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Stability of the (2+2)-fermionic system with zero-range interaction

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Abstract

We introduce a 3D model, and we study its stability, consisting of two distinct pairs of identical fermions coupled with a two-body interaction between fermions of different species, whose effective range is essentially zero (a so called (2+2)-fermionic system with zero-range interaction). The interaction is modelled by implementing the celebrated (and ubiquitous in the literature of this field) Bethe–Peierls contact condition with given two-body scattering length within the Krein–Višik–Birman theory of extensions of semi-bounded symmetric operators, in order to make the Hamiltonian a well-defined (self-adjoint) physical observable. After deriving the expression for the associated energy quadratic form, we show analytically and numerically that the energy of the model is bounded below, thus describing a stable system.

Keywords: point interactions, unitary gases, Bethe–Peierls contact condition, Ter-Martirosyan–Skornyakov operators, Krein–Višik–Birman theory of self-adjoint extensions, numerical minimization by basis truncation

(Some figures may appear in colour only in the online journal)

1. Motivation and background

In this work we discuss analytically and numerically a rigorous model in three dimensions for two identical fermions (atoms or nucleons) of one species coupled with two other identical

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fermions of another species, when the inter-species interaction range is zero. We shall refer to this kind of interaction also as a contact, point, or delta interaction, consistently with the common use in the literature.

While deferring to a later part of this introduction what exactly is meant here by 'rigorous', in comparison with other typical heuristic treatments, let us first discuss the physical context for this model and its relevance in a current, noticeably quite active, research mainstream.

Quantum systems in which the two-body inter-particle interaction has practically zero range and, possibly, very large or infinite scattering length have been under an intensive investigation in the recent years, both theoretically, numerically, and experimentally. Initially proposed by Bethe and Peierls [8, 9] and by Thomas [66] as theoretical models to solve the neutron–proton scattering in the approximation of very short interaction range, and subsequently developed by Fermi [31] and Breit [16] with the introduction of the so-called 'delta pseudo-potential', the interest towards them remained for some decades at the mere level of a formal first-order perturbation theory, at least within the English physical literature, whereas the subject flourished within the Russian physics and developed a literature that, albeit not fully rigorous, yet enlarged its applicability, from nuclear to atomic and molecular physics, mainly under the input of Faddeev. This includes, in particular: the work of Ter-Martirosyan and Skornyakov [65] on the celebrated equations, named after them, which model the physical heuristics for the behaviour of a wave-function when two particles with point interaction come on top of each other (which is in fact the momentum representation counterpart of the 'zero-range contact condition' of Bethe and Peierls); the work of Berezin and Faddeev [6] with the first rigorous analysis of a 3D two-particle system with point interaction; the work of Faddeev and Minlos [50, 51] with the first complete solution of the three-body problem in 3D with point interaction; a mainstream that ideally culminates with the late 1970s monograph of Demkov and Ostrovskii [27] on the 'zero-range potentials' and their application to atomic physics.

The subject remained active for some thirty more years on a theoretical and mathematical level also after the subsequent unification of the western and eastern research mainstreams, mainly due to the works of Albeverio *et al* in the 1980s [1], and produced an amount of results, some of which we shall quote and review in the following. The present renewed flurry of interest in condensed matter physics is due to the last decade's rapid progress in the manipulation techniques for ultra-cold atoms and, in particular, for tuning the effective *s*-wave scattering length by means of a magnetically induced Feshbach resonance [21]. From the experimental point of view, zero-range interactions are today far from being just an idealization of real-world two-body potentials with small support and in many realizations the zero-range, delta-like character of the interaction turns out to be an extremely realistic and in fact an unavoidable description. This is in particular the case in the so-called 'unitary regime' [20, 59, 71], i.e. the case of negligible interaction range and huge, virtually infinite, scattering length (both lengths being compared to a standard reference length such as the Bohr radius). In such a regime, unitary gases show properties, including superfluidity, with the remarkable feature of being universal in several respects [15], and are under active experimental and theoretical [34] investigation.

Hetheronuclear mixtures where the inter-species interaction can be effectively tuned by means of Feschbach resonances and can be modelled with zero-range interactions are today under current active investigation both experimentally and theoretically [13, 14, 24, 30, 33, 35, 38, 39, 53–55, 63, 64, 72, 73]. In this context, the 'building block' that one addresses first is the three-body system consisting of two identical particles of one species plus a third different particle, coupled with a point interaction. This '2+1' system is deceptively simple and its understanding is crucial for the comprehension of the many-body system, in particular

for what the stability is concerned. Indeed, an interaction supported only at the coincidence points between two particles may cause, under appropriate conditions, the collapse of the three-body system onto its barycenter known as the Thomas effect [66], owing to the occurrence of a sequence of energy levels accumulating at $-\infty$, a phenomenon that has clearly no counterpart in the case of ordinary potentials with finite interaction range. To make the picture even more striking, the Thomas effect turns out to depend on the masses and on the exchange symmetry of the particles constituting the three body system: it is proved theoretically to occur for three identical bosons [50, 51] or of three different particles [40, 41], and to be suppressed instead in a '2+1' system with two identical fermions if the third (different) particle is sufficiently heavy [43, 52, 62].

For these and other features, the three-body problem with point interaction, possibly at unitarity, is currently subject to an extensive investigation in the physical literature [3, 4, 15, 18–20, 29, 36, 38, 42, 53, 56, 69–71], which has been giving in turn new impulse and motivations to the already developing mathematical research on the subject [22, 23, 26, 32, 43, 45–49, 52, 62].

The next simplest structure with zero-range interaction, including the unitary regime, which one is naturally led to investigate is the four-body problem. This too is being thoroughly studied both physically [12, 17, 24, 28, 57, 58, 60, 68] and, to a somewhat lesser extent, also mathematically [22, 32, 45, 46]. With no pretension of surveying here this segment of the literature and all the most topical questions under study, it is worth mentioning that from the point of view of the *stability* of the system, the issue we are most concerned with in the present work, the four-body system is naturally expected to be unstable at least as long as one of its three-body subsystems is, which corresponds to a configuration in which the fourth particle is placed very far away from the unstable trimer. In view of the above-mentioned facts concerning the occurrence/non-occurrence of the Thomas effect in a three-body system [22, 40, 41, 43, 50–52, 62], if out of the four particles two are identical fermions, then the stability issue is non-trivial both in the '3+1' case of three identical fermions plus a fourth different particle and in the '2+2' case of two distinct couples of identical fermions. It is precisely the latter that we study in the present work (as for the former, let us mention that the only rigorous stability result known so far is the fact that the '3+1' fermionic system is stable at least for the same values of the mass ratio between the fourth particle and each fermion as the mass ratios of stability in the corresponding '2+1' fermionic system, [22]). In particular we focus on the case that is currently most open, namely the case of identical masses (if the fermions of one couple have different mass than the other two, at least one can argue that there is an instability regime when the masses are in the same instability regime of the '2+1' trimers).

Thus, setting for convenience all masses to 1 and working in units $\hbar = 1$, the system we study here is governed by the formal Hamiltonian

$$H_{2+2} \;=\; -\frac{1}{2}\Delta_{\mathbf{x}_1} - \frac{1}{2}\Delta_{\mathbf{x}_2} - \frac{1}{2}\Delta_{\mathbf{y}_1} - \frac{1}{2}\Delta_{\mathbf{y}_2} + \;\mu\sum_{i,j=1}^{2}\delta(\mathbf{x}_i - \mathbf{y}_j), \tag{1}$$

where μ is a real coupling constant. The interaction depending only on the spatial configuration of the system, the spin variable is factored out. $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{y}_1$ and $\mathbf{y}_2$ are three-dimensional variables for the position of the fermions in each couple. No delta-interaction terms are present between fermions of the same type, for fermionicity forces the wave-function to vanish when $\mathbf{x}_1 = \mathbf{x}_2$ or $\mathbf{y}_1 = \mathbf{y}_2$.

The expression (1) leads us to discuss what precise (unambiguous) meaning to give to H_{2+2} and hence in which sense the model under investigation here is *rigorous*. The well-

posedness of (1) will be the main object of the somewhat technical section 2: let us only develop here the main arguments.

The issue of making (1) a mathematically rigorous and physically unambiguous definition amounts to realize H_{2+2} as a self-adjoint (and not merely a symmetric) operator on the Hilbert space

$$\mathcal{H} \;=\; L^2_{\mathrm{f}}\big(\mathbb{R}^6,\, \mathrm{d}\mathbf{x}_1\mathrm{d}\mathbf{x}_2\big) \otimes L^2_{\mathrm{f}}\big(\mathbb{R}^6,\, \mathrm{d}\mathbf{y}_1\mathrm{d}\mathbf{y}_2\big), \tag{2}$$

(where the index 'f' stays for fermionic, thus wave-functions $\Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)$ that are antisymmetric under exchange $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$ or $\mathbf{y}_1 \leftrightarrow \mathbf{y}_2$). This is a mathematically non-trivial, but physically very clean issue: one has to declare a suitable set of physical states where H_{2+2} is meant to act on, hence a suitable domain of self-adjointness in $\mathcal{H}$. The fact that (1) can only make sense if a precise prescription is made concerning the wave-functions H_{2+2} is meant to be applied to is already evident in the one-variable case '$-\Delta + \delta(\mathbf{x})$': in order for $(-\Delta + \delta(\mathbf{x}))\psi$ to still be an ordinary (square-integrable) function, ψ needs to have a suitable singularity at $\mathbf{x} = \mathbf{0}$ so that the delta function and the term resulting from $-\Delta\psi$ cancel each other out. As well known, the self-adjointness of H_{2+2} is equivalent to the fact that the corresponding Schrödinger equation $\mathrm{i}\partial_t\Psi = H_{2+2}\Psi$, given an initial state at time $t = 0$, does have a unique norm-preserving solution (in the chosen domain of H_{2+2}) and thus that the dynamics exists unambiguously and is unitary. It is also equivalent to the fact that the spectrum of H_{2+2} is real, while the spectral properties of H_{2+2}, including eigenvalues and eigenfunctions, depend critically on what domain the Hamiltonian is associated with. In fact, self-adjointness is ultimately the mathematical transcription of certain boundary conditions that in turn encode the physics that takes place when the wave-function reaches the 'boundaries' of the allowed spatial configurations: in the present case, clearly, this is the behaviour of the wave-functions when two different fermions come on top of each other.

Because of the point-like character of the interaction, (1) certainly defines an operator that acts as the free Hamiltonian as long as it is applied to wave-functions that vanish around the '*coincidence hyperplanes*' $\{\mathbf{x}_i = \mathbf{y}_j\}$, $i, j \in \{1, 2\}$. In this respect, one customarily thinks of the actual Hamiltonian of the '2+2' model as *a self-adjoint extension* of the free Hamiltonian initially restricted to such wave-functions. Since a symmetric operator that admits more than one self-adjoint extension has necessarily infinitely many distinct extensions, the next non-trivial issue is to select the one that describes the correct physics of the interaction, and hence to declare the appropriate boundary condition at the coincidence hyperplanes.

To this aim, it has to be remarked that in the context of zero-range interactions, only scattering of very low energy occurs among particles (this is the case, experimentally, for gases of ultra-cold atoms). Driven by this assumption, Bethe and Peierls first identified, by means of some easy manipulation of the two-body Schrödinger equation at low energy, a specific boundary conditions that a point-like interaction forces in the wave-function, the celebrated '*Bethe–Peierls contact condition*' [8, 9] (subsequently refined, as already mentioned, by Fermi [31], Breit [16], Ter-Martirosyan and Skornyakov [65]). For a generic wave-function $\Psi(\mathbf{z}_1, \ldots, \mathbf{z}_N)$ of N particle coupled by a two-body interaction of zero range, the contact condition reads

$$\Psi(\mathbf{z}_1, \ldots, \mathbf{z}_N) \;=\; \left(\frac{1}{|\mathbf{z}_i - \mathbf{z}_j|} - \frac{1}{a_{ij}}\right) q_{ij} + o(1) \qquad \text{as } |\mathbf{z}_i - \mathbf{z}_j| \to 0. \tag{3}$$

Here (3) expresses the leading singularity in the variable $|\mathbf{z}_i - \mathbf{z}_j|$, thus each scalar q_{ij} accounts for the value of the rest of the wave-function as $\mathbf{z}_i = \mathbf{z}_j$; each coefficient a_{ij} is the s-wave scattering length in the interaction between particle i and particle j, allowing at this level

of generality the possibility that different particles be coupled by different two-body interactions.

The contact condition (3) is a ubiquitous tool in the treatment of zero-range interactions and in a large part of the physical literature quoted above it is adopted systematically for the study of the spectral, scattering, and dynamical properties of the system, often by-passing the self-adjointness issue itself. Such a condition allows for explicit computations and, even when exploited formally, it encodes stringent physical heuristics. For our purpose of discussing the stability of the system, instead, we need a complete characterization of the domain of self-adjointness of the Hamiltonian, for otherwise one might miss a sequence of admissible states whose energy is negative and arbitrarily large.

Summarizing, the correct (mathematically rigorous, physical unambiguous) model for the '2+2' fermionic system under investigation has to be a self-adjoint realization of the formal Hamiltonian H_{2+2} whose domain consists of wave-functions that display the contact condition (3) with given scattering length α^{-1}. We will carry on this construction in section 2 below. This way we will come to the correct expression for the energy of the system, whose minimization problem will be then set up in section 3 and addressed numerically in section 4.

2. The model

We introduce in this section the Hamiltonian of the '2+2' fermionic system with zero-range interaction. Following the discussion of the second part of section 1, we need to identify a Hamiltonian H_α that

(i) extends the free Hamiltonian defined on wave-functions vanishing around the coincidence hyperplanes,
(ii) has a domain consisting of wave-functions that satisfy the Bethe–Peierls contact condition (3) at the coincidence hyperplanes,
(iii) models a two-body point interaction with the prescribed value a for the s-wave scattering length (α will be nothing but a re-labelling of a),
(iv) and is self-adjoint.

Let us remark, before proceeding, that a Hamiltonian that satisfies (i)–(iii), and hence whose domain incorporates the boundary condition (3) at the coincidence hyperplanes, is customarily referred to as a Ter-Martirosyan–Skornyakov (TMS) Hamiltonian. TMS Hamiltonians are therefore the natural Hamiltonians for system of particles interacting at zero range. Unfortunately, whereas constructing a TMS Hamiltonian is relatively manageable, albeit somehow technical, proving that it is also *self-adjoint* is considerably more difficult and requires ad hoc techniques. This because the well-developed and powerful mathematical tools for the proof of self-adjointness of ordinary Schrödinger operators (e.g. Hamiltonians of atoms, molecules, gaseous systems, ...) are not applicable to the case in which there is no interaction potential in the ordinary sense. This problem was originally discovered by Danilov [25] and by Faddeev and Minlos [51] and so far it has been solved in few cases only [1, 23, 43, 45–49, 52], all concerning two-body and three-body systems! To make the problem even more intricate, as was found recently for the fermionic '2+1' system [23, 47–49], it may happen that the sole contact condition (3) prescribing the behaviour at the coincidence between any two particles is not enough to qualify the self-adjointness domain, meaning that further boundary conditions are needed, typically in the form of an asymptotic behaviour at a multiple coincidence point, that is, when several particles come simultaneously on top of each other.

Regrettably, *it is unproved at present whether the '2+2' fermionic TMS Hamiltonian is actually self-adjoint* and here will not prove (iv) above either. This does not invalidate our discussion on the stability of the system. Initially we will give (iv) for granted: as discussed later in the end of section 3, the way we organize the control of the lower boundedness of the energy quadratic form associated to H_α would also give, as a by-product, the self-adjointness of H_α. Eventually we will only give a numerical evidence that the energy form is bounded below (and hence a numerical basis in favour of the self-adjointness of H_α), a separate analysis being needed for a rigorous proof of it.

2.1. Free particles away from the coincidence hyperplanes

Let us start by considering the 'restricted free Hamiltonian' $\mathring{H}_0$

$$\mathring{H}_0 \;=\; -\frac{1}{2}\Delta_{\mathbf{x}_1} - \frac{1}{2}\Delta_{\mathbf{x}_2} - \frac{1}{2}\Delta_{\mathbf{y}_1} - \frac{1}{2}\Delta_{\mathbf{y}_2} \tag{4}$$

defined only on those states in which the four particles are far from each other, more precisely on the domain $\mathcal{D}(\mathring{H}_0)$ of wave-functions $\Psi \in \mathcal{H}$ (see (2) above) that are smooth and vanish in a neighbourhood of the inter-species coincidence hyperplanes

$$\Gamma_{ij} \;:=\; \{\,\mathbf{x}_i = \mathbf{y}_j\,\} \qquad i,j \in \{1,2\} \tag{5}$$

besides the obvious vanishing around the intra-species coincidence hyperplanes

$$\Gamma_{\mathrm{f}} \;:=\; \{\,\mathbf{x}_1 = \mathbf{x}_2\,\} \bigcup \{\,\mathbf{y}_1 = \mathbf{y}_2\,\} \tag{6}$$

which is due to fermionicity (anti-symmetry).

It is convenient to work in the Fourier space, with coordinates $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2)$ conjugate to $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)$. It is also convenient to work in the centre-of-mass frame, thus constraining the momentum variables to the manifold

$$\Gamma \;:=\; \left\{ \begin{array}{c} (\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) \in \left(\mathbb{R}^3\right)^4 \text{ such that} \\ \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{q}_1 + \mathbf{q}_2 = \mathbf{0} \end{array} \right\}. \tag{7}$$

Thus, when we write $\widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2)$ we actually mean that $\widehat{\Psi}$ is a function of three independent 3D variables, say, $\widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)$, the fourth being determined by the constraint in (7). The corresponding Hilbert space is $L^2(\Gamma) \cong L^2((\mathbb{R}^3)^3)$ and we are interested in its fermionic sector

$$\mathcal{H}_{\mathrm{f}} := \left\{ \begin{array}{c} \Psi \mid \widehat{\Psi} \in L^2(\Gamma) \text{ such that} \\ \widehat{\Psi}(\mathbf{p}_2, \mathbf{p}_1, \mathbf{q}_1, \mathbf{q}_2) = -\widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) = \widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_2, \mathbf{q}_1) \end{array} \right\} \tag{8}$$

which implements anti-symmetry separately for each couple $(\mathbf{p}_1, \mathbf{p}_2)$ and $(\mathbf{q}_1, \mathbf{q}_2)$.

In Fourier coordinates $\mathring{H}_0$ acts as

$$\begin{aligned} \widehat{\mathring{H}_0\Psi} &= \frac{1}{2}\left(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\right)\widehat{\Psi} \\ &= \left(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{p}_1\cdot\mathbf{p}_2 + \mathbf{p}_1\cdot\mathbf{q}_1 + \mathbf{p}_1\cdot\mathbf{q}_2\right)\widehat{\Psi} \end{aligned} \tag{9}$$

and its domain $\mathcal{D}(\mathring{H}_0)$ is characterized as follows: the vanishing conditions

$$\Psi\,|_{\Gamma_{ij}} \equiv\; 0, \qquad \Psi\,|_{\Gamma_{\mathrm{f}}} \equiv\; 0 \tag{10}$$

read, after a Fourier transform,

$$\int_{\mathbb{R}^3} \widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\, \mathrm{d}\mathbf{p}_2 \;=\; 0, \qquad \int_{\mathbb{R}^3} \widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\, \mathrm{d}\mathbf{q}_1 \;=\; 0 \tag{11}$$

(clearly one also has $\int_{\mathbb{R}^3} \widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\, \mathrm{d}\mathbf{p}_1 = 0$, etc, by anti-symmetry), where the vanishing in (11) is to be interpreted, rigorously speaking, in the dual sense

$$\begin{aligned} &\iiint \widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\, \widehat{h}\,(\mathbf{p}_1, \mathbf{q}_1)\, \mathrm{d}\mathbf{p}_1 \mathrm{d}\mathbf{p}_2 \mathrm{d}\mathbf{q}_1 \;=\; 0 \\ &\quad \forall\, h \in H^{-1/2}\big(\mathbb{R}^3 \times \mathbb{R}^3,\, \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y}\big). \end{aligned} \tag{12}$$

Here and in the following $H^s(\mathbb{R}^d)$ denotes the d-dimensional Sobolev space of order $s \in \mathbb{R}$: thus, the h's in (12) are such that

$$\iint \frac{\widehat{h}\,(\mathbf{p}, \mathbf{q})}{\big(1 + \mathbf{p}^2 + \mathbf{q}^2\big)^{1/2}}\, \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q} \;<\; +\infty. \tag{13}$$

Condition (11), or its 'smoothed' version (12), provides the correct characterization of $\mathcal{D}(\mathring{H}_0)$.

2.2. The adjoint $\mathring{H}_0^\dagger$

Since $\mathring{H}_0$ is positive (i.e., $\langle\Psi|\mathring{H}_0|\Psi\rangle \geqslant 0$ for any $\Psi \in \mathcal{D}(\mathring{H}_0)$), it is natural to use the Krein–Višik–Birman theory for bounded below symmetric operators [2, 5, 10, 11, 37, 61, 67] to determine the adjoint of $\mathring{H}_0$ and to characterize its self-adjoint extensions.

According to this elegant and efficient mathematical theory[6], $\mathring{H}_0^\dagger$ is identified through the following steps. First, one determines the subspace

$$U \;\coloneqq\; \ker\Big(\mathring{H}_0^\dagger + 1\Big) \;=\; \mathrm{ran}\Big(\mathring{H}_0 + 1\Big)^\perp, \tag{14}$$

(where hereafter ker ≡ kernel and ran ≡ range). The rhs above, in view of (9) and (12), is recognized to be the space of all functions u_h of the form

$$\widehat{u_h}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) \;=\; \frac{\widehat{h}\,(\mathbf{p}_1, \mathbf{q}_1) - \widehat{h}\,(\mathbf{p}_1, \mathbf{q}_2) - \widehat{h}\,(\mathbf{p}_2, \mathbf{q}_1) + \widehat{h}\,(\mathbf{p}_2, \mathbf{q}_2)}{\frac{1}{2}\big(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\big) + 1} \tag{15}$$

for arbitrary $h \in H^{-1/2}(\mathbb{R}^6)$ (the alternating signs for the h-functions are due to the anti-symmetry, moreover $\langle(\mathring{H}_0 + 1)\Psi, u_h\rangle = 0$).

Second step, owing to the strict positivity and hence invertibility of $\mathring{H}_0 + 1$, the theory shows that the domain of $\mathring{H}_0^\dagger + 1$ is given by the direct sum

$$\mathcal{D}\Big(\mathring{H}_0^\dagger + 1\Big) = \mathcal{D}\Big(\mathring{H}_0 + 1\Big) + (\mathring{H}_0 + 1)_F^{-1} U + U \tag{16}$$

(clearly $\mathcal{D}(\mathring{H}_0 + 1) = \mathcal{D}(\mathring{H}_0)$ and $\mathcal{D}(\mathring{H}_0^\dagger + 1) = \mathcal{D}(\mathring{H}_0^\dagger)$), where $(\mathring{H}_0 + 1)_F$ denotes a self-adjoint extension of $\mathring{H}_0 + 1$ with distinguished properties in the theory, the so-called '*Friedrichs extension*'. Apart from its certain existence, $(\mathring{H}_0 + 1)_F$ is only implicitly known at this level of abstraction: still, one knows for sure that $(\mathring{H}_0 + 1)_F$ too is strictly positive and hence invertible, its action on the wave-functions of $\mathcal{D}(\mathring{H}_0)$ is precisely the same as the action of $\mathring{H}_0 + 1$, and its inverse acts, up to a Fourier transform, as the multiplication by $\Big(\frac{1}{2}(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2) + 1\Big)^{-1}$.

[6] We will adapt our notation to the one of Birman's original work [11], a translation of which may be found in the SISSA preprint series 08/2015/MAT, url: http://urania.sissa.it/xmlui/handle/1963/34443.

Therefore, a generic $\Phi \in \mathcal{D}(\mathring{H}_0^\dagger)$ has the form

$$\widehat{\Phi} = \widehat{\Psi} + \frac{\widehat{u_\eta}}{\frac{1}{2}\big(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\big) + 1} + \widehat{u_\xi} \tag{17}$$

for some $\Psi \in \mathcal{D}(\mathring{H}_0)$ and some $\eta, \xi \in H^{-1/2}(\mathbb{R}^6)$. Recalling that $\mathring{H}_0^\dagger + 1$ is an operator extension both of $\mathring{H}_0 + 1$ and of $(\mathring{H}_0 + 1)_F$ (meaning that its action on wave-functions in the domain of any of the latter operators coincide with the action of the operator itself), and that u_ξ belongs to the kernel of $\mathring{H}_0^\dagger + 1$, we deduce from (17)

$$\Big(\mathring{H}_0^\dagger + 1\Big)\Phi \;=\; \Big(\mathring{H}_0 + 1\Big)\Psi + u_\eta \tag{18}$$

and hence

$$\begin{aligned}\widehat{\mathring{H}_0^\dagger \Phi}\,(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) = \frac{1}{2}\big(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\big)\,\widehat{\Phi} \\ - \Big(\hat{\xi}(\mathbf{p}_1, \mathbf{q}_1) - \hat{\xi}(\mathbf{p}_1, \mathbf{q}_2) - \hat{\xi}(\mathbf{p}_2, \mathbf{q}_1) + \hat{\xi}(\mathbf{p}_2, \mathbf{q}_2)\Big).\end{aligned} \tag{19}$$

Identities (17) and (19) characterize domain and action of the adjoint $\mathring{H}_0^\dagger$ of $\mathring{H}_0$. Our next task, following the programme stated at the beginning of this Section, is to select a suitable restriction of $\mathring{H}_0^\dagger$, hence a suitable extension of $\mathring{H}_0$, that be self-adjoint and encode, in the form of a contact condition like (3), a point interaction of given strength (s-wave scattering length) $-\alpha^{-1}$. This will be done in section 2.4 below.

2.3. 'Charges' on the coincidence hyperplanes

Let us highlight, for future purposes, a pictorial physical interpretation for wave-functions u_h of the form (15).

It is clear that u_h has low regularity: acting on it with the free Hamiltonian (that is, differentiating twice in each variable) does not yield a square-integrable function, as can be seen computing in the Fourier space

$$\begin{aligned}\left(\frac{1}{2}\big(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\big) + 1\right)\widehat{u_h}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) \\ = \hat{h}(\mathbf{p}_1, \mathbf{q}_1) - \hat{h}(\mathbf{p}_1, \mathbf{q}_2) - \hat{h}(\mathbf{p}_2, \mathbf{q}_1) + \hat{h}(\mathbf{p}_2, \mathbf{q}_2).\end{aligned} \tag{20}$$

The outcome, in the rhs of (20), can be seen (distributionally) as the Fourier transform of a sum of four terms of the form

$$h(\mathbf{x}_1, \mathbf{y}_1)\,\delta(\mathbf{x}_2)\delta(\mathbf{y}_2), \;\text{ etc}$$

or also, keeping into account that working in the centre-of-mass frame one of the four particle coordinates depend on the others,

$$h(\mathbf{x}_1, \mathbf{y}_1)\,\delta(\mathbf{x}_2 - \mathbf{y}_2), \;\text{ etc.}$$

It is natural to interpret the expressions above as functions (in a generalized sense, namely distributions) that 'live on each hyperplane' Γ_{11}, Γ_{12}, Γ_{21}, Γ_{22} (the one above, for instance, 'lives' on Γ_{22}).

This marks a fundamental difference between the wave-functions in the domain of $\mathring{H}_0$ and those in the domain of $\mathring{H}_0^\dagger$. The action of the free Hamiltonian on some of the latter produces a delta-like singularity that has the chance to compensate the formal delta-potential present in (1).

Importing to this context a typical terminology from electrostatics, it is customary to refer to h in (15) as a '*charge*' ('living on a hyperplane') and to the corresponding u_h as the '*potential generated by the charge h*'.

2.4. TMS Hamiltonian

To select one out of the, *a priori*, infinitely many distinct self-adjoint extensions of the restricted free Hamiltonian $\mathring{H}_0$, all mathematically legitimate, requires a physical prescription.

To this aim, we observe first of all that the behaviour of the wave-functions Φ of the domain of $\mathring{H}_0^\dagger$ in the vicinity of each coincidence hyperplane Γ_{ij} differs drastically from that of the wave-functions Ψ in $\mathcal{D}(\mathring{H}_0)$. The latter satisfy the vanishing conditions (10), or also (11) in momentum coordinates. For a generic $\Phi \in \mathcal{D}(\mathring{H}_0^\dagger)$, instead, it is straightforward to obtain by means of (11), (15), and (17)

$$\int_{|\mathbf{p}_2|<R} \widehat{\Phi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\, \mathrm{d}\mathbf{p}_2 \;=\; 4\pi^2 R\, \widehat{A_\xi}(\mathbf{p}_1, \mathbf{q}_1) + \widehat{B_{\eta,\xi}}(\mathbf{p}_1, \mathbf{p}_2) + o(1) \quad \text{as } R \to +\infty \tag{21}$$

(plus the obvious analogue when the integration and the cut-off in R is on the $\mathbf{p}_1$-variable), for some functions A_ξ and $B_{\eta,\xi}$ in $H^{-1/2}(\mathbb{R}^6)$ that is not difficult to express in terms of the charges η and ξ. Thus, $\int_{|\mathbf{p}_2|<R} \widehat{\Phi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\, \mathrm{d}\mathbf{p}_2$ diverges with R for $\Phi \in \mathcal{D}(\mathring{H}_0^\dagger)$ whereas $\int_{|\mathbf{p}_2|<R} \widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\, \mathrm{d}\mathbf{p}_2$ vanishes as $R \to \infty$ when $\Psi \in \mathcal{D}(\mathring{H}_0)$, a signature of the singular behaviour of Φ close to $\bigcup_{ij} \Gamma_{ij}$.

We now select a particular restriction of $\mathring{H}_0^\dagger$ to those Φ's of $\mathcal{D}(\mathring{H}_0^\dagger)$ which, instead of the generic asymptotics (21), satisfy the special asymptotics

$$\int_{|\mathbf{p}_2|<R} \widehat{\Phi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\, \mathrm{d}\mathbf{p}_2 \;=\; \big(4\pi^2 R + \alpha\big)\, \widehat{A_\xi}(\mathbf{p}_1, \mathbf{q}_1) + o(1) \quad \text{as } R \to +\infty \tag{22}$$

for some $\alpha \in \mathbb{R}$, which amounts to impose

$$B_{\eta,\xi} \;\equiv\; \alpha A_\xi \tag{23}$$

in (21), and hence, implicitly, amounts to impose a transformation that maps η to ξ. In particular, there is only one independent charge in the Φs of $\mathcal{D}(\mathring{H}_0^\dagger)$ that satisfy the constraint (22) or (23) above[7]. The physical meaning of the constant α and the reason of the choice (23) will be clarified in a moment.

[7] *A priori*, for these constrained Φs the regularity of their charge may need be higher than the $H^{-1/2}$-regularity needed to belong to $\mathcal{D}(\mathring{H}_0^\dagger)$. One can see that $h \in H^{-1/2}(\mathbb{R}^6)$ is the necessary and sufficient regularity for the corresponding u_h to be square-integrable (see e.g. lemma B.2 in [22]) and the square integrability of u_h is in turn needed for the expression (17) to define a Φ in the domain of $\mathring{H}_0^\dagger$, and for (18) to make $\mathring{H}_0^\dagger \Phi$ square-integrable. H_α being a restriction of $\mathring{H}_0^\dagger$, the regularity of the charge is *a priori* higher. This turns out to be indeed the case, although we do not develop the details here: for the restriction (23) to be possible, the charge ξ must actually belong to $H^1(\mathbb{R}^6)$. Keeping the indication $H^{-1/2}(\mathbb{R}^6)$ in the following expression (24) for $\mathcal{D}(H_\alpha)$ is still correct, because implicitly the additional constraint listed (24) makes the $H^{-1/2}$-regularity actually a H^1-regularity.

The resulting operator, that we denote by H_α, is thus defined on the domain

$$\mathcal{D}(H_\alpha) := \left\{ \begin{array}{c} \Phi \in \mathcal{H}_{\mathrm{f}} \ \text{ such that} \\ \widehat{\Phi} = \widehat{\Psi} + \dfrac{\widehat{u_\eta}}{\frac{1}{2}\big(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\big) + 1} + \widehat{u_\xi} \\ \text{with} \\ \Psi \in \mathcal{D}\big(\mathring{H}_0\big), \quad \eta,\, \xi \in H^{-1/2}\big(\mathbb{R}^6\big), \\ u_h \text{ given by (15)}, \quad \text{and (as } R \to +\infty) \\ \displaystyle\int_{|\mathbf{p}_2|<R} \widehat{\Phi}(\mathbf{p}_1,\, \mathbf{p}_2,\, \mathbf{q}_1)\, \mathrm{d}\mathbf{p}_2 = \big(4\pi^2 R + \alpha\big)\, \widehat{A_\xi}\,(\mathbf{p}_1,\, \mathbf{q}_1) + o(1) \end{array} \right\} \tag{24}$$

by the action

$$\begin{aligned} \widehat{H_\alpha \Phi}\,(\mathbf{p}_1,\, \mathbf{p}_2,\, \mathbf{q}_1,\, \mathbf{q}_2) = {} & \frac{1}{2}\big(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\big)\, \widehat{\Phi}(\mathbf{p}_1,\, \mathbf{p}_2,\, \mathbf{q}_1,\, \mathbf{q}_2) \\ & - \Big(\hat{\xi}(\mathbf{p}_1,\, \mathbf{q}_1) - \hat{\xi}(\mathbf{p}_1,\, \mathbf{q}_2) - \hat{\xi}(\mathbf{p}_2,\, \mathbf{q}_1) + \hat{\xi}(\mathbf{p}_2,\, \mathbf{q}_2)\Big) \end{aligned} \tag{25}$$

(i.e. the same action (19) of $\mathring{H}_0^\dagger$, of which H_α is a restriction). Clearly $\mathcal{D}(\mathring{H}_0) \subset \mathcal{D}(H_\alpha) \subset \mathcal{D}(\mathring{H}_0^\dagger)$.

Given a wave-function Φ in $\mathcal{D}(H_\alpha)$, the spatial version of the asymptotics (22) is retrieved as follows. In the expression of the inverse Fourier transform

$$\Phi(\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{y}_1) \;=\; \frac{1}{(2\pi)^{9/2}} \iiint \mathrm{e}^{\mathrm{i}\mathbf{p}_1\mathbf{x}_1}\, \mathrm{e}^{\mathrm{i}\mathbf{p}_2\mathbf{x}_2}\, \mathrm{e}^{\mathrm{i}\mathbf{q}_1\mathbf{y}_1}\, \widehat{\Phi}(\mathbf{p}_1,\, \mathbf{p}_2,\, \mathbf{q}_1)\, \mathrm{d}\mathbf{p}_1 \mathrm{d}\mathbf{p}_2 \mathrm{d}\mathbf{q}_1$$

we explore the limit $|\mathbf{x}_i - \mathbf{y}_j| \to 0$, say, for concreteness, $|\mathbf{x}_2 - \mathbf{y}_2| \lesssim R^{-1} \to 0$; recalling that the four variables are not independent (centre-of-mass frame), it is not restrictive to fix $\mathbf{y}_2 = \mathbf{0}$, $|\mathbf{x}_2| \lesssim R^{-1}$; the integral above can then be expressed as a limit $R \to \infty$ of the same integration, cut at $|\mathbf{p}_2| \lesssim R$; using (22) one is then left with the inverse Fourier transform of $(4\pi^2 R + \alpha)\widehat{A_\xi}\,(\mathbf{p}_1, \mathbf{q}_1)$, whence eventually

$$\Phi(\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{y}_1) = \left(\frac{4\pi^2}{|\mathbf{x}_2 - \mathbf{y}_2|} + \alpha\right) A_\xi(\mathbf{x}_1,\, \mathbf{y}_1) + o(1) \quad \text{as } |\mathbf{x}_2 - \mathbf{y}_2| \to 0. \tag{26}$$

(The analogous asymptotics in the vicinity of the other coincidence hyperplanes are obtained along the same line.)

We therefore conclude: first, being an extension of $\mathring{H}_0$, H_α acts on the wave-functions Φ's of $\mathcal{D}(\mathring{H}_0)$, i.e., those functions that vanish around the coincidence hyperplanes, precisely as the free Hamiltonian, thus encoding a point-like interaction supported on $\cup_{ij}\ \Gamma_{ij}$; furthermore, as it follows by comparing (26) with (3), such an interaction has precisely the form of a Bethe–Peierls-like contact condition with scattering length $-\alpha^{-1}$ (in appropriate units). As such, H_α is customarily referred to as the TMS operator for the '2+2' fermionic system with given interaction strength $-\alpha^{-1}$. This completes points (i)–(iii) of the programme announced at the beginning of this section.

2.5. Energy form for H_α

We complete our analysis of the point-interaction '2+2' fermionic model by discussing now the explicit expression for the energy of the system in a state Φ, namely the quadratic form[8] associated with the expectation $\langle\Phi|H_\alpha|\Phi\rangle$

In principle, given $\Phi \in \mathcal{D}(H_\alpha)$, it is harmless to compute $\langle\Phi|H_\alpha|\Phi\rangle$ by means of (17) and (25) and to isolate the contribution due to the Ψ-part, the η-part, and the ξ-part of Φ. In practice, though, there is an additional relation to implement, which links the charges η and ξ of any $\Phi \in \mathcal{D}(H_\alpha)$, and which is only implicitly encoded in (24). This condition was imposed in the (implicit) form (23) at the moment of selecting the asymptotics (22) that characterize the behaviour of Φ in the vicinity of the coincidence hyperplanes.

We shall now discuss the computation of $\langle\Phi|H_\alpha|\Phi\rangle$ refraining from writing the transformation $\eta \mapsto \xi$ explicitly[9], out of the implicit condition (23), highlighting instead how the result that one may find in the mathematical literature [32, 44] actually emerges in the form of (39) for the energy expectation[10].

As is customary in this context, it is convenient at this point to re-express the decomposition (24) of a wave-function $\Phi \in \mathcal{D}(H_\alpha)$ in the alternative form

$$\Phi \;=\; F + \mathcal{G}\xi, \tag{27}$$

where

$$\widehat{F} \;:=\; \widehat{\Psi} + \frac{\widehat{u_\eta}}{\frac{1}{2}\big(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\big) + 1} \tag{28}$$

and

$$\widehat{\mathcal{G}\xi} \;:=\; \widehat{u_\xi} \tag{29}$$

are referred to, respectively, as the '*regular part*' and the '*singular part*' of Φ. This terminology reflects the fact that $F \in H^2((\mathbb{R}^3)^4)$. It is also customary to refer to $\mathcal{G}\xi$ as the potential generated by the charge ξ (section 2.3).

In addition, condition (23) relating the charges η and ξ translates into a constraint between the behaviour of the regular and the singular part of Φ in the vicinity of the coincidence hyperplanes. In Fourier transform it can be seen (as done already in [22, 23, 32, 43, 45–49, 52, 62]) that this constraint takes the form

[8] For readability reasons we make an abuse of notation and we denote by $\langle\Phi|H_\alpha|\Phi\rangle$ both the expectation of H_α in a state $\Phi \in \mathcal{D}(H_\alpha)$ and the energy form of H_α. The latter is a quadratic form defined on a domain larger than $\mathcal{D}(H_\alpha)$ and it is this form to be proved to be lower-bounded (and closed, for the purposes of self-adjointness: see the concluding remark of section 3). In fact, we could adjust our notation showing that all our following arguments about the expectation $\langle\Phi|H_\alpha|\Phi\rangle$ do apply to the energy form as well. While not doing so, we are tacitly using the fact that the decompositions (27) and (33) in regular and singular part for the functions in $\mathcal{D}(H_\alpha)$ are also taken for the elements in the larger domain of the quadratic form, and we are omitting further domain specifications. This does not affect our conclusions.

[9] It is worth remarking that it is essentially in terms of the map $\eta \mapsto \xi$ that the Krein–Višik–Birman theory for self-adjoint extensions of semi-bounded symmetric operators studies the possible self-adjointness of a TMS Hamiltonian [2, 5, 10, 11, 37, 61, 67].

[10] An alternative approach is possible to establish the energy quadratic form for a system of particles with point interactions, originally due to Dell'Antonio *et al* [26], in which one starts from the energy of a delta-like interaction that is regularised by means of an ultra-violet cut-off, and one then goes through a suitable renormalization procedure (see details in the appendix in either [22, 32]). At least for 'N+1' fermionic models the resulting quadratic form can be then proved to be the energy form associated to a TMS Hamiltonian [22, 23, 32], that is, a Hamiltonian of point interaction forcing the wave-functions of its domain to satisfy the appropriate boundary condition at the coincidence hyperplanes.

$$\int_{\mathbb{R}^3} \widehat{F}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\,\mathrm{d}\mathbf{p}_2 \;=\; \alpha\hat{\xi}(\mathbf{p}_1, \mathbf{q}_1) + \widehat{T\xi}(\mathbf{p}_1, \mathbf{q}_1) \tag{30}$$

(plus the analogous identities in the other variables), for some operator T acting on charges, which we do not specify here.

For a given $\Phi \in \mathcal{D}(H_\alpha)$, the decomposition (27)–(29) allows one to re-write the action of H_α (see (18) and (25) above) in the alternative, instructive form

$$\widehat{(H_\alpha + 1)\Phi} \;=\; \left(\frac{1}{2}\left(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\right) + 1\right)\widehat{F} \tag{31}$$

that is, the action of $(H_\alpha + 1)$ on Φ is the same as the action of the free Hamiltonian on the regular part of Φ.

Before proceeding, let us now generalize the shift $\mathring{H}_0 + 1$, introduced in section 2.2 in order to get a strictly positive operator, to a generic shift $\mathring{H}_0 + \lambda$ with $\lambda > 0$. Allowing λ to be arranged at our convenience (sufficiently large, depending on α) will turn out to be useful when we will discuss the stability of H_α in section 3. In fact, in all our considerations there was nothing special in shifting $\mathring{H}_0$ by 1, instead of by λ. One can immediately generalize (14)–(18), (20), and (24) above by replacing '+1' with '+λ' (noticeably the form of (19) and (25) remains the same). In particular, u_h is now depends on λ in an obvious way, that is

$$\widehat{u_h}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) \;=\; \frac{\hat{h}(\mathbf{p}_1, \mathbf{q}_1) - \hat{h}(\mathbf{p}_1, \mathbf{q}_2) - \hat{h}(\mathbf{p}_2, \mathbf{q}_1) + \hat{h}(\mathbf{p}_2, \mathbf{q}_2)}{\frac{1}{2}\left(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\right) + \lambda}. \tag{32}$$

As a consequence, the decomposition (27)–(29) takes the λ-dependent form

$$\Phi \;=\; F_\lambda + \mathcal{G}_\lambda\xi, \tag{33}$$

where regular part

$$\begin{aligned}\widehat{F_\lambda}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) \coloneqq\; & \widehat{\Psi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) \\ & + \frac{\hat{\eta}(\mathbf{p}_1, \mathbf{q}_1) - \hat{\eta}(\mathbf{p}_1, \mathbf{q}_2) - \hat{\eta}(\mathbf{p}_2, \mathbf{q}_1) + \hat{\eta}(\mathbf{p}_2, \mathbf{q}_2)}{\left(\frac{1}{2}\left(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\right) + \lambda\right)^2}\end{aligned} \tag{34}$$

and singular part

$$\widehat{\mathcal{G}_\lambda\xi}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2) \;\coloneqq\; \frac{\hat{\xi}(\mathbf{p}_1, \mathbf{q}_1) - \hat{\xi}(\mathbf{p}_1, \mathbf{q}_2) - \hat{\xi}(\mathbf{p}_2, \mathbf{q}_1) + \hat{\xi}(\mathbf{p}_2, \mathbf{q}_2)}{\frac{1}{2}\left(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\right) + \lambda} \tag{35}$$

are linked by the λ-dependent version of the constraint (30) at the coincidence hyperplanes, that is

$$\int_{\mathbb{R}^3} \widehat{F_\lambda}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1)\,\mathrm{d}\mathbf{p}_2 \;=\; \alpha\hat{\xi}(\mathbf{p}_1, \mathbf{q}_1) + \widehat{T_\lambda\xi}(\mathbf{p}_1, \mathbf{q}_1) \tag{36}$$

for some operator T_λ acting on the charges. Note that each summand in the rhs of (33) depends on λ, but their sum does not: $\mathcal{D}(H_\alpha)$ is independent of the auxiliary parameter λ, which only indexes a convenient decomposition of the domain. Correspondingly, (31) is turned into

$$\widehat{(H_\alpha + \lambda)\Phi} \;=\; \left(\frac{1}{2}\left(\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{q}_1^2 + \mathbf{q}_2^2\right) + \lambda\right)\widehat{F_\lambda}. \tag{37}$$

Therefore (recall that $\mathbf{q}_2 = -\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{q}_1$),

$$\begin{aligned}\langle\Phi|\big(H_\alpha+\lambda\big)|\Phi\rangle &\overset{(33),(37)}{=} \Big\langle \widehat{F_\lambda} + \widehat{\mathcal{G}_\lambda\xi}\Big|\,\frac{1}{2}\big(\mathbf{p}_1^2+\mathbf{p}_2^2+\mathbf{q}_1^2+\mathbf{q}_2^2\big)+\lambda\,\Big|\,\widehat{F_\lambda}\Big\rangle\\ &\overset{(35)}{=}\Big\langle\widehat{F_\lambda}\Big|\,\frac{1}{2}\big(\mathbf{p}_1^2+\mathbf{p}_2^2+\mathbf{q}_1^2+\mathbf{q}_2^2\big)+\lambda\,\Big|\,\widehat{F_\lambda}\Big\rangle\\ &\quad+\iiint \mathrm{d}\mathbf{p}_1\mathrm{d}\mathbf{p}_2\mathrm{d}\mathbf{q}_1\Big(\hat\xi(\mathbf{p}_1,\mathbf{q}_1)^*-\hat\xi(\mathbf{p}_1,\mathbf{q}_2)^*\\ &\quad-\hat\xi(\mathbf{p}_2,\mathbf{q}_1)^*+\hat\xi(\mathbf{p}_2,\mathbf{q}_2)^*\Big)\,\widehat{F_\lambda}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}_1,\mathbf{q}_2)\end{aligned}$$

whence, using (34) and (12),

$$\langle\Phi|\,H_\alpha\,|\Phi\rangle = \Big\langle\widehat{F_\lambda}\Big|\,\frac{1}{2}\big(\mathbf{p}_1^2+\mathbf{p}_2^2+\mathbf{q}_1^2+\mathbf{q}_2^2\big)+\lambda\,\Big|\,\widehat{F_\lambda}\Big\rangle-\lambda\,\|\Phi\|^2+\big\langle u_\xi\big|\,u_\eta\big\rangle. \tag{38}$$

In the formula (38) for the energy of the state Φ one still has to express η in terms of ξ—by decomposition (33) the energy must depend only on the regular part and on the charge of the singular part of Φ. To do so one needs to implement (23), equivalently, (36) and write $\big\langle u_\xi\big|u_\eta\big\rangle$ in terms of the sole charge ξ—plus a dependence on the parameter α of the model and on the variational parameter λ. The final result [32, 44] reads

$$\begin{aligned}\langle\Phi|\,H_\alpha\,|\Phi\rangle &= \Big\langle\widehat{F_\lambda}\Big|\,\frac{1}{2}\big(\mathbf{p}_1^2+\mathbf{p}_2^2+\mathbf{q}_1^2+\mathbf{q}_2^2\big)+\lambda\,\Big|\,\widehat{F_\lambda}\Big\rangle\\ &\quad-\lambda\,\|\Phi\|^2+4\Big(\alpha\,\|\xi\|^2+\Phi_1^\lambda[\xi]+\Phi_2^\lambda[\xi]+\Phi_3^\lambda[\xi]+\Phi_4^\lambda[\xi]\Big),\end{aligned} \tag{39}$$

where

$$\Phi_1^\lambda[\xi] \;:=\; 2\pi^2\iint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\sqrt{\frac{3}{4}\big(\mathbf{p}^2+\mathbf{q}^2\big)+\frac{1}{2}\mathbf{p}\cdot\mathbf{q}+\lambda}\;|\hat\xi(\mathbf{p},\mathbf{q})|^2, \tag{40}$$

$$\Phi_2^\lambda[\xi] \;:=\; \iiint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{r}\,\frac{\hat\xi(\mathbf{p},\mathbf{q})^*\,\hat\xi(\mathbf{q},\mathbf{r})}{\mathbf{p}^2+\mathbf{q}^2+\mathbf{r}^2+\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{r}+\mathbf{q}\cdot\mathbf{r}+\lambda}, \tag{41}$$

$$\Phi_3^\lambda[\xi] \;:=\; \iiint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{r}\,\frac{\hat\xi(\mathbf{p},\mathbf{q})^*\,\hat\xi(\mathbf{r},\mathbf{p})}{\mathbf{p}^2+\mathbf{q}^2+\mathbf{r}^2+\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{r}+\mathbf{q}\cdot\mathbf{r}+\lambda}, \tag{42}$$

$$\Phi_4^\lambda[\xi] \;:=\; -\iiint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{r}\,\frac{\hat\xi(\mathbf{p},\mathbf{q})^*\,\hat\xi\big(\mathbf{r}-\frac{1}{2}(\mathbf{p}+\mathbf{q}),-\mathbf{r}-\frac{1}{2}(\mathbf{p}+\mathbf{q})\big)}{\mathbf{r}^2+\frac{3}{4}\big(\mathbf{p}^2+\mathbf{q}^2\big)+\frac{1}{2}\mathbf{p}\cdot\mathbf{q}+\lambda}. \tag{43}$$

The last contribution to $\langle\Phi|H_\alpha|\Phi\rangle$ in (39) is for obvious reasons referred to as the '*energy form of the charges*'. It is not surprising that for the finiteness of the form of the charges ξ need be more regular than the initial $H^{-1/2}$-regularity—the $\Phi_j^\lambda[\xi]$'s are finite only if $\xi\in H^{1/2}(\mathbb{R}^6)$: indeed, H_α is a restriction of $\mathring{H}_0^\dagger$ obtained by imposing the constrain (23), which forces the charges to be suitably more regular, as remarked in a footnote following (23).

3. The problem of stability

After having set up the model for the '2+2' fermionic system with zero-range interaction, we now turn to the problem of its *stability*, that is, the boundedness from below of the Hamiltonian H_α. The question is whether, for fixed α, there exists a finite lower bound $e_\circ > -\infty$ to the energy spectrum of H_α, meaning

$$\frac{\langle\Phi|H_\alpha|\Phi\rangle}{\langle\Phi|\Phi\rangle} \;\geqslant\; e_\circ \tag{44}$$

for all non-zero, finite-energy Φ's, or whether instead the lhs of (44) attains arbitrarily negative values.

Let us emphasize already at this stage that we will establish (44) with a lower bound $e_\circ$ that is certainly non-optimal—a precise determination of the actual ground state energy of H_α escapes our present methods. We will therefore focus on the question of stability, proving that the answer is affirmative, rather than quantify the optimal lower bound. For this reason we will refrain from keeping track of the precise magnitude of the constants in our estimates and from restoring the appropriate physical units in the end: this could be done easily and it would provide a final value of $e_\circ$ that nevertheless, as it will be clear in the course of our discussion, stays well below the actual ground state energy.

As a preparatory step for our discussion, we observe that (39) implies

$$\begin{aligned}\langle\Phi|\,H_\alpha\,|\Phi\rangle \;\geqslant\; &-\lambda\,\|\Phi\|^2 \,+\, \lambda\,\|F_\lambda\|^2 \,+\, 4\alpha\,\|\xi\|^2\\ &+\,4\Big(\,\Phi_1^\lambda[\xi]\,+\,\Phi_2^\lambda[\xi]\,+\,\Phi_3^\lambda[\xi]\,+\,\Phi_4^\lambda[\xi]\,\Big).\end{aligned} \tag{45}$$

Moreover, we re-write

$$\sum_{j=2}^{4}\Phi_j^\lambda[\xi] \;=\; \langle\xi|\,R_\lambda\,|\xi\rangle \,+\, \sum_{j=2}^{4}\Phi_j^0[\xi], \tag{46}$$

where each Φ_j^0 is the form Φ_j^λ with $\lambda = 0$ and R_λ is a *bounded* 'remainder' operator. A straightforward scaling argument shows that

$$\langle\xi|\,R_\lambda\,|\xi\rangle \;\leqslant\; \beta\sqrt{\lambda}\;\;\|\xi\|^2 \tag{47}$$

for some constant $\beta > 0$ that we do not need to compute here[11]. Thus

$$\begin{aligned}&\langle\Phi|H_\alpha|\Phi\rangle\\ &\geqslant -\lambda\,\|\Phi\|^2 \,+\, \lambda\,\|F_\lambda\|^2 \,+\, 4\alpha\,\|\xi\|^2 \,+\, 4\left(\,\Phi_1^\lambda[\xi]\,+\,\sum_{j=2}^{4}\Phi_j^0[\xi]\right) - 4\beta\sqrt{\lambda}\;\|\xi\|^2\,.\end{aligned} \tag{48}$$

[11] One has $\langle f\,|\,R_\lambda\,|g\rangle = \sum_{j=2}^{4}(\Phi_j^\lambda[\xi] - \Phi_j^0[\xi])$. In view of (41) the term $j = 2$ reads $\Phi_2^\lambda[\xi] - \Phi_2^0[\xi]$

$$=-\lambda\iiint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{r}\;\frac{\hat\xi(\mathbf{p},\,\mathbf{q})^*\,\hat\xi(\mathbf{q},\,\mathbf{r})}{\big(\mathbf{p}^2+\mathbf{q}^2+\mathbf{r}^2+\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{r}+\mathbf{q}\cdot\mathbf{r}+\lambda\big)\big(\mathbf{p}^2+\mathbf{q}^2+\mathbf{r}^2+\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{r}+\mathbf{q}\cdot\mathbf{r}\big)}$$

and the change of variables $(\mathbf{p},\,\mathbf{q},\,\mathbf{r}) \mapsto (\lambda^{-1/2}\mathbf{p},\,\lambda^{-1/2}\mathbf{q},\,\lambda^{-1/2}\mathbf{r})$ yields

$$\begin{aligned}&\big|\,\Phi_2^\lambda[\xi] - \Phi_2^0[\xi]\big|\\ &\leqslant \lambda^{7/2}\iiint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{r}\;\frac{\big|\,\hat\xi\big(\lambda^{1/2}\mathbf{p},\,\lambda^{1/2}\mathbf{q}\big)\big|\;\big|\,\hat\xi\big(\lambda^{1/2}\mathbf{q},\,\lambda^{1/2}\mathbf{r}\big)\big|}{\big(\mathbf{p}^2+\mathbf{q}^2+\mathbf{r}^2+\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{r}+\mathbf{q}\cdot\mathbf{r}+1\big)\big(\mathbf{p}^2+\mathbf{q}^2+\mathbf{r}^2+\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{r}+\mathbf{q}\cdot\mathbf{r}\big)}\\ &\leqslant C\;\lambda^{7/2}\iint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\;|\hat\xi\big(\lambda^{1/2}\mathbf{p},\,\lambda^{1/2}\mathbf{q}\big)|^2\\ &= C\lambda^{1/2}\,\|\xi\|^2\end{aligned}$$

for some constant C. The same can be argued for $j = 3,\,4$ and this yields (47).

Our *key point* is that for the '2+2'-fermionic system the homogeneous form of the charges, i.e., $\sum_{j=1}^{4}\Phi_j^0[\xi]$, is *positive* and, more precisely, is bounded below as

$$\sum_{j=1}^{4}\Phi_j^0[\xi] \;\geqslant\; \kappa\,\Phi_1^0[\xi] \qquad \text{for some } \kappa\in(0,\,1). \tag{49}$$

In this work we obtain (49) numerically: this will be the main subject of the following section 4 and to our knowledge no analytical proof of this fact is known so far. In the remainder of this section we will show how (49) implies the stability of the system, equation (44).

From (49) one has

$$\Phi_1^\lambda[\xi]+\sum_{j=2}^{4}\Phi_j^0[\xi] \;\geqslant\; \Phi_1^\lambda[\xi]-(1-\kappa)\Phi_1^0[\xi] \;\geqslant\; \kappa\,\Phi_1^\lambda[\xi]$$

and (48) then reads

$$\Big\langle\Phi|H_\alpha|\Phi\Big\rangle \;\geqslant\; -\lambda\,\|\Phi\|^2+\lambda\,\|F_\lambda\|^2+4\alpha\,\|\xi\|^2+4\kappa\Phi_1^\lambda[\xi]-4\beta\sqrt{\lambda}\,\|\xi\|^2. \tag{50}$$

Therefore in the 'repulsive' case $\alpha>0$ it is enough to choose the positive variational parameter λ so small as to have $\alpha>\beta\sqrt{\lambda}$: neglecting the other positive terms $\|F_\lambda\|^2$ and $\Phi_1^\lambda[\xi]$ in (50) one is left with $\Big\langle\Phi\Big|H_\alpha\Big|\Phi\Big\rangle\geqslant-\lambda\,\|\Phi\|^2$, whence the stability of the system (with lower bound $-\lambda$). Analogously, in the unitary regime (infinite scattering length, $\alpha=0$) the limit $\lambda\to0^+$ shows that $\langle\Phi|H_\alpha|\Phi\rangle\geqslant0$ for any Φ: the system is stable and the bottom of its spectrum is non-negative.

The 'attractive case' $\alpha<0$ requires a further analysis. The main observation is that

$$\|\xi\|^2 \;\leqslant\; a\,\|\mathcal{G}_\lambda\xi\|\,\Big(\Phi_1^\lambda[\xi]\Big)^{1/2} \tag{51}$$

for some $a>0$. Equation (51) follows from interpolating the L^2-norm of ξ between its $H^{-1/2}$-norm and its $H^{1/2}$-norm as we will now sketch. It is clear that inserting

$$1 \;=\; \frac{\Big(\frac{3}{4}\big(\mathbf{p}^2+\mathbf{q}^2\big)+\frac{1}{2}\mathbf{p}\cdot\mathbf{q}+\lambda\Big)^{1/4}}{\Big(\frac{3}{4}\big(\mathbf{p}^2+\mathbf{q}^2\big)+\frac{1}{2}\mathbf{p}\cdot\mathbf{q}+\lambda\Big)^{1/4}}$$

into the integrand in $\|\xi\|^2=\iint\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}\Big|\widehat{\xi}(\mathbf{p},\,\mathbf{q})\Big|^2$ and then performing a Schwartz inequality one gets

$$\|\xi\|^2 \;\leqslant\; \|\xi\|_{H_\lambda^{-1/2}}\|\xi\|_{H_\lambda^{1/2}} \overset{(40)}{=} \|\xi\|_{H_\lambda^{-1/2}}\Big(\Phi_1^\lambda[\xi]\Big)^{1/2},$$

where

$$\|\xi\|^2_{H_\lambda^m} \;=\; 2\pi^2\iint\mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\left(\frac{3}{4}\big(\mathbf{p}^2+\mathbf{q}^2\big)+\frac{1}{2}\mathbf{p}\cdot\mathbf{q}+\lambda\right)^m|\widehat{\xi}(\mathbf{p},\,\mathbf{q})|^2$$

is an equivalent Sobolev H^m-norm. The fact that $\|\mathcal{G}_\lambda\xi\|\sim\|\xi\|_{H_\lambda^{-1/2}}$, that is, that $\|\mathcal{G}_\lambda\xi\|$ is another equivalent $H^{-1/2}$-norm for ξ, is also standard (see, e.g., the detailed discussion in lemma B.2, equation (B.19), of [23]) and this completes the derivation of (51).

As a consequence, for an arbitrarily small $\varepsilon>0$ one can always find λ_ε large enough such that

$$4\beta\sqrt{\lambda}\,\|\xi\|^2 \;\leqslant\; \frac{\lambda}{2}\,\|\mathcal{G}_\lambda\xi\|^2+\varepsilon\,\Phi_1^\lambda[\xi] \tag{52}$$

for all $\lambda > \lambda_\varepsilon$. Since, on the other hand

$$\begin{aligned}\|\Phi\|^2 + \|F_\lambda\|^2 &\overset{(33)}{=} \|F_\lambda + \mathcal{G}_\lambda\xi\|^2 + \|F_\lambda\|^2 \\ &\geqslant 2\,\|F_\lambda\|^2 - 2\|F_\lambda\|\,\|\mathcal{G}_\lambda\xi\| + \|\mathcal{G}_\lambda\xi\|^2 \\ &\geqslant \frac{1}{2}\,\|\mathcal{G}_\lambda\xi\|^2, \end{aligned} \tag{53}$$

choosing $\varepsilon = 3\kappa$ one concludes from (52) and (53) that

$$\begin{aligned}&\lambda\,\|\Phi\|^2 + \lambda\,\|F_\lambda\|^2 + 4\kappa\Phi_1^\lambda[\xi] - 4\beta\sqrt{\lambda}\,\|\xi\|^2 \geqslant \\ &\geqslant \frac{\lambda}{2}\,\|\mathcal{G}_\lambda\xi\|^2 + 4\kappa\Phi_1^\lambda[\xi] - \frac{\lambda}{2}\,\|\mathcal{G}_\lambda\xi\|^2 - \varepsilon\,\Phi_1^\lambda[\xi] \\ &= \kappa\Phi_1^\lambda[\xi] \;\geqslant\; 2\pi^2\kappa\sqrt{\lambda}\,\|\xi\|^2. \end{aligned} \tag{54}$$

Last, plugging (54) into (50) yields

$$\langle\Phi|\,H_\alpha\,|\Phi\rangle \;\geqslant\; -2\lambda\,\|\Phi\|^2 - 4|\alpha|\|\xi\|^2 + 2\pi^2\kappa\sqrt{\lambda}\,\|\xi\|^2 \tag{55}$$

and it suffices to choose λ large enough to conclude that $\left\langle\Phi\middle|H_\alpha\middle|\Phi\right\rangle \geqslant -2\lambda\,\|\Phi\|^2$, that is, the stability of the system.

Let us conclude this section with a side remark on the self-adjointness of the Hamiltonian H_α. As already commented at the beginning of section 2 we do not have a proof here that H_α is indeed self-adjoint and hence a physically admissible Hamiltonian. Nevertheless, the scheme of control of the stability developed in this section is also a scheme for the proof of self-adjointness via quadratic forms: if one could give an analytical ground to the bound (49) that here we only obtain numerically, this would also complete the proof of self-adjointness. The reason, without entering details, is that a symmetric Hamiltonian is also self-adjoint if its energy form is a closed and bounded below quadratic form: the lower bound (44) is the first of these two conditions, as for the closedness of the energy form it can be proved by means of the estimates discussed in this section, in complete analogy to what done for the quadratic form and the Hamiltonian of the '2+1' fermionic model [22, 23].

4. Numerical analysis

This section is devoted to the numerical derivation of the bound (49).

4.1. Numerical set-up

In preparation for our numerics we first re-write (49) in a more manageable form. It is convenient to introduce a 'weighted' charge f defined in terms of the charge ξ by

$$f(\mathbf{p},\,\mathbf{q}) \;:=\; \left(\frac{3}{4}\big(\mathbf{p}^2+\mathbf{q}^2\big) + \frac{1}{2}\mathbf{p}\cdot\mathbf{q}\right)^{1/4}\widehat{\xi}\,(\mathbf{p},\,\mathbf{q}). \tag{56}$$

Correspondingly one re-writes

$$\Phi_j^0[\xi] \;=\; \mathcal{M}_j[f], \qquad j = 1,\,2,\,3,\,4, \tag{57}$$

where the $\mathcal{M}_j[f]$'s are obviously given by replacing $\widehat{\xi}$ with f in (40)–(43). In fact, it is also convenient to regard the $\mathcal{M}_j[f]$'s as quadratic forms associated with sesquilinear forms, i.e.,

$$\mathcal{M}_j[f] \;:=\; \mathcal{M}_j[f,f], \qquad j = 1,\,2,\,3,\,4, \tag{58}$$

where

$$\mathcal{M}_1[f,\,g] \;\coloneqq\; 2\pi^2 \iint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\overline{f\,(\mathbf{p},\,\mathbf{q})}\;g\,(\mathbf{p},\,\mathbf{q}), \tag{59}$$

$$\mathcal{M}_2[f,\,g] \;\coloneqq\; \iiint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{r}\;\frac{\overline{f\,(\mathbf{p},\,\mathbf{q})}\;g\,(\mathbf{q},\,\mathbf{r})}{P\,(\mathbf{p},\,\mathbf{q})\,D\,(\mathbf{p},\,\mathbf{q},\,\mathbf{r})\,P\,(\mathbf{q},\,\mathbf{r})}, \tag{60}$$

$$\mathcal{M}_3[f,\,g] \;\coloneqq\; \iiint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{r}\;\frac{\overline{f\,(\mathbf{p},\,\mathbf{q})}\;g\,(\mathbf{r},\,\mathbf{p})}{P\,(\mathbf{p},\,\mathbf{q})\,D\,(\mathbf{p},\,\mathbf{q},\,\mathbf{r})\,P\,(\mathbf{r},\,\mathbf{p})}, \tag{61}$$

$$\mathcal{M}_4[f,\,g] \;\coloneqq\; -\iiint \mathrm{d}\mathbf{p}\,\mathrm{d}\mathbf{q}\,\mathrm{d}\mathbf{r}\;\frac{\overline{f\,(\mathbf{p},\,\mathbf{q})}\;g\left(\mathbf{r}-\frac{\mathbf{p}+\mathbf{q}}{2},\,-\mathbf{r}-\frac{\mathbf{p}+\mathbf{q}}{2}\right)}{P\,(\mathbf{p},\,\mathbf{q})\left(\mathbf{r}^2+P\,(\mathbf{p},\,\mathbf{q})^4\right)P\left(\mathbf{r}-\frac{\mathbf{p}+\mathbf{q}}{2},\,-\mathbf{r}-\frac{\mathbf{p}+\mathbf{q}}{2}\right)}, \tag{62}$$

and

$$\begin{aligned} D\,(\mathbf{p},\,\mathbf{q},\,\mathbf{r}) &\coloneqq \mathbf{p}^2+\mathbf{q}^2+\mathbf{r}^2+\mathbf{p}\cdot\mathbf{q}+\mathbf{p}\cdot\mathbf{r}+\mathbf{q}\cdot\mathbf{r} \\ P\,(\mathbf{p},\,\mathbf{q}) &\coloneqq \left(\tfrac{3}{4}\left(\mathbf{p}^2+\mathbf{q}^2\right)+\tfrac{1}{2}\,\mathbf{p}\cdot\mathbf{q}\right)^{1/4}. \end{aligned} \tag{63}$$

In terms of the quadratic form

$$\mathcal{M}[f] \;\coloneqq\; \sum_{j=1}^{4}\mathcal{M}_j[f] \tag{64}$$

the bound (49) reads equivalently

$$\frac{\mathcal{M}[f]}{\langle f,f\rangle} \;\geqslant\; 2\pi^2\kappa \qquad \text{for some } \kappa\in(0,\,1), \tag{65}$$

to be valid for all non-zero, square-integrable f (ξ being in $H^{1/2}(\mathbb{R}^6)$ implies, by (56), that $f\in L^2(\mathbb{R}^6)$).

Our approach will be to control the form $\mathcal{M}[f,\,g]$ defined on the Hilbert space $L^2(\mathbb{R}^6)$ by truncating it on a finite-dimensional subspace, with dimension d large enough, spanned by a collection $\{f_1, f_2\,,...,f_d\}$ of explicitly chosen orthonormal functions. This results in computing a $d\times d$ symmetric matrix with entries $\mathcal{M}[f_i, f_j]$ and in calculating its lowest eigenvalue E_0. By ensuring that E_0 is acceptably stable with respect to d, for d large enough, with an asymptotic value in $(0,\,2\pi^2)$, we will then be in the condition of claiming (65).

We remark that $\mathcal{M}[f,\,g]$ being a *Hermitian* sesquilinear form, that is, $\mathcal{M}[f,\,g]=\overline{\mathcal{M}[g,f]}$ for all admissible f, g, so is the matrix $(\mathcal{M}[f_i, f_j])_{i,j=1\,,...,\,d}$, which then necessarily has *d real* eigenvalues. By standard variational arguments its lowest eigenvalue E_0 is an upper bound to the actual infimum of the quadratic form $\mathcal{M}[f,\,g]$ and it approximates such an infimum when the dimension d of the minimization subspace goes to ∞. The higher eigenvalues do not have *a priori* a special meaning concerning the spectral properties of the form $\mathcal{M}[f,\,g]$: they might or might not converge to critical values of the latter. It is also clear that the symmetry of $(\mathcal{M}[f_i, f_j])_{i,j=1\,,...,\,d}$ reduces by a factor $\frac{1}{2}$ the number of entries to be computed.

We also remark that the forms $\mathcal{M}_1[f,\,g]$, $\mathcal{M}_2[f,\,g]+\mathcal{M}_3[f,\,g]$, and $\mathcal{M}_4[f,\,g]$ too are Hermitian (while $\mathcal{M}_2[f,\,g]$ an $\mathcal{M}_3[f,\,g]$ separately are not). This give sense to computing also the lowest eigenvalue separately for each of the corresponding truncated matrices, in order to monitor the various contributions to the global E_0.

4.2. Choice of the basis

We introduce the functions

$$\begin{aligned}\Psi_{n,\ell,m}(\mathbf{r}) &:= f_n(r)\, Y_{\ell,m}(\theta,\,\phi),\\ n &= 0,\,1,\,2,\,3\,,...\\ \ell &= 0,\,1,\,2,\,3\,,...\\ m &= -\,\ell\,,...,\,\ell,\end{aligned} \tag{66}$$

where $\mathbf{r} \equiv (r\,\sin\theta\,\cos\phi,\, r\,\sin\theta\,\sin\phi,\, r\,\cos\theta) \in \mathbb{R}^3$, $r \equiv |\mathbf{r}| \geqslant 0$, $\theta \in\,]0,\,\pi)$, $\phi \in [0,\,2\pi)$, and

$$f_n(r) \;:=\; \mathrm{e}^{-r/2} \sum_{k=0}^{n} \frac{(\,-\,1)^k}{k!}, \tag{67}$$

$$Y_{\ell,m}(\theta,\,\phi) \;:=\; (-1)^{m\cdot\frac{\mathrm{sgn}(m)+1}{2}} \sqrt{\frac{2\ell\,+\,1}{4\pi}\,\frac{(\ell\,-\,|m|)\,!}{(\ell\,+\,|m|)\,!}}\;\; P_{\ell,|m|}(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}, \tag{68}$$

$$P_{\ell,|m|}(x) \;:=\; \frac{1}{2^\ell\,\ell!}\big(1\,-\,x^2\big)^{|m|/2}\,\frac{\mathrm{d}^{\ell+|m|}}{\mathrm{d}x^{\ell+|m|}}\big(x^2\,-\,1\big)^{\ell}. \tag{69}$$

As well known, the collection $\{\Psi_{n,\ell,m}\}_{n,\ell,m}$ is an orthonormal basis of $L^2(\mathbb{R}^3)$. As an orthonormal basis of $L^2(\mathbb{R}^6)$ we choose the collection of functions

$$\big(\Psi_{n,\ell,m}\Psi_{N,L,M}\big)(\mathbf{p},\,\mathbf{q}) \;\equiv\; \Psi_{n,\ell,m}(\mathbf{p})\Psi_{N,L,M}(\mathbf{q}). \tag{70}$$

We shall denote them with a collective index $k = 1,\,...,\,d$, say, $\{f_1\,,...,f_d\,\}$, where d is the dimension of the minimizing space. Our actual choice of the indices n, ℓ, m, N, L, M and our correspondence $\Psi_{n,\ell,m}\Psi_{N,L,M} \leftrightarrow f_k$ for the labelling of the basis functions is reported in table 1 (see appendix).

Let us stress that the charge ξ of our model does not inherits any special symmetry from the fermionic character of the particles involved. We recall from our discussion in section 2 that fermionicity resulted in the anti-symmetry of the regular part and in the alternating structure of the singular part of the admissible wave-functions (see (15), (34), and (35)). In particular the singular part is expressed in terms of *one* charge $\widehat{\xi}\,(\mathbf{p},\,\mathbf{q})$ that has no *a priori* symmetry such as symmetry under exchange of variables or under parity in each variable. For this reason also our chosen basis is not restricted to any such symmetry.

We see from (60)–(62) and (67)–(69) that in each multiple integral $\mathcal{M}_j[f_k\,,f_j\,]$ the integrand function has a (radial) exponential fall off that makes the integration only relevant in a suitable finite ball, plus an amount of oscillations that increase with the quantum numbers n, ℓ, m, N, L, M. Not to have too highly oscillatory integrals we keep ℓ, L not too close to their maximal values n, N and we neglect the less relevant oscillations due to m, M by choosing $m = M = 0$. This last choice in particular makes the *a priori* complex-valued entry $\mathcal{M}_j[f_k\,,f_j\,]$ a real number.

4.3. Minimization of $\mathcal{M}$

Let us present now the results of the minimization that has been set up in sections 4.1 and 4.2 above of the quadratic form $\mathcal{M}[f,\,g]$ truncated on a suitable finite-dimensional space, and let us defer to the following section 4.4 the discussion of the numerical algorithm and of the error analysis.

We considered pools of basis functions $\Psi_{n,\ell,m}\Psi_{N,L,M}$ of the form (70) with increasing number d of elements for each pool, ensuring the presence of increasing values of the radial

Table 1. Labelling of the 625 basis vectors and correspondence $\Psi_{n\ell m}\Psi_{NLM} \leftrightarrow f_k$, $k = 1,...,625$.

$n\ell m$/ NLM	000	100	110	210	300	410	420	200	530	310	510	600	610	710	810	910	330	430	630	400	500	700	220	800	900
000	1	2	3	4	38	39	40	55	69	99	117	135	153	271	297	323	349	375	401	427	453	479	505	535	581
100	5	6	7	8	41	42	43	57	71	101	119	137	155	273	299	325	351	377	403	429	455	481	507	537	583
110	9	10	11	12	44	45	46	59	73	103	121	139	157	275	301	327	353	379	405	431	457	483	509	539	585
210	13	14	15	16	47	48	49	61	75	105	123	141	159	277	303	329	355	381	407	433	459	485	511	541	587
300	26	27	28	29	17	18	19	63	77	107	125	143	161	279	305	331	357	383	409	435	461	487	513	543	589
410	30	31	32	33	20	21	22	65	79	109	127	145	163	281	307	333	359	385	411	437	463	489	515	545	591
420	34	35	36	37	23	24	25	67	81	111	129	147	165	283	309	335	361	387	413	439	465	491	517	547	593
200	54	56	58	60	62	64	66	50	51	113	131	149	167	285	311	337	363	389	415	441	467	493	519	549	595
530	68	70	72	74	76	78	80	52	53	115	133	151	169	287	313	339	365	391	417	443	469	495	521	551	597
310	98	100	102	104	106	108	110	112	114	82	83	85	87	289	315	341	367	393	419	445	471	497	523	553	599
510	116	118	120	122	124	126	128	130	132	84	89	90	92	291	317	343	369	395	421	447	473	499	525	555	601
600	134	136	138	140	142	144	146	148	150	86	91	94	95	293	319	345	371	397	423	449	475	501	527	557	603
610	152	154	156	158	160	162	164	166	168	88	93	96	97	295	321	347	373	399	425	451	477	503	529	559	605
710	270	272	274	276	278	280	282	284	286	288	290	292	294	170	171	173	175	177	179	181	183	185	187	561	607
810	296	298	300	302	304	306	308	310	312	314	316	318	320	172	189	190	192	194	196	198	200	202	204	563	609
910	322	324	326	328	330	332	334	336	338	340	342	344	346	174	191	206	207	209	211	213	215	217	219	565	611
330	348	350	352	354	356	358	360	362	364	366	368	370	372	176	193	208	221	222	224	226	228	230	232	567	613
430	374	376	378	380	382	384	386	388	390	392	394	396	398	178	195	210	223	234	235	237	239	241	243	569	615
630	400	402	404	406	408	410	412	414	416	418	420	422	424	180	197	212	225	236	245	246	248	250	252	571	617
400	426	428	430	432	434	436	438	440	442	444	446	448	450	182	199	214	227	238	247	254	255	257	259	573	619
500	452	454	456	458	460	462	464	466	468	470	472	474	476	184	201	216	229	240	249	256	261	262	264	575	621
700	478	480	482	484	486	488	490	492	494	496	498	500	502	186	203	218	231	242	251	258	263	266	267	577	623
220	504	506	508	510	512	514	516	518	520	522	524	526	528	188	205	220	233	244	253	260	265	268	269	579	625
800	534	536	538	540	542	544	546	548	550	552	554	556	558	560	562	564	566	568	570	572	574	576	578	530	531
900	580	582	584	586	588	590	592	594	596	598	600	602	604	606	608	610	612	614	616	618	620	622	624	532	533

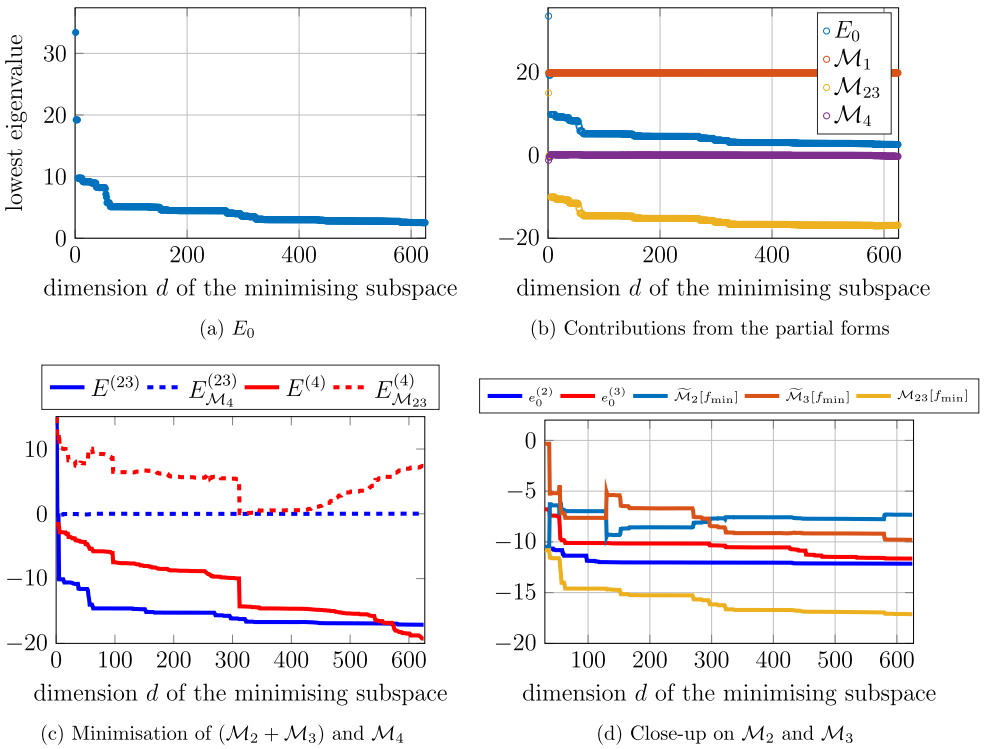

Figure 1. (a) Lowest eigenvalue of the global $d \times d$-matrix. (b) Contributions $\mathcal{M}_1[f_{\min}] = 2\pi^2$, $\mathcal{M}_{23}[f_{\min}] \equiv \mathcal{M}_2[f_{\min}] + \mathcal{M}_3[f_{\min}]$, and $\mathcal{M}_4[f_{\min}]$ from the partial forms to the global lowest eigenvalue. (c) Lowest eigenvalue $E^{(23)} \equiv \mathcal{M}_{23}[f^{(23)}_{\min}]$ of the partial form $\mathcal{M}_{23} \equiv \mathcal{M}_2 + \mathcal{M}_3$ and corresponding contribution $E^{(23)}_{\mathcal{M}_4} \equiv \mathcal{M}_4[f^{(23)}_{\min}]$ of the partial form $\mathcal{M}_4$ evaluated in the minimizer $f^{(23)}_{\min}$ of $\mathcal{M}_{23}$ (solid and dashed blue lines), compared with the lowest eigenvalue $E^{(4)} \equiv \mathcal{M}_4[f^{(4)}_{\min}]$ of the partial form $\mathcal{M}_4$ and corresponding contribution $E^{(4)}_{\mathcal{M}_{23}} \equiv \mathcal{M}_{23}[f^{(4)}_{\min}]$ of the partial form $\mathcal{M}_{23}$ evaluated in the minimizer $f^{(4)}_{\min}$ of $\mathcal{M}_4$ (solid and dashed red lines). (d) Lowest eigenvalue $e_0^{(j)}, j = 2$, 3, of the Hermitian form $\widetilde{\mathcal{M}}_j[f, g] \equiv \frac{1}{2}\big(\mathcal{M}_j[f, g] + \overline{\mathcal{M}_j[g, f]}\big)$ and contribution of each $\widetilde{\mathcal{M}}_j$ on the global minimizer $f_{\min}$. Clearly, $\mathcal{M}_{23}[f] = \widetilde{\mathcal{M}}_2[f] + \widetilde{\mathcal{M}}_3[f]$ for any f.

quantum numbers n, N and a balanced presence of odd and even angular quantum numbers ℓ, L. The dimension d of the minimization subspace ranged from $d = 1$ to $d = 625$.

Figure 1(a) shows the behaviour of the lowest eigenvalue E_0 of the resulting $d \times d$ matrices. Figure 1(b) shows also the separate contributions to E_0 coming from the matrices relative to the partial quadratic forms $\mathcal{M}_1$,$\mathcal{M}_2 + \mathcal{M}_3$, and $\mathcal{M}_4$, that is, the expectation of such matrices computed on the minimizing vector of the global matrix. Numerical values of E_0 for selected values of d are collected in table 2. We accept a value

$$E_0 \approx 2.5 \tag{71}$$

for the asymptotic lowest eigenvalue, which corresponds to

$$\kappa \approx 0.25 \tag{72}$$

in the stability bound (65).

Table 2. Lowest eigenvalue E_0 of the truncated matrix $\mathcal{M}$ for selected values of the dimension d of the minimizing subspace, and separate contributions $E_0^{\mathcal{M}_{23}}$ and $E_0^{\mathcal{M}_4}$ from the partial matrices $\mathcal{M}_{23} \equiv \mathcal{M}_2 + \mathcal{M}_3$ and $\mathcal{M}_4$ respectively. (Recall that the contribution of the partial matrix $\mathcal{M}_1$ is always identically equal to $E_0^{\mathcal{M}_1} \equiv 2\pi^2 \approx 19.74$).

d	E_0	$E_0^{\mathcal{M}_{23}}$	$E_0^{\mathcal{M}_4}$
1	33.4	14.9	−1.3
2	19.2	0.0	−0.5
5	9.7	−10.1	0.1
14	9.2	−10.6	0.0
38	8.3	−11.5	0.1
57	5.8	−14.0	0.0
63	5.1	−14.6	0.0
153	4.6	−15.1	0.0
338	3.0	−16.7	0.0
545	2.8	−16.9	−0.1
625	2.5	−16.9	−0.3

Our findings show that, irrespectively of the dimension d of the minimizing subspace (apart from the, obviously not particularly meaningful, very low values of d), the minimizing function $f_{\min}$ gives a negligible contribution

$$\mathcal{M}_4\big[f_{\min}\big] \;\approx\; 0 \tag{73}$$

from the partial quadratic form $\mathcal{M}_4$ to the lowest eigenvalue E_0. Thus, E_0 is essentially given by the difference of the fixed contribution $2\pi^2 = \mathcal{M}_1[f_{\min}]$ and a negative contribution $(\mathcal{M}_2 + \mathcal{M}_3)[f_{\min}]$ (figure 1(b)).

As a further check, we minimized separately also the partial forms $\mathcal{M}_2 + \mathcal{M}_3$ and $\mathcal{M}_4$, say, with minimizers respectively $f_{\min}^{(23)}$ and $f_{\min}^{(4)}$, and we computed the contributions of the other partial forms on such minimizers (figure 1(c)). The numerical evidence

$$\big(\mathcal{M}_2 + \mathcal{M}_3\big)\big[f_{\min}^{(23)}\big] \;\approx\; \big(\mathcal{M}_2 + \mathcal{M}_3\big)\big[f_{\min}\big] \tag{74}$$

is immediately explained by the fact that the global minimization favours configurations where $\mathcal{M}_4$ practically does not contribute, and hence the minimum of $\mathcal{M}$ is almost the same as the minimum of $\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$, i.e., $2\pi^2$ plus the minimum of $\mathcal{M}_2 + \mathcal{M}_3$. Figure 1(c) also shows that even though $\mathcal{M}_4[f_{\min}^{(4)}]$ attains negative values, they are compensated by positive values of $(\mathcal{M}_2 + \mathcal{M}_3)[f_{\min}^{(4)}]$ in such a way that

$$\begin{aligned}\big(\mathcal{M}_2 + \mathcal{M}_3\big)\big[f_{\min}^{(4)}\big] + \mathcal{M}_4\big[f_{\min}^{(4)}\big] &\geqslant \big(\mathcal{M}_2 + \mathcal{M}_3\big)\big[f_{\min}^{23}\big] + \mathcal{M}_4\big)\big[f_{\min}^{23}\big] \\ &\approx \big(\mathcal{M}_2 + \mathcal{M}_3\big)\big[f_{\min}\big],\end{aligned} \tag{75}$$

thus confirming once again that non-zero, negative (minimal) values of $\mathcal{M}_4$ are penalized in the global minimization.

As for the contribution from quadratic form $\mathcal{M}_2 + \mathcal{M}_3$, the analysis summarized in figure 1(d) shows that it is essentially equipartite between $\mathcal{M}_2$ and $\mathcal{M}_3$.

A closer inspection to the structure of the global minimizer $f_{\min}$ reveals a remarkable pattern, displayed in figures 2(a) and (b), that appears to be independent of the size of the minimization subspace. When we look at the representation

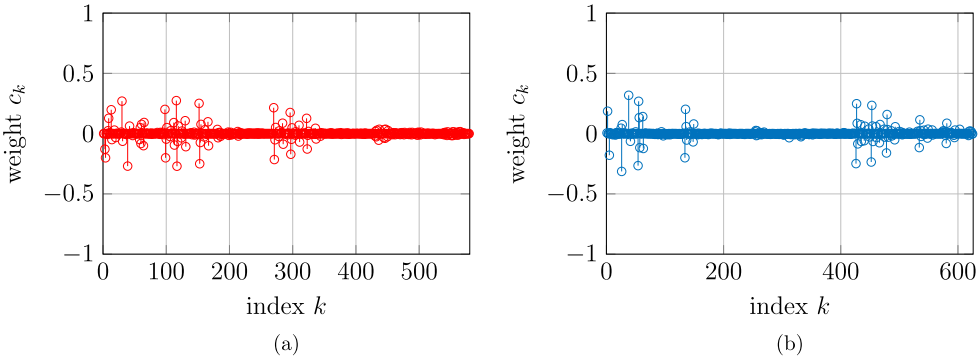

Figure 2. Coefficients of the approximate minimizer $f_{\min}$ in the representation $f_{\min} = \sum_{k=1}^{d} c_k f_k$ (76) for the first 579 and for all the 625 basis functions of the form (70) of the minimization pool of table 1.

$$f_{\min} = \sum_{k=1}^{d} c_k f_k \tag{76}$$

of the minimizer $f_{\min}$ in terms of the basis functions f_1 ,..., f_d spanning the chosen minimizing space (the c_k's are all real, as a consequence of the choice of our basis functions), we first of all observe that (76) is considerably sparse, that is, only a small number of weights c_k's are practically non-zero. Moreover, and most significantly, the non-zero contributions in (76) come typically in doublets (k, k') with $c_{k'} \approx -c_k$, that is, contributions of the form $c_k(f_k - f_{k'})$, where an explicit check of the labelling reveals that

$$f_k(\mathbf{p}, \mathbf{q}) = \Psi_{n,\ell,m}(\mathbf{p})\Psi_{N,L,M}(\mathbf{q})$$
$$f_{k'}(\mathbf{p}, \mathbf{q}) = \Psi_{n,\ell,m}(\mathbf{q})\Psi_{N,L,M}(\mathbf{p})$$

for certain indices n, ℓ, m, N, L, M. This implies that, albeit not evident *a priori*, *the minimizer is anti-symmetric in* $(\mathbf{p}, \mathbf{q})$.

Such a feature of $f_{\min}$ is present independently of the dimension d (large enough), although, not surprisingly, when we enlarge the pool of functions and hence the minimizing space, some of the doublets of basis functions that carry the (largest part of the) non-zero contribution to $f_{\min}$ do change with d. Explicitly, we checked that minimizing the quadratic form $\mathcal{M}$ over the first $d = 579$ basis functions yields a minimizer

$$f_{\min} \approx \widetilde{f}_{\min} \in \mathcal{A}\Big(\mathrm{Span}\big\{\Psi_{n10}\Psi_{N00}\big\}\Big),$$

where the linear span runs over all the considered indices n, N up to the 579th basis function and $\mathcal{A}$ is the operator of global anti-symmetrization. On the other hand, the minimization over all the $d = 625$ considered basis functions yields a minimizer

$$f_{\min} \approx \widetilde{f}_{\min} \in \mathcal{A}\Big(\mathrm{Span}\big\{\Psi_{n00}\Psi_{N00}\big\}\Big).$$

The numerical evidence that the minimizer $f_{\min}$ is an anti-symmetric function explains in turn the vanishing of $\mathcal{M}_4[f_{\min}]$, see (73). Indeed, it is immediate to see from (62) that the integrand function, say $I(\mathbf{p}, \mathbf{q}, \mathbf{r})$, in the integral that defines $\mathcal{M}_4[f]$ is an odd function of $\mathbf{r}$ whenever $f(\mathbf{p}, \mathbf{q}) = -f(\mathbf{q}, \mathbf{p})$, whence $\iiint I(\mathbf{p}, \mathbf{q}, \mathbf{r}) = 0$.

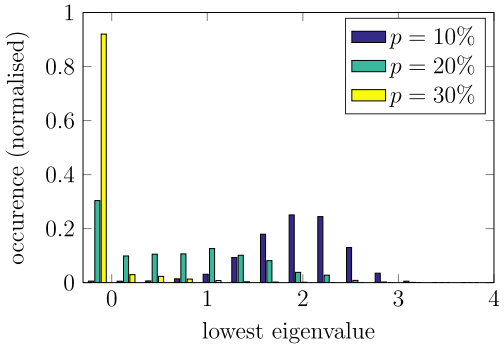

Figure 3. Histogram of the lowest eigenvalue when the computed matrix $(M_{ij}) \equiv (\mathcal{M}[f_i, f_j])_{i,j\,=\,1,\ldots,d}$ is perturbed with a random additive term according to (77). p denotes the amplitude of perturbation (the standard deviation of the normal-distributed perturbative term in units of the calculated value $\left|M_{ij}\right|$).

4.4. Numerical algorithm and error analysis

All matrix entries $\mathcal{M}[f_i, f_j]$ are estimated integral values. The integral procedure used is an implementation of an adaptive algorithm for the approximate calculation of multiple integrals, developed in [7] and implemented[12] in `Matlab`.

The algorithm returns the estimated integral value, an estimated error and the number of evaluations required to achieve the estimated value with the corresponding error. The two parameters that need be specified are the maximum number of evaluations and a stopping criterion to interrupt the calculation if the estimated error reaches the prescribed threshold before the maximum number of evaluations. We set this threshold to 10%.

The computed matrix $(\mathcal{M}[f_i, f_j])_{i,j=1,\ldots,d}$ resulted to be extremely sparse, with a small fraction of non-zero entries, scattered over the off-diagonal region, plus the obvious constant contribution $2\pi^2$ along the diagonal. For a large fraction of entries we have solid reasons not to trust the errors estimated by the algorithm[13].

In turn, as a check of the reliability of the numerical evaluations, we randomly perturbed the computed matrix entries $M_{ij} \equiv \mathcal{M}[f_i, f_j]$ by means of an additive term, normal-distributed around zero with standard deviation given by a chosen percentage p of the calculated value $\left|M_{ij}\right|$, i.e.

$$M_{ij}^{\text{pert}} = M_{ij} + N\big(0,\, p \cdot \left|M_{ij}\right|\big). \tag{77}$$

The outcoming distribution of the lowest eigenvalue of the perturbed matrix shows us that perturbations with standard deviation up to 20% of the calculated matrix value keep such a perturbed lower eigenvalue well above zero (figure 3).

[12] Available at http://math.wsu.edu/faculty/genz/software/matlab/adapt.m

[13] This is primarily the case for all those entries with *absolute* value $\leqslant 10^{-4}$, which were assigned an estimated relative error above the 100%. Since the values of such entries fall below 10^{-4} and are thus 2–5 orders of magnitude less than the other 'non-zero' entries, we were confident to consider such entries as actual zeroes, their nominal estimated error being not trustable. The other case includes those non-zero entries whose estimated error exceeds 100% and is not trustable for the reason that the 100% of the evaluated entry exceeds by far the theoretical bounds (78) on the matrix entries which are known in the literature and are mentioned in section 5.

5. Discussion and concluding remarks

Our analysis can be summarized by saying that the considered Hamiltonian H_α for a '2+2' fermionic system with a two-body interaction of zero range and s-wave two-body scattering length $-\alpha^{-1}$, constructed rigorously by implementing the Bethe–Peierls contact condition within the Krein–Višik–Birman theory of self-adjoint extensions, turns out to be stable in the sense that its energy form is bounded below.

In the attractive case $\alpha < 0$ we could only estimate quite crudely the negative contributions to the energy form and the resulting lower bound $-\lambda$ in $\left\langle \Phi \middle| H_\alpha \middle| \Phi \right\rangle \geqslant -\lambda \, \|\Phi\|^2$ is certainly neither optimal nor a good approximation to the actual energy minimum. It is therefore not particularly informative to restore the physical units in λ, which one could easily do anyway by following our arguments. Conversely, whenever $\alpha \geqslant 0$ we proved (and actually no numerics is needed for that) that the energy is non-negative, and in fact its minimum is precisely zero in the case of infinite scattering length ($\alpha = 0$), the so-called unitary limit.

Noticeably, our separate analysis on the lowest contribution to the minimum E_0 of the global quadratic form $\mathcal{M}$ coming from the partial forms $\mathcal{M}_2 + \mathcal{M}_3$ and $\mathcal{M}_4$ (figure 1(b)), as well as on the minimum of such partial forms (figure 1(c)), indicates that in units $2\pi^2$ (which is the fixed contribution given by $\mathcal{M}_1$) the actual magnitude of such partial contributions is strictly smaller than 1, and in fact almost zero in the case of $\mathcal{M}_4$. These values are significantly lower than the known theoretical bounds

$$|\mathcal{M}_j[f]| \;\leqslant\; b_j \, \mathcal{M}_1[f] \qquad j = 2, 3, 4, \tag{78}$$

that were recently established in the mathematical literature [32]. Those bounds only allowed us to conclude that

$$\mathcal{M}[f] \;\geqslant\; \mathcal{M}_1[f] - \sum_{j=2}^{4} b_j |\mathcal{M}_j[f]| \tag{79}$$

which in practice provides a negative, and hence not decisive, lower bound.

We are therefore confident that a closer inspection and further improvements of our numerical findings may indicate the appropriate way to keep into account the compensation effects between $\mathcal{M}_2$, $\mathcal{M}_3$, and $\mathcal{M}_4$ which in (79) are so crudely neglected, and lead to a fully analytical minimization of the global quadratic form $\mathcal{M}$.

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