September 2005

Aspects of Finite Temperature Quantum Field Theory in a Black Hole Background

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Abstract

We quantize a scalar field at finite temperature T in the background of a classical black hole, adopting 't Hooft's "brick wall" model with generic mixed boundary conditions at the brick wall boundary. We first focus on the exactly solvable case of two dimensional space-time. As expected, the energy density is integrable in the limit of vanishing brick wall thickness only for $T = T_H$ - the Hawking temperature. Consistently with the most general stress energy tensor allowed in this background, the energy density shows a surface contribution localized on the horizon. We point out that the usual divergences occurring in the entropy of the thermal atmosphere are due to the assumption that the third law of thermodynamics holds for the quantum field in the black hole background. Such divergences can be avoided if we abandon this assumption. The entropy density also has a surface term localized on the horizon, which is open to various interpretations. The extension of these results to higher space-time dimensions is briefly discussed.

SISSA 63/2005/EP
IFUP-TH 19/2005
hep-th/0509080

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1 Introduction and outline

Since the original proposal by Bekenstein [1] that a black hole carries an entropy proportional to its area and later discovery of the Hawking radiation [2], the microscopic origin of the thermodynamical behavior of black holes has been one of the main puzzles of theoretical physics and possibly one of the key problems for the understanding of quantum gravity. The literature is really vast and here we will give an account of just the references strictly relevant to our context.

A very simple but nonetheless instructive model to address the problem of black hole entropy is the so called "brick wall" by 't Hooft [3]. 't Hooft considers a quantum field in the background of a classical black hole and using the WKB approximation he derives the thermal entropy of the field outside the horizon of the black hole. In performing the computation, two spatial cutoffs are employed: a large distance one, needed to avoid large volume divergences in the asymptotically flat region, and a short distance one, the "brick wall", localized just outside the horizon and suppressing the divergences due to the growing number of modes close to the horizon. On the boundaries of the space slice arising in this way, Dirichlet boundary conditions are imposed. As noted in [4, 5], the finite temperature state used in this quantization is a thermal excitation of the Boulware vacuum [6]. The entropy obtained in this model is divergent in the limit of vanishing brick wall thickness. These divergences were later recognized as quantum corrections to the Bekenstein-Hawking formula which can be absorbed into renormalization of the one loop effective gravitational lagrangian [7, 8, 9, 10, 11, 12] (see also [13] for a recent perspective). Within this context the "brick wall" takes the role of a useful mathematical tool to regularize the theory. An interesting different interpretation has been recently proposed in [14, 15].

The introduction of the brick wall cutoff as a more physical device can be considered consistent with recent proposals arising in string theory in which the nature of space-time "inside" the horizon has deep quantum mechanical structure (see [16]).

In this paper we revisit the brick wall model. We first consider in detail the case of 1+1 dimensional Schwarzschild black hole, because it can be solved exactly without resorting to the WKB approximation. We then argue that our results can be extended to higher dimensions. In order to clarify the role of the boundary conditions, we adopt a generic mixed (Robin) boundary condition, which is the most general linear and local one. The cyclic state used in the quantization is a Kubo-Martin-Schwinger (KMS) state[3], with respect to the Schwarzschild time. It accounts for the thermal excitations over the Boulware vacuum. The Boulware vacuum polarization can be exactly calculated in 1+1 dimensions, via the energy-momentum conservation and the trace anomaly [5]. At this point one can derive the energy inside the "shell" between the brick wall and any fixed point outside the horizon. In the limit of vanishing brick wall thickness, the energy of the shell is divergent unless we choose the KMS state temperature T to coincide with the Hawking temperature T_H. In other words, the brick wall can be removed only provided that $T = T_H$. This somewhat expected result does not depend on the boundary conditions being in this sense universal [19, 20].

In all the above considerations one has to keep in mind that the expectation value of the energy-momentum tensor is a distribution rather than a function. We observe in this respect that when considering the horizon as part of the space-time in exam, the above expectation value admits in general a non-vanishing Dirac delta contribution, localized on the horizon. To our knowledge this term has not been previously taken into account and indeed appears in our computation in the limit of vanishing brick wall thickness. We will show that it will also lead to a corresponding surface term in the entropy density.

The entropy of a thermodynamical system is determined by its energy up to an arbitrary constant. In most of the cases this constant is fixed via the third principle of thermodynamics (Nernst theorem), which requires that the entropy vanishes at zero temperature. In our case, the computation of the entropy from the energy of a shell outside the horizon is however a subtle matter. The derivation of the entropy for generic T must be performed before removing the brick wall, since the removal is possible only for $T = T_H$. Any attempt to determine at this stage the arbitrary constant by the third principle, leads to divergences when later removing the brick wall. These divergences are of the same kind encountered in usual computation in the brick wall model. We can get rid of them if we do not ask the quantum fields on such a background to satisfy the third principle. We note here that the issue on the third principle is a priori non related to the observations regarding extremal black holes and Nernst theorem (see for example [21, 22]). Note also that abandoning the third principle turns out to be a different kind of regularization of the theory with respect to the one previously considered in the literature [8, 9, 10, 11, 12, 13] and involving infinite renormalization of the coupling constants.

Our calculations yield also a surface term for the entropy density localized

[3]See e.g. [17, 18]

on the horizon. In our setting this term, together with the corresponding surface term in the energy density, is understood as a boundary effect: taking the brick wall boundary as a pure mathematical tool, it can be considered as a (potentially finite) contribution to the one loop renormalization of Newton constant; on the other hand, taking the brick wall and the boundary condition as somewhat more physical, we can interpret these terms as an indication of the presence of physical degrees of freedom localized on the horizon. We leave some more detailed discussion on this for the last section of the paper.

The paper is organized as follows. The model is introduced in the next section. In Sect. 3 we show that the total energy of a shell outside the horizon is finite even in the limit of vanishing brick wall thickness, provided that the temperature of the thermalized field equals T_H. We also show the appearance of a surface term in the energy density. In Sect. 4 we derive the entropy and discuss the issue related to the third principle of thermodynamics. Sect. 5 is devoted to a comparison with 't Hooft results in the WKB approximation. In Sect. 6 we look for a possible generalization to more realistic models in higher dimensions. Finally, Sect. 7 contains a further discussion and our conclusions.

2 The model

We consider a free real scalar field in a generic $n+1$ dimensional space-time $\{\mathcal{M},\, g\}$ with a time-like boundary $\partial\mathcal{M}$. The action

$$S = \int_{\mathcal{M}} \mathrm{d}^{n+1}x\, \sqrt{|g|}\left(\frac{1}{2}\, g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\right) - \int_{\partial\mathcal{M}} \mathrm{d}^n x\, \sqrt{|g_{\mathrm{ind}}|}\frac{\eta}{2}\varphi^2\,, \quad (2.1)$$

where $|g_{\mathrm{ind}}|$ is the determinant of the induced (lorentzian) metric on $\partial\mathcal{M}$, implies both the Klein Gordon equation

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2)\varphi = 0 \quad (2.2)$$

and the Robin boundary condition

$$(g^{\mu\nu}N_\mu\partial_\nu\varphi - \eta\varphi)|_{\partial\mathcal{M}} = 0\,, \quad (2.3)$$

where N^μ is the unit vector normal to $\partial\mathcal{M}$. Using the boundary condition (2.3), one can also express the action as

$$S = \int_{\mathcal{M}} \mathrm{d}^{n+1}x\, \sqrt{|g|}\left[\frac{1}{2}\, g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{2}g^{\mu\nu}\nabla_\mu(\varphi\,\partial_\nu\varphi)\right]\,. \quad (2.4)$$

The stationarity condition for (2.4) and the boundary condition (2.3) are equivalent to the stationarity condition for the action (2.1). We note that the second expression for the action is suitable also for the Dirichlet boundary condition. From (2.4) one derives the energy-momentum tensor

$$\theta_{\mu\nu}(x) = \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta \left[g^{\mu\nu}(x) \right]} = -\varphi \nabla_\mu \nabla_\nu \varphi + \frac{1}{2} g_{\mu\nu} \left(g^{\rho\sigma} \varphi \nabla_\rho \nabla_\sigma \varphi + m^2 \varphi^2 \right) . \quad (2.5)$$

Let us focus now on a massless ($m = 0$) scalar field in the background of a $1+1$ dimensional classical black hole, with metric

$$\mathrm{d}s^2 = f(r) \mathrm{d}t^2 - \frac{1}{f(r)} \mathrm{d}r^2 . \quad (2.6)$$

We require asymptotic flatness

$$\lim_{r \to \infty} f(r) = 1$$

and assume that f has one and only one zero in $r = r_0$ (the horizon) with positive *surface gravity*

$$\kappa_0 \equiv \frac{1}{2} f'(r_0) > 0 . \quad (2.7)$$

We are thus considering a non-extremal black hole.

Following [3], we insert a brick wall at $r = \rho > r_0$ and study the region $r \geq \rho$, which represents a static space-time with time-like boundary. We consider there the dynamics with respect to the time-translation defined by the flux of $\frac{\partial}{\partial t}$, the time-like Killing vector of the metric. This is the Schwarzschild time for our model.

The classical equation of motion for the scalar field is

$$\frac{1}{f(r)} \partial_t^2 \varphi - \partial_r \left[f(r) \partial_r \varphi \right] = 0 . \quad (2.8)$$

The Robin boundary condition (2.3) takes the form

$$\left(\sqrt{f(r)}\, \partial_r \varphi - \eta \varphi \right) \Big|_{r=\rho} = 0 , \quad (2.9)$$

where η in general can depend on ρ:

$$\eta = \eta(\rho) .$$

In the coordinates (t, y), with the "tortoise coordinate" y defined by

$$y = y(r)\,, \qquad \frac{\mathrm{d}y}{\mathrm{d}r} = \frac{1}{f(r)}\,,$$

the metric (2.6) is conformally flat. Notice that $f(r_0) = 0$ and $f'(r_0) = 2\kappa_0 > 0$ imply that for $r \to r_0$, to leading order

$$y(r) \approx \frac{1}{2\kappa_0} \ln 2\kappa_0(r - r_0)\,, \qquad f(r) \approx \mathrm{e}^{2\kappa_0 y(r)}\,. \tag{2.10}$$

In the coordinates (t, y) the Klein-Gordon equation assumes its flat space form

$$(\partial_t^2 - \partial_y^2)\varphi = 0\,, \tag{2.11}$$

and one is left with the problem of a scalar field on the half-line $y > Y \equiv y(\rho)$ with the boundary condition

$$(\partial_y \varphi - \eta_\rho \varphi)\big|_{y=Y} = 0\,, \qquad \eta_\rho = \eta\sqrt{f(\rho)}\,. \tag{2.12}$$

In order to avoid imaginary energies we ask

$$\eta \geq 0\,. \tag{2.13}$$

The quantization of (2.11),(2.12) is easily performed. The initial conditions are fixed by the canonical equal-time commutation relations

$$[\varphi(t, y_1)\,,\, \varphi(t, y_2)] = 0\,, \qquad [\partial_t\varphi(t, y_1)\,,\, \varphi(t, y_2)] = -i\delta(y_1 - y_2)\,. \tag{2.14}$$

We introduce a class of quasi-free states $G_{\beta,\rho}$. Their two point functions satisfy the Kubo-Martin-Schwinger condition. They are Gibbs states at temperature $T = \beta^{-1}$ for our model with brick wall position ρ. The relative expectation values are denoted by $\langle\varphi(t_1, y_1)\cdots\varphi(t_n, y_n)\rangle_{\beta,\rho}$. The basic one is the two-point function [23, 24]

$$\langle\varphi(t_1, y_1)\varphi(t_2, y_2)\rangle_{\beta,\rho} =$$
$$\int_{\mathbb{R}} \frac{\mathrm{d}p}{4\pi} \frac{|p|_\ell^{-1}}{\mathrm{e}^{\beta|p|} - 1} \left[\mathrm{e}^{\beta|p|}\mathrm{e}^{-\mathrm{i}|p|(t_1 - t_2)} + \mathrm{e}^{\mathrm{i}|p|(t_1 - t_2)}\right] \left[\mathrm{e}^{-\mathrm{i}p(y_1 - y_2)} + B(p, \rho)\mathrm{e}^{\mathrm{i}p(y_1 + y_2 - 2Y)}\right] \tag{2.15}$$

where $B(p, \rho)$ is the reflection factor from the boundary

$$B(p, \rho) = \frac{p - \mathrm{i}\eta_\rho}{p + \mathrm{i}\eta_\rho} \tag{2.16}$$

and the distribution $|p|_\ell^{-1}$ is defined by

$$|p|_\ell^{-1} \equiv \frac{\mathrm{d}}{\mathrm{d}p}\,\varepsilon(p)\ln(|p|\ell)\,, \tag{2.17}$$

ε being the sign function. The derivative in (2.17) is understood in the sense of distributions. The scale parameter ℓ has a well-known infrared origin [25]. Note that

$$p\,|p|_\ell^{-1} = \varepsilon(p)\,, \tag{2.18}$$

which implies, as we shall see later on, that ℓ is irrelevant in the calculation of the energy density.

The states $G_{\infty,\rho}$ can be considered as the analogous of the Boulware vacuum. Indeed they appear as usual vacuum to an observer at rest in the r coordinate in the asymptotically flat region. They are annihilated by every destruction operator associated to a normal mode with respect to the Schwarzschild time.

3 Energy

In this Section we discuss in detail the derivation of the energy density for the states introduced in Section 2. At the Hawking temperature, the divergences occurring in the thermal energy of a shell outside the horizon are perfectly balanced by the Boulware vacuum polarization. From general considerations on the structure of the stress energy tensor, we cannot exclude a contribution to the energy density in the form of a surface term localized on the horizon. The introduction and following removal of a brick wall as a regularization confirms the presence of a finite term of this form. In general, the Boulware vacuum polarization could give a further surface contribution which cannot be determined in this setting. In principle a complete determination of such a term can be performed by some sort of measurement or deduced by a better understanding of this semiclassical picture in the context of a quantum theory of both matter and gravity.

3.1 Wald's axioms and definition

Given one of the Gibbs states described in the last section, we want to calculate the expectation value of the thermal excitations of the energy-momentum tensor over the Boulware-like vacuum. Wald showed [26, 27, 28] that a correctly renormalized energy-momentum tensor $T^\mu{}_\nu$ obeying certain assumptions is essentially unique. Wald's requirements are

1. *Conservation.* Given any state α

$$\nabla_\mu \langle T^\mu{}_\nu \rangle_\alpha = 0\,. \tag{3.1}$$

2. *Consistency.* Given any regular enough couple of states α_1, α_2 we have that $\langle T^\mu{}_\nu \rangle_{\alpha_1} - \langle T^\mu{}_\nu \rangle_{\alpha_1}$ is defined by the usual point-splitting procedure.

3. *Causality holds* in the form of a locality requirement.

4. *Normalization.* In Minkowski space-time, being Ω the usual Fock vacuum, $\langle T^\mu{}_\nu \rangle_\Omega = 0$.

In a generic $n+1$ dimensional space-time, from the contraction of a Killing vector K with $T^\mu{}_\nu$ we can construct a form J_K whose expectation values satisfy the following relations

$$\langle J_K \rangle = \sqrt{|g|}\epsilon_{\mu_1 \cdots \mu_n \nu} \langle T^\nu{}_\rho \rangle K^\rho \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}\,, \tag{3.2}$$

$$\mathrm{d}\langle J_K \rangle = 0\,. \tag{3.3}$$

In our model we consider the Killing vector $K = \partial_t$. For any state α, integrating the second relation above, we can define, the *energy* inside a "shell" (actually a segment) (r_1, r_2)

$$E(r_1, r_2) = \int_{r_1}^{r_2} \langle T^t{}_t(t, r) \rangle_\alpha \mathrm{d}r\,. \tag{3.4}$$

We define

$$\langle \theta^\mu{}_\nu(x) \rangle_{\beta,\rho} \equiv \tag{3.5}$$
$$\lim_{x' \to x} \left(-\nabla^\mu \nabla_\nu + \frac{1}{2} g^\mu{}_\nu g^{\rho\sigma} \nabla_\rho \nabla_\sigma \right) \left[\langle \varphi(x')\varphi(x) \rangle_{\beta,\rho} - \langle \varphi(x')\varphi(x) \rangle_{\infty,\rho} \right]\,.$$

Then, the second of Wald's requirements implies

$$\langle \theta^\mu{}_\nu \rangle_{\beta,\rho} = \langle T^\mu{}_\nu \rangle_{\beta,\rho} - \langle T^\mu{}_\nu \rangle_{\infty,\rho} \tag{3.6}$$

and thus the energy inside a shell for the Gibbs state at temperature β is given by

$$E_{\beta,\rho}(r_1, r_2) = \int_{r_1}^{r_2} \left[\langle \theta^t{}_t(r) \rangle_{\beta,\rho} + \langle T^t{}_t(r) \rangle_{\infty,\rho} \right] \mathrm{d}r\,. \tag{3.7}$$

In view of the point-splitting procedure (3.5), the expression (2.15) for the two point function and the property (2.18), an integration by parts and a change of variable give

$$E_{\beta,\rho}(r_1,r_2)=\int_{y(r_1)}^{y(r_2)}\varepsilon_{\beta,\rho}(y)\mathrm{d}y+\int_{r_1}^{r_2}\langle T^t_{\ t}(r)\rangle_{\infty,\rho}\mathrm{d}r\,, \tag{3.8}$$

where

$$\varepsilon_{\beta,\rho}(y)=\int_{\mathbb{R}}\frac{\mathrm{d}p}{2\pi}\frac{|p|}{\mathrm{e}^{\beta|p|}-1}\left[1+B(p,\rho)\mathrm{e}^{2\mathrm{i}p(y-Y)}\right]\,, \tag{3.9}$$

which can also be expressed in a manifestly real form as

$$\varepsilon_{\beta,\rho}(y)=\int_0^\infty\frac{\mathrm{d}p}{\pi}\frac{p}{\mathrm{e}^{\beta p}-1}\left\{1+\frac{p^2-\eta_\rho^2}{p^2+\eta_\rho^2}\cos[2p(y-Y)]+\frac{2p\eta_\rho}{p^2+\eta_\rho^2}\sin[2p(y-Y)]\right\}\,.$$

We can single out the usual Stefan-Boltzmann contribution to the thermal energy and a specific contribution due to the boundary

$$\varepsilon_{\beta,\rho}(y)=\frac{\pi}{6\beta^2}+h_{\beta,\rho}(y-Y)\,, \tag{3.10}$$

$$h_{\beta,\rho}(\xi)\equiv\frac{1}{\beta^2}\mathcal{F}\left[\frac{|p|}{(\mathrm{e}^{|p|}-1)}\frac{(p-\mathrm{i}\eta_\rho\beta)}{(p+\mathrm{i}\eta_\rho\beta)}\right]\left(\frac{2\xi}{\beta}\right)\,, \tag{3.11}$$

$\mathcal{F}$ being the Fourier transform

$$\mathcal{F}[g(p)](x)=\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}g(p)\,\mathrm{e}^{\mathrm{i}px}\,.$$

We note that since the function $\frac{|p|}{(\mathrm{e}^{|p|}-1)}\frac{(p+\mathrm{i}\eta_\rho\beta)}{(p-\mathrm{i}\eta_\rho\beta)}$ is continuous and L^1, its Fourier transform is continuous, L^1 and infinitesimal at infinity.

3.2 Vacuum polarization and boundary contribution

In order to get a complete expression for the energy in (3.7) we still have to determine $\langle T^\mu_{\ \nu}\rangle_{\infty,\rho}$, that is the Boulware vacuum polarization. In our 1+1 dimensional model the expectation value $\langle T^\mu_{\ \nu}\rangle$ for any state that does not imply transport, i. e. $\langle T^t_{\ r}\rangle=0$, is almost completely determined [5] by its conservation law (3.1) and the trace anomaly

$$\langle T^\mu_{\ \mu}\rangle_{\beta,\rho}=\frac{1}{24\pi}R\,, \tag{3.12}$$

where R is the scalar curvature and in our case $R = f''$. The integration gives

$$f(r)\langle T^r{}_r(r)\rangle = \frac{1}{24\pi}\kappa^2(r) - C\,, \tag{3.13}$$

where C is an integration constant and $\kappa(r) = \frac{1}{2}f'(r)$. Since this is an equation involving distributions, the general solution is given by

$$\langle T^r{}_r(r)\rangle = \frac{1}{f(r)}\left[\frac{1}{24\pi}\kappa^2(r) - C\right] - U\,\delta(r - r_0)\,,$$

where U is an arbitrary constant with the dimensions of an energy. We can call the $U\delta$ term a "boundary" term. Different values of the constants C and U identify different states. By means of the trace anomaly one thus derives

$$\langle T^t{}_t(r)\rangle = \frac{1}{f(r)}\left[C - \frac{1}{24\pi}\kappa^2(r)\right] + \frac{1}{24\pi}f''(r) + U\delta(r - r_0)\,. \tag{3.14}$$

As we noted in the previous section, for any ρ, the Boulware like state appears as vacuum to an observer in the asymptotically flat region; since $\kappa(r) \to 0$ when $r \to \infty$, it is identified by the choice $C = 0$. Moreover, since we are dealing with the $r \geq \rho > r_0$ region, the "boundary" term is irrelevant. We can thus write

$$\langle T^t{}_t(r)\rangle_{\infty,\rho} = -\frac{1}{24\pi}\frac{\kappa^2(r)}{f(r)} + \frac{1}{24\pi}f''(r)\,.$$

Note that it is actually ρ independent.

We now write the energy in a segment or "shell" $[r_1, r_2]$ as

$$E_{\beta,\rho}(r_1, r_2) = \int_{r_1}^{r_2}\langle T^t{}_t(r)\rangle_{\beta,\rho}\mathrm{d}r = E^{(b)}_{\beta,\rho}(r_1, r_2) + E^{HH}_{\beta,\rho}(r_1, r_2)\,, \tag{3.15}$$

$$E^{(b)}_{\beta,\rho}(r_1, r_2) = \int_{y(r_1)}^{y(r_2)} h_{\beta,\rho}(y - Y)\,, \tag{3.16}$$

$$E^{HH}_{\beta,r_0}(r_1, r_2) = \int_{r_1}^{r_2}\left[\frac{\pi}{6\beta^2}\frac{1}{f(r)} + \langle T^t{}_t(r)\rangle_{\infty,\rho}\right]\,. \tag{3.17}$$

We will show in the following that the first term can be regarded as a pure boundary contribution while the second term, with the appropriate choice for β, can be regarded as the contribution from the "Hartle-Hawking" state [19, 20].

3.3 Removing the brick wall

Now we study the limit $\rho \to r_0$, that is the limit of vanishing brick wall thickness. We first consider the case of fixed r_1 and r_2 satisfying $r_2 > r_1 > \rho$. As $\rho \to r_0$, $Y \to -\infty$ and so, as noted at the end of Section 3.1, $h_{\beta,\rho}(y-Y) \to 0$ pointwise inside all of the segment, and we get

$$\begin{aligned} E^{(b)}_{\beta,r_0}(r_1,r_2) &= 0\,, \\ E^{HH}_{\beta,r_0}(r_1,r_2) &= \int_{r_1}^{r_2}\left[\frac{\pi}{6\beta^2}\frac{1}{f(r)} + \langle T^t{}_t(r)\rangle_{\infty,r_0}\right]\mathrm{d}r. \end{aligned} \tag{3.18}$$

In the case of the segment $r_1 = \rho, r_2 = \sigma$ with σ independent on ρ, we have in the limit $\rho \to r_0$

$$\begin{aligned} E^{(b)}_{\beta,\rho}(\rho,\sigma) &= \int_Y^{y(\sigma)} h_{\beta,\rho}(y-Y)\mathrm{d}y = \\ &= \int_0^{y(\sigma)-Y} h_{\beta,\rho}(\xi)\mathrm{d}\xi \to \int_0^\infty h_{\beta,r_0}(\xi)\mathrm{d}\xi = E^{(b)}_\beta\,, \end{aligned} \tag{3.19}$$

where $E^{(b)}_\beta$ depends only on β and $\eta_{r_0} = \lim_{\rho\to r_0}\eta_\rho$:

$$\begin{aligned} E^{(b)}_\beta &= \frac{1}{\beta^2}\int_0^\infty \mathcal{F}\left[\frac{|p|}{\mathrm{e}^{|p|}-1}\frac{p-\mathrm{i}\eta_{r_0}\beta}{p+\mathrm{i}\eta_{r_0}\beta}\right]\left(\frac{2\xi}{\beta}\right)\mathrm{d}\xi = \\ &= \frac{1}{\beta}\int_0^\infty \mathcal{F}\left[\frac{|p|}{\mathrm{e}^{|p|}-1}\frac{p-\mathrm{i}\eta_{r_0}\beta}{p+\mathrm{i}\eta_{r_0}\beta}\right](2\xi)\mathrm{d}\xi. \end{aligned} \tag{3.20}$$

From equations (3.18),(3.19) we can deduce that the function $h_{\beta,\rho}$ determines a pure boundary term in the stress energy tensor localized on the horizon.

We consider now

$$\begin{aligned} E^{HH}_{\beta,\rho}(\rho,\sigma) &= \int_\rho^\sigma\left[\frac{\pi}{6\beta^2}\frac{1}{f(r)} + \langle T^t{}_t(r)\rangle_{\infty,\rho}\right]\mathrm{d}r = \\ &\int_\rho^\sigma\left[\frac{\pi}{6\beta^2}\frac{1}{f(r)} - \frac{1}{24\pi}\frac{\kappa^2(r)}{f(r)} + \frac{1}{24\pi}f''(r)\right]\mathrm{d}r\,. \end{aligned} \tag{3.21}$$

This term may diverge when $\sigma \to \infty$ or $\rho \to r_0$. The first one is a well understood volume divergence and it is not interesting to us. We thus keep σ constant and finite. Then,

$$E^{HH}_{\beta,\rho}(\rho,\sigma) = \left(\frac{\pi}{6\beta^2} - \frac{\kappa_0^2}{24\pi}\right)X(\rho,\sigma) + \frac{\kappa_0^2}{24\pi}\Delta(\rho,\sigma) + \frac{1}{12\pi}\left[\kappa(\sigma)-\kappa(\rho)\right]\,, \tag{3.22}$$

where

$$X(r_1,r_2) \equiv \int_{r_1}^{r_2} \frac{1}{f(r)} \mathrm{d}r = y(r_2) - y(r_1)\,, \tag{3.23}$$

$$\int_{r_1}^{r_2} \frac{1}{24\pi} \frac{\kappa(r)^2}{f(r)} \mathrm{d}r = \frac{\kappa_0^2}{24\pi} \left[X(r_1,r_2) - \Delta(r_1,r_2)\right] \tag{3.24}$$

In the limit $\rho \approx r_0$,

$$X(\rho,\sigma) \approx \frac{1}{2\kappa_0} \ln \frac{a}{\rho - r_0}\,, \tag{3.25}$$

$$\Delta(\sigma,\rho) \quad \text{is finite}\,, \tag{3.26}$$

where a depends on σ. The quantity in (3.22) is thus divergent in the limit $\rho \to r_0$, unless we put

$$\beta = \beta_H = \frac{1}{T_H} = \frac{2\pi}{\kappa_0}. \tag{3.27}$$

In this case the divergences due to thermal excitations are perfectly balanced by the Boulware vacuum polarization and we get a finite energy for every σ. This can be considered as a definition of the Hawking temperature T_H for our model.

At $T = T_H$ it is thus possible to remove the brick wall and we are left with the expression:

$$\begin{aligned} \langle T^t_t \rangle_{\beta_H,r_0} = \frac{1}{f(r)} \left[\frac{\pi}{6\beta_H^2} - \frac{\kappa^2(r)}{24\pi} \right] + \frac{f''(r)}{24\pi} + E^{(b)}_{\beta_H} \delta(r - r_0) = \\ = \langle T^t_t \rangle_{HH} + E^{(b)}_{\beta_H} \delta(r - r_0)\,, \end{aligned} \tag{3.28}$$

where $\langle T^\mu{}_\nu \rangle_{HH}$ is known as the expectation value of $T^\mu{}_\nu$ in the Hartle Hawking state[19, 20, 29, 30, 28]. Indeed, by definition, the expectation value of the energy-momentum tensor in the Hartle Hawking state is regular on both the past and the future horizon of the black hole; this corresponds to the choice

$$C = \frac{\pi}{6} \left(\frac{\kappa_0}{2\pi} \right)^2 = \frac{\pi}{6\beta_H^2}$$

in equation (3.14). To an observer at rest in the asymptotically flat region, the Hartle Hawking state appears as a thermal bath at the Hawking temperature T_H.

4 Entropy

In this Section we describe the derivation of the entropy of a shell of our space-time from the previously calculated energy density. Even at the Hawking temperature, the entropy in a shell attached to the horizon is divergent. The divergence can be resolved allowing that the system of the quantum field in this background does not satisfy Nernst theorem. The price for this is the presence of an arbitrary constant in the entropy of any shell. One can consider this freedom as a consequence of a breakdown of the semiclassical picture or as an intrinsic feature of the QFT in this background (analogous to renormalization effects), or as a combination of both.

The entropy density admits a surface term localized on the horizon. It indicates the presence of physical degrees of freedom there, which can give rise to a (potentially finite) renormalization of Newton constant.

4.1 Entropy from energy

We will now perform the calculation of the entropy inside a shell. Since

$$E = \frac{\partial(\beta F)}{\partial\beta}\,, \qquad F = E - S/\beta\,,$$

we have

$$S(\beta) = \beta E(\beta) - \bar{\beta}F(\bar{\beta}) - \int_{\bar{\beta}}^{\beta} E(b)\mathrm{d}b\,. \tag{4.1}$$

When the relations

$$E(\beta) \to E_0\,, \quad \beta[E(\beta) - E_0] \to 0 \qquad \text{for} \quad \beta \to \infty \tag{4.2}$$

are satisfied, the arbitrary constant $\bar{\beta}F(\bar{\beta})$ can be determined via the third principle of thermodynamics in the form

$$S(\beta) \to 0 \qquad \text{for} \quad \beta \to \infty\,,$$

that gives

$$F(\bar{\beta}) = -\frac{1}{\bar{\beta}}\int_{\bar{\beta}}^{\infty}(E(b) - E_0)\mathrm{d}b + E_0\,.$$

It is straightforward to show that the conditions in (4.2) are necessary and sufficient for the system to verify the third principle.

4.2 Removing the brick wall with and without third principle

Since the removal of the brick wall is not possible "off shell", i.e. for $\beta \neq \beta_H$, we are forced to perform the integration in (4.1) at non zero brick wall thickness and then to perform the limit $\rho \rightarrow r_0$.

Let's consider the energy in a shell (r_1, r_2). The analysis of the previous section gives

$$E_{\beta,\rho}(r_1,r_2) = \int_0^{X(r_1,r_2)} h_{\beta,\rho}(\xi)\mathrm{d}\xi + \left(\frac{\pi}{6\beta^2} - \frac{\kappa_0^2}{24\pi}\right) X(r_1,r_2) + \\ + \frac{\kappa_0^2}{24\pi}\Delta(r_1,r_2) + \frac{1}{24\pi}\left[2\kappa(r_2) - 2\kappa(r_1)\right] .$$

We recall that

$$h_{\beta,\rho}(\xi) = \frac{1}{\beta^2}\mathcal{F}\left[\frac{|p|}{(\mathrm{e}^{|p|}-1)}\frac{(p - \mathrm{i}\eta_\rho\beta)}{(p + \mathrm{i}\eta_\rho\beta)}\right]\left(\frac{2\xi}{\beta}\right) .$$

Using equation (4.1) and putting $\bar{\beta} = \beta_H = \frac{2\pi}{\kappa_0}$ we have

$$S_{\beta,\rho}(r_1,r_2) = \frac{\pi}{6\beta}X(r_1,r_2) + \beta\int_0^{X(r_1,r_2)} h_{\beta,\rho}(\xi)\mathrm{d}\xi + \\ + \beta_H\frac{1}{24\pi}\left[-\kappa_0^2 X(r_1,r_2) + \kappa_0^2\Delta(r_1,r_2) + 2\kappa(r_2) - 2\kappa(r_1)\right] + \\ - \beta_H F_H - X(r_1,r_2)\int_{\beta_H}^{\beta}\frac{\pi}{6b^2}\mathrm{d}b - \int_{\beta_H}^{\beta}\int_0^{X(r_1,r_2)} h_{b,\rho}(\xi)\mathrm{d}b\mathrm{d}\xi.$$

The third principle is satisfied if

$$\beta_H F_H = -X(r_1,r_2)\int_{\beta_H}^{\infty}\frac{\pi}{6b^2}\mathrm{d}b - \int_{\beta_H}^{\infty}\int_0^{X(r_1,r_2)} h_{b,\rho}(\xi)\mathrm{d}\xi\mathrm{d}b + \\ + \beta_H\frac{1}{24\pi}\left[-\kappa_0^2 X(r_1,r_2) + \kappa_0^2\Delta(r_1,r_2) + 2\kappa(r_2) - 2\kappa(r_1)\right] .$$

Hence, going "on shell", we have:

$$S_{\beta_H,\rho}(r_1,r_2) = \\ = \frac{\pi}{3\beta_H}X(r_1,r_2) + \beta_H\int_0^{X(r_1,r_2)} h_{\beta_H,\rho}(\xi)\mathrm{d}\xi + \int_{\beta_H}^{\infty}\int_0^{X(r_1,r_2)} h_{b,\rho}(\xi)\mathrm{d}\xi\mathrm{d}b . \quad (4.3)$$

We note here that the function

$$\int_{\beta_H}^{\infty} h_{b,\rho}(\xi)\mathrm{d}b = -\frac{1}{\beta_H}\mathcal{F}\left[\ln(1-\mathrm{e}^{-|p|})\frac{p-\mathrm{i}\eta_\rho\beta_H}{p+\mathrm{i}\eta_\rho\beta_H}\right]\left(\frac{2\xi}{\beta_H}\right)$$

is not L^1 but it is infinitesimal at infinity, since it is the Fourier transform of a non continuous L^1 function. One can analyze (4.3) in analogy of what we did in the previous section for the energy.

In the limit $\rho \to r_0$ with fixed r_1, r_2 satisfying $r_2 > r_1 > \rho$, the second and third term in expression (4.3) vanish and we are left with an usual volume term:

$$S_{\beta_H,\rho}(r_1, r_2) = \frac{\pi}{3\beta_H} X(r_1, r_2)\,. \tag{4.4}$$

When $r_1 = \rho$ and $r_2 = \sigma$, with σ constant and finite, we have in the limit $\rho \to r_0$

$$X(\rho,\sigma) \to \infty \quad , \quad \Delta(\rho,\sigma)\,\text{is finite}\,.$$

The first term in (4.3) is linearly divergent in X, and the last one can also be divergent. The function $\int_{\beta_H}^{\infty} h_{b,\rho}(\xi)\mathrm{d}b$ is not L^1 but is infinitesimal at infinity: the possible divergence from the last term in (4.3) can not be linear in X[4]. Its origin can be traced back to the β dependence of the so called "surface" term in the energy

$$E_{\beta,\rho}^{(b)} = \int_0^X h_{\beta,\rho}(\xi)\mathrm{d}\xi = \int_0^X \frac{1}{\beta^2}\mathcal{F}\left[\frac{|p|}{\mathrm{e}^{|p|}-1}\frac{p-\mathrm{i}\eta_\rho\beta}{p+\mathrm{i}\eta_\rho\beta}\right]\left(\frac{2\xi}{\beta}\right)\mathrm{d}\xi\,.$$

As we showed in the preceding section, in the limit $\rho \to r_0$ we have

$$E_{\beta}^{(b)} = \frac{1}{\beta}\int_0^{\infty}\mathcal{F}\left[\frac{|p|}{\mathrm{e}^{|p|}-1}\frac{p-\mathrm{i}\eta_{r_0}\beta}{p+\mathrm{i}\eta_{r_0}\beta}\right](2\xi)\mathrm{d}\xi\,.$$

When $\beta \to \infty$, or more precisely $\beta \gg \eta_{r_0}^{-1}$, we have, up to leading order:

$$E_{\beta}^{(b)} = -\frac{1}{\beta}\int_0^{\infty}\mathcal{F}\left[\frac{|p|}{\mathrm{e}^{|p|}-1}\right](2\xi)\mathrm{d}\xi \propto \frac{1}{\beta}$$

and the β dependence is incompatible with the third principle. Such a behavior can be qualitatively understood looking at the description of the system in the (t, y) coordinates: when $\rho \to r_0$ we have $Y \to -\infty$ and thus the specific boundary condition cannot appear independently in the leading order

[4]In principle, we can not be sure of the actual appearance of such a divergence since the integral could be convergent in improper way.

expansion of thermodynamical quantities. Due to the conformal symmetry of our two dimensional model, the only dimensional parameter relevant for the expansion is β and thus β^{-1} is the only possible leading term.

The different behavior of the two divergences reflects their different physical origin. The one linear in X is basically due to the thermal excitations of the model close to the horizon and we expect it to be present also in more realistic models; the other one can be regarded as a specific feature of our oversimplified model.

We have thus shown that the entropy of a shell contains two distinct terms:

$$S_{\beta_H,\rho}(r_1,r_2) = S^{HH}_{\beta_H,\rho}(r_1,r_2) + S^{(b)}_{\beta_H,\rho}(r_1,r_2) \tag{4.5}$$

$$S^{HH}_{\beta_H,\rho}(r_1,r_2) \equiv \frac{\pi}{3\beta_H} X(r_1,r_2) \tag{4.6}$$

$$S^{(b)}_{\beta_H,\rho}(r_1,r_2) \equiv \beta_H \int_0^{X(r_1,r_2)} h_{\beta_H,\rho}(\xi)\mathrm{d}\xi + \int_{\beta_H}^{\infty}\int_0^{X(r_1,r_2)} h_{b,\rho}(\xi)\mathrm{d}\xi\mathrm{d}b\,. \tag{4.7}$$

The first one is a volume term, which is also obtained in the usual treatment of the brick wall model (see Section 5). The second one is a surface term localized on the horizon which is inherited from the surface term in the energy density.

Both these terms are divergent in the limit $\rho \to r_0$ but both divergences can be avoided if we do not use the third principle in order to fix the arbitrariness in the determination of the entropy. With the substitution

$$\beta_H F_H \to \beta_H F_H + \frac{\pi}{3\beta_H}\frac{1}{2\kappa_0}X + \int_{\beta_H}^{\infty}\int_0^{X} h_{b,\rho}(\xi)\mathrm{d}\xi\mathrm{d}b - S^{\Omega}_{\beta_H}(\rho,\sigma)$$

we have

$$S_{\beta_H,r_0}(r_0,\sigma) = \beta_H \int_0^{\infty} \frac{1}{\beta_H^2}\mathcal{F}\left[\frac{|p|}{(\mathrm{e}^{|p|}-1)}\frac{(p-\mathrm{i}\eta_{r_0}\beta_H)}{(p+\mathrm{i}\eta_{r_0}\beta_H)}\right]\left(\frac{2\xi}{\beta_H}\right)\mathrm{d}\xi + S^{\Omega}_{\beta_H}(r_0,\sigma)\,, \tag{4.8}$$

where $S^{\Omega}_{\beta_H}(\rho,\sigma)$ is an arbitrary continuous function, which parameterize the inability of determining the exact value of the entropy without any external information input such as, for example, a derivation from a more fundamental theory which correctly identifies the degrees of freedom of the system. This picture recalls the standard situation one has in renormalization theory, where we pay the finiteness of the theory with an indetermination.

As we already noted, we expect that the divergences appearing in the surface term of the entropy are a specific feature of our extremely simplified model. Thus we expect that a surface term is generically present and potentially finite, showing an accumulation of degrees of freedom on the horizon which is different from the one usually considered as a consequence of the growing number of modes close to the horizon. This can be interpreted as a first order quantum correction to the Bekenstein-Hawking formula due to the interaction of the matter field with the gravitational field. Also this term can be considered as affected by some sort of indetermination since we cannot exclude a priori the presence of a surface contribution also in the arbitrary function $S^{\Omega}_{\beta_H}(r_0,\sigma)$. In general, the boundary contribution is non-vanishing whenever

$$S_{\beta_H,r_0}(r_0,r_0) = \beta_H \int_0^\infty h_{\beta_H,\rho}(\xi)\mathrm{d}\xi + S^{\Omega}_{\beta_H}(r_0,r_0) \neq 0\,. \tag{4.9}$$

The expression for the entropy in a shell far outside the horizon can be deduced from (4.8)

$$S_{\beta_H,r_0}(r_1,r_2) = S^{\Omega}_{\beta_H}(r_0,r_2) - S^{\Omega}_{\beta_H}(r_0,r_1)\,. \tag{4.10}$$

If we assume that the origin of the awkward behavior of the entropy is in some sort of interaction with the black hole and its horizon, it seems reasonable to ask for the expression (4.10) to become equal to the expected one in the asymptotically flat region, that is

$$S^{\Omega}_{\beta_H}(r_0,r_2) - S^{\Omega}_{\beta_H}(r_0,r_1) \to \frac{\pi}{3\beta}(r_2-r_1) \qquad \text{for} \quad r_1,r_2\to\infty\,.$$

5 The WKB approximation

It is instructive to reexamine the model described in the previuos sections within the WKB approximation. For this purpose we consider the shell (ρ,σ) and define the wave number $k(r)$

$$k(r) = \frac{E}{f(r)}\,.$$

The density of states in the WKB approximation is given by

$$n(E) = \frac{\partial N(E)}{\partial E}\,,$$
$$\pi N(E) = \int_\rho^\sigma \mathrm{d}r \frac{E}{f(r)}\,.$$

We note that for Neumann or Dirichlet boundary conditions in 1+1 dimensions the WKB approximation gives the right eigenvalues of the energy. The free energy F reads

$$F_{\rm WKB} = \int n(E) \ln(1 - {\rm e}^{-\beta E}) {\rm d}E = \int \frac{N(E)}{{\rm e}^{\beta E} - 1} = \frac{\pi}{6\beta^2} \int_\rho^\sigma \frac{{\rm d}r}{f(r)} \,.$$

Recalling that $S = \beta^2 \partial_\beta F$ and neglecting subleading terms in the limit of $\rho \approx r_0$, one obtains

$$S_{\rm WKB} = \frac{\pi}{3\beta_H} X(\rho, \sigma) = S^{HH}_{\beta_H,\rho}(\rho, \sigma) \,.$$

Again neglecting subleading terms and comparing with (4.3) one finds

$$S_{\beta_H,\rho}(\rho, \sigma) = S_{\rm WKB} + S^{(b)}_{\beta_H,\rho}(\rho, \sigma) \,,$$

showing that the the boundary term in the entropy is lost in the WKB approximation. In both cases however the entropy is divergent in the limit $\rho \to r_0$ and the considerations at the end of the previous section apply.

6 Higher dimensions

In the higher dimensional and massive extension of our model we have to deal with two main issues:

- resolving the spectral problem for the field, i.e. the determination of the its normal modes;
- determining the Boulware vacuum polarization.

These problems are not conceptual but technical. The following steps towards their solution can be made in the spherically symmetric case.

The metric for an $n + 1$ dimensional spherical black hole is given by

$$ {\rm d}s^2 = U(r){\rm d}t^2 - \frac{1}{U(r)}{\rm d}r^2 + r^2 \Omega^{(n-1)}_{ij} {\rm d}\theta^i {\rm d}\theta^j \,,$$

where U is assumed to have one and only one zero in r_0 and $U'(r_0) = 2\kappa_0 \neq 0$. We insert a brick wall and consider a scalar field only in the exterior of the sphere of radius $\rho = r_0$, with a boundary condition

$$\left(\sqrt{U} \partial_r \varphi = \frac{h}{r_0} \varphi \right) \Big|_{r=\rho} \,.$$

Using the "tortoise" coordinate r^* defined by

$$\frac{\mathrm{d}r^*}{\partial r} = \frac{1}{U}$$

the thermal energy in a spherical shell is given by

$$E_{\beta,\rho} = \int_{r_1}^{r_2} \langle \theta^t{}_t(r) \rangle_{\beta,\rho} \Sigma_{n-1} r^{n-1} \mathrm{d}r = \int_{r_1^*}^{r_2^*} \varepsilon(r^*)_{\beta,\rho} \Sigma_{n-1} (r(r^*))^{n-1} \mathrm{d}r^* \,,$$

where $\Sigma_{n-1} r^{n-1}$ is the area of the $(n-1)$ dimensional sphere of radius r, $\Sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$. At this point one can separate the angular dependence and pass to the reduced radial wave functions $f = r^{\frac{n-1}{2}}\psi$, where ψ is the original radial wave function. Then one is left with the study of the operators

$$D_l = -\partial_{r^*}^2 + \frac{n-1}{2r} U\,(\partial_r U) + \frac{(n-1)(n-3)}{4} \frac{U^2}{r^2} + U \frac{l(l+n-2)}{r^2} + U m^2 \,,$$

where $r = r(r^*)$, $l(l+n-2)$ is the eigenvalue of the angular part of the flat Laplace operator in n dimension (the squared total angular momentum) and m is the mass of the field. Now, one has to investigate the spectral problem for D_l with the boundary condition

$$\left(\frac{1}{\sqrt{U}} \partial_{r^*} f = \frac{h + \frac{n-1}{2}}{r_0} f \right)\Bigg|_{r^*=\rho^*} \,,$$

where $\rho^* = r^*(\rho)$. Being $f_{\lambda l}(r^*)$ the complete set of orthonormal (improper) eigenfunctions, i. e.

$$D_l f_{\lambda l} = \lambda^2 f_{\lambda_l} \,,$$
$$\int \sigma_{\lambda l} f_{\lambda l}(r_1^*) f_{\lambda l}(r_2^*) \mathrm{d}\lambda = \delta(r_1^* - r_2^*) \,,$$

we get

$$\varepsilon(r^*)_{\beta,\rho} = \sum_{l=0}^{\infty} \frac{d_l}{\Sigma_n} \frac{1}{r^{n-1}} \int \sigma_{\lambda l} \frac{\lambda}{\mathrm{e}^{\beta\lambda} - 1} f_{\lambda l}^2(r^*) \mathrm{d}\lambda \,, \tag{6.1}$$

where $d_l = \frac{(2l+n-2)(l+n-3)!}{l!(n-2)!}$ is the multiplicity of the eigenvalue $l(l+n-2)$ of the squared total angular momentum.

We expect that $\langle T^\mu{}_\nu \rangle_{\beta,\rho}$ has a structure analogous to the one arising in the two dimensional case: a volume term with singular behaviour for all values of β except $\beta = \beta_H$ and a boundary term, which appears when considering a shell attached to the boundary. When performing an analysis similar to the one of Section 4 we expect analogous diverging contribution from the bulk and a potentially finite (see discussion in Section 4) horizon contribution. Again, the leading divergence could be traced back to the third principle. These issues need a further investigation.

7 Outlook and conclusions

In this work we analysed the behavior of a quantum field at finite temperature T in the backgroud of a classical black hole. We applied the brick wall approach of 't Hooft, introducing in addition a parameter η, which fixes the boundary conditions for the field on the brick wall and which can be interpreted as a parametrization of the interaction of the field with gravity close and behind the horizon. Focusing mainly on a 1+1 dimensional black hole space-time, we computed both the energy and the entropy densities. The energy density contains an η-dependent boundary term, which is localized on the horizon and respects the conservation and the trace anomaly of the energy-momentum tensor. Taking into account the Boulware vacuum polarization, we have shown that the energy density remains finite in the limit of vanishing brick wall thickness, provided that T equals the Hawking temperature $T_H = \frac{\kappa_0}{2\pi}$. We recall in this respect that in the original brick wall model the determination of the brick wall thickness is made by requiring that the full classical black hole entropy is due to the thermal atmosphere of all the fundamental fields at the Hawking temperature[5], which enters the model as an external input.

The entropy density shows analogous regular behavior for vanishing brick wall thickness, provided we release the third principle of thermodynamics in deriving the thermal entropy from the thermal energy. We note here that there are several examples of incomplete models that do not verify Nernst theorem. Likely the most known and simple one is the perfect Boltzmann gas, where the entropy diverges when $T \to 0$. In this case the problem is solved observing that for sufficiently low temperatures the particle interactions and quantum statistical effects cannot be neglected and the model is no longer valid. This means that the physical degrees of freedom of the system are not correctly identified by the model - a situation which looks similar to ours.

The entropy we obtained has two main features. First, it is also endowed with a boundary term localized on the horizon and determined by the bound-

[5]For specific issues concerning the two dimensional case we refer to [31].

ary condition. Second, the entropy in the bulk is determined up to a function, since a new input, substituting the third principle, is needed for its complete determination. We believe that this input should come from a better understanding of the interplay between quantum mechanics and general relativity in systems containing horizons and black holes.

For example, in the context of string theory, a recently quite popular proposal [16] suggests that the structure of space-time inside the horizon is deeply quantum mechanical. In this picture a model in which the semiclassical picture is confined outside the horizon can be consistent. The specific boundary condition can thus mimic some sort of interaction with the physics inside the horizon. However, it could appear non reasonable to perform the limit of vanishing brick wall thickness since we cannot expect the transition between quantum and semiclassical behaviour to happen sharply at the horizon. Also in this picture some of our results are relevant: in particular the term that gives rise to the "would be" boundary contribution, still remain as a an indication of a peculiar accumulation of degrees of freedom in the near horizon region.

Acknowledgements

We would like to thank S. Liberati and V. Moretti for encouragement and useful discussions. This research is supported by the Italian MIUR under the program, "Teoria dei Campi, Superstringhe e Gravità".

References

[1] J. D. Bekenstein, "Black holes and entropy," *Phys. Rev.* **D7** (1973) 2333–2346.

[2] S. W. Hawking, "Particle creation by black holes," *Commun. Math. Phys.* **43** (1975) 199–220.

[3] G. 't Hooft, "On the quantum structure of a black hole," *Nucl. Phys.* **B256** (1985) 727.

[4] S. Mukohyama and W. Israel, "Black holes, brick walls and the Boulware state," *Phys. Rev.* **D58** (1998) 104005, `gr-qc/9806012`.

[5] S. Mukohyama, "The origin of black hole entropy," `gr-qc/9812079`.

[6] D. G. Boulware, "Quantum field theory in Schwarzschild and Rindler spaces," *Phys. Rev.* **D11** (1975) 1404.

[7] L. Susskind and J. Uglum, "Black hole entropy in canonical quantum gravity and superstring theory," *Phys. Rev.* **D50** (1994) 2700–2711, `hep-th/9401070`.

[8] J.-G. Demers, R. Lafrance, and R. C. Myers, "Black hole entropy without brick walls," *Phys. Rev.* **D52** (1995) 2245–2253, `gr-qc/9503003`.

[9] D. V. Fursaev and S. N. Solodukhin, "On one loop renormalization of black hole entropy," *Phys. Lett.* **B365** (1996) 51–55, `hep-th/9412020`.

[10] G. Cognola, L. Vanzo, and S. Zerbini, "One loop quantum corrections to the entropy for a four- dimensional eternal black hole," *Class. Quant. Grav.* **12** (1995) 1927–1936, `hep-th/9502006`.

[11] S. P. de Alwis and N. Ohta, "Thermodynamics of quantum fields in black hole backgrounds," Phys. Rev. D **52** (1995) 3529 [arXiv:hep-th/9504033].

[12] E. Winstanley, "Renormalized black hole entropy in anti de-Sitter space via the "brick wall" method," *Phys. Rev.* **D 63** (2001).

[13] R. Brustein, M. B. Einhorn, and A. Yarom, "Entanglement interpretation of black hole entropy in string theory," `hep-th/0508217`.

[14] J. L. F. Barbon and E. Rabinovici, "Very long time scales and black hole thermal equilibrium," *JHEP* **11** (2003) 047, `hep-th/0308063`.

[15] S. N. Solodukhin, "Restoring unitarity in BTZ black hole," *Phys. Rev.* **D71** (2005) 064006, `hep-th/0501053`.

[16] S. D. Mathur, "The fuzzball proposal for black holes: An elementary review," *Fortsch. Phys.* **53** (2005) 793–827, `hep-th/0502050`.

[17] R. Haag, *Local Quantum Physics.* Springer, second revised and enlarged ed., 1996.

[18] O. Brattelli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, vol. II. Springer-Verlag, New York, 1981.

[19] J. B. Hartle and S. W. Hawking, "Path integral derivation of black hole radiance," *Phys. Rev.* **D13** (1976) 2188–2203.

[20] T. Jacobson, "A Note on Hartle-Hawking vacua," *Phys. Rev.* **D50** (1994) 6031–6032, gr-qc/9407022.

[21] R. M. Wald, "The Nernst theorem and black hole thermodynamics," *Phys. Rev.* **D56** (1997) 6467–6474, gr-qc/9704008.

[22] F. Belgiorno and M. Martellini, "Black holes and the third law of thermodynamics," Int. J. Mod. Phys. D **13** (2004) 739 [arXiv:gr-qc/0210026].

[23] A. Liguori, M. Mintchev, and L. Zhao, "Boundary Exchange Algebras and Scattering on the Half Line," *Commun. Math. Phys.* **194** (1998) 569–589, hep-th/9607085.

[24] M. Mintchev and P. Sorba, "Finite temperature quantum field theory with impurities," *JSTAT* **0407** (2004) P001, hep-th/0405264.

[25] G. Grignani and M. Mintchev, "Renormalization and conformal properties of sigma models on Riemannian spaces with torsion," *Phys. Rev.* **D38** (1988) 3163.

[26] R. M. Wald, "The Back Reaction Effect In Particle Creation In Curved Space-Time," Commun. Math. Phys. **54** (1977) 1.

[27] R. M. Wald, "Trace Anomaly Of A Conformally Invariant Quantum Field In Curved Space-Time,' Phys. Rev. D **17** (1978) 1477.

[28] R. M. Wald, *Quantum Field Theory in Curved Space-time and Black Hole Termodynamics.* The University of Chicago Press, London and Chicago, 1994.

[29] G. L. Sewell, "Relativity of temperature and the Hawking effect," *Physics Letters A* **79** (Sept., 1980) 23–24.

[30] K. Fredenhagen and R. Haag, "On the derivation of Hawking radiation associated with the formation of a black hole," *Commun. Math. Phys.* **127** (1990) 273.

[31] R. B. Mann, L. Tarasov, and A. Zelnikov, "Brick walls for black holes," *Class. Quant. Grav.* **9** (1992) 1487–1494.