Periodic solutions of forced Kirchhoff equations

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Abstract

We consider Kirchhoff equations for vibrating strings and elastic membranes under the action of an external forcing of period $2\pi/\omega$ and small amplitude ε. We prove existence, regularity and local uniqueness of $2\pi/\omega$-periodic solutions of order ε by means of a Nash-Moser iteration scheme; the results hold for parameters (ω, ε) in a Cantor-like set which has asymptotically full measure for $\varepsilon \to 0$.

1 Introduction

In this paper we deal with Kirchhoff equations for strings and membranes under the action of periodic, small amplitude forcing terms.

First, we consider the problem

$$(1) \quad u_{tt} - \Delta u \Big(1 + \int_\Omega |\nabla u|^2 \, dx\Big) = \varepsilon g(x,t), \qquad u(x,t) = 0 \ \ \forall x \in \partial\Omega$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain, $d \geq 1$, Δ, ∇ refer to the x variable, ε is a positive parameter, g is $2\pi/\omega$-periodic in time, $g = 0$ on $\partial\Omega$, and $u : \Omega \times \mathbb{R} \to \mathbb{R}$ is the unknown.

In dimension $d = 2$, (1) models an elastic membrane which is fixed at the boundary $\partial\Omega$, under the external forcing εg; $u(x,t)$ represents the displacement of the point $x \in \Omega$ with respect to its rest position.

In dimension $d = 1$, $\Omega = (0, \pi)$, (1) is the equation of a string with normalized length π and fixed endpoints. The nonlinear nonlocal term $u_{xx} \int_0^\pi u_x^2 dx$, proposed by Kirchhoff [17], is a purely geometric correction which keep into account the increase of the string length caused by the displacement u and, as a consequence, the increase of the string tension. Independently, Carrier [9] and Narasimha [21] rediscovered the same equation as the final result of approximations of the exact model for the stretched string.

We consider also the equation for a string in the 3-dimensional space

$$(2) \qquad u_{tt} - u_{xx}\Big(1 + \int_0^{\pi} |u_x|^2 \, dx\Big) = \varepsilon g(x,t), \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where the forcing g and the displacement $u \in \mathbb{R}^2$ are vectors belonging to the plane orthogonal to the rest position of the string (see [9, 21]), so that nonplanar vibrations of the string are permitted.

Many authors studied Kirchhoff equations and their generalizations like

$$u_{tt} - m\Big(\int_{\Omega} |\nabla u|^2 \, dx\Big) \Delta u + \alpha u_t = g(x,t,u)$$

both with a dumping $\alpha \neq 0$ and in the conservative case $\alpha = 0$, with m Lipschitz positive, from the point of view of the Cauchy problem, about questions like global and blow-up solutions, stability and regularity, first in [5], then [13, 22, 18, 2, 12, 3] and many others, up to now [19]; see [1] for a survey.

On the other hand, to the best of our knowledge, almost nothing is known about the existence of periodic solutions for equations (1),(2). Kirchhoff himself observed that the free string equation

$$u_{tt} - u_{xx}\Big(1 + \int_0^{\pi} u_x^2 \, dx\Big) = 0$$

possesses a sequence of normal modes, that is solutions of the form $u(x,t) = u_j(t)\sin jx$, $j = 1, 2, \dots$ where $u_j(t)$ is periodic and solves the Duffing ODE

$$u_j'' + j^2 u_j \Big(1 + \frac{\pi}{2} j^2 u_j^2\Big) = 0 \, ;$$

u_j can be written explicitly by means of the Jacobi elliptic sine. Analogously, the free membrane admits normal modes $u(x,t) = u_j(t)\varphi_j(x)$ where $\varphi_j(x)$ is the j-th eigenfunction of the Laplacian in the domain Ω, and the 3D free string equation has normal modes $u(x,t) = u_j(t)\sin jx$ with $u_j = (u_{j,1}(t), u_{j,2}(t)) \in \mathbb{R}^2$; the difference in the temporal phase between the two component $u_{j,1}$ and $u_{j,2}$ can determine both planar oscillations and whirling motions of the string, corresponding to

$$u_{j,1}(t) = \cos\big(j\sqrt{1 + (\pi/2)\,j^2}\ t\big), \quad u_{j,2}(t) = \sin\big(j\sqrt{1 + (\pi/2)\,j^2}\ t\big).$$

We indicate [14, 16] about questions of stability for the normal modes of (1). Moreover, we report that recently an equation similar to (2) has been studied experimentally in [20] for the particular case when the forcing $g(x,t) = g(t)\sin j_0 x$ excites only the j_0-th mode.

In this paper we prove existence, regularity and local uniqueness of $2\pi/\omega$-periodic solutions of (1),(2) of order ε, when the frequency ω and the amplitude ε of g belong to a "nonresonant" Cantor set of positive measure,

asymptotically full for $\varepsilon \to 0$. As a byproducts, the results apply for the autonomous case $g = g(x)$ and the free case $g = 0$, see Theorem 1 and the following remarks in the next section. Our proof is based on the modern version of the KAM theory for infinite-dimensional dynamical systems, which was developed in the Nineties mainly by Bourgain, Craig, Kuksin, Pöschel, Wayne (see [10]). In particular, our Nash-Moser iteration scheme is similar to that in [6, 4].

With respect to wave equations of the type $u_{tt} - u_{xx} = f(x, t, u)$ like in [4], in Kirchhoff equations there is an interesting difference: the presence of derivatives in the nonlinearity. In general, little is known about periodic solutions of equations of the form

$$u_{tt} - u_{xx} = f(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}).$$

This problem was studied by Rabinowitz [24] in the dissipative case $\alpha u_t \neq 0$, by Craig [10] in the pseudoderivatives case

$$u_{tt} - u_{xx} = a(x)u + b(x, |\partial_x|^\beta u) = 0, \quad \beta < 1,$$

by Bourgain [8] in cases like $u_{tt} - u_{xx} + \rho u + u_t^2 = 0$ and, for quasiperiodic solutions, [7] $u_{tt} - u_{xx} = a(x)u + \varepsilon \partial_x^{1/2}(h(x, u))$. We remark that, in general, the presence of derivatives in the nonlinearity makes uncertain the existence of global (not only periodic) solutions; see for example the non-existence result of Lax, Klainerman and Majda in [10, Thm. 2.2] for the equation $u_{tt} - a(u_x)u_{xx} = 0$ with $a > 0$, $a(v) = O(v^p)$ near 0, $p \geq 1$.

In the present integro-differential case, luckily, the structure of the nonlinearity is very special. The integral $\int_\Omega |\nabla u|^2 \, dx$ and the second-order derivative Δu induce to develop solutions in space Fourier series; the differential of the Kirchhoff nonlinearity

$$h \mapsto \Delta h \int_\Omega |\nabla u|^2 \, dx + \Delta u \int_\Omega 2\nabla u \circ \nabla h \, dx$$

in the basis $\{\varphi_j(x)\}_{j=1,2,\dots}$ is the sum of a diagonal part and a "projection" part (see section 6), so that the proof of the inversion of the linearized operator — the crucial part of the Nash-Moser method — is exceptionally simple.

2 Functional setting and main results

Let $2\pi/\omega$ be the period of g; we look for solutions u with the same period. Normalizing the time $t \to \omega t$ and rescaling $u \to \varepsilon^{1/3} u$, (1) becomes

(3)
$$\omega^2 u_{tt} - \Delta u = \mu\Big(\Delta u \int_\Omega |\nabla u|^2 \, dx + g(x, t)\Big)$$

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where $\mu := \varepsilon^{2/3}$ and g, u are 2π-periodic. Expanding in Fourier series in x, we set the problem in the Hilbert spaces X_{σ,s,H^r} of the functions $u : \Omega \times \mathbb{R} \to \mathbb{R}$ which are 2π-periodic in time and have regularity (σ, s)-analytic in space and Sobolev $H^r(\mathbb{T})$ in time, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$,

$$X_{\sigma,s} = \left\{ u(x,t) = \sum_{j=1}^{+\infty} u_j(t)\varphi_j(x) : \|u\|_{\sigma,s}^2 := \sum_{j=1}^{+\infty} \|u_j\|_{H^r}^2 j^{2s} e^{2\sigma j} < \infty \right\}$$

where $\sigma \geq 0$, $s \geq 0$, and $\{\varphi_j(x)\}_{j=1,2,\ldots}$ are the eigenfunctions of $-\Delta$ on Ω,

$$-\Delta\varphi_j = \lambda_j\varphi_j, \quad \varphi_j|_{\partial\Omega} = 0$$

normalized by $\int_\Omega \varphi_j^2 \, dx = 1$. Setting $q := 1/d$, the Weyl formula for the asymptotic behaviour of the eigenvalues gives $\lambda_j = O(j^{2q})$ as $j \to \infty$. We define $p_j := \sqrt{\lambda_j}$; there holds $Cj^q \leq p_j \leq C'j^q$ for all $j = 1, 2, \ldots$, for some C, C' depending on the domain Ω.

We write X_{σ,s,H^r}, $\| \ \|_{\sigma,s,H^r}$ if the time-regularity r has to be emphasized. Where subscripts s, H^r do not appear, we mean $s = 1$, $H^r = H^1$.

Remark. In general, $X_{\sigma,s}$ is not an algebra. Indeed, for $\sigma > 0$, $\Omega = (0,\pi)$, $\sin x \in X_{\sigma,s}$ but $\sin^2 x \notin X_{\sigma,s}$ because its coefficients are of order $O(1/j)$,

$$\sin^2 x = \sum_{j \ odd} \frac{2j}{j^2 - 4} \sin jx \notin X_{\sigma,s} \, .$$

Similarly, products of an even number of sine factors have, in general, polynomial decay for the coefficients, while the norms $\| \ \|_{\sigma,s}$ require an exponential decay.

On the contrary, in the case of spatial periodic boundary conditions $u(x,t) = u(x + 2\pi, t)$, $\Omega = \mathbb{R}$, one can develop in the exponential basis (e^{ijx}), and the spaces

$$Y_{\sigma,s} = \left\{ u(x,t) = \sum_{j \in \mathbb{Z}} u_j(t) e^{ijx} : \|u\|_{\sigma,s}^2 := \sum_{j \in \mathbb{Z}} \|u_j\|_{H^r}^2 j^{2s} e^{2\sigma j} < \infty \right\}$$

are algebras for $\sigma \geq 0$, $s > 1/2$. This permits the study of general nonlinearities which are written in power series in u, that is, $g(x,t,u) = \sum_{k \geq 0} \sum_{j \in \mathbb{Z}} g_{kj}(t) e^{ijx} u^k$. For g regular enough, $g(u) \in Y_{\sigma,s}$ for all $u \in Y_{\sigma,s}$, see [4].

Conventions. $\mathbb{N} = \{0, 1, \ldots\}$. We say that a constant is "universal" if it does not depend on any parameter. When we write "some constant C", we mean a constant depending on the problem data d, Ω, on the 2π-periodic function g (after the time rescaling g does not depend on ω) and on the fixed parameter $\tau > 1$; the dependence on other quantities is indicated explicitly. If A is a set, we denote by $|A|$ its Lebesgue measure.

Theorem 1. *Suppose that $g \in X_{\sigma_0}$ for some $\sigma_0 > 0$. There are positive constants γ_0, δ, C with the following properties. For every $\gamma \in (0, \gamma_0)$ there exists a Cantor set $\mathcal{A}_\gamma \subset (0, +\infty) \times (0, \delta\gamma)$ of parameters such that for every $(\omega, \mu) \in \mathcal{A}_\gamma$ there exists a classical solution $u(\omega, \mu) \in X_{\sigma_0/2}$ of (3). Such a solution satisfies*

$$\|u(\omega, \mu)\|_{\sigma_0/2} \leq \frac{\mu}{\gamma} C,$$

it is unique in the ball $\{\|u\|_{\sigma_0/2} < 1\}$ and if $g \in X_{\sigma_0, H^r}$ then $u(\omega, \mu) \in X_{\sigma_0/2, H^{r+2}}$.

   The Cantor set satisfies the following measure property: for every $0 < \bar{\omega}_1 < \bar{\omega}_2 < \infty$ there exists a constant $\bar{C}$ independent on γ such that in the rectangular region $\mathcal{R}_\gamma := (\bar{\omega}_1, \bar{\omega}_2) \times (0, \delta\gamma)$ there holds

$$\frac{|\mathcal{R}_\gamma \cap \mathcal{A}_\gamma|}{|\mathcal{R}_\gamma|} > 1 - \bar{C}\gamma.$$

Remarks. (i) Coming back to equation (1), the solution $u(\omega, \mu)$ of (3) found in Theorem 1 corresponds to a $2\pi/\omega$-periodic solution $u(\omega, \varepsilon)$ of (1) of order ε, unique among the $2\pi/\omega$-periodic functions of order $\varepsilon^{1/3}$.

   (ii) In the autonomous case when $g = g(x)$ does not depend on time, we can consider ω in (3) as a "free" parameter, so that, for a given static forcing term $\varepsilon g(x)$ of fixed amplitude ε, the theorem implies the existence of many periodic solutions of order ε, namely for every ω such that $(\omega, \varepsilon^{2/3}) \in \mathcal{A}_\gamma$ there is a solution of period $2\pi/\omega$. The set $\mathcal{A}_\gamma(\varepsilon) := \{\omega : (\omega, \mu) \in \mathcal{A}_\gamma\}$ is a large set because it satisfies the 1D version of the estimate for $\mathcal{A}_\gamma$, that is, $|(\bar{\omega}_1, \bar{\omega}_2) \cap \mathcal{A}_\gamma(\varepsilon)|/(\bar{\omega}_2 - \bar{\omega}_1) > 1 - \bar{C}\gamma$ (see Lemma 8 and its proof).

   (iii) In the free case when $g = 0$ we can see both ω and ε as free parameters. We know that the free equation $u_{tt} - \Delta u(1 + \int_\Omega |\nabla u|^2 dx) = 0$ possesses the trivial solution $u = 0$; the theorem implies that, if $(\omega, \varepsilon^{2/3}) \in \mathcal{A}_\gamma$, the only $2\pi/\omega$-periodic solution of order $\varepsilon^{1/3}$ is the trivial one.

   (iv) For problem (2) we need only to set $\|u_j\|_{H^r}^2 := \|u_{j,1}\|_{H^r}^2 + \|u_{j,2}\|_{H^r}^2$ in the definition of the spaces $X_{\sigma, s}$; all the computations in the paper hold, so that Theorem 1 holds also for the 3-dimensional string equation.

Related problems. (i) Can we construct solutions when the forcing is $O(1)$, namely $\varepsilon = 1$, and $\omega \to +\infty$?

   (ii) The solutions found in Theorem 1 have small amplitude, namely they are perturbations of the trivial solution. Can we find also large solutions which are perturbations of the normal modes of the free Kirchhoff equation? Since the j-th normal mode can vibrate with every frequency $\omega > p_j$ provided it has the corresponding amplitude, what frequencies can survive when a forcing appears?

   (iii) Can we consider forcing terms $g(x, t, u)$ depending also on the unknown u, at least in the case of periodic boundary conditions?

(iv) What is the link between periodic solutions results like Theorem 1 and Cauchy problems?

The rest of the paper is devoted to the proof of the theorem.

3   The iteration scheme

The setting for our iterative scheme is a chain of finite dimensional subspaces of $X_{\sigma,s}$. We fix a positive integer N_0, we denote $N_n := 2^n N_0$ and split the spaces $X_{\sigma,s}$ defining

$$X^{(n)} = \Big\{ u(x,t) = \sum_{j=1}^{N_n} u_j(t)\varphi_j(x) \Big\}, \quad X^{(n)\perp} = \Big\{ u(x,t) = \sum_{j>N_n} u_j(t)\varphi_j(x) \Big\}.$$

We indicate P_n, $P_n^\perp$ the projections onto $X^{(n)}, X^{(n)\perp}$ respectively. We will write $X_{H^r}^{(n)}$, $X_{H^r}^{(n)\perp}$ when the regularity of the coefficients $u_j(t)$ has to be indicated.

We define $L_\omega := \omega^2 \partial_{tt} - \Delta$ and $f(u) := \Delta u \int_\Omega |\nabla u|^2\, dx$, so that (3) can be written as

(4) $$L_\omega u = \mu(f(u) + g).$$

Note that f is not a Nemitski operator, because of its nonlocal nature.

Since the integral $\int_\Omega |\nabla u|^2 dx$ depends only on time, we have that $f(u) \in X^{(n)}$ for all $u \in X^{(n)}$. The map f is cubic, in the sense that $f(u) = F[u,u,u]$ where $F : X^{(n)} \to X^{(n)}$ is the three-linear map $F[u,v,w] = \Delta u \int_\Omega \nabla v \circ \nabla w\, dx$. Therefore f is analytic on $X^{(n)}$ w.r.t. any norm $\| \ \|_{\sigma,s}$. The quadratic remainder at u is

$$\begin{aligned} Q(u,h) &= f(u+h) - f(u) - f'(u)[h] \\ &= \Delta u \int_\Omega |\nabla h|^2 dx + \Delta h \int_\Omega \left(2\nabla u \circ \nabla h + |\nabla h|^2\right) dx. \end{aligned}$$

We observe that, if $a(t)$ does not depend on x, then $\|a(t)u(x,t)\|_{\sigma,s} \leq \|a\|_{H^r}\|u\|_{\sigma,s}$. By Hölder inequality it follows that for $u,h,z \in X^{(n)}$

(5) $$\|f(u)\|_{\sigma,s} \leq N_n^{4q} \|u\|_{\sigma,s}^3$$
(6) $$\|f'(u)[h]\|_{\sigma,s} \leq N_n^{4q}\, 3\|u\|_{\sigma,s}^2 \|h\|_{\sigma,s}$$
(7) $$\|f''(u)[h,z]\|_{\sigma,s} \leq N_n^{4q}\, 6\|u\|_{\sigma,s}\|h\|_{\sigma,s}\|z\|_{\sigma,s}$$
(8) $$\|Q(u,h)\|_{\sigma,s} \leq N_n^{4q}\left(3\|u\|_{\sigma,s} + \|h\|_{\sigma,s}\right)\|h\|_{\sigma,s}^2$$

(we omit a factor given by the algebra constant of H^r times a constant depending on Ω; every derivation ∂_x gives a factor N_n. In the infinite-dimensional spaces $X_{\sigma,s}$ similar calculations give

$$\|f(u)\|_{\sigma,s} \leq \|u\|_{\sigma,s+2q}^3 \quad \forall\, u \in X_{\sigma,s+2q}$$

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and so on; the loss of derivatives is in the norm parameter s.

The iteration scheme consists of the construction of a sequence (u_n) of approximate solutions $u_n \in X^{(n)}$ of (4), which converges to an exact solution. The convergence has to be so rapid that it overcomes the loss of derivatives. To do this in the present analytic setting, we define a decreasing sequence (σ_n) starting from σ_0,

$$(9) \qquad \sigma_{n+1} := \sigma_n - \frac{\vartheta}{(n+1)^2}\,,$$

where ϑ is a fixed positive constant such that

$$\sum_{n=0}^{+\infty} \frac{\vartheta}{(n+1)^2} < \frac{\sigma_0}{2}\,, \qquad \frac{\sigma_0}{2} < \ldots < \sigma_{n+1} < \sigma_n < \ldots < \sigma_0\,.$$

At every step the approximate solution will lose an amount $\vartheta/(n+1)^2$ of analyticity, so that, at the end of the iteration, the limit solution will belong to $X_{\sigma_0/2}$.

We start (Lemma 1) finding a solution u_0 of the finite-dimensional "truncated" equation in $X^{(0)}$

$$(10) \qquad L_\omega u_0 = \mu(f(u_0) + P_0 g)\,.$$

Then we construct inductively (Lemma 3) the sequence (u_n) in the following way: if $u_n \in X^{(n)}$ is a solution of the n-th truncated equation

$$(11) \qquad L_\omega u_n = \mu(f(u_n) + P_n g)\,,$$

we find $u_{n+1} = u_n + h_{n+1} \in X^{(n+1)}$ solving the $(n+1)$th truncated equation for the unknown h_{n+1}, which can be written as

$$(12) \qquad \mathcal{L}(u_n)h_{n+1} = \mu\big[Q(u_n, h_{n+1}) + (P_{n+1} - P_n)g\big]$$

where the linearized operator $\mathcal{L}(u)$ is defined as

$$(13) \qquad \mathcal{L}(u)h := L_\omega h - \mu f'(u)[h]\,.$$

We solve (12) by means of the standard contraction mapping theorem; for this purpose we have to prove the invertibility of $\mathcal{L}(u_n)$ (Lemma 2). In this inversion problem a small divisors difficulty appears; the linearized operator can be inverted provided the parameters (ω, μ) belong to a "nonresonant" set. At each step we have to excise new "resonant" parameters, so that we construct a decreasing sequence (A_n) of parameter sets resulting by a sequence of excisions. We will prove (Lemma 8) that after infinitely many excisions it remains a set of positive, large measure.

We fix $\tau > 1$ once for all. Let $\gamma \in (0,1)$ and

$$A_0 := \left\{ (\omega, \mu) \in (0, +\infty) \times (0,1) \, : \, |p_j - \omega l| > \frac{\gamma}{j^\tau} \quad \forall j = 1, \dots, N_0, \ l \in \mathbb{N} \right\}.$$

By induction, suppose we have u_n and A_n. We denote

$$(14) \qquad\qquad a_n(t) := \int_\Omega |\nabla u_n|^2 \, dx,$$

we consider the Hill's eigenvalue problem

$$(15) \qquad\qquad \begin{cases} y'' + \lambda\big(1 + \mu a_n(t)\big)\, y = 0 \\ y(t) = y(t + 2\pi) \end{cases}$$

and indicate $\lambda_l^{(n)}$ the eigenvalues of (15), $l \in \mathbb{N}$, and $p_l^{(n)} := (\lambda_l^{(n)})^{1/2}$. Let

$$A_{n+1} := \left\{ (\omega, \mu) \in A_n \, : \, |p_j - \omega p_l^{(n)}| > \frac{\gamma}{j^\tau} \quad \forall j = 1, \dots, N_{n+1}, \ l \in \mathbb{N} \right\}.$$

Lemma 1. (First step). *There exists two positive constants δ_0, C, both depending on N_0, such that, if $(\omega, \mu) \in A_0$ and $\mu/\gamma < \delta_0$, then there exists a unique solution $u_0 \in X^{(0)}$ of (10) satisfying*

$$(16) \qquad\qquad \|u_0\|_{\sigma_0} < \frac{\mu}{\gamma}\, C.$$

Moreover $u_0 \in X_{H^3}^{(0)}$ and $\|\partial_{tt} u_0\|_{\sigma_0} < C\mu/\gamma\omega^2$.

Proof. L_ω is the diagonal operator $\lambda_j - \omega^2 l^2$ w.r.t. the basis $\{e^{ilt}\varphi_j(x) : l \in \mathbb{Z}, \ j = 1, \dots, N_0\}$. By the definition of A_0, since $p_j \geq Cj^q$, we have $|\lambda_j - \omega^2 l^2| > C\gamma/j^{\tau-q}$, therefore

$$(17) \qquad\qquad \|L_\omega^{-1} u\|_{\sigma_0} \leq \frac{CN_0^{\tau-q}}{\gamma}\|u\|_{\sigma_0} \quad \forall u \in X^{(0)}$$

and (10) becomes the fixed point problem

$$(18) \qquad\qquad u = \mu L_\omega^{-1}[f(u) + P_0 g]\,.$$

By (5),(6), the term on the right is a contraction in the ball $\{u \in X^{(0)} : \|u\|_{\sigma_0} < CN_0^{\tau-q}\|g\|_{\sigma_0}\mu/\gamma\}$ provided μ/γ is smaller than some quantity which depends on N_0.

Since u_0 satisfies the equality $\omega^2 u_{tt} = \Delta u + \mu(f(u) + P_0 g)$, the estimate for $\partial_{tt} u_0$ follows. $\qquad\square$

We fix a positive constant R such that, if $u \in X_{0,1}$, $a(t) := \int_\Omega |\nabla u|^2 \, dx$ and $\|u\|_{0,1} < R$, then $\|a\|_{H^1} < 1$ and $\|a\|_\infty < 1/2$.

Lemma 2. (Inversion of $\mathcal{L}(u_n)$). *There exist two constants C, C' such that the following holds. Let $u_n \in X^{(n)}$, $\|u_n\|_{0,1} < R$ and $\sigma \geq \sigma_0/2$. Let $(\omega, \mu) \in A_{n+1}$. If*

$$(19) \qquad \frac{\mu}{\gamma} \|u_n\|_{\sigma, 1+2q}^2 < C',$$

then $\mathcal{L}(u_n)$ is invertible, $\mathcal{L}(u_n)^{-1} : X^{(n+1)} \to X^{(n+1)}$ and

$$(20) \qquad \|\mathcal{L}(u_n)^{-1} h\|_\sigma \leq \frac{C N_{n+1}^{\tau - q}}{\gamma} \|h\|_\sigma \quad \forall h \in X^{(n+1)}.$$

Proof. In section 6. □

Lemma 3. (Inductive step). *There exist a choice for N_0 and a positive constant δ_1 such that the following holds. Suppose that*

(i) for every $k = 0, \dots, n$, $h_k \in X^{(k)}$ satisfies

$$(21) \qquad \|h_k\|_{\sigma_k} < \frac{\mu}{\gamma} K \exp[-(3/2)^k]$$

for some K;

(ii) for every $k = 0, \dots, n$, the sum $u_k := h_0 + \dots + h_k$ satisfies $\|u_k\|_{0,1} < R$ and it solves the k-th truncated equation

$$(22) \qquad L_\omega u_k = \mu(f(u_k) + P_k g);$$

(iii) the parameters $(\omega, \mu) \in A_n$, where each A_{k+1} is defined by means of A_k, u_k as showed above.

If $(\omega, \mu) \in A_{n+1}$ and $\mu/\gamma < \delta_1$, then the induction goes on, namely: there exists a unique $h_{n+1} \in X^{(n+1)}$ satisfying

$$(23) \qquad \|h_{n+1}\|_{\sigma_{n+1}} < \frac{\mu}{\gamma} K \exp[-(3/2)^{n+1}]$$

such that $u_{n+1} = u_n + h_{n+1}$ solves $L_\omega u_{n+1} = \mu(f(u_{n+1}) + P_{n+1} g)$. Moreover $\|u_{n+1}\|_{0,1} < R$, $h_{n+1} \in X_{H^3}^{(n+1)}$ and

$$(24) \qquad \|\partial_{tt} h_{n+1}\|_{\sigma_{n+1}} \leq \frac{\mu}{\gamma \omega^2} C \exp[-(4/3)^{n+1}]$$

for some C.

Proof. We have to verify condition (19) in order to invert $\mathcal{L}(u_n)$. We note that $\|u_n\|_{\sigma_{n+1}, 1+2q} \leq \sum_{k=0}^{n} \|h_k\|_{\sigma_{n+1}, 1+2q}$. For every $k \leq n$, $j \geq 1$ we have

$$(25) \qquad \frac{j^{4q}}{\exp[2(\sigma_k - \sigma_{n+1})j]} \leq \left(\frac{2q}{e\vartheta}\right)^{4q} (k+1)^{8q},$$

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so that $\|h_k\|_{\sigma_{n+1},1+2q} \leq (2q/e\vartheta)^{2q}(k+1)^{4q}\|h_k\|_{\sigma_k}$ and by (i)

$$(26) \qquad \|u_n\|_{\sigma_{n+1},1+2q} \leq \left(\frac{2q}{e\vartheta}\right)^{2q} \sum_{k=0}^{n}(k+1)^{4q}\|h_k\|_{\sigma_k} \leq \frac{KC\mu}{\gamma}\,.$$

Then (19) is verified provided $(\mu/\gamma)^3 K^2$ is smaller than a constant depending on N_0. In such a case, by Lemma 2 $\mathcal{L}(u_n)$ is invertible and (12) becomes the fixed point problem

$$(27) \qquad h_{n+1} = \mu\mathcal{L}(u_n)^{-1}\big[Q(u_n, h_{n+1}) + (P_{n+1} - P_n)g\big] =: \Phi(h_{n+1})\,.$$

We prove that Φ is a contraction in the ball $B_{n+1} = \{h \in X^{(n+1)} : \|h\|_{\sigma_{n+1}} < r_{n+1}\}$ with $r_{n+1} := (\mu/\gamma)K \exp[-(3/2)^{n+1}]$. By (i), $\|u_n\|_{\sigma_{n+1}} \leq \|u_n\|_{\sigma_n} < (\mu/\gamma)KC$ for some universal C, so that by (8) for $h \in B_{n+1}$

$$\|Q(u_n, h)\|_{\sigma_{n+1}} \leq \frac{\mu}{\gamma}\, KC\|h\|_{\sigma_{n+1}}^2\,.$$

Since $(P_{n+1} - P_n)g \in X^{(n+1)} \cap X^{(n)\perp}$,

$$\|(P_{n+1} - P_n)g\|_{\sigma_{n+1}} \leq \|g\|_{\sigma_n}\, \exp[-(\sigma_n - \sigma_{n+1})N_n]\,.$$

Then, by Lemma 2, $\Phi(B_{n+1}) \subseteq B_{n+1}$ if

$$(28)\quad \frac{\mu^2 N_{n+1}^{\tau-q}C}{\gamma^2}\, r_{n+1}^2 < \frac{r_{n+1}}{2}\,, \qquad \frac{\mu N_{n+1}^{\tau-q}C\|g\|_{\sigma_0}}{\gamma}\, \exp\left(-\frac{\vartheta N_0\, 2^n}{(n+1)^2}\right) < \frac{r_{n+1}}{2}\,.$$

The second inequality is verified for every $n \in \mathbb{N}$ provided N_0 is choosen large enough; we fix N_0 in such a way. Then the first inequality is verified for every n if μ/γ is small enough. The estimate for $\|\Phi(v) - \Phi(w)\|$, $v, w \in B_{n+1}$ is similar and we get the fixed point h_{n+1}.

Since h_{n+1} solves (12), that is

$$\omega^2 h_{tt} = \Delta h + \mu(f(u_n + h) - f(u_n)) + \mu P_{n+1}P_n^\perp g,$$

it follows that $h_{n+1} \in X_{H^3}^{(n+1)}$. Since $\|u_{n+1}\|_{\sigma_{n+1}}$ is bounded, by (6) and the second inequality in (28) we have

$$\omega^2\|(h_{n+1})_{tt}\|_{\sigma_{n+1}} \leq \frac{\mu CK N_{n+1}^{4q}}{\gamma}\, \exp[-(3/2)^{n+1}] \leq \frac{\mu C'}{\gamma}\, \exp[-(4/3)^{n+1}]$$

for some C, C'. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

4 The solution

Lemma 4. (Existence of a solution). *Suppose $A_\infty := \cap_{n\in\mathbb{N}} A_n \neq \emptyset$. If $(\omega,\mu) \in A_\infty$ and $\mu/\gamma < \delta_1$, then u_n converges in $X_{\sigma_0/2}$ to $u_\infty := \sum_{k=0}^{+\infty} h_k$. u_∞ is a classical solution of (4) and*

$$\tag{29} \|u_\infty\|_{\sigma_0/2} \leq \frac{\mu}{\gamma} C$$

for some C.

Proof. By (25)

$$\|h_k\|_{\sigma_0/2,\, 1+2q, H^3} \leq C\, (k+1)^{4q}\, \|h_k\|_{\sigma_k, H^3},$$

so that by (24) the series $\sum \|h_k\|_{\sigma_0/2,\, 1+2q, H^3}$ converges. Both L_ω and f are continuous from $X_{\sigma_0/2,\, 1+2q, H^3}$ into $X_{\sigma_0/2, 1, H^1}$, so

$$\|L_\omega[u_\infty - u_n]\|_{\sigma_0/2}, \quad \|f(u_\infty) - f(u_n)\|_{\sigma_0/2} \to 0 \quad (n \to \infty).$$

Also $\|P_n g - g\|_{\sigma_0} \to 0$. Since $L_\omega u_n = \mu(f(u_n) + P_n g)$ for every n, it follows that u_∞ solves (4). The estimates (16),(21) imply (29). $\qquad\square$

Lemma 5. (Uniqueness of the solution). *Suppose $A_\infty := \cap_{n\in\mathbb{N}} A_n \neq \emptyset$. For every fixed radius $\bar{R} > 0$ there exists $\bar{\delta}$ such that, for $(\omega,\mu) \in A_\infty$ and $\mu/\gamma < \min\{\bar{\delta}, \delta_1\}$, u_∞ is the unique solution of (4) in the ball $\{\|u\|_{\sigma_0/2} < \bar{R}\}$.*

Proof. Suppose v is another solution of (4), $\|v\|_{\sigma_0/2} < \bar{R}$. We denote $v_n := P_n v$. Projecting the equation $L_\omega v = \mu(f(v) + g)$ on $X^{(n)}$ we get

$$\tag{30} L_\omega v_n = \mu(f(v_n) + R_n(v) + P_n g), \quad R_n(v) := \Delta v_n \int_\Omega |\nabla(v - v_n)|^2\, dx.$$

Since u_n solves (11), the difference $w_n := v_n - u_n$ satisfies

$$\tag{31} \mathcal{L}(u_n) w_n = \mu(Q(u_n, w_n) + R_n(v)).$$

It holds $\|R_n(v)\|_{\sigma_0/2} \leq \bar{R}^3 C N_n^{2q} \exp(-\sigma_0 N_n)$ because

$$\|\Delta v_n\|_{\sigma_0/2} \left\| \int_\Omega |\nabla(v - v_n)|^2\, dx \right\|_{H^1} \leq N_n^{2q} \|v\|_{\sigma_0/2} C \sum_{j > N_n} \lambda_j \|v_j\|_{H^1}^2$$

and, by (20), $w \mapsto \mu \mathcal{L}(u_n)^{-1}[Q(u_n, w) + R_n(v)]$ is a contraction in the ball $\{w \in X^{(n)} : \|w\|_{\sigma_0/2} < (\mu/\gamma)\bar{R}^3 N_n^{q+\tau} \exp(-\sigma_0 N_n)\}$ if $\mu/\gamma < \bar{\delta}$ for some $\bar{\delta}$ depending on $\bar{R}$. Therefore $\|v_n - u_n\|_{\sigma_0/2} \to 0$ as $n \to \infty$. Since $v_n \to v$ and $u_n \to u_\infty$ in $X_{\sigma_0/2}$, it follows that $v = u_\infty$. $\qquad\square$

By Lemma (5), in particular, we can choose δ_1 so small that the solution u_∞ is unique in the ball $\{\|u\|_{\sigma_0/2} < 1\}$.

Lemma 6. (Regularity of the solution). *If $g \in X_{\sigma_0, H^r}$, $r \geq 1$, then the solution u_∞ constructed in Lemma 4 belongs to $X_{\sigma_0/3, H^{r+2}}$.*

Proof. Suppose $g \in X_{\sigma_0, H^r}$ and let u be a solution of (3). By Lemma 4 $u \in X_{\sigma_0/2, 1+2q, H^3}$ so that $\Delta u(1 + \int |\nabla u|^2 dx)$ belongs to $X_{\sigma_0/2, H^3}$. By (3) it follows that $u_{tt} \in X_{\sigma_0/2, H^p}$ where $p = \min\{3, r\}$, and $u \in X_{\sigma_0/2, H^{p+2}}$. If $r \leq 3$ we have finished.

Let $r \geq 4$. Since $u \in X_{\sigma_0/2, H^5}$, losing a small amount of analyticity we can get back $2q$ x-derivatives, namely $u \in X_{\sigma', 1+2q, H^5}$ where $\sigma' < \sigma_0/2$. By (3) it follows $u_{tt} \in X_{\sigma', H^p}$ where $p = \min\{5, r\}$.

After a finite number of similar bootstraps we conclude. $\qquad\square$

Clearly, choosing a smaller ϑ in (9), the solution u_∞ of Lemma 4 belongs to $X_{3\sigma_0/4}$, so that we have actually $u_\infty \in X_{\sigma_0/2, H^{r+2}}$ for every r.

5   The Cantor set of parameters

Lemma 7. (Regular dependence on the parameter ω). *There exist $\delta_2 \leq \delta_1$ such that the map*

$$h_n : A_n \cap \{(\omega, \mu) : \mu/\gamma < \delta_2\} \to X^{(n)}, \quad (\omega, \mu) \mapsto h_n(\omega, \mu)$$

is differentiable in ω for every n and

$$(32) \qquad \qquad \|\partial_\omega u_n\|_{\sigma_n} \leq \frac{\mu}{\gamma^2 \omega} C$$

for some C.

Proof. $h_0 = u_0$ is the solution of (18), so it is differentiable in $(\omega, \mu) \in A_0$ by the implicit function theorem. Differentiating (10) w.r.t. ω we have

$$\big(L_\omega - \mu f'(u_0)\big)[\partial_\omega u_0] = -2\omega\, \partial_{tt} u_0\,.$$

$L_\omega - \mu f'(u_0)$ is invertible in $X^{(0)}$ because L_ω is invertible and, if μ/γ is small enough, $\|L_\omega^{-1} \mu f'(u_0)[z]\|_{\sigma_0} \leq (1/2)\|z\|_{\sigma_0}$ $\forall z$. By the estimate for $\partial_{tt} u_0$

$$\|\partial_\omega h_0\|_{\sigma_0} \leq \frac{\mu}{\gamma^2 \omega} C\,.$$

By induction, suppose h_k regular on A_k for $k \leq n$. Since h_{n+1} is the solution of (27), it is differentiable by the implicit function theorem. We write the $(n+1)$th truncated equation as

$$L_\omega h_{n+1} = \mu\big[f(u_n + h_{n+1}) - f(u_n) + (P_{n+1} - P_n)g\big]\,;$$

differentiating it w.r.t. ω we get

$$(33) \qquad \mathcal{L}(u_{n+1})[\partial_\omega h_{n+1}] = \mu\big(f'(u_{n+1}) - f'(u_n)\big)[\partial_\omega u_n] - 2\omega\, \partial_{tt} h_{n+1}$$

where $\mathcal{L}(u_{n+1})z = L_\omega z - \mu f'(u_{n+1})[z]$ according to (13). By Lemma 2 $\mathcal{L}(u_n)$ is invertible for $(\omega, \mu) \in A_{n+1}$, therefore $\mathcal{L}(u_{n+1})$ is invertible provided

$$\left\|\mathcal{L}(u_n)^{-1}(\mathcal{L}(u_{n+1}) - \mathcal{L}(u_n))z\right\|_{\sigma_{n+1}} \leq \frac{1}{2}\|z\|_{\sigma_{n+1}}.$$

By (20),(7),(26),(23), this holds true for μ/γ small enough. It follows that $\|\mathcal{L}(u_{n+1})^{-1}\| \leq 2\|\mathcal{L}(u_n)^{-1}\|$. After calculations by induction, since $\|\partial_\omega u_n\| \leq \sum_{k=0}^{n} \|\partial_\omega h_k\|$, by (33) we have the estimate for $\|\partial_\omega h_{n+1}\|$. So for every n

(34) $$\|\partial_\omega h_n\|_{\sigma_n} \leq \frac{\mu}{\gamma^2 \omega} C \exp[-(5/4)^n]$$

and summing (34) we conclude. $\qquad\square$

Remark. In fact, u_n is C^∞; similar calculations give $\|\partial_\mu u_n\|_{\sigma_n} \leq C/\gamma$.

We prove that the amount of all the excisions in the parameters set is of order γ, so that A_∞ is a large, positive measure set for γ small.

Lemma 8. (The Cantor set). *There exist δ_3, γ_0 such that for $\gamma < \gamma_0$ the Cantor set $\mathcal{A}_\gamma := A_\infty \cap \{(\omega, \mu) : \mu < \delta_3 \gamma\}$ has the following measure property.*

For every interval $I = (\bar{\omega}_1, \bar{\omega}_2)$ with $0 < \bar{\omega}_1 < \bar{\omega}_2 < \infty$ there is a constant $\bar{C}$ depending on I such that, denoted by $\mathcal{R}_\gamma$ the rectangular region $\mathcal{R}_\gamma = I \times (0, \delta_3\gamma)$,

$$\frac{|\mathcal{R}_\gamma \cap \mathcal{A}_\gamma|}{|\mathcal{R}_\gamma|} > 1 - \bar{C}\gamma.$$

Proof. We fix μ and define

$$\Omega_{j,l}^n := \left\{\omega : |p_j - \omega p_l^{(n)}(\omega, \mu)| \leq \frac{\gamma}{j^\tau}\right\}.$$

Since $p_l^{(n)} = 0$ for $l = 0$, taking γ small enough $\Omega_{j,0}^n = \emptyset$ for all j, n.

By definition of A_n, the $(n+1)$th excision E_{n+1} is the union

$$E_{n+1} := \{\omega : (\omega, \mu) \in A_n \setminus A_{n+1}\} = \bigcup_{j \leq N_{n+1},\ l \geq 1} \Omega_{j,l}^n.$$

We have to prove that $\cup_{n \geq 1} E_n$ has small measure; as a consequence, its complementary set A_∞ will be a large set.

We observe that if $\omega \in \Omega_{j,l}^n$, then by (40) $Cp_j < \omega l < C'p_j$, and by (41) and (32)

$$|\partial_\omega p_l^{(n)}(\omega, \mu)| \leq C\, l\mu \|\partial_\omega a_n\|_{H^1} \leq C'\, l\mu \|\partial_\omega u_n(\omega, \mu)\|_{\sigma_n} \leq C''\, l\frac{\mu^2}{\gamma^2 \omega}$$

for some C, C', C''. It follows by (40) that

(35) $$\partial_\omega(\omega\, p_l^{(n)}(\omega, \mu)) \geq p_l^{(n)}(\omega, \mu) - \omega C'' l\frac{\mu^2}{\gamma^2 \omega} > \frac{l}{4}$$

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provided μ/γ is small enough. As a consequence, fixed $0 < \bar{\omega}_1 < \bar{\omega}_2 < \infty$, we have the following: if $\Omega_{j,l}^n \cap (\bar{\omega}_1, \bar{\omega}_2)$ is nonempty, then, independently of n, the index $l \in \Lambda(j) := [C j^q / \bar{\omega}_2, \, C' j^q / \bar{\omega}_1]$ and

$$|\Omega_{j,l}^n| < \frac{8\gamma}{lj^\tau} < C \frac{\gamma \bar{\omega}_2}{j^{\tau+q}} \, .$$

So the estimate for E_1 is

$$(36) \qquad |E_1 \cap (\bar{\omega}_1, \bar{\omega}_2)| \leq \sum_{j \leq N_1} \sum_{l \in \Lambda(j)} |\Omega_{l,j}^0| < \gamma C(\bar{\omega}_1, \bar{\omega}_2) \sum_{j \leq N_1} \frac{1}{j^\tau}$$

for some $C(\bar{\omega}_1, \bar{\omega}_2)$ depending on $(\bar{\omega}_1, \bar{\omega}_2)$.

Now, suppose we have an estimate for $|\cup_{k=1}^n E_k \cap (\bar{\omega}_1, \bar{\omega}_2)|$, $n \geq 1$; we study what "actually new" excision is made at the $(n+1)$th step, namely $(E_{n+1} \setminus \cup_{k=1}^n E_k) \cap (\bar{\omega}_1, \bar{\omega}_2)$. We separate

$$E_{n+1} \cap (\bar{\omega}_1, \bar{\omega}_2) = \bigcup_{\substack{1 \leq j \leq N_n \\ l \in \Lambda(j)}} \Omega_{j,l}^n \quad \cup \bigcup_{\substack{N_n < j \leq N_{n+1} \\ l \in \Lambda(j)}} \Omega_{j,l}^n$$

and observe that the measure of the union on the right is smaller than $\gamma C(\bar{\omega}_1, \bar{\omega}_2) \sum_{j=N_n}^{N_{n+1}} 1/j^\tau$ where the constant is that of (36). To estimate the union on the left, suppose that $\omega \in \Omega_{j,l}^n$ for some $j \leq N_n$. Then by (41)

$$
\begin{aligned}
|p_j - \omega p_l^{(n-1)}(\omega, \mu)| &\leq |p_j - \omega p_l^{(n)}(\omega, \mu)| + \omega \, |p_l^{(n-1)}(\omega, \mu) - p_l^{(n)}(\omega, \mu)| \\
&\leq \frac{\gamma}{j^\tau} + C \, j^q \mu \|h_n\|_{\sigma_n}
\end{aligned}
$$

because $u_n - u_{n-1} = h_n$. Since $\Omega_{j,l}^{n-1}$ is contained in E_n, the new excision here is only the set

$$\left\{ \omega \, : \, \frac{\gamma}{j^\tau} < |p_j - \omega p_l^{(n-1)}(\omega, \mu)| \leq \frac{\gamma}{j^\tau} + C \, j^q \mu \|h_n\|_{\sigma_n} \right\}$$

which has measure smaller than $C \bar{\omega}_2 \mu \|h_n\|_{\sigma_n}$ by (35) and definition of $\Lambda(j)$.

Recalling (21), we have proved that

$$\left| \bigcup_{k=1}^n E_k \cap (\bar{\omega}_1, \bar{\omega}_2) \right| < C\gamma \quad \forall n$$

where C depends on $\bar{\omega}_1, \bar{\omega}_2, \tau$.

Integrating in μ we conclude. $\qquad \square$

Remark. It is possible to construct a smooth function $\tilde{u} : A_0 \to X_{\sigma_0/2}$ which coincides with u_∞ for parameters (ω, μ) in a subset B of A_∞, where B satisfies the measure property described in Lemma 8 (see e.g. [6]). Such a function is called a "Whitney extension" of u_∞.

6 Inversion of the linearized operator

We prove in this section Lemma 2. Let $u \in X^{(n)}$, $h \in X^{(n+1)}$; the linearized operator is

$$\mathcal{L}(u)h = L_\omega h - \mu f'(u)[h] = Dh + Sh$$

where we distinguish a diagonal part

$$(37) \qquad Dh := \omega^2 h_{tt} - \Delta h\Big(1 + \mu \int_\Omega |\nabla u|^2 \, dx\Big)$$

and a "projection" part

$$(38) \qquad Sh := -\mu \Delta u \int_\Omega 2\nabla u \circ \nabla h \, dx \, .$$

We recall here some results on Hill's problems; the proof is in the Appendix.

Lemma 9. (Hill's problems). *Let $\alpha(t)$ be 2π-periodic, $\alpha \in L^\infty$ and $\|\alpha\|_\infty < 1/2$. The eigenvalues of the periodic problem*

$$(39) \qquad \begin{cases} y'' + \lambda\,(1 + \alpha(t))\,y = 0 \\ y(t) = y(t + 2\pi) \end{cases}$$

form an increasing sequence $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \ldots \to +\infty$. Let $p_l := \sqrt{\lambda_l}$. There holds

$$(40) \qquad \frac{1}{3}\,l < p_l < 2l \quad \forall l \in \mathbb{N}$$

and for $\alpha, \beta \in L^\infty$

$$(41) \qquad \big|p_l(\alpha) - p_l(\beta)\big| \leq Cl\,\|\alpha - \beta\|_\infty \quad \forall l \in \mathbb{N}$$

where C is a universal constant.

The eigenfunctions $\psi_l(t)$ of (39) form an orthonormal basis of $L^2(\mathbb{T})$ w.r.t. the scalar product

$$(u, v)_{L^2_\alpha} = \int_0^{2\pi} uv(1 + \alpha)\,dt$$

and also an orthogonal basis of $H^1(\mathbb{T})$ w.r.t. the scalar products

$$(u, v)_{H^1_\alpha} = \int_0^{2\pi} u'v'\,dt + (u, v)_{L^2_\alpha} \, .$$

The corresponding norms are equivalent to the standard Sobolev norms,

$$(42) \qquad \frac{1}{2}\,\|y\|_{L^2} \leq \|y\|_{L^2_\alpha} \leq 2\,\|y\|_{L^2}, \quad \frac{1}{2}\,\|y\|_{H^1} \leq \|y\|_{H^1_\alpha} \leq 2\,\|y\|_{H^1},$$

and $\|\psi_l\|^2_{H^1_\alpha} = \lambda_l + 1$.

Lemma 10. (Inversion of D). *Let $u \in X^{(n)}$, $a(t) := \int_\Omega |\nabla u|^2\, dx \in H^1(\mathbb{T})$, $\|a\|_{H^1} < 1$, $\|a\|_\infty < 1/2$. Let λ_l be the eigenvalues of the Hill's problem*

$$(43) \qquad \begin{cases} y'' + \lambda(1 + \mu a(t))\, y = 0 \\ y(t) = y(t + 2\pi) \end{cases}$$

and $p_l = \sqrt{\lambda_l}$. If (ω, μ) satisfy the nonresonant condition

$$(44) \qquad |p_j - \omega p_l| > \frac{\gamma}{j^\tau} \quad \forall j = 1, \ldots, N_{n+1}, \quad l \in \mathbb{N},$$

then D is invertible, $D^{-1} : X^{(n+1)} \to X^{(n+1)}$ and

$$(45) \qquad \|D^{-1}h\|_{\sigma,s} \leq \frac{C}{\gamma} \|h\|_{\sigma, s+\tau-q} \quad \forall h \in X^{(n+1)}$$

for some C.

Proof. If $h \in X^{(n+1)}$, it holds $Dh = \sum_j D_j h_j(t)\, \varphi_j(x)$, where

$$D_j z(t) = \omega^2 z''(t) + \lambda_j z(t)\rho(t), \quad \rho(t) := 1 + \mu a(t).$$

Using the eigenfunctions $\psi_l(t)$ of (43) as a basis of $H^1(\mathbb{T})$,

$$D_j z(t) = \sum_{l \in \mathbb{N}} (\lambda_j - \omega^2 \lambda_l)\hat{z}_l\, \psi_l(t)\, \rho(t), \qquad z = \sum_{l \in \mathbb{N}} \hat{z}_l\, \psi_l(t),$$

and $K_j := (1/\rho)D_j$ is the diagonal $\{\lambda_j - \omega^2\lambda_l\}_{l \in \mathbb{N}}$. Since $|\lambda_j - \omega^2\lambda_l| > C\gamma/j^{\tau-q}$, K_j is invertible and

$$\|K_j^{-1}z\|_{H^1_{\mu a}}^2 = \sum_{l \in \mathbb{N}} \left(\frac{\hat{z}_l}{\lambda_j - \omega^2\lambda_l} \right)^2 (1 + \lambda_l) \leq \frac{j^{2(\tau-q)}}{(C\gamma)^2} \|z\|_{H^1_{\mu a}}^2.$$

By (42) $\|K_j^{-1}z\|_{H^1} \leq 4(j^{\tau-q}/C\gamma)\|z\|_{H^1}$. Since $D_j^{-1}z = K_j^{-1}(z/\rho)$ and $\|1/\rho\|_{H^1}$ is smaller than 5,

$$\|D_j^{-1}z\|_{H^1} \leq \frac{C'j^{\tau-q}}{\gamma} \|z\|_{H^1}$$

for some C'. Finally $D^{-1}h = \sum_j D_j^{-1}h_j(t)\varphi_j(x)$ and we conclude. $\qquad\square$

Lemma 11. (Control of S). *$S : X_{\sigma,s} \to X_{\sigma,s}$ is bounded and*

$$(46) \qquad \|Sh\|_{\sigma,s} \leq \mu C\|u\|_{\sigma,s+2q}^2 \|h\|_{0,0} \quad \forall h \in X_{\sigma,s}$$

for some C.

Proof. Since $\int_\Omega \nabla u \circ \nabla h \, dx$ does not depend on x,

$$\left\| \Delta u \int_\Omega \nabla u \circ \nabla h \, dx \right\|_{\sigma,s} \leq \|\Delta u\|_{\sigma,s} \left\| \int_\Omega \nabla u \circ \nabla h \, dx \right\|_{H^1}.$$

$\int_\Omega \nabla u \circ \nabla h \, dx = \sum_j \lambda_j u_j(t) h_j(t)$, so $\| \int_\Omega \nabla u \circ \nabla h \, dx \|_{H^1} \leq C \|u\|_{0,2q} \|h\|_{0,0}$ by Hölder inequality. $\quad\square$

Proof of Lemma 2. $\mathcal{L}(u) = D + S = (I + SD^{-1})D$ where I is the identity map of X_σ. To prove the invertibility of $I + SD^{-1}$, by Neumann series it is sufficient to show that

(47) $$\|SD^{-1}h\|_\sigma \leq \frac{1}{2} \|h\|_\sigma \quad \forall h \in X_\sigma.$$

By Lemmas 10 and 11

$$\|SD^{-1}h\|_\sigma \leq \frac{C\mu}{\gamma} \|u\|_{\sigma,1+2q}^2 \|h\|_{0,\tau-q}.$$

Since for $\sigma \geq \sigma_0/2$

$$\max_{j \geq 1} \frac{j^{\tau-q}}{j^2 \, e^{\sigma j}} \leq C$$

we have $\|h\|_{0,\tau-q} \leq C\|h\|_\sigma$. Then condition (19) implies (47) and by Neumann series $\|(I + SD^{-1})^{-1}h\|_\sigma \leq 2\|h\|_\sigma$. $\quad\square$

7 Appendix

Proof of Lemma 9. The proof is a consequence of classical results in [15, 23]. First, if $y'' + \lambda(1+\alpha)y = 0$ and $\|\alpha\|_\infty < 1/2$,

$$\int_0^{2\pi} y'^2 \, dt = \lambda \int_0^{2\pi} (1+\alpha)y^2 \, dt,$$

so that $\lambda \geq 0$. $\lambda_0 = 0$ is an eigenvalue, the corresponding eigenfunctions are the constants, and $0 < \lambda_1 \leq \dots$.

By [15, Theorem 2.2.2, p.23], for every $m \in \mathbb{N}$ both λ_{2m+1} and λ_{2m+2} satisfy

$$\frac{2}{3}(m+1)^2 \leq \lambda \leq 2(m+1)^2$$

and (40) follows.

The eigenvalues $\lambda_l(\alpha)$ and the eigenfunctions $\psi_l(\alpha)$ of (39) depend in a differentiable way on $\alpha \in L^2$. Indeed, we consider the solution $y_1(t,\lambda,\alpha)$ of the Cauchy problem

$$y'' + \lambda(1+\alpha)y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

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and the solution $y_2(t, \lambda, \alpha)$ of the same equation with $y(0) = 0$, $y'(0) = 1$. Adapting the calculations in [23, ch. 1] one proves that y_1, y_2 are real analytic functions of $(\lambda, \alpha) \in \mathbb{C} \times L^2_{\mathbb{C}}(\mathbb{T})$. By [15, ch. 2] λ is an eigenvalue for (39) if $D(\lambda, \alpha) := y_1(2\pi, \lambda, \alpha) + y'_2(2\pi, \lambda, \alpha) = 2$, and by the properties of D it follows the analyticity of $\lambda_l(\alpha)$, $\psi_l(\alpha)$ on L^2.

So we can differentiate the equation $\psi''_l + \lambda_l(1 + \alpha)\psi_l = 0$ w.r.t. α; multiplying by ψ_l and integrating

$$(48) \qquad d\lambda_l(\alpha)[\varphi] = -\lambda_l(\alpha) \int_0^{2\pi} \psi_l^2(\alpha)\, \varphi\, dt \,.$$

If $\|\alpha\|_\infty, \|\beta\|_\infty < 1/2$,

$$\lambda_l(\beta) - \lambda_l(\alpha) = \int_0^1 d\lambda_l(\alpha + \xi(\beta - \alpha))[\beta - \alpha]\, d\xi \,,$$

therefore applying (48) and (40) we get $|\lambda_l(\beta) - \lambda_l(\alpha)| \leq Cl^2 \|\alpha - \beta\|_\infty$ for some universal constant C. By (40) we have (41).

The equivalence (42) and the orthogonality of (ψ_l) w.r.t. $(\,,)_{H^1_\alpha}$ can be verified by direct calculations. $\qquad \square$

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