Multiplicity of solutions for a mean field equation on compact surfaces

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ABSTRACT

We consider a scalar field equation on compact surfaces which has variational structure. When the surface is a torus and a physical parameter ρ belongs to $(8\pi, 4\pi^2)$ we show under some extra assumptions that, as conjectured in [8], the functional admits at least three saddle points other than a local minimum.

Key Words: Scalar field equations, Geometric PDE's, Multiplicity result.

1 Introduction

Let (Σ, g) be a compact Riemannian surface (without boundary), $h \in C^2(\Sigma)$ be a positive function and ρ a positive real parameter. We consider the equation

$$-\triangle_g u + \rho = \rho \frac{h(x)e^u}{\int_\Sigma h(x)e^u dV_g} \qquad x \in \Sigma,\ u \in H_g^1(\Sigma), \tag{$*$}$$

where $\triangle_g$ is the Laplace-Beltrami operator on Σ.

When (Σ, g) is a flat torus equation $(*)$ is related to the study of some Chern–Simons–Higgs models; indeed via its solutions it is possible to describe the asymptotic behavior of a class of condensates (or multivortex) solutions which are relevant in theoretical physics and which were absent in the classical (Maxwell-Higgs) vortex theory (see [23], [25], [26] and references therein). This PDE arises also in conformal geometry; when (Σ, g) is the standard sphere and $\rho = 8\pi$, the geometric meaning of this problem is that from a solution u we can obtain a new conformal metric $e^u g$ which has curvature $\frac{\rho}{2}h$; the latter is known as the Kazdan-Warner problem, or as the Nirenberg problem and has been studied for example in [3], [4] and [17]. Moreover this problem arises in statistical mechanics. Indeed when formulated on bounded domains of $\mathbb{R}^2$ with Dirichlet boundary conditions equation $(*)$ was considered in [1] and [16] as the mean field limit as point vortices for the two–dimensional Euler equation.

Problem $(*)$ has a variational structure and solutions can be found as critical points of the functional

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dV_g + \rho \int_\Sigma u\, dV_g - \rho \log \int_\Sigma h(x)e^u dV_g \qquad u \in H_g^1(\Sigma). \tag{1.1}$$

Since equation $(*)$ is invariant when adding constants to u, we can restrict ourselves to the subspace of the functions with zero average $\bar{H}_g^1(\Sigma) := \left\{u \in H_g^1(\Sigma) : \int_\Sigma u\, dv_g = 0\right\}$.

Because of the Moser-Trudinger inequality (see Lemma 2.2) one can easily prove the compactness and the coercivity of I_ρ when $\rho < 8\pi$ and thus one can find solutions of $(*)$ by minimization.

If $\rho = 8\pi$ the situation is more delicate since I_ρ still has a lower bound but it is not coercive anymore; in general when ρ is an integer multiple of 8π, the existence problem of $(*)$ is much harder (a far from complete list of references on the subject includes works by Chang and Yang [4], Chang, Gursky and Yang [3], Chen and Li [5], Nolasco and Tarantello [23], Ding, Jost, Li and Wang [11] and Lucia [20]).

For $\rho > 8\pi$, as the functional I_ρ is unbounded from below and from above, solutions have to be found as saddle points.

In [10] Ding, Jost, Li and Wang proved that, assuming $\rho \in (8\pi, 16\pi)$ and assuming that the genus of the surface is greater or equal than 1, there exists a solution to $(*)$. In [18] Yan Yan Li initiated a program to find solutions for $\rho > 8\pi$ by using the topological degree theory. He proved an uniform bound for solutions to equation $(*)$ whenever ρ is contained in a compact set of $(8m\pi, 8(m+1)\pi)$ where $m \geq 0$ is an integer. Therefore, the LeraySchauder degree for $(*)$ remains the same when ρ is in the interval $(8m\pi, 8(m+1)\pi)$. Few years ago this program was completed by Chen and Lin in [7] using a finite-dimensional reduction to compute the jump values. The authors obtained a complete degree-counting formula, extending the results in [19], where the case $\Sigma = S^2$ and $k = 1$ was studied. Finally, when $\rho \notin 8\mathbb{N}\pi$, Djadli [12] generalized these previous results establishing the existence of a solution for any (Σ, g); to do that he deeply investigated the topology of low sublevels of I_ρ in order to perform a min-max scheme (already introduced in Djadli and Malchiodi [13]).

Not much is known about multiplicity. Recently the author in [9], via Morse inequalities, improved significantly the multiplicity estimate which can be deduced from the degree-counting formula in [7].

Besides, the case of the flat torus, which is a relevant situation from the physical point of view, has been treated by Struwe and Tarantello under the assumptions that $h \equiv 1$ and $\rho \in (8\pi, 4\pi^2)$. In these hypotheses, $u = 0$ is clearly a critical point for I_ρ. Moreover, $u = 0$ is a strict local minimum, since the second variation in the direction $v \in \bar{H}^1_g(T)$ can be estimated as follows

$$D^2 I_\rho(0)[v,v] = \|v\|^2 - \rho \int_\Sigma v^2 dx \geq \left(1 - \frac{\rho}{4\pi^2}\right) \|v\|^2 . \tag{1.2}$$

Under these conditions, the functional possesses a mountain pass geometry and by thanks to this structure the existence of a saddle point of I_ρ has been detected by Struwe and Tarantello.

Theorem 1.1. *([24]) Let Σ be the flat torus and $h \equiv 1$. Then, for any $\rho \in (8\pi, 4\pi^2)$, there exists a non-trivial solution u_ρ of $(*)$ satisfying $I_\rho(u_\rho) \geq (1 - \rho/4\pi^2)c_0$ for some constant $c_0 > 0$ independent of ρ.*

As g is the flat metric and h is constant, if u is a solution of $(*)$, the functions $u_{x_0}(x) := u(x - x_0)$ still solve $(*)$, for any $x_0 \in T$; so from Theorem 1.1 we can deduce the existence of an infinite number of solutions of $(*)$.

Perturbing g and h there is still a local minimum, $\bar{u}$, close to $u = 0$ and the same procedure of [24] ensures the presence of a saddle point, but on the other hand, if u is a non-trivial solution, the criticality of the translated functions u_{x_0} is not anymore guaranteed. In [8] the author improved this result stating that apart from $\bar{u}$ there are at least two critical points, see Theorem 3.1 in Section 3.

The strategy of the proof consists in defining a deformed functional $\tilde{I}_\rho$, having the same saddle points of I_ρ but a greater topological complexity of its low sublevels, and in estimating from below the number of saddle points of $\tilde{I}_\rho$ using the notion of Lusternik-Schnirelmann relative category (roughly speaking a natural number measuring how a set is far from being contractible, when a subset is fixed).

Always in [8] the author conjectured that apart from the minimum and the two saddle points another critical point should exist. In fact this turns out to be true.

Theorem 1.2. *If $\rho \in (8\pi, 4\pi^2)$ and $\Sigma = T$ is the torus, if the metric g is sufficiently close in $C^2(T; S^{2\times 2})$ to dx^2 and h is uniformly close to the constant 1, I_ρ admits a point of strict local minimum and at least three different saddle points.*

In the above statement $S^{2\times 2}$ stands for the symmetric matrices on T. To prove Theorem 1.2 we exploit the following inequality derived in [8]:

$$\#\{\text{solutions of } (*)\} \geq \text{cat}_{X,\partial X}\, X,$$

where X is the topological cone over T. Next we apply a result (Theorem 1.2) which allows to estimate from below the previous relative category by one plus the cuplength of the pair $(X, \partial X)$, (the cuplength of the pair $(X, \partial X)$, denoted by $\text{CL}(X, \partial X)$, is the maximum number of elements of $\tilde{H}^*(X, \partial X)$ which when multiplied give a non-zero result). Finally we prove that $\text{CL}(X, \partial X) \geq \text{CL}(T) = 2$, which concludes the proof.

Remark 1.3. *Since all the arguments only use the presence of a strict local minimum and the fact that X is the topological cone over T, whenever on some (Σ, g) the functional I_ρ possesses a strict local minimum, the theorem holds true, more precisely I_ρ has at least $CL(\Sigma)+1$ critical points other than the minimum.*

In section 2 we collect some useful material concerning the topological structure of I_ρ and we recall some definitions and some classical results in algebraic topology; besides, we focus on the notion of Lusternik-Schnirelman relative category and its relation with the cuplength. In section 3 we present briefly the result in [8] and prove our multiplicity result.

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2 Notation and preliminaries

In this section we collect some facts needed in order to obtain the multiplicity result. First of all we consider some improvements of the Moser-Trudinger inequality which are useful to study the topological structure of the sublevels of I_ρ. Next, we collect some basic notions in algebraic topology and we recall the definition of Lusternik-Schnirelman relative category stating also some results relating the category to both the cup-length and the existence of critical points.

Let now fix our notation. The symbol $B_r(p)$ denotes the metric ball of radius r and center p. As already specified we set $\bar{H}^1_g(\Sigma) := \left\{u \in H^1_g(\Sigma) \,:\, \int_\Sigma u\, dv_g\right\}$.

Large positive constants are always denoted by C, and the value of C is allowed to vary from formula to formula. Finally, given a smooth functional $I : H^1_g(\Sigma) \to \mathbb{R}$ and a real number c, we set $I^c := \{u \in H^1_g(\Sigma) \,|\, I(u) \leq c\}$.

2.1 Variational Structure

Even though the Palais-Smale is not known to hold for our functional, employing together a deformation lemma proved by Lucia in [21] and a compactness result due to Li [18] it is possible to establish for I_ρ a strong result though and through analogous to the classical deformation lemma.

Proposition 2.1. *If $\rho \neq 8k\pi$ and if I_ρ has no critical levels inside some interval $[a, b]$, then $\{I_\rho \leq a\}$ is a deformation retract of $\{I_\rho \leq b\}$.*

To understand the topology of sublevels of I_ρ it is useful to recall the well-known Moser-Trudinger inequality on compact surfaces.

Lemma 2.2 (*Moser-Trudinger inequality*)**.** *There exists a constant C, depending only on (Σ, g) such that for all $u \in H^1_g(\Sigma)$*

$$\int_\Sigma e^{\frac{4\pi(u-\bar{u})^2}{\int_\Sigma |\nabla_g u|^2 dV_g}} \leq C. \tag{2.1}$$

where $\bar{u} := \fint_\Sigma u dV_g$. *As a consequence one has for all* $u \in H^1_g(\Sigma)$

$$\log \int_\Sigma e^{(u-\bar{u})} dV_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla_g u|^2 dV_g + C. \tag{2.2}$$

Chen and Li [6] from this result showed that if e^u has integral controlled from below (in terms of $\int_\Sigma e^u dV_g$) into $(l+1)$ distinct regions of Σ, the constant $\frac{1}{16\pi}$ can be basically divided by $(l+1)$. Since we are interested in the behavior of the functional when $\rho \in (8\pi, 16\pi)$, it is sufficient to consider the case $l = 1$.

Lemma 2.3. *[6] Let* Ω_1, Ω_2 *be subsets of* Σ *satisfying* $\mathrm{dist}(\Omega_1, \Omega_2) \geq \delta_0$, *where* δ_0 *is a positive real number, and let* $\gamma_0 \in (0, \frac{1}{2})$. *Then, for any* $\tilde{\varepsilon} > 0$ *there exists a constant* $C = C(\tilde{\varepsilon}, \delta_0, \gamma_0)$ *such that* $\log \int_\Sigma e^{(u-\bar{u})} dV_g \leq C + \frac{1}{32\pi - \tilde{\varepsilon}} \int_\Sigma |\nabla_g u|^2 dV_g$ *for all the functions satisfying* $\frac{\int_{\Omega_i} e^u dV_g}{\int_\Sigma e^u dV_g} \geq \gamma_0$, *for* $i = 1, 2$.

Therefore if $\rho \in (8\pi, 16\pi)$ Lemma 2.3 implies that if "e^u" is spread in at least two regions then the functional I_ρ stays uniformly bounded from below. Qualitatively if I_ρ attains large negative values, $\frac{e^u}{\int_\Sigma e^u}$ has to concentrate at a point of Σ. Indeed, using the previous Lemma and a covering argument, Ding, Jost, Li and Wang obtained (see [10] or [12]) the following result.

Lemma 2.4. *Assuming* $\rho \in (8\pi, 16\pi)$, *the following property holds. For any* $\varepsilon > 0$ *and any* $r > 0$ *there exists a large positive constant* $L = L(\varepsilon, r)$ *such that for every* $u \in H^1_g(\Sigma)$ *with* $I_\rho(u) \leq -L$, *there exist a point* $p_u \in \Sigma$ *such that* $\int_{\Sigma \setminus B_r(p_u)} e^u dV_g / \int_\Sigma e^u dV_g < \varepsilon$.

By means of Lemma 2.4 it is possible to map continuously low sublevels of the Euler functional into Σ, roughly speaking associating to u the point p_u (see [12] for details); in the following we will denote this map $\Psi : I_\rho^{-L} \to \Sigma$. Viceversa, one can map Σ into arbitrarily low sublevels, associating to $x \in \Sigma$ the function $\varphi_{\lambda,x} := \tilde{\varphi}_{\lambda,x} - \overline{\tilde{\varphi}_{\lambda,x}}$, where $\tilde{\varphi}_{\lambda,x}(y) := \log \left(\frac{\lambda}{1 + \lambda^2 \, \mathrm{dist}^2(x,y)} \right)^2$ and λ is a sufficiently large positive real parameter. The composition of the former map with the latter can be taken to be homotopic to the identity on Σ, and hence the following result holds true.

Proposition 2.5. *[22] If* $\rho \in (8\pi, 16\pi)$, *there exists* $L > 0$ *such that* $\{I_\rho \leq -L\}$ *has the same homology as* Σ.

On the other hand in [22] Proposition 2.1 is used to prove that, since I_ρ stays uniformly bounded on the solutions of $(*)$ (again by the compactness result due to Li), it is possible to retract the whole Hilbert space $\bar{H}^1_g(\Sigma)$ onto a high sublevel $\{I_\rho \leq b\}$, $b \gg 0$. More precisely:

Proposition 2.6. *[22] If* $\rho \in (8\pi, 16\pi)$ *for some* $k \in \mathbb{N}$ *and if* b *is sufficiently large positive, the sublevel* $\{I_\rho \leq b\}$ *is a deformation retract of* X, *and hence it has the same homology of a point.*

Remark 2.7. *Let notice that, since* Σ *is not contractible, Proposition 2.5 together with Proposition 2.6 and Proposition 2.1 permits to derive an alternative proof of the general existence result due to Djadli.*

2.2 Notions in algebraic topology

In this subsection we recall some well known definitions in algebraic topology (see [15] for further details). Throughout, the sign $\simeq$ will refer to homotopy equivalences, while $\cong$ will refer to homeomorphisms between topological spaces or isomorphisms between rings.
Given a pair of spaces (X, A) we will denote by $H^q(X, A)$ the relative q-th cohomology group with coefficient in $\mathbb{Z}$ and by $\tilde{H}^q(X) := H^q(X, x_0)$ the reduced cohomology, where $x_0 \in X$. We recall that it is possible to endow the direct sum of the cohomology groups $\tilde{H}^*(X) = \bigoplus_q \tilde{H}^q(X)$ with a multiplication, namely the cup product $\bigcup : \tilde{H}^q(X) \times \tilde{H}^l(X) \to \tilde{H}^{q+l}(X)$. This multiplication turns

$\tilde{H}^*(X)$ into a ring; in fact it is naturally a $\mathbb{Z}$-graded ring with the integer q serving as degree and the cup product respects this grading. In de Rham cohomology the cup product of differential forms is also known as the wedge product.

Cup–length. A numerical invariant derived from the cohomology ring is the cup-length, which means the maximum number of graded elements of degree ≥ 1, which when multiplied give a non-zero result. For example the 2-torus has cup-length equal to 2 (to see it just think to the volume form in de Rham cohomology). For any space X we will denote by $\mathrm{CL}(X)$ its cup-length.
More generally, we define the cup length for a topological pair (X, Y).

$$\mathrm{CL}(X,Y) = \max\{\, l \in \mathbb{N} \,|\, \exists c_0 \in \tilde{H}^*(X,Y),\ c_1, \ldots, c_l \in \tilde{H}^*(X),\ \text{with degree}(c_i) > 0,\ i = 1, 2, \ldots, l, \\ \text{such that } c_0 \cup c_1 \cup \ldots \cup c_l \neq 0\}$$

Wedge sum. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \amalg Y$ obtained by identifying x_0 and y_0 to a single point. If $\{x_0\}$ (resp. $\{y_0\}$) is a closed subspace of X (resp. Y) that is a deformation retract of some neighbourhood in X (resp. Y), then $\tilde{H}^*(X \vee Y) \cong \tilde{H}^*(X) \bigoplus \tilde{H}^*(Y)$, provided that the wedge sum is formed at basepoints x_0 and y_0. In particular for any (c_1, d_1), $(c_2, d_2) \in \tilde{H}^*(X \vee Y)$, where c_1, $c_2 \in \tilde{H}^*(X)$ and d_1, $d_2 \in \tilde{H}^*(Y)$, the following identity holds:

$$(c_1, d_1) \cup (c_2, d_2) = (c_1 \cup c_2, d_1 \cup d_2). \tag{2.3}$$

Smash Product. Inside a product space $X \times Y$ there are copies of X and Y, namely $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $x_0 \in X$ and $y_0 \in Y$. These two copies of X and Y in $X \times Y$ intersect only at the point (x_0, y_0), so their union can be identified with the wedge sum $X \vee Y$. The smash product $X \bigwedge Y$ is then defined to be the quotient $X \times Y / X \vee Y$.

Proposition 2.8. *([15] page 223) Given two spaces X and Y, if $\tilde{H}^*(X)$ or $\tilde{H}^*(Y)$ is free and finitely generated in each dimension, the rings $\tilde{H}^*(X) \otimes_{\mathbb{Z}} \tilde{H}^*(Y)$ and $\tilde{H}^*(X \wedge Y)$ are isomorphic.*

Suspension. For a space X, the suspension SX is the quotient of $X \times [0, 1]$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. Let $\{x_0\}$ a point of X; inside the suspension SX we have the line segment $\{x_0\} \times [0, 1]$, and collapsing this to a point yields a space ΣX, called the reduced suspension of X. The reduced suspension is actually the same as the smash product $X \wedge S^1$.

Attaching spaces. A general definition of attaching one space to another is the following. We start with a space X and another space Y that we wish to attach to X by identifying the points in a subspace $A \subset Y$ with points of X. The data needed to do this is a map $f : A \to X$, for then we can form a quotient space of $X \amalg Y$ by identifying each point $a \in A$ with its image $f(a) \in X$. Let us denote this quotient space by $X \amalg_f Y$, the space X with Y attached along A via f.

Proposition 2.9. *([15] page 11) If A is a contractible subcomplex of a CW complex X, then the quotient map $X \to X/A$ is a homotopy equivalence.*

2.3 Relative Category

We recall the definition of Lusternik-Schnirelman relative category. We will see that this is a powerful tool in critical point theory to obtain multiplicity results.

Definition 2.10. Let X be a topological space and Y a closed subset of X. A closed subset A of X *is of the k-th category relative to Y* (we write $\mathrm{cat}_{X,Y} A = k$) if k is the least positive integer such that there exist $A_i \subset A$ closed and $h_i : A_i \times [0,1] \to X$, $i = 0, \dots, k$, such that

(i) $A = \cup_{i=0}^{k} A_i$,

(ii) $h_i(x,0) = x \quad \forall\, x \in A_i \;\; 0 \leq i \leq m$,

(iii) $\forall\, i \geq 1 \;\; \exists x_i \in X$ such that $h_i(x,1) = x_i$ with $h_i(A_i \times [0,1]) \cap Y = \emptyset$,

(iv) $h_0(x,1) \in Y \quad \forall x \in A_0$ and $h_0(y,t) = y \quad \forall y \in Y \quad \forall t \in [0,1]$.

If one such k does not exist, then we set $\mathrm{cat}_{X,Y} A = \infty$.

This notion is employed to find critical points of functionals in connection with the topology of their sublevels. In particular a Theorem in [14] can be adapted to our functional I_ρ (see [8]).

Theorem 2.11. *If $-\infty < a < b < +\infty$ and a, b are regular value for I_ρ, then*

$$\#\,\{\textit{critical points of } I_\rho \textit{ in } a \leq I_\rho \leq b\} \geq \mathrm{cat}_{I_\rho^b, I_\rho^a} I_\rho^b.$$

Besides, the relative category is related to the cup-length for the pair (X,Y) in the following way:

Theorem 2.12. *([2] Theorems 3.6 and 1.1)*

$$\mathrm{cat}_{X,Y} X \geq \mathrm{CL}(X,Y) + 1.$$

We have to point out that the definition of relative category given in [2] (see page 109) is slightly different from Definition 2.10; but since the requests in the definition above are stronger, Theorem 2.12 still holds for our notion of relative category.

3 Proof of Theorem 1.2

Before proving Theorem 1.2 we recall the previous result in [8] and we summarize its proof.

Theorem 3.1. *[8] If $\rho \in (8\pi, 4\pi^2)$ and $\Sigma = T$ is the torus, if the metric g is sufficiently close in $C^2(T; S^{2\times 2})$ to dx^2 and h is uniformly close to the constant 1, I_ρ admits a point of strict local minimum and at least two different saddle points.*

Let consider a new functional $\tilde{I}_\rho$ which coincides with I_ρ out of a small neighborhood of $\bar{u}$ and assumes large negative values near $\bar{u}$ (here we are exploiting the existence of a strict local minimum), then fix b and L conveniently, in particular such that $I_\rho^b = \tilde{I}_\rho^b$ and $\tilde{I}_\rho^{-L} = I_\rho^{-L} \amalg \{\text{neighb. of } 0\}$, I_ρ and $\tilde{I}_\rho$ have the same critical points of saddle type in $\tilde{I}_\rho^b \setminus \tilde{I}_\rho^{-L}$.

Let X denote the contractible cone over T and let ∂X be its boundary; they can be represented as $X = \frac{T\times[0,1]}{T\times\{0\}}$, $\partial X = \frac{T\times(\{0\}\cup\{1\})}{T\times\{0\}}$. To get the thesis it is sufficient to establish the following chain of inequalities:

$$\begin{aligned} \#\{\text{critical points of } \tilde{I}_\rho \text{ in } -L \leq \tilde{I}_\rho \leq b\} &\overset{1}{\geq} \mathrm{cat}_{\tilde{I}_\rho^b, \tilde{I}_\rho^{-L}} \tilde{I}_\rho^b \overset{2}{\geq} \mathrm{cat}_{\tilde{I}_\rho^b, \phi(\partial X)} \tilde{I}_\rho^b \\ &\overset{3}{\geq} \mathrm{cat}_{\tilde{I}_\rho^b, \phi(\partial X)} \phi(X) \overset{4}{\geq} \mathrm{cat}_{\phi(X), \phi(\partial X)} \phi(X) \\ &\overset{5}{\geq} \mathrm{cat}_{X, \partial X} X \overset{6}{\geq} 2, \end{aligned} \tag{3.1}$$

where ϕ is the homeomorphism on the image defined as follows:

$$\begin{aligned} \phi :\; X &\longrightarrow \bar{H}_g^1(T) \\ (x,t) &\longmapsto t\,\varphi_{\lambda,x}, \end{aligned}$$

with $\varphi_{\lambda,x}$ defined in Section 2.1 and L, λ, b suitable constants, clearly depending on ρ.

The first inequality follows immediately from Theorem 2.11, which as showed in [8] holds true also for $\tilde{I}_\rho$, while the third and the fifth can be easily derived from the properties of the relative category.
To prove 2, the map $\Psi : I_\rho^{-L} \to \Sigma$ has to be composed with the map which realizes the deformation of $\bar{H}_g^1(T)$ on $\tilde{I}_\rho^b$, in order to obtain a deformation retraction (in $\tilde{I}_\rho^b$) of $\tilde{I}_\rho^{-L}$ onto $\phi(\partial X)$; this is enough to conclude.
Moreover, just perturbing Ψ, it is possible to obtain a new continuous map $\tilde{\Psi} : \tilde{I}_\rho^{-L} \to \phi(\partial X)$ verifying $\tilde{\Psi}_{|\phi(\partial X)} = \mathrm{Id}_{|\phi(\partial X)}$. The key point is that applying again (2.1), one is able to extend $\tilde{\Psi}$ to $\tilde{I}_\rho^b \setminus B_R$, $R = R(\rho, b)$. Finally by means of $\tilde{\Psi}$, one can construct a new map $r : \tilde{I}_\rho^{-L} \to \phi(X)$ such that $r_{|\phi(X)} = \mathrm{Id}_{|\phi(X)}$ and $r^{-1}(\phi(\partial X)) = \phi(\partial X)$. Therefore, once again, category's properties allow to conclude.
At last the sixth inequality has been tackled using a direct topological argument.
Proof of Theorem 1.2 Our aim will be to improve the last inequality of (3.1), proving that $\mathrm{cat}_{X,\partial X} X \geq 3$. To do that we first apply Theorem 2.12, obtaining $\mathrm{cat}_{X,\partial X} X \geq \mathrm{CL}(X, \partial X) + 1$. Then to get the thesis it is sufficient to establish that $\mathrm{CL}(X, \partial X) \geq 2$. So we have to study the cohomology ring $H^*(X, \partial X)$. First of all, we observe that $H^*(X, \partial X)$ and $\tilde{H}^*(X/\partial X)$ are isomorphic as rings, since ∂X is a closed subspace of X that is a deformation retract of some neighborhood and because of the naturality of the cup product (see [15], Exercise 8 pag 205 and Proposition 3.10). Therefore, once we denote by A the quotient $X/\partial X$, we are interested in the understanding of $\tilde{H}^*(A)$.
We first claim that $S^1 \vee \Sigma T$, where ΣT is the reduced suspension of the torus.
Let consider $ST \amalg_f [0,1]$, where ST is the suspension of T, and $f : \partial([0,1]) \to ST$ is defined by $f(0) := T \times \{0\}/(T \times \{0\} \amalg T \times \{1\})$ and $f(1) := T \times \{1\}/(T \times \{0\} \amalg T \times \{1\})$.
Clearly since $f([0,1])$ is a contractible subcomplex of $ST \amalg_f [0,1]$, by Proposition 2.9 $ST \amalg_f [0,1]$ is homotopically equivalent to $ST \amalg_f [0,1]/f([0,1])$ which is nothing but A. On the other hand if we fix $y \in T$, then $\{y\} \times [0,1]$ is another contractible subcomplex of $ST \amalg_f [0,1]$ and then, again by Proposition 2.9, $ST \amalg_f [0,1]$ is also homotopically equivalent to $ST \amalg_f [0,1]/(\{y\} \times [0,1])$ which is exactly $S^1 \vee \Sigma T$.
This concludes the proof of the claim, which is presented schematically in Figure 1.

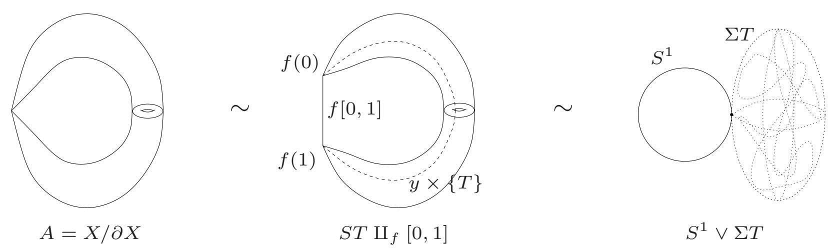

Figure 1: A chain of homotopy equivalences.

Taking advantage of the Claim we just proved and of the decomposition of the cohomology of a wedge sum, we have that $\tilde{H}^*(A) \cong \tilde{H}^*(S^1) \oplus \tilde{H}^*(\Sigma T)$. It is crucial to notice that thanks to formula (2.3) if we find a chain of length 2 in $\tilde{H}^*(\Sigma T)$ then we immediately get our thesis, i.e. $\tilde{H}^*(A) \geq 2$. Then applying Proposition 2.8 we obtain

$$\tilde{H}^*(\Sigma T; \mathbb{Z}) \cong \tilde{H}^*(T; \mathbb{Z}) \otimes_{\mathbb{Z}} \tilde{H}^*(S^1; \mathbb{Z}) \cong \tilde{H}^*(T; \mathbb{Z}).$$

Finally, collecting all these facts and recalling that $\mathrm{CL}(T) = 2$, we are able to conclude, indeed:

$$\mathrm{cat}_{X,\partial X} X \geq \mathrm{CL}(X, \partial X) + 1 = \mathrm{CL}(A) + 1 \geq \mathrm{CL}(\Sigma T) + 1 = \mathrm{CL}(T) + 1 = 3.$$

■

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