Nonlinear Schrödinger Equations with vanishing and decaying potentials

ANTONIO AMBROSETTI & Z.-Q. WANG

Abstract. We study the existence and the asymptotic behavior of solutions of $-\epsilon^2 \Delta u + V(x)u = K(x)u^p$, $u \in W^{1,2}(\mathbf{R}^n)$, $u > 0$, when V can vanish and decay to zero at infinity.

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Abstract

We study the existence and the asymptotic behavior of solutions of (1), when V can vanish and decay to zero at infinity.

1 Introduction and main results

This paper deals with Nonlinear Schrödinger Equations (NLS, for short) with potentials like

$$(1) \qquad \begin{cases} -\varepsilon^2 \Delta u + V(x) u = K(x) u^p, & x \in \mathbb{R}^n, \\ u \in W^{1,2}(\mathbb{R}^n), \quad u > 0. \end{cases}$$

We are motivated by the recent works [1, 3, 5, 6]. The two former papers deal with the case that V is positive and have an appropriate decay to zero at infinity. The two latter ones, with the case in which V vanishes at some set $\mathcal{Z}$ (referred in [5] as the case of the critical frequency), but $\liminf_{|x|\to\infty} V(x) > 0$. The main purpose of the present Note is to show that, using some ideas of [1, 3] jointly with some arguments of [5, 6], it is possible to extend the results proved in the aforementioned papers to potentials V that can both vanish and decay to zero at infinity.

First of all, we will show that for ε small there exists a ground state solution of (1) (semiclassical state), so that the fact that V can vanish does not affect the existence resuts of [1, 3]. By a ground state we mean a solution which is a mountain pass critical point of the energy functional associated to (1).

Next, we study the concentration of these ground states as $\varepsilon \to 0$, a phenomenon which is important for its implications in Quantum Mechanics. Loosely speaking, we say that a solution v_ε of (1) concentrates at a point x^* if v_ε tends to zero uniformly out of x^*. Concerning concentration, we show that the ground states

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concentrate on points of zero set of V. Moreover the behavior of the solutions near these points is similar to the case studied in [5], of V being positive away from zero at infinity. It is not affected by the fact that V decays to zero at inifity, but depends only on the local behaviors of V near the points of concentration where V is zero. However, the decay rates of the solutions at infinity do depend on the decay property of V.

We assume that V and K satisfy

(V) $V \in C(\mathbb{R}^n, \mathbb{R})$, and there exist $R_0, k_1, \alpha > 0$ such that

$$V(x) \geq \frac{k_1}{1+|x|^\alpha}, \qquad |x| \geq R_0.$$

(K) $K \in C(\mathbb{R}^n, \mathbb{R})$, and there exist $k_2, \beta > 0$ such that

$$0 < K(x) \leq \frac{k_2}{1+|x|^\beta}, \qquad x \in \mathbb{R}^n.$$

Let

$$\sigma = \begin{cases} \frac{n+2}{n-2} - \frac{4\beta}{\alpha(n-2)}, & \text{if } 0 < \beta < \alpha \\ 1 & \text{otherwise.} \end{cases} \tag{2}$$

and set

$$\mathcal{Z} = \{x \in \mathbb{R}^n : V(x) = 0\}.$$

Let us remark that (V) implies that $\mathcal{Z}$ is bounded. We will be interested in the case that $\mathcal{Z} \neq \emptyset$. Our main existence result is the following one.

Theorem 1 *Suppose that* (V) *and* (K) *hold and let* $0 < \alpha < 2$, $\beta > 0$ *and* $\sigma < p < \frac{n+2}{n-2}$. *Moreover, assume that* $\mathcal{Z} \neq \emptyset$. *Then for* ε *sufficiently small,* (1) *has a ground state* $v_\varepsilon \in W^{1,2}(\mathbb{R}^n)$, *concentrating at some point* $x^* \in \mathcal{Z}$, *as* $\varepsilon \to 0$. *Moreover, there holds*

$$\lim_{\epsilon \longrightarrow 0} ||v_\varepsilon||_\infty = 0, \qquad \textit{and} \qquad \liminf_{\varepsilon \longrightarrow 0} \varepsilon^{\frac{-2}{p-1}} \|v_\varepsilon\|_\infty > 0. \tag{3}$$

Remarks 2 (i) In [1] it is proved that the growth restriction $\sigma < p < \frac{n+2}{n-2}$ is necessary in order to get a ground state.

(ii) If $\mathcal{Z} = \emptyset$, related existence results can be found in [1, 3]. In the fomer, any $\varepsilon > 0$ is allowed and $\sigma < p < \frac{n+2}{n-2}$ is assumed. In the latter, it has been proved that for $\varepsilon \ll 1$ (1) has a solution (possibly not a ground state) for all $1 < p < \frac{n+2}{n-2}$.

(iii) If $\mathcal{Z} \neq \emptyset$ and $\liminf_{|x| \to \infty} V > 0$, a result similar to the preceding theorem is contained in [5, Thm. 2.1].

(iv) The fact that v_ε concentrates at some point $x^* \in \mathcal{Z}$ agrees with the results of [1], where it is proved that concentration arises at a global minimum of the auxiliary potential $\mathcal{A} := V^{\frac{p+1}{p-1} - \frac{n}{2}} K^{-\frac{2}{p-1}}$. Obviously, in the present case each point in $\mathcal{Z}$ is a global minimum of $\mathcal{A}$, because $\mathcal{A}$ vanishes on $\mathcal{Z}$. But here, in contrast with [1], the ground state does not remain bounded away from zero. Actually, the behavior proved in (3) is just the behavior of solutions found in [5]. ■

As in [5, 6], one can be more precise about the asymptotic profile of the concentrating solutions, provided one makes some further assumption on the behavior of V near $\mathcal{Z}$. However, here we do not consider all the cases discussed in [5] but we shall focus on the one (which is referred as finite case in [5]) where V has a polynomial decay to zero near a zero point of V. The other cases studied in [5] could be considered similarly but we will not carry out the details.

Without loss of generality we assume $V(0) = 0$. $P(x)$ is said to be of homogenous degree $m > 0$ if $P(\lambda x) = |\lambda|^m P(x)$.

The following theorem shows that also the asymptotic profile of the ground states is quite similar to the one established in [5]. Actually one can prove:

Theorem 3 *Suppose that* (V) *and* (K) *hold and let* $0 < \alpha < 2$, $\beta > 0$ *and* $\sigma < p < \frac{n+2}{n-2}$. *Let* $\mathcal{Z} = \{0\}$ *and suppose that for some* $m > 0$, $V(x) = P_m(x) + Q(x)$ *satisfies* $\lim_{|x| \to 0} |x|^{-m} Q(x) = 0$, *where* P_m *is homogeneous of degree* $m > 0$. *Let* v_ε *be a solution of 8, localized near* 0, *given in Theorem 1. Then for any* $\varepsilon_n \to 0$ *there is a subsequence (denoted still by* ε_n*) such that* $\varepsilon_n^{-\frac{2}{p-1}\frac{m}{m+2}} v_{\varepsilon_n}(\varepsilon_n^{\frac{2}{m+2}} x)$ *converges uniformly to a ground state solution of*

$$-\Delta w + P_m(x) w = K(0) w^p, x \in \mathbb{R}^N. \tag{4}$$

Remark 4 The behavior of the solution v_ε found above depends on the fact that the concentration point is a zero of V. If there exists a solution concentrating on a critical point of $\mathcal{A}$ with $V > 0$, its behavior would be like a usual spike that one finds in problems where $\inf_{\mathbb{R}^n} V > 0$. This has been proved in [4] dealing with the radial problem

$$-\varepsilon^2 \Delta u + V(|x|) u = u^p, \quad u \in W^{1,2}(\mathbb{R}^n), \quad u > 0, \tag{5}$$

where $p > 1$ and V is radial and satisfies (V). If the weighted potential $M(r) = r^{n-1} V^\ell(r)$, $\ell = \frac{p+1}{p-1} - \frac{1}{2}$, has a minimum or maximum at some $r^* > R_0$, then it is proved that (5) has, for $\varepsilon \ll 1$, a radial solution v_ε which concentrates on the sphere of radius r^*. In such a case, $v_\varepsilon \sim U(\frac{r - r^*}{\varepsilon})$, where U is the positive, radial solution of $-U'' + U = U^p$ such that $U'(0) = 0$. This is related to the fact that $V(r^*) > 0$. ■

The proofs of Theorems 1 and 3 are carried out in Section 2 and 3, respectively.

2 Proof of Theorem 1

The proof of Theorem 1 is divided into several steps.

A. The functional setting. Let

$$\|u\|_\varepsilon = \int_{\mathbb{R}^n} \left[|\nabla u|^2 + V(\varepsilon x) u^2 \right] dx,$$

and let E_ε denote the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to $\|\cdot\|_\varepsilon$. According to the results of [7], E_ε is embedded (respectively, compactly embedded) into the weighted Lebesgue space

$$L_K^{q+1}(\mathbb{R}^n) := \{u \in L^{q+1}(\mathbb{R}^n) : \int_{\mathbb{R}^n} K(\varepsilon x)|u|^{q+1}dx\},$$

provided $0 < \alpha \leq 2$, $\beta > 0$ and $\sigma \leq q \leq \frac{n+2}{n-2}$, resp. $\sigma < q < \frac{n+2}{n-2}$. To be more precise, the results in [7] are proved under the further assumption that $V(x) > 0$ on $\mathbb{R}^n$. These results have been also proved in [1], see in particular Remark 10, and it is easy to check that the arguments carried out in [1] rely only on the behavior of V and K for $|x| \gg 1$, namely on the assumptions (V) and (K).

In particular, one has that

$$\int_{\mathbb{R}^n} K(\varepsilon x)|u|^{p+1}dx < +\infty.$$

If $A \subset \mathbb{R}^n$ we set $A_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon x \in A\}$ and denote by A^δ the δ-neighborhood of A. For simplicity we denote $(A^\delta)_\varepsilon$ as A_ε^δ. Fixed $\delta > 0$ small enough, let us consider the following constrained minimization problem

$$m_\varepsilon = \inf\{\|u\|_\varepsilon^2 : \int_{\mathbb{R}^n} K(\varepsilon x)|u|^{p+1}dx = 1, \quad \int_{\mathbb{R}^n\setminus \mathcal{Z}_\varepsilon^\delta} K(\varepsilon x)|u|^{p+1}dx \leq \varepsilon^{\frac{3(p+1)}{p-1}}\}.$$

Since, as pointed out before, the embedding of E_ε into L_K^{p+1} is compact provided $\sigma < q < \frac{n+2}{n-2}$, it follows that m_ε is achieved at some $u_\varepsilon \in E_\varepsilon$. Hence $m_\varepsilon > 0$ and there exist λ_ε, $\mu_\varepsilon \in \mathbb{R}$ such that

$$-\Delta u_\varepsilon + V(\varepsilon x)u_\varepsilon = \lambda_\varepsilon K(\varepsilon x)u_\varepsilon^p + \mu_\varepsilon \chi_{\mathbb{R}^n\setminus \mathcal{Z}_\varepsilon^{4\delta}} K(\varepsilon x)u_\varepsilon^p, \quad u_\varepsilon > 0. \tag{6}$$

We want to show that $u_\varepsilon \in W^{1,2}(\mathbb{R}^n)$ and there holds

$$\int_{\mathbb{R}^n\setminus \mathcal{Z}_\varepsilon^\delta} K(\varepsilon x)|u|^{p+1}dx < \varepsilon^{\frac{3(p+1)}{p-1}}. \tag{7}$$

If this is the case, then $\widetilde{u}_\varepsilon = m_\varepsilon^{\frac{1}{p-1}} u_\varepsilon$ is a solution of

$$-\Delta \widetilde{u}_\varepsilon + V(\varepsilon x)\widetilde{u}_\varepsilon = K(\varepsilon x)\widetilde{u}_\varepsilon^p, \qquad u_\varepsilon > 0, \tag{8}$$

and $v_\varepsilon(x) := \widetilde{u}_\varepsilon(\varepsilon^{-1}x) = m_\varepsilon^{\frac{1}{p-1}} u_\varepsilon(\varepsilon^{-1}x)$ solves (1).

B. Some estimates. We first show

Lemma 5 *There holds:* $m_\varepsilon = o(1)$ *as* $\varepsilon \to 0$.

Since a can be taken arbitrarily small, the last infimum tends to zero as $\varepsilon \to 0$ and the lemma follows. ■

Next, we turn our attention to (6). By similar arguments to [5] one finds that $\mu_\varepsilon \leq 0 \leq \lambda_\varepsilon$. We claim that there is a constant Λ such that

$$\limsup_{\varepsilon \to 0} \lambda_\varepsilon \leq \Lambda. \tag{9}$$

If not, for $\varepsilon_n \to 0$ one has $\lim_{n\to\infty} \lambda_{\varepsilon_n} = +\infty$. Take a cut-off function φ_n such that

$$\varphi_n(x) \begin{cases} 0 & \text{if } x \notin \mathcal{Z}^\delta_{\varepsilon_n} \\ 1 & \text{if } x \in \mathcal{Z}^{\delta/2}_{\varepsilon_n}, \end{cases}$$

$0 \leq \varphi_n \leq 1$, $|\nabla \varphi_n(x)| \leq 2\varepsilon_n/\delta$. Set $u_n = u_{\varepsilon_n}$ and $\lambda_n = \lambda_{\varepsilon_n}$. Multiplying (6) by $\varphi_n u_n$ and integrating by parts, we get

$$\begin{aligned} \lambda_n \int_{\mathcal{Z}^{\delta/2}_{\varepsilon_n}} K(\varepsilon_n x) u_n^{p+1} dx &\leq \int_{\mathbb{R}^n} \left[|\nabla u_n \cdot \nabla(u_n \varphi_n)| + V(\varepsilon_n x) u_n^2 \varphi_n \right] dx \\ &\leq c_1 \int_{\mathbb{R}^n} \left[|\nabla u_n|^2 + |\nabla \varphi_n|^2 |u_n|^2 + V(\varepsilon_n x) u_n^2 \right] dx. \end{aligned}$$

Since $\inf\{V(x) : x \in \mathcal{Z}^\delta \setminus \mathcal{Z}^{\delta/2}\} > 0$ and $|\nabla \varphi_n(x)| \leq 2\varepsilon_n/\delta$, the above inequality implies, for $n \gg 1$,

$$\lambda_n \int_{\mathcal{Z}^{\delta/2}_{\varepsilon_n}} K(\varepsilon_n x) u_n^{p+1} dx \leq c_2 \int_{\mathbb{R}^n} \left[|\nabla u_n|^2 + V(\varepsilon_n x) u_n^2 \right] dx = c_2 m_{\varepsilon_n}.$$

Since $m_{\varepsilon_n} \to 0$ and $\lambda_n \to \infty$, it follows

$$\int_{\mathcal{Z}^{\delta/2}_{\varepsilon_n}} K(\varepsilon_n x) u_n^{p+1} dx \to 0, \qquad n \to \infty. \tag{10}$$

Choose another cut-off function

$$\psi_n(x) \begin{cases} 0 & \text{if } x \in \mathcal{Z}^{\delta/2}_{\varepsilon_n}, \text{ or } x \notin \mathcal{Z}^{5\delta/4}_{\varepsilon_n} \\ 1 & \text{if } x \in \mathcal{Z}^{\delta}_{\varepsilon_n} \setminus \mathcal{Z}^{3\delta/4}_{\varepsilon_n}, \end{cases}$$

such that $0 \leq \psi_n \leq 1$, $|\nabla \psi_n(x)| \leq 4\varepsilon_n/\delta$. Taking $n \gg 1$ such that $\lambda_n \geq 1$, and using arguments similar to the previous ones, we get

$$\begin{aligned} \int_{\mathbb{R}^n} K(\varepsilon_n x) u_n^{p+1} \psi_n dx &\leq \lambda_n \int_{\mathbb{R}^n} K(\varepsilon_n x) u_n^{p+1} \psi_n dx \\ &= \int_{\mathbb{R}^n} \left[\nabla u_n \cdot \nabla(u_n \psi_n) + |u_n \psi_n|^2 \right] dx \\ &\leq c_1 \int_{\mathbb{R}^n} \left[|\nabla u_n|^2 |\psi_n|^2 + |u_n|^2 |\nabla \psi_n|^2 + |u_n \psi_n|^2 \right] dx \\ &\leq c_2 \int_{\mathbb{R}^n} \left[|\nabla u_n|^2 + V(\varepsilon_n x) u_n^2 \right] dx \to 0, \end{aligned}$$

where we have used again that $\inf\{V(x) : x \in \mathcal{Z}^{5\delta/4} \setminus \mathcal{Z}^{\delta/2}\} > 0$. But,

$$\int_{\mathbb{R}^n} K(\varepsilon_n x)\psi_n u_n^{p+1} dx \geq \int_{\mathcal{Z}^{\delta}_{\varepsilon_n} \setminus \mathcal{Z}^{3\delta/4}_{\varepsilon_n}} K(\varepsilon_n x)|u_n|^{p+1} dx.$$

Using (16) it follows that

$$\int_{\mathcal{Z}^{\delta}_{\varepsilon_n} \setminus \mathcal{Z}^{3\delta/4}_{\varepsilon_n}} K(\varepsilon_n x)|u_n|^{p+1} dx \to 1, \quad n \to \infty.$$

Hence,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^n} K(\varepsilon_n x)|\psi_n u_n|^{p+1} dx \geq \text{Const.} > 0,$$

a contradiction. This proves that (9) holds.

C. Exponential decay.

Lemma 6 *There exist $R_1 > 0$, $C = C(p, n) > 0$ and $d = d(p, \alpha, \beta, n) > 0$, such that for $|x| \geq \frac{2R_1+C}{\varepsilon}$ there holds*

$$|u_\varepsilon(x)| \leq C|x|^d \exp\left\{ -\tfrac{1}{4}\left|\log \tfrac{3}{4}\right| \frac{\left(|x|^{\frac{2-\alpha}{2}} - (\frac{R_1}{\varepsilon})^{\frac{2-\alpha}{2}}\right)}{\varepsilon^{\alpha/2}} \right\}. \tag{11}$$

This is essentially Lemma 22 of [1]. For the reader convenience, let us outline below the proof, referring to [1] for more details. The main two steps in the proof of [1] are the following (i) and (ii) below.

(i) For all $\delta_1 > 0$ there exists $\overline{R} > 0$ such that, for all $R \geq \overline{R}$ and all $u \in E_\varepsilon$

$$\begin{aligned} &\int_{|x|>\frac{R}{\varepsilon}} K(\varepsilon x)|u|^{p+1}(x) dx \\ &\qquad \leq \delta_1 \left(\int_{|x|>\frac{R}{\varepsilon}} \left[|\nabla u(x)|^2 + V(\varepsilon x)u^2(x)\right] dx \right)^{(p+1)/2}. \end{aligned} \tag{12}$$

(ii) let u_ε satisfy (6) and set $\Omega_n = \{|x| > n^{2/(2-\alpha)}\}$. Then for $n \gg 1$,

$$\int_{\Omega_{n+1}} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x)u_\varepsilon^2\right] dx \leq \tfrac{3}{4} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x)u_\varepsilon^2\right] dx, \tag{13}$$

As for (i), this is nothing but the counterpart of [1, Prop. 11] in our setting and it is easy to check that in the proof carried out in [1] can be repeated here. Actually the assumption that $V \geq c > 0$ does not play any role, because in (12) all the integrals are evaluated for $|x| \gg 1$, only.

To prove (13) we modify the arguments used in [1, Lemma 17] as follows. Let $\phi_n(r)$ be a piecewise affine functions such that

$$\phi_n(r) \equiv 0, \quad \forall\, r \leq n^{2/(2-\alpha)}, \quad \phi_n(r) \equiv 1, \quad \forall\, r \geq (n+1)^{2/(2-\alpha)}.$$

Using (6), that $\mu_\varepsilon \leq 0$ and that $\phi_n \leq 1$ we find

$$\int_{\Omega_n} \left[\nabla u_\varepsilon \cdot \nabla(u_\varepsilon \phi_n) + V(\varepsilon x) u_\varepsilon^2 \phi_n\right] dx \leq \lambda_n \int_{\Omega_n} K(\varepsilon_n x) u_\varepsilon^{p+1} dx.$$

By calculation similar to that in [1] and using (9), we get

$$\begin{aligned}\int_{\Omega_{n+1}} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx &\leq \int_{\Omega_n} \phi_n \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \\ &\leq \lambda_n \int_{\Omega_n} K(\varepsilon x) u_\varepsilon^{p+1} dx - \int_{\Omega_n} (\nabla u_\varepsilon \cdot \nabla \phi_n) dx \\ &\leq \Lambda \int_{\Omega_n} K(\varepsilon x) u_\varepsilon^{p+1} dx + \tfrac{1}{2} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + |\nabla \phi_n|^2 u_\varepsilon^2\right] dx.\end{aligned}$$

Since $|\nabla \phi_n| \sim n^{-\alpha/(2-\alpha)}$, we infer that $|\nabla \phi_n|^2 \leq V(\varepsilon x)$ in Ω_ε for $n \gg 1$, and hence

$$\begin{aligned}&\int_{\Omega_{n+1}} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \\ &\qquad \leq \Lambda \int_{\Omega_n} K(\varepsilon x) u_\varepsilon^{p+1} dx + \tfrac{1}{2} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx.\end{aligned}$$

This and (12) yield, for $\varepsilon \ll 1$,

$$\begin{aligned}&\int_{\Omega_{n+1}} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \\ &\leq \delta_1 \Lambda \left(\int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx\right)^{\frac{p+1}{2}} + \tfrac{1}{2} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx.\end{aligned}$$

One also has

$$\begin{aligned}&\left(\int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx\right)^{\frac{p+1}{2}} \\ &\quad = \left(\int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx\right)^{\frac{p-1}{2}} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \\ &\quad \leq m_\varepsilon^{\frac{p-1}{2}} \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx.\end{aligned}$$

Then we find

$$\int_{\Omega_{n+1}} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx \leq \left(\delta_1 \Lambda m_\varepsilon^{\frac{p-1}{2}} + \tfrac{1}{2}\right) \int_{\Omega_n} \left[|\nabla u_\varepsilon|^2 + V(\varepsilon x) u_\varepsilon^2\right] dx,$$

proving (13).

Once that $(i) - (ii)$ hold, one can repeat the arguments in [1] proving Lemma 6.

D. Proof of Theorem 1 completed. To complete the proof of Theorem 1 it remains to show that $u_\varepsilon \in W^{1,2}(\mathbb{R}^n)$ and that (7) holds.

Fixed R_1 and C as in the Lemma 6, let $\rho \geq 2R_1 + C$. Then (11) implies

$$\int_{\mathbb{R}^n \setminus B_{\rho/\varepsilon}} K(\varepsilon x) u_\varepsilon^{p+1} dx \sim c \exp\left(-\frac{c}{\varepsilon^{\alpha/2}}\right), \quad c > 0. \tag{14}$$

Letting $\Omega_\varepsilon := B_{2\rho/\varepsilon} \setminus \mathcal{Z}_\varepsilon^{2\delta}$, there results $\inf_{\Omega_\varepsilon} K(\varepsilon x) \geq$ const. > 0. This and the fact that

$$\int_{\Omega_\varepsilon} K(\varepsilon x) u_\varepsilon^{p+1} dx \leq \int_{\mathbb{R}^n \setminus \mathcal{Z}_\varepsilon^\delta} K(\varepsilon x) u_\varepsilon^{p+1} dx \leq \varepsilon^{\frac{3(p+1)}{p-1}},$$

imply that $\int_{\Omega_\varepsilon} u_\varepsilon^{p+1} dx \to 0$ as $\varepsilon \to 0$. Then by elliptic estimates, $\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \to 0$. Taking δ possibly smaller, we can assume that $\mathcal{Z}^{2\delta} \subset B_{2\rho}$ and hence there is $\gamma > 0$ such that $\inf_{\Omega_\varepsilon} V(\varepsilon x) \geq 2\gamma$. For ε small, we have

$$\lambda_\varepsilon \left(\sup_{\Omega_\varepsilon} [K(\varepsilon x) u_\varepsilon(x)]\right)^{p-1} \leq \gamma, \quad \sup_{\Omega_\varepsilon} u_\varepsilon(x) \leq 1.$$

Then there holds

$$\begin{cases} -\Delta u_\varepsilon + [V(\varepsilon x) - \gamma] u_\varepsilon \leq 0, & \text{in } \Omega_\varepsilon, \\ u_\varepsilon(x) \leq 1 & \text{on } \partial \mathcal{Z}_\varepsilon^{2\delta}, \\ u_\varepsilon(x) \leq c \exp(-c\, \varepsilon^{-\alpha/2}) & \text{in } \partial B_{2\rho/\varepsilon}. \end{cases}$$

Let Ψ_ε denote the solution of

$$\begin{cases} -\Delta \Psi_\varepsilon + \gamma \Psi_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ \Psi_\varepsilon(x) = 1 & \text{on } \partial \mathcal{Z}_\varepsilon^{2\delta}, \\ \Psi_\varepsilon(x) = c \exp(-c\, \varepsilon^{-\alpha/2}) & \text{in } \partial B_{2\rho/\varepsilon}. \end{cases}$$

Then by the comparison principle,

$$u_\varepsilon(x) \leq \Psi_\varepsilon(x), \qquad \forall x \in \Omega_\varepsilon.$$

Since Ψ_ε decays exponentially to zero at infinity, there exists $C > 0$ such that

$$u_\varepsilon(x) \leq C \exp\left(-\frac{C}{\varepsilon}\right), \qquad \forall x \in B_{\rho/\varepsilon} \setminus \mathcal{Z}_\varepsilon^{4\delta}.$$

Hence $u_\varepsilon \in W^{1,2}(\mathbb{R}^n)$. Moreover, the preceding inequality and (14) yield, for $\varepsilon \ll 1$,

$$\int_{\mathbb{R}^n \setminus \mathcal{Z}_\varepsilon^\delta} K(\varepsilon x) u_\varepsilon^{p+1} dx \leq C_1 \exp\left(-\frac{C_1}{\varepsilon}\right) < \varepsilon^{\frac{3(p+1)}{p-1}},$$

proving (7).

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proving (7).

Now define $v_\varepsilon(x) = m_\varepsilon^{\frac{1}{p-1}} u_\varepsilon(\varepsilon^{-1} x)$. Then v_ε solves equation (1). By Lemma 5 we have $||v_\varepsilon||_\infty \to 0$ as $\varepsilon \to 0$.

Finally we show $\liminf_{\varepsilon \to 0} \varepsilon^{\frac{-2}{p-1}} ||v_\varepsilon||_\infty > 0$. Set $\widehat{v}_\varepsilon = \varepsilon^{\frac{-2}{p-1}} v_\varepsilon$, and note that $\widehat{v}_\varepsilon$ satisfies

$$-\Delta \widehat{v}_\varepsilon + \frac{1}{\varepsilon^2} V(x) \widehat{v}_\varepsilon = K(x) \widehat{v}_\varepsilon^p. \tag{15}$$

Choosing a cut-off function $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\phi(x) = 1$ for $x \in B_R(0)$ such that $R > \max\{R_0, 2R_1 + C\}$ and $\mathcal{Z}^{4\delta} \subset B_R(0)$, where R_0 is from (V) and $2R_1 + C$ is from Lemma 6 so that (11) holds. We have

$$\int_{\mathbb{R}^n} K(x) \widehat{v}_\varepsilon^{p+1} \le k_2 \int_{\mathbb{R}^n} \widehat{v}_\varepsilon^{p-1} (\phi^2 \widehat{v}_\varepsilon^2 + (1-\phi)^2 \widehat{v}_\varepsilon^2). \tag{16}$$

There holds, for some $C > 0$,

$$\int_{\mathbb{R}^n} \widehat{v}_\varepsilon^{p-1} \phi^2 \widehat{v}_\varepsilon^2 \le C ||\widehat{v}_\varepsilon||_\infty^{p-1} \int_{\mathbb{R}^n} |\nabla(\phi \widehat{v}_\varepsilon)|^2 \le C \int_{\mathbb{R}^n} |\nabla \widehat{v}_\varepsilon|^2 + \frac{1}{\varepsilon^2} V \widehat{v}_\varepsilon^2. \tag{17}$$

Furthermore, choosing $\delta > 0$ such that $p - 1 - \delta > 0$ we use (V) to infer that

$$\begin{aligned} \int_{\mathbb{R}^n} \widehat{v}_\varepsilon^{p-1} (1-\phi)^2 \widehat{v}_\varepsilon^2 &\le \|\widehat{v}_\varepsilon\|_\infty^{p-1-\delta} \int_{\mathbb{R}^n} \widehat{v}_\varepsilon^\delta (1-\phi)^2 \widehat{v}_\varepsilon^2 \\ &\le k_1^{-1} \|\widehat{v}_\varepsilon\|_\infty^{p-1-\delta} \int_{\mathbb{R}^n} (1 + |x|^\alpha) \widehat{v}_\varepsilon^\delta V(x) \widehat{v}_\varepsilon^2. \end{aligned}$$

Using Lemma 6, we find that $(1 + |x|^\alpha) \widehat{v}_\varepsilon^\delta \le 1$, provided $|x| > R$ and ε is sufficiently small. From this and the previous equation we get

$$\int_{\mathbb{R}^n} \widehat{v}_\varepsilon^{p-1} (1-\phi)^2 \widehat{v}_\varepsilon^2 \le k_1^{-1} \|\widehat{v}_\varepsilon\|_\infty^{p-1-\delta} \int_{\mathbb{R}^n} V(x) \widehat{v}_\varepsilon^2. \tag{18}$$

Using (16 - 17 - 18) and the equation (15) we infer that $\|\widehat{v}_\varepsilon\|_\infty \ge C > 0$ and thus $\liminf_{\varepsilon \to 0} \varepsilon^{-\frac{2}{p-1}} ||v_\varepsilon||_\infty > 0$. This completes the proof of Theorem 1

Remark 7 We may also consider localized solutions concentrating near an isolated subset A of the set of zeros of V, $\mathcal{Z}$. I.e., we require $d(A, \mathcal{Z} \setminus A) > 0$. This was done in [6] for the case $\liminf_{|x| \to \infty} V$ is positive. Slightly refined arguments above give results of this type. We refer [6] for details. ■

3 Proof of Theorem 3

In this section we will give the proof of Theorem 3. By [6], equation has a least energy solution v_ε which has an exponential decay at infinity.

By a scaling depending on m, we define

$$w_\varepsilon(x) = \varepsilon^{-\frac{2}{p-1}\frac{m}{m+2}} v_\varepsilon(\varepsilon^{\frac{2}{m+2}} x)$$

Then w_ε satisfies

$$-\Delta w_\varepsilon(x) + (P_m(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x)) w_\varepsilon = K(\varepsilon^{\frac{2}{m+2}} x) w_\varepsilon^p.$$

By Lemma 6 and the construction of the solution we have that for each $\delta_1 > 0$ there exist $C, c > 0$ such that for $|\varepsilon^{\frac{2}{m+2}} x| \geq \delta_1$,

$$w_\varepsilon(x) \leq C\varepsilon^{-\frac{2}{p-1}\frac{m}{m+2}} \exp(-c\varepsilon^{-\frac{m}{m+2}}|x|). \tag{19}$$

By using (V), we first claim that for $|x| \geq \varepsilon^{-\frac{2}{m+2}} R_0$,

$$\varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{-\frac{2m}{m+2}}) \geq \frac{k_1}{|x|^\alpha}. \tag{20}$$

By the property of Q there exists $\delta_2 > 0$ such that for $|\varepsilon^{\frac{2}{m+2}} x| \leq \delta_2$, $P_m(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x) \geq \frac{1}{2} P_m(x)$. Thus there is $R_2 > 0$ such that for $|x| \geq R_2$

$$\varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{-\frac{2m}{m+2}}) \geq \frac{k_1}{|x|^\alpha}. \tag{21}$$

Since a ground state solution w to (4) is exponentially decay at infinity, we have

$$\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^n} |\nabla w_\varepsilon|^2 + (P_m(x) + \varepsilon^{-\frac{2m}{m+2}} Q(\varepsilon^{\frac{2}{m+2}} x)) w_\varepsilon^2 \leq \int_{\mathbb{R}^n} |\nabla w|^2 + P_m(x) w^2.$$

From these and the elliptic estimates we get that the L^∞-norm of w_ε is uniformly bounded for ε small. By the fact $P_m(x) \to \infty$ as $|x| \to \infty$ and elliptic estimate again, w_ε tends to zero as $|x| \to \infty$ uniformly for ε small. Then by this and (19) we have $R_3 \geq R_2$ such that for $|x| \geq R_3$,

$$K(\varepsilon^{\frac{2}{m+2}} x) w_\varepsilon^{p-1}(x) \leq \frac{1}{2} \frac{k_1}{|x|^\alpha}.$$

Thus for $|x| \geq R_3$, $\Delta w_\varepsilon - \frac{1}{2}\frac{k_1}{|x|^\alpha} w_\varepsilon \geq 0$. By Lemma 6 of [3] and comparison principle, we get for some constant $C, c > 0$

$$w_\varepsilon(x) \leq C \exp(-c|x|^{\frac{2-\alpha}{2}})$$

Next we show $\liminf_{\varepsilon \to 0} ||w_\varepsilon||_\infty > 0$. If not, using the above estimate and similar to the end of the last section, we have

$$\int_{\mathbb{R}^n} |\nabla w_\varepsilon|^2 + \varepsilon^{-\frac{2m}{m+2}} V(\varepsilon^{\frac{2}{m+2}} x) w_\varepsilon^2 \leq C \exp(-c\varepsilon^{\frac{(2-\alpha)(p+1)}{m+2}})$$

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