Subdivision Analysis of Topological

Z_p Lattice Gauge Theory

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Abstract

We analyze the subdivision properties of certain lattice gauge theories for the discrete abelian groups Z_p, in four dimensions. In these particular models we show that the Boltzmann weights are invariant under all (k, l) subdivision moves, when the coupling scale is a pth root of unity. For the case of manifolds with boundary, we demonstrate analytically that Alexander type 2 and 3 subdivision of a bounding simplex is equivalent to the insertion of an operator which equals a delta function on trivial bounding holonomies. The four dimensional model then gives rise to an effective gauge invariant three dimensional model on its boundary, and we compute the combinatorially invariant value of the partition function for the case of S^3 and $S^2 \times S^1$.

1 Introduction

In [1, 2, 3], a class of lattice gauge theories was introduced which had appealing topological features. The program was essentially to study lattice models which were based on a discrete Chern form type action; the setting was a general simplicial complex which modeled a 4-manifold with boundary. With the particular action functional that was adopted there, it was possible to prove several remarkable subdivision properties when the coupling parameter took on quantized values, for the gauge groups Z_2 and Z_3. A complete analysis of the subdivision properties under moves which subdivided the 4-dimensional simplex, as well as for the 3-dimensional boundary, was presented for these two models. It was found that the model was subdivision invariant under all 4-dimensional moves which left the boundary unchanged, while a subdivision of a bounding 3-simplex was shown to be equivalent to the insertion of certain delta functions which trivialized bounding holonomies.

In this paper, we consider a variation of the model presented in [1, 2, 3] which allows for a uniform extension to all finite cyclic groups. The previous models discussed for Z_2 and Z_3 groups will appear as special cases of the general program. The change is essentially to use angles to measure deviations from trivial holonomy in roughly the same way that heat kernel methods are used as an alternative to the usual Wilson action in Yang-Mills theory. We will also be able to provide a complete analytic proof of the boundary subdivision properties that were earlier proved by computer techniques in [3].

Given the subdivision properties of the bounding 3-dimensional simplex, we will be able to identify a topological invariant as the continuum limit of these models, and compute it for complexes which model S^3 and $S^2 \times S^1$. This model, which differs from the naive partition function by a scale factor associated with the number of 3-simplices in the bounding complex, is related to structures recently considered in [4].

2 Action for Z_p Groups

The models defined in [1, 2, 3] are based upon an action which depends on two independent holonomies sharing a common vertex, conveniently referred to as a bow-tie structure. The action, evaluated on a 4-simplex $[0, 1, 2, 3, 4]$, was then given by a sum over permutations of the vertices, a typical term being of the form: $(U-1)_{012} (U-1)_{034}$ for the Z_2 case, and $(U-U^{-1})_{012} (U-U^{-1})_{034}$ for all other models. Here, U_{ijk} is the holonomy around the 2-simplex determined by the vertices i, j, and k. However, since $(U-1)$ and $(U-U^{-1})$ both measure deviations from trivial holonomy, one can consider simply replacing this by an angle while keeping the bow-tie structure of the Boltzmann weight intact. This, in fact, allows one to streamline the analysis for the general Z_p case, and we confine our attention to these actions in the following.

The cyclic group Z_p is represented multiplicatively by the p roots of unity $\exp[2\pi i n/p]$, where n is an integer in the set $\{0, \cdots, p-1\}$. We will take the link variables of our theory to be this set of integers, and the fundamental "holonomy" combination is defined by

$$n_{ijk} = n_{ij} + n_{jk} + n_{ki} \quad mod \quad p \ . \tag{1}$$

We also have the usual rule associated with a reversal of link orientation:

$$n_{ji} = -n_{ij} \quad mod \quad p \ . \tag{2}$$

Our theory is defined by specifying the Boltzmann weight evaluated on the 4-simplex $[0, 1, 2, 3, 4]$;

$$W[0, 1, 2, 3, 4] = \exp[\beta S[0, 1, 2, 3, 4]] \ , \tag{3}$$

where the action is given by

$$
\begin{aligned}
S[0, 1, 2, 3, 4] \ = \ & (n_{012}\, n_{034} + n_{013}\, n_{042} + n_{014}\, n_{023} \\
+ \ & n_{102}\, n_{143} + n_{103}\, n_{124} + n_{104}\, n_{132} \\
+ \ & n_{201}\, n_{234} + n_{203}\, n_{241} + n_{204}\, n_{213} \\
+ \ & n_{301}\, n_{342} + n_{302}\, n_{314} + n_{304}\, n_{321} \\
+ \ & n_{401}\, n_{423} + n_{402}\, n_{431} + n_{403}\, n_{412}) \ ,
\end{aligned}
\tag{4}
$$

and β is the coupling parameter. We are concerned in this paper with the behavior of the theory when the scale factor $s = \exp[\beta]$ is a pth root of unity:

$$s = \exp[2\,\pi\,i\,k/p] \quad , \tag{5}$$

for $k = \{0, 1, \cdots, p-1\}$. For later convenience, we denote the Boltzmann weight for a given vertex ordering by:

$$B[0, 1, 2, 3, 4] = s^{n_{012}\,n_{034}} \quad . \tag{6}$$

3   Behavior Under the (k, l) Moves

We begin by considering the behavior of the theory under subdivision moves of (k, l) type. For closed manifolds of dimension less than or equal to four, these moves have been shown to be equivalent to the Alexander type subdivisions [5, 6].

Lemma: The Boltzmann weights of the Z_p theory, for a given vertex ordering, satisfy the conditions,

$$B[0, 1, 2, 3, 4]\, B[0, 1, 2, 4, 5]\, B[0, 1, 2, 5, 3] = s^{n_{012}\,n_{345}} \quad , \tag{7}$$

$$B[0, 1, 2, 3, 4]\, B[1, 2, 0, 4, 3] = s^{n_{012}\,n_{014}}\, s^{-n_{012}\,n_{013}} \quad , \tag{8}$$

when the scale factor is a pth root of unity, $s^p = 1$.

To prove this, one first notes that the relations are trivially satisfied when $n_{012} = 0$, so we can concentrate on the non-trivial values for this holonomy. In fact, it suffices to study only the case when $n_{012} = 1$; the remaining values lead to less restrictive constraints on the scale parameter. Since the holonomy angle n_{ijk} is defined as the modular sum of three link variables, we immediately have the relation

$$n_{034} + n_{045} + n_{053} - n_{345} = 0 \quad mod \quad p \quad , \tag{9}$$

and the results follow.

This leads us to our first theorem:

Theorem 1: The Boltzmann weights of the Z_p model satisfy the relation,

$$
\begin{aligned}
W[0,1,2,3,4]\,W[0,1,2,4,5]\,W[0,1,2,5,3] \;=\;\; & W[0,1,3,4,5]\,W[1,2,3,4,5] \\
& W[2,0,3,4,5]\ , \quad\quad\quad (10)
\end{aligned}
$$

when $s^p = 1$.

The proof of this is again straightforward, though a little tedious. In verifying the result, one can use the relation

$$
\begin{aligned}
B[0,1,2,3,4]\,B[0,1,2,4,5]\,B[0,1,2,5,3] \;=\;\; & B[3,4,5,0,1]\,B[3,4,5,1,2] \\
& B[3,4,5,2,0]\ , \quad\quad\quad (11)
\end{aligned}
$$

which is a trivial consequence of (7), to eliminate all but 18 of the 90 terms. The identity (8) then comes into play, and the result is secured.

This theorem now allows us to analyze completely the remaining (k, l) moves, which we state as a corollary.

Corollary: The Boltzmann weights of the Z_p model satisfy the following two relations:

$$
\begin{aligned}
W[0,1,2,3,4]\,W[0,1,2,5,3] \;=\;\; & W[1,2,3,4,5]\,W[2,0,3,4,5]\,W[0,1,3,4,5] \\
& W[1,0,2,4,5]\ , \quad\quad\quad\quad\quad\quad (12)
\end{aligned}
$$

$$
\begin{aligned}
W[0,1,2,3,4] \;=\;\; & W[5,1,2,3,4]\,W[0,5,2,3,4]\,W[0,1,5,3,4]\,W[0,1,2,5,4] \\
& W[0,1,2,3,5]\ , \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (13)
\end{aligned}
$$

at the points $s^p = 1$.

The proof here is a simple application of Theorem 1, together with the fact that,

$$
W[0,1,2,3,4]^{-1} = W[0,1,2,4,3] \tag{14}
$$

holds in this theory at the points $s^p = 1$.

Having established invariance with respect to the (k, l) moves, we can state that the partition function of the Z_p model is a combinatorial invariant for all closed 4-manifolds. However, these identities also allow us to conclude

triviality of the invariant for all manifolds in this class. In particular, we can show that these four dimensional models actually reduce to behavior on the boundary.

To establish the triviality for closed 4-manifolds, we consider the identity of theorem 1, in the form:

$$1 = \quad W[0,1,2,3,4]\, W[0,1,2,4,5]\, W[0,1,2,5,3] \qquad (15)$$
$$W[1,0,3,4,5]\, W[2,1,3,4,5]\, W[0,2,3,4,5] \ .$$

Written in this way, one can recognize that the 4-simplices in this identity are actually the boundary of a 5-simplex $[0,1,2,3,4,5]$;

$$\partial\,[0,1,2,3,4,5] = \quad [1,2,3,4,5] - [0,2,3,4,5] + [0,1,3,4,5] - \qquad (16)$$
$$[0,1,2,4,5] + [0,1,2,3,5] - [0,1,2,3,4] \ .$$

If t is any 5-simplex, we can write this compactly as:

$$W[\partial\, t] = 1 \ . \qquad (17)$$

Let K denote a simplicial complex which models a 4-manifold, possibly with boundary. For our purposes, the 4-simplices $\{s_i\}$ in K are most important,

$$K = \sum_i s_i \ . \qquad (18)$$

Now consider the abstract simplicial complex called the cone over K [7], which is obtained by adding a new vertex x to the simplicial complex K, and linking it to all other vertices; we denote this simplicial complex by $x * K$. Computing the boundary of that complex, one sees

$$\partial\,(x * K) = K - x * \partial K \ . \qquad (19)$$

Given that the Boltzmann weight of the left hand side is just 1, we have then

$$W[K] = W[x * \partial K] \ , \qquad (20)$$

where we mean, more precisely, that

$$W[K] = \prod_i W[s_i] \ . \qquad (21)$$

The simplicity of equation (20) is striking; the implication is that the Boltzmann weight of any four dimensional simplicial complex K is identical to that of the cone over its boundary. Essentially, the cone construction is giving a canonical presentation - or framing - of the boundary of K. Moreover, if ∂K has several disjoint components M_α, then K is a cobordism connecting them, and we immediately have that

$$Z[K] = \prod_\alpha Z[x * M_\alpha] \ . \tag{22}$$

This is one of the axioms for a topological field theory [8].

Having established that we are dealing with a four dimensional gauge theory which essentially reduces to something on the boundary, it is natural to wonder about its interpretation as an intrinsically three dimensional theory. As a gauge theory, we can gauge fix the links on any maximal tree, and one such tree is given by the links which spew from the vertex x. The value of the partition function is independent of how we fix them, so we could always set those link variables to 1 say. However one chooses to gauge fix these link variables, we can consider the result to be a three dimensional lattice theory. If there is any residual gauge invariance left, then it is not at all manifest, but this is also reminiscent of the continuum Chern-Simons theories [9].

4   Analysis of Boundary Subdivision

In this section, we will undertake an analysis of the theory when a bounding simplex is subdivided by an Alexander move of type 2 or 3.

Consider what happens to the Boltzmann weight of the theory when a bounding 3-simplex $[0, 1, 2, 3]$ is subdivided. Let x be the cone vertex discussed in the previous section, so that the partition function would have the factor $W[0, 1, 2, 3, x]$. Under subdivision where we add a new vertex c to the center of the tetrahedron, we would then consider a new set of Boltzmann weights which are unchanged except that we would replace the factor $W[0, 1, 2, 3, x]$ by the quantity,

$$W[c, 1, 2, 3, x] \, W[0, c, 2, 3, x] \, W[0, 1, c, 3, x] \, W[0, 1, 2, c, x] \tag{23}$$

in the partition function, and sum over the new link variables. However, the main identity that we established above, namely (10), says that this product of four weights is equal to

$$W[0, 1, 2, 3, x] \, W^{-1}[0, 1, 2, 3, c] \quad . \tag{24}$$

Here, W^{-1} denotes the inverse value which, for the Boltzmann weights in our construction, is equivalent to simply an odd permutation of the vertices; $W^{-1}[0, 1, 2, 3, 4] = W[0, 1, 2, 4, 3]$. We see then that the subdivided Boltzmann weights represented by (23) are precisely equivalent to having introduced an extra factor $W^{-1}[0, 1, 2, 3, c]$ into the original assembly of Boltzmann weights. It is then crucial to understand how the theory behaves under insertions of the kind:

$$I[0, 1, 2, 3] = \frac{1}{|G|^4} \sum_{n_{ci}} W^{-1}[0, 1, 2, 3, c] \quad , \tag{25}$$

where the sum is over the four link variables connected to c, and $|G|$ is the order of the gauge group. At the trivial points where (10) holds, namely $s = 1$, this quantity is manifestly 1. We are interested in investigating the nontrivial roots of unity. Let $\delta(n)$ denote the Mod p delta function which is 1 for $n = 0$ mod p, and zero otherwise.

Theorem 2: The insertion $I[0, 1, 2, 3]$ is equal to:

$$\delta(n_{012}) \, \delta(n_{013}) \, \delta(n_{023}) \, \delta(n_{123}) \tag{26}$$

at the non-trivial roots of unity, $s^p = 1$, in the Z_p model.

In order to establish this result, we require the formula

$$\frac{1}{|G|} \sum_{k=0}^{p-1} \exp[2\pi i k n/p] = \delta(n) \quad . \tag{27}$$

One can now sum over the four link variables in succession to yield the four delta functions. In addition, one finds that the remaining portion of the Boltzmann weight reduces to 1, upon implementation of the delta function constraints.

Thus, the insertion is 1 if all holonomies on the bounding 3-simplex are trivial, and zero otherwise.

It is also interesting to consider Alexander type 2 subdivision of a 2-simplex which belongs to the bounding 3-manifold. Since the bounding space is a manifold, a given 2-simplex, say $[0, 1, 2]$, will be shared by precisely two 3-simplices; we denote their sum by $[0, 1, 2, 3] - [0, 1, 2, 4]$. The requirement of the relative minus sign is dictated by the fact that we have a 3-manifold without boundary. Under type 2 subdivision, we add a new vertex c to the center of the $[0, 1, 2]$ face, and link it to the other vertices,

$$[0, 1, 2] \rightarrow [c, 1, 2] + [0, c, 2] + [0, 1, c] \ . \tag{28}$$

Now, in the Boltzmann weights appropriate to the original complex, one will find the product,

$$W[0, 1, 2, 3, x] \, W^{-1}[0, 1, 2, 4, x] \ . \tag{29}$$

In the subdivided situation, each of these two factors will be replaced by a product of three Boltzmann weights according to the structure of (28). If we again use the identity (10), one finds that the subdivided situation is equivalent to the insertion of the following factor in the original product of Boltzmann weights:

$$I'[0, 1, 2, 3, 4] = \frac{1}{|G|^5} \sum_{n_{ci}} W^{-1}[0, 1, 2, 3, c] \, W[0, 1, 2, 4, c] \ . \tag{30}$$

Let $n_{ijkl} = n_{ij} + n_{jk} + n_{kl} + n_{li}$ denote the holonomy angle through four vertices; we then have the following result.

Theorem 3: The quantity $I'[0, 1, 2, 3, 4]$ is equal to,

$$\delta(n_{012}) \, \delta(n_{0314}) \, \delta(n_{1324}) \, \delta(n_{2304}) \ , \tag{31}$$

at the same points as in Theorem 2.

Again, the proof here requires only a straightforward application of the summation rule (27). As before, the remainder of the Boltzmann weight reduces to 1 after the delta function constraints are applied.

Notice that there is one 3-vertex holonomy around the 2-simplex $[0, 1, 2]$ which is the face common to the two 3-simplices that have been glued together; the other 4-vertex holonomies are just products of the more elementary holonomies. Since any 2-simplex on the boundary is common to precisely

two bounding 3-simplices, the restriction imposed by type 2 subdivision is actually equivalent to that from the type 3 move in the full partition function.

The above relations allow us to obtain the continuum limit of these models in an economical fashion. Starting from any finite simplicial complex, we simply perform a single Alexander subdivision of type 3 on each of the bounding 3-simplices. The Boltzmann weight is then given solely by an assembly of delta functions. In fact, since these insertions are themselves gauge invariant, we have distilled a gauge invariant model from the original four dimensional theory. Actually, in order to ensure that the value of the partition function is invariant under further subdivision, a scaling of the original partition function must be performed. Defining the quantity

$$J[0,1,2,3] = \frac{1}{|G|} \sum_{n_{ci}} \delta(n_{c01})\delta(n_{c02})\delta(n_{c03})\delta(n_{c12})\delta(n_{c13})\delta(n_{c23}) \ , \qquad (32)$$

and performing the integrations with the aid of the formula

$$\sum_{k=0}^{p-1} \delta(k) = 1 \ , \qquad (33)$$

we obtain the result:

$$J[0,1,2,3] = \delta(n_{012})\delta(n_{013})\delta(n_{023})\delta(n_{123}) \ . \qquad (34)$$

Similarly, the quantity

$$\begin{aligned} J'[0,1,2,3,4] &= \frac{1}{|G|} \sum_{n_{ci}} \delta(n_{c01})\delta(n_{c02})\delta(n_{c03})\delta(n_{c12})\delta(n_{c13})\delta(n_{c23}) \\ &\quad \cdot \ \delta(n_{c04})\delta(n_{c14})\delta(n_{c24}) \ , \end{aligned} \qquad (35)$$

can easily be shown to be:

$$J'[0,1,2,3,4] = \delta(n_{012})\delta(n_{013})\delta(n_{023})\delta(n_{123})\delta(n_{014})\delta(n_{024})\delta(n_{124}) \ . \qquad (36)$$

The quantities J and J' represent an assembly of delta functions after Alexander subdivision of type 3 and 2. Since the number of 3-simplices increases by 3 and 4 under these moves, it is a simple exercise to conclude that the partition function

$$Z = |G|^{(N_3(\partial K) - N_1)} \sum_{n_{ij}} W[K] \ , \qquad (37)$$

is subdivision invariant, where $W[K]$ is a Boltzmann weight trivializing all holonomies on the bounding 3-simplex. Here, $N_3(\partial K)$ is the number of bounding 3-simplices, and N_1 is the number of link variables in K.

9

5 Computations

Having established the fact that one can extract a subdivision invariant model from the four dimensional setting, we now turn to some explicit examples. In particular, we compute the partition function for the case of a four manifold with S^3 or $S^2 \times S^1$ boundary.

Consider first the case of S^3. As we have seen, one can gauge fix the link variables which spew from the cone vertex, leaving an effective theory on the boundary which is itself gauge invariant. We can thus confine our attention to a simplicial complex for S^3. A suitable choice is given by the boundary of the 4-simplex $[0, 1, 2, 3, 4]$ as follows:

$$K = [0, 1, 2, 3] - [0, 1, 2, 4] + [0, 1, 3, 4] - [0, 2, 3, 4] + [1, 2, 3, 4] \ . \qquad (38)$$

In order to compute the partition function, one can gauge the links on a maximal tree, an example being:

$$n_{01} = n_{12} = n_{23} = n_{34} = 0 \ . \qquad (39)$$

Corresponding to the simplicial complex (38), we have ten holonomy constraints; however, the Bianchi identity can be used to eliminate all but six of these. The remaining set can now be analyzed in the presence of the gauge fixing, and one sees that all 10 link variables assume a value of 0. The subdivision invariant value for the partition function (37) is then given by:

$$Z[S^3] = \frac{1}{|G|} \ . \qquad (40)$$

Turning now to the case of a 4-manifold with $S^2 \times S^1$ boundary, we have the following simplicial complex for $S^2 \times S^1$:

$$
\begin{aligned}
K \ = \ & [0, 1, 2, 4] \ - \ [1, 2, 3, 5] \ - \ [0, 1, 3, 4] \ + \ [0, 2, 3, 4] \\
+ \ & [1, 2, 4, 5] \ - \ [2, 3, 5, 6] \ - \ [1, 3, 4, 5] \ + \ [2, 3, 4, 6] \\
+ \ & [2, 4, 5, 6] \ - \ [3, 5, 6, 7] \ - \ [3, 4, 5, 7] \ + \ [3, 4, 6, 7] \\
+ \ & [4, 5, 6, 4'] \ - \ [5, 6, 7, 5'] \ - \ [4, 5, 7, 4'] \ + \ [4, 6, 7, 4'] \\
+ \ & [5, 6, 4', 5'] \ - \ [6, 7, 5', 6'] \ - \ [5, 7, 4', 5'] \ + \ [6, 7, 4', 6']
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & [6, 4', 5', 6'] - [7, 5', 6', 7'] - [7, 4', 5', 7'] + [7, 4', 6', 7'] \\
+ \quad & [4', 5', 6', 0] - [5', 6', 7', 1] - [4', 5', 7', 0] + [4', 6', 7', 0] \\
+ \quad & [5', 6', 0, 1] - [6', 7', 1, 2] - [5', 7', 0, 1] + [6', 7', 0, 2] \\
+ \quad & [6', 0, 1, 2] - [7', 1, 2, 3] - [7', 0, 1, 3] + [7', 0, 2, 3] \quad . \tag{41}
\end{aligned}
$$

It requires a little more work to verify that this is indeed a suitable simplicial complex, but it can be obtained by first constructing a complex for $S^2 \times I$, where I is the unit interval. The two S^2 boundaries are then identified, yielding (41).

In this case, one has 48 link variables, and a maximal tree involves the gauge fixing of 11 of these, to 0 say. As before, the Bianchi identity is used to obtain the independent holonomy constraints, and upon implementation of the gauge fixing conditions, these can be resolved. In fact, one finds a single constraint which specifies that 10 of the link variables are equal, with the remaining ones being set to zero. The value of the partition function (37) is therefore

$$
Z[S^2 \times S^1] = 1 \quad . \tag{42}
$$

We should note that these are equal to the values obtained in the models presented in [10]; however, the value of the partition function on RP^3 (which was 0 in those models) must necessarily be different.

6 Concluding Remarks

As we have seen from our analysis of boundary subdivision, the original four dimensional theory spawns a gauge invariant model on its boundary. It would be interesting to determine to what extent these results can be extended to the nonabelian situation; one might expect the quantum group case to give some insight in this regard. We would also like to examine the duality properties of these models, and to explore more fully the relationship with those considered in [10, 11].

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