2D NAVIER-STOKES EQUATION: CONTINUITY PROPERTIES FOR CONTROLLABILITY.†

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Abstract. We study some continuity properties of the Navier-Stokes Equations on plane bounded C^3 domains. These properties are important for the study of controllability issues as we may see in the works done in [1] for periodic boundary conditions and, in [7] for Lions boundary conditions. Here we show that the same continuity properties hold in the C^3 connected bounded domains having a boundary with a finite number of connected components. Navier and no-slip boundary conditions are considered. As a corollary of the continuity properties it follows that the existence of a finite saturating set of vector fields implies controllability on observed component and $\mathbf{L}^2$-approximate controllability of the equation. These controllabilities are both obtained by means of finite dimensional controls taking values in the space spanned by the vector fields in the saturating set.

Contents

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1. Introduction

Following part of the work done in [1] and [7] we study continuity properties of the 2D Navier-Stokes (N-S) equation. We deal with the system

$$u_t + (u \cdot \nabla)u + \nabla p = \nu\Delta u + F(x_1, x_2) + v(t, x_1, x_2); \tag{1.1}$$

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega; \tag{1.2}$$

with either Navier boundary conditions

$$u \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega; \tag{1.3}$$

$$\nabla^\perp \cdot u = (\alpha - 2\kappa)u \cdot \mathbf{t} \quad \text{on} \quad \partial\Omega; \tag{1.4}$$

or no-slip boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

Where Ω is a bounded connected plane domain, its boundary $\partial\Omega$ is an 1-dimensional manifold of class C^3 with a finite number of connected components, $\nabla^\perp := \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}$, $\mathbf{n}$ is the unit (outward) normal to the boundary, $\mathbf{t}$ is the unit tangent to the boundary vector, α is a C^1 function in $\partial\Omega$ and κ is the curvature of $\partial\Omega$.

In the equation (1.1) u is the *velocity of the fluid "particle"*; p is the *pressure*; the only nonlinear term of the equation — $(u\cdot\nabla)u$ — is called the *inertial term*; $\nu\Delta u$ is called the *viscosity term*, $\nu > 0$ is the *coefficient of viscosity*; F is an *external force* and; the term v, in the study of controllability issues, is a control at our disposal. We are interested in the case where v is a degenerate forcing, i.e., v takes values in a finite-dimensional subspace $\mathbb{J}$ of $\mathbf{H}^2(\Omega)$.

We note that the Lions boundary conditions that we considered in [7] are a particular case of Navier boundary conditions taking α equal to 2κ. [1]

A natural way to study the N-S equation is to study its evolution on subspaces of Sobolev spaces. We shall denote by $L^2(\Omega)$ the space of Lebesgue measurable square integrable real functions defined on Ω and by $\mathbf{L}^2(\Omega)$ the product space $L^2(\Omega)^2$. Similarly
$H^1(\Omega) := \{f \in L^2(\Omega) \mid \frac{\partial f}{\partial x_j} \in L^2(\Omega),\, j = 1, 2\}$, $\mathbf{H}^1(\Omega) := H^1(\Omega)^2$ and,
$H^2(\Omega) := \{f \in H^1(\Omega) \mid \frac{\partial^2 f}{\partial x_j x_i} \in L^2(\Omega),\, i,\, j = 1, 2\}$, $\mathbf{H}^2(\Omega) := H^2(\Omega)^2$.

The goal of the present work is to study the continuity issues for the present cases analogously as we did in [7]:

In section 2 we present the spaces where we shall consider the evolution of the equation on. Such spaces depend on the boundary conditions.

In section 3 we deal with the existence of weak and strong solutions for (1.1)-(1.2), as well as its uniqueness and continuous dependence in the initial data. Both Navier and no-slip boundary conditions are considered.

In section 4 we deal with a variation of the N-S equation resulting of a change of variables and repeat the study of section 3 for that equation. This variation is important for the proof of continuity of the N-S equation in so-called relaxation metric that is an important property for the study of controllability issues.

In section 5 we show that we can realize any given curve $q \in W^{1,\infty}(t_i,\, t_f,\, \mathbb{J})$ in a suitable finite dimensional space $\mathbb{J}$ using a suitable control taking values in $\mathbb{J}$. We study the continuity dependence of the control in the map q. From q we find some U^J taking values in the space orthogonal to $\mathbb{J}$ and then, we find the control.

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[1] Note that in [2] α is considered to be positive and, in that case Lions boundary conditions is a particular case of Navier boundary conditions only if the domain Ω has a boundary with positive curvature, i.e., if Ω is convex. Here we do not impose, like in [5], any conditions on the sign of α.

In section 5.1 we deal with the L^2-norm of the derivative U_t^J of the curve U^J found in section 5. In particular we show that that norm does not depend in the derivative q_t.

Finally, in section 6 we present a sufficient condition for controllability on observed component and $\mathbf{L}^2$-approximate controllability of N-S equation.

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2. Linear Operators. The Spaces.

We denote by $\mathbb{N}$ the set of natural numbers: $\{0, 1, 2, \cdots\}$ and, $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

Definition 2.1. *An open and connected subset $\tilde{\Omega}$ in $\mathbb{R}^N$, is called a* **domain** *in $\mathbb{R}^N$.*

Definition 2.2. *Let $\tilde{\Omega}$ be a bounded set in $\mathbb{R}^N$, $N \in \mathbb{N}_0$ and $k \in \mathbb{N}$. We say that $\tilde{\Omega}$ is of class $\mathfrak{R}^{k,1}$ if locally (up to a change of coordinates), its boundary Γ is the graph of a C^k function f with $D^\alpha f$ Lipschitz for all $|\alpha| = k$ and; locally $\tilde{\Omega}$ is located on one side of Γ.*

We recall also the definitions of C^k and Lipschitz sets in $\mathbb{R}^N$:

Definition 2.3. *Let $\tilde{\Omega}$ be a set in $\mathbb{R}^N$. We say that $\tilde{\Omega}$ is of class $\boldsymbol{C^k}$ (resp.* **Lipschitz***) if locally, its boundary Γ is the graph of a C^k function (resp. a Lipschitz function) and; locally $\tilde{\Omega}$ is located on one side of Γ.*

In particular a bounded Lipschitz set is a $\mathfrak{R}^{0,1}$ set and; a $\mathfrak{R}^{k,1}$ set is a bounded C^k set.

2.1. Recollection of Auxiliary Material on Sobolev Spaces $\mathbf{H}^m$.

Let us fix a plane bounded domain $\Omega \subset \mathbb{R}^2$ and put $\Gamma := \partial\Omega$. We assume that

$$\begin{aligned} &\bullet\, \Omega \text{ is of class } C^3 \text{ and;} \\ &\bullet\, \Gamma \text{ has a finite number of connected components denoted} \\ &\quad \Gamma_1, \Gamma_2, \cdots, \Gamma_k \quad (k \geq 1). \end{aligned} \tag{2.1}$$

Recall that since $L^2(\Omega)$ is a Hilbert space for the scalar product

$$(u, v) := (u, v)_0 := \int_R uv\, dx, \tag{2.2}$$

then $\mathbf{L}^2(\Omega)$ is a Hilbert space for the product topology and the scalar product is

$$(u, v) := \int_R u \cdot v\, dx. \tag{2.3}$$ [2]

We note that for $u, v \in \mathbf{L}^2(\Omega)$

$$(u, v) = \int_R u \cdot v\, dx = \sum_{i=1}^{2} \int_R u_i v_i\, dx = (u_1, v_1) + (u_2, v_2).$$

[2] We will use the same notation for the scalar products and norms in $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$. It will be clear, in the statements, when functions are real or vector so, no ambiguity will appear. For the same reason below we use the same notation for the usual scalar products and norms of $H^m(\Omega)$ and $\mathbf{H}^m(\Omega)$, $\quad m \geq 1$.

The norms associated with the previous scalar products shall be represented by

$$|u| := (u,\, u)^{\frac{1}{2}}. \tag{2.4}$$

We note that

$$|u|^2 := |u_1|^2 + |u_2|^2.$$

Similarly, the Sobolev space $H^1(\Omega)$ is a Hilbert space for the scalar product

$$(u,\, v)_1 := (u,\, v) + \sum_{i=1}^{2}(\frac{\partial u}{\partial x_i},\, \frac{\partial v}{\partial x_i}) = (u,\, v) + (\nabla u,\, \nabla v). \tag{2.5}$$

then $\mathbf{H}^1(\Omega)$ is a Hilbert space for the product topology and the scalar product is

$$(u,\, v)_1 := \sum_{j=1}^{2}(u_j,\, v_j)_1 = (u,\, v) + \sum_{i=1}^{2}(\nabla u_i,\, \nabla v_i) \tag{2.6}$$

The norms associated with the previous scalar products shall be represented by

$$|u|_1 := (u,\, u)_1^{\frac{1}{2}}. \tag{2.7}$$

We note that

$$|u|_1^2 := |u_1|_1^2 + |u_2|_1^2.$$

Similarly we denote the usual scalar product in $H^m(\Omega)$ by

$$(u,\, v)_m := (u,\, v)_{m-1} + \sum_{|\alpha|=m}(\frac{\partial^{|\alpha|}u}{\partial x^\alpha},\, \frac{\partial^{|\alpha|}v}{\partial x^\alpha}). \tag{2.8}$$

where, as usual, $\alpha = (\alpha_1,\, \alpha_2) \in \mathbb{N}^2$, $|\alpha| = \alpha_1 + \alpha_2$ and ∂x^α stays for $\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}$. Then $\mathbf{H}^m(\Omega)$ is a Hilbert space for the product topology and the scalar product is

$$(u,\, v)_m := \sum_{j=1}^{2}(u_j,\, v_j)_m \tag{2.9}$$

The norms associated with the previous scalar products shall be represented by

$$|u|_m := (u,\, u)_m^{\frac{1}{2}}. \tag{2.10}$$

We note that

$$|u|_m^2 := |u_1|_m^2 + |u_2|_m^2.$$

2.2. Characterization of $H^m(\Omega)$.

Remark 1. *Our domain, satisfying* (2.1)*, may be simply connected or multi connected; in the latter case, as referred in [*[11]*, Appendix I], we can make it simply connected with a finite number of cuts:*

$$\begin{aligned} &\textit{There are } \Sigma_1,\, \Sigma_2,\, \cdots,\, \Sigma_N,\ N \quad (N \geq 0) \textit{ disjoint manifolds}\\ &\textit{of dimension } 1 \textit{ of class } C^2 \textit{ such that}\\ &\dot{\Omega} := \Omega \setminus \Sigma \quad (\Sigma = \cup_{i=1}^N) \textit{ is simply connected and Lipschitzian}\\ &(\textit{i.e., the } \Sigma_i\textit{'s are not tangent to } \Gamma.) \end{aligned} \tag{2.11}$$

In [[11], Appendix I] (Proposition 1.4) we have the following:

Proposition 2.1. *Assume that* Ω *satisfies* (2.1) [*and then* (2.11)] *and also that* Ω *is of class* $C^r \quad (r \geq m+1)$*. Then*

$$\mathbf{H}^m(\Omega) = \{u \in \mathbf{L}^2(\Omega) \mid \nabla \cdot u \in H^{m-1},\, \nabla^\perp \cdot u \in H^{m-1},\, u \cdot \mathbf{n} \in H^{m-\frac{1}{2}}(\Gamma)\}$$

and, there exists a constant $C_0 = C_0(m,\, \Omega)$ *such that*

$$|u|_m \leq C_0\Big(|u| + |\nabla \cdot u|_{m-1} + |\nabla^\perp \cdot u|_{m-1} + |u \cdot \mathbf{n}|_{H^{m-\frac{1}{2}}(\Gamma)}\Big)$$

for every $u \in \mathbf{H}^m(\Omega)$.

In particular, for our domain satisfying (2.1) we have

Corollary 2.2. *There is a constant* C_0 $(= max\{C_0(1, \Omega), C_0(2, \Omega)\})$ *such that*

$$|u|_1 \leq C_0\Big(|u| + |\nabla \cdot u| + |\nabla^\perp \cdot u| + |u \cdot \mathbf{n}|_{H^{1-\frac{1}{2}}(\Gamma)}\Big)$$

for every $u \in \mathbf{H}^1(\Omega)$ *and;*

$$|u|_2 \leq C_0\Big(|u| + |\nabla \cdot u|_1 + |\nabla^\perp \cdot u|_1 + |u \cdot \mathbf{n}|_{H^{2-\frac{1}{2}}(\Gamma)}\Big)$$

for every $u \in \mathbf{H}^2(\Omega)$.

We have the following trace theorem (see [6] section 2.5.4):

Lemma 2.3. *Let* $\tilde{\Omega} \subseteq \mathbb{R}^N$ *be a* $\mathfrak{R}^{k-1,1}$ *open set,* $p > 1$, $k \in \mathbb{N}_0$, $u \in W^{k,p}(\tilde{\Omega})$. *Then for* $l \leq k-1$ *there holds:*

$$\Big|\frac{\partial^l u}{\partial u^l}\Big|_{W^{k-l-\frac{1}{p}}(\partial\tilde{\Omega})} \leq C|u|_{W^{k,p}(\tilde{\Omega})}.$$

In particular we have that

Corollary 2.4. *For our fixed domain* $\Omega \subset \mathbb{R}^2$*, there exists a constant* C_0 *such that*

$$|u|_{H^{1-\frac{1}{2}}(\Gamma)} \leq C_0|u|_1$$

and

$$|v|_{H^{2-\frac{1}{2}}(\Gamma)} \leq C_0|v|_2.$$

every $u \in H^1(\Omega)$, $v \in H^2(\Omega)$.

Corollary 2.5. *The norms*

$$|u|_1 \text{ and } \Big(|u|^2 + |\nabla \cdot u|^2 + |\nabla^\perp \cdot u|^2 + |u \cdot \mathbf{n}|^2_{H^{1-\frac{1}{2}}}\Big)^{\frac{1}{2}}$$

are equivalent norms in $\mathbf{H}^1(\Omega)$ *and, the norms*

$$|u|_2 \text{ and } \Big(|u|^2 + |\nabla \cdot u|_1^2 + |\nabla^\perp \cdot u|_1^2 + |u \cdot \mathbf{n}|^2_{H^{2-\frac{1}{2}}}\Big)^{\frac{1}{2}}$$

are equivalent norms in $\mathbf{H}^2(\Omega)$.

Proof. We prove the equivalence of the first pair of norms, for the second pair the proof is similar.
We have that all the terms $|u|$, $|\nabla \cdot u|$, $|\nabla^\perp \cdot u|$ and $|u \cdot \mathbf{n}|_{H^{1-\frac{1}{2}}(\Gamma)}$ are less than $C_2|u|_1$ for a suitable constant C_2 (that can be taken the same for all terms). Indeed by definition of the norm $|u|_1$ all the three summands $|u|$, $|\nabla \cdot u|$ and $|\nabla^\perp \cdot u|$ are less than $2|u|_1$, the remaining summand $|u \cdot \mathbf{n}|_{H^{1-\frac{1}{2}}(\Gamma)}$, by Corollary 2.4, is less than $C_1|u|_1$ for some constant C_1. In fact, since Ω is a C^3 domain, $\mathbf{n} \in C^2(\Gamma)$ and then, $u \cdot \mathbf{n} \in H^{1-\frac{1}{2}}(\Gamma)$. Moreover

$$\begin{aligned}|u \cdot \mathbf{n}|_{H^{1-\frac{1}{2}}(\Gamma)} = |u_1 n_1 + u_2 n_2|_{H^{1-\frac{1}{2}}(\Gamma)} &\leq |n_1||u_1|_{H^{1-\frac{1}{2}}(\Gamma)} + |n_2||u_2|_{H^{1-\frac{1}{2}}(\Gamma)} \\ &\leq |u_1|_{H^{1-\frac{1}{2}}(\Gamma)} + |u_2|_{H^{1-\frac{1}{2}}(\Gamma)}\end{aligned}$$

because both $|n_1|$, $|n_2|$ are not bigger than 1. Hence, by Corollary 2.4 we have

$$\begin{aligned}|u \cdot \mathbf{n}|_{H^{1-\frac{1}{2}}(\Gamma)} &\leq C_0\big(|u_1|_1 + |u_2|_1\big) \leq C_0\Big(2\big(|u_1|_1^2 + |u_2|_1^2\big)\Big)^{\frac{1}{2}} \\ &\leq C_1|u|_1.\end{aligned}$$

Set, for example, $C_2 = C_1 + 2$.
By Corollary 2.2 the norms $|u|_1$ and $|u| + |\nabla\cdot u| + |\nabla^\perp\cdot u| + |u\cdot\mathbf{n}|_{H^{1-\frac{1}{2}}(\Gamma)}$ are equivalent norms in $\mathbf{H}^1(\Omega)$. The Corollary follows from

$$\begin{aligned}\Big(|u|^2 + |\nabla\cdot u|^2 + |\nabla^\perp\cdot u|^2 + |u\cdot\mathbf{n}|^2_{H^{1-\frac{1}{2}}}\Big)\\ \le \Big(|u| + |\nabla\cdot u| + |\nabla^\perp\cdot u| + |u\cdot\mathbf{n}|_{H^{1-\frac{1}{2}}(\Gamma)}\Big)^2\\ \le 2^3\Big(|u|^2 + |\nabla\cdot u|^2 + |\nabla^\perp\cdot u|^2 + |u\cdot\mathbf{n}|^2_{H^{1-\frac{1}{2}}}\Big).\end{aligned}$$

□

Remark 2. *In the simply connected case [looking at* [11]*, Appendix I, Equation (1.28)] we have the inequality*

$$|u|_m \le C_1\Big(|\nabla\cdot u|_{m-1} + |\nabla^\perp\cdot u|_{m-1} + |u\cdot\mathbf{n}|_{H^{m-\frac{1}{2}}(\Gamma)}\Big).$$

and then we conclude the equivalence of the norms

$$|u|_1 \text{ and } \Big(|\nabla\cdot u|^2 + |\nabla^\perp\cdot u|^2 + |u\cdot\mathbf{n}|^2_{H^{1-\frac{1}{2}}}\Big)^{\frac{1}{2}}$$

in $\mathbf{H}^1(\Omega)$ *and, of the norms*

$$|u|_2 \text{ and } \Big(|\nabla\cdot u|_1^2 + |\nabla^\perp\cdot u|_1^2 + |u\cdot\mathbf{n}|^2_{H^{2-\frac{1}{2}}}\Big)^{\frac{1}{2}}$$

in $\mathbf{H}^2(\Omega)$.

2.3. Linear Operators. The Spaces H, V and $D(A)$.

2.3.1. *Navier Boundary Conditions.* We follow subsection II.2.1 of [10], applying to our case: Put

$$\begin{aligned} H &:= \{u\in\mathbf{L}^2(\Omega) \mid \nabla\cdot u = 0 \;\;\&\;\; u\cdot\mathbf{n} = 0 \text{ on } \Gamma\};\\ &\text{and}\\ V &:= \{u\in\mathbf{H}^1(\Omega) \mid \nabla\cdot u = 0 \;\;\&\;\; u\cdot\mathbf{n} = 0 \text{ on } \Gamma\}.\end{aligned}\tag{2.12}$$

In H we consider the scalar product induced by $\mathbf{L}^2(\Omega)$ and respective norm.

Since Γ is of class C^3 we have that both α and κ are of class $C^1(\Gamma)$. We may extend both α and κ to a $C^1(\tilde\Omega)$ function in a neighborhood $\tilde\Omega$ of $\overline{\Omega}$. We may also extend the normal $\mathbf{n}$ to a $C^2(\tilde\Omega)$ function. We fix one extension of α, one of κ and one of $\mathbf{n}$ that we will denote again by α, κ and $\mathbf{n}$. Note that in this way the "tangent" $\mathbf{t} := \begin{pmatrix}-\mathbf{n}_2\\ \mathbf{n}_1\end{pmatrix} \in C^2(\tilde\Omega)$ is an extension of the tangent $\mathbf{t}\in C^2(\Gamma)$.

Put $\beta := \alpha - 2\kappa$ and define on V the bilinear form

$$((u,\, v)) := (\nabla^\perp\cdot u,\, \nabla^\perp\cdot v) + D_0(u,\, v) - (\beta u\cdot\mathbf{t},\, \nabla^\perp\cdot v) - (\beta v\cdot\mathbf{t},\, \nabla^\perp\cdot u).$$

Now we show that this bilinear form is a scalar product on V for big enough D_0:

Let D_0 satisfy

$$(D_0 - \frac{1}{2})|u|^2 \ge 2|\beta u\cdot\mathbf{t}|^2, \quad \text{for all} \quad u\in H \quad [3] \tag{2.13}$$

The symmetry and bilinearity of $((\cdot,\,\cdot))$ is clear. For $((u,\, u))$ we find

$$((u,\, u)) = |\nabla^\perp\cdot u|^2 + D_0|u|^2 - 2(\beta u\cdot\mathbf{t},\, \nabla^\perp\cdot u) \tag{2.14}$$

[3]Note that $2|\beta u\cdot\mathbf{t}|^2 \le C|u|^2$ where C depends only in the $C^0(\Omega)$-norm of the previously fixed functions β and $\mathbf{t}$. Then set $D_0 = C + \frac{1}{2}$.

where the last term satisfy

$$2(\beta u\cdot \mathbf{t},\, \nabla^\perp\cdot u) \le 2|\beta u\cdot \mathbf{t}||\nabla^\perp\cdot u| \le 2|\beta u\cdot \mathbf{t}|^2 + \frac{1}{2}|\nabla^\perp\cdot u|^2.$$

Then from (2.13) and (2.14):

$$\begin{aligned}((u,\,u)) &\ge |\nabla^\perp\cdot u|^2 + \frac{1}{2}|u|^2 + (D_0 - \frac{1}{2})|u|^2 - 2|\beta u\cdot \mathbf{t}|^2 - \frac{1}{2}|\nabla^\perp\cdot u|^2\\ &\ge \frac{1}{2}\big(|\nabla^\perp\cdot u|^2 + |u|^2\big).\end{aligned}$$

On the other side

$$\begin{aligned}((u,\,u)) &\le |\nabla^\perp\cdot u|^2 + \frac{1}{2}|u|^2 + (D_0 - \frac{1}{2})|u|^2 + 2|\beta u\cdot \mathbf{t}|^2 + \frac{1}{2}|\nabla^\perp\cdot u|^2\\ &\le \frac{3}{2}|\nabla^\perp\cdot u|^2 + \frac{1}{2}|u|^2 + 2(D_0 - \frac{1}{2})|u|^2\\ &\le (2D_0+1)(|\nabla^\perp\cdot u|^2 + |u|^2).\end{aligned}$$

Hence

$$\frac{1}{2}\big(|\nabla^\perp\cdot u|^2 + |u|^2\big) \le ((u,\,u)) \le (2D_0+1)(|\nabla^\perp\cdot u|^2 + |u|^2) \tag{2.15}$$

from which, using Corollary 2.5, we conclude that $((u,\,u))$ is a scalar product on V and, its associated norm

$$\|\cdot\| := ((\cdot,\,\cdot))^{\frac{1}{2}}$$

is equivalent to the norm induced in V by the usual norm of $\mathbf{H}^1(\Omega)$ defined in (2.7).

From now we consider V endowed with the scalar product $((\cdot,\,\cdot))$ and respective norm. Since H and V are closed subspaces of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$ respectively, they are Hilbert spaces.

We denote by A the canonical isomorphism between V and V' associated to $((\cdot,\,\cdot))$, i.e., $A : V \to V'$

$$((u,\,v)) =: < Au,\, v >_{V',V}\ .$$

The inclusions (identifying H with its dual)

$$V \subset H \subset V'$$

are both continuous and dense. The first one is also compact. For $v \in V$ and $u \in H$ we have $< u,\, v >_{V',V} = (u,v)$.
We may define the domain $D(A)$ of the operator A in H as

$$D(A) := \{u \in V \mid\ Au \in H\}$$

and consider A as an unbounded linear operator in H with domain $D(A)$. The operator A is strictly positive [$(Au,\,u) = \|u\|^2 > 0$, for all $u \in D(A)\setminus\{0\}$].
We endow $D(A)$ with the scalar product $(u,\,v)_{[2]} := (Au,\,Av)$ and respective norm $|u|_{[2]} = |Au|$. A turns out to be an isomorphism of $D(A)$ onto H.

Since the injection $V \to H$ is compact the operator A^{-1} may be considered as a compact operator in H. We infer that there exists a complete orthonormal basis

$$\mathcal{W} := \{W_j \mid\ j \in \mathbb{N}_0\}$$

where $\mathbb{N}_0 := \mathbb{N}\setminus\{0\}$.

$$\begin{aligned}A^{-1}W_j = \mu_j W_j \quad j \in \mathbb{N}_0;\\ \mu_j \text{ a decreasing sequence;}\\ \mu_j \to 0 \quad \text{as} \quad j\to\infty\end{aligned}$$

It is clear that for each $j \in \mathbb{N}_0$ we have $W_j \in D(A)$ and, setting $k_j = \mu_j^{-1}$ we obtain

$$AW_j = k_j W_j \quad j \in \mathbb{N}_0$$
$$0 < k_1 \leq k_2 \leq \cdots \leq k_j \leq \cdots$$
$$k_j \to \infty \quad \text{as} \quad j \to \infty.$$

The family $\mathcal{W}$ is orthogonal in H, V and $D(A)$:

$$(W_j,\, W_i) = \delta_{ji};$$
$$((W_j,\, W_i)) =< AW_j,\, W_i >= (AW_j,\, W_i) = k_j \delta_{ji};$$
$$(W_j,\, W_i)_{[2]} = (AW_j,\, AW_i) = k_j k_i (W_j,\, W_i) = k_j^2 \delta_{ji}$$

We also have

$$((W_j,\, W_i))_{V'} = ((W_j,\, W_i))_* = ((A^{-1} W_j,\, W_i)) = k_j^{-1} \delta_{ji}.$$

We may also define the powers A^s of A for $s \in \mathbb{R}$. For $s > 0$, A^s is an unbounded self adjoint operator in H with dense domain $D(A^s) \subseteq H$. A^s is strictly positive and $D(A^s)$ is endowed with the scalar product $(u,\, v)_{D(A^s)} := (A^s u,\, A^s v)$ and norm $|u|_{D(A^s)} := |A^s u|$. A^s is an isomorphism of $D(A^s)$ onto H.

For $s = 1$ we recover $D(A)$ and for $s = \frac{1}{2}$ we have $D(A^{\frac{1}{2}}) = V$.

For $s = 0$ we put $A^0 = I$, $D(A^0) = H = H'$. For $s > 0$ we put

$$D(A^{-s}) := \text{ dual of } D(A^s).$$

In this case A^s can be extended as an isomorphism from H onto $D(A^{-s})$. We endow $D(A^{-s})$ with the scalar product $(u,\, v)_{D(A^{-s})} := (A^{-s} u,\, A^{-s} v)$ and norm $|u|_{D(A^{-s})} := |A^{-s} u|$.

For every $s_1 > s_0$ the inclusion

$$D(A^{s_1}) \subset D(A^{s_0})$$

is dense continuous and compact. The map $A^{s_1 - s_0}$ is an isomorphism of $D(A^{s_1})$ onto $D(A^{s_0})$.

For $s \geq 0$ we have the characterizations:

$$(u,\, v)_{D(A^s)} = \sum_{j=1}^{+\infty} k_j^{2s} (u,\, W_j)(v,\, W_j)$$

$$|u|^2_{D(A^s)} = \sum_{j=1}^{+\infty} k_j^{2s} (u,\, W_j)^2$$

and then

$$D(A^s) = \{u \in H \mid \sum_{j=1}^{+\infty} k_j^{2s} (u,\, W_j)^2 < +\infty\}.$$

For $s < 0$, $D(A^s)$ is the completion of H for the norm

$$\Big(\sum_{j=1}^{+\infty} k_j^{2s} (u,\, W_j)^2 \Big)^{\frac{1}{2}}.$$

For $s \in \mathbb{R}$ if

$$u = \sum_{j=1}^{+\infty} (u,\, W_j) W_j \in D(A^s),$$

then

$$A^s u = \sum_{j=1}^{+\infty} k_j^s (u,\, W_j) W_j.$$

Characterization of $D(A)$. In the study of continuity issues, for the norm of the space $D(A)$, is important to have suitable properties. Here we arrive to the following characterization:

$$D(A) = D_A := \{u \in \mathbf{H}^2(\Omega) \mid \nabla \cdot u = 0,\ (\nabla^\perp \cdot u = \beta u \cdot \mathbf{t} \wedge u \cdot \mathbf{n} = 0 \text{ on } \Gamma)\} \tag{2.16}$$

and prove that the norm $|u|_{[2]}$ in $D(A)$, defined above by $|u|_{[2]} = |Au|$ is equivalent to the norm induced by the usual norm $|u|_2$ of $\mathbf{H}^2(\Omega)$.

Define the operator

$$\begin{aligned} L : V &\to H \\ u &\mapsto Lu := P^\nabla[(\nabla^\perp \cdot u)\beta \mathbf{t}]. \end{aligned}$$

So $(Lu,\, v) = ((\nabla^\perp \cdot u)\beta\mathbf{t},\, v) = (\nabla^\perp \cdot u,\, \beta v \cdot \mathbf{t})$, for all $v \in H$. Note that $v \mapsto (Lu,\, v)$ is linear and continuous as v varies in H.

For every test function

$$\varphi \in (\mathcal{D}(\Omega))^2;\ \mathcal{D}(\Omega) := \{u \in C^\infty(\Omega) \mid \ supp(u) \subset \Omega\},$$ [4]

we write $\varphi = P^\nabla\varphi + \nabla\psi$, where P^∇ stays for the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto H and, $\nabla\phi$ belongs to the space

$$H^\perp = \{\nabla u \mid \ u \in H^1(\Omega)\} \tag{2.17}$$ [5]

orthogonal to H (in $\mathbf{L}^2(\Omega)$).

It is known [see for example in ([11], Section I.1.4, proof of Theorem 1.5)] that ϕ is the solution of the Neumann problem

$$\begin{aligned} \Delta\phi &= \nabla \cdot \varphi \\ \frac{\partial\phi}{\partial\mathbf{n}} &= \varphi \cdot \mathbf{n}; \end{aligned}$$

thus for $P^\nabla\varphi$ we have:

$$P^\nabla\varphi \in \mathbf{L}^2(\Omega); \qquad \nabla^\perp \cdot P^\nabla\varphi = \nabla^\perp \cdot \varphi \in L^2(\Omega);$$

$$\nabla \cdot P^\nabla\varphi = \nabla \cdot \varphi - \Delta\phi = 0 \in L^2(\Omega); \qquad P^\nabla\varphi \cdot \mathbf{n} = \varphi \cdot \mathbf{n} - \frac{\partial\phi}{\partial\mathbf{n}} = 0 \in H^{1-\frac{1}{2}}(\Gamma);$$

from which, using Proposition 2.1, we obtain $P^\nabla\varphi \in V$.

Let $u \in D(A)$ and compute

$$\begin{aligned} &\langle -\Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu,\, \varphi\rangle \\ =&(\nabla^\perp \cdot u,\, \nabla^\perp \cdot \varphi) + D_0(u,\, \varphi) - (\beta u \cdot \mathbf{t},\, \nabla^\perp \cdot \varphi) + (Lu,\, \varphi) \\ =&(\nabla^\perp \cdot u,\, \nabla^\perp \cdot P^\nabla\varphi) + D_0(u,\, P^\nabla\varphi) - (\beta u \cdot \mathbf{t},\, \nabla^\perp \cdot P^\nabla\varphi) + (Lu,\, P^\nabla\varphi) \\ =&(Au,\, P^\nabla\varphi) = (Au,\, \varphi); \end{aligned} \tag{2.18}$$

where $\langle\cdot,\,\cdot\rangle$ stays for the scalar product in the duality between $((\mathcal{D}(\Omega))^2)' = (\mathcal{D}'(\Omega))^2$ and $(\mathcal{D}(\Omega))^2$ and $(\cdot,\,\cdot)$ stays for the scalar product in $\mathbf{L}^2(\Omega)$. Note that $\Delta u = \nabla(\nabla \cdot u) + \nabla^\perp(\nabla^\perp \cdot u) = \nabla^\perp(\nabla^\perp \cdot u)$ because, $\nabla \cdot u = 0$ for $u \in V$.

Therefore we conclude that $-\Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu$ and $Au \in H \subset \mathbf{L}^2(\Omega)$ are the same distribution in $(\mathcal{D}'(\Omega))^2$:

$$-\Delta u + D_0 u + \nabla^\perp(\beta u \cdot \mathbf{t}) - Lu = Au \in \mathbf{L}^2(\Omega); \quad u \in D(A). \tag{2.19}$$

[4] Here $supp(u)$ stays for the support of u defined by: $supp(u) :=$ closure of $\{x \in \Omega \mid \ u(x) \neq 0\}$.
[5] See [11], Section I.1.4.

From $-\Delta u + D_0 u + \nabla^\perp(\beta u\cdot \mathbf{t}) - Lu \in \mathbf{L}^2(\Omega)$ and $u \in D(A) \subset V$ we obtain:

$$u \in \mathbf{L}^2(\Omega);\quad \nabla^\perp\cdot u \in L^2(\Omega);\quad \nabla\cdot u = 0 \in H^1(\Omega);$$
$$(u\cdot\mathbf{n})\,|_\Gamma = 0 \in H^{2-\frac{1}{2}}(\Gamma);\quad \nabla(\nabla^\perp\cdot u) = \begin{pmatrix}\Delta u_2\\ -\Delta u_1\end{pmatrix} \in \mathbf{L}^2(\Omega).$$ [6]

Hence from Proposition 2.1 we have $u \in \mathbf{H}^2(\Omega)$. In particular $\nabla^\perp\cdot u \in H^1(\Omega)$ and so, the trace $(\nabla^\perp\cdot u)\mathbf{t}$ belong to $(H^{1-\frac{1}{2}}(\Gamma))^2$. [7]

Now, for $v \in \mathbf{H}^1(\Omega)$, considering that $P^\nabla v$ and $\nabla\phi$ are the orthogonal projections of v onto H and $H^\perp$ respectively; the Green formula gives

$$\begin{aligned}
&(-\Delta u + D_0 u + \nabla^\perp(\beta u\cdot\mathbf{t}) - Lu,\, v)\\
=&(\nabla^\perp\cdot u,\, \nabla^\perp\cdot v) + D_0(u,\, v) - (\beta u\cdot\mathbf{t},\, \nabla^\perp\cdot v) - (Lu,\, v)\\
&\qquad + \int_\Omega \nabla^\perp\cdot((-\nabla^\perp\cdot u + \beta u\cdot\mathbf{t})v)\,dx\\
=&(\nabla^\perp\cdot u,\, \nabla^\perp\cdot P^\nabla v) + D_0(u,\, P^\nabla v) - (\beta u\cdot\mathbf{t},\, \nabla^\perp\cdot P^\nabla v) - (Lu,\, P^\nabla v)\\
&\qquad + \int_\Gamma (-\nabla^\perp\cdot u + \beta u\cdot\mathbf{t})v\cdot\mathbf{t}\,d\Gamma\\
=&(Au,\, P^\nabla v) + \int_\Gamma (-\nabla^\perp\cdot u + \beta u\cdot\mathbf{t})v\cdot\mathbf{t}\,d\Gamma.
\end{aligned}\tag{2.20}$$

Note that since ϕ is the solution of the Neumann problem

$$\Delta\phi = \nabla\cdot v$$
$$\frac{\partial\phi}{\partial\mathbf{n}} = v\cdot\mathbf{n};$$

we have:

$$P^\nabla v \in \mathbf{L}^2(\Omega);\qquad \nabla^\perp\cdot P^\nabla v = \nabla^\perp\cdot v \in L^2(\Omega);$$
$$\nabla\cdot P^\nabla v = \nabla\cdot v - \Delta\phi = 0 \in L^2(\Omega);\qquad P^\nabla v\cdot\mathbf{n} = v\cdot\mathbf{n} - \frac{\partial\phi}{\partial\mathbf{n}} = 0 \in H^{1-\frac{1}{2}}(\Gamma);$$

from which, using Proposition 2.1, we obtain $P^\nabla v \in V$.
Since

$$(Au,\, v) = (Au,\, P^\nabla v)$$

by (2.19) and (2.20) we conclude that

$$\int_\Gamma (\nabla^\perp\cdot u - \beta u\cdot\mathbf{t})v\cdot\mathbf{t}\,d\Gamma = 0,\qquad \forall v \in \mathbf{H}^1(\Omega),$$

in particular for $v = (\nabla^\perp\cdot u - \beta u\cdot\mathbf{t})\mathbf{t}$ we obtain $(\nabla^\perp\cdot u - \beta u\cdot\mathbf{t})^2 = 0$ on Γ.

Up to now we have concluded that

$$D(A) \subseteq D_A;$$

next we prove the reverse inclusion and so we have the characterization (2.16) for $D(A)$.

First we note that for $b \in D_A$ and for $v \in V$ we find

$$(-\Delta b + D_0 b + \nabla^\perp(\beta b\cdot\mathbf{t}) - Lb,\, v) =< Ab,\, v >_{V',V}$$

then, to prove that $D(A) \supseteq D_A$, is enough to prove that $-\Delta b + D_0 b + \nabla^\perp(\beta b\cdot\mathbf{t}) - Lb \in H$ because, in that case we have necessarily $-\Delta b + D_0 b + \nabla^\perp(\beta b\cdot\mathbf{t}) - Lb = Ab$.

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[6] From $-\Delta u + D_0 u + \nabla^\perp(\beta u\cdot\mathbf{t}) - Lu \in \mathbf{L}^2(\Omega)$ we obtain $\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix} \in \mathbf{L}^2(\Omega)$.

[7] Note that $\mathbf{t} \in (C^2(\Gamma))$

On Ω we have

$$\nabla\cdot(-\Delta b+D_0b+\nabla^\perp(\beta b\cdot\mathbf{t})-Lb)=0$$

and, on Γ we have

$$\mathbf{n}\cdot(-\Delta b+D_0b+\nabla^\perp(\beta b\cdot\mathbf{t})-Lb)=(\mathbf{n}\cdot\nabla^\perp)(-\nabla^\perp\cdot u+\beta u\cdot\mathbf{t})=0$$

because, since $-\nabla^\perp\cdot u+\beta u\cdot\mathbf{t}$ is constant on Γ, we have that $\nabla^\perp(-\nabla^\perp\cdot u+\beta u\cdot\mathbf{t})$ is tangent to Γ. [8]

Therefore $-\Delta b+D_0b+\nabla^\perp(\beta b\cdot\mathbf{t})-Lb\in H$ and, then $b\in D(A)$.

Remark 3. *Since* $\nabla^\perp(\nabla^\perp\cdot b)\in\nabla^\perp(H^1(\Omega))$ *and*

$$\nabla^\perp(H^1(\Omega))=\{u\in\mathbf{L}^2(\Omega)\mid\nabla\cdot u=0,\ \int_{\Gamma_i}u\cdot\mathbf{n}\,d\Gamma=0,\ i=1,\,\dots,\,k\}$$

(see ([11], Appendix I, Prop. 1.3 and Remark 1.5)), where Γ_i *are the connected components of* Γ *(see (2.1)), we may conclude immediately that* $\int_{\Gamma_i}[-\Delta u+\nabla^\perp(\beta u\cdot\mathbf{t})]\cdot\mathbf{n}\,d\Gamma=0$ *for all* $i=1,\,2,\,\dots,\,k$ *but, to conclude that* $[-\Delta u+\nabla^\perp(\beta u\cdot\mathbf{t})]\cdot\mathbf{n}=0$ *on* Γ *we have needed the fact that* $\nabla^\perp\cdot u$ *is constant on* Γ.

Remark 4. *Defining*

$$\mathcal{D}_1(\Omega):=\{\varphi\in C^\infty(\overline{\Omega})\mid\ \nabla\cdot\varphi=0,\,(\varphi\cdot\mathbf{n}=0\wedge\nabla^\perp\cdot\varphi=\beta u\cdot\mathbf{t}\ \mathit{on}\,\Gamma)\};$$

we have the following characterizations:

$$\begin{aligned} H=&\ \textit{closure of}\,\mathcal{D}_1(\Omega)\,\textit{in}\,\mathbf{L}^2(\Omega);\\ V=&\ \textit{closure of}\,\mathcal{D}_1(\Omega)\,\textit{in}\,\mathbf{H}^1(\Omega);\\ D(A)=&\ \textit{closure of}\,\mathcal{D}_1(\Omega)\,\textit{in}\,\mathbf{H}^2(\Omega).\end{aligned}$$

Indeed it is known that H *is the closure of* $\mathcal{V}:=\{\varphi\in\mathcal{D}(\Omega)\mid\ \nabla\cdot\varphi=0\}$ *in* $\mathbf{L}^2(\Omega)$ [9] *and, from* $\mathcal{V}\subset\mathcal{D}_1(\Omega)\subset H$, *follows that* H *is the closure of* $\mathcal{D}_1(\Omega)$ *in* $\mathbf{L}^2(\Omega)$. *It is also clear that* $D(A)$ *is the closure of* $\mathcal{D}_1(\Omega)$ *in* $\mathbf{H}^2(\Omega)$ *and then, by the density, and continuity of the inclusion, of* $D(A)$ *into* V *we can conclude the density of* $\mathcal{D}_1(\Omega)$ *in* V.

Now, for $u\in D(A)$, we compute $|Au|$ to find some useful properties of the norm of $D(A)$:

$$\begin{aligned}(Au,\,Au)&=(-\Delta u+D_0u+\nabla^\perp(\beta u\cdot\mathbf{t})-Lu,\,Au)\\
&=(-\Delta u,\,Au)+(D_0u,\,Au)+(\nabla^\perp(\beta u\cdot\mathbf{t})-Lu,\,Au)\\
&=|\Delta u|^2-(\Delta u,\,D_0u)-(\Delta u,\,\nabla^\perp(\beta u\cdot\mathbf{t}))+(\Delta u,\,Lu)+D_0\|u\|^2\\
&\quad+(\nabla^\perp(\beta u\cdot\mathbf{t}),\,-\Delta u)-(Lu,\,-\Delta u)+D_0(\nabla^\perp(\beta u\cdot\mathbf{t}),\,u)-D_0(Lu,\,u)\\
&\quad+|\nabla^\perp(\beta u\cdot\mathbf{t})-Lu|^2\\
&=|\Delta u|^2+D_0\|u\|^2+|\nabla^\perp(\beta u\cdot\mathbf{t})-Lu|^2-2(\Delta u,\,\nabla^\perp(\beta u\cdot\mathbf{t})-Lu)\\
&\quad+D_0[(-\Delta u,\,u)+(\nabla^\perp(\beta u\cdot\mathbf{t}),\,u)-(Lu,\,u)]\\
&=|\Delta u|^2+D_0\|u\|^2+|\nabla^\perp(\beta u\cdot\mathbf{t})-Lu|^2-2(\Delta u,\,\nabla^\perp(\beta u\cdot\mathbf{t})-Lu)\\
&\quad+D_0\Big[|\nabla^\perp\cdot u|^2-\int_\Gamma(\nabla^\perp\cdot u)u\cdot\mathbf{t}\,d\Gamma-(\nabla^\perp\cdot u,\,\beta u\cdot\mathbf{t})\\
&\quad+\int_\Gamma(\beta u\cdot\mathbf{t})u\cdot\mathbf{t}\,d\Gamma-(Lu,\,u)\Big];\end{aligned}$$

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[8] Indeed it is well known that ∇g is normal to the curve γ if g is constant on γ; on the other side $\nabla^\perp g$ is orthogonal to ∇g.

[9] See [11], Section 1.1.4.

i.e.,

$$\begin{aligned}|Au|^2 = &|\Delta u|^2 + D_0\|u\|^2 + |\nabla^\perp(\beta u\cdot\mathbf{t}) - Lu|^2 + D_0|\nabla^\perp\cdot u|^2\\ &- 2(\Delta u,\, \nabla^\perp(\beta u\cdot\mathbf{t}) - Lu) - 2D_0(Lu,\, u).\end{aligned}\tag{2.21}$$

From

$$2(\Delta u,\, \nabla^\perp(\beta u\cdot\mathbf{t}) - Lu) \le 2|\Delta u||\nabla^\perp(\beta u\cdot\mathbf{t}) - Lu| \le \frac{1}{2}|\Delta u|^2 + 2|\nabla^\perp(\beta u\cdot\mathbf{t}) - Lu|^2$$

we obtain

$$\begin{aligned}|Au|^2 \ge &\frac{1}{2}|\Delta u|^2 + D_0\|u\|^2 + D_0|\nabla^\perp\cdot u|^2\\ &- |\nabla^\perp(\beta u\cdot\mathbf{t}) - Lu|^2 - 2D_0(Lu,\, u).\end{aligned}\tag{2.22}$$

Now for D_0 big enough, namely if D_0 satisfies both (2.13) and

$$\frac{D_0}{2}|u|^2 \ge \frac{9}{4}|\beta u\cdot\mathbf{t}|^2, \quad \text{for all} \quad u\in V\tag{2.23}$$

we have

$$\begin{aligned}\frac{D_0}{2}\|u\|^2 + \frac{D_0}{2}|\nabla^\perp\cdot u|^2 =&\frac{D_0}{2}\Big(|\nabla^\perp\cdot u|^2 + D_0|u|^2 - 2(Lu,\, u)\Big) + \frac{D_0}{2}|\nabla^\perp\cdot u|^2\\ =&D_0\Big(|\nabla^\perp\cdot u|^2 + \frac{D_0}{2}|u|^2 - (Lu,\, u)\Big)\\ \ge&D_0\Big(|\nabla^\perp\cdot u|^2 + \frac{9}{4}|\beta u\cdot\mathbf{t}|^2 - (Lu,\, u)\Big)\end{aligned}$$ [10]

and, since

$$3(Lu,\, u) = 3(\beta u\cdot\mathbf{t},\, \nabla^\perp\cdot u) \le |\nabla^\perp\cdot u|^2 + \frac{9}{4}|\beta u\cdot\mathbf{t}|^2$$

we have

$$\frac{D_0}{2}\|u\|^2 + \frac{D_0}{2}|\nabla^\perp\cdot u|^2 \ge 2D_0(Lu,\, u).$$

Hence from (2.22) we obtain

$$|Au|^2 \ge \frac{1}{2}|\Delta u|^2 + \frac{D_0}{2}\|u\|^2 + \frac{D_0}{2}|\nabla^\perp\cdot u|^2 - |\nabla^\perp(\beta u\cdot\mathbf{t}) - Lu|^2.\tag{2.24}$$

We already know, by the discussion after (2.14), that the norm $\|\cdot\|$ is equivalent to norm induced by the usual norm $|\cdot|_1$ of $\mathbf{H}^1(\Omega)$ in V. Then since there is a constant C_1 such that, for all $u\in\mathbf{H}^1(\Omega)$ holds $|\nabla^\perp(\beta u\cdot\mathbf{t}) - Lu| \le C_1|u|_1$, we may choose D_0 satisfying

$$\frac{D_0-1}{2}\|u\|^2 \ge |\nabla^\perp(\beta u\cdot\mathbf{t}) - Lu|^2, \quad \text{for all} \quad u\in V.\tag{2.25}$$

From now on we consider D_0 satisfying all the conditions (2.13), (2.23) and (2.25), i.e.,

$$\left.\begin{aligned}(D_0 - \tfrac{1}{2})|u|^2 \ge 2|\beta v\cdot\mathbf{t}|^2, \quad \text{for all} \quad u\in H\\ \tfrac{D_0}{2}|u|^2 \ge \tfrac{9}{4}|\beta u\cdot\mathbf{t}|^2 \quad \text{for all} \quad u\in V\\ \tfrac{D_0-1}{2}\|u\|^2 \ge |\nabla^\perp(\beta u\cdot\mathbf{t}) - Lu|^2 \quad \text{for all} \quad u\in V\end{aligned}\right\}\tag{2.26}$$

For such a choice, from (2.24) and (2.15), we obtain

$$\begin{aligned}|Au|^2 &\ge \frac{1}{2}|\Delta u|^2 + \frac{1}{2}\|u\|^2 + \frac{D_0}{2}|\nabla^\perp\cdot u|^2\\ &\ge \frac{1}{2}|\Delta u|^2 + \frac{1}{2}\|u\|^2 \ge \frac{1}{2}|\Delta u|^2 + \frac{1}{4}(|\nabla^\perp\cdot u|^2 + |u|^2)\\ &\ge \frac{1}{4}\Big(|\Delta u|^2 + |\nabla^\perp\cdot u|^2 + |u|^2\Big).\end{aligned}\tag{2.27}$$

[10] See (2.14) for the definition of $((\cdot,\,\cdot))$.

On the other hand, by (2.21), we have

$$\begin{aligned}
|Au|^2 &\le |\Delta u|^2 + D_0\|u\|^2 + |\nabla^\perp(\beta u\cdot \mathbf{t}) - Lu|^2 + D_0|\nabla^\perp\cdot u|^2\\
&\quad + 2|\Delta u||\nabla^\perp(\beta u\cdot \mathbf{t}) - Lu| + 2D_0|(Lu,\, u)|\\
&\le |\Delta u|^2 + D_0\|u\|^2 + |\nabla^\perp(\beta u\cdot \mathbf{t}) - Lu|^2 + D_0|\nabla^\perp\cdot u|^2\\
&\quad + |\Delta u|^2 + |\nabla^\perp(\beta u\cdot \mathbf{t}) - Lu|^2 + D_0|\nabla^\perp\cdot u|^2 + D_0|\beta u\cdot \mathbf{t}|^2\\
&\le 2|\Delta u|^2 + D_0\|u\|^2 + 2|\nabla^\perp(\beta u\cdot \mathbf{t}) - Lu|^2 + 2D_0|\nabla^\perp\cdot u|^2 + D_0|\beta u\cdot \mathbf{t}|^2\\
&\le 2|\Delta u|^2 + (2D_0-1)\|u\|^2 + 2D_0|\nabla^\perp\cdot u|^2 + \frac{D_0}{2}(D_0 - \frac{1}{2}))|u|^2\\
&\le 2|\Delta u|^2 + (2D_0-1)(2D_0+1)(|\nabla^\perp\cdot u|^2 + |u|^2)\\
&\quad + 2D_0|\nabla^\perp\cdot u|^2 + \frac{D_0}{2}(D_0 - \frac{1}{2})|u|^2
\end{aligned}$$

and since

$$2 + (2D_0-1)(2D_0+1) + 2D_0 + \frac{D_0}{2}(D_0 - \frac{1}{2}) \le 1 + 4D_0^2 + 2D_0 + D_0^2 \le 5(D_0+1)^2$$

we obtain

$$|Au|^2 \le 5(D_0+1)^2\big(|\Delta u|^2 + |\nabla^\perp\cdot u|^2 + |u|^2\big). \tag{2.28}$$

Hence from (2.27), (2.28) and Corollary 2.5 we have that the norm $|\cdot|_{[2]} := |Au|$ is equivalent to the norm induced by the usual norm $|\cdot|_2$ in $D(A)$. Note that $|\Delta u| = |\nabla\nabla^\perp\cdot u|$ for each solenoidal (i.e., $\nabla\cdot u = 0$) $u \in \mathbf{H}^2(\Omega)$.

2.3.2. *No-slip boundary conditions.* For the case of no-slip boundary conditions (1.5), well studied in [11](see also [4]), the respective subspaces and operator A are

$$\begin{aligned}
H &:= \{u \in \mathbf{L}^2(\Omega) : \nabla\cdot u = 0 \ \ \&\ \ u\cdot\mathbf{n} = 0 \,\mathrm{on}\, \Gamma\};\\
V &:= \{u \in \mathbf{H}^1(\Omega) : \nabla\cdot u = 0 \ \ \&\ \ u = 0 \,\mathrm{on}\, \Gamma\} = \{u \in \mathbf{H}^1_0(\Omega) : \nabla\cdot u = 0\};\\
A &: V \to V' \qquad < Au,\, v > := (\nabla u,\, \nabla v), \quad v \in V;\\
D(A) &:= \{u \in V \mid\ Au \in H\} = \mathbf{H}^2(\Omega)\cap V = \{u \in \mathbf{H}^2_0(\Omega) : \nabla\cdot u = 0\}.
\end{aligned}$$

The bilinear form

$$((u,\, v)) := < Au,\, v >, \quad u, v \in V,$$

is a scalar product on V and its associated norm $\|u\| := ((u,\, u))^{\frac{1}{2}}$ is equivalent to the norm induced on V by the usual norm of $\mathbf{H}^1(\Omega)$.

The operator $A : D(A) \to H$ is called the stokes operator and the norm of $D(A)$ defined by $|u|_{[2]} := |Au|$ is equivalent to the norm induced in $D(A)$ by the usual norm of $\mathbf{H}^2(\Omega)$.

3. Existence, Uniqueness and Continuity.

3.1. **The Operator B.** In this section we present some theorems whose proofs are similar to those of similar theorems we can find in [8]. Mainly we follow the techniques from [11] and [9].

We start by defining a trilinear form b by

$$(u,\, v,\, w) \mapsto \sum_{i,j=1}^{2} \int_R u_i(\partial_i v_j) w_j \, dx \tag{3.1}$$

for which we have the estimates

$$|b(u,\, v,\, w)| \le C_1 K$$

where C_1 is a constant and K is one of the following products

$$\|u\|\|v\|\|w\| \qquad u,\, v,\, w \in \mathbf{H}^1(\Omega), \tag{3.2}$$

$$|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|v\|_2^{\frac{1}{2}}|w| \qquad u \in \mathbf{H}^1(\Omega),\, v \in \mathbf{H}^2(\Omega),\, w \in \mathbf{L}^2(\Omega), \tag{3.3}$$

$$|u|^{\frac{1}{2}}\|u\|_2^{\frac{1}{2}}\|v\||w| \qquad u \in \mathbf{H}^2(\Omega),\, v \in \mathbf{H}^1(\Omega),\, w \in \mathbf{L}^2(\Omega), \tag{3.4}$$

$$|u|\|v\||w|^{\frac{1}{2}}\|w\|_2^{\frac{1}{2}} \qquad u \in \mathbf{L}^2(\Omega),\, v \in \mathbf{H}^1(\Omega),\, w \in \mathbf{H}^2(\Omega), \tag{3.5}$$

$$|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}} \qquad u,\, v,\, w \in \mathbf{H}^1(\Omega). \tag{3.6}$$

In particular, by (3.2) we have

Corollary 3.1. *The form b is continuous on the product space $(\mathbf{H}^1(\Omega))^3$.*

The form b, being continuous in $(\mathbf{H}^1(\Omega))^3$, is continuous in V^3 and, for each pair $(u,\, v) \in V^2$ we define the operator $B(u,\, v) \in V'$ by

$$\begin{aligned} B(u,\, v) : V &\to \mathbb{R} \\ w &\mapsto \,< B(u,\, v),\, w >\, = b(u,\, v,\, w) \end{aligned}$$

and we set

$$B(u) := B(u,\, u) \in V', \quad \forall u \in V.$$

Lemma 3.2. *Fixing the first variable in V, the form b defined in* (3.1) *results skew-symmetric in the last two variables, i.e.,*

$$\forall u \in V \forall v,\, w \in \mathbf{H}^1[\, b(u,\, v,\, w) = -b(u,\, w,\, v)\,].$$

Due to the trilinearity of b the previous Lemma is equivalent to the following corollary which proof can be found in ([8], Section 2.6.2).

Corollary 3.3. *Fixing the first variable in V, we have*

$$\forall u \in V \forall v \in \mathbf{H}^1[\, b(u,\, v,\, v) = 0\,].$$

3.2. **Weak Solutions.**

3.2.1. *Existence.* Classically the existence problem for (1.1)–(1.2), renaming $\tilde{F} := F + v$ amounts to finding a vector function

$$u : \overline{\Omega} \times [0,\, T] \to \mathbb{R}^2$$

and a scalar function

$$p : \overline{\Omega} \times [0,\, T] \to \mathbb{R},$$

such that

$$u_t + (u \cdot \nabla)u + \nabla p = \nu \Delta u + \tilde{F} \qquad \text{in} \quad \Omega \times]0,\, T[; \tag{3.7}$$

$$\nabla \cdot u = 0 \qquad \text{in} \quad \Omega \times]0,\, T[; \tag{3.8}$$

$$u(x,\, 0) = u_0(x) \qquad \text{in} \quad \Omega \tag{3.9}$$

satisfying suitable boundary conditions that in the case of Navier boundary conditions are

$$u \cdot \mathbf{n} = 0 \qquad \text{on} \quad \Gamma \times]0,\, T[; \tag{3.10}$$

$$\nabla^{\perp} \cdot u - \beta u \cdot \mathbf{t} = 0 \qquad \text{on} \quad \Gamma \times]0,\, T[; \tag{3.11}$$

and, in the case of no-slip boundary conditions are

$$u = 0 \quad \text{on} \quad \Gamma \times]0,\, T[. \tag{3.12}$$

Where $\tilde{F}$ and u_0 are given and defined in $\Omega \times [0,\, T]$ and Ω respectively. u_0 is the position at time 0 of the system.

Rewriting (3.7) as

$$u_t + (u\cdot\nabla)u + \nabla p = \nu(\Delta u - Cu) + \nu Cu + \tilde{F};$$

where, for $u \in V$, in the no-slip case we have $Cu = 0$ and, in the Navier case

$$Cu = P^{\nabla}[D_0 u + \nabla^{\perp}(\beta u\cdot\mathbf{t}) - Lu] = D_0 u + P^{\nabla}[\nabla^{\perp}(\beta u\cdot\mathbf{t})] - Lu;$$

a weak formulation of problem (3.7)–(3.9), with either (3.10)-(3.11) or (3.12), is:

Problem 3.1. *Given*

$$\tilde{F} \in L^2(0,\,T,\,V'), \tag{3.13}$$

&

$$u_0 \in H, \tag{3.14}$$

to find

$$u \in L^2(0,\,T,\,V), \qquad u' \in L^1(0,\,T,\,V') \tag{3.15}$$

satisfying

$$u' + \nu Au + Bu = \nu Cu + \tilde{F} \quad on \quad]0,\,T[, \tag{3.16}$$ [11]

and

$$u(0) = u_0. \tag{3.17}$$

Lemma 3.4. *The operator C from V to H is symmetric in $V \times V$, when seen as $C(u,\,v) := (Cu,\,v)$, and satisfies*

$$(Cu,\,v) \le K\|u\||v|; \quad u \in V,\ v \in H.$$

Proof. In the no-slip case the statement of the Lemma is trivial. In the Navier case the symmetry follows from

$$\begin{aligned}(Cu,\,v) &= (P^{\nabla}[D_0 u + \nabla^{\perp}(\beta u\cdot\mathbf{t}) - Lu],\,v) = (D_0 u + \nabla^{\perp}(\beta u\cdot\mathbf{t}) - Lu,\,v)\\ &= D_0(u,\,v) - (\beta u\cdot\mathbf{t},\,\nabla^{\perp}\cdot v) - (\nabla^{\perp}\cdot u,\,\beta v\cdot\mathbf{t}) + \int_\Gamma \beta u\cdot\mathbf{t} v\cdot\mathbf{t}\,d\Gamma\end{aligned}$$

and, the estimate is easy because

$$|D_0 u + \nabla^{\perp}(\beta u\cdot\mathbf{t}) - Lu| \le D_0|u| + K_1|u|_1 + K_2|\nabla^{\perp}\cdot u| \le K\|u\|.$$

□

In the proof of existence of weak solutions we use mainly Gronwall and Young inequalities:

Lemma 3.5 (Gronwall Inequality)**.** *Let $g,\,h,\,y,\,\frac{dy}{dt}$ be locally integrable functions satisfying*

$$\frac{dy}{dt} \le gy + h \quad for\ t \ge t_0. \tag{3.18}$$

Then

$$y(t) \le y(t_0)\exp\left(\int_{t_0}^t g(\tau)\,d\tau\right) + \int_{t_0}^t h(s)\exp\left(-\int_t^s g(\tau)\,d\tau\right)\,ds, \quad t \ge t_0.$$

[The proof can be found in [10]*].*

Lemma 3.6 (Young Inequality)**.** *Given $a,\,b,\,\varepsilon > 0$, $1 < p < +\infty$, we have*

$$ab \le \varepsilon a^p + C^Y_{\varepsilon,p} b^{p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $C^Y_{\varepsilon,p} = \frac{p-1}{(p^{p'})(\varepsilon^{\frac{1}{p-1}})}$.

[11] To be seen as an equality in V'.

Theorem 3.7. *Given $\tilde{F}$ and u_0 satisfying* (3.13) *and* (3.14). *There is at least one function u satisfying* (3.15)-(3.17).

Proof. The proof is similar to that of Theorem 2.6.4 of [8], it differs only in the expressions of the "a priori" estimates. Following that proof:
We start by defining, for each $m \in \mathbb{N}_0$, an approximate solution u^m of (3.16):

$$u^m := \sum_{i \le m} u_i^m(t) W_i, \tag{3.19}$$

$$\begin{aligned} &((u^m)'(t), W_j) + \nu((u^m(t), W_j)) + b(u^m(t), u^m(t), W_j) \\ &\qquad = \nu(Cu^m(t), W_j) + < \tilde{F}(t), W_j >, \quad t \in [0, T], \quad j \le m, \end{aligned} \tag{3.20}$$

with

$$u^m(0) = u_0^m. \tag{3.21}$$

Where u_0^m is the orthogonal projection of u_0 onto $span\{W_i \mid i \le m\}$.
From (3.20) we obtain the nonlinear system of differential equations in the functions $u_i^m, \quad i \le m$:

$$\begin{aligned} &\sum_{i \le m} (u_i^m)'(t)(W_i, W_j) + \nu \sum_{i \le m} u_i^m((W_i, W_j)) + \sum_{\substack{i \le m \\ l \le m}} u_i^m u_l^m b(W_i, W_l, W_j) \\ &\qquad = \nu \sum_{i \le m} u_i^m(CW_i, W_j) + < \tilde{F}(t), W_j >, \end{aligned}$$

that reduces to the ODE's system

$$\begin{aligned} &(u_j^m)'(t) + \nu u_j^m k_j + \sum_{\substack{i \le m \\ l \le m}} u_i^m u_l^m b(W_i, W_l, W_j) \\ &\qquad = \nu \sum_{i \le m} u_i^m(CW_i, W_j) + < \tilde{F}(t), W_j >, \qquad j \le m. \end{aligned} \tag{3.22}$$

Note that (3.21) is the same as the m scalar conditions

$$u_j^m(0) = \text{the projection of } u_0 \text{ onto } span\{W_j\} = u_{0j}. \tag{3.23}$$

The rest of the proof is similar to that of Theorem 2.6.4 of [8] relative to the case the rectangle with Lions boundary conditions. The only difference is the term Cu (or Cu^m for approximate solutions) but it does not carry any problem because it can be "absorbed" by the Gronwall and Young inequalities when we compute the estimates we need. Following the proof presented in [8] we will find the inequality

$$\frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 \le 2\nu|(Cu^m(t), u^m(t))| + 2 < \tilde{F}(t), u^m(t) > .$$

Working a little the new term $2\nu|(Cu^m(t), u^m(t))|^2$ we obtain

$$\begin{aligned} \frac{d}{dt}|u^m(t)|^2 + 2\nu\|u^m(t)\|^2 &\le 2\nu K\|u^m(t)\||u^m(t)| + 2 < \tilde{F}(t), u^m(t) > \quad ^{12} \\ &\le \nu\|u^m(t)\|^2 + \frac{1}{\nu}K^2\nu^2|u^m(t)|^2 + 2 < \tilde{F}(t), u^m(t) > . \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}|u^m(t)|^2 + \nu\|u^m(t)\|^2 &\le K^2\nu|u^m(t)|^2 + 2 < \tilde{F}(t), u^m(t) > \\ &\le K^2\nu|u^m(t)|^2 + \frac{2}{\nu}\|\tilde{F}(t)\|_{V'}^2 + \frac{\nu}{2}\|u^m(t)\|^2 \end{aligned}$$

[12] Here K is the constant appearing in Lemma 3.4.

so,

$$\frac{d}{dt}|u^m(t)|^2 + \frac{\nu}{2}\|u^m(t)\|^2 \le K^2\nu|u^m(t)|^2 + \frac{2}{\nu}\|\tilde{F}(t)\|^2_{V'} \tag{3.24}$$

and then, by the Gronwall inequality, for $s \in [0, T]$

$$\begin{aligned} |u^m(s)|^2 &\le |u_0^m|^2 \exp\Big(\int_0^s K^2\nu\, d\tau\Big) + \int_0^s \frac{2}{\nu}\|\tilde{F}(t)\|^2_{V'} \exp\Big(-\int_s^t K^2\nu\, d\tau\Big)\, dt \\ &\le \exp(K^2T\nu)\Big(|u_0|^2 + \frac{2}{\nu}\int_0^T \|\tilde{F}(t)\|^2_{V'}\, dt\Big). \end{aligned} \tag{3.25}$$

Therefore

$$|u^m(s)|^2 \le K_1$$

with K_1 independent of m, i.e.,

(3.26) *the sequence* (u^m) *remains in a bounded set of* $L^\infty(0, T, H)$.

As we have seen, first we have used Young inequality to "absorb" the term $\nu\|u^m(t)\|^2$ coming from the new term $2\nu|(Cu^m(t), u^m(t))|^2$. Then the remaining term $K^2\nu|u^m(t)|^2$ has been "absorbed" by the Gronwall inequality.

From (3.24) and (3.26) we also have

(3.27) *the sequence* (u^m) *remains in a bounded set of* $L^2(0, T, V)$.

The rest of the proof is completely analogous to that of Theorem 2.6.4 of [8]. □

3.2.2. *Uniqueness.*

Theorem 3.8. *The solution of Problem 3.1 given by Theorem 3.7 is unique. Moreover it is a.e. equal to a continuous function from* $[0, T]$ *into* H *and,*

$$u(t) \to u(t_1), \quad \text{in } H \quad \text{as } t \to t_1 \quad t_1 \in [0, T]. \tag{3.28}$$

In particular

$$u(t) \to u_0, \quad \text{in } H \quad \text{as } t \to 0;$$
$$u(t) \to u(T), \quad \text{in } H \quad \text{as } t \to T.$$

The proof follows is analogous to that of Theorem 3.6 of [8] we need only to absorb the new term by Young and Gronwall inequalities.

3.2.3. *Continuity on the Initial Data.*

Theorem 3.9. *The map*

$$\mathbb{S} : H \times L^2(0, T, V')\times]0, +\infty[\to C([0, T], H)$$
$$(u_0, \tilde{F}, \nu) \mapsto u$$

is continuous. Here $u \in C([0, T], H)$ *is the unique solution of Problem 3.1 (see Theorem 3.8).*

Theorem 3.10. *The map*

$$\mathbb{S}_2 : H \times L^2(0, T, V')\times]0, +\infty[\to L^2(0, T, V)$$
$$(u_0, \tilde{F}, \nu) \mapsto u$$

is continuous. Here $u \in L^2(0, T, V)$ *is the unique solution of Problem 3.1 (see Theorem 3.8).*

The proofs are analogous to those of Theorems 3.8 and 3.9 of [8]. Again we need only to absorb the new terms by Young and Gronwall inequalities.

3.3. **Strong Solutions.** Asking for some regularity in the initial condition and external force we consider the following strong formulation of problem (3.7)–(3.9):

Problem 3.2. *Given*

$$\tilde{F} \in L^2(0,\, T,\, H), \tag{3.29}$$

&

$$u_0 \in V, \tag{3.30}$$

to find

$$u \in L^2(0,\, T,\, D(A)) \cap L^\infty(0,\, T,\, V),\; u' \in L^2(0,\, T,\, H) \tag{3.31}$$

satisfying

$$u' + \nu A u + B u = \nu C u + \tilde{F} \quad \text{on} \quad]0,\, T[, \tag{3.32}$$ [13]

and

$$u(0) = u_0. \tag{3.33}$$

3.3.1. *Existence.*

Theorem 3.11. *Given $\tilde{F}$ and u_0 satisfying* (3.29) *and* (3.30)*. There is at least one function u satisfying* (3.31)-(3.33)*.*

For the proof we define an approximate solution u^m for each $m \in \mathbb{N}_0$, like in (3.19)–(3.21), and arrive to the estimate (3.24) and conclusions (3.26) and (3.27) exactly in the same way. For the present Theorem we need only some more estimates that we obtain from the equation

$$\begin{aligned} ((u^m(t))',\, Au^m(t)) + \nu((u^m(t),\, Au^m(t))) + Bu^m(t)(Au^m(t)) \\ = \nu(Cu^m(t),\, Au^m(t)) + (\tilde{F}(t),\, Au^m(t)). \end{aligned}$$

First of all, comparing with the study done in [8] the only new term is the term $\nu(Cu^m(t),\, Au^m(t))$. Like we have done in the weak case we absorb this term putting

$$2\nu(Cu^m,\, Au^m) \leq 2\nu K \|u^m\| |Au^m| \leq \nu |Au^m|^2 + \frac{1}{\nu}\nu^2 K^2 \|u^m\|^2$$

The term $\nu|Au^m|^2 = \nu|u^m|^2_{[2]}$ will be absorbed by the term $2\nu((u^m(t),\, Au^m)) = 2\nu|u^m|^2_{[2]}$ that will appear in the first member and the term $\nu K^2\|u^m\|^2$ will be absorbed by the Gronwall inequality when this inequality is applied to obtain an estimate in $L^\infty([0,\, T],\, V)$. Since we can absorb the new term the proof becomes analogous to that of Theorem 3.10 of [8].

3.3.2. *Uniqueness.*

Theorem 3.12. *The solution of Problem 3.2 is unique. Moreover it is a.e. equal to a continuous function from $[0,\, T]$ into V.*

The proof is analogous to that of Theorem 3.11 of [8].

3.3.3. *Continuity on Initial Data.*

Theorem 3.13. *The map*

$$\begin{aligned} \mathbb{S}_s : V \times L^2(0,\, T,\, H) \times]0,\, +\infty[&\to C([0,\, T],\, V) \\ (u_0,\, \tilde{F},\, \nu) &\mapsto u \end{aligned}$$

is continuous. Where $u \in C([0,\, T],\, V)$ is the unique solution of Problem 3.2 (see Theorem 3.12).

[13] Like in (3.16), to be seen as an equality in V'.

Theorem 3.14. *The map*

$$\mathbb{S}_{2s} : V \times L^2(0,\, T,\, H) \times]0,\, +\infty[\,\to L^2(0,\, T,\, D(A))$$
$$(u_0,\, \tilde{F},\, \nu) \mapsto u$$

is continuous. Where $u \in L^2(0,\, T,\, D(A))$ *is the unique solution of Problem 3.2 (see Theorem 3.12).*

The proofs are analogous to those of Theorems 3.12 and 3.13 of [8].

4. Change of Variables. Relaxation.

In the previous section we have presented the classical, weak and strong formulations of (3.7)-(3.9), with either Navier or no-slip boundary conditions. For the controlled version with $F + v$ in the place of $\tilde{F}$ we have existence, uniqueness and continuity in the data $(u_0,\, F + v,\, \nu)$, because our control will be an essentially bounded function taking values on $D(A)$ and so, $F + v$ will belong where $\tilde{F}$ did: ($L^2(0,\, T,\, V')$ or $L^2(0,\, T,\, H)$). If we consider the initial data as $(u_0,\, \tilde{F},\, v,\, \nu)$, by Theorems 3.7, 3.9 and 3.10 we easily conclude that (considering again the external force depending on time):

Corollary 4.1. *Given*

$$(u_0,\, \tilde{F},\, v,\, \nu) \in H \times L^2(0,\, T,\, V') \times L^\infty(0,\, T,\, V') \times]0,\, +\infty[,$$

there is at least one weak solution $u \in C([0,\, T],\, H)$ *for Problem 3.1 with* $\tilde{F} + v$ *in the place of* $\tilde{F}$*;*

Corollary 4.2. *The maps*

$$\mathbb{S} : H \times L^2(0,\, T,\, V') \times L^\infty(0,\, T,\, V') \times]0,\, +\infty[\,\to C([0,\, T],\, H)$$
$$(u_0,\, \tilde{F},\, v,\, \nu) \mapsto u$$

and

$$\mathbb{S}_2 : H \times L^2(0,\, T,\, V') \times L^\infty(0,\, T,\, V') \times]0,\, +\infty[\,\to L^2(0,\, T,\, V)$$
$$(u_0,\, \tilde{F},\, v,\, \nu) \mapsto u$$

are continuous.

Similarly, by Theorems 3.11, 3.13 and 3.14 we can easily see that

Corollary 4.3. *Given*

$$(u_0,\, \tilde{F},\, v,\, \nu) \in V \times L^2(0,\, T,\, H) \times L^\infty(0,\, T,\, H) \times]0,\, +\infty[,$$

there is at least one weak solution $u \in C([0,\, T],\, V)$ *for Problem 3.2 with* $\tilde{F} + v$ *in the place of* $\tilde{F}$*;*

Corollary 4.4. *The maps*

$$\mathbb{S}_s : V \times L^2(0,\, T,\, V') \times L^\infty(0,\, T,\, H) \times]0,\, +\infty[\,\to C([0,\, T],\, V)$$
$$(u_0,\, \tilde{F},\, v,\, \nu) \mapsto u$$

and

$$\mathbb{S}_{2s} : V \times L^2(0,\, T,\, V') \times L^\infty(0,\, T,\, H) \times]0,\, +\infty[\,\to L^2(0,\, T,\, D(A))$$
$$(u_0,\, \tilde{F},\, v,\, \nu) \mapsto u$$

are continuous.

4.1. **Change of Variables.** If we make the change of variables

$$u = y + \mathbb{I}v$$

where $\mathbb{I}$ is the primitive operator — $[\mathbb{I}v](t) = \int_0^t v(\tau)\,d\tau$, from

$$u' = -\nu Au - Bu + \nu Cu + \tilde{F} + v$$

we arrive to the equation

$$y' = -\nu A(y + \mathbb{I}v) - B(y + \mathbb{I}v) + \nu C(y + \mathbb{I}v) + \tilde{F}.$$

Note that the function v appears only implicitly in the last equation. Now we forget that $\mathbb{I}v$ is a primitive of an essentially bounded function and replace it by P in the equation. Since v is a low modes forcing it takes value in a finite-dimensional space and, $\mathbb{I}v$ being a primitive we have $\mathbb{I}v \in C([0,\,T],\,D(A))$. But we take P in the larger space $L^4(0,\,T,\,D(A))$.

4.2. **Weak Case.** Similarly as we have done in [8] we consider the weak problem:

Problem 4.1. *Given*

(4.1) $$\tilde{F} \in L^2(0,\,T,\,V'),, \quad P \in L^4(0,\,T,\,D(A))$$

&

(4.2) $$y_0 \in H,$$

to find

(4.3) $$y \in L^2(0,\,T,\,V), \qquad y' \in L^1(0,\,T,\,V')$$

satisfying

(4.4) $$y' + \nu A(y + P) + B(y + P) = \nu C(y + P) + \tilde{F} \quad \text{on} \quad]0,\,T[,$$

and

(4.5) $$y(0) = y_0.$$

4.2.1. *Existence.* We have the theorem:

Theorem 4.5. *Given $\tilde{F}$, P and y_0 satisfying* (4.1) *and* (4.2)*. There is at least one function y satisfying* (4.3)-(4.5).

The proof is analogous to that of Theorem 4.4.1 in [8]. Defining an approximate solution

$$y^m = \sum_{i \le m} y_i^m(t) W_i$$

for each $m \in \mathbb{N}_0$ and arrive to the equation

(4.6) $$< (y^m)',\, y^m > + \nu < A(y^m + P^m),\, y^m > + < B(y^m + P^m),\, y^m > \\ = \nu < C(y^m + P^m),\, y^m > + < \tilde{F},\, y^m > .$$ [14]

From which we derive

$$\frac{d}{dt}|y^m|^2 + 2\nu\|y^m\|^2 = -2\nu((P^m,\, y^m)) \\ - 2b(y^m,\, P^m,\, y^m) + 2b(P^m,\, y^m,\, P^m) + 2\nu(C(y^m + P^m),\, y^m) + 2 < \tilde{F},\, y^m >$$

[14]Where P^m is the projection of P onto $span\{W_i \mid i \le m\}$.

that differs from the respective equation appearing in Theorem 4.4.1 in [8] only in the new term $2\nu(C(y^m + P^m), y^m)$. From the estimate

$$\begin{aligned}2\nu(C(y^m + P^m), y^m) \leq & 2\nu K\|y^m\||y^m| + 2\nu K\|P^m\||y^m| \\ \leq & \nu\|y^m\|^2 + \nu K^2|y^m|^2 + \nu\|P^m\|^2 + \nu K^2|y^m|^2 \\ = & \nu\|y^m\|^2 + 2\nu K^2|y^m|^2 + \nu\|P^m\|^2,\end{aligned}$$

we see that the term $\nu\|y^m\|^2$ will be absorbed by the term $2\nu\|y^m\|^2$ in the first member and, after applying the Gronwall inequality the term $2\nu K^2|y^m|^2$ will be also absorbed while the term $\nu\|P^m\|^2$ bring no problem because by hypothesis $\|P\|$, and then also $\|P^m\|$, is square integrable in $]0, T[$.
The rest of the proof is completely analogous to that of Theorem 4.4.1 in [8].

4.2.2. *Uniqueness.*

Theorem 4.6. *The solution of Problem 4.1 given by Theorem 4.5 is unique. Moreover it is a.e. equal to a continuous function from* $[0, T]$ *into* H*.*

Proof analogous to that of Theorem 4.5.1 in [8].

4.2.3. *Continuity.*

Theorem 4.7. *The map*

$$\begin{aligned}\mathbb{Y} : H \times L^2(0, T, V') \times L^4(0, T, D(A)) \times]0, +\infty[&\to C([0, T], H) \\ (y_0, \tilde{F}, P, \nu) &\mapsto y\end{aligned}$$

is continuous. Where y *is the unique solution of Problem* (4.1) *corresponding to the data* $(y_0, \tilde{F}, P, \nu)$*.*

The proof is analogous to that of Theorem 4.6.1 in [8]. In that proof at some point we arrive to the equation [15]

$$< w', w >= -\eta\|w\|^2 + f, \qquad [\text{or } \frac{d}{dt}|w|^2 = -2\eta\|w\|^2 + 2f]$$

where w is the difference of two solutions and f is an expression depending on the initial data. If we follow that proof in the present case we arrive to the equation

$$\frac{d}{dt}|w|^2 = -2\eta\|w\|^2 + 2f + 2\eta(Cw, w) + 2\eta(C(Q-P), w) - 2(\nu - \eta)(C(y+P), w).$$

For the new term we find

$$\begin{aligned}&2\eta(Cw, w) + 2\eta(C(Q-P), w) - 2(\nu - \eta)(C(y+P), w) \\ \leq & \eta\|w\|^2 + \eta K^2|w|^2 + \eta\|Q-P\|^2 + \eta K^2|w|^2 + |\nu - \eta|^2\|y+P\|^2 + K^2|w|^2 \\ = & \eta\|w\|^2 + 2\eta K^2|w|^2 + \eta\|Q-P\|^2 + |\nu - \eta|^2\|y+P\|^2 + K^2|w|^2.\end{aligned}$$

The term $\eta\|w\|^2$ is absorbed by $-2\eta\|w\|^2$ and, after Gronwall inequality, if we choose η not far from η (for example $|\eta - \nu| < \frac{\nu}{2}$ like in [8]), the terms $2\eta K^2|w|^2$ and $K^2|w|^2$ is absorbed while the other terms will appear under an integral and then bringing no problem: If the initial data is close those integrals are small. Therefore the proof is analogous.

We also have

[15] Equation 4.6.1 in [8].

Theorem 4.8. *The map*

$$\mathbb{Y}_2 : H \times L^2(0,\,T,\,V') \times L^4(0,\,T,\,D(A)) \times]0,\,+\infty[\to L^2(0,\,T,\,V)$$
$$(y_0,\,\tilde{F},\,P,\,\nu) \mapsto y$$

is continuous. Where y is the unique solution of Problem (4.1) *corresponding to the data* $(y_0,\,\tilde{F},\,P,\,\nu)$.

Proof analogous to that of Theorem 4.6.2 in [8].

4.3. **Strong Case.** Consider the strong problem:

Problem 4.2. *Given*

$$\tilde{F} \in L^2(0,\,T,\,H),, \quad P \in L^4(0,\,T,\,D(A)) \tag{4.7}$$

&

$$y_0 \in V, \tag{4.8}$$

to find

$$y \in L^2(0,\,T,\,D(A)) \cap L^\infty(0,\,T,\,V), \qquad y' \in L^2(0,\,T,\,H) \tag{4.9}$$

satisfying

$$y' + \nu A(y+P) + B(y+P) = \nu C(y+P) + \tilde{F} \quad \textit{on} \quad]0,\,T[, \tag{4.10}$$

and

$$y(0) = y_0. \tag{4.11}$$

4.3.1. *Existence.*

Theorem 4.9. *Given $\tilde{F}$, P and u_0 satisfying* (4.7) *and* (4.8)*. There is at least one function y satisfying* (4.9)-(4.11).

The proof is analogous to that of Theorem 4.7.1 in [8]. We will find a new term:

$$2\nu(Cy^m,\,Ay^m)$$

that satisfies

$$2\nu(Cy^m,\,Ay^m) \le \nu K^2 \|y^m\|^2 + \nu |y^m|_{[2]}$$

and then, will be absorbed by Young and Gronwall inequalities.

4.3.2. *Uniqueness.*

Theorem 4.10. *The solution of Problem 4.2 given by Theorem 4.9 is unique. Moreover it is a.e. equal to a continuous function from* $[0,\,T]$ *into* V.

The proof is analogous to that of Theorem 4.7.2 in [8].

4.3.3. *Continuity.*

Theorem 4.11. *The map*

$$\mathbb{Y}_s : V \times L^2(0,\,T,\,H) \times L^4(0,\,T,\,D(A)) \times]0,\,+\infty[\to C([0,\,T],\,V)$$
$$(y_0,\,\tilde{F},\,P,\,\nu) \mapsto y$$

is continuous. Where y is the unique solution of Problem (4.2) *corresponding to the data* $(y_0,\,\tilde{F},\,P,\,\nu)$.

Theorem 4.12. *The map*

$$\mathbb{Y}_{2s} : V \times L^2(0,\,T,\,H) \times L^4(0,\,T,\,D(A)) \times]0,\,+\infty[\to L^2([0,\,T],\,D(A))$$
$$(y_0,\,\tilde{F},\,P,\,\nu) \mapsto y$$

is continuous. Where y is the unique solution of Problem (4.2) *corresponding to the data* $(y_0,\,\tilde{F},\,P,\,\nu)$.

The proofs are analogous to those of Theorems 4.7.3 and 4.7.4 in [8]. The new expression appearing in the estimates is

$$\eta(Cw,\, Aw) + \eta(C(Q-P),\, Aw) - (\nu-\eta)(C(y+P),\, Aw)$$

that satisfies

$$\begin{aligned}
&\eta(Cw,\, Aw) + \eta(C(Q-P),\, Aw) - (\nu-\eta)(C(y+P),\, Aw)\\
&\le \frac{\eta}{3}|w|^2_{[2]} + 3\eta K^2\|w\|^2 + \frac{\eta}{3}|w|^2_{[2]} + 3\eta K^2\|Q-P\|^2 + \frac{\eta}{3}|w|^2_{[2]} + \frac{3}{\eta}|\nu-\eta|^2 K^2\|y+P\|^2\\
&= \eta|w|^2_{[2]} + 3\eta K^2\|w\|^2 + 3\eta K^2\|Q-P\|^2 + \frac{3}{\eta}|\nu-\eta|^2K^2\|y+P\|^2
\end{aligned}$$

and then, will be absorbed by Young and Gronwall inequalities. Note that the last two terms will be small if the initial data is close.

4.4. **Continuity in Relaxation Metric.** We begin with a definition:

Definition 4.1. *Given a finite dimensional normed space $\mathbb{F} \subset H$ and a basis $\{e_i \mid i = 1, \ldots, p\}$ for $\mathbb{F}$; the* **relaxation metric** *in $L^1([0,T],\mathbb{F})$ is defined by the norm*

$$\|\tilde{g}\|_{rx} := \max_{t_1,\, t_2 \in [0,\, T]} \left\| \int_{t_1}^{t_2} g(\tau)\, d\tau \right\|_{l_1}; \quad g = (g_1, \ldots, g_p),\ \tilde{g} = \sum_{i=1}^{p} g_i e_i. \tag{4.12}$$ [16]

Consider, also, the **w-relaxation metric** on $L^1([0,T],\mathbb{F})$ defined by the norm

$$\|\tilde{g}\|_{wrx} := \max_{t\in[0,\, T]} \left\| \int_0^t g(\tau)\, d\tau \right\|_{l_1}. \tag{4.13}$$

Remark 5. *It is easy to see that the identity map*

$$\Big(L^1([0,T],\mathbb{F}),\, \|\cdot\|_{rx}\Big) \to \Big(L^1([0,T],\mathbb{F}),\, \|\cdot\|_{wrx}\Big)$$

and the map

$$\begin{aligned}\mathbb{I} : L^\infty_{wrx}([0,T],\mathbb{F}) &\to C([0,T],\mathbb{F})\\ v &\mapsto \mathbb{I}v\end{aligned}$$

are continuous. Where the subscript "wrx" means that we are considering w-relaxation metric on the set $L^\infty([0,T],\mathbb{F})$. Since all the norms in $\mathbb{F}$ are equivalent, in the last space $C([0,T],\mathbb{F})$ we may consider in $\mathbb{F}$ any norm.

Recall that by definition, the map $\mathbb{S}$ of Corollary 4.2 gives us the weak solution, belonging to $C([0,T],H)$, of the N-S equation for an initial data in $\Pi := H \times L^2(0,T,V') \times L^\infty([0,T],\mathbb{F}) \times \mathbb{R}^+$. Changing the topology on the third factor of the previous product to the w-relaxation one, we arrive to the space $L^\infty_{wrx}([0,T],\mathbb{F})$ and we define the function $\mathbb{S}_{wrx}$ as the function defined in the product $\Pi_{wrx} := H \times L^2(0,T,V') \times L^\infty_{wrx}([0,T],\mathbb{F}) \times \mathbb{R}^+$ and taking the same values as $\mathbb{S}$.

$\diamond$ **The case $\mathbb{F} \subset D(A)$:**

Proposition 4.13. *The map $\mathbb{S}_{wrx}$ is continuous.*

Proof. Put $\mathbb{I}_\circ(u_0, \tilde{F}, v, \nu) := \mathbb{I}v$. By Remark 5 and Theorem 4.7 the map

$$\begin{aligned}\mathbb{Y}_{wrx} : \Pi_{wrx} &\to C([0,T],H)\\ (u_0, \tilde{F}, v, \nu) &\mapsto \mathbb{Y}(u_0, \tilde{F}, \mathbb{I}v, \nu) = \mathbb{Y}\circ\mathbb{I}^\circ(u_0, \tilde{F}, v, \nu)\end{aligned}$$

is continuous; where $\mathbb{I}^\circ(u_0, \tilde{F}, v, \nu) := (u_0, \tilde{F}, \mathbb{I}v, \nu)$.
By the equality $\mathbb{S}_{wrx} = \mathbb{Y}_{wrx} + \mathbb{I}_\circ$ we conclude the continuity of $\mathbb{S}_{wrx}$. ∎

[16] Recall that for $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, $\|x\|_{l_1} := \sum_{i=1}^p |x_i|$.

Analogously, by Remark 5 and theorems 4.8, 4.11 and 4.12, we can prove the continuity on relaxation metric of the maps $\mathbb{S}_2$, $\mathbb{S}_s$ and $\mathbb{S}_{2s}$ arriving to the Proposition

Proposition 4.14. *The maps $\mathbb{S}_{wrx}$, $\mathbb{S}_{2wrx}$, $\mathbb{S}_{swrx}$ $\mathbb{S}_{2swrx}$ are all continuous.*

By Remark 5 we obtain

Corollary 4.15. *The maps $\mathbb{S}_{rx}$, $\mathbb{S}_{2rx}$, $\mathbb{S}_{srx}$ $\mathbb{S}_{2srx}$ are all continuous.* [17]

⋄ **The case $\mathbb{F} \subset H$:**

Corollary 4.16. *Let $\mathbb{F} \subset H$ be a finite-dimensional subspace and let $\{e_i \mid i = 1, \ldots, p\}$ be a basis for $\mathbb{F}$. Let $\mathcal{V} := \{v_b \in L^\infty([0, T], \mathbb{F}) \mid b \in B\}$ be a uniformly l_1-bounded family of controls, say $|v_b|_{L^\infty([0,T],(\mathbb{R}^p,l_1))} \le M$ where $M > 0$ is a constant independent of the parameter b. Then the map*

$$(u_0, \tilde{F}, v, \nu) \mapsto \mathbb{S}(u_0, \tilde{F}, v, \nu)$$

is (X, Y)-continuous. Here $\mathbb{S}(u_0, \tilde{F}, v, \nu)$ is the solution of the N-S equation for the given data and the pair (X, Y) is one of the following

$$(H \times L^2(0, T, V') \times \mathcal{V}_{rx} \times \mathbb{R}^+, Y_1);$$
$$(V \times L^2(0, T, H) \times \mathcal{V}_{rx} \times \mathbb{R}^+, Y_2);$$

where

$$Y_1 \in \{L^2(0, T, V), C([0, T], H)\};$$
$$Y_2 \in \{L^2(0, T, D(A)), C([0, T], V)\}.$$

Proof. Let $\varepsilon > 0$ be a real number. Set $f_i \in D(A)$ such that $|e_i - f_i| < \varepsilon$ for any $v_b = \sum_{i=1}^p v_b^i e_i \in \mathcal{V}$ define $w_b = \sum_{i=1}^p v_b^i f_i$. Note that $|v_b - w_b|$ and $|v_b - w_b|_{V'}$ are small if so is ε.

For a target space Y, corresponding to the data, we have:

$$\begin{aligned}
&|\mathbb{S}(u_0, \tilde{F}, v_b, \nu) - \mathbb{S}(u_0, \tilde{F}, v_a, \nu)|_Y \\
\le &|\mathbb{S}(u_0, \tilde{F}, v_b, \nu) - \mathbb{S}(u_0, \tilde{F}, w_b, \nu)|_Y \\
&+ |\mathbb{S}(u_0, \tilde{F}, w_b, \nu) - \mathbb{S}(u_0, \tilde{F}, w_a, \nu)|_Y \\
&+ |\mathbb{S}(u_0, \tilde{F}, w_a, \nu) - \mathbb{S}(u_0, \tilde{F}, v_a, \nu)|_Y
\end{aligned}$$

and, since $|w_b - w_a|_{rx} = |v_b - v_a|_{rx}$ we have that, for small ε and small $|v_b - v_a|_{rx}$, the norm

$$|\mathbb{S}(u_0, \tilde{F}, v_b, \nu) - \mathbb{S}(u_0, \tilde{F}, v_a, \nu)|_Y$$

is small. □

5. Realization onto the Projection.

In this section we show that every curve in $W^{1,\infty}([t_i, t_f], \mathbb{J})$, with J finite-dimensional, is the projection of a strong solution of the Navier-Stokes system.

For $u \in D(A) \subset \mathbf{H}^2(\Omega)$ we have $u_i, \partial_i u_j \in L^4(\Omega)$ for all $i, j \in \{1, 2\}$ and, then $u_i \partial_i u_j \in L^2(\Omega)$ and $(u \cdot \nabla)u \in \mathbf{L}^2(\Omega)$. Denoting by P^∇ the orthogonal projection from $\mathbf{L}^2(\Omega)$ onto H and writing $(u \cdot \nabla)u = P^\nabla[(u \cdot \nabla)u] + \nabla p, \quad p \in H^1(\Omega)$ (see (2.17)), we have, for every $v \in V$, that

$$\begin{aligned}
b(u, u, v) &= ((u \cdot \nabla)u, v) = (P^\nabla[(u \cdot \nabla)u], v) + (\nabla p, v) \\
&= (P^\nabla[(u \cdot \nabla)u], v) = \langle P^\nabla[(u \cdot \nabla)u], v \rangle_{V',V};
\end{aligned}$$

[17] These "rx"-maps are defined similarly as the "wrx" ones, just considering the "rx"-topology in the factor of essentially bounded functions.

Since by definition Bu is the element of V' such that, for every $v \in V$, $Bu(v) = b(u, u, v)$ we have that

$$Bu = P^{\nabla}[(u \cdot \nabla)u].$$

Lemma 5.1. *Given:*

- *A finite subspace $\mathbb{J} \subset D(A)$;*
- *A function $q \in W^{1,\infty}([t_i, t_f], \mathbb{J})$ with $q(t_i) = q_i$;*
- *An element $Q_i \in \mathbb{J}_V^{\perp} := \mathbb{J}_H^{\perp} \cap V$; where $\mathbb{J}_H^{\perp}$ denotes the orthogonal space to $\mathbb{J}$ in H.* [18]

Suppose $\tilde{F} \in L^2(t_i, t_f, H) \cap L^{\infty}([t_i, t_f], D(A^s))$ for some $s < -\frac{1}{2}$. [19] *Then there exists a control $v^J(q, Q_i) \in L^{\infty}([t_i, t_f], \mathbb{J})$ depending on q and Q_i such that the projection onto $\mathbb{J}$ of the solution of the N-S equation*

$$u_t = -\nu Au - Bu + \nu Cu + \tilde{F} + v^J(q, Q_i), \qquad u(t_i) = q_i + Q_i$$

equals q on $[t_i, t_f]$.
Moreover the map $v^J : (q, Q_i) \mapsto v^J(q, Q_i)$ can be set $(W^{1,2} \times \mathbb{J}_V^{\perp}, L^2(t_i, t_f, \mathbb{J})$-continuous.

Before the proof we remark that, due to the continuity of the inclusions $D(A^t) \subset D(A^s)$ for $t \geq s$, the condition "$\tilde{F} \in L^{\infty}([t_i, t_f], D(A^s))$ for some $s < -\frac{1}{2}$" is equivalent to "$\tilde{F} \in L^{\infty}([t_i, t_f], D(A^s))$ for some $s \in \mathbb{R}$".

The above Lemma generalizes Lemma 4.10.4 in [8] where J is considered as the linear spanning of a finite number of eigenfunctions of the operator $A : D(A) \to H$. Now J is any finite dimensional space and, in such a case we have an important difference: The operator A is not commuting with the orthogonal projection onto J. This carries some additional difficulties and then, we have to repeat all the proof.

If nothing is said in the contrary $\mathbb{J}$ is considered endowed with the norm induced by $D(A)$.

Proof. We proceed as in the proof of Lemma 4.10.4 in [8]: Let q, q_i, Q_i and $\mathbb{J}$ be like in the statement of the Lemma. Consider the (non controlled) N-S equation: $u_t = -\nu Au - Bu + \nu Cu + \tilde{F}$ with initial condition $u(t_i) = q_i + Q_i =: u_i$ and split it into

$$\begin{aligned} P^J(u_t) &= P^J(-\nu Au - Bu + \nu Cu + \tilde{F}) \\ P^{-J}(u_t) &= P^{-J}(-\nu Au - Bu + \nu Cu + \tilde{F}) \\ u(t_i) &= u_i \end{aligned}$$

with the same initial condition, where P^J (resp. P^{-J}) is the orthogonal projection $H \to \mathbb{J}$ (resp. $H \to \mathbb{J}_H^{\perp}$). If we put $u^J := P^J u$ and $U^J := P^{-J} u$ we arrive to the

[18] Note that $\mathbb{J}_V^{\perp}$ does not coincide necessarily with the orthogonal space to $\mathbb{J}$ in V, because the scalar products in H and V are different.

[19] We need the condition $\tilde{F} \in L^2(t_i, t_f, H)$ because we are interested in strong solutions. The condition $\tilde{F} \in L^{\infty}([t_i, t_f], D(A^s))$ will be used to guarantee that our control is an essentially bounded function. Actually as we will see it would be enough to have $P^J\tilde{F} \in L^{\infty}([t_i, t_f], D(A^s))$, but since in the following we shall apply this Lemma to an increasing and "dense" sequence of subspaces (the sequence of Definition 6.1) we will need $\tilde{F} \in L^{\infty}([t_i, t_f], D(A^s))$.

systems

$$\begin{cases} u^J_t & = P^J(u_t) = -\nu P^J Au - P^J Bu + \nu P^J Cu + P^J \tilde{F} \\ u^J(t_i) & = q_i \end{cases}$$ [20]

$$\begin{cases} U^J_t & = P^{-J}(u_t) = -\nu P^{-J} Au - P^{-J} Bu + \nu P^{-J} Cu + P^{-J} \tilde{F} \\ U^J(t_i) & = Q_i. \end{cases} \tag{5.1}$$

In the system (5.1), since we want to realize q in $\mathbb{J}$ we replace u by $U^J + q$ arriving, in this way, to the (closed) system

$$\begin{cases} U^J_t & = -\nu P^{-J} A(U^J + q) - P^{-J} B(U^J + q) + \nu P^{-J} C(U^J + q) + P^{-J} \tilde{F} \\ U^J(t_i) & = Q_i. \end{cases} \tag{5.2}$$

This system leads us to a formulation of a "weak" problem similar to Problem 3.1. Now the problem is that of finding a (weak) solution

$$U^J \in L^\infty([t_i,\, t_f],\, \mathbb{J}^\perp_H) \cap L^2(t_i,\, t_f,\, \mathbb{J}^\perp_V);\; U^J_t \in L^1(t_i,\, t_f,\, (\mathbb{J}^\perp_V)')$$

for

$$\begin{aligned} U^J_t &= \Big[-\nu A(U^J + q) - B(U^J + q) + \nu C(U^J + q) + \tilde{F} \Big]_{V'} \\ &=: \mathbb{L}(U^J,\, q), \qquad\qquad \text{on }]t_i,\, t_f[; \\ U^J(t_i) &= Q_i. \end{aligned}$$

where the first identity must be satisfied in $(\mathbb{J}^\perp_V)'$. More precisely $\mathbb{L}(U^J,\, q)$ is the element of $(\mathbb{J}^\perp_V)'$ defined by

$$\begin{aligned} &< \mathbb{L}(U^J,\, q),\, v >_{V',\,V} \\ &:=< -\nu A(U^J + q) - B(U^J + q) + \nu C(U^J + q) + \tilde{F},\, v >_{V',\,V}; \\ &v \in \mathbb{J}^\perp_V. \end{aligned}$$

Remark 6. *Note that* $V = \mathbb{J} \oplus \mathbb{J}^\perp_V$. *The inclusion* $V \supseteq \mathbb{J} \oplus \mathbb{J}^\perp_V$ *follows from* $V \supseteq \mathbb{J} \cup \mathbb{J}^\perp_V$. *On the other hand, if* $v \in V \subset H$ *we may write* $v = P^J v + P^{-J} v$ *from which, we obtain* $P^{-J} v = v - P^J v \in V$ *and then,* $V \subset \mathbb{J} \oplus \mathbb{J}^\perp_V$.
Since $\mathbb{J} \subset D(A) \subset V'$, *we may write*

$$V' = (\mathbb{J} \oplus \mathbb{J}^\perp_V)' = \mathbb{J}' \oplus (\mathbb{J}^\perp_V)' = \mathbb{J} \oplus (\mathbb{J}^\perp_V)'.$$

$\diamond$ Existence.

We can prove existence and uniqueness of a strong solution, defined in $[t_i,\, t_f]$, for this system as we prove existence and uniqueness for the "full" equation. We give a sketch of the proof:

We start by choosing an orthonormal basis $\{e_i \mid i \in \mathbb{N}_0\}$ in the space $\mathbb{J}^\perp_H$. Since $D(A)$ is dense in H, $P^{-J} D(A)$ is dense in $P^{-J} H = \mathbb{J}^\perp_H$. So we can suppose that $\{e_i \mid i \in \mathbb{N}_0\}$ is contained in $P^{-J} D(A)$. Then for each element e_i there is $E_i \in D(A)$ such that $E_i = e_i + P^J E_i$, in particular $e_i \in D(A)$.

Defining approximate solutions

[20] Note that $\frac{\partial}{\partial t}$ commutes with $P^{\pm J}$ because $P^{\pm J}$ is linear and the elements of a basis $\{e_i\}$ for H do not depend on t. In fact we can not conclude the same with A in the place of $\frac{\partial}{\partial t}$.

$$
\begin{aligned}
&U^{J,L} := \sum_{j\le L} U_j^{J,L} e_j;\\
&< U_t^{J,L},\, e_j > = < \mathbb{L}(U^{J,L},\, q),\, e_j >\\
&= -\,\nu < A(U^{J,L}+q),\, e_j > - < B(U^{J,L}+q),\, e_j >\\
&+ \nu < C(U^{J,L}+q),\, e_j > + < \tilde F,\, e_j >; \qquad \forall j \le L\\
&U^{J,L}(t_i) = Q_i^L, = \text{ projection of } Q_i \text{ onto } span\{e_i \mid i \le L\},
\end{aligned}
\tag{5.3}
$$

from which we obtain the ODE

$$
\begin{aligned}
\dot U_j^{J,L} &= -\nu((U^{J,L}+q,\, e_j)) - \sum_{m,n\le L} u_m^{J,L} u_n^{J,L} b(e_m,\, e_n,\, e_j)\\
&\quad + \nu(C(U^{J,L}+q),\, e_j) + < \tilde F,\, e_j >;\\
U_j^{J,L} &= (Q_i^L)_j;
\end{aligned}
$$

that has a maximal solution defined on $[t_i, t_{max}[$. Now we compute some estimates that, in particular, imply $t_{max} = t_f$:
Multiplying, for each j, the equation (5.3) by $U_j^{J,L}$ and summing up we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}|U^{J,L}|^2 &\le -\nu\|U^{J,L}\|^2 + \nu\|q\|\|U^{J,L}\|\\
&\quad + |(B(U^{J,L}+q), U^{J,L})| + \nu K(\|U^{J,L}\| + \|q\|)|U^{J,L}| + |\tilde F||U^{J,L}|.
\end{aligned}
$$

Hence

$$
\frac{d}{dt}|U^{J,L}|^2 + \nu\|U^{J,L}\|^2 \le C_0\|q\|^2 + C_0|U^{J,L}|^2\|q\|^2 + C_0\|q\|^4 + C_0|U^{J,L}|^2 + C_0|\tilde F|^2 \tag{5.4}
$$

from which we obtain

$$
\begin{aligned}
&|U^{J,L}(s)|^2\\
&\le \exp\Big(C_0\int_{t_i}^{t_f} \|q(t)\|^2 + 1\, dt\Big)\Big(|Q_i|^2 + C_0\int_{t_i}^{t_f} \|q(t)\|^4 + \|q(t)\|^2 + |\tilde F(t)|^2\, dt\Big)\\
&\le C_2(\|Q_i\|^2 + 1), \qquad s \in [t_i,\, t_f]
\end{aligned}
$$

or

$$
\|U^{J,L}\|_{L^\infty(t_i,\, t_f,\, \mathbb{J}_H^\perp)} \le D_1. \tag{5.5}
$$

for some constant D_1 depending only in $\|q\|_{L^4(t_i,\, t_f,\, \mathbb{J}}$ and $\|Q_i\|$. [21]
From (5.4) we can also obtain

$$
\begin{aligned}
\int_{t_i}^{t_f} \|U^{J,L}(t)\|^2\, dt &\le C_3\|U^{J,L}\|^2_{L^\infty(t_i,\, t_f,\, \mathbb{J}_H^\perp)}\Big(1 + \int_{t_i}^{t_f} \|q(t)\|^2\, dt\Big)\\
&\quad + C_3\int_{t_i}^{t_f} \|q(t)\|^2 + \|q(t)\|^4 + |\tilde F(t)|^2\, dt.
\end{aligned}
$$

Hence for a constant D_2 depending only in $\|q\|_{L^4([t_i,\, t_f],\, \mathbb{J})}$ and $\|Q_i\|$ we have

$$
\|U^{J,L}\|_{L^2(t_i,\, t_f,\, \mathbb{J}_V^\perp)} \le D_2. \tag{5.6}
$$

[21]Note that t_i, t_f and $\tilde F$ are fixed, otherwise D_1 would depend also on $\|\tilde F\|_{L^2(t_i,\, t_f,\, V')}$ and on the length $|t_f - t_i|$. We will need to apply this Lemma to cases where the force $\tilde F$ is fixed and the lengths $|t_f - t_i|$ are upper-bounded by some $T > 0$ (in fact in the applications we will have $t_i,\, t_f \in [0,\, T]$), so the constant can be taken independent of $\tilde F$, t_i and t_f. Note that in $\mathbb{J}$ all norms are equivalent.

From (5.5) and (5.6) we have that

$$\begin{cases} (U^{J,L})_L \text{ remains in a bounded subset of } L^\infty([t_i,\, t_f],\, H)\\ (U^{J,L})_L \text{ remains in a bounded subset of } L^2(t_i,\, t_f,\, V). \end{cases} \tag{5.7}$$

Similarly as we have done in ([8], proof of Theorem 2.6.4) we extend $U^{J,L}$ to the entire real line putting

$$\tilde{U}^{J,L}(t) := \begin{cases} U^{J,L}(t) & \text{if } t \in [t_i,\, t_f]\\ 0 & \text{if } t \notin [t_i,\, t_f] \end{cases}$$

and the Fourier transform of $\tilde{U}^{J,L}$ will be denoted by $\hat{U}^{J,L}$.
Now we estimate the integral

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} |\hat{U}^{J,L}(\tau)|^2 \, d\tau, \quad 0 < \gamma < \frac{1}{4}. \tag{5.8}$$

From equation (5.3) with $U^{J,L}$ replaced by $\tilde{U}^{J,L}$ we derive

$$\frac{d}{dt}(\tilde{U}^{J,L}(t),\, e_j) = < \tilde{f}^{J,L},\, e_j > + (U^{J,L}(t_i),\, e_j)\delta_{t_i} - (U^{J,L}(t_f),\, e_j)\delta_{t_f}; \quad j \le L \tag{5.9}$$

where δ_{t_i}, δ_{t_f} are the Dirac distributions at t_i and t_f and,

$$\begin{aligned} f^{J,L} &= P^{-J}\Big[-\nu A(U^{J,L}+q) - B(U^{J,L}+q) + \nu C(U^{J,L}+q) + \tilde{F}\Big],\\ \tilde{f}^{J,L}(t) &:= \begin{cases} f^{J,L}(t) & \text{on } [t_i,\, t_f]\\ 0 & \text{outside } [t_i,\, t_f] \end{cases}. \end{aligned}$$

Using the Fourier Transform, (5.9) becomes

$$\begin{aligned} 2\pi i\tau(\hat{U}^{J,L}(\tau),\, e_j) &= < \hat{f}^{J,L}(\tau),\, e_j >\\ &\quad + (u^{J,L}(t_i),\, e_j)\exp(-2\pi i t_i\tau) - (U^{J,L}(t_f),\, e_j)\exp(-2\pi i t_f\tau). \end{aligned} \tag{5.10}$$

Multiplying (5.10) by $\hat{U}_j^{J,L}(\tau)$ and adding the obtained L equations we arrive to

$$\begin{aligned} 2\pi i\tau|\hat{U}^{J,L}(\tau)|^2 &= < \hat{f}^{J,L}(\tau),\, \hat{U}^{J,L}(\tau) >\\ &\quad + (U^{J,L}(t_i),\, \hat{U}^{J,L}(\tau))\exp(-2\pi i t_i\tau) - (U^{J,L}(t_f),\, \hat{U}^{J,L}(\tau))\exp(-2\pi i t_f\tau). \end{aligned} \tag{5.11}$$

From

$$\begin{aligned} \|f^{J,L}(t)\|_{V'} &\le \|\tilde{F}(t)\|_{V'} + C_1\Big(\|(U^{J,L}+q)(t)\| + \|(U^{J,L}+q)(t)\|^2 + |U^{J,L}(t)|\Big)\\ &\le \|\tilde{F}(t)\|_{V'} + C_2\Big(1 + \|(U^{J,L}(t)\|^2 + \|q(t)\|^2\Big) \end{aligned}$$

we have that the integral

$$\int_{t_i}^{t_f} \|f^{J,L}(t)\|_{V'} \, dt$$

is uniformly bounded and so, for some constant C_0

$$\sup_{\tau\in\mathbb{R}} \|\hat{f}^{J,L}(\tau)\|_{V'} \le C_0.$$

Now by (5.11) we have

$$|\tau||\hat{U}^{J,L}(\tau)|^2 \le C_1\big(\|\hat{U}^{J,L}(\tau)\| + |\hat{U}^{J,L}(\tau)| \le D\|\hat{U}^{J,L}(\tau)\| \tag{5.12}$$

then, analogously as we have done in ([8], proof of Theorem 2.6.4), we can conclude the uniform boundedness of (5.8) and then,

(5.13) *the sequence* $(U^{J,L})$ *remains in a bounded set of* $\mathcal{H}^\gamma(\mathbb{R},\, \mathbb{J}_V^\perp,\, \mathbb{J}_H^\perp)$.

The existence of a weak solution

$$U^J \in L^\infty([t_i,\, t_f],\, \mathbb{J}_H^\perp) \cap L^2(t_i,\, t_f,\, \mathbb{J}_V^\perp) \tag{5.14}$$

follows from standard compactness results.

We extend $\mathbb{L}(U^J+q)$ to $\overline{\mathbb{L}}(U^J+q) \in V'$, as usually, by $<\mathbb{L}(U^J+q),\, v>:=<\mathbb{L}(U^J+q),\, P^{-J}v>$ for every $v \in V$, i.e., $\overline{\mathbb{L}}(U^J+q) = \mathbb{L}(U^J+q) \circ P^{-J}$.

Computing $(U^J+q)_t$ we obtain the following equalities in V'

$$\begin{aligned}
&(U^J+q)_t = \overline{\mathbb{L}}(U^J+q) + q_t \\
&= -\nu A(U^J+q) - B(U^J+q) + \nu C(U^J+q) + \tilde{F} \\
&\quad - \Big(-\nu A(U^J+q) - B(U^J+q) + \nu C(U^J+q) + \tilde{F}\Big) \circ P^J + q_t.
\end{aligned}$$

Put $\tilde{G} := -\Big(-\nu A(U^J+q) - B(U^J+q) + \nu C(U^J+q) + \tilde{F}\Big) \circ P^J + q_t$. Then for $|\tilde{G}|$ we obtain the estimate

$$\begin{aligned}
|\tilde{G}| &\le |q_t| + \Big|\Big(-\nu A(U^J+q) - B(U^J+q) + \nu C(U^J+q) + \tilde{F}\Big) \circ P^J\Big| \\
&= |q_t| + \Big|\Big(-\nu A(U^J+q) - B(U^J+q) + \nu C(U^J+q) + \tilde{F}\Big) \circ P^J\Big|_{H'} \\
&\Big(\text{putting } G := -\nu A(U^J+q) - B(U^J+q) + \nu C(U^J+q) + \tilde{F}\Big) \\
&\le |q_t| + \sup_{x \in H\setminus\{0\}} \frac{|G \circ P^J(x)|}{|x|} = |q_t| + \sup_{x \in H\setminus\{0\}} \frac{|G(P^J(x))|}{|x|};
\end{aligned}$$

since the orthogonal projection $P^J : H \to \mathbb{J}$ is continuous, we have

$$\frac{1}{C_0}|P^J(x)| \le |x|$$

for some constant $C_0 > 0$ and, since all norms are equivalent in $\mathbb{J}$:

$$\begin{aligned}
|\tilde{G}| &\le |q_t| + \sup_{\substack{\in H\setminus\{0\} \\ P^J(x)\neq 0}} C_0 \frac{|G(P^J(x))|}{|P^J(x)|} = |q_t| + \sup_{y \in \mathbb{J}\setminus\{0\}} C_0 \frac{|G(y)|}{|y|} \\
&\le |q_t| + \sup_{y \in \mathbb{J}\setminus\{0\}} C_1 \frac{|G(y)|}{\|y\|} \le |q_t| + \sup_{y \in V\setminus\{0\}} C_1 \frac{|G(y)|}{\|y\|},
\end{aligned}$$

i.e., $|\tilde{G}| \le |q_t| + C_1 \|G\|_{V'}$. Hence

$$\begin{aligned}
|\tilde{G}| &\le D_1\Big(1 + \|U^J+q\| + |U^J+q|\|U^J+q\| + |U^J+q| + |\tilde{F}|_{V'}\Big) \\
&\le D_2\Big(|\tilde{F}| + 1 + \|U^J+q\|\Big).
\end{aligned}$$

Hence

$$|\tilde{G}|^2 \le D\Big(|\tilde{F}|^2 + 1 + \|U^J+q\|^2\Big),$$

i.e., $\tilde{G} \in L^2(t_i,\, t_f,\, H)$. Therefore U^J+q is the unique solution given by Theorem 3.8 for the external force $\tilde{G}$ and initial condition $u_i = q(t_i) + Q_i$. [22] By Theorem 3.12 U^J+q is strong, i.e., $U^J+q \in L^\infty([t_i,\, t_f],\, V) \cap L^2(t_i,\, t_f,\, D(A))$ and then

$$U^J \in L^\infty([t_i,\, t_f],\, \mathbb{J}_V^\perp) \cap L^2(t_i,\, t_f,\, \mathbb{J}_{D(A)}^\perp) \tag{5.15}$$

where $\mathbb{J}_{D(A)}^\perp := \mathbb{J}_H^\perp \cap D(A)$.

[22] Note that in Theorem 3.8 $t_i = 0$ but, if we change the initial time to $t_i \neq 0$ that Theorem follows analogously.

◇ Uniqueness.

For the uniqueness we consider the difference w of two solutions $V^J = V^J(q,\, Q_i)$ and $U^J = U^J(q,\, Q_i)$ — $w := V^J - U^J$. From the equations

$$U_t^J = -\nu P^{-J} A(U^J + q) - P^{-J} B(U^J + q) + \nu P^{-J} C(U^J + q) + P^{-J}\tilde{F};$$
$$V_t^J = -\nu P^{-J} A(V^J + q) - P^{-J} B(V^J + q) + \nu P^{-J} C(V^J + q) + P^{-J}\tilde{F};$$

we derive

$$w_t = -\nu P^{-J} Aw - P^{-J} B(V^J + q) + P^{-J} B(U^J + q) + \nu P^{-J} Cw;$$

multiplying by w:

$$(w_t,\, w) = -\nu(Aw,\, w) - (B(V^J + q),\, w) + (B(U^J + q),\, w) + \nu(Cw,\, w);$$

so

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 \le |-(B(V^J + q),\, w) + (B(U^J + q),\, w)| + \nu K\|w\||w|$$

and,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2 + \nu\|w\|^2 &\le C_0|(B(w,\, w,\, U^J + q)| + \nu K\|w\||w|\\ &\le C_1|w|\|w\|\|U^J + q\| + \nu K\|w\||w|;\\ \frac{d}{dt}|w|^2 + \nu\|w\|^2 &\le C_2|w|^2\|U^J + q\|^2 + C_2|w|^2;\end{aligned}$$

$$|w(s)|^2 \le |w(0)|^2 \exp\Big(C_2\int_{t_i}^{t_f} \|U^J(t) + q(t)\|^2 + 1\, dt\Big) = 0.$$

Hence the weak solution is unique, and then so is the strong one.

◇ Continuity.

We have just proved that the map $(q,\, Q_i) \mapsto U^J(q,\, Q_i)$ is well defined. We claim that it is $(L^4(t_i,\, t_f,\, \mathbb{J}) \times (\mathbb{J}_V^\perp),\, X)$-continuous, where X is either $L^\infty([t_i,\, t_f],\, \mathbb{J}_V^\perp)$ or $L^2(t_i,\, t_f,\, \mathbb{J}_{D(A)}^\perp)$. To prove these continuities we proceed as usually: Fix a pair $(q,\, Q_i) \in (W^{1,\infty} \times \mathbb{J}_V^\perp)$ and consider another one $(p,\, P_i)$ in the same product space. Define $w := U^J(q,\, Q_i) - U^J(p,\, P_i)$. To simplify the writing we put $Q := U^J(q,\, Q_i)$ and $P := U^J(p,\, P_i)$. Then w satisfies the following equation:

$$\begin{aligned} w_t &= -\nu P^{-J} A(Q + q) + \nu P^{-J} A(P + p)\\ &\quad - P^{-J} B(Q + q) + P^{-J} B(P + p) + \nu P^{-J} C(Q + q) - \nu P^{-J} C(P + p)\\ &= -\nu P^{-J} Aw - \nu P^{-J} A(q - p)\\ &\quad - P^{-J} B(Q + q) + P^{-J} B(P + p) + \nu P^{-J} Cw + \nu P^{-J} C(q - p) \end{aligned} \tag{5.16}$$

and, multiplying by w:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|^2 \le &-\nu\|w\|^2 + \nu\|q - p\|\|w\| + |b(P,\, P,\, w) - b(Q,\, Q,\, w)|\\ &+ |b(P,\, p,\, w) - b(Q,\, q,\, w)| + |b(p,\, P,\, w) - b(q,\, Q,\, w)|\\ &+ |b(p,\, p,\, w) - b(q,\, q,\, w)| + \nu K\|w\||w| + \nu K\|q - p\||w|\\ \le &-\nu\|w\|^2 + \nu\|q - p\|\|w\| + |b(w,\, w,\, Q)|\\ &+ \Big\{|b(-w,\, p,\, w) + b(Q,\, p - q,\, w)|\Big\} + |b(p - q,\, Q,\, w)|\\ &+ \Big\{|b(p - q,\, p,\, w) + b(q,\, p - q,\, w)|\Big\} + \nu K\|w\||w| + \nu K\|q - p\||w|;\end{aligned}$$

from which we obtain

$$\frac{d}{dt}|w|^2+\nu\|w\|^2 \le C_0\|p-q\|^2+C_0|w|^2\|Q\|^2+C_0\|Q\|^2\|p-q\|^2+C_0|w|^2\|p\|^2$$
$$+C_0(\|p\|^2+\|q\|^2)\|p-q\|^2+C_0|w|^2. \tag{5.17}$$

Then by Gronwall Inequality

$$\|w\|^2_{C([t_i,\,t_f],\,\mathbb{J}^\perp_H)} \le \exp\Big[C_0\int_{t_i}^{t_f}\|Q(t)\|^2+\|p(t)\|^2+1\,dt\Big]\bigg(|w(t_i)|^2$$
$$+C_0\int_{t_i}^{t_f}\|p-q\|^2(1+\|Q(t)\|^2+\|p(t)\|^2+\|q(t)\|^2)\,dt\bigg).$$

For $\|p-q\|_{L^4}\le 1$ we obtain[23]

$$\|w\|^2_{C([t_i,\,t_f],\,\mathbb{J}^\perp_H)} \le C_1\|Q_i-P_i\|^2+C_1\|p-q\|^2_{L^4} \tag{5.18}$$

with C_1 independent of $(p,\,P_i)$.
We conclude that the map U^J is $(L^4\times(\mathbb{J}^\perp_V),\,C([t_i,\,t_f],\,\mathbb{J}^\perp_H))$-continuous.

From (5.17) we can also obtain

$$\int_{t_i}^{t_f}\|w(t)\|^2\,dt$$
$$\le C_0\|p-q\|^2_{L^2}+C_0\|w\|^2_{C([t_i,\,t_f],\,\mathbb{J}^\perp_H)}\Big(1+\int_{t_i}^{t_f}\|Q(t)\|^2+\|p(t)\|^2\,dt\Big)$$
$$+C_0\|p-q\|^2_{L^4}\Big(\|Q\|^2_{L^4}+\|p\|^2_{L^4}+\|q\|^2_{L^4}\Big)$$
$$\le C_2\|w\|^2_{C([0,T],\,\mathbb{J}^\perp_H)}+C_2\|p-q\|^2_{L^4},$$

for $\|p-q\|_{L^4}\le 1$. Hence by (5.18) we arrive to

$$\Big[\|p-q\|_{L^4}\le 1\Big]\Rightarrow\Big[|w|^2_{L^2(t_i,\,t_f,\,\mathbb{J}^\perp_V)}\le C_3\|Q_i-P_i\|^2+C_3\|p-q\|^2_{L^4}\Big] \tag{5.19}$$

with C_3 independent of $(p,\,P_i)$.
We conclude that the map U^J is $(L^4\times(\mathbb{J}^\perp_V),\,L^2(t_i,\,t_f,\,\mathbb{J}^\perp_V))$-continuous.

Multiplying (5.16) by $P^{-J}Aw$ we obtain

$$(w_t,\,Aw)=-\nu(Aw,\,Aw-P^JAw)-\nu(A(q-p),\,P^{-J}Aw)$$
$$-(B(Q+q),\,P^{-J}Aw)+(B(P+p),\,P^{-J}Aw)$$
$$+\nu(Cw,\,P^{-J}Aw)+\nu(C(q-p),\,P^{-J}Aw).$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|w\|^2+\nu|w|^2_{[2]}$$
$$\le\nu\|w\|\|P^JAw\|+\nu|q-p|_{[2]}|P^JAw|+\nu K\|w\||P^{-J}Aw|+\nu K\|q-p\||P^{-J}Aw|$$
$$+|(B(P+p),\,P^{-J}Aw)-(B(Q+q),\,P^{-J}Aw)|$$
$$\le C_0\|w\||P^JAw|_{D(A^{-\frac{1}{2}})}+\nu|q-p|_{[2]}|w|_{[2]}+\nu K\|w\||w|_{[2]}+\nu K\|q-p\||w|_{[2]}$$
$$+|(B(P+p),\,P^{-J}Aw)-(B(Q+q),\,P^{-J}Aw)|$$

[23] Of course we could ask for $\|p-q\|_{L^4}\le D$ for any $D>0$. What we need is a first bound for $\|p\|_{L^4}$.

From which, using the equalities

$$\begin{aligned}
&b(P,\, P,\, x) - b(Q,\, Q,\, x) &&= b(w,\, w,\, x) - b(w,\, Q,\, x) - b(Q,\, w,\, x);\\
&b(P,\, p,\, x) - b(Q,\, q,\, x) &&= -b(w,\, p,\, x) + b(Q,\, p-q,\, x);\\
&b(p,\, P,\, x) - b(q,\, Q,\, x) &&= -b(p,\, w,\, x) + b(p-q,\, Q,\, x);\\
&b(p,\, p,\, x) - b(q,\, q,\, x) &&= b(p-q,\, p,\, x) + b(q,\, p-q,\, x);
\end{aligned} \tag{5.20}$$

we derive, by appropriate Young inequalities,

$$\begin{aligned}
&\frac{d}{dt}\|w\|^2 + \nu |w|^2_{[2]}\\
&\leq D\|w\|^2 + D|p-q|^2_{[2]} + D\|w\|^2 + D\|q-p\|^2 \qquad (5.21)\\
&\quad + D\Big(|w|^2\|w\|^4 + \|w\|^2\|Q\|^4\Big) + D\Big(\|w\|^2|p|^2_{[2]} + \|Q\|^2|p-q|^2_{[2]}\Big)\\
&\quad + D\Big(|p|^2_{[2]}\|w\|^2 + |p-q|^2_{[2]}\|Q\|^2\Big) + D\Big(|p-q|^2_{[2]}\|p\|^2 + \|q\|^2|p-q|^2_{[2]}\Big). \qquad (5.22)
\end{aligned}$$

Thus

$$\begin{aligned}
&\|w\|^2_{C([t_i,\,t_f],\,\mathbb{J}^\perp_V)}\\
&\leq \exp\Big[D_0\int_{t_i}^{t_f} 1 + |w(t)|^2\|w(t)\|^2 + \|Q(t)\|^4 + |p(t)|^2_{[2]}\,dt\Big]\Big(\|w(t_i)\|^2\\
&\qquad + D_0\int_{t_i}^{t_f} |p(t)-q(t)|^2_{[2]}(1 + \|Q(t)\|^2 + \|p(t)\|^2 + \|q(t)\|^2)\,dt\Big).
\end{aligned}$$

Then using (5.18) and (5.19), we have

$$\left\{\begin{array}{l} \|p-q\|_{L^4} \leq 1\\ \|P_i - Q_i\|_{\mathbb{J}^\perp_V} \leq 1 \end{array}\right. \Rightarrow \Big[\|w\|^2_{C([t_i,\,t_f],\,\mathbb{J}^\perp_V)} \leq D_1\|Q_i - P_i\|^2 + D_1\|p-q\|^2_{L^4}\Big]. \tag{5.23}$$

with D_1 independent of $(p,\, P_i)$.
We conclude that the map U^J is $(L^4 \times (\mathbb{J}^\perp_V),\, C([t_i,\, t_f],\, \mathbb{J}^\perp_V))$-continuous.

From (5.22) we can also obtain

$$\begin{aligned}
&\int_{t_i}^{t_f} |w(t)|^2_{[2]}\,dt\\
&\leq D_2\|w\|^2_{C([0,\,T],\,\mathbb{J}^\perp_V)}\Big(1 + \int_{t_i}^{t_f} |w(t)|^2\|w(t)\|^2 + \|Q(t)\|^4 + |p(t)|^2_{[2]}\,dt\Big)\\
&+ D_2\|p-q\|^2_{L^4}\Big(1 + \|p\|^2_{L^4} + \|q\|^2_{L^4} + \|Q\|^2_{L^4(t_i,\,t_f,\,\mathbb{J}^\perp_V)}\Big).
\end{aligned}$$

Hence by (5.18) and (5.19), we have

$$\left\{\begin{array}{l} \|p-q\|_{L^4} \leq 1\\ \|P_i - Q_i\|_{\mathbb{J}^\perp_V} \leq 1 \end{array}\right.$$
$$\Rightarrow \Big[\|w\|^2_{L^2(t_i,\,t_f,\,\mathbb{J}^\perp_{D(A)})} \leq D_3\|w\|^2_{C([t_i,\,t_f],\,\mathbb{J}^\perp_V)} + D_3\|p-q\|^2_{L^4}\Big]$$

and, using (5.23),

$$\left\{\begin{array}{l} \|p-q\|_{L^4} \leq 1\\ \|P_i - Q_i\|_{\mathbb{J}^\perp_V} \leq 1 \end{array}\right. \Rightarrow \Big[\|w\|^2_{L^2(t_i,\,t_f,\,\mathbb{J}^\perp_{D(A)})} \leq D_4\|Q_i - P_i\|^2 + D_4\|p-q\|^2_{L^4}\Big] \tag{5.24}$$

with D_4 independent of $(p,\, P_i)$.
We conclude that the map U^J is $(L^4 \times (\mathbb{J}^\perp_V),\, L^2(t_i,\, t_f,\, \mathbb{J}^\perp_{D(A)}))$-continuous.

$\diamond$ The map Γ.

Now we define another map in the product $W^{1,\infty}([t_i,\, t_f],\, \mathbb{J}) \times \mathbb{J}_V^{\perp}$, taking values on H for a.e. $t \in]t_i,\, t_f[$:

$$\Gamma^J : (q,\, Q_i) \mapsto -\nu A(U^J(q,\, Q_i) + q) - B(U^J(q,\, Q_i) + q) + \nu C(U^J(q,\, Q_i) + q) + \tilde{F}$$

Fix $(q,\, Q_i) \in W^{1,\infty}([t_i,\, t_f],\, \mathbb{J}) \times \mathbb{J}_V^{\perp}$ and, consider another pair $(p,\, P_i)$ in the same space. Again, as we have done before, put $Q = U^J(q,\, Q_i)$ and $P = U^J(p,\, P_i)$. For the norm of the difference we can obtain the estimate

$$\begin{aligned}
&\left|\Gamma^J(q,\, Q_i) - \Gamma^J(p,\, P_i)\right| \\
=&\Big|\nu A(P+p) + B(P+p) - \nu C(P+p) \\
&\qquad\qquad - \nu A(Q+q) - B(Q+q) + \nu C(Q+q)\Big| \\
\le&\nu|P-Q|_{[2]} + \nu|p-q|_{[2]} + D|P-Q|_{[2]}(\|P\| + \|Q\|) \\
+&D\Big(|P-Q|_{[2]}\|q\| + \|P\||p-q|_{[2]}\Big) + D\Big(\|P\||p-q|_{[2]} + |P-Q|_{[2]}\|q\|\Big) \\
+&D|p-q|_{[2]}(\|p\| + \|q\|) + \nu K\|Q-P\| + \nu K\|p-q\|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&|\Gamma^J(q,\, Q_i) - \Gamma^J(p,\, P_i)| \\
\le& C_0|P-Q|_{[2]}\Big(1 + \|P\| + \|Q\| + \|q\|\Big) + C_0|p-q|_{[2]}\Big(1 + \|P\| + \|p\| + \|q\|\Big).
\end{aligned}$$

For $\|p-q\|_{L^4} \le 1$, and $\|P_i - Q_i\| \le 1$, using (5.23) we obtain $\|P(t) - Q(t)\|^2 \le 2D_1$ and then $\|P(t)\| \le \sqrt{2D_1} + \|Q(t)\|$. So we can arrive to

$$|\Gamma^J(q,\, Q_i) - \Gamma^J(p,\, P_i)| \le C_2|P-Q|_{[2]} + C_2|p-q|_{[2]}\Big(1 + \|p\|\Big) \tag{5.25}$$

from which we obtain

$$\begin{aligned}
&\int_{t_i}^{t_f} |\Gamma^J(q,\, Q_i) - \Gamma^J(p,\, P_i)|^2\, dt \\
\le& 2C_2^2 \int_{t_i}^{t_f} |P-Q|_{[2]}^2\, dt + 2C_2^2 \int_{t_i}^{t_f} |p-q|_{[2]}^2 \Big(1 + \|p\|\Big)^2 dt \\
\le& 2C_2^2\|P-Q\|^2_{L^2(t_i,\, t_f,\, D(A))} + 2C_2^2 \int_{t_i}^{t_f} |p-q|_{[2]}^2 2\Big(1 + \|p\|^2\Big)\, dt \\
\le& C_4\|P-Q\|^2_{L^2(t_i,\, t_f,\, D(A))} + C_4\|p-q\|^2_{L^4}(1 + \|p\|^2_{L^4}).
\end{aligned}$$

Therefore, using (5.24):

$$\begin{aligned}
&\begin{cases} \|p-q\|_{L^4} \le 1 \\ \|P_i - Q_i\|_{\mathbb{J}_V^{\perp}} \le 1 \end{cases} \\
&\Rightarrow \|\Gamma^J(q,\, Q_i) - \Gamma^J(p,\, P_i)\|^2_{L^2(t_i,\, t_f,\, H)}\, dt \le C_5\|P_i - Q_i\|^2 + C_5\|p-q\|^2_{L^4}
\end{aligned} \tag{5.26}$$

with C_5 independent of $(p,\, P_i)$.
We conclude the $(L^4 \times (\mathbb{J}_V^{\perp}),\, L^2(t_i,\, t_f,\, H))$-continuity of Γ^J.

$\diamond$ THE CONTROL v^J.

Now we can indicate which is the control v^J appearing in the statement of the proposition: In fact

$$v^J(q,\, Q_i) := \dot{q} - P^J\Gamma^J(q,\, Q_i)$$

satisfies the statement. Indeed its $(W^{1,2}([t_i,\, t_f],\, \mathbb{J}) \times (\mathbb{J}_V^{\perp}),\, L^2(t_i,\, t_f,\, \mathbb{J})$-continuity follows from the $(L^4 \times (\mathbb{J}_V^{\perp}),\, L^2(t_i,\, t_f,\, H)$-continuity of Γ^J and from the $(W^{1,2}([t_i,\, t_f],\, \mathbb{J}),\, L^2(t_i,\, t_f,\, \mathbb{J})$-continuity of $q \mapsto \dot{q}$.

To prove that the projection of the solution of the system

$$u_t = -\nu Au - Bu + \nu u + \tilde{F} + v^J(q,\, Q_i), \qquad u_i = q(t_i) + Q_i \tag{5.27}$$

coincides with q we differentiate $q + U^J(q,\, Q_i)$:

$$\begin{aligned}
&[q + U^J(q,\, Q_i)]_t \\
&= \dot{q} - \nu P^{-J} A(U^J(q,\, Q_i)) - P^{-J} B(U^J(q,\, Q_i) + q) \\
&\quad + \nu P^{-J} C(U^J(q,\, Q_i) + q) + P^{-J}\tilde{F} \\
&= -\nu A(q + U^J(q,\, Q_i)) - B(q + U^J(q,\, Q_i)) + \nu u + \tilde{F} \\
&\quad + \dot{q} - \Big[-\nu P^J A(q + U^J(q,\, Q_i)) - P^J B(q + U^J(q,\, Q_i)) \\
&\qquad\qquad + \nu P^J C(U^J(q,\, Q_i) + q) + P^J \tilde{F} \Big] \\
&= -\nu A(q + U^J(q,\, Q_i)) - B(q + U^J(q,\, Q_i)) + \tilde{F} + v^J(q,\, Q_i);
\end{aligned}$$

showing that $q + U^J(q,\, Q_i)$ is the (unique) solution of (5.27).

To finish the prove remains to verify that $v^J \in L^\infty([t_i,\, t_f],\, \mathbb{J})$. Let $s < -\frac{1}{2}$ be like in the statement of the Lemma. Since $q \in W^{1,\infty}([t_i,\, t_f],\, \mathbb{J})$ we have $\dot{q} \in L^\infty([t_i,\, t_f],\, \mathbb{J})$ and, by

$$\begin{aligned}
\|\Gamma^J(q,\, Q_i)\|_{D(A^s)} &\le \|\Gamma^J(q,\, Q_i) - \tilde{F}\|_{D(A^s)} + \|\tilde{F}\|_{D(A^s)} \\
&\le D_1 \|\Gamma^J(q,\, Q_i) - \tilde{F}\|_{V'} + \|\tilde{F}\|_{D(A^s)} \\
&\le D\|Q + q\| + D\|Q + q\|^2 + D|Q + q| + \|\tilde{F}\|_{D(A^s)} \\
&\le C_0 \Big(\|Q + q\|^2 + 1 \Big) \le C_1,
\end{aligned} \tag{5.28}$$

we have that $\Gamma^J(q,\, Q_i) \in L^\infty([t_i,\, t_f],\, \mathbb{D}(A^s))$ and then, $P^J\Gamma^J(q,\, Q_i)$ belongs to $L^\infty([t_i,\, t_f],\, \mathbb{J}_{D(A^s)})$, where $\mathbb{J}_{D(A^s)}$ denotes the space $\mathbb{J}$ endowed with the norm of $D(A^s)$; so, $P^J\Gamma^J(q,\, Q_i) \in L^\infty([t_i,\, t_f],\, \mathbb{J})$, for any norm we consider in $\mathbb{J}$. Hence

$$\|v^J(q,\, Q_i)\|_{L^\infty([t_i,\, t_f],\, \mathbb{J})} \le \|\dot{q}\|_{L^\infty([t_i,\, t_f],\, \mathbb{J})} + \|P^J\Gamma^J(q,\, Q_i)\|_{L^\infty([t_i,\, t_f],\, \mathbb{J})} \le D_2, \text{ [24]}$$

where D_2 depends only on $\|\dot{q}\|_{L^\infty([t_i,\, t_f],\, \mathbb{J})}$, $\|q\|_{L^\infty([t_i,\, t_f],\, \mathbb{J})}$ and $\|Q\|_{L^\infty([t_i,\, t_f],\, \mathbb{J}_V^{\perp})}$.

Using (5.23) the constant D_2 can be chosen depending only on the norm $\|q\|_{W^{1,\infty}([t_i,\, t_f],\, \mathbb{J})}$ and $\|Q_i\|$.

□

Remark 7 (Uniform estimates)**.** *From the computations of the previous Lemma, we can see that, given a uniformly bounded family*

$$\{(q_\lambda,\, Q_\lambda) \in C([t_i,\, t_f],\, \mathbb{J}) \times \mathbb{J}_V^{\perp} \mid \lambda \in \Lambda\},$$

i.e,

$$\exists C_0 > 0 \forall \lambda \in \Lambda \begin{cases} \|q_\lambda\|_{C([t_i,\, t_f],\, \mathbb{J})} \le C_0 \\ \|Q_\lambda\|_{\mathbb{J}_V^{\perp})} \le C_0; \end{cases}$$

[24] We can see that, for fixed $\mathbb{J}$ as in the present case, if we repeat the computations in (5.28) for $P^J\Gamma^J(q,\, Q_i) = P^J\Gamma^J(q,\, Q_i) - P^J\tilde{F} + P^J\tilde{F}$, we need only to have $\tilde{F} \in L^\infty([t_i,\, t_f],\, D(A^s))$ to obtain $|P^J\Gamma^J(q,\, Q_i)|_{D(A^s)} \le C_1$.

then all the following families are uniformly bounded (in the respective spaces):

$$\{U(q_\lambda,\, Q_\lambda) \in C(t_i,\, t_f,\, \mathbb{J}_V^\perp) \mid \lambda \in \Lambda\};$$
$$\{U(q_\lambda,\, Q_\lambda) \in L^2(t_i,\, t_f,\, \mathbb{J}_{D(A)}^\perp) \mid \lambda \in \Lambda\}$$
$$\{\Gamma(q_\lambda,\, Q_\lambda) \in L^\infty(t_i,\, t_f,\, D(A^s)) \mid \lambda \in \Lambda\}$$

and, the uniform bound can be taken depending only on C_0.
If we have

$$\exists C_1 > 0 \forall \lambda \in \Lambda \begin{cases} \|q_\lambda\|_{W^{1,\infty}([t_i,\, t_f],\, \mathbb{J})} \le C_1 \\ \|Q_\lambda\|_{\mathbb{J}_V^\perp)} \le C_1; \end{cases}$$

then also the family

$$\{v^J(q_\lambda,\, Q_\lambda) \in L^\infty([t_i,\, t_f],\, \mathbb{J}) \mid \lambda \in \Lambda\}.$$

is uniformly bounded and, the bound can be taken depending only on C_1.

5.1. **The $\mathbf{L}^2(t_i,\, t_f,\, H)$-norm of U_t^J .** From equation (5.2) we have

$$\begin{aligned} |U_t^J| &\le \nu|P^{-J}A(U^J+q)| + |P^{-J}B(U^J+q)| + \nu|P^{-J}C(U^J+q)| + |P^{-J}\tilde{F}| \\ &\le C_0\Big(|U^J+q|_{[2]} + \|U^J+q\||U^J+q|_{[2]} + \nu K\|U^J+q\| + |\tilde{F}|\Big); \end{aligned}$$

hence

$$|U_t^J|^2 \le C_1\Big(|U^J+q|_{[2]}^2 + 1 + |\tilde{F}|^2\Big)$$

from which we obtain that

$$\|U_t^J\|_{L^2(t_i,\, t_f,\, H)} \le C_2.$$

Moreover if we have a family of curves and points such that

$$\exists C > 0 \forall \lambda \in \Lambda \begin{cases} \|q_\lambda\|_{C([t_i,\, t_f],\, \mathbb{J})} \le D \\ \|Q_\lambda\|_{\mathbb{J}_V^\perp)} \le D; \end{cases}$$

then also the family

$$\{(U^J(q_\lambda,\, Q_\lambda))_t \mid \lambda \in \Lambda\};$$

is uniformly bounded and, the bound can be taken depending only on D. In particular we can take a bound independent of the derivative q_t; this fact was important in [8, 7] when we imitated the action of a given low modes control by another one, taking values in a smaller space, and compared projections of trajectories in the infinite dimensional component $\mathbb{J}_H^\perp$.

6. Saturating sets

In [7] a finite saturating set for system

$$u_t = -\nu Au - Bu + \nu Cu + F; \quad u(0) = u_0,$$

was defined as a finite subset $g \subset D(A)$ such that the sequence $(G^j)_{n\in\mathbb{N}}$ of subspaces of $D(A)$ defined recursively by

(1) $G^0 := span(g)$;
(2) $G^{j+1} := \Big(G^j + Conv\{BY \mid Y \in G^j\}\Big) \cap \Big(G^j - Conv\{BY \mid Y \in G^j\}\Big) \cap D(A)$

satisfies

$$\overline{\bigcup_{i\in\mathbb{N}} G^i} = H.$$

Now we present a more flexible definition of w-saturating set. Put $\mathbf{L}^4(\Omega) := (L^4(\Omega))^2$ and $H_4 := H \cap \mathbf{L}^4(\Omega)$.

Definition 6.1. *A finite set of vectors $g \subset H_4$ is said* **w-saturating** *for system*

$$u_t = -\nu Au - Bu + \nu Cu + F; \quad u(0) = u_0, \tag{6.1}$$

if the sequence $(G^j)_{n\in\mathbb{N}}$ of finite dimensional subspaces of H defined recursively by

(1) $G^0 := span(g)$;

(2) $G^{j+1} := \Big(G^j + Conv\overline{\{BY \mid Y \in G^j\} \cap H_4}\Big) \cap \Big(G^j - Conv\overline{\{BY \mid Y \in G^j\} \cap H_4}\Big)$

satisfies

$$\overline{\bigcup_{i\in\mathbb{N}} G^i} = H.$$

Here $\overline{\{BY \mid Y \in G^j\} \cap H_4}$ stays for the closure of $\{BY \mid Y \in G^j\} \cap H_4$ in H.[25]

Remark 8. *In the previous definition, the condition*

$$(A) \qquad \overline{\bigcup_{i\in\mathbb{N}} G^i} = H$$

is equivalent to

$$(B) \qquad \forall x \in H \, [j \to \infty \text{ only if } |x - \Pi_j x| \to 0]$$

Remark 9. *The linear space G^{j+1} is contained in $G_{j+1} := span\{B\gamma_n,\, B(\gamma_n,\, \gamma_m) + B(\gamma_m,\, \gamma_n) \mid n,\, m = 1,\, \dots,\, r\}$ where $\{\gamma_n \mid n = 1,\, \dots,\, r\}$ is any basis for G^j: for write $X \in G^j$ as $X = \sum_{i=1}^{r} X_i\gamma_i$ then*

- $B(X_1\gamma_1) = X_1^2 B\gamma_1 \in G_{j+1}$ *and;*
- *if* $B\Big(\sum_{i=1}^{p} X_i\gamma_i\Big) \in G_{j+1}$ *and* $1 \le p \le r-1$, *then*

$$B\Big(\sum_{i=1}^{p+1} X_i\gamma_i\Big) = B\Big(\sum_{i=1}^{p} X_i\gamma_i\Big) + X_{p+1}^2 B\gamma_{p+1} + \sum_{i=1}^{p} X_{p+1}X_i\Big(B(\gamma_{p+1},\, \gamma_i) + B(\gamma_i,\, \gamma_{p+1})\Big)$$

so, $B\Big(\sum_{i=1}^{p+1} X_i\gamma_i\Big) \in G_{j+1}$.

Hence $\{BX \mid X \in G^j\} \subseteq G_{j+1}$ and, consequently $G^{j+1} \subseteq G_{j+1}$.

In the case of Lions boundary conditions for $Y \in D(A)$ we have $\nabla^\perp \cdot BY = \nabla^\perp \cdot [(Y \cdot \nabla)Y + \nabla\psi]$ for some $\psi \in H^1(\Omega)$ so, $\nabla^\perp \cdot BY = Y \cdot \nabla[\nabla^\perp \cdot Y]$ and, since $\nabla^\perp \cdot Y = 0$ on Γ, we have that $\nabla^\perp \cdot BY$ vanishes on Γ. Let $g \subset D(A) \cap C^\infty(\overline{\Omega})$ be a finite set. Then, for a C^∞ domain, in the definition of saturating set presented in [8] the sequence stops (i.e., for some order $j \in \mathbb{N}$ $G^{j+1} = G^j$) if, and only if, $Conv\{BY \mid Y \in G^j\} \subseteq G^j$. Unfortunately we do not know if for other boundary conditions, like no-slip or Navier, we also have $BY \in D(A)$ whenever $Y \in D(A)$. If that is not so, the sequence may even not "start": if $Conv\{BY \mid Y \in G^0\} \cap D(A) = \emptyset$ we have $G^1 = G^0$. At each step we loose all the (possibly new) directions on $(Conv\{BY \mid Y \in G^j\} \setminus D(A)) \cap H$.

The Definition 6.1 does not depend on $D(A)$, i.e., does not depend on the boundary conditions.

The theory developed in [1] and used in [8] to achieve the controllability results for the N-S equation works well for saturating sets with $G^j \subset D(A)$ as in the case of Lions boundary conditions and a C^∞ domain. If the spaces G^j are not contained in $D(A)$ we have some difficulties, with that theory, related with the

[25] Note that $\{BY \mid Y \in G^j\} \cap H_4$ and its closure are cones so, $Conv\overline{\{BY \mid Y \in G^j\} \cap H_4}$ is a convex cone and then G^{j+1} is a linear space.

application of Lemma 5.1: we would need a similar lemma with the finite subspace $\mathbb{J} \subset D(A)$ replaced by a finite subspace $\mathbb{J} \subset H$. Unfortunately we do not know if that replacement is possible. Later, in order to apply Lemma 5.1 we will take a "good" approximation $\tilde{\mathbb{J}} \subset D(A)$ for a given $\mathbb{J} \subset H$.

7. Density, traces and gradients

To continue the studying of controllability we will need some auxiliary material on function spaces.

7.1. **Density theorems.** We start by, proceeding analogously as in ([11], section 1.1.2), defining the following auxiliary space

$$E^{p,q}(\tilde{\Omega}) := \{u \in \mathbf{L}^p(\tilde{\Omega}) \mid \nabla \cdot u \in L^q(\tilde{\Omega})\}$$

where $\mathbf{L}^p(\tilde{\Omega}) := L^p(\tilde{\Omega})^N$, $N \in \mathbb{N}_0$. This space is a Banach space for the norm

$$|u|_{E^{p,q}} := |u|_{\mathbf{L}^p(\tilde{\Omega})} + |\nabla \cdot u|_{L^q(\tilde{\Omega})}.$$

Lemma 7.1. *Let $\tilde{\Omega}$ be a Lipschitz open set of $\mathbb{R}^n$. Then the set $\mathcal{D}(\overline{\tilde{\Omega}})$ of smooth vector functions with compact support contained in $\overline{\tilde{\Omega}}$ is dense in $E^{p,q}(\tilde{\Omega})$.*

The proof is analogous to that of Theorem 1.1 in ([11], subsection 1.1.2) where is considered the case $p = q = 2$.

By Lemma 2.3 the set $W_0^{k,p}(\tilde{\Omega}) := \{u \in W^{k,p}(\tilde{\Omega}) \mid u = 0 \text{ on } \Gamma\}$ is closed in $W^{k,p}(\tilde{\Omega})$. Another classical density theorem we will need, and can be found in ([6], section 2.4.3) is:

Lemma 7.2. *Let $\tilde{\Omega}$ be a bounded Lipschitz domain in $\mathbb{R}^N$. Then the set of test functions $\mathcal{D}(\tilde{\Omega})$ is dense in $W_0^{k,p}(\tilde{\Omega})$.*

7.2. **About traces.** In [[6], Section 2.5.7] we can find the following Lemma that says that the "restriction to the boundary" trace from Lemma 2.3, has a continuous right inverse (some more regularity is asked for $\tilde{\Omega}$ when $k \geq 2$):

Lemma 7.3. *Let $\tilde{\Omega} \subset \mathbb{R}^N$ be of class $\mathfrak{R}^{k,1}$ with boundary Γ, $p > 1$ and $h_l \in W^{k-l-\frac{1}{p}}(\Gamma)$ for $l = 0, 1, \ldots, k-1$. Then there exists a continuous linear map*

$$Z : \prod_{i=0}^{k-1} W^{k-l-\frac{1}{p}}(\Gamma) \to W^{k,p}(\tilde{\Omega})$$

such that $\begin{cases} (h_0, h_1, \ldots, h_{k-1}) \in \prod_{i=0}^{k-1} W^{k-l-\frac{1}{p}}(\Gamma) \\ Z(h_0, h_1, \ldots, h_{k-1}) = u \end{cases}$ *implies* $\frac{\partial^l u}{\partial u^l} = h_l$ *on* Γ.

If $k = 1$, is enough to take $\tilde{\Omega}$ of class $\mathfrak{R}^{0,1}$, i.e., bounded Lipschitz, instead of $\mathfrak{R}^{1,1}$.

Put $E^p(\tilde{\Omega}) := E^{p,p}(\tilde{\Omega})$, $p > 1$.

Proposition 7.4. *Let $\tilde{\Omega} \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let q satisfy $\frac{1}{q} + \frac{1}{p} = 1$. Then there exists a linear continuous trace operator γ_n from $E^p(\tilde{\Omega})$ to the dual $W^{-1+\frac{1}{q}}(\Gamma)$ of $W^{1-\frac{1}{q}}(\Gamma)$ that coincides with $u \cdot \mathbf{n}$ for every $u \in \mathcal{D}(\overline{\tilde{\Omega}})$. Here $\mathbf{n}$ is the unit outward vector normal to the boundary Γ of $\tilde{\Omega}$.*

Proof. For every $u \in E^p(\tilde{\Omega})$ and $\phi \in W^{1-\frac{1}{q}}(\Gamma)$, define $\gamma_n(u)(\phi) := \int_{\tilde{\Omega}} u \cdot \nabla\varphi \, dx + \int_{\tilde{\Omega}} (\nabla \cdot u)\varphi \, dx$, where $\varphi \in W^q(\Gamma)$ satisfies $\varphi \mid_\Gamma = \phi$. Using the previous lemmas the proof is analogous to that of Theorem 1.2 of ([11], section 1.1.3). From now we will use the following notations:

$$u \cdot \mathbf{n} := \gamma_n(u) \qquad \& \qquad \int_{\tilde{\Omega}} \phi u \cdot \mathbf{n} := \gamma_n(u)(\phi).$$

□

7.3. **The gradient of a distribution.** Here we present the characterization of a gradient of a distribution and some of its properties. Let $\mathcal{V} = \{\phi \in \mathcal{D}(\tilde{\Omega}) \mid \nabla\cdot\phi = 0\}$ be the set of solenoidal test functions.

In ([11], section 1.1.4) we may find the following:

Lemma 7.5. *Let $\tilde{\Omega}$ be an open set in $\mathbb{R}^N$ and $\mathbf{f} = (f_1, f_2, \ldots, f_N) \in \mathcal{D}'(\tilde{\Omega})^N$. A necessary and sufficient condition that*

$$\mathbf{f} = \nabla\phi, \text{ for some } \phi \in \mathcal{D}'(\tilde{\Omega})$$

is that

$$\langle \mathbf{f}, \psi \rangle, \text{ for all } \psi \in \mathcal{V}.$$

In ([3], Corollary 2.1) we can find:

Lemma 7.6. *Let $\tilde{\Omega}$ be a domain in $\mathbb{R}^N$ and $1 \leq q$. If T is a distribution such that $\nabla T \in L^q_{loc}(\tilde{\Omega})$, then T is a.e. equal to a function in L^q_{loc}.*

7.4. **More density.** For $1 < p < +\infty$, define the following space

$$H_p := \text{ closure of } \mathcal{V} \text{ in } \mathbf{L}^p(\tilde{\Omega}).$$

We have already seen (see (2.12) and Remark 4) that for $p = 2$ we have

$$H_2 = H = \{u \in \mathbf{L}^2(\tilde{\Omega}) \mid \nabla\cdot\phi = 0, u\cdot\mathbf{n} = 0 \, on \, \Gamma\}.$$

Let $\tilde{\Omega}$ be a bounded Lipschitz domain in $\mathbb{R}^N$. By Theorem 7.4 the $E^p(\tilde{\Omega})$-closure H_p of the set $\mathcal{V}$ is contained in $H \cap \mathbf{L}^p(\tilde{\Omega})$, i.e.,

$$H_p \subseteq \{u \in \mathbf{L}^p(\tilde{\Omega}) \mid \nabla\cdot\phi = 0, u\cdot\mathbf{n} = 0 \, on \, \Gamma\}. \tag{7.1}$$

In $H \cap \mathbf{L}^p(\tilde{\Omega})$ the norm induced by $E^p(\tilde{\Omega})$ is equivalent to that induced by $\mathbf{L}^p(\tilde{\Omega})$ so, both H_p and $H \cap \mathbf{L}^p(\tilde{\Omega})$ are closed in $\mathbf{L}^p(\tilde{\Omega})$. Let q satisfy $\frac{1}{q} + \frac{1}{p} = 1$ and let $\mathbf{f} \in \mathbf{L}^q(\tilde{\Omega}) = \mathbf{L}^p(\tilde{\Omega})'$ vanish in H_p, i.e.,

$$\int_{\tilde{\Omega}} \mathbf{f}\cdot v \, dx = 0, \quad \forall v \in H^p. \tag{7.2}$$

Then by Lemmas 7.5 and 7.6 we have that

$$\mathbf{f} = \nabla\phi, \quad \phi \in W^{1,q}(\tilde{\Omega}),$$

By Proposition 7.4, for every $v \in H \cap \mathbf{L}^p(\tilde{\Omega})$ we may write

$$\int_{\tilde{\Omega}} \mathbf{f}\cdot v \, dx = \int_{\tilde{\Omega}} \nabla\phi\cdot v \, dx = -\int_{\tilde{\Omega}} \phi(\nabla\cdot v) \, dx + \int_{\Gamma} \phi v\cdot\mathbf{n} \, d\Gamma = 0.$$

Then, by (7.1) and (7.2) we have necessarily the equality $H_p = H \cap \mathbf{L}^p(\tilde{\Omega})$. Hence:

Proposition 7.7. *If $\tilde{\Omega} \subseteq \mathbb{R}^N$ is a bounded Lipschitz domain, then for any $1 < p < +\infty$ there holds*

$$H_p = H \cap \mathbf{L}^p(\tilde{\Omega}) = \{u \in \mathbf{L}^p(\tilde{\Omega}) \mid \nabla\cdot u = 0, u\cdot\mathbf{n} = 0 \, on \, \Gamma\}.$$

8. Controllability: a sufficient condition

In this section we continue the study of the N-S equation on bounded domain $\Omega \subset \mathbb{R}^2$ satisfying (2.1).

8.1. **Controllability in observed component.**

Definition 8.1. *Let $\phi^0 : M^1 \to M^2$ be a continuous map between two finite dimensional C^0-manifolds, $B \subset M^1$ be an open subset with compact closure and, $S \subseteq M^2$ be any subset. We say that $\phi^0(B)$* **covers** *S* **solidly**, *if for some C^0-neighborhood $\mathcal{N}$ of $\phi^0 \mid_{\overline{B}}$ there holds: $S \subseteq \phi(B)$.*

Let $\mathcal{O} \subset H$ be the finite set of directions we want to observe and, P^O be the orthogonal projection map from H onto $\mathcal{O}_s := span\{x \mid x \in \mathcal{O}\} = \mathbb{R}^{\#\mathcal{O}}$. Define, for each $T > 0$ and each finite subset $\mathbb{F} \subset H$, the "end point" map

$$\mathbb{E}_T : V \times L^\infty([0,\, T],\, \mathbb{R}^{\#\mathbb{F}}) \to \mathcal{O}_s$$
$$(u_0,\, v) \mapsto P^O \circ \mathbb{S}_s(u_0,\, F,\, Mv,\, \nu)(T),$$

where $\mathbb{S}_s(u_0,\, F,\, Mv,\, \nu)$ is the strong solution of the N-S system

$$u_t = -\nu Au - Bu + \nu Cu + F + Mv; \quad u(0) = u_0; \qquad v \in \mathbb{R}^{\#\mathbb{F}}, \tag{8.1}$$

and M is a matrix whose columns are the elements of $\mathbb{F}$.

Definition 8.2. *We say that system* (8.1) *is* **time-T solidly controllable in observed component** *if for any $u_0 \in V$ and $R > 0$ there exists a family*

$$\mathcal{V}_{u_0,R} := \{v_b \in L^\infty([0,\, T], \mathbb{R}^{\#\mathbb{F}}) \mid b \in B_{u_0,R}\}$$

such that $\mathbb{E}_T(u_0, B_{u_0,R}) := \mathbb{E}_T(u_0, \mathcal{V}_{u_0,R})$ covers $\overline{\mathcal{O}}_R(P^O u_0)$ solidly. $B_{u_0,R}$ is an open relatively compact subset of a C^0-manifold and; $\overline{\mathcal{O}}_R(y)$ is the closed ball

$$\{x \in \mathcal{O}_s \mid \|x - y\|_{l_1} \le R\} := \{x \in \mathbb{R}^{\#\mathcal{O}} \mid \|x - y\|_{l_1} \le R\}.$$

The respective open balls will be denoted by $\mathcal{O}_R(y)$.

Again let us put $H_4 := \mathbf{L}^4(\Omega)$. As we did in [8] we can prove (we give an idea below) that

Proposition 8.1. *Let $g \subset H_4$ be a w-saturating set for the N-S system* (6.1)*. Then the system*

$$u_t = -\nu Au - Bu + \nu Cu + F + gv; \quad u(0) = u_0; \qquad v \in \mathbb{R}^{\#g}, \tag{8.2}$$

where g is a matrix whose columns are the elements of the set $g \subset H$ and, v is our control, is time-T solidly controllable in observed component.

For any $N \in \mathbb{N}_0$ define the system

$$[N]: \qquad u_t = -\nu Au - Bu + \nu Cu + F + g_N v; \quad u(0) = u_0; \qquad v \in \mathbb{R}^{\#g_N}, \tag{8.3}$$

where g_N is a matrix whose columns are vectors spanning G^N.

Proposition 8.2.

(1) *For some $T^0 > 0$, every $0 < T \le T^0$ and every $N \in \mathbb{N}_0$ the system* [(8.3).N] *is time-T solid controllable in observed component;*

(2) *For each pair $(u_0,\, R) \in V \times [0,\, +\infty[$ the family*

$$\mathcal{V}_{u_0,R} := \{v_b \mid b \in B_{u_0,R}\}$$

can be chosen satisfying:

- *The map $b \mapsto v_b$ is $(B,\, L^2(0,\, T,\, \mathbb{R}^{\#g_N}))$-continuous and;*
- *The controls $v_b(t)$ are uniformly (w.r.t. b and t) l_1-bounded:*

$$\|v_b(t)\|_{l_1} \le A = A(T, R, u_0).$$

Proposition 8.1 foloows from Proposition 8.2: fix $T_1 > 0$. If $T_1 \leq T^0$ Proposition 8.1 is contained in Proposition 8.2 (N=1); if $T_1 > T^0$ we apply any admissible control in G^0 for time $T_1 - T^0$ and then apply Proposition 8.2 (N=1).

Following the proof of Proposition 4.10.2 of [8] we prove Proposition 8.2 in two steps:

First step: The Proposition holds for big enough N.

Analogously as we did in [8] we may prove that, for a fixed $\gamma > 1$, the family of constant controls

$$\mathcal{V} := \{v_p := pT^{-1} \mid p \in \mathcal{O}_{\gamma R}(P^O u_0)\}$$

satisfy the second point of the Proposition 8.2 and; defining the map $\phi(p) := P^O \circ \mathbb{S}_s(u_0,\, F,\, v_p,\, \nu)(T)$, $\phi(\mathcal{O}_{\gamma R}(P^O u_0))$ covers $\overline{\mathcal{O}}_R(P^O u_0)$ solidly.

Let $\{e_i \mid i = 1,\, \ldots,\, r\}$ be a basis for $\mathcal{O}$ and let δ be a small positive real number. Set $N \in \mathbb{N}$ such that for each element e_i of this basis we have $|e_i - P^N e_i| < \delta$ and; for each $p = \sum_{i=1}^{r} p_i e_i$ put $\tilde{p} = \sum_{i=1}^{r} p_i P^N e_i$ and $v_{\tilde{p}} := \tilde{p}T^{-1}$. Here $P^N : H \to G^N$ denotes the orthogonal projection from H onto G^N.

For small δ we have that the controls $v_{\tilde{p}}$ and v_p are closed in H so, at final time T, also $\mathbb{S}_s(u_0,\, F,\, v_{\tilde{p}},\, \nu)(T)$ and $\mathbb{S}_s(u_0,\, F,\, v_p,\, \nu)(T)$ are closed in H. Then for any $p \in \mathcal{O}_{\gamma R}(P^O u_0)$ we have that $\mathbb{E}(u_0,\, \tilde{p})$ is close to $\phi(p)$. Hence, for small δ, $\mathbb{E}(u_0,\, \tilde{p})$ covers $\overline{\mathcal{O}}_R(P^O u_0)$ solidly, i.e., system [(8.3).N] is solidly controllable in observed component.

Moreover we have the following:

Corollary 8.3. *For $t \in [0,\, T]$ there holds*

$$|\mathbb{S}_s(u_0,\, F,\, v_{\tilde{p}},\, \nu)(t) - (u_0 + t\tilde{p})| \leq [T \exp T]^{\frac{1}{2}} K,$$

with K a constant (independent of T).

Second step: If the Proposition holds for N, then it holds $N - 1$.

Suppose that Proposition 8.2 holds for a given N. We are given a family of controls $v(t,\, b)$ taking values on G^N and satisfying the Proposition. By definition and, by the fact that G^{N-1} is a linear space we have that

$$G^N \subseteq G^{N-1} + Conv\overline{\{BY \mid Y \in G^{N-1}\} \cap H_4} = Conv\left(G^{N-1} + \overline{\{BY \mid Y \in G^{N-1}\} \cap H_4}\right)$$

so, for a give basis $\{\delta_i \mid i = 1,\, \ldots,\, M\}$ of G^N we have that

$$\delta_i = \sum_{p=1}^{P_i^+} \lambda_{i,p}^+ (e_{i,p}^+ + f_{i,p}^+); \qquad -\delta_i = \sum_{p=1}^{P_i^-} \lambda_{i,p}^- (e_{i,p}^- + f_{i,p}^-)$$

where $e_{i,p}^\pm \in G^{N-1}$; $f_{i,p}^\pm \in \overline{\{BY \mid Y \in G^{N-1}\} \cap H_4}$; $\lambda_{i,p}^\pm \geq 0$ and; $\sum_{p=1}^{P_i^+} \lambda_{i,p}^+ = \sum_{p=1}^{P_i^-} \lambda_{i,p}^- = 1$.

Put $R := \{e_{i,p}^+ + f_{i,p}^+,\, e_{i,q}^- + f_{i,q}^- \mid i = 1,\, \ldots,\, M;\, p = 1,\, \ldots,\, P_i^+;\, q = 1,\, \ldots,\, P_i^-\}$ and for simplicity write

$$R := \left\{e_r + f_r,\, r = 1,\, \ldots,\, s := \sum_{i=1}^{M} P_i^+ + P_i^-\right\}.$$

Note that some of the $e_r \in G^{N-1}$ or $f_r \in \overline{\{BY \mid Y \in G^{N-1}\} \cap H_4}$ may vanish.

Since the family of controls is uniformly bounded we have that for some constant $\Xi > 0$

$$v(t,\, b) \in \Xi Conv\{\pm\delta_i \mid i = 1,\, \ldots,\, M\} \subseteq \Xi Conv R.$$

Relaxation. Due to the continuity of the N-S equation on relaxation metric we may approximate the family $v(t,\, b)$ by a family of piecewise constant controls $z(t,\, b)$ taking values in $\Xi\{e_r + f_r \mid r = 1,\, \dots,\, s\}$. The solutions of the N-S equations

$$u_t = -\nu Au - Bu + \nu Cu + F + v(t,\, b); \quad u(0) = u_0 \in V$$

and

$$u_t = -\nu Au - Bu + \nu Cu + F + z(t,\, b); \quad u(0) = u_0 \in V$$

are close in $C([0,\, T],\, H)$-norm.

Leaving the boundary. Let $\tilde{f}_i$ be an element of G^{N-1} such that $|B\tilde{f}_i - f_i|$ is small. Put $\tilde{z}(t,\, b) := \Xi(e_i + B\tilde{f}_i)$ if $z(t,\, b) := \Xi(e_i + f_i)$. Then the solutions of the N-S equations

$$u_t = -\nu Au - Bu + \nu Cu + F + z(t,\, b); \quad u(0) = u_0 \in V$$

and

$$u_t = -\nu Au - Bu + \nu Cu + F + \tilde{z}(t,\, b); \quad u(0) = u_0 \in V$$

are close in $C([0,\, T],\, H)$-norm.

Going from H to $D(A)$. Let $\{\gamma_i \mid i = 1,\, \dots,\, M\}$ be a basis of G^{N-1}. By Proposition 7.7 we may choose $\hat{\gamma}_i$ in $\mathcal{V} \subset D(A)$ such that $\|\hat{\gamma}_i - \gamma_i\|_{\mathbf{L}^4(\Omega)}$ is small. Put $\hat{f}_i = \sum_{i=1}^{M} a_i \hat{\gamma}_i$ if $\tilde{f}_i = \sum_{i=1}^{M} a_i \gamma_i$ and, $\hat{z}(t,\, b) = \Xi(e_i + B\hat{f}_i)$ if $\tilde{z}(t,\, b) = \Xi(e_i + B\tilde{f}_i)$. Then the solutions of the N-S equations

$$u_t = -\nu Au - Bu + \nu Cu + F + \tilde{z}(t,\, b); \quad u(0) = u_0 \in V$$

and

$$u_t = -\nu Au - Bu + \nu Cu + F + \hat{z}(t,\, b); \quad u(0) = u_0 \in V$$

are close in $C([0,\, T],\, H)$-norm: note that for some constant C,

$$|B\hat{f}_i - B\tilde{f}_i|_{V'} \le C|\hat{f}_i - \tilde{f}_i|_{\mathbf{L}^4(\Omega)} \left(|\hat{f}_i|_{\mathbf{L}^4(\Omega)} |\tilde{f}_i|_{\mathbf{L}^4(\Omega)} \right).$$

Imitation. We may proceed as in [8] and prove that the solutions of the N-S equations

$$u_t = -\nu Au - Bu + \nu Cu + F + \hat{z}(t,\, b); \quad u(0) = u_0 \in V$$

and

$$u_t = -\nu Au - Bu + \nu Cu + F + z^w(t,\, b); \quad u(0) = u_0 \in V$$

are close in $C([0,\, T],\, H)$-norm, where w is a big enough positive real number and the control $z^w(t,\, b)$ is defined as follows: let $\alpha_1,\, \dots,\, \alpha_{m-1}$ be the switching points of $\hat{z}(t,\, b)$, i.e., putting $\alpha_0 := 0$ and $\alpha_m = T$, $\hat{z}(t,\, b)$ is constant in $[\alpha_p,\, \alpha_{p+1}]$, $p = 0,\, \dots,\, m-1$.

- In the first interval of constancy $[\alpha_0,\, \alpha_1]$:

  $$z^w(\cdot,\, b) := \Xi e_r + v^J(q_1^\infty(\cdot,\, b) + \sqrt{2\Xi}\phi^w(\cdot,\, b)(\hat{f}_r),\, U_0)$$

  if $\hat{z}(\cdot,\, b) = \Xi(e_r + B\hat{f}_r)$. Here v^J is the control given by Lemma 5.1, for $\mathbb{J} = span\{\hat{f}_r \mid r = 1,\, \dots,\, s\}$; U_0 is the projection of u_0 onto $\mathbb{J}_V^\perp$; $q^\infty(\cdot,\, b)$ is the orthogonal projection $P^{\mathbb{J}}\mathbb{S}_s(u_0,\, F,\, \hat{z}(\cdot,\, b),\, \nu)$ of the solution of N-S equation controlled by $\hat{z}(\cdot,\, b)$ onto $\mathbb{J}$ and; $q_1^\infty(\cdot,\, b)$ is the restriction of $q^\infty(\cdot,\, b)$ to $[\alpha_0,\, \alpha_1]$;
- If the control $z^w(\cdot,\, b)$ is already defined in the first $p-1$ intervals of constancy (up to α_{p-1}), we define it in the p^{th} interval $[\alpha_{p-1},\, \alpha_p]$ by:

  $$z^w(\cdot,\, b) := \Xi e_r + v^J(q_1^\infty(\cdot,\, b) + \sqrt{2\Xi}\phi^w(\cdot,\, b)(\hat{f}_r),\, U^w(\alpha_{p-1}))$$

if $\hat{z}(\cdot, b) = \Xi(e_r + B\hat{f}_r)$. Here U^w is the projection onto $(\mathbb{R}^{\kappa_N})^{\perp}_{V}$ of the solution of the equation

$$u_t^w(\cdot, b) = -\nu A u^w - B u^w + F + z^w(\cdot, b), \quad u(0) = u_0,\, t \in [0, \alpha_{p-1}]$$

and, $q_p^\infty(\cdot, b)$ is the restriction of $q^\infty(\cdot, b)$ to $[\alpha_{p-1}, \alpha_p]$.

Here $\phi^w(\cdot, b) \in W^{1,\infty}([0, T], \mathbb{R})$ if a function satisfying the following points:

- $\phi_w(\cdot, b)$ vanishes at the points α_i, $\quad i = 0, \ldots, m$;
- $\|\phi_w(\cdot, b)\|_{C([0,T],\mathbb{R})} \leq 1; \qquad \|\dot{\phi}_w(\cdot, b)\|_{L^\infty([0,T],\mathbb{R})} \leq \frac{w(1+\theta)}{\theta}$;
- $\phi_w(t, b)$ differs from $\sin(wt)$ on a set of measure not bigger than $\frac{2T}{w}$ and;
- For fixed w, the map $A(b) \mapsto \phi_w(\cdot, b)$ is $(\mathcal{A}_\theta, W^{1,2}(0, T, \mathbb{R}))$-continuous (where $\mathcal{A}_\theta$ is endowed with the topology induced by $\mathbb{R}^{m+1}$).

Coming back to H. Of course the solutions of the N-S equations

$$u_t = -\nu A u - B u + \nu C u + F + z^w(t, b); \quad u(0) = u_0 \in V$$

and

$$u_t = -\nu A u - B u + \nu C u + F + y(t, b); \quad u(0) = u_0 \in V$$

are close in $C([0, T], H)$-norm if $y(t, b) := \sum_{r=1}^{s} a_r(t) e_r + \sum_{r=1}^{s} b_r(t) \tilde{f}_r$ whenever $z^w(t, b) := \sum_{r=1}^{s} a_r(t) e_r + \sum_{r=1}^{s} b_r(t) \hat{f}_r$ and; $y(t, b)$ takes its values on G^{N-1}.

8.2. **$\mathbf{L}^2(\Omega)$-approximate controllability.** We have that for any $T > 0$, system (8.2) is **time-T approximately controllable** in $\mathbf{L}^2$-norm, i.e.,

Proposition 8.4. *Let $g \subset H_4$ be a w-saturating set for the N-S system* (6.1)*. Then for any $u_0 \in V$ and $T > 0$, the attainable set at time T, from u_0, of system* (8.2) *is dense in H.*

the proof is completely analogous to that of Proposition 5.0.6 of [8]: we want to drive the system from u_0 to some neighborhood of u_1. First we set N big enough such that both $|u_0 - P^N u_0|$ and $|u_1 - P^N u_1|$ are small. Then from Corollary 8.3 if T is small, the control $(P^N u_1 - P^N u_0)T^{-1}$ drives the system in time T to a point close to $u_0 + (P^N u_1 - P^N u_0)$, i.e., close to u_1.

We may imitate the dynamics given by a control taking values in G^i by the dynamics of a control taking values in G^{i-1} (as we did before in the proof of Proposition 8.1) in such a way that at final time T the two dynamics are close.

Hence we can arrive to a control taking values in $G^0 = span\{g\}$ driving the system to the neigborhood of u_1.

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