$$G_\varepsilon(u) = \frac{1}{\varepsilon}\int_{(0,1)} W(u)dt + \varepsilon^{2s_\varepsilon - 1}\iint_{(0,1)\times(0,1)} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s_\varepsilon}}dxdy.$$

To find the right scaling factor, we start by considering a sequence of functions u_ε which converges in measure to the function

$$u_0 = \begin{cases} 1 & \text{in } [0,1] \\ -1 & \text{in } [-1,0). \end{cases}$$

We let $\eta \in (0, \frac{1}{4})$ and define

$$\sigma_\varepsilon = |\{|u_\varepsilon| \leq 1-\eta\}|, \quad C_\eta = \inf_{|z|\leq 1-\eta} W(z) > 0,$$

$$A_\varepsilon = \{u_\varepsilon > 1-\eta\}, \quad B_\varepsilon = \{u_\varepsilon < \eta - 1\}.$$

Then, we have the estimate:

$$G_\varepsilon(u_\varepsilon) \geq \frac{\sigma_\varepsilon C_\eta}{\varepsilon} + 2\varepsilon^{2s_\varepsilon - 1}\iint_{A_\varepsilon \times B_\varepsilon} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s_\varepsilon}}dxdy$$

$$\geq \frac{\sigma_\varepsilon C_\eta}{\varepsilon} + 2(2-2\eta)^2\varepsilon^{2s_\varepsilon - 1}\iint_{A_\varepsilon \times B_\varepsilon} \frac{1}{|x-y|^{1+2s_\varepsilon}}dxdy.$$

Since the function $\psi(|x-y|) = \frac{1}{|x-y|^{1+2s_\varepsilon}}$ is monotonically decreasing in $|x-y|$, we obtain a lower bound by increasing the distance between x and y (see also Lemma 2 of [1]). This leads to

$$G_\varepsilon(u_\varepsilon) \geq \frac{\sigma_\varepsilon C_\eta}{\varepsilon} + 2(2-2\eta)^2 \varepsilon^{2s_\varepsilon - 1} \int_{-1}^{-1+|A_\varepsilon|} \int_{1-|B_\varepsilon|}^{1} \frac{1}{|x-y|^{1+2s_\varepsilon}} dxdy$$

$$= \frac{\sigma_\varepsilon C_\eta}{\varepsilon} + \frac{(2-2\eta)^2 \varepsilon^{2s_\varepsilon - 1}}{s_\varepsilon(2s_\varepsilon - 1)} \left(2^{1-2s_\varepsilon} - (2-|B_\varepsilon|)^{1-2s_\varepsilon} \right.$$
$$\left. -(2-|A_\varepsilon|)^{1-2s_\varepsilon} + (2-|B_\varepsilon|-|A_\varepsilon|)^{1-2s_\varepsilon}\right),$$

which simplifies to

$$\frac{\sigma_\varepsilon C_\eta}{\varepsilon} + C\varepsilon^{2s_\varepsilon - 1} \frac{\sigma_\varepsilon^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} + O(\varepsilon^{2s_\varepsilon - 1})$$

Now we minimize the principal part with respect to $\sigma = \sigma_\varepsilon$. The minimum is attained for

$$\sigma_\varepsilon = K\varepsilon \quad \text{with } K = \left(\frac{C}{C_\eta}\right)^{\frac{1}{2s_\varepsilon}},$$

which leads to

$$F_\varepsilon(u_\varepsilon) \geq C_1 C_\eta + C_2 \frac{\varepsilon^{2s_\varepsilon - 1}}{2s_\varepsilon} \left(\frac{(K\varepsilon)^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1}\right) - C_3 \varepsilon^{2s_\varepsilon - 1}.$$

Since $K^{1-2s_\varepsilon} \to 1$ as $s_\varepsilon \to \frac{1}{2}$, the leading term in the RHS is

$$\frac{\varepsilon^{2s_\varepsilon - 1} - 1}{1 - 2s_\varepsilon}.$$

This computation suggests the scaling factor

$$\lambda_+(\varepsilon) = \frac{1 - 2s_\varepsilon}{\varepsilon^{2s_\varepsilon - 1} - 1}.$$

Note that, since we are interested in vanishing perturbations of the double-well functional, we ought to make sure that the scaling factor we have found behaves correctly (that is, we want the coefficient in front of the double well potential to diverge as $\varepsilon \to 0$ and the one in front of the Gagliardo seminorm to tend to 0). Let us check the validity of such conditions: in this case, the functional has the form

$$F_\varepsilon^{s_\varepsilon}(u) = \frac{\lambda_+(\varepsilon)}{\varepsilon} \int_{(0,1)} W(u)dt + \lambda_+(\varepsilon)\varepsilon^{2s_\varepsilon - 1}[u]_{s_\varepsilon}^2,$$

thus, the conditions are

$$\lambda_+(\varepsilon)\varepsilon^{2s_\varepsilon - 1} \to 0 \quad \text{and} \quad \frac{\lambda_+(\varepsilon)}{\varepsilon} \to +\infty.$$

Here, the first condition is equivalent to

$$0 \leftarrow \frac{2s_\varepsilon - 1}{\varepsilon^{1-2s_\varepsilon} - 1} = \frac{(2s_\varepsilon - 1)\,|\log(\varepsilon)|}{e^{(2s_\varepsilon - 1)|\log(\varepsilon)|} - 1} \cdot \frac{1}{|\log(\varepsilon)|},$$

which, since the function $f(x) = \frac{e^x - 1}{x}$ is bounded from below by a positive constant for all $x \in [0, +\infty)$, is satisfied.

Remark 2.1. *The condition*

$$\lambda_+(\varepsilon)\varepsilon^{2s_\varepsilon-1}\to 0$$

is also satisfied if $s_\varepsilon<\frac{1}{2}$. *Indeed, if* $|\log(\varepsilon)|\,(2s_\varepsilon-1)$ *is bounded the reasoning above still applies. On the other hand, if* $|\log(\varepsilon)|\,(2s_\varepsilon-1)\to-\infty$*, one has*

$$\frac{2s_\varepsilon-1}{e^{(2s_\varepsilon-1)|\log(\varepsilon)|}-1}\to 0.$$

The reason why study the cases $s_\varepsilon>\frac{1}{2}$ *and* $s_\varepsilon<\frac{1}{2}$ *separately lies in the behaviour of the term in front of the double well potential, which may not diverge for* $s<\frac{1}{2}$ *with this scaling factor.*

We now check the second condition, namely:

$$\frac{\lambda_+(\varepsilon)}{\varepsilon}\to+\infty$$

this, after simplifications, leads to

$$\frac{(2s_\varepsilon-1)\,|\log(\varepsilon)|}{1-e^{(1-2s)|\log(\varepsilon)|}}\cdot\frac{1}{\varepsilon^{2s_\varepsilon}\,|\log(\varepsilon)|}\to+\infty.$$

Here, the first term is bounded from below whenever $|\log(\varepsilon)|\,(2s_\varepsilon-1)$ is bounded, while $\frac{1}{\varepsilon^{2s_\varepsilon}|\log(\varepsilon)|}\to+\infty$. On the other hand, if $|\log(\varepsilon)|\,(2s_\varepsilon-1)\to+\infty$, both terms in the product tend to $+\infty$.

If $s_\varepsilon\to\frac{1}{2}^-$, it suffices to repeat the same computation above, considering the unscaled functionals

$$G_\varepsilon(u)=\frac{1}{\varepsilon}\int_{(0,1)}W(u)dt+\iint_{(0,1)\times(0,1)}\frac{|u(x)-u(y)|^2}{|x-y|^{1+2s_\varepsilon}}dx\,dy.$$

In this case, the optimal value in the minimization procedure is attained for $\sigma\approx\varepsilon^{\frac{1}{2s_\varepsilon}}$, which leads to

$$F_\varepsilon(u_\varepsilon)\geq C_1C_\eta\varepsilon^{\frac{2s_\varepsilon-1}{2s_\varepsilon}}+\frac{C_2}{2s_\varepsilon}\left(\frac{(K\varepsilon)^{\frac{1-2s_\varepsilon}{2s_\varepsilon}}-1}{2s_\varepsilon-1}\right)-C_3.$$

Again, the value K satisfies $K^{1-2s_\varepsilon}\to 1$ as $s_\varepsilon\to\frac{1}{2}$. Hence, the leading term in the RHS is

$$\frac{\varepsilon^{\frac{1-2s_\varepsilon}{2s_\varepsilon}}-1}{2s_\varepsilon-1}$$

This computation suggests the scaling factor

$$\lambda_-(\varepsilon)=\frac{2s_\varepsilon-1}{\varepsilon^{\frac{1-2s_\varepsilon}{2s_\varepsilon}}-1}.$$

In this case, the conditions become

$$\lambda_-(\varepsilon)\to 0\quad\text{and}\quad\frac{\lambda_-(\varepsilon)}{\varepsilon}\to+\infty.$$

The first, being equivalent to

$$\frac{(2s_\varepsilon-1)\,|\log(\varepsilon)|}{e^{(2s_\varepsilon-1)|\log(\varepsilon)|}-1}\cdot\frac{1}{|\log(\varepsilon)|}\to 0$$

is satisfied by Remark 2.1. On the other hand, the second condition leads to

$$\frac{2s_\varepsilon - 1}{\varepsilon(\varepsilon^{\frac{1-2s_\varepsilon}{2s_\varepsilon}} - 1)} \to +\infty$$

which is equivalent to

$$\frac{2s_\varepsilon - 1}{\varepsilon(\varepsilon^{1-2s_\varepsilon} - 1)} \to +\infty.$$

Once again, we write it as

$$+\infty \leftarrow \frac{2s_\varepsilon - 1}{\varepsilon(\varepsilon^{1-2s_\varepsilon} - 1)} = \frac{(2s_\varepsilon - 1)\,|\log(\varepsilon)|}{e^{(2s_\varepsilon-1)|\log(\varepsilon)|} - 1} \cdot \frac{1}{\varepsilon\,|\log(\varepsilon)|},$$

where the previous term is bounded from below by a positive constant if $(2s_\varepsilon - 1)\,|\log(\varepsilon)|$ is bounded and tends to $+\infty$ if $(2s_\varepsilon - 1)\,|\log(\varepsilon)| \to -\infty$. The second term, on the other hand, always tends to $+\infty$, which implies that the second condition is satisfied for any regime of $s < \frac{1}{2}$.

Remark 2.2. *For $\varepsilon \in (0,1)$ fixed,*

$$\lim_{s \to \frac{1}{2}^+} \frac{1-2s}{\varepsilon^{2s-1} - 1} \varepsilon^{2s-1} = \lim_{s \to \frac{1}{2}^+} \frac{2s-1}{\varepsilon^{1-2s} - 1} = \frac{1}{|\log(\varepsilon)|}$$

and

$$\lim_{s \to \frac{1}{2}^-} \frac{2s-1}{\varepsilon^{\frac{1-2s}{2s}} - 1} = \frac{1}{|\log(\varepsilon)|}.$$

This allows us to recover the scaling factor for the critical regime $s = \frac{1}{2}$.

3. The Gamma limit

In this section we prove that, for any sequence $s_\varepsilon \to \frac{1}{2}$, the functionals

$$F_\varepsilon(u) = \begin{cases} \frac{\lambda_+(\varepsilon)}{\varepsilon} \int_{(0,1)} W(u)dt + \lambda_+(\varepsilon)\varepsilon^{2s_\varepsilon - 1}[u]^2_{s_\varepsilon} & \text{if } s_\varepsilon > \frac{1}{2} \\ \frac{1}{|\log \varepsilon|\varepsilon} \int_{(0,1)} W(u)dt + \frac{1}{|\log \varepsilon|}[u]^2_{s_\varepsilon} & \text{if } s_\varepsilon = \frac{1}{2} \\ \frac{\lambda_-(\varepsilon)}{\varepsilon} \int_{(0,1)} W(u)dt + \lambda_-(\varepsilon)[u]^2_{s_\varepsilon} & \text{if } s_\varepsilon < \frac{1}{2} \end{cases}$$

with $\lambda_-(\varepsilon)$ and $\lambda_-(\varepsilon)$ being the scaling previously defined, Γ-converge to the functional $F_0^{\frac{1}{2}}$ found in [1]. In this section we prove the following convergence result.

Theorem 3.1. *Assume that W is a double-well potential as above, let $\varepsilon > 0$ and $s_\varepsilon \in (0,1)$ be such that*

$$\lim_{\varepsilon \to 0} s_\varepsilon = \frac{1}{2}$$

Then the family of functionals $(F_\varepsilon)_\varepsilon$ defined above Γ-converge to the functional

$$F_0 = \begin{cases} 8\#S(u) & \text{if } u \in BV((0,1), \{-1,1\}) \\ +\infty & \text{otherwise} \end{cases}.$$

Before proceeding to the proof of the theorem we state and prove the corresponding compactness result. Its proof follows the technique employed in [14]. For simplicity, given any subset $E \subset (0,1)$, we define the restricted energy

$$F_\varepsilon(u;E) = \begin{cases} \frac{\lambda_+(\varepsilon)}{\varepsilon} \int_E W(u)dt + \lambda_+(\varepsilon)\varepsilon^{2s_\varepsilon - 1}[u]^2_{H^{s_\varepsilon}(E)} & \text{if } s_\varepsilon > \frac{1}{2} \\ \frac{\lambda_-(\varepsilon)}{\varepsilon} \int_E W(u)dt + \lambda_-(\varepsilon)[u]^2_{H^{s_\varepsilon}(E)} & \text{if } s_\varepsilon < \frac{1}{2}, \end{cases}$$

where

$$[u]^2_{H^{s_\varepsilon}(E)} = \iint_{E\times E} \frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}} dxdy.$$

Moreover, since the case $s_\varepsilon = \frac{1}{2}$ has already been addressed in [1], we will only prove the theorem in the case $s_\varepsilon \neq \frac{1}{2}$.

Lemma 3.2. *Assume that*

$$\sup_\varepsilon F_\varepsilon(u_\varepsilon) \leq S < +\infty.$$

Then there exists $u \in BV((0,1),\{-1,1\})$ such that, up to subsequences, $u_\varepsilon \to u$ in measure.

Proof. By renaming if necessary, we can replace u_ε by its truncation at any height $\pm k$ (where $k \geq 1$), denoted by $\mathcal{T}_k(u_\varepsilon)$, since the energy F_ε decreases by truncation. Here, the truncation function $\mathcal{T}_k$ at height $\pm k$ is defined as follows:

$$\mathcal{T}_k(s) = \begin{cases} k\,\mathrm{sgn}(s) & \text{if } |s| \geq k \\ s & \text{otherwise.} \end{cases}$$

We claim that the set u_ε is totally bounded in $L^1(0,1)$, i.e. for any $\delta > 0$ there exists a finite set $\mathcal{A} \subset L^1$ such that for any small ε there exists $v_\varepsilon \in \mathcal{A}$ with

$$\|u_\varepsilon - v_\varepsilon\|_{L^1} \leq \delta.$$

Let $\eta \in (0,\frac{1}{4})$ and $\rho > 0$. We begin by partitioning the interval $[0,1]$ in $\frac{1}{\rho}$ (which we assume to be an integer) subintervals I_i^ρ of length ρ. We will refer to the set of indexes in such partition as $\mathcal{J}_\varepsilon$ and further classify these intervals as follows:

$$\mathcal{J}^1 = \{i \in \mathcal{J}_\varepsilon \quad : \quad |\{u_\varepsilon < 1-\eta\} \cap I_i^\rho| < \eta\rho\}$$

$$\mathcal{J}^{-1} = \{i \in \mathcal{J}_\varepsilon \quad : \quad |\{u_\varepsilon > -1+\eta\} \cap I_i^\rho| < \eta\rho\}$$

$$\mathcal{J}^0 = \mathcal{J}_\varepsilon \setminus (\mathcal{J}_\varepsilon^1 \cup \mathcal{J}_\varepsilon^{-1}).$$

For simplicity, we also call

$$K_+ = \bigcup_{\mathcal{J}^1} I_i^\rho \quad K_- = \bigcup_{\mathcal{J}^{-1}} I_i^\rho \quad K_0 = \bigcup_{\mathcal{J}^0} I_i^\rho$$

and define v_ε to be equal to 1 in K_+, and -1 in $K_- \cup K_0$.

We also point out that, since $s_\varepsilon < \frac{3}{2}$, the space $H^{s_\varepsilon}(0,1)$ is closed under truncation and $F_\varepsilon(v)$ decreases by the truncation of v at height $h \geq 1$, we can replace the u_ε by their truncations at height $1+2\eta$.

Now we estimate from below the energy $F_\varepsilon(u; I_i^\rho)$ for $i \in \mathcal{J}^0$. First, note that, for such intervals, we have

$$|I_i^\rho \cap \{|u_\varepsilon| < 1-\eta\}| \geq (1-2\eta)\rho.$$

Thus, if we define

$$C_\eta = \min_{[-1+\eta,1-\eta]} W,$$

we have

$$F_\varepsilon(u_\varepsilon; I_i^\rho) \geq \frac{\lambda_\pm(\varepsilon)}{\varepsilon} \int_{I_i^\rho} W(u_\varepsilon) dt \geq \frac{\lambda_\pm(\varepsilon)}{\varepsilon} C_\eta (1-2\eta)\rho.$$

This implies

$$S \geq \sum_{\mathcal{J}^0} F_\varepsilon(u_\varepsilon; I_i^\rho) \geq \sum_{\mathcal{J}^0} \frac{\lambda_\pm(\varepsilon)}{\varepsilon} \int_{I_i^\rho} W(u_\varepsilon)dt \geq \frac{\lambda_\pm(\varepsilon)}{\varepsilon} C_\eta(1-2\eta)\rho \left|\mathcal{J}^0\right|.$$

Hence

$$\left|\mathcal{J}^0\right| \leq \frac{\varepsilon}{\lambda_\pm(\varepsilon)} \frac{S}{C_\eta(1-2\eta)\rho}. \tag{3.1}$$

This, for η and ε small enough, leads to

$$|K_0| = \sum_{\mathcal{J}^0} |I_i^\rho| \leq \frac{\varepsilon}{\lambda_\pm(\varepsilon)} \frac{S}{C_\eta(1-2\eta)} \tag{3.2}$$

Now we estimate the L^1 distance between u_ε and v_ε outside of K_0. We will do so by estimating such distance in the subsets of K_+ (resp. K_-) where u_ε is away from ± 1, when it is close to -1 and when it is close to 1. First of all, we have

$$|\{|u_\varepsilon| \leq 1-\eta\}| \leq \frac{1}{C_\eta} \int_0^1 W(u_\varepsilon)dt \leq \frac{\varepsilon}{\lambda_\pm(\varepsilon)} \frac{1}{C_\eta} S.$$

Thus,

$$\int_{\{|u_\varepsilon| \leq 1-\eta\}} |u_\varepsilon - v_\varepsilon| \, dt \leq 3 \left|\{|u_\varepsilon| \leq 1-\eta\}\right| \leq \frac{\varepsilon}{\lambda_\pm(\varepsilon)} \tilde{C}_\eta. \tag{3.3}$$

Regarding the set where u_ε is close to -1, we have

$$|K_+ \cap \{u_\varepsilon \leq \eta - 1\}| = \sum_{\mathcal{J}^1} |I_i^\rho \cap \{u_\varepsilon \leq \eta - 1\}| \leq \left|\mathcal{J}^1\right| \eta\rho = \eta \left|K_+\right|$$

which implies

$$\int_{K_+ \cap \{u_\varepsilon \leq \eta - 1\}} |u_\varepsilon - v_\varepsilon| \, dt \leq 3\eta \left|K_+\right|.$$

Moreover, we have

$$\int_{K_+ \cap \{u_\varepsilon \geq 1-\eta\}} |u_\varepsilon - v_\varepsilon| \, dt = \int_{K_+ \cap \{u_\varepsilon \geq 1-\eta\}} |u_\varepsilon - 1| \, dt \leq 2\eta \left|K_+\right|.$$

This and (3.3) lead to

$$\int_{K_+} |u_\varepsilon - v_\varepsilon| \, dt \leq \tilde{C} \left(\eta + \frac{\varepsilon}{\lambda_\pm(\varepsilon)}\right). \tag{3.4}$$

Performing the same computations for K_- (up to exchanging the roles of the sets where u_ε is closed to 1 and -1), we obtain

$$|K_- \cap \{u_\varepsilon \geq 1-\eta\}| \leq \eta \left|K_-\right| \quad \text{and} \quad \int_{K_- \cap \{u_\varepsilon \leq \eta - 1\}} |u_\varepsilon - v_\varepsilon| \, dt \leq 2\eta \left|K_+\right|,$$

which, as before, implies

$$\int_{K_-} |u_\varepsilon - v_\varepsilon| \, dt \leq \tilde{C} \left(\eta + \frac{\varepsilon}{\lambda_\pm(\varepsilon)}\right). \tag{3.5}$$

Putting (3.2), (3.4) and (3.5) together we obtain, for η and ε small enough,

$$\int_{[0,1]} |u_\varepsilon - v_\varepsilon|\, dt \leq \int_{K_+} |u_\varepsilon - v_\varepsilon|\, dt + \int_{K_-} |u_\varepsilon - v_\varepsilon|\, dt + \int_{K_0} |u_\varepsilon - v_\varepsilon|\, dt$$
$$\leq 2\tilde{C}\left(\eta + \frac{\varepsilon}{\lambda_\pm(\varepsilon)}\right) + 3\,|K_0| \leq \delta.$$

It follows that u_ε has a converging subsequence to some function $u \in L^1(0,1)$. Moreover, since $|v_\varepsilon| = 1$ for all ε, it also follows that $u = 2\chi_E - 1$ for some set $E \subset (0,1)$. Moreover, this proves that the choice of the limit u is not affected by the height $k \geq 1$ of the truncation we applied at the beginning of the proof. Moreover, since $\mathcal{T}_k(u_\varepsilon)$ converges in measure to u, we have

$$u_\varepsilon \to u \quad \text{in measure.}$$

Now we claim that $\#S(u)$ is finite. To prove it, define the functions ψ_ρ to be equal to 1 in I_i^ρ if $|E \cap I_i^\rho| \geq \frac{\rho}{2}$ and -1 otherwise. As $\psi_\rho \to u$ in measure as $\rho \to 0^+$, it suffices to show that $\#S(\psi_\rho)$ is equibounded. Let I_i^ρ be an interval such that ψ_ρ has a jump point, denoted by t_i, either at the beginning or the end of I_i^ρ. consider the interval $\mathcal{I}_i = (t_i - \rho, t_i + \rho)$.

Note that, by convergence in measure,

$$|\{u_\varepsilon \geq 1 - \eta\} \cap \mathcal{I}_i| =: a_\varepsilon \rho \geq \frac{\rho}{2} \quad \text{and} \quad |\{u_\varepsilon \geq \eta - 1\} \cap \mathcal{I}_i| =: b_\varepsilon \rho \geq \frac{\rho}{2}$$

for ε small enough. Then we have

$$[u_\varepsilon]_{H^{s_\varepsilon}(\mathcal{I}_i)} \geq \int_{-\rho}^{-\rho + a_\varepsilon \rho} \int_{\rho - b_\varepsilon \rho}^{\rho} \frac{(2-2\eta)^2}{|x-y|^{1+2s_\varepsilon}} dxdy$$
$$\geq (2-2\eta)^2 \frac{\rho^{(1-2s_\varepsilon)}}{2s_\varepsilon}\left[\frac{(2-(a_\varepsilon^i + b_\varepsilon^i))^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} - C\right].$$

We now define $\sigma_\varepsilon = 2 - (a_\varepsilon^i + b_\varepsilon^i)$ (which tends to zero as $\varepsilon \to 0$, by convergence in measure) and repeat the minimization argument from Section 2. For $s_\varepsilon > \frac{1}{2}$, we obtain

$$F_\varepsilon(u_\varepsilon, \mathcal{I}_i) \geq \frac{2s_\varepsilon - 1}{\varepsilon^{1-2s_\varepsilon} - 1}(2-2\eta)^2 \frac{\rho^{(1-2s_\varepsilon)}}{2s_\varepsilon}\left[\frac{\sigma_\varepsilon^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} - C\right] \geq C_0 > 0$$

and for $s_\varepsilon < \frac{1}{2}$ we obtain

$$F_\varepsilon(u_\varepsilon, \mathcal{I}_i) \geq \frac{2s_\varepsilon - 1}{\varepsilon^{\frac{1-2s_\varepsilon}{2s}} - 1}(2-2\eta)^2 \frac{\rho^{(1-2s_\varepsilon)}}{2s_\varepsilon}\left[\frac{\sigma_\varepsilon^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} - C\right] \geq C_0 > 0.$$

Here, C_0 is a positive constant.

Note that the intervals I_i may note be disjoint. However, by choosing them alternately, we can divide them in two family of disjoint intervals (indexed, say, by J_1 and J_2). In particular, we have that

$$S \geq F_\varepsilon(u_\varepsilon) \geq \sum_{i \in J_1} F_\varepsilon(u_\varepsilon, \mathcal{I}_i),$$
$$S \geq F_\varepsilon(u_\varepsilon) \geq \sum_{i \in J_2} F_\varepsilon(u_\varepsilon, \mathcal{I}_i).$$

Thus

$$2S \geq 2F_\varepsilon(u_\varepsilon) \geq \sum_{t_i \in S(\psi_\rho)} F(u_\varepsilon, \mathcal{I}_i) \geq C_0 \# S(\psi_\rho),$$

which concludes the proof. $\square$

The next result deals with the lower bound.

Lemma 3.3. *Let $u_\varepsilon \to u$ in measure. Then $u \in BV((0,1),\{-1,1\})$ and*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq m_{\frac{1}{2}} \# S(u). \tag{3.6}$$

Proof. By the compactness assumption, u_ε converges in measure to a function $v \in BV((0,1),\{-1,1\})$. This implies that $u \in BV((0,1),\{-1,1\})$ by the uniqueness of the limit.

Note that for every $t_i \in S(u)$ there exists an interval $I^i = (t_i - \delta, t_i + \delta)$ such that $I^i \cap I^j = \emptyset$ if $i \neq j$. Furthermore, let $\eta \in (0, \frac{1}{4})$ and define

$$A^i_\varepsilon = \{u_\varepsilon > 1 - \eta\} \cap I^i, \quad B^i_\varepsilon = \{u_\varepsilon < \eta - 1\} \cap I^i.$$

Then, repeating the computations from Section 2, we obtain

$$\begin{aligned}
\iint_{[0,1]\times[0,1]} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s_\varepsilon}} dxdy &\geq 2 \sum_{i=1}^{\#S(u)} \iint_{A^i_\varepsilon \times B^i_\varepsilon} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s_\varepsilon}} dxdy \\
&\geq (2-2\eta)^2 \int_{t_i-\delta}^{t_i-\delta+|A^i_\varepsilon|} \int_{t_i+\delta-|B^i_\varepsilon|}^{t_i+\delta} \frac{1}{|x-y|^{1+2s_\varepsilon}} dxdy \\
&\geq (2-2\eta)^2 \frac{\delta^{(1-2s_\varepsilon)}}{2s_\varepsilon} \left[\frac{(2 - \frac{|A^i_\varepsilon| + |B^i_\varepsilon|}{\delta})^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} - C \right].
\end{aligned}$$

Moreover, by convergence in measure, we have

$$\sigma_\varepsilon = 2 - \frac{|A^i_\varepsilon| + |B^i_\varepsilon|}{\delta} \to 0.$$

Now we consider the case $s_\varepsilon > \frac{1}{2}$: here we obtain

$$F_\varepsilon(u_\varepsilon) \geq \sum_{i=1}^{\#S(u)} \left[\lambda_+(\varepsilon) C_\eta \frac{\sigma_\varepsilon}{\varepsilon} + \lambda_+(\varepsilon) 8(1-\eta)^2 \varepsilon^{2s_\varepsilon - 1} \frac{\delta^{(1-2s_\varepsilon)}}{2s_\varepsilon} \left(\frac{\sigma_\varepsilon^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} - C \right) \right].$$

From the minimization argument in the first section, we know that that the minimum over σ_ε of each addendum in the RHS is attained when $\sigma_\varepsilon \approx \varepsilon$. Thus, we obtain

$$F_\varepsilon(u_\varepsilon, \mathcal{I}_i) \geq \sum_{i=1}^{\#S(u)} \left[\lambda_+(\varepsilon) C_\eta \tilde{C} + \frac{2s_\varepsilon - 1}{\varepsilon^{1-2s_\varepsilon} - 1} 8(1-\eta)^2 \frac{\rho^{(1-2s_\varepsilon)}}{2s_\varepsilon} \left(\frac{\varepsilon^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} - C \right) \right].$$

Now we take the limit as $\eta \to 0^+$ (and thus $C_\eta \to 0$), which leads to

$$F_\varepsilon(u_\varepsilon) \geq \sum_{i=1}^{\#S(u)} 8 \frac{2s_\varepsilon - 1}{\varepsilon^{1-2s_\varepsilon} - 1} \frac{\rho^{(1-2s_\varepsilon)}}{2s_\varepsilon} \left(\frac{\varepsilon^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} - C \right).$$

and for $s_\varepsilon < \frac{1}{2}$ we obtain (by the same procedure)

$$F_\varepsilon(u_\varepsilon) \geq \sum_{i=1}^{\#S(u)} 8 \frac{2s_\varepsilon - 1}{\varepsilon^{\frac{1-2s_\varepsilon}{2s}} - 1} \frac{\rho^{(1-2s_\varepsilon)}}{2s_\varepsilon} \left(\frac{\varepsilon^{\frac{1-2s_\varepsilon}{2s_\varepsilon}} - 1}{2s_\varepsilon - 1} - C \right).$$

In both cases, taking the lower limit as $\varepsilon \to 0^+$, we obtain

$$\liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \geq 8\#S(u),$$

which concludes the proof. □

Lemma 3.4. *Let $u \in BV((0,1), \{-1,1\})$, then there exists a sequence $(u_\varepsilon)_\varepsilon$ such that $u_\varepsilon \in H^{s_\varepsilon}(0,1)$ such that $u_\varepsilon \to u$ in $L^2(0,1)$ and*

$$\lim_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) = m_{\frac{1}{2}} \#S(u). \tag{3.7}$$

Proof. For any jump point $t_i \in S(u)$, we consider the piecewise affine function defined as follows:

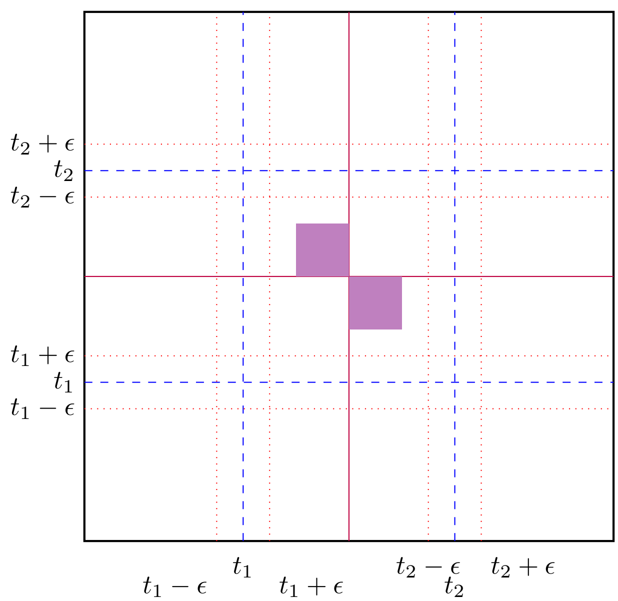
$$u_\varepsilon(x) = \begin{cases} \mathcal{T}_1(\frac{x-t_i}{\varepsilon})u(t_i^+) & \text{if } |x - t_i| < \varepsilon \\ u(x) & \text{otherwise.} \end{cases}$$

We also define the points $x_0 = 0$, $x_i = \frac{t_i + t_{i+1}}{2}$ for $1 \leq i \leq \#S(u) - 1$ and $x_{\#S(u)} = 1$.

First, note that $u_\varepsilon \to u$ in measure as $\varepsilon \to 0$. This implies that

$$\int_0^1 W(u_\varepsilon) \to \int_0^1 W(u) = 0.$$

To compute the Gagliardo seminorm of u_ε, we proceed as follows: let us divide the square $(0,1) \times (0,1)$ in many rectangles, as in the following picture (for reference, consider the solid line to be at the middle point between t_1 and t_2).



Our aim is to show that the contribution to Gagliardo seminorm $[u_\varepsilon]_{H^{s_\varepsilon}}$ given by the interaction of points at a distance greater than a given threshold can be bounded from above by a constant, so that the scaling factor (either $\lambda_-(\varepsilon)$ or $\lambda_+(\varepsilon)\varepsilon^{2s_\varepsilon - 1}$) would make their contribution to $F_\varepsilon(u_\varepsilon)$ infinitesimal.

First, note that on the purple squares (which can be taken with sides smaller than $\frac{1}{4}\inf_i |t_i - t_i + 1|$) we have $u(x) - u(y) = 0$. Moreover, calling

$$Q_i = [x_i, x_{i+1}] \times [x_i, x_{i+1}],$$

we have

$$\iint_{[0,1]\times[0,1]} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s_\varepsilon}} dxdy$$

$$= \sum_{i=0}^{\#S(u)-1} \iint_{Q_i} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s_\varepsilon}} dxdy + \iint_{[0,1]\times[0,1]\setminus\bigcup Q_i} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s_\varepsilon}} dxdy.$$

Working term by term, we have

$$\iint_{Q_i} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s_\varepsilon}} dxdy = 2\int_{x_i}^{t_i-\varepsilon} \int_{t_i+\varepsilon}^{x_{i+1}} \frac{4}{|x-y|^{1+2s_\varepsilon}} dxdy$$

$$+ \int_{t_i-\varepsilon}^{t_i+\varepsilon} \int_{t_i-\varepsilon}^{t_i+\varepsilon} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s_\varepsilon}} dxdy + 4\int_{t_i+\varepsilon}^{x_{i+1}} \int_{t_i-\varepsilon}^{t_i+\varepsilon} \frac{|u_\varepsilon(x) - 1|^2}{|x-y|^{1+2s_\varepsilon}} dxdy$$

Which, after computations, leads to

$$\iint_{Q_i} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s_\varepsilon}} dxdy = 8\frac{1}{2s}\frac{(2\varepsilon)^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} + O(1).$$

On the other hand

$$\iint_{[0,1]\times[0,1]\setminus\bigcup Q_i} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^{1+2s_\varepsilon}} dxdy \leq \frac{C}{\inf_i |t_i - t_i + 1|^{1+2s_\varepsilon}} \leq \tilde{C} < +\infty.$$

Now we estimate the energy contribution from the double-well potential. In particular, we have:

$$\int_0^1 W(u)dt = \sum_{i=1}^{\#S(u)} \int_{t_i-\varepsilon}^{t_i+\varepsilon} W\left(\frac{t-t_i}{\varepsilon}\right) dt \leq \#S(u) 2\varepsilon \max_{[-1,1]} W.$$

Putting everything together we have, for $s_\varepsilon > \frac{1}{2}$,

$$F_\varepsilon(u_\varepsilon) \leq C_1\lambda_+(\varepsilon) + 8\#S(u)\lambda_+(\varepsilon)\varepsilon^{2s_\varepsilon-1}\frac{1}{2s}\left(\frac{(2\varepsilon)^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} + C_2\right),$$

and, for $s_\varepsilon < \frac{1}{2}$,

$$F_\varepsilon(u_\varepsilon) = F_\varepsilon(u_\varepsilon) \leq C_1\lambda_-(\varepsilon) + 8\#S(u)\lambda_-(\varepsilon)\frac{1}{2s}\left(\frac{(2\varepsilon)^{1-2s_\varepsilon} - 1}{2s_\varepsilon - 1} + C_2\right).$$

Either way, we proved that

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq m_{\frac{1}{2}}\#S(u)$$

which concludes the proof. □

Remark 3.5. *Assume that* $(2s_\varepsilon - 1)|\log\varepsilon| \to C \in \mathbb{R}$. *Then*

$$\frac{1-2s_\varepsilon}{\varepsilon^{2s_\varepsilon-1} - 1}\varepsilon^{2s_\varepsilon-1} \approx \frac{C}{e^C - 1}\frac{1}{|\log\varepsilon|}$$

and

$$\frac{2s_\varepsilon - 1}{\varepsilon^{\frac{1-2s_\varepsilon}{2s_\varepsilon}} - 1} \approx \frac{C}{e^C - 1}\frac{1}{|\log\varepsilon|}.$$

This computation highlights that in the regime

$$|2s_\varepsilon - 1| << \frac{1}{|\log\varepsilon|},$$

we have the separation of scales *effect discussed in the introduction. Namely, the* Γ*-limit of* $\lambda(\varepsilon,s)F^s_\varepsilon$ *is the same as the one obtained first letting* $s\to\frac{1}{2}$ *with* $\varepsilon>0$ *fixed, which gives* $\frac{1}{|\log\varepsilon|}F^{1/2}_\varepsilon$*, and then letting* $\varepsilon\to 0$*.*

4. On the behaviour of optimal profiles

In this section we briefly discuss the properties of the optimal values

(4.1)
$$m_s=\inf\left\{\int_{\mathbb{R}}W(u)dt+\iint_{\mathbb{R}\times\mathbb{R}}\frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}dxdy\mid u\in H^s(\mathbb{R}),u(\pm\infty)=\pm1\right\}$$

as $s\to\frac{1}{2}^+$. In particular, we prove that the value $m_{\frac{1}{2}}$ is the solution of a (suitably rescaled) optimal profile problem and that

$$\lim_{s\to\frac{1}{2}^+}(2s-1)m_s=m_{\frac{1}{2}}.$$

Both properties are strongly related to the fact that, while the Γ-limit of F^s_ε with $s>\frac{1}{2}$ involves the optimal constant m_s, the case $s=\frac{1}{2}$ requires an additional scaling argument.

For any $T\geq 0$ and $s\geq\frac{1}{2}$ consider the family $\mathcal{H}^s_T$ of real valued functions v defined on $\mathbb{R}$ such that $v\in H^s(-T,T)$, $v(\pm T)=\pm1$, $v(t)=1$ if $t>T$ and $v(t)=-1$ if $t<-T$. Notably, it was shown in [13] that for any $s\in\left(\frac{1}{2},1\right)$

$$m_s=\lim_{T\to+\infty}\inf\left\{\int_{-T}^T W(u)dt+\int_{-T}^T\int_{-T}^T\frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}dxdy\mid v\in\mathcal{H}^s_T\right\}. \tag{4.2}$$

The following lemma gives a similar result for $s=\frac{1}{2}$

Lemma 4.1. *The quantity* $m_{\frac{1}{2}}$ *satisfies the following property:*

(4.3)
$$m_{\frac{1}{2}}=\lim_{T\to+\infty}\frac{1}{\log(2T)}\inf\left\{\int_{-T}^T W(u)dt+\int_{-T}^T\int_{-T}^T\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy\mid v\in\mathcal{H}^{\frac{1}{2}}_T\right\}$$

Proof. Let

$$\tilde{m}^T_{\frac{1}{2}}=\inf\left\{\int_{-T}^T W(u)dt+\int_{-T}^T\int_{-T}^T\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy\mid v\in\mathcal{H}^{\frac{1}{2}}_T\right\}.$$

For any function $v\in\mathcal{H}^s_T$, we fix? $\eta\in(0,\frac{1}{4})$ and consider

$$A=\{v>1-\eta\}\cap[-T,T],\quad B=\{v<\eta-1\}\cap[-T,T],$$

$$a=\frac{|A|}{2T},\quad b=\frac{|B|}{2T},\quad C_\eta=\min_{|z|\leq 1-\eta}W(z).$$

Following the techniques used in Section 2, we have:

$$\int_{-T}^T W(v)dt\geq C_\eta 2T(1-a-b)$$

and

$$\int_{-T}^T\int_{-T}^T\frac{|v(x)-v(y)|^2}{|x-y|^2}dxdy\geq 2\iint_{A\times B}\frac{|v(x)-v(y)|^2}{|x-y|^2}dxdy$$

$$\geq 8(1-\eta)^2 \iint_{A\times B} \frac{1}{|x-y|^2} dxdy \geq 8(1-\eta)^2 \int_{-T}^{-T+|B|} \int_{T-|A|}^{T} \frac{1}{|x-y|^2} dxdy$$
$$= 8(1-\eta)^2 [\log(2T-|A|) + \log(2T-|B|) - \log(2T) - \log(2T-|A|-|B|)]$$
$$= 8(1-\eta)^2 \left[\log(1-a) + \log(1-b) - \log(1-a-b)\right].$$

Hence

$$\int_{-T}^{T} W(v)dt + \int_{-T}^{T}\int_{-T}^{T} \frac{|v(x)-v(y)|^2}{|x-y|^2} dxdy$$
$$\geq C_\eta 2T(1-a-b) + 8(1-\eta)^2 [\log(1-a) + \log(1-b) - \log(1-a-b)]$$
$$= 8(1-\eta)^2 \Bigg\{ \frac{C_\eta}{8(1-\eta)^2} 2T(1-a-b) + [\log(1-a) + \log(1-b) - \log(1-a-b)] \Bigg\}.$$

This, using the inequality

$$-\log x + Mx \geq \log M$$

with

$$x = (1-a-b) \quad \text{and} \quad M = \frac{C_\eta}{8(1-\eta)^2} 2T,$$

we obtain

$$\int_{-T}^{T} W(v)dt + \int_{-T}^{T}\int_{-T}^{T} \frac{|v(x)-v(y)|^2}{|x-y|^2} dxdy$$
$$\geq 8(1-\eta)^2 \left\{ \log\left(\frac{C_\eta}{8(1-\eta)^2} 2T\right) + \log(1-a) + \log(1-b) \right\}$$

In conclusion, we have

$$\frac{1}{\log(2T)} \left(\int_{-T}^{T} W(v)dt + \int_{-T}^{T}\int_{-T}^{T} \frac{|v(x)-v(y)|^2}{|x-y|^2} dxdy \right)$$
$$\geq \frac{\tilde{C}(\eta)}{\log(2T)} + 8(1-\eta)^2.$$

Hence, taking the limit as $T \to +\infty$ and then letting $\eta \to 0^+$, we obtain

$$\lim_{T\to+\infty} \tilde{m}^T_{\frac{1}{2}} \geq 8.$$

For the opposite inequality, we consider the test function $u(t) = \mathcal{T}_1(t)$. Indeed, for the potential term, we have

$$\int_{T}^{T} W(u)dt = \int_{1}^{1} W(t)dt.$$

On the other hand, regarding the Gagliardo seminorm

$$\int_{-T}^{T}\int_{-T}^{T} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy,$$

we split the integral as in the proof of Lemma 3.4, obtaining

$$\int_{-1}^{1}\int_{-1}^{1}\frac{|x-y|^2}{|x-y|^2}dxdy+4\int_{1}^{T}\int_{-1}^{1}\frac{|x-1|^2}{|x-y|^2}dxdy+2\int_{1}^{T}\int_{1}^{T}\frac{4}{(x+y)^2}dxdy$$

which is equal to

$$C+8[2\log(1+T)-\log(2T)-\log(2)].$$

Hence

$$\frac{1}{\log(2T)}\left(\int_{-T}^{T}W(u)dt+\int_{-T}^{T}\int_{-T}^{T}\frac{|u(x)-u(y)|^2}{|x-y|^2}dxdy\right)=8+O\left(\frac{1}{\log(2T)}\right).$$

Taking the limit as $T\to+\infty$ concludes the proof. □

Lemma 4.2. *Given m_s as in (4.1), one has*

$$\lim_{s\to\frac{1}{2}^+}(2s-1)m_s=m_{\frac{1}{2}}. \tag{4.4}$$

Proof. Let $v_s\in H^{\frac{1}{2}}(\mathbb{R})$ and $T>0$ such that $v_s(\pm T)=\pm1$ and v_s is constant outside of $[-T,T]$. For any $\eta\in(0,\frac{1}{4})$, define:

$$\delta=\delta_s(\eta)=|\{|v_s|\le 1-\eta\}|\,,$$

$$A_\eta=\{v_s\ge 1-\eta\}\quad B_\eta=\{v_s\le\eta-1\}.$$

The energy

$$F_1^s(v_s,\mathbb{R})=\int_{\mathbb{R}}W(v_s)dt+[u]^2_{H^s(\mathbb{R})}$$

satisfies:

$$\begin{aligned}F_1^s(v_s,\mathbb{R})&\ge C_\eta\delta+2\iint_{A_\eta\times B_\eta}\frac{|v_s(x)-v_s(y)|^2}{|x-y|^{1+2s}}dxdy\\&\ge C_\eta\delta+8(1-\eta)^2\iint_{A_\eta\times B_\eta}\frac{1}{|x-y|^{1+2s}}dxdy.\end{aligned}$$

Since $|\mathbb{R}\setminus(A_\eta\cup B_\eta)|=\delta$, translation invariance implies

$$\begin{aligned}F_1^s(v_s,\mathbb{R})&\ge C_\eta\delta+8(1-\eta)^2\int_{\frac{\delta}{2}}^{+\infty}\int_{\frac{\delta}{2}}^{+\infty}\frac{1}{(x+y)^{1+2s}}dxdy\\&\ge C_\eta\delta+8(1-\eta)^2\frac{\delta^{1-2s}}{2s(2s-1)}.\end{aligned}$$

Taking the infimum over all possible values of δ, which is attained for $\delta=\left(\frac{8(1-\eta)^2}{2sC_\eta}\right)^{\frac{1}{2s}}$, yields

$$F_1^s(v_s,\mathbb{R})\ge C_\eta\left(\frac{8(1-\eta)^2}{2sC_\eta}\right)^{\frac{1}{2s}}+\frac{1}{2s}\left(\frac{8(1-\eta)^2}{2sC_\eta}\right)^{\frac{1-2s}{2s}}(1-\eta)^2\frac{8}{2s-1}.$$

Taking the infimum over all choices of v_s and then multiplying both sides by $2s-1$ leads to

$$(2s-1)m_s\ge(2s-1)C_\eta\left(\frac{8(1-\eta)^2}{2sC_\eta}\right)^{\frac{1}{2s}}+\frac{1}{2s}\left(\frac{8(1-\eta)^2}{2sC_\eta}\right)^{\frac{1-2s}{2s}}(1-\eta)^2 8.$$

Now we take the limit as $s\to\frac{1}{2}^+$ and then as $\eta\to0^+$ to conclude:

$$\lim_{s\to\frac{1}{2}^+}(2s-1)m_s\ge m_{\frac{1}{2}}.$$

We now prove the converse inequality: let $u(x) = \mathcal{T}_1(x)$. Then

$$F_1^s(u, \mathbb{R}) = \int_1^1 W(x)dx + 8\int_1^{+\infty}\int_1^{+\infty} \frac{1}{(x+y)^{1+2s}}dxdy$$
$$+ \iint_{[-1,1]^2} |x-y|^{1-2s}\, dxdy + \int_{-1}^1 |x-1|^2 \int_1^{+\infty} \frac{1}{|x-y|^{1+2s}}dydx.$$

Since, as $s \to \frac{1}{2}^+$, one has

$$\int_1^1 W(x)dx + \iint_{[-1,1]^2} |x-y|^{1-2s}\, dxdy$$
$$+ \int_{-1}^1 |x-1|^2 \int_1^{+\infty} \frac{1}{|x-y|^{1+2s}}dydx = O(1),$$

the leading term is

$$8\int_1^{+\infty}\int_1^{+\infty} \frac{1}{(x+y)^{1+2s}}dxdy = 8\frac{2^{\frac{1}{2s}}}{2s(2s-1)}.$$

Hence we have

$$m_s \le F_1^s(u, \mathbb{R}) = 8\frac{2^{\frac{1}{2s}}}{2s(2s-1)} + O(1),$$

which implies

$$\lim_{s \to \frac{1}{2}^+} (2s-1)m_s \le m_{\frac{1}{2}}.$$

□

5. Continuity and regular points

Consider the scaling factor, which is continuous in both variables

$$\lambda(\varepsilon, s) = \begin{cases} \frac{1}{|\log \varepsilon|} & s = \frac{1}{2} \\ \frac{1-2s}{\varepsilon^{(2s-1)}-1} & s \in (\frac{1}{2}, 1) \end{cases}$$

and the family of functionals

$$\mathcal{F}_\varepsilon^s(u) = \lambda(\varepsilon, s)\left(\frac{1}{\varepsilon}\int W(u)dt + \varepsilon^{2s-1}[u]_{H^s}^2\right) \quad s \in \left[\frac{1}{2}, 1\right).$$

We can reinterpret our previous results through the lens of regular values introduced in [3]. Namely, for any $s_0 \in \left[\frac{1}{2}, 1\right)$ and any pair of sequences $(s_j, \varepsilon_j) \to (s_0, 0)$, $(s_j', \varepsilon_j') \to (s_0, 0)$, we have

$$\Gamma\text{-}\lim_{j \to +\infty} \mathcal{F}_{\varepsilon_j}^{s_j} = \Gamma\text{-}\lim_{j \to +\infty} \mathcal{F}_{\varepsilon_j'}^{s_j'}.$$

Moreover, we point out that

$$\Gamma\text{-}\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon^s(u) = \begin{cases} m_{\frac{1}{2}} F_0(u) & \text{if } s = \frac{1}{2} \\ (2s-1)m_s F_0(u) & \text{if } s \in \left(\frac{1}{2}, 1\right), \end{cases}$$

where

$$F_0(u) = \begin{cases} \#S(u) & \text{if } u \in BV((0,1), \{-1,1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

and $\#S(u)$ denotes the number of jump points of u in $BV((0,1),\{-1,1\})$. Notably, the regularity of the point $s=\frac{1}{2}$ and the continuity of the Γ-limits with respect to $s\geq\frac{1}{2}$ only occur thanks to the presence of the scaling factor $\lambda(\varepsilon,s)$. This highlights another separation of scales. Specifically, in the regime

$$|2s_\varepsilon-1|>>\frac{1}{|\log\varepsilon|},$$

the Γ-limit of $\lambda(\varepsilon,s)F^s_\varepsilon$ coincides with the one obtained by first taking $\varepsilon\to 0$ with $s>\frac{1}{2}$ fixed, yielding $(2s-1)m_s\#S(u)$, and then letting $s\to\frac{1}{2}$.

In the terminology of Γ-expansions [3] we can state that $\int W(u)dt+\varepsilon^{2s}[u]^2_{H^s}$ is uniformly equivalent to

$$\begin{cases}\varepsilon|\log\varepsilon|m_{\frac{1}{2}}\#S(u) & \text{if } s=\frac{1}{2}\\ \varepsilon\big(1-\varepsilon^{2s-1}\big)m_s\#S(u) & \text{if } s\in\big(\frac{1}{2},1\big)\end{cases}$$

for s varying in compact sets of $[\frac{1}{2},1)$.

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