ON CLASSIFICATION AND CONSTRUCTION OF ALGEBRAIC FROBENIUS MANIFOLDS

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Abstract. We develop the theory of generalized bi-Hamiltonian reduction. Applying this theory to the loop algebra proved to be equivalent to a generalized Drinfeld-Sokolov reduction. This gives a way to construct new examples of algebraic Frobenius manifolds.

Contents

1. Introduction

This work was intended as an attempt to prove the Dubrovin conjecture [15].

The conjecture: Massive irreducible algebraic Frobenius manifolds with positive degrees d_i correspond to primitive conjugacy classes in Coxeter groups.

A **Frobenius manifold** is a manifold M with the structure of Frobenuis algebra on the tangent space T_t at any point $t \in M$ with certain compatibility conditions [15]. We say M is **massive** if T_t is semisimple for generic t. In general this structure locally corresponds to a potential $F(t^1, ..., t^n)$

2000 *Mathematics Subject Classification.* Primary 37K10; Secondary 35D45.

Key words and phrases. Integrable systems, bi-Hamiltonian manifolds, Frobenius manifolds.

satisfying the WDVV equations and the quasihomogeneity

$$\begin{aligned} c_{ijk}(t)\eta^{kp}c_{pqr}(t) &= c_{rjk}(t)\eta^{kp}c_{pqi}(t) \\ \sum_{i=1}^{n} d_i t_i \partial_i F(t) &= (3-d)\, F(t) \end{aligned} \tag{1.1}$$

where $c_{ijk}(t) = \partial_i\partial_j\partial_k F(t)$ and $\eta_{ij}^{-1} = \partial_n\partial_i\partial_j F(t)$ is a constant matrix, $d_n = 1$. If $F(t)$ is an algebraic function we call M an **algebraic Frobenius manifold**.

Hertling [20] proved that any irreducible massive polynomial Frobenius manifold with positive degrees d_i is isomorphic to the Frobenius structure defined by Dubrovin on the orbit spaces of a Coxeter group [12]. The next step in the classification problem begins from the observation in [13] that algebraic solutions to equations of the isomonodromic deformation of an algebraic Frobenius manifold correspond to finite orbits of the braid group action on tuples of reflection. Then Stefanov [26] proved this finite orbits correspond to primitive conjugacy classes in Coxeter groups, this result was proved independently in [23]. The problem of existence of this algebraic Frobenius manifolds remains open.

Our main idea is to use the theory of infinite dimensional bi-Hamiltonian manifolds to construct all massive algebraic Frobenius manifolds. **A bi-Hamiltonian manifold** is a manifold endowed with two Poisson tensors P_1 and P_2 such that $P_\lambda = P_2 + \lambda P_1$ is a Poisson tensor for any constant λ. The dispersionless limit of a bi-Hamiltonian structure on the loop space $\mathfrak{L}(M)$ of finite dimensional manifold M always gives a bi-Hamiltonian structure of hydrodynamic type:

$$\{t^i(x), t^j(y)\}_{1,2} = g_{1,2}^{ij}(t(x))\delta'(x-y) + \Gamma_{1,2;k}^{ij}(t(x))t_x^k\delta(x-y), \tag{1.2}$$

defined on the loop space $\mathfrak{L}(M)$. This in turn gives a flat pencil of metrics $g_{1,2}^{ij}$ on M which under some assumptions corresponds to a Frobenius structure on M [14].

The paper is divided into three parts. In the next section we develop the theory of generalized bi-Hamiltonian reduction. The idea goes back to [6] where a bi-Hamiltonian reduction is given for every bi-Hamiltonian manifold, using Marsden-Ratiu theorem, by taking a level surface S of all the Casimirs of the first Poisson brackets and a distribution D defined by the second Poisson bracket.

In section 3 we apply the generalized bi-Hamiltonian reduction to Lie-Poisson bracket of a loop algebra of simple Lie algebra. We give a general bi-hamiltonian reduction for any nilpotent element e with an associated good grading (in the sense of [18]). Section (3.1) contains a brief summary of the theory of nilpotent elements and gradings on a simple Lie algebra and we set up notations and terminology. In section 3.4 we indicate how the bi-Hamiltonian reduction associated with a nilpotent element may be used to obtain the primary fields of classical W-algebras first found in [1] by using the Drinfeld-Sokolov reduction. In [4] they construct the classical W_n-algebra of the Lie algebra of type A_n by studying the relation between bi-Hamiltonian and Drinfeld-Sokolov reductions. Our method is more straightforward and does not depend on a particular type of the Lie algebra.

Section 4 provides a detailed exposition of a generalized Drinfeld-Sokolov reduction which is a special case of the more general Drinfeld-Sokolov reduction scheme given in [19]. In section 4.1 we establish our main theorem, the equivalence between the generalized bi-Hamiltonian and generalized Drinfeld-Sokolov reductions. The equivalence was obtained by Pedroni [25] in the special case of the reduction associated with the principal nilpotent element.

Finally section 5 is devoted to our main aim which is constructing algebraic Frobenius manifolds. We apply bi-Hamiltonian reduction to distinguished nilpotent elements in the Lie algebra of type F_4. This gives rise to four algebraic Frobenius manifolds in agreement with the Dubrovin's conjecture.

2. Bi-Hamiltonian Reduction and Transversal Manifolds

A bi-Hamiltonian manifold M is a manifold endowed with two Poisson tensors P_1 and P_2 such that $P_\lambda = P_2 + \lambda P_1$ is a Poisson tensor for any constant λ. The Jacobi identity for P_λ gives the relation

$$\begin{aligned}\{\{F,G\}_1,H\}_2+\{\{G,H\}_1,F\}_2+\{\{H,F\}_1,G\}_2+\\ \{\{F,G\}_2,H\}_1+\{\{G,H\}_2,F\}_1+\{\{H,F\}_2,G\}_1=0\end{aligned} \tag{2.1}$$

for any functions F, G and H on M. The main implication of this identity is that the set of Casimirs of P_1 is a Lie algebra with respect to P_2. Our basic assumption is the following. There is a set

$$\Xi = \{K_1, K_2, ..., K_n\} \tag{2.2}$$

of independent Casimirs of P_1 (n is not necessary equal to the corank of P_1) closed with respect to P_2. Let us denote by S a level set of Ξ and define the integrable distribution D on M generated by the Hamiltonian vector fields

$$X_{K_i} = P_2(dK_i), \qquad i = 1, ..., n. \tag{2.3}$$

The following lemma makes it legitimate to apply the Marsden-Ratiu theorem.

Lemma 2.1. *For any constant* λ

1. *The functions which are constant along* D *form a Lie subalgebra with respect to* P_λ.
2. $v \in D^0$ *if and only if* $P_\lambda(v) \in TS$. *Here* $D^0 \subset T^*M$ *is the annihilator of* D.

Proof. The first condition is easily deduced from the relation (2.1) and Jacobi identity for P_2. Since $P_1(T^*M) \subset TS$ the statement (2) is equivalent to proving that

$$v \in D^0 \text{ if and only if } P_\lambda(v) \in TS.$$

To this end, let $v \in D^0$. Then

$$\begin{aligned} (v, D) = 0 &\iff (v, P_2(dK_i)) = 0,\ i = 1, ..., n \\ &\iff (P_2(v), dK_i) = 0,\ i = 1, ..., n \\ &\iff P_2(v) \in TS, \end{aligned} \tag{2.4}$$

and the proof is complete. □

In the remainder of this section we assume there is a submanifold $Q \subset S$ transversal to $E = D \cap TS$, i.e

$$T_qS = E_q \oplus T_qQ, \quad \text{for all } q \in Q. \tag{2.5}$$

Following [6], Q has a natural bi-Hamiltonian structure P_1^Q, P_2^Q from P_1, P_2 respectively. Let $i : Q \hookrightarrow M$ be the canonical immersion. Then the pencil P_λ^Q is defined, for any functions f, g on Q, by

$$\{f, g\}_\lambda^Q = \{F, G\} \circ i \tag{2.6}$$

where F, G are functions on M extending f, g and constant along D.

Our next purpose is to find a way to write the reduced Poisson pencil tensor. Here the advantage of having a transversal manifold Q becames clear.

Lemma 2.2. *For any $q \in Q$ and $w \in T_q^*Q$ there exists $v \in T_q^*M$ such that:*

1. *v is an extension of w, i.e $(v, \dot{q}) = (w, \dot{q})$ for any $\dot{q} \in T_qQ$.*
2. *$P_\lambda(v) \in T_qQ$, i.e $(v, P_\lambda(TQ)^0) = 0$.*

Then the Poisson tensor $P_\lambda^Q(w)$ is given by

$$P_\lambda^Q w = P_\lambda v \tag{2.7}$$

for any extension v satisfying conditions (1) and (2).

Proof. $w \in T_q^*Q$ has an extension $v \in T_qM$ satisfying (2) if $T_qQ \cap P_\lambda(TQ)^0 = 0$. Assume $\dot{q} \in T_qQ \cap P_\lambda(TQ)^0$. Then $\dot{q} = P_\lambda(r)$ for $r \in (TQ)^0$. Since $P_\lambda(r) \in T_qS$ by lemma (2.1), $r \in D^0$. Then

$$r \in (TQ)^0 \cap D^0 \subset (TQ + D)^0 \subset (TS)^0. \tag{2.8}$$

This implies $P_2(r) \in D$ which gives $\dot{q} \in E$. But $\dot{q} \in T_qQ$. This contradiction proves the first part. Let $v_1, v_2 \in T_q^*M$ be extensions of $w_1, w_2 \in T_q^*Q$ satisfying the condition (2). Then

$$\begin{aligned} (w_1, P_\lambda^Q w_2) &= (v_1, P_\lambda v_2) \\ &= (w_1, P_\lambda v_2) \end{aligned} \tag{2.9}$$

where the first equality is obtained by definition and the second one follows from condition (2). □

We will see in the examples that the Poisson tensor P_λ on the manifold Q takes the form of a Lax operator, i.e considering λ as spectral parameter and studying the set of covectors $\{w \in T_q^*M \ : \ P_\lambda(w) \in TQ\}$ leads to integrable hierarchies on Q. This examples include AKNS hierarchies or Zakarov-Shabat hierarchies and the generalized Drinfeld-Sokolov hierarchy (see [11], [10] and [9] resp.).

3. Examples for Lie Poisson brackets

3.1. **Nilpotent elements and gradings in Lie algebras.**

Here we introduce some notations and a basic facts from the theory of nilpotent elements in simple Lie algebras.

Let $\mathfrak{g}$ be a simple Lie algebra over complex numbers with a nondegenerate invariant bilinear form $\langle .|.\rangle$. For a vector subspace $V \subset \mathfrak{g}$ we denote by $V^\perp$ its orthogonal complement and by $\mathfrak{L}(V)$ its loop space, i.e the space of smooth maps from the circle S^1 to V.

Introduce the following bilinear form on the loop algebra $\mathfrak{L}(\mathfrak{g})$:

$$(u|v) = \int_{s^1} \langle u(x)|v(x)\rangle dx, \ \ u,v \in \mathfrak{L}(M). \tag{3.1}$$

Then identify $(\mathfrak{L}(\mathfrak{g}))^*$ with $\mathfrak{L}(\mathfrak{g})$ using this bilinear form. For a functional F on $\mathfrak{L}(\mathfrak{g})$ we define the gradient $\delta H(q)$ to be the unique element in $\mathfrak{L}(\mathfrak{g})$ such that

$$\frac{d}{d\theta}F(q+\theta\dot{s}) \mid_{\theta=0} = \int_{s^1} \langle \delta F|\dot{s}\rangle \text{ for all } \dot{s} \in \mathfrak{L}(\mathfrak{g}). \tag{3.2}$$

We introduce the following Poisson tensors

$$\begin{aligned} P_2(v) &= v_x + [q,v] \\ P_1(v) &= [a,v] \end{aligned} \tag{3.3}$$

given at a point $q \in \mathfrak{L}(\mathfrak{g})$ and for every $v \in (\mathfrak{L}(\mathfrak{g}))^* \simeq \mathfrak{L}(\mathfrak{g})$, here $a \in \mathfrak{g}$ is constant element. It is well known that the pair in (3.3) defines a bi-Hamiltonian structure on $\mathfrak{L}(\mathfrak{g})$ [22]. Our first examples of bi-Hamiltonian reduction will be constructed from (3.3) by choosing an appropriate element $a \in \mathfrak{g}$ and a set of Casimirs of P_1.

Let

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i. \tag{3.4}$$

be a $\mathbb{Z}$-grading on $\mathfrak{g}$, i.e $[g_i, g_j] \subset \mathfrak{g}_{i+j}$ (We will omit the letter $\mathbb{Z}$ since any grading considered here is a $\mathbb{Z}$-grading). Since all derivations of $\mathfrak{g}$ are inner this grading is defined as eigenspaces of ad $\widetilde{h}$ for some element $\widetilde{h}$.

$$\mathfrak{g}_i = \{a \in \mathfrak{g} | \mathrm{ad} h(a) = ia\}. \tag{3.5}$$

Hence $\widetilde{h}$ is a semisimple.

An element $e \in \mathfrak{g}_2$ is called *good* if it satisfies the following condition

$$\text{ad } e : \mathfrak{g}_j \to \mathfrak{g}_{j+2} \text{ is injective for } j \leq -1. \tag{3.6}$$

A grading is called *good* if it admits a good element. All good gradings on simple Lie algebras up to conjugation are classified in [18].

Notation 3.1. *For any good grading we introduce the following subalgebras* $\mathfrak{b}^- = \bigoplus_{i\leq 0} \mathfrak{g}_i$, $\mathfrak{n}^- = \bigoplus_{i\leq -1} \mathfrak{g}_i$, $\mathfrak{g}^- = \bigoplus_{i\leq -2} \mathfrak{g}_i$ *and* $\mathfrak{n}^+ = \bigoplus_{i\geq 1} \mathfrak{g}_i$.

A subspace $\mathcal{C} \subset \mathfrak{g}$ is called a *transversal subspace* of e if

$$\mathrm{ad}e(\mathfrak{g}^-) \oplus \mathcal{C} = \mathfrak{b}^-. \tag{3.7}$$

Let $e \in \mathfrak{g}$ be an arbitrary nilpotent element. By Jacobson-Morozov theorem there exist h and $f \in \mathfrak{g}$ such that $\{e, h, f\}$ is an sl_2-triple, where

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{3.8}$$

From representation theory of sl_2 it is easy to see that h defines a grading on $\mathfrak{g}$ with e a good element. We call the grading thus obtained the Dynkin grading. It turns out that two nilpotent elements are conjugate if and only if they have the same Dynkin grading. A nilpotent orbit is the conjugacy class of a nilpotent element under the action of the adjoint group. See [8] for more information and the classification tables of the nilpotent orbits which is given in the form of weighted Dynkin diagrams.

Example 3.2. The principal nilpotent orbit in $\mathfrak{g}$ is the unique nilpotent orbit of codimension $r(=\text{rank } \mathfrak{g})$. Any representative e of this nilpotent orbit is regular, i.e the centralizer of e in $\mathfrak{g}$ is abelain and of dimension r. The Dynkin grading is the only good grading associated to e.

A nilpotent element is called *distinguished* iff $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_2)$ in the Dynkin grading associated to e. It follows then that $\mathfrak{g}_1 = 0$ [18].

Throughout the paper we assume all gradings are good with a fixed good element denoted by e.

3.2. **Bi-Hamiltonian reduction for a semisimple element.** We take on $\mathfrak{L}(\mathfrak{g})$ the bi-Hamiltonian structure (3.3) with $a \in \mathfrak{g}$ a semisimple regular element. Let $\mathfrak{h}$ be Cartan subalgebra containing a and

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \tag{3.9}$$

is a root space decomposition of $\mathfrak{g}$. The set of all Casimirs Ξ of P_1 corresponds to

$$\ker P_1 = \mathfrak{L}(\mathfrak{g}_a) := \{v \in \mathfrak{L}(\mathfrak{g}) : [v, a] = 0\} = \mathfrak{L}(\mathfrak{h}). \tag{3.10}$$

Thus Ξ forms an abelain Lie algebra under P_2. Consider the level set $S := \mathfrak{L}(\mathcal{C}) = \mathfrak{h}^\perp$ where

$$\mathcal{C} = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha. \tag{3.11}$$

The distribution D on $\mathfrak{L}(\mathfrak{g})$ defined by P_2 is the set

$$D_q := \{h_x + [q, h] \text{ where } h \in \mathfrak{L}(\mathfrak{h})\}. \tag{3.12}$$

It follows that $E := TS \cap D = P_2(\mathfrak{h})$ is a distribution on S. We choose the space $\mathfrak{L}(\mathcal{C})$ itself as the transversal space of E. According to lemma 2.2 the reduced Poisson tensors are obtained with the help of covectors $w \in \mathfrak{L}(\mathfrak{g})$ such that

$$w_x + [q, w] + \lambda[a, w] \in \mathfrak{L}(\mathcal{C}). \tag{3.13}$$

Note that the integrable hierarchy on $\mathfrak{L}(\mathcal{C})$ is given in terms of solutions to the equation

$$w_x + [q, w] + \lambda[a, w] = 0 \tag{3.14}$$

as functions of the spectral parameter λ (see e.g [10],[9] and [11]).

3.3. **Bi-Hamiltonian reduction for a nilpotent element.** Take on $\mathfrak{L}(\mathfrak{g})$ the Poisson pencil (3.3) with $a \in \mathcal{C}$ a homogenous element of minimal degree. Let Ξ be a subset of the set of Casimirs of P_1 corresponding to $\mathfrak{L}(\mathfrak{n}^-) \subset \operatorname{Ker} P_1$. Since $\mathfrak{n}^-$ is a Lie subalgebra, it is easy to verify that Ξ is closed under P_2. Following the bi-Hamiltonian reduction we take as a level surface the affine space

$$S := \mathfrak{b}^- + e. \tag{3.15}$$

The following proposition gives a nice Lie algebra theoretic meaning to the distribution E on S which is defined by

$$E := P_2(\mathfrak{L}(\mathfrak{n}^-)) \cap \mathfrak{L}(\mathfrak{b}^-). \tag{3.16}$$

Proposition 3.3.

$$E = P_2(\mathfrak{L}(\mathfrak{g}^-)). \tag{3.17}$$

Proof. From

$$E = P_2(\{v \in \mathfrak{L}(\mathfrak{n}^-) : v_x + [q, v] + [e, v] \in \mathfrak{L}(\mathfrak{b}^-) \text{ for } q \in \mathfrak{L}(\mathfrak{b}^-)\}) \tag{3.18}$$

and the gradation (3.4), it is obvious that $v \in E$ if and only if $[e, v] \in \mathfrak{L}(\mathfrak{b}^-)$. Since $v \in \mathfrak{L}(\mathfrak{n}^-)$ and ad e is injective we have $v \in \mathfrak{L}(\mathfrak{g}^-)$. ☐

Fix a transversal space $\mathcal{C}$ and define the submanifold

$$Q := e + \mathfrak{L}(\mathcal{C}) \tag{3.19}$$

of S.

Lemma 3.4. *The manifold Q is transversal to E on S.*

Proof. We must prove that at any point $q \in \mathfrak{L}(\mathcal{C})$ and $\dot{s} \in \mathfrak{L}(\mathfrak{b}^-)$ there is $v \in \mathfrak{L}(\mathfrak{g}^-)$ and $\dot{w} \in \mathfrak{L}(\mathcal{C})$ such that

$$\dot{s} = P_2(v) + \dot{w}. \tag{3.20}$$

We write this equation using the gradation (3.4) of $\mathfrak{g}$. We obtain

$$\dot{s}_i = v_i' + [e, v_{i-2}] + \dot{w}_i + \sum_k [q_k, v_{i-k}]. \tag{3.21}$$

Then for $i = 0$ we have

$$\dot{s_0} = [e, v_{-2}] + \dot{w_0} \tag{3.22}$$

which can be solved uniquely since

$$\mathfrak{L}(\mathcal{C}) \oplus [e, \mathfrak{L}(\mathfrak{g}^-)] = \mathfrak{L}(\mathfrak{b}^-). \tag{3.23}$$

Inductively in this way for $i < 0$ we obtain a recursive relation to determine v and $\dot{s}$ uniquely. □

Let us explain the procedure of finding the reduced Poisson pencil following [7]. We first choose a basis $\xi_1, ..., \xi_n$ for $\mathfrak{g}$ with $\xi_1, ..., \xi_m$ a basis for $\mathcal{C}$ for $m < n$. Let $\xi_1^*, ..., \xi_n^* \in \mathfrak{g}$ be a dual basis satisfying $\langle \xi_i | \xi_j^* \rangle = \delta_{ij}$. Then a point in the space Q will have the form $q = q^i \xi_i + e$. For a covector $w = (w_1, ..., w_m) \in T_q^* Q$ a lift $v \in T_q^* \mathfrak{L}(\mathfrak{g})$ satisfies the first condition in (2.2) if and only if

$$\langle \xi_i | v \rangle = w_i, \qquad i = 1, ..., m. \tag{3.24}$$

From lemma (2.1) the second condition gives the constrain

$$P_\lambda(V) \in \mathfrak{L}(\mathcal{C}). \tag{3.25}$$

Using the grading we can prove this lift is unique. Then the Poisson pencil P_λ^Q is given by

$$\dot{q^i} := \langle P_\lambda(v) | \xi_i^* \rangle. \tag{3.26}$$

Its independence from the choice of a basis follows From lemma 2.2.

Example 3.5. (**Fractional KdV**) Consider $\mathfrak{g} = sl_3$ with its standard representation. We denote by $e_{i,j}$ the fundamental matrix defined by $(e_{i,j})_{s,t} = \delta_{i,s}\delta_{j,t}$. Take the minimal nilpotent element $e := e_{1,3}$. It is a good element for the grading defined by

$$\widetilde{h} := \frac{4}{3} e_{1,1} - \frac{2}{3} e_{2,2} - \frac{2}{3} e_{3,3} \tag{3.27}$$

Take the Poisson tensors (3.3) with $a = e_{2,1}$. Here $\mathfrak{n}^-$ is generated by $\{e_{2,1}, e_{3,1}\}$ and a point $b \in S$ will have the form

$$b = \begin{pmatrix} * & 0 & 1 \\ * & * & * \\ * & * & * \end{pmatrix}. \tag{3.28}$$

We define a transversal space $\mathcal{C}$ such that a point $q \in Q$ takes the form

$$q = \begin{pmatrix} (\alpha - \beta) q_1 & 0 & 1 \\ q_2 & -\alpha q_1 & 0 \\ q_4 & q_3 & \beta q_1 \end{pmatrix} \tag{3.29}$$

for arbitrary $q_1, ..., q_4$ and nonzero constants α, β. Then the reduced Poisson pencil P_λ^Q has the following form

$$\begin{aligned}
\{q_1(x), q_1(y)\}_\lambda &= \frac{2\,\delta'(x-y)}{3\,\alpha^2}\\
\{q_1(x), q_2(y)\}_\lambda &= -\frac{(\lambda + q_2(x))\,\delta(x-y)}{\alpha}\\
\{q_1(x), q_3(y)\}_\lambda &= \frac{q_3(x)\delta(x-y)}{\alpha}\\
\{q_1(x), q_4(y)\}_\lambda &= -\frac{(\alpha-2\,\beta)^2\,{q_1}'\delta(x-y)}{3\,\alpha^2} - \frac{(\alpha-2\,\beta)^2\,q_1(x)\,\delta'(x-y)}{3\,\alpha^2}\\
&\quad -\frac{(\alpha-2\,\beta)\;\delta''(x-y)}{3\,\alpha^2}\\
\{q_2(x), q_3(y)\}_\lambda &= \left(\left(2\,\alpha^2+\alpha\,\beta-\beta^2\right)\,{q_1}^2(x) - q_4(x) - (\alpha+\beta)\;{q_1}'\right)\delta(x-y)\\
&\quad -3\,\alpha\,q_1(x)\,\delta'(x-y) + \delta''(x-y)\\
\{q_2(x), q_4(y)\}_\lambda &= -\frac{\left(2\,\left(\alpha^2-\alpha\,\beta+\beta^2\right)\,q_1(x)\;(\lambda+q_2(x)) - \alpha\,{q_2}'\right)\delta(x-y)}{\alpha}\\
&\quad +\frac{(\alpha+\beta)\;(\lambda+q_2(x))\;\delta'(x-y)}{\alpha}\\
\{q_3(x), q_4(y)\}_\lambda &= \frac{\left(2\,\left(\alpha^2-\alpha\,\beta+\beta^2\right)\,q_1(x)\,q_3(x) + \alpha\,{q_3}'\right)\delta(x-y)}{\alpha}\\
&\quad \frac{(2\,\alpha-\beta)\;q_3(x)\,\delta'(x-y)}{\alpha}\\
\\
\{q_4(x), q_4(y)\}_\lambda &= \frac{(2\,(\alpha-2\,\beta)^2\,\left(\alpha^2-\alpha\,\beta+\beta^2\right)\,q_1(x)\,{q_1}')\delta(x-y)}{3\alpha^2}\\
&\quad +\frac{(3\,\alpha^2\,{q_4}' - 2\,\left(\alpha^3-3\,\alpha^2\,\beta+3\,\alpha\,\beta^2-2\,\beta^3\right)\,{q_1}'')\delta(x-y)}{3\alpha^2}\\
&\quad +2\frac{\left(3\,\alpha^2\,q_4(x) - 2\,\left(\alpha^3-3\,\alpha^2\,\beta+3\,\alpha\,\beta^2-2\,\beta^3\right)\,{q_1}'\right)\delta'(x-y)}{3\,\alpha^2}\\
&\quad +2\frac{(\alpha-2\,\beta)^2\,\left(\alpha^2-\alpha\,\beta+\beta^2\right)\,{q_1}^2(x)\,\delta'(x-y)}{3\,\alpha^2}\\
&\quad -2\frac{\left(\alpha^2-\alpha\,\beta+\beta^2\right)\,\delta^{(3)}(x-y)}{3\,\alpha^2}.
\end{aligned}$$

The vector field defined by a covector $w \in T_q^*Q$ is written in the form

$$\dot{q}_\lambda = [v, L] \tag{3.30}$$

where v is an extension of w and L is the matrix operator

$$L = \partial_x + \begin{pmatrix} (\alpha-\beta)q_1 & 0 & 1\\ q_2 & -\alpha q_1 & 0\\ q_4 & q_3 & \beta q_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1\\ \lambda & 0 & 0\\ 0 & 0 & 0 \end{pmatrix} \tag{3.31}$$

In the case $\alpha = 2\beta$, $q_4(x)$ is a Virasoro density, i.e

$$\{q_4(x), q_4(y)\}_2 = 2\, q_4(x)\, \delta'(x-y) + \delta(x-y)\, {q_4}' - \frac{\delta^{(3)}(x-y)}{2} \tag{3.32}$$

and the second Poisson bracket is the W_3^2-algebra (see e.g. [9]).

Remark 3.6. Perform the bi-Hamiltonian reduction on sl_3 by taking the symplectic leaf of P_1 defined by setting $a = e_{2,1} + e_{2,3}$ in (3.3) and fixing the transversal manifold to have the form (3.29). The reduced second Poisson tensor on this manifold is equal to the one of the example above, (see e.g. [5]). The form of the operator L will change to

$$L := \partial_x + \begin{pmatrix} (\alpha-\beta)q_1 & 0 & 1 \\ q_2 & -\alpha q_1 & 0 \\ q_4 & q_3 & \beta q_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix} \tag{3.33}$$

which is the Lax operator considered in [2] to obtain integrable hierarchy associated to W_3^2-algebra.

3.4. **Classical W-Algebras from bi-Hamiltonian reduction.**

Classical W-algebras will be obtained for the principal nilpotent element with an appropriate choice of a basis and a transversal subspace constructed using representation theory of sl_2-algebra.

Let e be a principal nilpotent element and $\{h, e, f\}$ is the associated sl_2-triple. We denote by $A \subset \mathfrak{g}$ the subalgebra generated by this triple. Then we have a decomposition of $\mathfrak{g}$ as irreducible A-submodules:

$$\mathfrak{g} = A \oplus \bigoplus_{\alpha=1}^{m} V_\alpha. \tag{3.34}$$

Let $n_\alpha + 1$ be the dimension of V_α. Fix a basis in V_α

$$X_\alpha^j, \qquad j = 0, ..., n_\alpha,\ \alpha = 1, ..., m. \tag{3.35}$$

From the representation theory of sl_2 these vectors satisfy the following commutation relations

$$\begin{aligned} [h, X_\alpha^j] &= (n_\alpha - 2j) X_\alpha^j \\ [f, X_\alpha^j] &= (j+1) X_\alpha^{j+1} \\ [e, X_\alpha^j] &= (n_\alpha - j + 1) X_\alpha^{j-1}. \end{aligned} \tag{3.36}$$

It is easy to prove that $\mathcal{C} =$ Ker(ad f) is a transversal subspace associated to e. We will apply the generalized bi-Hamiltonian reduction using this transversal subspace. Let

$$v := v_j^\alpha X_\alpha^j + v_h h + v_e e + v_f f \tag{3.37}$$

be a general covector in $\mathfrak{L}(\mathfrak{g})^*$ and

$$q := q^\alpha X_\alpha^{n_\alpha} + q_f f + e \tag{3.38}$$

a point in Q. Using (3.36) the second Poisson tensor at $q \in Q$ reads

$$\begin{aligned} P_2(v) &= [\frac{d}{dx} + q^\alpha X_\alpha^{n_\alpha} + q_f f + e, v_j^\alpha X_\alpha^j + v_h h + v_e e + v_f\ f] \\ &= \Psi + (v_j^\alpha)_x\ X_\alpha^j + (v_h)_x\ h + (v_e)_x\ e + (v_f)_x\ f + \\ &\quad n_\alpha\ q^\alpha\ v_h\ X_\alpha^{n_\alpha} - q^\alpha\ v_e\ X_\alpha^{n_\alpha - 1} + (j+1)\ q_f\ v_j^\alpha\ X_\alpha^{j+1} + 2v_h\ q_f\ f \\ &\quad -q_f\ v_e\ h + (n_\alpha - j + 1)\ v_j^\alpha\ X_\alpha^{j-1} - 2v_h\ e + v_f\ h \end{aligned} \tag{3.39}$$

where

$$\Psi = q^\alpha v_j^\lambda [X_\alpha^{n_\alpha}, X_\lambda^j]. \tag{3.40}$$

To find the reduced Poisson tensor P_2^Q one must solve the recursion relations equating to zero the coefficients of X_α^j , for $j = 0, ..., n_\alpha - 1$ and of e and h (using the procedure explained after lemma 3.4).

Proposition 3.7. *The brackets with q_f will be given as follows*

$$\begin{aligned} \{q_f(x), q_f(y)\} &= -c_1\big(\frac{1}{2}\delta'''(x-y) + 2q_f(x)\delta'(x-y) \\ &\quad +(q_f)_x \delta(x-y)\big) \end{aligned} \tag{3.41}$$

$$\{q^\alpha(x), q_f(y)\} = c^\alpha\big(q_x^\alpha \delta(x-y) + \frac{(n_\alpha+2)}{2}\ q^\alpha(x)\delta'(x-y)\big), \tag{3.42}$$

where c_1 and c^α are some constants depending on the the choice of the basis (they are unique up to multiplication of X_α^0 by a nonzero constant).

Proof. The main idea of the proof is to study the contribution of v_e and its derivatives on the solutions of (3.39). This gives the Poisson brackets with $q_f(x)$. First we put $v_0^\alpha = 0,\ \alpha = 1, \dots, m$. It follows easily from Dynkin grading that

$$v_i^\alpha = 0,\ i = 1, ..., \frac{n_\alpha}{2},\ \forall \alpha$$

(recall $n_\alpha\ \forall\alpha$ is even for principal nilpotent elements). It follows that the expansion of Ψ does not contain h, f or e. Therefore equating the coefficient of e to zero we have

$$v_h = 1/2(v_e)_x \tag{3.43}$$

and the coefficient of h gives

$$v_f = v_e q_f - 1/2(v_e)_{xx} \tag{3.44}$$

and the coefficient of f reads

$$\begin{aligned} (v_f)_x &+ 2v_h q_f = \\ -1/2(v_e)_{xxx} &+ 2(v_e)_x q_f + v_e (q_f)_x \end{aligned} \tag{3.45}$$

Observe that v_e appears explicitly only when equating the coefficient of $X_\alpha^{n_\alpha - 1}$ to zero which gives

$$v_{n_\alpha}^\alpha = q^\alpha v_e + \text{other terms} \tag{3.46}$$

The next step is using the fact that ker(ad f) is abelain subalgebra (since f is a principal nilpotent element) to rewrite

$$\Psi = q^\alpha v_j^\lambda [X_\alpha^{n_\alpha}, X_\lambda^j], \quad j \neq n_\lambda. \tag{3.47}$$

Thus solving the equation (3.39) recursively we have

$$v_j^\alpha = 0,\ j = \frac{n_\alpha}{2} + 1, \ldots, n_\alpha$$

Then we have

$$v_{n_\alpha}^\alpha = q^\alpha v_e \tag{3.48}$$

Finally the coefficient of $X_\alpha^{n_\alpha}$ leads to the expression

$$\begin{aligned}(v_{n_\alpha}^\alpha)_x + n_\alpha q^\alpha v_h = \\ q_x^\alpha v_e \quad + \quad \frac{(n_\alpha + 2)}{2} q^\alpha (v_e)_x\end{aligned} \tag{3.49}$$

We substitute $\delta^k(x-y)$ for $\partial_x^k(v_e)$, and the proof is complete. □

Thus we proved the reduced second Poisson brackets is classical W-algebra as defined in [1] where $q_f(x)$ is a Virasoro density and $q^\alpha(x)$ are primary fields of weights $\frac{c^\alpha}{2}(n_\alpha + 2)$.

In [1] they obtained the same brackets from the Drinfeld-Sokolov reduction associated to e and the transversal space $\mathcal{C}$. We will discuss the Drinfeld-Sokolov reduction and its relation with bi-Hamiltonain reduction in the next section.

Remark 3.8. In a similar manner one can obtain the Virasoro density using arbitrary distinguished nilpotent element e' (see also [2]). Our methods fail to produce the primary fields since the Poisson bracket with Virasoro density will depend on the structure constants of the transversal space $\mathcal{C}' = \ker(\text{ad } f')$ where $\{e', h', f'\}$ is the associated sl_2-triple.

4. DRINFELD-SOKOLOV REDUCTION

In this section we will recall briefly the Drinfeld-Sokolov reduction which is another procedure to obtain a bi-Hamiltonian manifold. In the next section we will show its equivalence to our bi-Hamiltonian reduction. We use the notations and terminology of section 3.1.

Let us denote by S the manifold consisting of operators of the form

$$L = \frac{d}{dx} + b + e \qquad \text{where } b \in \mathfrak{L}(\mathfrak{b}^-). \tag{4.1}$$

The adjoint group G^- of $\mathfrak{L}(\mathfrak{g}^-)$ acts on S by

$$(n, L) \to \exp(\text{ad } n)\ L \text{ for all } n \in \mathfrak{L}(\mathfrak{g}^-) \text{ and } L \in S. \tag{4.2}$$

Proposition 4.1. *For any operator $L \in S$ there is a unique element $s \in \mathfrak{L}(\mathfrak{g}^-)$ such that the operator $L^c = \exp ad\, sL$ has the form*

$$L^c := \frac{d}{dx} + q + e \tag{4.3}$$

where $q \in \mathfrak{L}(\mathcal{C})$. Hence q and s are differential polynomials in b. The entries of q are generators of the ring R of differential polynomials invariant under the action of the group G^- on S.

Proof. We write $L^c = \exp(\mathrm{ad}\ s)\ L$ in the grading associated to e. Then inductively for $i \le 0$ the equation has the form

$$b_i + [e, s_{i-2}] = ...$$

where the right-hand side is a differential expression in q and s of the degree greater than i. The result follows by noticing that

$$\mathcal{C}_i \oplus \mathrm{ad}e(\mathfrak{g}^-_{i-2}) = \mathfrak{b}^-_i .$$

□

From the above lemma we define the space $\widetilde{Q} := S/G^-$. The set $\mathcal{R}$ of functionals on $\widetilde{Q}$ can be realized as functionals on S which have densities in the ring R. Consider the space S as a subspace of $\mathfrak{g}$. Then for a functional H on S we define the gradient $\delta H(q)$ to be the unique element in $\mathfrak{L}(\mathfrak{b}^+)$ such that

$$\frac{d}{d\theta} H(q + \theta \dot{s}) \mid_{\theta=0} = \int_{s^1} \langle \delta H | \dot{s} \rangle \text{ for all } \dot{s} \in \mathfrak{L}(\mathfrak{b}^-) \tag{4.4}$$

and

$$\int_{s^1} \langle \delta H | \dot{s} \rangle = 0 \text{ for all } \dot{s} \in \mathfrak{L}(\mathfrak{n}^-). \tag{4.5}$$

We define on $\mathfrak{L}(\mathfrak{g})$ the Poisson pencil (3.3) with $a \in \mathfrak{g}$ a homogenous element of minimal degree. Then this Poisson pencil P_λ is reduced on $\widetilde{Q}$ using the following

Lemma 4.2. *$\mathcal{R}$ is closed subalgebra with respect to the Poisson pencil P_λ.*

Proof. Note that if

$$L = \frac{d}{dx} + q + e \in S \tag{4.6}$$

and

$$\widetilde{L} := \frac{d}{dx} + \widetilde{q} + e = \exp(\mathrm{ad}\ n)\ L \tag{4.7}$$

then for $F \in \mathcal{R}$ we have $\delta F(\widetilde{q}) = \exp(\mathrm{ad}\ (-n))\ \delta F(q)$. The proof is easily obtained by using any faithful matrix representation of $\mathfrak{g}$. The result follows by substituting into the bracket

$$\{F, H\}_\lambda(q) = \int_{s^1} \langle [\delta H, \delta F] | \frac{d}{dx} + q + \lambda a \rangle \tag{4.8}$$

and using the invariance of the bilinear form $\langle .|. \rangle$ under the adjoint action. □

4.1. Drinfeld-Sokolov and bi-Hamiltonian reductions.

In this section we will be mainly following the spirit of [25].

Let M be the space of operators of the form

$$Z = \frac{d}{dx} + q \qquad \text{where } q \in \mathfrak{L}(\mathfrak{g}). \tag{4.9}$$

The adjoint group N^- of $\mathfrak{L}(\mathfrak{n}^-)$ acts on M by

$$(n, Z) \to \exp(\text{ad } n)\ Z \text{ for all } n \in \mathfrak{n}^- \text{and } Z \in M. \tag{4.10}$$

Introduce on M the bihamiltonian structure (3.3) with $a \in \mathcal{C}$ an homogenous element of minimal degree.

Proposition 4.3. *The action of N^- on M with Poisson tensor P_λ is Hamiltonian for all λ. It admits a momentum map J to be the projection*

$$J : \mathfrak{g} \to \mathfrak{n}^+.$$

Moreover, J is Ad-equivariant.*

Proof. We consider a faithful matrix representation of $\mathfrak{g}$. Then the action on M has the form

$$\Psi_n : q \to nqn^{-1} - n_x n^{-1} \qquad q \in \mathfrak{L}(\mathfrak{g}), n \in N^-. \tag{4.11}$$

For $\xi \in \mathfrak{L}(\mathfrak{n}^-)$ the fundamental vector field will have the form

$$\begin{aligned} X_{-\xi} &= \frac{d}{dt}\big(\exp(-t\xi)\ q\ \exp(t\xi) - (\exp(-t\xi))_x \exp(t\xi)\big) \\ &= \xi_x + [q, \xi]. \end{aligned} \tag{4.12}$$

Define the functional

$$H_\xi(q) = \int \langle q, \xi \rangle = \int \langle J(q), \xi \rangle. \tag{4.13}$$

Then

$$P_\lambda \delta H_\xi = \xi_x + [\xi, q + \lambda a] = \xi_x + [\xi, q], \tag{4.14}$$

which proves the action is Hamiltonian. The momentum map is Ad*-equivariant iff

$$J(\Psi_n(q)) = \text{Ad}^*_n J(q).$$

Since the moment map is just the projection we have

$$\begin{aligned} J(\Psi_n(q)) &= J(nqn^{-1} - n_x n^{-1}) \\ &= J(nqn^{-1}) \\ &= J(nJ(q)n^{-1}) \\ &= \text{Ad}^*_n J(q) \end{aligned}$$

where the last equality follows from the fact that the coadjoint action of N^- on $\mathfrak{n}^+ \simeq \mathfrak{n}^{-*}$ is given by

$$\mathrm{Ad}_n^* v = J(nvn^{-1}), \qquad v \in \mathfrak{n}^{-*}. \tag{4.15}$$

□

We take the nilpotent element e as a regular value of J. Define the space

$$S := J^{-1}(e) = \frac{d}{dx} + \mathfrak{b}^- + e. \tag{4.16}$$

and let D denote the distribution defined by the group action

$$D := P_\lambda(\mathfrak{L}(\mathfrak{n}^-)) = P_2(\mathfrak{L}(\mathfrak{n}^-)). \tag{4.17}$$

Let $E := D \cap TS$. Then P_λ, S and D satisfy Marsden-Ratiu theorem [21].

Consider the space

$$\widetilde{Q} := S/G^- \simeq S/E \tag{4.18}$$

where $G^- \subset N^-$ is the isotropy subgroup of e under the action of N^-. From the properties of the grading G^- is the adjoint group of $\mathfrak{g}^-$. This obviously leads to Drinfeld-Sokolov reduction.

Define the space

$$Q := \frac{d}{dx} + \mathfrak{L}(\mathcal{C}) + e. \tag{4.19}$$

Then Q is transversal to the distribution $E := D \cap TS = P_2(\mathfrak{L}(\mathfrak{b}^-))$ on S by lemma 3.4 and 3.18 and we end with the generalized bi-Hamiltonian reduction.

Thus we have proved the following

Theorem 4.4. *The generalized Drinfeld-Sokolov and generalized bi-Hamiltonian reductions are equivalent.*

As we mentioned in the introduction, the special case of principal nilpotent element the equivalence is obtained in [25]. Using generalized bi-Hamiltonian reduction the proof is more simple even in this case.

5. Applications to Frobenius manifolds

Let M be a manifold with local coordinates $R = (R^1, ..., R^n)$. On the loop space $\mathfrak{L}(M)$ a local Poisson bracket can be written in the form [17]

$$\{R^i(x), R^j(y)\} = \sum_{k=-1}^{\infty} \epsilon^k \{R^i(x), R^j(y)\}^{[k]} \tag{5.1}$$

here ϵ are just a parameter and

$$\{R^i(x), R^j(y)\}^{[k]} = \sum_{s=0}^{k+1} A_{k,s}^{i,j} \delta^{(k-s+1)}(x-y), \tag{5.2}$$

where $A^{i,j}_{k,s}$ are homogenous polynomials in $\partial_x^m u^i(x),\ i=1,...,m,\ m\geq 0$, of degree s (we assign $\partial_x^m u^i(x)$ degree m). The first terms can be written as follows

$$\begin{aligned} \{R^i(x),R^j(y)\}^{[-1]} &= F^{ij}(R)\delta(x-y) && (5.3)\\ \{R^i(x),R^j(y)\}^{[0]} &= g^{ij}(R)\delta'(x-y)+\Gamma^{ij}_k(R)R^k_x\delta(x-y) && (5.4)\end{aligned}$$

where F^{ij}, g^{ij} and Γ^{ij}_k are smooth functions in R^i. The matrix F^{ij} defines a Poisson structure on M. Assume $F^{ij}=0$ then $\{R^i(x),R^j(y)\}^{[0]}$ defines a Poisson bracket on $\mathfrak{L}(M)$ known as Poisson bracket of Hydrodynamic type. By nondegenerate Poisson bracket of Hydrodynamic type we mean the metric g^{ij} is nondegenerate. In this case its inverse defines a flat metric on the space M and Γ^{ij}_k is the contravariant Levi Civita coefficients of g^{ij} [16]. If there is a bi-Hamiltonian structure P_λ on $\mathfrak{L}(M)$ such that $F^{ij}_{1;2}=0$ then g^{ij}_λ is called a flat pencil of metrics if $\det(g_\lambda)\neq 0,\ \forall\lambda$. Under some assumption of quasihomogeniuty and regularity a flat pencil of metrics is equivalent to a Frobenius structure on M [14]. In the notations of (1.1) from a Frobenuis structure on M the flat pencil of metric is found from the relations

$$\begin{aligned} \eta^{ij} &= g^{ij}_1 \\ g^{i,j}_2 &= (d-1+d_i+d_j)\eta^{i\alpha}\eta^{j\beta}\partial_\alpha\partial_\beta F \end{aligned} \qquad (5.5)$$

We apply generalized bi-Hamiltonian reduction to distinguished nilpotent elements of the Lie algebra F_4. Our aim is to obtain a Frobenius manifold from the reduced Poisson pencil. There is four distinguished nilpotent orbits on F_4 which yields four algebraic Frobenius manifolds. Two of them give a polynomial Frobenius manifolds isomorphic to Frobenius structure on the orbit spaces of Coxeter group of type F_4 and B_4. One of the remaining distinguished nilpotent orbits is likely to give algebraic Frobenius structure isomorphic to the one found in [24] by applying Drinfeld-Sokolov reduction on Lie algebra of type D_4. We end with one class of nilpotent elements which give a new algebraic Frobenius manifold.

Example 5.1. (Algebraic Frobenius manifold) Assume X_α with $H_\alpha\in\mathfrak{h}$, $\alpha\in\Psi$, form Weyl-Chevalley basis of F_4. For the following computations we use the minimal representation of F_4 of dimension 26 (see e.g [3]). We apply the generalized bi-Hamiltonian reduction with the nilpotent element

$$\begin{aligned} e \ :=\ & X_{\alpha_2}+X_{\alpha_1+\alpha_2}+X_{\alpha_2+\alpha_3}+X_{\alpha_1+\alpha_2+\alpha_3}+X_{\alpha_2+2\,\alpha_3}\\ &+X_{\alpha_1+\alpha_2+2\,\alpha_3}+X_{\alpha_4}+X_{\alpha_3+\alpha_4} \end{aligned} \qquad (5.6)$$

which corresponds to the nilpotent orbit $F_4(a_2)$ in the notations of [8]. We fix the associated Dynkin grading and define the first Poisson bracket with

$a = X_{-2\,\alpha_1-3\,\alpha_2-4\,\alpha_3-2\,\alpha_4}$. Define the transversal manifold Q to be of the form

$$
\begin{aligned}
q \;=\;& U_2\,X_{-\alpha_2-2\,\alpha_3} + U_3\,X_{-\alpha_1-\alpha_2-2\,\alpha_3} + U_1\,X_{-\alpha_1-\alpha_2-\alpha_3}\\
&+U_8\,X_{-2\,\alpha_1-3\,\alpha_2-4\,\alpha_3-2\,\alpha_4} + U_7\,X_{-\alpha_1-3\,\alpha_2-4\,\alpha_3-2\,\alpha_4}\\
&+U_5\,X_{-\alpha_1-\alpha_2-2\,\alpha_3-2\,\alpha_4} + U_4\,X_{-\alpha_1-\alpha_2-2\,\alpha_3-\alpha_4}\\
&+U_6\,X_{-\alpha_1-2\,\alpha_2-3\,\alpha_3-2\,\alpha_4}. \qquad (5.7)
\end{aligned}
$$

Write the reduced Poisson pencil in the notations of (5.2). Then the coefficient F_λ^{ij} of ϵ^{-1} does not vanish. Indeed, the coefficient F_1^{ij} of the first Poisson bracket reads

(5.8)

$$
F_1^{ij} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0\\
* & 0 & 0 & 0 & 0 & \frac{5}{72} & 0 & 0\\
* & * & 0 & 0 & 0 & -\frac{11}{72} & 0 & 0\\
* & * & * & 0 & -\frac{4}{5} & 0 & \frac{10\,U_1+23\,U_2+5\,U_3}{120} & \frac{-6\,U_1+43\,U_2+U_3}{120}\\
* & * & * & * & 0 & \frac{-17\,U_1-198\,U_2-18\,U_3}{180} & \frac{-U_4}{32} & \frac{-U_4}{160}\\
* & * & * & * & * & 0 & F_1^{6,7} & F_1^{6,8}\\
* & * & * & * & * & * & 0 & \frac{5\,U_1\,U_4-24\,U_2\,U_4}{1920}\\
* & * & * & * & * & * & * & 0
\end{pmatrix}
$$

where

$$
\begin{aligned}
F_1^{6,7} &= \frac{-15\,{U_1}^2 - 380\,U_1\,U_2 - 700\,{U_2}^2 - 60\,U_1\,U_3 - 328\,U_2\,U_3 - 60\,{U_3}^2 + 150\,U_5}{5760}\\
F_1^{6,8} &= \frac{69\,{U_1}^2 + 772\,U_1\,U_2 - 1356\,{U_2}^2 - 28\,U_1\,U_3 - 424\,U_2\,U_3 - 76\,{U_3}^2 + 110\,U_5}{5760}.
\end{aligned}
$$

Our next aim is to use Dirac reduction. For this we introduce the coordinates

$$
\begin{aligned}
W_1 &= \frac{U_1}{2} + U_2 + U_3\\
W_2 &= 11\,U_2 + 5\,U_3\\
W_3 &= \frac{U_1\,\left(-205\,{U_1}^2 + 1908\,U_1\,U_2 - 7416\,{U_2}^2 + 360\,\left(U_1 + 2\,U_2\right)\,U_3 + 360\,{U_3}^2\right)}{1080}\\
&\quad +\frac{5\,U_1\,U_5}{12} - 2\,U_2\,U_5 + U_7 - 5\,U_8
\end{aligned}
$$

$$\begin{aligned}
W_4 &= \frac{95\,U_1{}^3 - 849\,U_1{}^2\,U_2 + 3948\,U_1\,U_2{}^2 - 195\,U_1{}^2\,U_3 - 120\,U_1\,U_2\,U_3 - 180\,U_1\,U_3{}^2}{720}\\
&\quad + \frac{U_4{}^2}{32} + \left(\frac{-U_1}{12} + \frac{19\,U_2}{12} + \frac{U_3}{12}\right) U_5 + U_7 + 3\,U_8\\
W_5 &= U_1\\
W_6 &= U_4\\
W_7 &= U_5\\
W_8 &= U_6
\end{aligned}$$

where the first four are the Casimirs of F_1^{ij}. Following [24], it turns out that the Poisson pencil P_λ^Q can be reduced only along the submanifold N given by

$$\begin{aligned}
W_6 &= W_8 = 0\\
W_5 &= Z\\
W_7 &= \frac{-5\,Z^2 + 150\,Z\,W_1 - 100\,W_1{}^2 - 33\,Y\,W_2 + 64\,W_1\,W_2 - 7\,W_2{}^2}{150}
\end{aligned}$$

where $Z(W_1, ..., W_4)$ is a solution of a cubic equation to be given below. The leading term of the reduced Poisson pencil P_λ^N gives a nondegenerate Poisson bracket of Hydrodynamic type satisfying the quasihomogeneity and regularity conditions of [14]. In a flat coordinates (s_1, s_2, s_3, s_4) the Frobenius structure will have the potential

$$\begin{aligned}
\mathbb{F} &= \frac{9\,Z^2\,s_1{}^5}{44800} + \frac{3\,Z^2\,s_1{}^4\,s_2}{89600} - \frac{3\,Z^2\,s_1{}^3\,s_2{}^2}{89600} - \frac{3\,Z^2\,s_1{}^2\,s_2{}^3}{640000} + \frac{153\,Z^2\,s_1\,s_2{}^4}{89600000}\\
&\quad + \frac{1107\,Z^2\,s_2{}^5}{4480000000} + \frac{81\,Z^2\,s_1{}^2\,s_3}{2800} + \frac{243\,Z^2\,s_1\,s_2\,s_3}{14000} + \frac{729\,Z^2\,s_2{}^2\,s_3}{280000}\\
&\quad + \frac{409\,Z\,s_1{}^6}{2419200} - \frac{191\,Z\,s_1{}^5\,s_2}{1344000} + \frac{187\,Z\,s_1{}^4\,s_2{}^2}{5376000} + \frac{67\,Z\,s_1{}^3\,s_2{}^3}{13440000}\\
&\quad - \frac{319\,Z\,s_1{}^2\,s_2{}^4}{179200000} + \frac{529\,Z\,s_1\,s_2{}^5}{13440000000} + \frac{1247\,Z\,s_2{}^6}{38400000000} + \frac{27\,Z\,s_1{}^3\,s_3}{560}\\
&\quad - \frac{117\,Z\,s_1{}^2\,s_2\,s_3}{5600} + \frac{9\,Z\,s_1\,s_2{}^2\,s_3}{56000} + \frac{369\,Z\,s_2{}^3\,s_3}{560000} + \frac{243\,Z\,s_3{}^2}{70}\\
&\quad - \frac{29459\,s_1{}^6\,s_2}{580608000} + \frac{6089\,s_1{}^5\,s_2{}^2}{276480000} - \frac{254609\,s_1{}^4\,s_2{}^3}{34836480000} + \frac{152263\,s_1{}^3\,s_2{}^4}{116121600000}\\
&\quad - \frac{300457\,s_1{}^2\,s_2{}^5}{5806080000000} - \frac{1973651\,s_1\,s_2{}^6}{174182400000000} + \frac{292289\,s_2{}^7}{193536000000000} + \frac{17\,s_1{}^4\,s_3}{44800}\\
&\quad - \frac{2647\,s_1{}^3\,s_2\,s_3}{336000} + \frac{6059\,s_1{}^2\,s_2{}^2\,s_3}{2240000} - \frac{18223\,s_1\,s_2{}^3\,s_3}{33600000}\\
&\quad + \frac{11443\,s_1{}^7}{174182400} + \frac{60131\,s_2{}^4\,s_3}{1344000000} + \frac{s_1\,s_3{}^2}{20} + \frac{3\,s_2\,s_3{}^2}{200}\\
&\quad -2\,s_1\,s_3\,s_4 + \frac{3\,s_2\,s_3\,s_4}{5} + 2\,s_1\,s_4{}^2 \qquad (5.9)
\end{aligned}$$

where Z is a solution of the cubic equation

$$Z^3 - Z\left(\frac{{s_1}^2}{48} + \frac{s_1\,s_2}{80} + \frac{3\,{s_2}^2}{1600}\right) - \frac{{s_1}^3}{96} + \frac{13\,{s_1}^2\,s_2}{2880} - \frac{s_1\,{s_2}^2}{28800} - \frac{41\,{s_2}^3}{288000} - \frac{3\,s_3}{2} = 0 \tag{5.10}$$

It is straightforward to check validity of the WDVV equations for this potential. The quasihomogeneity reads

$$s_4\partial_1 F(s) + s_3\partial_3 F(s) + \frac{1}{3}s_2\partial_2 F(s) + \frac{1}{3}s_1\partial_1 F(s) = 2F(s). \tag{5.11}$$

In a subsequent publications we will consider further examples of Frobenius structures and integrable hierarchies on bi-Hamiltonian manifolds produced by applying the reduction methods introduced in this paper.

Acknowledgments.
The author is extremely grateful to his supervisor B.Dubrovin for posing him this problem, his constant encouragement and support. The author also thanks P. Lorenzoni for stimulating discussions. The present work is partially supported by the European Science Foundation Programme "Methods of Integrable Systems, Geometry, Applied Mathematics" (MISGAM), Marie Curie RTN "European Network in Geometry, Mathematical Physics and Applications" (ENIGMA). The work is also partially supported by Italian Ministry of Universities and Researches (MUR) research grant PRIN 2006 "Geometric methods in the theory of nonlinear waves and their applications". The author would like to thank "The Canon Foundation for Scientific Research" for supporting his travel to attend "Poisson 2006" conference.

REFERENCES

[1] Balog, J., Feher, L., O'Raifeartaigh, L., Forgacs, P., and Wipf, A., Toda theory and W algebra from a gauged WZNW point of view, Ann.Phys.203, 76 (1990).

[2] Burroughs, N., de Groot, M., Hollowood, T. and Miramontes, J., Generalized Drinfeld-Sokolov hierarchies II: the Hamiltonian structures, Comm. Math. Phys.153, 187 (1993).

[3] Carter, R., Lie algebras of finite and affine type. Cambridge University Press,ISBN 978-0-521-85138-1(2005).

[4] Casati, P., Falqui, G., Magri, F., Pedroni, M., Bi-Hamiltonian reductions and W_n-algebras, J. Geom. Phys. 26 no. 3-4, (1998).

[5] Casati, P., Falqui, G., Magri, F., Pedroni, M., A note on fractional KdV hierarchies, J. Math. Phys. 38 , no. 9, 4606 (1997).

[6] Casati, P., Magri, F., and Pedroni, M., bi-Hamiltonian Manifolds and τ-Function, Comm. Math. Phys.153.187(1993).

[7] Casati,P., Pedroni, M., Drinfeld-Sokolov Reduction on a Simple Lie Algebra from the Bihamiltonain Point of View, Lett. Math. Phys. 25 (1992).

[8] Collingwood, D., McGovern, W., Nilpotents Orbits in Semisimple Lie Algebras, Van Nostrand Reinhold, New York ISBN 0534188346,(1992).

[9] De Groot, M., Hollowood, T. and Miramontes, J., Generalized Drinfeld-Sokolov hierarchies, Comm. Math. Phys. 145 157 (1992).

[10] Drinfeld, V., Sokolov, V., Lie algebras and equations of Korteweg de Vries type, Jour. Sov. Math. 30, 1975 (1985).

[11] Dubrovin, B. A, Completely integrable Hamiltonian systems that are associated with matrix operators, and Abelian varieties, (Russian) Funkcional. Anal. i Priložen. 11 no. 4, 28 (1977).

[12] Dubrovin, B., Differential geometry of the space of orbits of a Coxeter group, Surveys in differential geometry, integral systems [integrable systems],181, ISBN 1-57146-066-7 (1998).

[13] Dubrovin, B.: Geometry of 2D topological field theories. Springer Lecture Notes in Math. **1620**, 120 (1996).

[14] Dubrovin, B., Flat pencils of metrics and Frobenius manifolds, Inegrable systems and algebraic geometry, World Sci. Publ., River Edge, NJ,ISBN 981-02-3266-7 (1998).

[15] Dubrovin, B., Painlev transcendents in two-dimensional topological field theory. The Painlev property, 287, ISBN 0-387-98888-2 (1999).

[16] Dubrovin, B., Novikov, S. , Poisson brackets of hydrodynamic type. (Russian) Dokl. Akad. Nauk SSSR 279 (1984), no. 2, 294.

[17] Dubrovin, B. , Zhang, Y., Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants, www.arxiv.org math/0108160.

[18] Elashvili, A., Kac, V., Classification of good gradings of simple Lie algebras. Lie groups and invariant theory, 85–104, Amer. Math. Soc. Transl. Ser. 2, 213 (2005).

[19] Fehr, L., O'Raifeartaigh, L., Ruelle, P., Tsutsui, I., Wipf, A., On Hamiltonian reductions of the Wess-Zumino-Novikov-Witten theories. Phys. Rep. 222 , no. 1, 64 (1992).

[20] Hertling, C., Frobenius manifolds and moduli spaces for singularities. Cambridge Tracts in Mathematics, Cambridge University Press, ISBN 0-521-81296-8 (2002).

[21] Marsden, J., Ratiu, T., Reductions on Poisson manifolds, Lett. Math. phys. 11, 161(1986).

[22] Marsden, J., Ratiu, T., Introduction to mechanics and symmetry, Texts in Applied Mathematics, 17. Springer-Verlag, New York, ISBN: 0-387-97275-7; (1994).

[23] Michel, J., Hurwitz action on tuples of Euclidean reflections. J. Algebra 295 , no. 1, 289–292 (2006).

[24] Pavlyk, O., Solutions to WDVV from generalized Drinfeld-Sokolov hierarchies, www.arxiv.org math-ph/0003020 (2003).

[25] Pedroni, M., Equivalence of Drinfeld-Sokolov Reduction to a Bi-Hamiltonian Reduction, Lett. Math. Phys. 35 (1995).

[26] Stefanov, A., Finite orbits of the braid group action on sets of reflections, www.arxiv.org math-ph/0409026 (2004).

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