

Gysin map and Atiyah-Hirzebruch spectral sequence

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Abstract

We study the relations between the Atiyah-Hirzebruch spectral sequence and the Gysin map for a multiplicative cohomology theory on spaces having the homotopy type of a finite CW-complex. In particular, let us fix such a multiplicative cohomology theory h^* and let us consider a smooth manifold X of dimension n and a compact submanifold Y of dimension r ; we require that X is orientable and that the normal bundle of Y is orientable with respect to h^* . Then we prove that, starting the Atiyah-Hirzebruch spectral sequence with the Poincaré dual of Y in X , which, in our setting, is a simplicial cohomology class with coefficients in $h^{n-r}\{*\}$, if such cocycle survives until the last step, its class in $E_\infty^{n-r,0}$ is represented by the image via the Gysin map of the unit cohomology class of Y . We then prove the analogous statement for a generic cohomology class on Y .

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1 Introduction

Given a multiplicative cohomology theory, under suitable hypotheses we can define the Gysin map, i.e., a natural push-forward in cohomology. Moreover, for a finite CW-complex or any space homotopically equivalent to it, we can construct the Atiyah-Hirzebruch spectral sequence, which relates cellular cohomology with the fixed cohomology theory. In particular, the groups of the starting step of the spectral sequence $E_1^{p,q}(X)$ are canonically isomorphic to the groups of cellular cochains $C^p(X, h^q\{*\})$ for $\{*\}$ a fixed space with one point. Since the first coboundary $d_1^{p,q}$ coincides with the cellular coboundary δ^p , the groups $E_2^{p,q}(X)$ are canonically isomorphic to the cellular cohomology groups $H^p(X, h^q\{*\})$. The sequence stabilizes to $E_\infty^{p,q}(X)$ and, denoting by X^p the p -skeleton of X , there is a canonical isomorphism:

$$E_\infty^{p,q}(X) \simeq \frac{\text{Ker}(h^{p+q}(X) \longrightarrow h^{p+q}(X^{p-1}))}{\text{Ker}(h^{p+q}(X) \longrightarrow h^{p+q}(X^p))} \quad (1)$$

i.e., $E_\infty^{p,q}$ is made by $(p+q)$ -classes on X which are 0 when pulled back to X^{p-1} , up to classes which are 0 when pulled back to X^p . Let us now consider an n -dimensional smooth manifold X and a compact r -dimensional submanifold Y . For $i : Y \rightarrow X$ the embedding, we can thus define the Gysin map:

$$i_! : h^*(Y) \longrightarrow \tilde{h}^{*+n-r}(X)$$

which in particular gives a map $i_! : h^0(Y) \longrightarrow \tilde{h}^{n-r}(X)$. We assume that we have an oriented triangulation of X restricting to a triangulation of Y : since, for an abelian group G , one has $C_p(X, G) = C_p(X, \mathbb{Z}) \otimes_{\mathbb{Z}} G$, given an element $g \in G$ we can consider $Y \otimes g$ as a chain in $C_r(X, G)$; if G is a ring with unit, we consider Y as $Y \otimes 1$. Since, for a multiplicative cohomology theory, $h^0\{*\}$ is a ring with unit, we can ask that Y is a cycle in $C_r(X, h^0\{*\})$. Then, for $1 \in h^0(Y)$ defined as the pull-back of the unit $1 \in h^0\{*\}$ via the unique map $P : Y \rightarrow \{*\}$, we prove that $i_!(1)$ belongs to the numerator of (1) for $p = n - r$ and $q = 0$ and, if Y survives until the last step, its class in $E_\infty^{n-r,0}$ is represented exactly by $i_!(1)$.

Similarly, for $\eta \in h^0\{*\}$, if the Poincaré dual of $Y \otimes \eta \in C_r(X, h^0\{*\})$ survives until $E_\infty^{n-r,0}$, its class is represented by $i_!((P^*)^0(\eta))$. More generally, without assuming $q = 0$, if $Y \otimes \alpha$ is a cycle in $C_r(X, h^q\{*\})$ for $\alpha \in h^q\{*\}$ and if it survives until $E_\infty^{n-r,q}$, then its class in (1) is represented by $i_!((P^*)^q(\alpha))$. All the classes on Y considered in these examples are pull-back of classes in $h^*\{*\}$: we will see that all the other classes give no more information.

The study of the relations between Gysin map and Atiyah-Hirzebruch spectral sequence was originally treated in [6] for K-theory, arising from the physical problem of relating two different classifications of D-brane charges in string theory. In this article the result is generalized to any multiplicative cohomological theory.

The paper is organized as follows: in chapter 2 we briefly recall the basic theory of spectral sequences in order to explicitly construct the maps needed in the following; in chapter 3 we discuss orientability, Thom isomorphism and Gysin map for a multiplicative cohomology theory; in chapter 4 we state and prove the theorems providing the link between Gysin map and Atiyah-Hirzebruch spectral sequence; finally, in chapter 5 we discuss some possible directions for further studies.

2 Spectral sequences

2.1 Review of Cartan-Eilenberg version

We consider spectral sequences in the version of [3] chap. XV. We briefly recall the general theory to fix the notations and to construct the map we will need in the following. We consider a graded abelian group $K = \bigoplus_{n \in \mathbb{Z}} K^n$ provided with a *finite* filtration, i.e., with a sequence of nested subgroups $\{F^p K^n\}_{p \in \{0, \dots, l\}}$ such that $K^n = F^0 K^n \supset \dots \supset F^l K^n = 0$. We then construct the groups $E_0^{p,q} K = F^p K^{p+q} / F^{p+1} K^{p+q}$ whose direct sum $\bigoplus_{p \in \mathbb{Z}} E_0^{p,q} K$ is called the *associated graded group* of the filtration $\{F^p K^{p+q}\}_{p \in \mathbb{Z}}$. Let $d^n : K^n \rightarrow K^{n+1}$ be a coboundary with associated cohomology $H^n(K^\bullet)$. Let us also suppose that d *preserves the filtration*, i.e., $d^n(F^p K^n) \subset F^p K^{n+1}$: in this case we have a cohomology $H^n(F^p K^\bullet)$ for every p . We also put $ZK^n = \text{Ker } d^n$, $BK^n = \text{Im } d^{n-1}$, $Z^p K^n = \text{Ker}(d^n|_{F^p K^n})$ and $B^p K^n = \text{Im}(d^{n-1}|_{F^p K^{n-1}})$. The inclusions $i_{p,n} : F^p K^n \hookrightarrow K^n$ induces morphisms in cohomology $i_{p,n}^\# : H^n(F^p, \bullet K) \rightarrow H^n(K^\bullet)$ whose image is given by equivalence classes of cocycles in $F^p K^n$ up to coboundaries coming from elements of all K^{n-1} . We define $F^p H^n(K^\bullet) = \text{Im}(i_{p,n}^\#)$. In this way, we obtain a filtration of $H^n(K^\bullet)$ given by $H^n(K^\bullet) = F^0 H^n(K^\bullet) \supset \dots \supset F^l H^n(K^\bullet) = 0$, whose associated graded group is $\bigoplus_p E_0^{p,q} H(K^\bullet) = \bigoplus_p F^p H^{p+q}(K^\bullet) / F^{p+1} H^{p+q}(K^\bullet)$.

We extend the filtration index to \mathbb{Z} declaring $F^p K^n = K^n$ for $n \leq 0$ and $F^p K^n = 0$ for $n \geq l$, and the same for $F^p H^n(K^\bullet)$. We define for $r \geq 1$:

$$\begin{aligned} Z_r^{p,q} K &= \{a \in F^p K^{p+q} \mid d(a) \in F^{p+r} K^{p+q+1}\} \\ \overline{\mathfrak{B}}_r^{p,q} K &= \langle B^{p-r+1} K^{p+q}, i^r F^{p+1} K^{p+q} \rangle \\ \overline{Z\mathfrak{B}}_r^{p,q} K &= \langle i^{r-1} Z_r^{p,q} K, B^{p-r+1} K^{p+q}, i^r F^{p+1} K^{p+q} \rangle \\ E_r^{p,q} K &= \overline{Z\mathfrak{B}}_r^{p,q} K / \overline{\mathfrak{B}}_r^{p,q} K \end{aligned} \tag{2}$$

which for $r = l$ become:

$$\begin{aligned} \overline{\mathfrak{B}}_l^{p,q} K &= \langle BK^{p+q}, i^{p+1} F^{p+1} K^{p+q} \rangle \\ \overline{Z\mathfrak{B}}_l^{p,q} K &= \langle i^p Z^{p,q} K, BK^{p+q}, i^{p+1} F^{p+1} K^{p+q} \rangle \\ E_l^{p,q} K &= E_0^{p,q} H(K) \end{aligned}$$

thus the sequence stabilizes to $E_\infty^{p,q} K = E_0^{p,q} H(K^\bullet) = F^p H^{p+q}(K^\bullet) / F^{p+1} H^{p+q}(K^\bullet)$. We put $E_r K = \bigoplus_{p \in \mathbb{Z}} E_r^p K$. We have a natural boundary $d_r^{p,q} : E_r^{p,q} K \rightarrow E_r^{p+r, q-r+1} K$. In fact, let $[a] \in E_r^{p,q} K$. Then:

- $a = z + b + x$, with $z \in Z_r^{p,q} K$, $b \in B^{p-r+1} K^{p+q}$ and $x \in F^{p+1} K^{p+q}$, and $[a] = [z]$ in $E_r^{p,q} K$;
- $d(z) \in Z^{p+r} K^{p+q+1} \subset Z_r^{p+r, q-r+1} K \subset \overline{Z\mathfrak{B}}_r^{p+r, q-r+1} K$, $d(b) = 0$ and $d(x) \in B^{p+1} K^{p+q} \subset \overline{\mathfrak{B}}_r^{p+r, q-r+1} K$;
- hence $[d(a)] = [i^{r-1}(d(z))]$ in $E_r^{p+r, q-r+1} K$, so that we can define $d_r^{p,q}[a] = [d(a)]$.

It is well defined, since if $[z_1] = [z_2]$, then $z_1 - z_2 = x + dy$ with $x \in F^{p+1}K^{p+q}$, hence $dz_1 - dz_2 = dx$ and $[dx] = 0$ in $E_r^{p+r, q-r+1}K$. In this setting, we can see that $E_{r+1}K \simeq H(E_rK, d_r)$ canonically. In fact:

- $\text{Ker}(d_r^{p,q}) = \langle i^{r-1}Z_{r+1}^{p,q}K, \overline{\mathfrak{B}}_r^{p,q}K \rangle / \overline{\mathfrak{B}}_r^{p,q}K \subset \overline{Z\mathfrak{B}}_r^{p,q}K / \overline{\mathfrak{B}}_r^{p,q}K$;
- $\text{Im}(d_r^{p-r, q+r-1}) = \langle \overline{\mathfrak{B}}_r^{p,q}K, i^{r-1}(B^{p-r}K^{p+q} \cap F^pK^{p+q}) \rangle / \overline{\mathfrak{B}}_r^{p,q}K \subset \overline{Z\mathfrak{B}}_r^{p,q}K / \overline{\mathfrak{B}}_r^{p,q}K$.

We can now describe the spectral sequence in a different way. Using the convention $F^{-\infty}K^n = K^n$ and $F^{+\infty}K^n = 0$, we define, for $-\infty \leq p \leq t \leq +\infty$:

$$\begin{aligned} H^n(p, t) &= H^n(F^pK^\bullet / F^tK^\bullet) \\ H^n(p) &= H^n(p, +\infty) = H^n(F^pK^\bullet). \end{aligned}$$

For $p \leq t \leq u$, $a, b \geq 0$, $p+a \leq t+b$, we define two maps:

$$\begin{aligned} \Psi^n : H^n(p+a, t+b) &\longrightarrow H^n(p, t) \\ \Delta^n : H^n(p, t) &\longrightarrow H^{n+1}(t, u) \end{aligned} \quad (3)$$

where:

- Ψ^n is induced in cohomology by the natural map $F^{p+a}K / F^{t+b}K \rightarrow F^pK / F^tK$, induced by inclusions of the numerators and the denominators;
- Δ^n is the composition of the bockstein map $\beta : H^n(p, t) \rightarrow H^{n+1}(t)$ and the map induced in cohomology by $\pi : F^tK \rightarrow F^tK / F^uK$.

Let us consider $H^{p+q}(p, p+r)$: it is given by cocycles in $F^pK^{p+q} / F^{p+r}K^{p+q}$, i.e. $Z_r^{p,q}K / F^{p+r}K^{p+q}$, up to $\langle B^pK^{p+q}, i^r(F^{p+r}K^{p+q}) \rangle$, so that we have a canonical isomorphism:

$$H^{p+q}(p, p+r) \simeq Z_r^{p,q}K / \langle B^pK^{p+q}, i^r(F^{p+r}K^{p+q}) \rangle. \quad (4)$$

We remark that $F^{p+r}K^{p+q} \subset Z_r^{p,q}K$ (so we do not need the explicit intersection) since every $x \in F^{p+r}K^{p+q}$ is such that $d(x) \in F^{p+r}K^{p+q}$. We have that for $r \geq 1$:

$$\text{Im}(H^{p+q}(p, p+r) \xrightarrow{\Psi^{p+q}} H^{p+q}(p-r+1, p+1)) \simeq \overline{Z\mathfrak{B}}_r^{p,q}K / \overline{\mathfrak{B}}_r^{p,q}K = E_r^{p,q}K. \quad (5)$$

In fact, considering (4), the image of $(\Psi^{p+q})_{p-r+1, p+1}^{p, p+r}$ can be described as:

$$\begin{aligned} \text{Im} \left(Z_r^{p,q}K / \langle B^pK^{p+q}, i^r(F^{p+r}K^{p+q}) \rangle \xrightarrow{\Psi^{p+q}} \right. \\ \left. Z_r^{p-r+1, q+r-1}K / \langle B^{p-r+1}K^{p+q}, i^r(F^{p+1}K^{p+q}) \rangle \right) \end{aligned}$$

which is:

$$\langle i^{r-1}Z_r^{p,q}K, B^{p-r+1}K^{p+q}, i^r(F^{p+1}K^{p+q}) \rangle / \langle B^{p-r+1}K^{p+q}, i^r(F^{p+1}K^{p+q}) \rangle$$

i.e. $\overline{Z\mathfrak{B}}_r^{p,q}K / \overline{\mathfrak{B}}_r^{p,q}K$.

We have a commutative diagram:

$$\begin{array}{ccc}
H^{p+q}(p, p+r) & \xrightarrow{\Psi_1^{p+q}} & H^{p+q}(p-r+1, p+1) \\
\downarrow \Delta_1^{p+q} & & \Delta_2^{p+q} \downarrow \\
H^{p+q+1}(p+r, p+2r) & \xrightarrow{\Psi_2^{p+q+1}} & H^{p+q+1}(p+1, p+r+1)
\end{array} \tag{6}$$

and:

- $\text{Im}(\Psi_1^{p+q}) = E_r^{p,q}K$ and $\text{Im}(\Psi_2^{p+q}) = E_r^{p+r,q}K$;
- $d_r^{p,q} = \Delta_2^{p+q} \big|_{\text{Im}(\Psi_1^{p+q})} : E_r^{p,q}K \longrightarrow E_r^{p+r,q-r+1}K$.

We have already proven the first part. For the boundary, let us consider $[a] \in E_r^{p,q}K$ with $a = z + b + x \in \langle i^{r-1}Z_r^{p,q}K, B^{p-r+1}K^{p+q}, i^r(F^{p+1}K^{p+q}) \rangle$. Then we know that $d_r^{p,q}[a] = [d(z)] \in E_r^{p+r,q-r+1}K$. Let us compute $\Delta_2^{p+q}([a])$: first we compute the Bockstein map to $H^{p+q+1}(p+1)$, which consists of applying the boundary to get $d(z) + d(x)$ and considering the class $[d(z)] \in H^{p+q+1}(p+1)$; then we compose with the map in cohomology induced by $\pi : F^{p+1}K^{p+q+1} \rightarrow F^{p+1}K^{p+q+1}/F^{p+r+1}K^{p+q+1}$, to get $[d(z)] \in H^{p+q+1}(p+1, p+r+1) = Z_r^{p+1,q} / \langle B^{p+1}K^{p+q+1}, i^r(F^{p+r+1}K^{p+q+1}) \rangle$. But, being $d(z)$ a boundary, we have $d(z) \in B^pK^{p+q+1} \cap Z_r^{p+1,q}$, thus $d(z) \in i^{r-1}Z_r^{p+r,q-r+1}$ so that we can consider $[d(z)] \in \text{Im}(\Psi_2^{p+q}) = E_r^{p+r,q-r+1}$.

This approach works for $r \geq 1$, since, for $r = 0$, we get 0 in (4), so that the lhs of (5) is zero and not equal $E_0^{p,q}$. Thus, we start from $r = 1$. The limit of the sequence can be obtained putting $r = +\infty$ in (5):

$$E_\infty^{p,q}K = E_0^{p,q}H(K) = \text{Im}(H^{p+q}(p, +\infty) \xrightarrow{\Psi^{p+q}} H^{p+q}(0, p+1)). \tag{7}$$

In fact, since $H^{p+q}(p, +\infty) = H^{p+q}(F^pK^\bullet)$ and $H^{p+q}(0, +\infty) = H^{p+q}(K^\bullet)$ we have that:

$$F^p H^{p+q}(K^\bullet) = \text{Im}(H^{p+q}(p, +\infty) \xrightarrow{\Psi^{p+q}} H^{p+q}(0, +\infty)). \tag{8}$$

Let us see that the associated graded group of this filtration of $H^{p+q}(K^\bullet) = H^{p+q}(0, +\infty)$ is given by (7). In fact, by (4) we have $H^{p+q}(0, +\infty) = Z_{+\infty}^{0,p+q}/BK^{p+q} = ZK^{p+q}/BK^{p+q} = H^{p+q}(K^\bullet)$ and $H^{p+q}(0, p+1) = Z_{p+1}^{0,p+q}/\langle BK^{p+q}, i^{p+1}(F^{p+1}K^{p+q}) \rangle = H^{p+q}(K^\bullet/F^{p+1}K^\bullet)$; then (7) is obtained by (8) via the immersion $ZK^{p+q} \rightarrow Z_{p+1}^{0,p+q}$, so that we get $\langle ZK^{p+q}, i^{p+1}(F^{p+1}K^{p+q}) \rangle / \langle BK^{p+q}, i^{p+1}(F^{p+1}K^{p+q}) \rangle$ which is exactly $F^p H^{p+q}(K^\bullet)/F^{p+1}H^{p+q}(K^\bullet)$.

Using this new language, we never referred to the groups F^pK^n , but only to $H^n(p, q)$: thus, we can give the groups $H^n(p, q)$ axiomatically, without referring to the filtered groups K^n . The main advantage of this axiomatization is the possibility to build a spectral sequence for a generic cohomology theory, not necessarily induced by a coboundary.

2.1.1 Description of the isomorphisms

We now explicitly describe, in this language of cohomology of quotients, the canonical isomorphisms involved in the definition of a spectral sequence, i.e. $E_{r+1}^{p,q} K \simeq \text{Ker}d_r^{p,q} / \text{Im}d_r^{p-r,q+r-1}$. Considering (6), from the two diagrams:

$$\begin{array}{ccc}
H^{p+q-1}(p-r, p) & \xrightarrow{\Psi_0^{p+q}} & H^{p+q-1}(p-2r+1, p-r+1) \\
\downarrow \Delta_{-1}^{p+q} & & \Delta_0^{p+q} \downarrow \\
H^{p+q}(p, p+r) & \xrightarrow{\Psi_1^{p+q}} & H^{p+q}(p-r+1, p+1) \\
\downarrow \Delta_1^{p+q} & & \Delta_2^{p+q} \downarrow \\
H^{p+q+1}(p+r, p+2r) & \xrightarrow{\Psi_2^{p+q+1}} & H^{p+q+1}(p+1, p+r+1) \\
\\
H^{p+q}(p, p+r+1) & \xrightarrow{\Psi_3^{p+q}} & H^{p+q}(p-r, p+1) .
\end{array} \tag{9}$$

we have that:

- $\text{Im}(\Psi_1^{p+q}) = E_r^{p,q} K$;
- $d_r^{p,q} = \Delta_2^{p+q} \big|_{\text{Im}(\Psi_1^{p+q})} : E_r^{p,q} K \longrightarrow E_r^{p+r,q-r+1} K$ and $d_r^{p-r,q+r-1} = \Delta_0^{p+q} \big|_{\text{Im}(\Psi_0^{p+q})} : E_r^{p-r,q+r-1} K \longrightarrow E_r^{p,q} K$;
- $\text{Im}(\Psi_3^{p+q}) = E_{r+1}^{p,q} K$.

To find the isomorphism $E_{r+1}^{p,q} K \simeq \text{Ker}d_r^{p,q} / \text{Im}d_r^{p-r,q+r-1}$ we thus need a map $\varphi_r^{p,q} : H^{p+q}(p-r+1, p+1) \rightarrow H^{p+q}(p-r, p+1)$ which is naturally given by the immersion $i : F^{p-r+1} K^{p+q} / F^{p+1} K^{p+q} \rightarrow F^{p-r} K^{p+q} / F^{p+1} K^{p+q}$. We claim that this map induces a surjection:

$$\varphi_r^{p,q} : \text{Ker}(\Delta_2^{p+q} \big|_{\text{Im}(\Psi_1^{p+q})}) \longrightarrow \text{Im}\Psi_3^{p+q} . \tag{10}$$

In fact:

- $\text{Im}\Psi_1^{p+q} = \langle i^{r-1} Z_r^{p,q} K, i^r F^{p+1} K^{p+q}, B^{p-r+1} K^{p+q} \rangle / \langle i^r F^{p+1} K^{p+q}, B^{p-r+1} K^{p+q} \rangle$;
- $\text{Ker}(\Delta_2^{p+q} \big|_{\text{Im}(\Psi_1^{p+q})}) = \langle i^{r-1} Z_{r+1}^{p,q} K, i^r F^{p+1} K^{p+q}, B^{p-r+1} K^{p+q} \rangle / \langle i^r F^{p+1} K^{p+q}, B^{p-r+1} K^{p+q} \rangle$;
- $\text{Im}\Psi_3^{p+q} = \langle i^r Z_{r+1}^{p,q} K, i^{r+1} F^{p+1} K^{p+q}, B^{p-r} K^{p+q} \rangle / \langle i^{r+1} F^{p+1} K^{p+q}, B^{p-r} K^{p+q} \rangle$.

The map $\varphi_r^{p,q}$ is induced by the immersion of the numerators, and it is surjective since the only elements in the numerator of $\text{Im}\Psi_3^{p+q}$ which can not to be in the image are the elements of $B^{p-r} K$ which are quotiented out. Moreover, the kernel of $\varphi_r^{p,q}$ is made by classes of elements in $i^{r-1} Z_{r+1}^{p,q} K$ which, after the immersion, belongs to $B^{p-r} K$, i.e.:

$$\begin{aligned}
\text{Ker}\varphi_r^{p,q} &= \langle i^{r-1} (Z_{r+1}^{p,q} K) \cap B^{p-r} K^{p+q}, i^r F^{p+1} K^{p+q}, B^{p-r+1} K^{p+q} \rangle / \\
&\quad \langle i^r F^{p+1} K^{p+q}, B^{p-r+1} K^{p+q} \rangle \\
&= \langle i^{r-1} (F^p K^{p+q}) \cap B^{p-r} K^{p+q}, i^r F^{p+1} K^{p+q}, B^{p-r+1} K^{p+q} \rangle / \\
&\quad \langle i^r F^{p+1} K^{p+q}, B^{p-r+1} K^{p+q} \rangle
\end{aligned} \tag{11}$$

but the latter is exactly $\text{Im}(\Delta_0^{p+q}|_{\text{Im}(\Psi_0^{p+q})})$, thus $\varphi_r^{p,q}$ induces the isomorphism $E_{r+1}^{p,q}K \simeq \text{Ker}d_r^{p,q} / \text{Im}d_r^{p-r,q+r-1}$.

Let us consider $E_1^{p,q} = H^{p+q}(p, p+1)$. Some elements will lie in $\text{Ker}d_1^{p,q}$, and they are mapped to $H^{p+q}(p-1, p+1)$ by $\varphi_1^{p,q}$, which is induced by $i_1 : F^pK^{p+q} / F^{p+1}K^{p+q} \rightarrow F^{p-1}K^{p+q} / F^{p+1}K^{p+q}$. We iterate the procedure: some elements will lie in $\text{Ker}d_2^{p,q}$ and are mapped to $H^{p+q}(p-2, p+1)$ by $\varphi_2^{p,q}$, which is induced by $i_2 : F^{p-1}K^{p+q} / F^{p+1}K^{p+q} \rightarrow F^{p-2}K^{p+q} / F^{p+1}K^{p+q}$. Thus, in the original group $E_1^{p,q} = H^{p+q}(p, p+1)$ we can consider the elements that survives to both these steps and maps them directly to $H^{p+q}(p-2, p+1)$ via $i_{1,2} : F^pK^{p+q} / F^{p+1}K^{p+q} \rightarrow F^{p-2}K^{p+q} / F^{p+1}K^{p+q}$. This procedure stops after l steps where l is the lenght of the filtration. In particular, we obtain a subset $A^{p,q} \subset E_1^{p,q}$ of *surviving elements*, and a map:

$$\varphi^{p,q} : A^{p,q} \subset E_1^{p,q} \longrightarrow E_\infty^{p,q} \quad (12)$$

assigning to each surviving element its class in the last step. The map is simply induced by $i_{1,\dots,l} : F^pK^{p+q} / F^{p+1}K^{p+q} \longrightarrow K^{p+q} / F^{p+1}K^{p+q}$. We now prove that the surviving elements are classes in $H^{p+q}(F^pK^\bullet / F^{p+1}K^\bullet)$ represented by elements which are in $F^{p+1}K^{p+q}$ or by elements whose boundary is 0 (not in $F^{p+1}K^{p+q+1}$), or more generally that the elements surviving for r steps (thus from 1 to $r+1$) are represented by elements of $F^{p+1}K^{p+q}$ or by elements whose boundary is in $F^{p+r+1}K^{p+q+1}$. In fact, the first boundary is given by $\Delta^{p,q} : H^{p+q}(p, p+1) \rightarrow H^{p+q}(p+1, p+2)$ so that the elements in its kernel are classes in $H^{p+q}(p, p+1)$ represented by elements of $F^{p+1}K^{p+q}$ or by elements whose boundary lives in $F^{p+2}K^{p+q}$. Then the isomorphism (10) sends such elements to $H^{p+q}(p-1, p+1)$ by immersion and the second boundary is given by $\Delta^{p,q} : H^{p+q}(p-1, p+1) \rightarrow H^{p+q}(p-1, p+3)$ restricted to the image, thus the elements in its kernel must have boundary in $F^{p+3}K^{p+q}$, and so on. Thus we have that:

$$A^{p,q} = \text{Im}(H^{p+q}(p, +\infty) \longrightarrow H^{p+q}(p, p+1))$$

and we have a commutative diagram:

$$\begin{array}{ccc} H^{p+q}(p, +\infty) & \xrightarrow{\Psi} & H^{p+q}(0, p+1) \\ & \searrow \pi^* & \nearrow i^* \\ & & H^{p+q}(p, p+1) \end{array} \quad (13)$$

with $A^{p,q} = \text{Im} \pi^*$ and $i^*|_{\text{Im} \pi^*} = \varphi^{p,q}$.

2.1.2 Axiomatization

We can now give the axiomatic version of spectral sequences, considering directly cohomology without referring to graded groups. Let us consider the following assignments, for $p, t, u \in \mathbb{Z} \cup \{-\infty, +\infty\}$:

- for $-\infty \leq p \leq t \leq \infty$ and $n \in \mathbb{Z}$ an abelian group $h^n(p, t)$, such that $h^n(p, t) = h^n(0, t)$ for $p \leq 0$ and there exists $l \in \mathbb{N}$ such that $h^n(p, t) = h^n(p, +\infty)$ for $t \geq l$;

- for $p \leq t \leq u$, $a, b \geq 0$, two maps:

$$\begin{aligned}\Psi^n &: h^n(p+a, t+b) \rightarrow h^n(p, t) \\ \Delta^n &: h^n(p, t) \rightarrow h^{n+1}(t, u)\end{aligned}$$

satisfying the axioms of [3] (page 334). Then we define for $r \geq 1$:

$$\begin{aligned}E_r^{p,q} &= \text{Im}(h^{p+q}(p, p+r) \xrightarrow{\Psi^{p+q}} h^{p+q}(p-r+1, p+1)) \\ d_r^{p,q} &= (\Delta^{p+q})^{p-r+1, p+1, p+r+1} \Big|_{\text{Im}((\Psi^{p+q})_{p-r+1, p+1}^{p, p+r})} : E_r^{p,q} K \longrightarrow E_r^{p+r, q-r+1} K\end{aligned}\tag{14}$$

$$F^p h^{p+q} = \text{Im}(h^{p+q}(p, +\infty) \xrightarrow{\Psi} h^{p+q}(0, +\infty)).$$

In this way:

- the groups $F^p h^n$ are a filtration of $h^n(0, +\infty)$;
- for $E_r = \bigoplus_{p,q} E_r^{p,q}$ and $d_r = \bigoplus_{p,q} d_r^{p,q}$ one has $E_{r+1} = h(E_r, d_r)$;
- for every $n = p + q$ fixed, the sequence $\{E_r^{p,q}\}_{r \in \mathbb{N}}$ stabilizes to $F^p h^{p+q} / F^{p+1} h^{p+q}$.

The canonical isomorphisms $E_{r+1}^{p,q} K \simeq \text{Ker} d_r^{p,q} / \text{Im} d_r^{p-r, q+r-1}$ are induced by the Ψ -map $\varphi_r^{p,q} : h^{p+q}(p-r+1, p+1) \rightarrow h^{p+q}(p-r, p+1)$, which induces a surjection in the diagram (9):

$$\varphi_r^{p,q} : \text{Ker}(\Delta_2^{p+q} \Big|_{\text{Im}(\Psi_1^{p+q})}) \longrightarrow \text{Im} \Psi_3^{p+q}.$$

Moreover, we have the commutative diagram (13), with $\varphi^{p,q}$ sending the surviving elements, i.e. $\text{Im} \pi^*$, to $E_\infty^{p,q}$, i.e., to $\text{Im} \Psi$.

2.2 Atiyah-Hirzebruch spectral sequence

The Atiyah-Hirzebruch spectral sequence (see [1]) relates the cellular cohomology of a finite CW-complex (or any space homotopically equivalent to it) to a generic cohomological theory h^\bullet satisfying the axioms of Eilenberg and Steenrod except in general dimension axiom (see [5]). We start from finite simplicial complexes. For a finite simplicial complex X we consider the natural filtration:

$$\emptyset = X^{-1} \subset X^0 \subset \dots \subset X^m = X$$

where X^i is the i -th skeleton of X . We define the following groups and maps:

- $h^n(p, t) = h^n(X^{t-1}, X^{p-1})$;
- $\Psi^n : h^n(p+a, t+b) \longrightarrow h^n(p, t)$ is induced in cohomology by the map of couples $i : (X^{t-1}, X^{p-1}) \longrightarrow (X^{t+b-1}, X^{p+a-1})$;

- $\Delta^n : h^n(p, t) \rightarrow h^n(t, u)$ is the composition of the map $\pi^* : h^n(X^{t-1}, X^{p-1}) \rightarrow h^n(X^{t-1})$ induced by $\pi : (X^{t-1}, \emptyset) \rightarrow (X^{t-1}, X^{p-1})$, and the Bockstein map $\beta : h^n(X^{t-1}) \rightarrow h^{n+1}(X^{u-1}, X^{t-1})$.

With these definitions all the axioms of section 2.1.2 are satisfied, so that we can consider the corresponding spectral sequence $E_r^{p,q}(X)$. We briefly recall the structure of the first two and the last steps of such a sequence, as described in [6].

2.2.1 The first two steps

From equation (14) with $r = 1$:

$$E_1^{p,q}(X) = \text{Im}(h^{p+q}(p, p+1) \xrightarrow{\Psi^{p+q}} h^{p+q}(p, p+1)) = h^{p+q}(p, p+1)$$

where the last equality is due to axiom 1. We now have, calling A_p the set indicizing the p -simplices:

$$\begin{aligned} h^{p+q}(p, p+1) &= h^{p+q}(X^p, X^{p-1}) \simeq \tilde{h}^{p+q}(X^p/X^{p-1}) \simeq \bigoplus_{A_p} \tilde{h}^{p+q}(S^p) \\ &\simeq \bigoplus_{A_p} \tilde{h}^q(S^0) \simeq \bigoplus_{A_p} h^q\{*\} = C^p(X, h^q\{*\}) \end{aligned}$$

where $C^p(X, h^q\{*\})$ is the group of simplicial cochains with coefficients in $h^q\{*\}$.

From equation (14) we get:

$$d_1^{p,q} = (\Delta^{p+q})|_{\text{Im}(\Psi^{p+q})} : E_1^{p,q}K \rightarrow E_1^{p+1,q}K$$

i.e.:

$$d_1^{p,q} = \Delta^{p+q} : h^{p+q}(p, p+1) \rightarrow h^{p+q+1}(p+1, p+2) \quad (15)$$

which becomes:

$$d_1^{p,q} : C^p(X, h^q\{*\}) \rightarrow C^{p+1}(X, h^q\{*\}) . \quad (16)$$

It follows from (15) that this is exactly the coboundary of cellular cohomology, thus for simplicial complexes it coincides with the simplicial coboundary.

We can write down functorially the canonical isomorphism $(E_1^{\bullet,q}(X), d_1^{\bullet,q}) \simeq (C^\bullet(X, h^q\{*\}), \delta^\bullet)$. In fact:

- we consider the index set of p -simplices A_p , and we consider it as a topological space with the discrete topology; thus, we have $h^q(A_p) = \bigoplus_{i \in A_p} h^q\{*\} = C^p(X, h^q\{*\})$;
- we consider the p -fold suspension of A_p^+ , i.e. $S^p A_p^+ = S^p \times (A_p \sqcup \{\infty\}) / (S^p \times \{\infty\} \cup \{N\} \times A_p)$: we have the homeomorphisms $S^p \simeq D^p / \partial D^p \simeq \Delta^p / \dot{\Delta}^p$ sending $N \in S^p$ to $\dot{\Delta}^p / \dot{\Delta}^p \in \Delta^p / \dot{\Delta}^p$ for $\dot{\Delta}^p$ the $(p-1)$ -skeleton of Δ^p , thus we have a canonical homeomorphism $S^p A_p^+ \simeq \Delta^p \times A^p / \dot{\Delta}^p \times A_p$;

- we have a canonical homeomorphism $\varphi : \Delta^p \times A^p / \dot{\Delta}^p \times A_p \rightarrow \Delta^p X / \Delta^{p-1} X$ obtained applying the CW-complex map φ_i^p to each $\Delta^p \times \{i\}$;
- thus we have canonical isomorphisms $h^q(A_p) \simeq \tilde{h}^q(A_p^+) \simeq \tilde{h}^{p+q}(S^p A_p^+) \simeq \tilde{h}^{p+q}(\Delta^p \times A^p / \dot{\Delta}^p \times A_p) \simeq \tilde{h}^{p+q}(\Delta^p X / \Delta^{p-1} X)$.

In a diagram:

$$\begin{array}{ccc}
A_p & \xrightarrow{S^p} & S^p A^+ \xrightarrow{\simeq} \frac{\Delta^p \times A^p}{\dot{\Delta}^p \times A_p} \xrightarrow{\sqcup \varphi_n^i} \frac{\Delta^p X}{\Delta^{p-1} X} \\
\downarrow h^q & & \downarrow \tilde{h}^{p+q} \\
C^p(X, h^q\{*\}) & \xrightarrow{\simeq} & \tilde{h}^{p+q}(\Delta^p X / \Delta^{p-1} X).
\end{array} \tag{17}$$

From equation (14) with $r = 2$:

$$\begin{aligned}
E_2^{p,q}(X) &= \text{Im}(h^{p+q}(p, p+2) \xrightarrow{\Psi^{p+q}} h^{p+q}(p-1, p+1)) \\
&= \text{Im}(h^{p+q}(X^{p+1}, X^{p-1}) \xrightarrow{\Psi^{p+q}} h^{p+q}(X^p, X^{p-2}))
\end{aligned} \tag{18}$$

and for what we have seen about the first coboundary we have a canonical isomorphism:

$$E_2^{p,q}(X) \simeq H^p(X, h^q\{*\}).$$

For a more accurate description of cocycles and coboundaries we refer to [6].

2.2.2 The last step

Notation: we denote $i^p : X^p \rightarrow X$ and $\pi^p : X \rightarrow X/X^p$ for any p .

We recall equation (7):

$$E_\infty^{p,q} = \text{Im}(h^{p+q}(p, +\infty) \xrightarrow{\Psi^{p+q}} h^{p+q}(0, p+1))$$

which, for Atiyah-Hirzebruch spectral sequence, becomes:

$$E_\infty^{p,q} = \text{Im}(h^{p+q}(X, X^{p-1}) \xrightarrow{\Psi} h^{p+q}(X^p)) \tag{19}$$

where Ψ is obtained by the pull-back of $i : X^p \rightarrow X/X^{p-1}$. Since $i = \pi_{p-1} \circ i_p$, the following diagram commutes:

$$\begin{array}{ccc}
\tilde{h}^{p+q}(X/X^{p-1}) & \xrightarrow{\Psi} & \tilde{h}^{p+q}(X^p) \\
\searrow \pi_{p-1}^* & & \nearrow i_p^* \\
& \tilde{h}^{p+q}(X) &
\end{array} \tag{20}$$

Remark: in the previous triangle we cannot say that $i_p^* \circ \pi_{p-1}^* = 0$ by exactness, since by exactness $i_p^* \circ \pi_p^* = 0$ at the same level p , as follows by $X^p \rightarrow X \rightarrow X/X^p$.

By exactness of $h^{p+q}(X, X^{p-1}) \xrightarrow{\pi_{p-1}^*} h^{p+q}(X) \xrightarrow{i_{p-1}^*} h^{p+q}(X^{p-1})$, we deduce that:

$$\text{Im } \pi_{p-1}^* = \text{Ker } i_{p-1}^*.$$

Since trivially $\text{Ker } i_p^* \subset \text{Ker } i_{p-1}^*$, we obtain that $\text{Ker } i_p^* \subset \text{Im } \pi_{p-1}^*$. Moreover:

$$\text{Im } \Psi = \text{Im } (i_p^* \circ \pi_{p-1}^*) = \text{Im } \left(i_p^* \Big|_{\text{Im } \pi_{p-1}^*} \right) \simeq \frac{\text{Im } \pi_{p-1}^*}{\text{Ker } i_p^*} = \frac{\text{Ker } i_{p-1}^*}{\text{Ker } i_p^*}$$

hence, finally:

$$E_{\infty}^{p,q} = \frac{\text{Ker}(h^{p+q}(X) \longrightarrow h^{p+q}(X^{p-1}))}{\text{Ker}(h^{p+q}(X) \longrightarrow h^{p+q}(X^p))} \quad (21)$$

i.e., $E_{\infty}^{p,q}$ is made by $(p+q)$ -classes on X which are 0 on X^{p-1} , up to classes which are 0 on X^p . In fact, the direct sum over p of (21) is the associated graded group of the filtration $F^p h^{p+q} = \text{Ker}(h^{p+q}(X) \longrightarrow h^{p+q}(X^{p-1}))$.

2.2.3 From the first to the last step

We now see how to link the first and the last step of the sequence. In the diagram (13), we know that an element $\alpha \in E_1^{p,q}$ survives until the last step if and only if $\alpha \in \text{Im } \pi^*$ and its class in $E_{\infty}^{p,q}$ is $\varphi^{p,q}(\alpha)$. We thus define, for $\alpha \in A^{p,q} = \text{Im } \pi^* \subset E_1^{p,q}$:

$$\{\alpha\}_{E_{\infty}^{p,q}}^{(1)} := \varphi^{p,q}(\alpha)$$

where the upper 1 means that we are starting from the first step.

For AHSS this becomes:

$$E_1^{p,q} = h^{p+q}(X^p, X^{p-1}) \quad E_{\infty}^{p,q} = \text{Im}(h^{p+q}(X, X^{p-1}) \xrightarrow{\Psi} h^{p+q}(X^p))$$

and the map which was called π^* in diagram (13) here becomes:

$$(i^p)^* : h^{p+q}(X, X^{p-1}) \longrightarrow h^{p+q}(X^p, X^{p-1})$$

for $i^p : X^p/X^{p-1} \rightarrow X/X^{p-1}$. Thus, the classes in $E_1^{p,q} = \tilde{h}^{p+q}(X^p/X^{p-1})$ surviving until the last step are the ones which are restrictions of a class defined on all X/X^{p-1} . Moreover, the map which in diagram (13) was called i^* here is $(\pi^p)^*$ for $\pi^p : X^p \rightarrow X^p/X^{p-1}$. Hence, for $\alpha \in \text{Im}(i^p)^* \subset E_1^{p,q}$:

$$\{\alpha\}_{E_{\infty}^{p,q}}^{(1)} = (\pi^p)^*(\alpha). \quad (22)$$

Remark: the spectral sequence does not depend on the triangulation chosen, and it can be generalized to any space homotopically equivalent to a finite CW-complex, as stated in [1] (page 18). We will deal with compact manifolds, thus we can suppose they are finite simplicial complexes: what is important for us is the independence on the triangulation.

3 Thom isomorphism and Gysin map

3.1 Multiplicative cohomology theories

We now introduce the notion of product in a cohomology theory following [4].

Definition 3.1 *A cohomology theory h^* on an admissible category \mathcal{A} is called multiplicative if there exists an exterior product, i.e., a natural map:*

$$\times : h^i(X, A) \times h^j(Y, B) \longrightarrow h^{i+j}(X \times Y, X \times B \cup A \times Y) \quad (23)$$

satisfying the following axioms:

- it is bilinear with respect to the sum in h^* ;
- it is associative and, for $(X, A) = (Y, B)$, graded-commutative;
- it admits a unit $1 \in h^0\{*\}$, for $\{*\} \in \text{Ob}\mathcal{A}$ a fixed space with one point;
- it is compatible with the Bockstein homomorphisms, i.e., the following diagram commutes:

$$\begin{array}{ccc} h^i(A) \times h^j(Y, B) & \xrightarrow{\times} & h^{i+j}(A \times Y, A \times B) \\ \downarrow \beta^i \times 1 & & \downarrow \text{exc} \\ & & h^{i+j}(X \times B \cup A \times Y, X \times B) \\ & & \downarrow \beta^{i+j} \\ h^{i+1}(X, A) \times h^j(Y, B) & \xrightarrow{\times} & h^{i+j+1}(X \times Y, X \times B \cup A \times Y) . \end{array}$$

In this case, we define the interior product:

$$\cdot : h^i(X, A) \times h^j(X, A) \longrightarrow h^{i+j}(X, A)$$

as $\alpha \cdot \beta := \Delta^*(\alpha \times \beta)$ for $\Delta : X \rightarrow X \times X$ the diagonal map.

Remarks:

- The interior product makes $h^*(X, A)$ a ring with unit.
- Let $(X, \{x_0\}), (Y, \{y_0\}) \in \mathcal{A}^+$ be spaces with marked point which are also good pairs and such that $(X \times Y, X \vee Y)$ is a good pair. Then the exterior product induces a map:

$$\tilde{h}^i(X)_{x_0} \times \tilde{h}^j(Y)_{y_0} \longrightarrow \tilde{h}^{i+j}(X \wedge Y) . \quad (24)$$

In fact, by (23) we have $h^i(X, \{x_0\}) \times h^j(Y, \{y_0\}) \longrightarrow h^{i+j}(X \times Y, X \vee Y)$ which is exactly (24).

Lemma 3.1 *If h^* is a multiplicative cohomology theory the coefficient group $h^0\{*\}$ is a commutative ring with unit.*

Proof: By the canonical homeomorphism $\{*\} \rightarrow \{*\} \times \{*\}$ we have a product $h^0\{*\} \times h^0\{*\} \rightarrow h^0\{*\}$ which is associative. Moreover, skew-commutativity in this case coincides with commutativity, and 1 is a unit also for this product. \square

Given a path-wise connected space X , we consider any map $p : \{*\} \rightarrow X$: by the path-wise connectedness of X two such maps are homotopic, thus the pull-back $p^* : h^*(X) \rightarrow h^*\{*\}$ is well defined.

Definition 3.2 For X a path-connected space we call rank of a cohomology class $\alpha \in h^n(X)$ the class $\text{rk}(\alpha) := (p^*)^n(\alpha) \in h^n\{*\}$ for any map $p : \{*\} \rightarrow X$.

Let us consider the unique map $P : X \rightarrow \{*\}$.

Definition 3.3 We call a cohomology class $\alpha \in h^n(X)$ trivial if there exists $\beta \in h^n\{*\}$ such that $\alpha = (P^*)^n(\beta)$. We denote by 1 the class $(P^*)^0(1)$.

Lemma 3.2 For X a path-wise connected space, a trivial cohomology class $\alpha \in h^n(X)$ is the pull-back of its rank.

Proof: Let $\alpha \in h^n(X)$ be trivial. Then $\alpha = (P^*)^n(\beta)$ so that $\text{rk}(\alpha) = (p^*)^n(P^*)^n(\beta) = (P \circ p)^{*n}(\beta) = \beta$, thus $\alpha = (P^*)^n(\text{rk}(\alpha))$. \square

3.2 Fiber bundles and module structure

Let $\pi : E \rightarrow B$ be a fiber bundle with fiber F and let h^* be a multiplicative cohomology theory. Then $h^*(E)$ has a natural structure of $h^*(B)$ -module given by:

$$\begin{aligned} \cdot : h^i(B) \times h^j(E) &\longrightarrow h^{i+j}(E) \\ a \cdot \alpha &:= (\pi^*a) \cdot \alpha. \end{aligned} \quad (25)$$

In general this is not an algebra structure since, because of skew-commutativity, one has $((\pi^*a)\alpha)\beta = \pm\alpha((\pi^*a)\beta)$.

We have an analogous module structure for relative fiber bundles, i.e., for pairs (E, E') with E' a sub-bundle of E with fiber $F' \subset F$. In fact, we have a natural diagonal map $\Delta : (E, E') \rightarrow (E \times E, E \times E')$ given by $\Delta(e) = (e, e)$, so that we can define the following module structure:

$$\begin{aligned} h^i(B) \times h^j(E, E') &\xrightarrow{\pi^* \times 1} h^i(E) \times h^j(E, E') \xrightarrow{\times} h^{i+j}(E \times E, E \times E') \\ &\xrightarrow{\Delta^*} h^{i+j}(E, E'). \end{aligned} \quad (26)$$

Similarly, we can consider the map $\Delta_\pi : (E, E') \rightarrow (B \times E, B \times E')$ given by $\Delta_\pi(e) = (\pi(e), e)$ and define the module structure:

$$h^i(B) \times h^j(E, E') \xrightarrow{\times} h^{i+j}(B \times E, B \times E') \xrightarrow{\Delta_\pi^*} h^{i+j}(E, E'). \quad (27)$$

To see that these two definitions are equivalent, we consider the following diagram:

$$\begin{array}{ccc} h^i(E) \times h^j(E, E') & \xrightarrow{(2)} & h^{i+j}(E \times E, E \times E') \\ \uparrow (1) & & \uparrow (4) \\ h^i(B) \times h^j(E, E') & \xrightarrow{(3)} & h^{i+j}(B \times E, B \times E') \end{array} \quad \begin{array}{c} \searrow (5) \\ \nearrow (6) \\ h^{i+j}(E, E') \end{array}$$

in which the structure (26) is given by (1)-(2)-(5) and the structure (27) by (3)-(6). The commutativity of the square, i.e. (1)-(2) = (3)-(4), follows from the naturality of the product, while the commutativity of the triangle, i.e. (6) = (4)-(5), follows from the fact that (4) = $(\pi \times 1)^*$, (5) = Δ^* , (6) = Δ_π^* , and $\Delta_\pi = (\pi \times 1) \circ \Delta$.

Lemma 3.3 *The module structure (26) or (27) is unitary, i.e. $1 \cdot \alpha = \alpha$ for 1 defined by 3.3. More generally, for a trivial class $t = P^*(\eta)$, with $\eta \in h^*\{*\}$, one has $t \cdot \alpha = \eta \cdot \alpha$.*

Proof: We prove for (27). The thesis follows from the commutativity of the following diagram:

$$\begin{array}{ccc} h^i(B) \times h^j(E, E') & \xrightarrow{\times} & h^{i+j}(B \times E, B \times E') \xrightarrow{\Delta_\pi^*} h^{i+j}(E, E') \\ (P^*)^{i \times 1^j} \uparrow & & \uparrow \text{((P \times 1)^*)^{i+j}} \quad \nearrow \simeq \\ h^i\{*\} \times h^j(E, E') & \xrightarrow{\times} & h^{i+j}(\{*\} \times E, \{*\} \times E') \end{array}$$

where the commutativity of the square follows directly from the naturality of the product while the commutativity of triangle follows from the fact that $(P \times 1) \circ \Delta_\pi$ is exactly the natural map $(E, E') \rightarrow (\{*\} \times E, \{*\} \times E')$ inducing the isomorphism \simeq . \square

Let us consider a real *vector* bundle $\pi : E \rightarrow B$ with fiber \mathbb{R}^n . In this case π^* is an isomorphism, since E retracts on B , thus the module structure (25) is just the product in $h^*(B)$ up to isomorphism. Let us instead consider the zero section $B_0 \simeq B$ and its complement $E_0 = E \setminus B_0$: then (26) or (27) gives a non-trivial module structure on $h^*(E, E_0)$. Defining the cohomology with compact support $h_{\text{cpt}}^*(X) := \tilde{h}^*(X^+)$ for X^+ the one-point compactification of X , we have:

$$h^*(E, E_0) \simeq h_{\text{cpt}}^*(E).$$

In fact, let us put a metric on E and consider the fiber bundles D_E and S_E obtained taking respectively the unit disc and the unit sphere in each fiber. Then we have:

$$\begin{aligned} h^*(E, E_0) &\stackrel{(1)}{\simeq} h^*(D_E, (D_E)_0) \stackrel{(2)}{\simeq} h^*(D_E, \partial D_E) \stackrel{(3)}{\simeq} \tilde{h}^*(D_E/\partial D_E) \\ &\stackrel{(4)}{\simeq} \tilde{h}^*(E^+) = h_{\text{cpt}}^*(E) \end{aligned} \quad (28)$$

where (1) follows by excision on the open set $U = E \setminus D_E$, (3) from the fact that $(D_E, \partial D_E)$ is a good pair and (4) from the homeomorphism sending $\text{Int}(D_E)$ to E and ∂D_E to ∞ .

We can also describe a natural module structure:

$$h_{\text{cpt}}^i(B) \times h_{\text{cpt}}^j(E) \longrightarrow h_{\text{cpt}}^{i+j}(E)$$

which, for B compact, coincides with the previous under the isomorphism (28). In fact, we consider:

$$h^i(B^+, \{\infty\}) \times h^j(E^+, \{\infty\}) \xrightarrow{\times} h^{i+j}(B^+ \times E^+, B^+ \vee E^+) \xrightarrow{(\Delta_\pi^+)^*} h^{i+j}(E^+, \{\infty\}) \quad (29)$$

for $\Delta_\pi^+ : (E^+, \{\infty\}) \rightarrow (B^+ \times E^+, B^+ \vee E^+)$ defined by $\Delta_\pi^+(e) = (\pi(e), e)$ and $\Delta_\pi^+(\infty) = \{\infty\} \times \{\infty\}$. For B compact, the module structure (29) becomes:

$$h^i(B) \times h^j(E^+, \{\infty\}) \xrightarrow{\times} h^{i+j}(B \times E^+, B \times \{\infty\}) \xrightarrow{(\Delta_\pi^+)^*} h^{i+j}(E^+, \{\infty\}). \quad (30)$$

We now see that (30) coincides with (27) under the isomorphism (28). In fact, we consider the following diagram (the arrows with -1 are inversions of natural isomorphisms):

$$\begin{array}{ccccc}
h^i(B) \times h^j(E, E_0) & \xrightarrow{\times} & h^{i+j}(B \times E, B \times E_0) & \xrightarrow{\Delta_\pi^*} & h^{i+j}(E, E_0) \\
\downarrow & & \downarrow & & \downarrow \\
h^i(B) \times h^j(D_E, (D_E)_0) & \xrightarrow{\times} & h^{i+j}(B \times D_E, B \times (D_E)_0) & \xrightarrow{\Delta_\pi^*} & h^{i+j}(D_E, (D_E)_0) \\
\downarrow & & \downarrow & & \downarrow \\
h^i(B) \times h^j(D_E, \partial D_E) & \xrightarrow{\times} & h^{i+j}(B \times D_E, B \times \partial D_E) & \xrightarrow{\Delta_\pi^*} & h^{i+j}(D_E, \partial D_E) \\
\downarrow -1 & & \downarrow -1 & & \downarrow -1 \\
h^i(B) \times h^j(D_E/\partial D_E, \partial D_E/\partial D_E) & \xrightarrow{\times} & h^{i+j}(B \times (D_E/\partial D_E), B \times (\partial D_E/\partial D_E)) & \xrightarrow{\Delta_\pi^*} & h^{i+j}(D_E/\partial D_E, \partial D_E/\partial D_E) \\
& & \downarrow & & \downarrow \\
& & h^{i+j}(B \times (D_E/\partial D_E), B \times (\partial D_E/\partial D_E)) & \xrightarrow{(\Delta_\pi^+)^*} & h^{i+j}(D_E/\partial D_E, \partial D_E/\partial D_E)
\end{array}$$

where the first line is (27) and the sequence made by the last element of each column is (30).

We remark for completeness that there is a homeomorphism $B^+ \wedge E^+ \simeq (B \times E)^+$: in fact, $B^+ \wedge E^+ = (B \sqcup \{\infty\}) \times (E \sqcup \{\infty\}) / ((\{\infty\} \times E) \cup (B \times \{\infty\}))$ and, at the quotient, $B \times E$ remains unchanged while the denominator $B^+ \vee E^+$ becomes a point which is the $\{\infty\}$ of $(B \times E)^+$. Thus the homeomorphism is $\varphi(b, e) = (b, e)$ and $\varphi(\infty, e) = \varphi(b, \infty) = \infty$. We then consider the map $\Delta_\pi^+ : E^+ \rightarrow (B \times E)^+$ given by $\Delta_\pi^+(e) = (\pi(e), e)$ and $\Delta_\pi^+(\infty) = \infty$. Thus, under the hypotheses that $(B^+, \{\infty\})$ and $(B^+ \times E^+, B^+ \vee E^+)$ are good pairs¹, (29) can also be written as:

$$\tilde{h}^i(B^+) \times \tilde{h}^j(E^+) \xrightarrow{(1)} \tilde{h}^{i+j}(B^+ \wedge E^+) \simeq \tilde{h}^{i+j}((B \times E)^+) \xrightarrow{(\Delta_\pi^+)^*} \tilde{h}^{i+j}(E^+) \quad (31)$$

where (1) is given by formula (24).

3.3 Orientability and Thom isomorphism

We now define *orientable* vector bundles with respect to a fixed *multiplicative* cohomology theory. By hypothesis, there exists a unit $1 \in h^0\{*\} = \tilde{h}^0(S^0)$. Since S^n is homeomorphic to the n -th suspension of S^0 , such homeomorphism defines (by the suspension isomorphism) an element $\gamma^n \in \tilde{h}^n(S^n)$ such that $\gamma^n = S^n(1)$ (clearly γ^n is not the unit class since the latter does not belong to $\tilde{h}^n(S^n)$). Moreover, given a vector bundle $E \rightarrow B$ with fiber \mathbb{R}^k , we have the canonical isomorphism (28) which, in each fiber $F_x = \pi^{-1}(x)$, restricts to:

$$h^k(F_x, (F_x)_0) \simeq h^k(D_x^k, \partial D_x^k) \simeq h^k(D_x^k/\partial D_x^k, \partial D_x^k/\partial D_x^k) \simeq h^k(S^k, N) \quad (32)$$

¹These hypotheses are surely satisfied when B is compact, since $\{\infty\}$ is a neighborhood of itself

where the last isomorphism is non-canonical since it depends on the local chart (N is the north pole of the sphere). However, since the homotopy type of a map from S^k to S^k is uniquely determined by its degree (see [9]) and a homeomorphism must have degree ± 1 , it follows that the last isomorphism of (32) is canonical up to an overall sign, i.e., up to a multiplication by -1 in $h^k(S^k, N)$.

Definition 3.4 *Let $\pi : E \rightarrow B$ be a vector bundle of rank k and h^* a multiplicative cohomology theory in an admissible category \mathcal{A} containing π . The bundle E is called h -orientable if there exists a class $u \in h^k(E, E_0)$ such that for each fiber $F_x = \pi^{-1}(x)$ it satisfies $u|_{F_x} \simeq \pm \gamma^k$ under the isomorphism (32). The class u is called orientation.*

We now discuss some properties of h -orientations. The following lemma is very intuitive and can be probably deduced by a continuity argument; however, since we have not discussed topological properties of the cohomology groups, we give a proof not involving such problems. For a rank- k vector bundle $\pi : E \rightarrow B$, let $(U_\alpha, \varphi_\alpha)$ be a contractible local chart for E , with $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$. Let us consider the compactification $\varphi_\alpha^+ : \pi^{-1}(U_\alpha)^+ \rightarrow (U_\alpha \times \mathbb{R}^k)^+$, restricting, for $x \in U_\alpha$, to $(\varphi_\alpha^+)_x : E_x^+ \rightarrow S^k$. Then we can consider the map:

$$\hat{\varphi}_{\alpha,x} := ((\varphi_\alpha^+)^{-1})^{*k} : \tilde{h}^k(E_x^+) \longrightarrow \tilde{h}^k(S^k). \quad (33)$$

Lemma 3.4 *Let u be an h -orientation of a rank- n vector bundle $\pi : E \rightarrow B$, let $(U_\alpha, \varphi_\alpha)$ be a contractible local chart for E and let $\hat{\varphi}_{\alpha,x}$ be defined by (33). Then $\hat{\varphi}_{\alpha,x}(u|_{E_x^+})$ is constant in x with value γ^k or $-\gamma^k$.*

Proof: Let us consider the map $(\varphi_\alpha^+)^{-1})^{*k} : \tilde{h}^k(\pi^{-1}(U_\alpha)^+) \longrightarrow \tilde{h}^k((U_\alpha \times \mathbb{R}^k)^+)$ and let call $\xi := (\varphi_\alpha^+)^{-1})^{*k}(u|_{\pi^{-1}(U_\alpha)^+})$. Since $(U_\alpha \times \mathbb{R}^k)^+ \simeq U_\alpha \times S^k / U_\alpha \times \{N\}$ canonically, we can consider the projection $\pi_\alpha : U_\alpha \times S^k \rightarrow U_\alpha \times S^k / U_\alpha \times \{N\}$. Then $\hat{\varphi}_{\alpha,x}(u|_{E_x^+}) = \xi|_{(\{x\} \times \mathbb{R}^k)^+} \simeq \pi_\alpha^*(\xi)|_{\{x\} \times S^k}$. But, since U_α is contractible, the projection $\pi : U_\alpha \times S^k \rightarrow S^k$ induces an isomorphism in cohomology, so that $\pi_\alpha^*(\xi) = \pi^*(\eta)$ for $\eta \in h^k(S^k)$, so that $\pi_\alpha^*(\xi)|_{\{x\} \times S^k} = \pi^*(\eta)|_{\{x\} \times S^k} \simeq \eta$, i.e., it is constant in x . By definition of orientation, its value must be $\pm \gamma^k$. \square

Theorem 3.5 *If a vector bundle $\pi : E \rightarrow B$ of rank k is h -orientable, then given trivializing contractible charts $\{U_\alpha\}_{\alpha \in I}$ it is always possible to choose trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$ such that $(\varphi_\alpha^+)_x^{*k}(\gamma^k) = u|_{E_x^+}$. In particular, for $x \in U_{\alpha\beta}$ the homeomorphism $(\varphi_\beta \varphi_\alpha^{-1})_x^+ : (\mathbb{R}^k)^+ \simeq S^k \longrightarrow (\mathbb{R}^k)^+ \simeq S^k$ satisfies $((\varphi_\beta \varphi_\alpha^{-1})_x^+)^*(\gamma^k) = \gamma^k$.*

Proof: Chosen any local trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$, it verifies $(\varphi_\alpha^+)_x^{*k}(\gamma^k) = \pm u|_{E_x^+}$ by lemma 3.4. If the minus sign holds, it is enough compose φ_α to the pointwise reflection by an axes in \mathbb{R}^k , so that the compactified map has degree -1 . \square

Definition 3.5 *An atlas satisfying the conditions of theorem 3.5 is called h -oriented atlas.*

Remark: the classical definition of orientability, i.e., the existence of an atlas with transition functions of pointwise positive determinant, coincides with H -orientability for H the singular cohomology with \mathbb{Z} -coefficients, as stated in [4]. Similarly, an oriented atlas is an H -oriented atlas.

Lemma 3.6 *Let $\pi : E \rightarrow B$ be a rank- k vector bundle which is orientable both for H^* and for a multiplicative cohomology theory h^* , and let u be an orientation with respect to h^* . Then an H -oriented atlas is h -oriented with respect to u or $-u$.*

Proof: by lemma 3.4 the value of u is constant in x for each chart, and it is $\pm\gamma^k$. Moreover, the compactified transition functions of an H -oriented atlas must have degree 1, thus they send γ^k in γ^k for every cohomology theory. Hence, the value of u must be γ^k or $-\gamma^k$ for each chart. The thesis immediately follows. \square

We now state Thom isomorphism following [4].

Theorem 3.7 *Let $(E, E') \rightarrow B$ be a relative fiber bundle with fiber (F, F') . Suppose that there exist $a_1, \dots, a_r \in h^*(E, E')$ such that, for every $x \in B$, their restrictions to $F_x = \pi^{-1}(x)$ form a base of $h^*(F_x, F'_x)$ as a $h^*\{*\}$ -module under the module structure (27). Then a_1, \dots, a_r form a base of $h^*(E, E')$ as a $h^*(B)$ -module. \square*

For the proof see [4] page 7.

Theorem 3.8 (Thom isomorphism) *Let $\pi : E \rightarrow B$ be a h -orientable vector bundle of rank k , and let $u \in h^k(E, E_0)$ be an orientation. Then, the map induced by the module structure (27):*

$$\begin{aligned} T : h^*(B) &\longrightarrow h^*(E, E_0) \\ T(\alpha) &:= \alpha \cdot u \end{aligned}$$

is an isomorphism of abelian groups.

Proof: The map $T : h^*\{*\} \longrightarrow \tilde{h}^*(S^n)_N$ given by $T(\alpha) = \alpha \cdot \gamma^n$ is an isomorphism since, up to the suspension isomorphism, it coincides with $T' : h^*\{*\} \longrightarrow h^*\{*\}$ given by $T'(\alpha) = 1 \cdot \alpha = \alpha$. Thus, γ^n is a base of $h^*(S^n, N)$ as a $h^*\{*\}$ -module. By definition of h -orientability and theorem 3.7, it follows that u is a base of $h^*(E, E_0)$ as a $h^*(B)$ -module, i.e., T is an isomorphism. \square

3.4 Gysin map

Let X be a compact smooth n -manifold and $Y \subset X$ a compact embedded r -dimensional submanifold such that the normal bundle $N(Y) = (TX|_Y)/TY$ is h -orientable. Then, since Y is compact, there exists a tubular neighborhood U of Y in X , i.e., there exists an homeomorphism $\varphi_U : U \rightarrow N(Y)$.

If $i : Y \rightarrow X$ is the embedding, from this data we can naturally define an homomorphism, called *Gysin map*:

$$i_! : h^*(Y) \longrightarrow \tilde{h}^*(X) .$$

In fact:

- we first apply the Thom isomorphism $T : h^*(Y) \longrightarrow h_{\text{cpt}}^*(N(Y)) = \tilde{h}^*(N(Y)^+)$;
- then we naturally extend φ_U to $\varphi_U^+ : U^+ \rightarrow N(Y)^+$ and apply $(\varphi_U^+)^* : h_{\text{cpt}}^*(N(Y)) \rightarrow h_{\text{cpt}}^*(U)$;
- there is a natural map $\psi : X \rightarrow U^+$ given by:

$$\psi(x) = \begin{cases} x & \text{if } x \in U \\ \infty & \text{if } x \in X \setminus U \end{cases}$$

hence we apply $\psi^* : \tilde{h}^*(U^+) \longrightarrow \tilde{h}^*(X)$.

Summarizing:

$$i_!(\alpha) = \psi^* \circ (\varphi_U^+)^* \circ T(\alpha). \quad (34)$$

Remark: One could try to use the immersion $i : U^+ \rightarrow X^+$ and the retraction $r : X^+ \rightarrow U^+$ to have a splitting $h(X) = h(U) \oplus h(X, U) = h(Y) \oplus K(X, U)$. But this is false, since the immersion $i : U^+ \rightarrow X^+$ is not continuous: *since X is compact*, $\{\infty\} \subset X^+$ is open, but $i^{-1}(\{\infty\}) = \{\infty\}$, and $\{\infty\}$ is not open in U^+ since U is non-compact.

4 Gysin map and Atiyah-Hirzebruch spectral sequence

Let X be a compact orientable manifold and Y a compact embedded submanifold. We choose a finite triangulation of X which restricts to a triangulation of Y (v. [10]). We use the following notation:

- we denote the triangulation of X by $\Delta = \{\Delta_i^m\}$, where m is the dimension of the simplex and i enumerates the m -simplices;
- we denote by X_Δ^p the p -skeleton of X with respect to Δ .

We refer to [8] (chapter 0.4) for the definition of dual decomposition and the statement of Poincaré duality in this setting.

Theorem 4.1 *Let X be an n -dimensional compact manifold and $Y \subset X$ an r -dimensional embedded compact submanifold. Let:*

- $\Delta = \{\Delta_i^m\}$ be a triangulation of X which restricts to a triangulation $\Delta' = \{\Delta'_i^m\}$ of Y ;
- $D = \{D_j^{n-m}\}$ be the dual decomposition of X with respect to Δ ;
- $\tilde{D} \subset D$ be subset of D made by the duals of simplices in Δ' .

Then:

- the interior of $|\tilde{D}|$ is a tubular neighborhood of Y in X ;
- the interior of $|\tilde{D}|$ does not intersect X_D^{n-r-1} , i.e.:

$$|\tilde{D}| \cap X_D^{n-r-1} \subset \partial|\tilde{D}|.$$

Proof: The n -simplices of \tilde{D} are the dual of the vertices of Δ' . Let $\tau = \{\tau_i^m\}$ be the first barycentric subdivision of Δ . For each vertex $\Delta_{i'}^0$ (thought as an element of Δ), its dual is:

$$\tilde{D}_{i'}^n = \bigcup_{\Delta_{i'}^0 \in \tau_k^n} \tau_k^n. \quad (35)$$

Moreover, if $\tau' = \{\tau_{i'}^m\}$ is the first barycentric subdivision of Δ' and D' is the dual of Δ' in Y , then

$$D'_{i'}^r = \bigcup_{\Delta_{i'}^0 \in \tau_{k'}^n} \tau_{k'}^n \quad (36)$$

and

$$\tilde{D}_{i'}^n \cap Y = D'_{i'}^r.$$

Moreover, let us consider the $(n-r)$ -simplices in \tilde{D} contained in $\partial\tilde{D}_{i'}^n$ (for the fixed i' of formula (35)), i.e. $\tilde{D}^{n-r} \cap \tilde{D}_{i'}^n$: it intersects Y transversally in the barycenters of each r -simplex of Δ' containing $\Delta_{i'}^0$: we call such barycenters $\{b_1, \dots, b_k\}$ and the intersecting $(n-r)$ -cells $\{\tilde{D}_j^{n-r}\}_{j=1, \dots, k}$. Since (for a fixed i') $\tilde{D}_{i'}^n$ retracts on $\Delta_{i'}^0$, we can consider a local chart $(U_{i'}, \varphi_{i'})$, with $U_{i'} \subset \mathbb{R}^n$ neighborhood of 0, such that:

- $\varphi_{i'}^{-1}(U_{i'})$ is a neighborhood of $\tilde{D}_{i'}^n$;
- $\varphi_{i'}(D'_{i'}^r) \subset U \cap (\{0\} \times \mathbb{R}^r)$, for $0 \in \mathbb{R}^{n-r}$ (v. eq. (36));
- $\varphi_{i'}(\tilde{D}_j^{n-r}) \subset U \cap (\mathbb{R}^{n-r} \times \pi_r(\varphi(b_j)))$, for $\pi_r : \mathbb{R}^n \rightarrow \{0\} \times \mathbb{R}^r$ the projection.

We now consider the natural foliation of U given by the intersection with the hyperplanes $\mathbb{R}^{n-r} \times \{x\}$ and its image via φ^{-1} : in this way, we obtain a foliation of $\tilde{D}_{i'}^n$ transversal to Y . If we do this for any i' , by construction the various foliations glue on the intersections, since such intersections are given by the $(n-r)$ -cells $\{\tilde{D}_j^{n-r}\}_{j=1, \dots, k}$, and the interior gives a C^0 -tubular neighborhood of Y .

Moreover, a $(n-r-t)$ -cell of \tilde{D} , for $t > 0$, cannot intersect the brane since it is contained in the boundary of a $(n-r)$ -cell, and such cells intersect Y , which is done by r -cells, only in their interior points b_j .

□

We now consider quadruples (X, Y, D, \tilde{D}) satisfying the following condition:

(#) X is an n -dimensional compact manifold and $Y \subset X$ an r -dimensional embedded compact submanifold such that $N(Y)$ is h -orientable. Moreover, D and \tilde{D} are defined as in theorem 4.1.

Lemma 4.2 *Let (X, Y, D, \tilde{D}) be a quadruple satisfying (#), $U = \text{Int}|\tilde{D}|$ and $\alpha \in h^*(Y)$. Then:*

- *there exists a neighborhood V of $X \setminus U$ such that $i_!(\alpha)|_V = 0$;*
- *in particular, $i_!(\alpha)|_{X_D^{n-r-1}} = 0$.*

Proof: By equation (34) at page 18:

$$i_!(\alpha) = \psi^*\beta \quad \beta = (\varphi_U^+)^* \circ T(\alpha) \in \tilde{h}^*(U^+).$$

Let $V_\infty \subset U^+$ be a contractible neighborhood of ∞ , which exists since U is a tubular neighborhood of a smooth manifold, and let $V = \psi^{-1}(V_\infty)$. Then $\tilde{h}^*(V_\infty) \simeq \tilde{h}^*\{*\} = 0$, thus $\beta|_{V_\infty} = 0$ so that $(\psi^*\beta)|_V = 0$. By theorem 4.1 X_D^{n-r-1} does not intersect the tubular neighborhood $\text{Int}|\tilde{D}|$ of Y , hence $X_D^{n-r-1} \subset V$, so that $(\psi^*\beta)|_{X_D^{n-r-1}} = 0$. \square

4.1 Unit class

We start by considering the case of the unit class $1 \in h^0(Y)$ (see def. 3.3). We first notice that Y , being a simplicial complex, in order to be a cycle in $C_r(X, R)$, for $R = h^0\{*\}$ a ring, must be oriented if $\text{char } R \neq 2$, while it is always a cycle if $\text{char } R = 2$. Since X is orientable, for $\text{char } R \neq 2$ the normal bundle is also orientable², thus it is orientable in $H^p(X, \mathbb{Z})$, thus in $H^p(X, R)$. Since for $\text{char } R = 2$ any bundle is orientable in $H^p(X, R)$, we conclude that the normal bundle is always orientable in $H^p(X, R)$ for any R .

Theorem 4.3 *Let (X, Y, D, \tilde{D}) be a quadruple satisfying (#) and $\Phi_D^{n-r} : C^{n-r}(X, h^q\{*\}) \longrightarrow h^{n-r+q}(X_D^{n-r}, X_D^{n-r-1})$ be the isomorphism stated in equation (17). Let us define the natural projection and immersion:*

$$\pi^{n-r} : X_D^{n-r} \longrightarrow X_D^{n-r}/X_D^{n-r-1} \quad i^{n-r} : X_D^{n-r} \longrightarrow X$$

and let $\text{PD}_\Delta(Y)$ be the representative of $\text{PD}_X Y$ given by the sum of the cells dual to the p -cells of Δ covering Y . Then:

$$(i^{n-r})^*(i_!(1)) = (\pi^{n-r})^*(\Phi_D^{n-r}(\text{PD}_\Delta(Y))).$$

Proof: Let U be the tubular neighborhood of Y in X stated in theorem (4.1). We define the space $(U^+)_D^{n-r}$ obtained considering the interior of the $(n-r)$ -cells intersecting Y

²In fact, $w_1(TX|_Y) = w_1(TY) + w_1(NY)$ and w_1 is a 2-torsion class.

transversally and compactifying to one point this space. The interior of such cells form exactly the intersection between the $(n-r)$ -skeleton of D and U , since in the (open) tubular neighborhood the only $(n-r)$ -cells are the ones intersecting Y as stated in theorem 4.1, i.e. we consider $X_D^{n-r}|_U$. If we close this space in X we obtain the closed cells intersecting Y transversally, whose boundary lies entirely in X_D^{n-r-1} . Thus the one-point compactification of the interior is:

$$(U^+)_D^{n-r} = \frac{\overline{X_D^{n-r}|_U}^X}{X_D^{n-r-1}|\partial U}$$

so that $(U^+)_D^{n-r} \subset U^+$ sending the denominator to ∞ (the numerator is exactly \tilde{D}^{n-r} of theorem 4.1). We also define:

$$\psi^{n-r} = \psi|_{X_D^{n-r}} : X_D^{n-r} \longrightarrow (U^+)_D^{n-r} .$$

ψ^{n-r} is well-defined since the $(n-r)$ -simplices outside U and all the $(n-r-1)$ -simplices are sent to ∞ by ψ . Moreover:

$$\pi^{n-r}(X_D^{n-r}) \simeq \bigvee_{i \in I} S_i^{n-r} .$$

We denote by $\{S_j^{n-r}\}_{j \in J}$, with $J \subset I$, the set of $(n-r)$ -spheres corresponding to $\pi^{n-r}(\overline{X_D^{n-r}|_U}^X)$. We define:

$$\rho : \bigvee_{i \in I} S_i^{n-r} \longrightarrow \bigvee_{j \in J} S_j^{n-r}$$

as the projection, i.e., ρ is the identity of S_j^{n-r} for every $j \in J$ and sends all the spheres in $\{S_i^{n-r}\}_{i \in I \setminus J}$ to the attachment point. We have that:

$$\psi^{n-r} = \rho \circ \pi^{n-r} .$$

In fact, the boundary of the $(n-r)$ -cells intersecting U is contained in ∂U , hence it is sent to ∞ by ψ^{n-r} , and also all the $(n-r)$ -cells outside U are sent to ∞ : hence, the image of ψ^{n-r} is homeomorphic to $\bigvee_{j \in J} S_j^{n-r}$ sending ∞ to the attachment point. Thus:

$$(\psi^{n-r})^* = (\pi^{n-r})^* \circ \rho^* .$$

We put $N = N(Y)$ and $\tilde{u}_N = (\varphi_U^+)^*(u_N)$. By lemma 3.3 and equation (34) at page 18, it is $i_!(1) = \psi^* \circ (\varphi_U^+)^*(u_N)$. Then:

$$(i^{n-r})^*(i_!(1)) = (i^{n-r})^*\psi^*(\tilde{u}_N) = (\psi^{n-r})^*(\tilde{u}_N|_{(U^+)_D^{n-r}})$$

and

$$\rho^*(\tilde{u}_N|_{(U^+)_D^{n-r}}) = \Phi_D^{n-r}(\text{PD}_\Delta(Y))$$

since:

- $\text{PD}_\Delta(Y)$ is the sum of the $(n-r)$ -cells intersecting U , oriented as the normal bundle;

- hence $\Phi_D^{n-r}(\text{PD}_\Delta(Y))$ gives a γ^{n-r} factor to each sphere S_j^{n-r} for $j \in J$ and 0 otherwise, orienting the sphere orthogonally to Y ;
- but this is exactly $\rho^*(\tilde{u}_N |_{(U^+)_D^{n-r}})$ since by definition of orientability the restriction of $\tilde{\lambda}_N$ must be $\pm\gamma^n$ for each fiber of N^+ . We must show that the sign ambiguity is fixed: this follows from lemma 3.6 since the normal bundle is H -orientable and the atlas naturally arising from the tubular neighborhood as in theorem 4.1 is H -oriented. For the spheres outside U , that ρ sends to ∞ , we have that:

$$\begin{aligned} \rho^*(\tilde{u}_N |_{(U^+)_D^{n-r}}) |_{\bigvee_{i \in I \setminus J} S_i^{n-r}} &= \rho^*(\tilde{u}_N |_{\rho(\bigvee_{i \in I \setminus J} S_i^{n-r})}) \\ &= \rho^*(\tilde{u}_N |_{\{\infty\}}) = \rho^*(0) = 0. \end{aligned}$$

Hence:

$$\begin{aligned} i_!(1) |_{X_D^{n-r}} &= (\psi^{n-r})^*(\tilde{u}_N |_{(U^+)_D^{n-r}}) \\ &= (\pi^{n-r})^* \circ \rho^*(\tilde{u}_N |_{(U^+)_D^{n-r}}) \\ &= (\pi^{n-r})^* \Phi_D^{n-r}(\text{PD}_\Delta(Y)). \end{aligned}$$

□

Let us now consider any trivial class $P^*\eta \in h^q(Y)$. By lemma 3.3 at page 14, we have that $P^*\eta \cdot u_N = \eta \cdot u_N$, hence theorem 4.3 becomes:

$$(i^{n-r})^*(i_!(P^*\eta)) = (\pi^{n-r})^*(\Phi_D^{n-r}(\text{PD}_\Delta(Y \otimes \eta))).$$

In fact, the same proof apply considering that $\eta \cdot u_N$ gives a factor of $\eta \cdot \gamma^{n-r}$ instead of γ^{n-r} for each sphere of N^+ , with $\eta \in h^q\{*\} \simeq \tilde{h}^q(S^q)$.

The following theorem encodes the link between Gysin map and AHSS: since the groups $E_r^{p,q}$ for $r \geq 2$ and the filtration $\text{Ker}(h^{p+q}(X) \rightarrow h^{p+q}(X^{n-p}))$ of $h^{p+q}(X)$ does not depend on the particular simplicial structure chosen (v. [1]), we can drop the dependence on D . We recall all the hypotheses in order to state the complete result:

Theorem 4.4 *Let us consider the following data:*

- an n -dimensional orientable compact manifold X and an r -dimensional embedded compact submanifold $Y \subset X$;
- a multiplicative cohomology theory h^* such that $N(Y)$ is h -orientable;
- for $\{(E_r^{p,q}, d_r^{p,q})\}$ be the Atiyah-Hirzebruch spectral sequence relative to h , the canonical isomorphism $\Phi^{n-r} : C^{n-r}(X, h^q\{*\}) \xrightarrow{\cong} E_1^{n-r,q}(X)$;
- a representative $\widetilde{\text{PD}}(Y)$ of $\text{PD}(Y)$ as a cochain in the cellular complex relative to any finite CW-complex structure of X .

For $\eta \in h^q\{*\}$, if $\Phi^{n-r} \widetilde{\text{PD}}(Y \otimes \eta)$ is contained in the kernel of all the boundaries $d_r^{n-r,q}$ for $r \geq 2$, we can define a class:

$$\{\Phi^{n-r} \widetilde{\text{PD}}(Y \otimes \eta)\}_{E_\infty^{n-r,q}}^{(1)} \in \frac{\text{Ker}(h^{n-r+q}(X) \longrightarrow h^{n-r+q}(X^{n-r-1}))}{\text{Ker}(h^{n-r+q}(X) \longrightarrow h^{n-r+q}(X^{n-r}))}.$$

Then:

$$\{\Phi^{n-r} \widetilde{\text{PD}}(Y \otimes \eta)\}_{E_\infty^{n-r,q}}^{(1)} = [i_!(P^*\eta)]_{E_\infty^{n-r,q}}.$$

Proof: We use the cellular decomposition D considered in the previous theorems. By equations (19) and (20) we have:

$$E_\infty^{n-r,q} = \text{Im}(\tilde{h}^{n-r+q}(X/X_D^{n-r-1}) \xrightarrow{\Psi} \tilde{h}^{n-r+q}(X_D^{n-r})) \quad (37)$$

$$\begin{array}{ccc} & \xrightarrow{\Psi} & \\ \pi_{n-r-1}^* \searrow & & \nearrow i_{n-r}^* \\ & \tilde{h}^{n-r+q}(X) & \end{array}$$

and, given a representative $\alpha \in \text{Ker}(h^{n-r+q}(X) \longrightarrow h^{n-r+q}(X_D^{n-r-1})) = \text{Im} \pi_{n-r-1}^*$, we have that $[\alpha]_{E_\infty^{n-r,q}} = i_{n-r}^*(\alpha) = \alpha|_{X_D^{n-r}}$. Moreover, from (13) we have the diagram:

$$E_\infty^{n-r,q} = \text{Im}(\tilde{h}^{n-r+q}(X/X_D^{n-r-1}) \xrightarrow{\Psi} \tilde{h}^{n-r+q}(X_D^{n-r})) \quad (38)$$

$$\begin{array}{ccc} & \xrightarrow{\Psi} & \\ (i^{n-r})^* \searrow & & \nearrow \varphi^{n-r,q} = (\pi^{n-r})^* \\ & \tilde{h}^{n-r+q}(X_D^{n-r}/X_D^{n-r-1}) & \end{array}$$

(Note that we used lower indices for the maps in (37) and upper indices for (38)). We have that:

- by formula (22) the class $\{\Phi_D^{n-p} \widetilde{\text{PD}}_\Delta(Y \otimes \eta)\}_{E_\infty^{n-p,q}}^{(1)}$ is given in diagram (38) by $(\pi^{n-r})^*(\Phi_D^{n-r} \widetilde{\text{PD}}_\Delta(Y \otimes \eta))$;
- by lemma 4.2 it is $i_!(1) \in \text{Ker}(h^{n-r+q}(X) \rightarrow h^{n-r+q}(X_D^{n-r-1}))$, hence $[i_!(P^*\eta)]_{E_\infty^{n-r,q}}$ is well-defined, and, by exactness, $i_!(P^*\eta) \in \text{Im} \pi_{n-r-1}^*$;
- by theorem 4.3, it is $i_{n-r}^*(i_!(P^*\eta)) = (\pi^{n-r})^*(\Phi_D^{n-r}(\widetilde{\text{PD}}_\Delta(Y \otimes \eta)))$;
- hence $\{\Phi_D^{n-r} \widetilde{\text{PD}}_\Delta(Y \otimes \eta)\}_{E_\infty^{n-r,q}}^{(1)} = [i_!(P^*\eta)]_{E_\infty^{n-r,q}}$.

□

Corollary 4.5 *Assuming the same data of the previous theorem, the fact that Y has orientable normal bundle with respect to h^* is a sufficient condition for $\widetilde{\text{PD}}(Y)$ to survive until the last step of the spectral sequence. Thus, any cohomology class $[Y] \in E_2^{n-r,0}$ having a smooth representative with h -orientable normal bundle survives until the last step.*

Proof: Let us put together the diagrams (37) and (38):

$$\begin{array}{ccc}
\tilde{h}^{n-r}(X/X_D^{n-r-1}) & \xrightarrow{\pi_{n-r-1}^*} & \tilde{h}^{n-r}(X) \\
\downarrow (i^{n-r})^* & \searrow \Psi & \downarrow i_{n-r}^* \\
\tilde{h}^{n-r}(X_D^{n-r}/X_D^{n-r-1}) & \xrightarrow{(\pi_{n-r-1}^*)^*} & \tilde{h}^{n-r}(X_D^{n-r})
\end{array} \tag{39}$$

so that the diagram commutes being $\pi^{n-r} \circ i^{n-r} = i_{n-r} \circ \pi_{n-r-1}$. Under the hypotheses stated, we have that $i_!(1) \in \text{Im } \pi_{n-r-1}^*$, so that $i_!(1) = \pi_{n-r-1}^*(\alpha)$. Then $(i^{n-r})^*(\alpha) \in A^{n-r,0}$, so that it survives until the last step giving a class $(i^{n-r})^*(\pi^{n-r})^*(\alpha)$ in the last step. \square

4.2 Generic cohomology class

If we consider a generic class α over Y of rank $\text{rk}(\alpha)$, we can prove that $i_!(E)$ and $i_!(P^*\text{rk}(\alpha))$ have the same restriction to X_D^{n-p} : in fact, the Thom isomorphism gives $T(\alpha) = \alpha \cdot u_N$ and, if we restrict $\alpha \cdot u_N$ to a *finite* family of fibers, which are transversal to Y , the contribution of α becomes trivial, so it has the same effect of the trivial class $P^*\text{rk}(\alpha)$. We now prove this.

Lemma 4.6 *Let (X, Y, D, \tilde{D}) be a quadruple satisfying (#) and $\alpha \in h^*(Y)$ a class of rank $\text{rk}(\alpha)$. Then:*

$$(i^{n-r})^*(i_! \alpha) = (i^{n-r})^*(i_!(P^*\text{rk} \alpha)).$$

Proof: Since X_D^{n-r} intersects the tubular neighborhood in a finite number of cells corresponding under φ_U^+ to a finite number of fibers of the normal bundle N attached to one point, it is sufficient to prove that, for any $y \in Y$, $(\alpha \cdot u_N)|_{N_y^+} = P^*\text{rk}(\alpha) \cdot u_N|_{N_y^+}$. Let us consider the following diagram for $y \in B$:

$$\begin{array}{ccc}
h^i(Y) \times h^n(N_y, N'_y) & \xrightarrow{\times} & h^{i+n}(Y \times N, Y \times N') \\
\downarrow (i^*)^i \times (i^*)^n & & \downarrow (i \times i)^{i+n} \\
h^i\{y\} \times h^n(N_y, N'_y) & \xrightarrow{\times} & h^{i+n}(\{y\} \times N_y, \{*\} \times N'_y)
\end{array}$$

The diagram commutes by naturality of the product, thus $(\alpha \cdot u_N)|_{N_y^+} = \alpha|_{\{y\}} \cdot u_N|_{N_y^+}$. Thus, we just have to prove that $\alpha|_{\{y\}} = (P^*\text{rk}(\alpha))|_{\{y\}}$, i.e. that $i^*\alpha = i^*P^*p^*\alpha = (p \circ P \circ i)^*\alpha$. This immediately follows from the fact that $p \circ P \circ i = i$. \square

5 Conclusions and perspectives

We proved the link between Gysin map and Atiyah-Hirzebruch spectral sequence stated in the introduction: the Gysin map from a submanifold gives a representative of the class in $E_\infty^{p,q}$ of its Poincaré dual. Since the definition of the Gysin map requires orientability of the normal bundle, considering corollary 4.5 it is natural to suppose that the submanifolds

not verifying this condition are selected by some boundaries of the spectral sequence. This is actually consistent with the K-theory case, where $d_3^{p,q}$ is exactly the integral Steenrod-square $Sq_{\mathbb{Z}}^3$, which is related to the integral Stiefel-Whitney class W_3 of the normal bundle: in fact, a bundle is orientable in K-theory if and only if it admits a Spin^c -structure, i.e., if its class W_3 is zero. Similarly, for real K-theory $d_3^{p,q}$ is related to w_2 (see [7]), and a bundle is orientable in real K-theory if and only if it admits a Spin structure, i.e., if its class w_2 is zero. Thus, it seems reasonable that this result extends to any cohomological theory, and it will be a subject of a future study.

Another possible generalization is the statement of a similar result for a more general spectral sequence, i.e. the Segal one (see [11]), which does not require that the space X is homotopic to a finite CW-complex. Finally, it is reasonable that this result extends to the homological version of Atiyah-Hirzebruch spectral sequence, considering that in homology Thom isomorphism is defined via a generalization of the cap product and the Gysin map is a pull-back.

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