4D Effective Theory and Geometrical Approach

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Abstract. We consider the 4D effective theory for the light Kaluza-Klein (KK) modes. The heavy KK mode contribution is generally needed to reproduce the correct physical predictions: an equivalence, between the effective theory and the D-dimensional (or *geometrical*) approach to spontaneous symmetry breaking (SSB), emerges only if the heavy mode contribution is taken into account. This happens even if the heavy mode masses are at the Planck scale. In particular, we analyze a 6D Einstein-Maxwell model coupled to a charged scalar and fermions. Moreover, we briefly review non-Abelian and supersymmetric extensions of this theory.

INTRODUCTION

The low energy limit of a higher dimensional theory is usually studied by taking into account only the light mode contribution. The masses are derived from the bilinear part of the effective action and the role of the heavy modes in the actual values of the masses and the couplings of the effective theory for the light modes are seldom taken into account. However, we know that the process of "integrating out" the heavy modes [1] has the effect of modifying the couplings of the light modes or introducing additional terms that are suppressed by inverse powers of the heavy masses [2].

In a first part of this contribution we will summarize the study of the heavy mode contribution to the low energy dynamics of higher dimensional models, performed in Ref. [3]. There two methods have been used. The first one, which is called the *4D effective theory approach*, starts from a solution of a higher dimensional theory and develops an action functional for the light modes. This effective action generally has a local symmetry that should be broken by Higgs mechanism. Our interest is in the broken phase of the effective theory. The procedure is essentially what is adopted in the effective description of higher dimensional theories including superstring and M-theory compactifications. In this construction the heavy KK modes are generally ignored simply by reasoning that their masses are of the order of the compactification mass and this can be as heavy as the Planck mass.

In the second approach, which we shall call the *geometrical approach*, we shall find a solution of the higher dimensional equations with the same symmetry group as the one of the broken phase of the 4D effective theory. We shall then study the physics of the light modes around this solution. The result for the low energy physics will turn out to be different from the first approach. The difference is precisely due to the fact that in constructing the effective theory along the lines of the first approach the contribution of the heavy KK modes has been ignored.

This statement has been explicitly proved, at the classical level, for a quite general

higher dimensional scalar model with lagrangian $\mathscr{L} = -\frac{1}{2}\partial_M \Phi \partial^M \Phi + V(\Phi)$, where Φ is a set of scalar fields, and then extended to a more interesting (Abelian) gauge and gravitational theory [3]. Here we will briefly report the latter case, which, in the low energy limit, reduces to a framework that is similar to the electroweak part of the Standard Model. In this framework, the heavy KK mode contribution can be geometrically interpreted as the deformation of the internal space.

Moreover, in a final section, we shall review possible extensions of these results to non-Abelian theories without fundamental scalars or to supersymmetric versions of 6D gauge and gravitational theories. In the former framework the Higgs field is identified with the internal components of the non-Abelian gauge field [4] and the complete 6D gauge symmetry relaxes the dependence of the Higgs mass on the ultraviolet cutoff. The latter class of theories can be used as toy models for string theory compactifications [5] and has shown some promise in addressing the cosmological constant problem [6].

6D EINSTEIN-MAXWELL-SCALAR MODEL

Here we analyze a 6D model, which includes the Einstein-Hilbert gravity, a Maxwell field A and a complex charged scalar ϕ. The bosonic action reads[1]

$$S_B = \int d^6 X \sqrt{-G} \left[\frac{1}{\kappa^2} R - \frac{1}{4} F^2 - |\nabla \phi|^2 - V(\phi) \right].$$

where $F = dA$, $\nabla \phi = (d + ieA)\phi$. We choose $V(\phi) = m^2 |\phi|^2 + \xi |\phi|^4 + \lambda$, where m^2 is a real mass squared, ξ is a real and positive parameter and λ represents a 6D cosmological constant. This system is a simple generalization of the 6D Einstein-Maxwell model of Ref. [7], where it was proved that the space-time $(Minkowsky)_4 \times S^2$ is a solution of the equations of motion (EOMs), in the presence of a monopole background. This solution is

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + a^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right), \\
A &= \frac{n}{2e}(cos\theta \mp 1)d\varphi, \\
\phi &= 0,
\end{aligned}
\tag{1}
$$

where a is the radius of S^2 and n is the monopole number ($n = 0, \pm 1, \ldots$). Besides 4D Poincaré invariance, this background preserves an $SU(2) \times U(1)$ symmetry, which turns out to be the gauge symmetry of the low energy 4D effective theory [7]. The group factor $SU(2)$ has a geometrical origin as the isometry group of the internal space, whereas $U(1)$ represents the bulk gauge symmetry.

Moreover it is possible to introduce a couple of fermions $\psi_\pm$, with 6D chirality ± 1, and standard Yukawa couplings: $\mathscr{L}_{Yuk} = g_Y \overline{\psi_+} \psi_- \phi^\dagger + g_Y \phi \overline{\psi_-} \psi_+$, where g_Y is assumed to be real for simplicity. Furthermore we assume that the corresponding $U(1)$ fermion

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[1] Our conventions are $\eta_{MN} = (-1, +1, +1, +1, \ldots)$ and $R_{MN} = \partial_P \Gamma^P_{MN} - \partial_M \Gamma^P_{PN} + \ldots$.
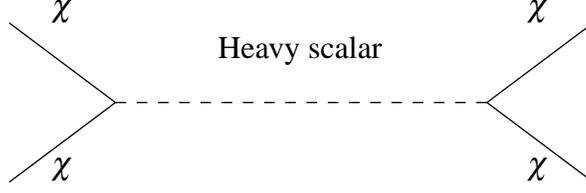
FIGURE 1. Heavy mode contribution to the quartic couplings. Reprinted from [3].

charges are $e_+ = e/2$ and $e_- = 3e/2$, because this corresponds to a simple fermion harmonic expansion over S^2.

A complete study of the fluctuations around Solution (1) shows that the low energy 4D bosonic spectrum presents the following states: the graviton, the $SU(2) \times U(1)$ gauge fields and a complex scalar field χ, coming from ϕ, in the $(|n|+1)$-dimensional representation of $SU(2)$. For example, for $n = 2$ we obtain a triplet and, henceforth, we will assume $n = 2$, as the geometrical approach turns out to be simple in this case. Moreover, in the fermion sector, we have a right-handed singlet and a left-handed triplet.

4D Higgs Mechanism. If the gauge symmetry is unbroken, the states that we have mentioned above are exactly massless, apart from χ. Indeed we can introduce a small[2] mass μ for χ by choosing a suitable value of m. In order to create a small mass for fermions and gauge fields, usually one computes an action functional for the light modes, including bilinear terms and interactions, and then studies the Higgs mechanism in the corresponding 4D theory. In our case, this can be achieved by generalizing the zero-mode ansatz method of Ref. [7], to include the light scalar χ. In particular, the lagrangian for χ turns out to be of the form $\mathscr{L}_\chi = -(D_\mu \chi)^\dagger D^\mu \chi - \mathscr{U}(\chi)$, where D_μ is the $SU(2) \times U(1)$ covariant derivative and $\mathscr{U}$ is the scalar potential for χ. In our model the $U(1)$ and $SU(2)$ gauge constants, which appear in D_μ, are respectively given by

$$g_1 = \frac{e}{\sqrt{4\pi}a}, \quad g_2 = \sqrt{\frac{3}{16\pi}} \frac{\kappa}{a^2}, \tag{2}$$

whereas the potential is $\mathscr{U}(\chi) = \mu^2 \chi^\dagger \chi + \lambda_1 (\chi^\dagger \chi)^2 + \lambda_2 |\chi^T \chi|^2 + \ldots$, where the dots represent higher order operators and the λ_i are the quartic coupling constants allowed by the $SU(2) \times U(1)$ gauge symmetry. We have $\lambda_1 = \lambda_H + c_1 \lambda_G$, and $\lambda_2 = -(\lambda_H + c_2 \lambda_G)/3$, where

$$\lambda_H = \frac{9}{20\pi a^2} \xi, \qquad \lambda_G = \frac{9\kappa^2}{80\pi a^4}.$$

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[2] It is small in the sense that $\mu << 1/a$.

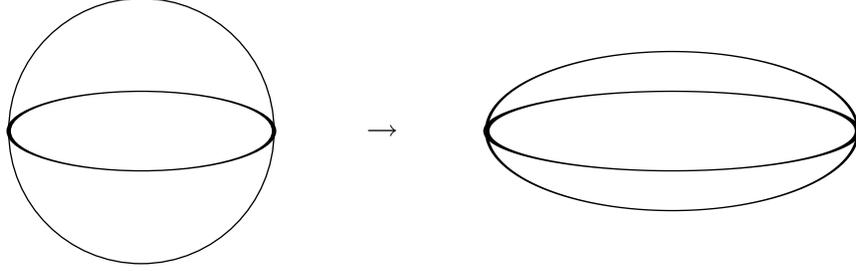**Sphere ($SU(2)$ symmetry)**   **Ellipsoid ($U(1)$ symmetry)**

FIGURE 2. Electroweak symmetry breaking in the geometrical approach.

We observe that λ_H and λ_G represent respectively the light mode and the heavy mode contribution to λ_i. The constants c_i parametrize the latter contribution, and can be explicitly computed by evaluating diagrams of the form given in Fig. 1.

Now we focus on the SSB of $SU(2) \times U(1)$ down to $U(1)$, and we assume $c_i = 0$. In this phase ($\mu^2 < 0$), χ acquires a non-vanishing vacuum expectation value (VEV), which, for $c_i = 0$, is given by

$$|<\chi>|^2 = \frac{3}{4}\frac{-\mu^2}{\lambda_H}. \tag{3}$$

The corresponding vector, fermion and scalar spectrum, at the leading non trivial order in $\sqrt{-a^2\mu^2}$, is shown in the second column of Table 1, apart from the massless gauge field associated to the residual $U(1)$.

Geometrical Approach. Now we want to compare the 4D effective theory with the geometrical approach to SSB. By definition the latter involves a solution of the higher dimensional EOMs that has the same symmetry as the effective theory in the broken phase. At the leading non trivial order in the small parameter $\eta^{1/2} \equiv \sqrt{-a^2\mu^2}$, we find[3]

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + a^2\left[(1+|\eta|\beta\sin^2\theta)d\theta^2 + \sin^2\theta d\varphi^2\right], \\
A &= \frac{1}{e}(\cos\theta \mp 1)d\varphi, \\
\phi &= \eta^{1/2}\alpha\exp(i\varphi)\sin\theta,
\end{aligned}
\tag{4}
$$

where $\beta = \kappa^2|\alpha|^2$ and

$$|\alpha|^2 = \frac{9}{32\pi a^4}\frac{1}{|\lambda_H - \lambda_G|}.$$

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[3] This solution was discussed in Ref. [8], but incorrectly.

TABLE 1. The vector (V), fermion (F) and scalar (S) spectra.

Mass squared	4D Effective Theory	Geometrical Approach
M_V^2	$\frac{3e^2}{8\pi a^2}\frac{-\mu^2}{\lambda_H}$	$\frac{3e^2}{8\pi a^2}\frac{-\mu^2}{\lambda_H-\lambda_G}$
$M_{V_\pm}^2$	$\frac{9e^2}{16\pi a^2}\frac{-\mu^2}{\lambda_H}$	$\frac{9e^2}{16\pi a^2}\frac{-\mu^2}{\lambda_H-\lambda_G}$
M_F^2	$\frac{3g_Y^2}{16\pi a^2}\frac{-\mu^2}{\lambda_H}$	$\frac{3g_Y^2}{16\pi a^2}\frac{-\mu^2}{\lambda_H-\lambda_G}$
$M_{F\pm}^2$	0	0
M_S^2	$-2\mu^2$	$-2\mu^2$
$M_{S\pm}^2$	$-\mu^2$	$-\mu^2\frac{\lambda_H+\lambda_G}{\lambda_H-\lambda_G}$

Consistency requires that if $\mu^2 > 0$ then $\lambda_H < \lambda_G$, whereas if $\mu^2 < 0$ then $\lambda_H > \lambda_G$. We are interested in $\mu^2 < 0$, as it corresponds to the gauge symmetry breaking in the 4D effective theory approach. Therefore we assume $\lambda_H > \lambda_G$. The VEV of χ, which corresponds to Solution (4), is

$$|<\chi>|^2 = \frac{3}{4}\frac{-\mu^2}{\lambda_H-\lambda_G}$$

and we observe that it is equal to (3), apart from the shift $\lambda_H \to \lambda_H - \lambda_G$. Moreover, the metric that appears in Configuration (1) describes an S^2, whereas in (4) we have the metric of an ellipsoid. This distortion corresponds to the electroweak symmetry breaking in the geometrical approach, as it is shown in Fig. 2.

   The low energy vector, fermion and scalar spectrum[4], which corresponds to Solution (4) is presented in the third column of Table 1, apart from the massless gauge field. We observe that, for vectors and fermions, the only difference between the 4D effective theory and the geometrical approach is the shift $\lambda_H \to \lambda_H - \lambda_G$, as for the VEV of χ. However, concerning the scalar spectrum, we have $M_S^2/M_{S\pm}^2 = 2$, in the 4D effective theory approach, whereas $M_S^2/M_{S\pm}^2 = 2(1-\delta)/(1+\delta)$, where $\delta \equiv \lambda_G/\lambda_H$, in the geometrical approach. Since a ratio of masses is a measurable quantity, there is a physical disagreement between the two approaches. The error is measured by λ_G/λ_H and we can roughly estimate its magnitude: if we require g_1 and g_2 in (2) to be of the order of 1, and we also consider the relation between κ and the 4D Planck length, we obtain that λ_G is of order of 1. Therefore the condition $\lambda_G/\lambda_H \ll 1$ becomes $\lambda_H \gg 1$, which is a strong coupling regime. Probably this range is not allowed if one requires to study the 4D effective theory by using perturbation theory. We conclude that the heavy mode contribution to the low energy dynamics is in general non negligible even

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[4] This spectrum has been computed by using the formalism presented in Ref. [9].

in standard KK theories, where the heavy mode masses are naturally at the Planck scale. Finally, we observe that this contribution can be interpreted in a geometrical way, as the internal space deformation of the 6D solution: indeed, if we put $\beta = 0$ but we keep $\alpha \neq 0$ in (4), which corresponds to neglecting the S^2 deformation, the spectra in Table 1 turn out to be equal.

NON-ABELIAN AND SUPERSYMMETRIC EXTENSIONS

Gauge-Higgs Unification. This scenario consists of models without fundamental scalars, which, in some sense, geometrize the Higgs mechanism. Explicit realizations, which include dynamical gravity, are presented in Ref. [4]. In particular, the authors analyzed a 6D Einstein-Yang-Mills model, which is a non-Abelian extension, without bulk scalars, of our theory. In a simple set up the bulk gauge group is chosen to be $SU(3)$ and a non-Abelian generalization of Solution (1) can break $SU(3)$ down to $SU(2) \times U(1)$. The internal components of the bulk gauge fields contain a doublet of $SU(2)$, which can be naturally interpreted as a Higgs field. In this way the Higgs mass is protected from dangerous power-law radiative corrections by the bulk gauge symmetry.

In the 4D effective theory approach the Higgs doublet triggers the SSB of $SU(2) \times U(1)$ down to the electromagnetic $U(1)$. The results presented in the present paper and in [3] suggest that this method provides the correct 6D predictions for the observable quantities. This is because, in our model, the solution of the EOMs of the 4D effective theory can be lifted back to a solution of the complete 6D theory, if the heavy modes are properly taken into account.

6D Supergravities. Other extensions of our work can be done in the context of supersymmetric versions of 6D gauge and gravitational theories. In particular, 6D gauged[5] supergravities have attracted much interest for several reasons. One of them is that the flat 6D space-time *is not* a solution of the corresponding EOMs and the most symmetric solution is $(Minkowski)_4 \times S^2$, which has been shown recently to be the *unique* maximally symmetric solution of such models [10]. Therefore, these theories provide a theoretical explanation for the background that we have considered in the previous section.

Moreover, 6D gauged supergravity compactifications share some properties with 10D supergravity compactifications, whilst remaining relatively simple, and so it can be used as a toy model for 10D string theory compactifications [5], in particular they can give rise to chiral fermions in 4D. Furthermore, like in string theory, the requirement of anomaly freedom is a strong guiding principle to construct consistent models. Indeed the minimal version of such gauged supergravity, *the Salam-Sezgin model* [11], suffers from the breakdown of local symmetries due to the presence of gravitational, gauge and mixed anomalies, which render this model inconsistent at the quantum level [12]; but it can be transformed in an anomaly free model by choosing the gauge group and the supermultiplet in a suitable way [13]. Therefore, the extension of our analysis to

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[5] "gauged" means that a subgroup of the R-symmetry group is promoted to a local symmetry.

this context could be a first step towards the study of the heavy modes in string theory compactifications.

Moreover, such 6D supergravities have been recently investigated in connection with attempts to find a solution to the cosmological dark energy problem [6]. Some 3-branes solutions and their perturbations, which can be relevant for this scenario, have been studied in Refs. [10, 14]. These backgrounds are deformations of Background (1), like our ellipsoid solution in the geometrical approach, but involving a warp factor and conical defects. This similarity suggests that our computation can be extended to 6D gauged supergravities expanded around these 3-brane solutions. If the heavy mode contribution is physically relevant, the underlying 6D physics should manifest itself in the low energy dynamics.

ACKNOWLEDGMENTS

REFERENCES

1. K. G. Wilson, Rev. Mod. Phys. **47**, 773 (1975); S. Weinberg, Phys. Lett. B **91** (1980) 51.
2. T. Appelquist and J. Carazzone, Phys. Rev. D **11** (1975) 2856.
3. S. Randjbar-Daemi, A. Salvio and M. Shaposhnikov, Nucl. Phys. B **741** (2006) 236 [arXiv:hep-th/0601066].
4. G. R. Dvali, S. Randjbar-Daemi and R. Tabbash, Phys. Rev. D **65** (2002) 064021 [arXiv:hep-ph/0102307].
5. Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, JHEP **0303** (2003) 032 [arXiv:hep-th/0212091].
6. Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, Nucl. Phys. B **680** (2004) 389 [arXiv:hep-th/0304256]; C. P. Burgess, arXiv:hep-th/0510123.
7. S. Randjbar-Daemi, A. Salam and J. A. Strathdee, Nucl. Phys. B **214** (1983) 491.
8. J. Sobczyk, Class. Quant. Grav. **4** (1987) 37.
9. S. Randjbar-Daemi and M. Shaposhnikov, Nucl. Phys. B **645** (2002) 188 [arXiv:hep-th/0206016].
10. G. W. Gibbons, R. Guven and C. N. Pope, Phys. Lett. B **595** (2004) 498 [arXiv:hep-th/0307238];
11. A. Salam and E. Sezgin, Phys. Lett. B **147** (1984) 47.
12. L. Alvarez-Gaume and E. Witten, Nucl. Phys. B **234** (1984) 269.
13. S. Randjbar-Daemi, A. Salam, E. Sezgin and J. A. Strathdee, Phys. Lett. B **151** (1985) 351; S. D. Avramis, A. Kehagias and S. Randjbar-Daemi, JHEP **0505** (2005) 057 [arXiv:hep-th/0504033]; S. D. Avramis and A. Kehagias, JHEP **0510** (2005) 052 [arXiv:hep-th/0508172]; R. Suzuki and Y. Tachikawa, J. Math. Phys. **47** (2006) 062302 [arXiv:hep-th/0512019].
14. Y. Aghababaie *et al.*, JHEP **0309** (2003) 037 [arXiv:hep-th/0308064]; C. P. Burgess, F. Quevedo, G. Tasinato and I. Zavala, JHEP **0411** (2004) 069 [arXiv:hep-th/0408109]; A. J. Tolley, C. P. Burgess, D. Hoover and Y. Aghababaie, JHEP **0603** (2006) 091 [arXiv:hep-th/0512218]; A. J. Tolley, C. P. Burgess, C. de Rham and D. Hoover, arXiv:hep-th/0608083; H. M. Lee and A. Papazoglou, Nucl. Phys. B **747**, 294 (2006) [arXiv:hep-th/0602208]; S. L. Parameswaran, S. Randjbar-Daemi and A. Salvio, arXiv:hep-th/0608074.