FORWARD UNTANGLING AND APPLICATIONS TO THE UNIQUENESS PROBLEM FOR THE CONTINUITY EQUATION

STEFANO BIANCHINI AND PAOLO BONICATTO

Abstract. We introduce the notion of forward untangled Lagrangian representation of a measure-divergence vector-measure $\rho(1, \boldsymbol{b})$, where $\rho \in \mathcal{M}^+(\mathbb{R}^{d+1})$ and $\boldsymbol{b}\colon \mathbb{R}^{d+1} \to \mathbb{R}^d$ is a ρ-integrable vector field with $\mathrm{div}_{t,x}(\rho(1, \boldsymbol{b})) = \mu \in \mathcal{M}(\mathbb{R} \times \mathbb{R}^d)$: forward untangling formalizes the notion of forward uniqueness in the language of Lagrangian representations. We identify *local* conditions for a Lagrangian representation to be forward untangled, and we show how to derive *global* forward untangling from such local assumptions. We then show how to reduce the PDE $\mathrm{div}_{t,x}(\rho(1, \boldsymbol{b})) = \mu$ on a partition of $\mathbb{R}^+ \times \mathbb{R}^d$ obtained concatenating the curves seen by the Lagrangian representation. As an application, we recover known well posedeness results for the flow of monotone vector fields and for the associated continuity equation.

Keywords: transport equation, continuity equation, uniqueness, Superposition Principle.

Preprint SISSA 10/2020/MATE

Contents

Introduction

This paper is devoted to the study of a possible variation of the concept of *untangling of trajectories*. This notion has been introduced in [BB19] within the framework of linear transportation theory and it has been used to investigate the Lagrangian structure of 1-dimensional normal currents in $\mathbb{R}^d$ and to establish well-posedness results for the continuity/transport equation drifted by non-smooth vector fields.

The starting point of the present work is the *continuity equation* which we write in the form

$$\mathrm{div}_{t,x}\, \rho(1, \boldsymbol{b}) = \mu \tag{PDE}$$

Date: May 13, 2020.
2010 *Mathematics Subject Classification.* 35F10, 35L03, 28A50, 35D30.

where $\rho \in \mathcal{M}^+(\mathbb{R}^{d+1})$ is a non-negative measure on $\mathbb{R}^{d+1}$, $\boldsymbol{b}\colon \mathbb{R}^{d+1} \to \mathbb{R}^d$ is a ρ-integrable vector field, $\mu \in \mathcal{M}(\mathbb{R}^{d+1})$ is a measure on $\mathbb{R}^{d+1}$, and the divergence operator in (PDE) is understood in the sense of distributions on $\mathbb{R}^{d+1}$. The dimension $d \geq 1$ is an integer number.

It is by now well-known that non-negative measure-valued solutions ρ to (PDE) admit a *representation*: one can find a suitable set of continuous curves in $\mathbb{R}^d$ such that the solution ρ can be thought of as their superposition. In other words, the evolution of a non-negative measure ρ solving (PDE) can be described following a set of continuous trajectories γ, which can be chosen to be characteristic curves of the vector field $\boldsymbol{b}$, i.e.

$$\gamma'(t) = \boldsymbol{b}(t, \gamma(t)), \qquad \text{for a.e. } t. \tag{ODE}$$

The mathematical, precise formulation of this representability property passes through the introduction of a measure η on the space of continuous curves Γ, telling which curves are selected to transport the mass. We highlight that it is also possible to represent, by means of η, the divergence measure μ as a superposition of Dirac masses (without cancellation) at the starting/ending points of the curves. A measure like η is called *Lagrangian representation of* $\rho(1, \boldsymbol{b})$. Its existence is ensured by general results which go back to earlier works by Ambrosio (in the context of the continuity equation) and by Smirnov (in the framework of normal currents).

From the Lagrangian viewpoint, it turns out that the lack of uniqueness of solutions to (PDE) is due to the presence of *intersections* among the trajectories selected by a Lagrangian representation. Aiming at establishing well-posedness results for continuity equation (PDE), one has thus to seek some conditions which prevent the presence of crossings among the curves γ. This was, in a nutshell, what led us to introduce the concept of (local) *cylinders of approximate flow* in [BB19]. It was proven that the existence of such cylinders was enough to ensure that the trajectories were disjoint (or, in the language of [BB19], *untangled*). By means of a disintegration argument, it was then shown how to use this disjoint family of trajectories to recover uniqueness of solutions to (PDE).

In this paper we follow a similar, parallel strategy, but with a major difference: we want the direction of time to come into play. We are thus led to discuss the notion of *forward untangling*: the curves seen by a Lagrangian representation have disjoint graphs only in one direction of time.

Definition 1 (Forward untangled Lagrangian representation)**.** A Lagrangian representation η of $\rho(1, \boldsymbol{b})$ is said to be *forward untangled* if the following condition holds true: η is concentrated on a set $\Delta \subset \Gamma$ made up of trajectories such that for every $(\gamma, \gamma') \in \Delta \times \Delta$ the following implication holds:

$$\text{if there exists } t \in \big(\max\{t^-_\gamma, t^-_{\gamma'}\}, \min\{t^+_\gamma, t^+_{\gamma'}\}\big] \text{ such that } \gamma(t) = \gamma'(t) \text{ then}$$
$$\mathrm{Graph}\gamma\llcorner_{[t, \min\{t^+_\gamma, t^+_{\gamma'}\}]} \text{ coincides with } \mathrm{Graph}\,\gamma'\llcorner_{[t, \min\{t^+_\gamma, t^+_{\gamma'}\}]},$$

where (t^-_γ, t^+_γ) denotes the time interval where the curve γ is defined and $\mathrm{Graph}\,\gamma$ is the graph of the curve γ (similarly for γ').

In other words, η is forward untangled when it is concentrated on a set of curves which may intersect but, if they do, then they remain the same curve in the future: trajectories can thus bifurcate only in the past. Having introduced the notion of forward untangled Lagrangian representation, we now want to follow the path of [BB19]: the first natural step is to identify some *local* conditions ensuring that a "large amount" of trajectories seen by η satisfies the forward untangling condition. It is worth mentioning that "local" means, for us, "inside a *proper set*". These sets were studied in details in [BB19] and their peculiar property is that they make possible the operation of *restriction* of a Lagrangian representation. In other words, we can always localize, in a suitable sense, the measure η inside a proper set; in addition, these sets can be suitably perturbed to adapt to the vector field under consideration. We refer the reader to [BB19, Section 4] for the full treatment of this topic (in the class of a.c. solutions ρ) and to see Section 2.1 below (or [BS]) for the general case.

We thus formulate the following assumption, which replaces the existence of cylinders of approximate flow of [BB19]:

Assumption 2. Let η be a Lagrangian representation of $\rho(1,\boldsymbol{b})$ in a (perturbed) proper set $\Omega \subset \mathbb{R}^{d+1}$. Let $\varpi > 0$ and assume that for all $R > 0$ there exists $r = r(R) > 0$ such that

$$\int_\Gamma \frac{1}{\eta^{\mathrm{in}}(\Sigma^r(\gamma))} \eta^{\mathrm{in}}(\mathcal{A}_{r,R}(\gamma))\, \eta^{\mathrm{in}}(d\gamma) \le \varpi,$$

where:

- η^{in} is the restriction of η to the set of curves which are entering in Ω;
- $\Sigma^r(\gamma)$ is the set of curves entering in Ω at a point z' such that $|z - z'| < r$, being z the entering point of γ in Ω;
- $\mathcal{A}_{r,R}(\gamma)$ is the subset of $\Sigma^r(\gamma)$ given by those curves which are not contained in the "tubular neighbourhood" of radius R around $\operatorname{Graph}\gamma$.

The measure η^{in} is well defined thanks to the fact that Ω is proper. Secondly, *tubular neighobourhood of radius* $R > 0$ *around* $\operatorname{Graph}\gamma$ means

$$\operatorname{Graph}\gamma + B^d_R(0) := \big\{(t,x) : t \in [t^-_\gamma, t^+_\gamma], \quad x = \gamma(t) + z \quad \text{for some } z \in B^d_R(0)\big\},$$

where $B^d_R(0)$ denotes the ball of radius R in around the origin in $\mathbb{R}^d$. The meaning of assumption is thus the following: for a fixed entering trajectory γ, we consider the amount of curves which enter in Ω close to γ (at a distance less than r), but which do not remain uniformly close to γ (i.e. at a distance R). We then average this quantity with respect to the total measure of trajectories entering in Ω close to γ; we ask this final quantity to be controlled (in an $L^1(\eta^{\mathrm{in}})$ sense w.r.t. γ) by ϖ.

It should be clear, from the informal explanation above, that Assumption 2 yields, in the limit as $R \to 0$, a control on the amount of trajectories which present bifurcation in the future, i.e. do *not* satisfy the forward untangling condition. The link between Assumption 2 and forward untangling is mathematically formalized in Proposition 3.2 and Proposition 3.7 of the present work. In the former, we show that a forward untangled representation always fulfills the condition written in Assumption 2. In the latter, instead, a quantitative version of the converse implication is proved. More precisely, postulating the validity of Assumption 2, it is possible to select (locally) a relatively "large" set of trajectories which satisfy the forward untangling condition and are all subset of some maximal curves; the measure of the complement of this set can be controlled by ϖ and by the (mass of the negative part of the) divergence measure μ. Proposition 3.7 can thus be seen as a local "almost forward untangling" result starting from Assumption 2. It is worth mentioning that, in the proof, we exploit some refined tools borrowed from optimal transportation theory (in particular, Kellerer's Duality Theorem).

Having settled the local theory, sticking to the plan of [BB19], we turn to show the global theory of forward untangling. We suppose Assumption 2 holds in a family of small balls around every point and we introduce the *forward untangling functional*: it measures the amount of curves to be removed from a proper set in such a way the remaining ones satisfy the forward untangling condition. As in [BB19], we show that this functional is subadditive w.r.t. union of proper sets; the subadditivity property suggests to replace the constant ϖ in Assumption 2 with $\varpi^\tau(\Omega)$, where ϖ^τ is a measure whose mass is $\tau > 0$. By means of a covering argument, in Corollary 4.5, we can finally show how to obtain a globally forward untangled Lagrangian representation.

The last step is to show how the forward untangling condition is related to (forward) uniqueness of solutions to (PDE). Once a set of forward untangled trajectories is selected, we construct a partition of the space-time made up of "one-dimensional" sets, obtained merging the trajectories whenever they meet. We then implement a reduction argument on this partition, by means of the disintegration theorem. It is possible to write the equation (PDE) as a one-dimensional balance on each element of the partition. The one-dimensional equation obtained on each set can be written in Eulerian coordinates, i.e. without resorting to the Lagrangian representation: this implies (see Theorem 5.5) that, if every representation is forward untangled, then the partition obtained above is unique. We then exploit this universality property of the partition to reduce also the equation $\operatorname{div}(u\rho(1,\boldsymbol{b})) = \nu \in \mathcal{M}(\mathbb{R}^{d+1})$ for $u \in L^\infty(\rho)$ and we use this to give a formula for the measure $\operatorname{div}(u\beta(\rho)(1,\boldsymbol{b}))$, for a convex, Lipschitz function $\beta \colon \mathbb{R} \to \mathbb{R}$.

Finally, being the notion of forward untangling clearly one-sided (in time), it is natural to apply it to vector fields satisfying asymmetric bounds, e.g. monotone vector fields. As an application, in the last section of this paper, we are able to recover some well-posedness results for monotone vector fields (see [BG11]).

We conclude this introduction with a couple of remarks. We believe that this technical variation on the theme of untangling could be of interest in the linear transportation theory. In particular, by imposing a control on the intersection of the curves only *forward in time* many estimates and propositions of the approach presented in [BB19] simplify: Assumption 2 (in comparison with its analogue concerning cylinders of approximate flow in [BB19]) has the advantage of making more transparent and easier some of the proofs, although the tools we exploit are essentially the same. Furthermore, the language and the formulation of forward untangling can be used to deduce, in a rather direct way, two-sided well-posedness results for other classes of vector fields: as a single example, in [Bon17, Chapter 9], it is shown that Assumption 2 is fulfilled by a class of vector fields whose difference quotients can be estimated in terms of particular weak-L^1 functions. In that case, the estimates are of the same form as the one in Assumption 2, but hold both backward and forward in time, thus yielding to full untangling of the representation (and hence uniqueness to (PDE), following the analysis of [BB19]). Notice that such class contains vector fields arising from convolutions of singular kernels with L^1 functions (and it is relevant in view of the results of [BC13, CNSS17]).

Synopsis of the paper. The paper contains six sections organized in the following way.

Section 1 explains the notation adopted throughout the rest of the article and should be used as a reference.

Section 2 collects useful preliminary material, mostly taken from [BB19]: we recall the definitions of Lagrangian representations, Ambrosio-Smirnov Theorem (see Subsection 2.1); then, in Subsection 2.2 we present a possible generalization of the concept of proper set, which was studied in [BB19] only for a.c. measures ρ. Finally, in Subsection 2.3, we recall the tools from optimal transportation theory we will need and in particular Kellerer's Duality Theorem.

Section 3 develops the local theory of the forward untangling: we define forward untangled Lagrangian representation in Definition 3.1 and then we prove, in Proposition 3.2 and Proposition 3.7, the mutual relationship of Assumption 2 with forward untangled representations.

We then face, in Section 4, the global theory: we introduce the *forward untangling functional* in Definition 4.1 and we prove it is subadditive in Proposition 4.2. Corollary 4.5 shows how to pass to a global untangling representation (starting from the local estimates like in Assumption 2).

In Section 5 we construct a partition of the space-time $\mathbb{R} \times \mathbb{R}^d$ on which we reduce the equation (PDE). Definition 5.1 introduces the concept of *concatenated* family of trajectories and, by means of such notion, we then define the sets E^t_x and F^t_x, which make up the partitions, respectively, of $\varDelta$ and of $\mathbb{R}^{d+1}$. Proposition 5.4 illustrates how to reduce the equation (PDE) on each set F^t_x and, in Theorem 5.5, it is shown that, if *every* Lagrangian representation of $\rho(1, \boldsymbol{b})$ is forward untangled, then the partition into sets F^t_x is essentially unique. Finally, Theorem 5.12 deals with the *composition measure*: we consider a function $u \in L^\infty(\rho)$ and such that $\operatorname{div}(u\rho(1, \boldsymbol{b})) = \nu \in \mathcal{M}(\mathbb{R}^{d+1})$ and we want to find a formula for $\operatorname{div}(\beta(u)\rho(1, \boldsymbol{b}))$, being $\beta \colon \mathbb{R} \to \mathbb{R}$ a Lipschitz, convex function. To do this, we first write the balances on time stripes (see Lemma 5.7), and then on more general sets (see Lemma 5.8). By a covering argument, we finally obtain in Theorem 5.12 a representation formula for the measure $\operatorname{div}(\beta(u)\rho(1, \boldsymbol{b}))$ in terms of μ, ν and an error term, the measure $\mathrm{j}_{\beta,u}$, which takes into account the possible merging among the trajectories.

Finally, Section 6 is devoted to the analysis of monotone vector fields: forward untangling is applied to the study of the differential inclusion $\dot{x}(t) \in -A(t)$, for a maximal monotone operator $A(t)$ on $\mathbb{R}^d$.

Acknowledgements. The work of the second author was supported by ERC Starting Grant 676675 FLIRT.

1. Notation

We collect in this section the notations we will be using throughout the paper.

Euclidean spaces. For an integer $d \geq 1$, the d-dimensional euclidean real vector space will be written as $\mathbb{R}^d$. In the following we will often consider the space $\mathbb{R}^{d+1}$ or the space $\mathbb{R}^+ \times \mathbb{R}^d$, whose coordinates will be denoted by t (time) and x (space), with $t \in \mathbb{R}$ (or $t \in \mathbb{R}^+$) and $x \in \mathbb{R}^d$. The open ball in $\mathbb{R}^d$ centered at a point $x \in \mathbb{R}^d$ with radius r is

$$B^d_r(x) := \big\{ y \in \mathbb{R}^d : |y - x| < r \big\}.$$

When $x = 0$ and there is no risk of confusion we will simply write B^d_r to denote $B^d_r(0)$. If not otherwise stated, Ω will stand for a generic open set in $\mathbb{R}^d$.

Metric spaces and topology. If (X, d) is a metric space, the ball centered in $x \in X$ with radius r will denoted $B^X_r(x)$, and $B_r(x)$ when no confusion occurs about X. If $E \subset X$ then $\operatorname{dist}(E, x)$ is the distance of x from the set E, defined as the infimum of $\mathsf{d}(x, y)$ as y varies in E. The closure of a set A is denoted by $\operatorname{clos} A$ usually being clear the ambient topological space. The relative closure of A in the topological space B is $\operatorname{clos}(A, B)$. Similarly, the interior will be written as $\operatorname{Int} A$ or $\operatorname{Int}(A, B)$. The boundary will be written as $\operatorname{Fr}(A)$ or $\operatorname{Fr}(A, B)$ and, in some cases (mainly for $\Omega \subset \mathbb{R}^d$), we will use the more conventional notation $\partial\Omega$. We will say that A has compact closure in B, writing $A \Subset B$, if $\operatorname{clos} A$ is a compact set contained in B. In a normed space X, given two sets $A, B \subset X$ we denote their sum by $A + B$, which is the set defined by

$$A + B := \Big\{ x \in X : x = a + b \text{ for some } a \in A, b \in B \Big\}.$$

In most of the cases B will be taken as a ball. The power set of set X will be denoted by $\mathcal{P}(X)$. Given a product space $X \times Y$, we denote the projection on the space X by $\mathtt{p}_X$: sometimes we will also write $\mathtt{p}_j : \prod_i X_i \to X_j$ to denote the projection on the j-component X_j. In the product space $X \times Y$, for a set $A \subset X \times Y$ we will denote its sections as

$$A(x) = \big\{ y : (x, y) \in A \big\}, \quad A(y) = \big\{ x : (x, y) \in A \big\}.$$

We say that the family $\{A_\alpha\}_{\alpha \in I}$ (I some set of index) is a *covering* of A if

$$A \subset \bigcup_\alpha A_\alpha,$$

and, if the elements of the family are disjoint, i.e. $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$, we say it is a *partition* of A.

Functions, distributions and differential operators. If A is a set, we will denote by $\mathbb{1}_A$ the characteristic function

$$\mathbb{1}_A(x) := \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

The identity function is denoted by $\operatorname{id}(x) = x$. The graph of a function f is denoted as $\operatorname{Graph} f$, and the support by $\operatorname{supp} f$. The restriction of a function f to a set A will be written as $f \llcorner_A$. In general, given X, Y topological spaces, $C(X, Y)$ will stand for the space of continuous functions $f : X \to Y$. The space of distributions over Ω will be $\mathscr{D}'(\Omega)$. The duality pairing between a distribution $f \in \mathscr{D}'(\Omega)$ and a smooth test function $\psi \in C^\infty_c(\Omega)$ will be written as $\langle f, \psi \rangle$. In the case of 1-dimensional BV functions f (or, in general, whenever the limits exist), we will write

$$f(\bar{x}\pm) = \lim_{x \to \bar{x}\pm} f(x) \tag{1.1}$$

for the right/left limit. The divergence of a vector field $\boldsymbol{b}$ will be denoted by $\operatorname{div} \boldsymbol{b}$. The same notation will be used also for the distributional counterpart, while $\operatorname{div}_{t,x} \rho(1, \boldsymbol{b}) := \partial_t \rho + \operatorname{div}_x(\rho \boldsymbol{b})$.

Measure theory. We will denote the Lebesgue measure in $\mathbb{R}^d$ by $\mathcal{L}^d$ and by $\mathcal{H}^d$ the d-dimensional Hausdorff measure in $\mathbb{R}^{d+k}$, $k > 0$. The *Dirac mass at* x will be written as δ_x. For a generic signed Radon measure μ on $\mathbb{R}^d$ we will write $|\mu|$ to denote the associated total variation measure and $\|\mu\| := |\mu|(\mathbb{R}^d)$ for its mass. The *push-forward* of a measure μ on X with respect to a Borel function $f : X \to Y$ is defined as the measure on Y given by $f_\sharp\mu(B) := \mu(f^{-1}(B))$ for all Borel set $B \subset Y$. Notice that for a Borel map $g\colon Y \to \mathbb{R}$ it holds

$$\int_Y g(y)\,(f_\sharp\mu)(dy) = \int_X (g \circ f)(x)\,\mu(dx).$$

The restriction of a measure μ to a set A will be written as $\mu \llcorner_A$. The positive/negative part of μ will be denoted by $\mu^\pm$: recall that it holds $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. The set of signed Radon measures over X is denoted by $\mathcal{M}(X)$, the non-negative Radon measures with $\mathcal{M}^+(X)$ and the bounded Radon measures by $\mathcal{M}_b(X)$. Since all results in this paper are local in space-time, we will not distinguish between weak and narrow convergence, and sometime we will just write weak (or weak*) convergence of measures to denote both of them.

Integration theory. Usually the integral of a Borel function f w.r.t. a measure μ will be written (when it exists) as

$$\int f(x)\mu(dx) \quad \text{or} \quad \int f\,\mu,$$

We will sometimes avoid to write the set of integration, being implicitly characterized by the measure w.r.t. we are integrating. The disintegration of a measure μ w.r.t. a partition $\{A_\alpha\}_\alpha$ will be written as

$$\mu = \int \mu_\alpha f_\sharp\mu(\alpha),$$

where f is the partition function, i.e. $f^{-1}(\alpha) = A_\alpha$.

2. Preliminaries on Lagrangian representations, proper sets and Optimal Transport

We present in this section some preliminary material we will use in the paper.

2.1. Lagrangian representations.

Consider a vector field of the form

$$\rho(1, \boldsymbol{b}) \in \mathcal{M}(\mathbb{R}^{d+1}, \mathbb{R}^{d+1}), \tag{2.1}$$

where

$$\rho \in \mathcal{M}^+(\mathbb{R}^{d+1}), \qquad \boldsymbol{b} \in L^1(\rho; \mathbb{R}^d). \tag{2.2}$$

We assume that ρ is compactly supported and that it holds

$$\operatorname{div}(\rho(1, \boldsymbol{b})) = \mu \in \mathcal{M}(\mathbb{R}^{d+1}) \tag{2.3}$$

in the sense of distribution, i.e. $\rho(1, \boldsymbol{b})$ is a *measure-divergence vector(-valued measure) field*. To avoid dealing with sets of ρ-negligible measure, we will assume that $\boldsymbol{b}$ is defined pointwise everywhere as Borel function.

An absolutely continuous curve $\gamma\colon I \to \mathbb{R}^d$, where $I \subset \mathbb{R}$ is an open time interval, is a *characteristic of the vector field* $\boldsymbol{b}$ if it solves the ODE

$$\frac{d}{dt}\gamma(t) = \boldsymbol{b}(t, \gamma(t)),$$

the equality holding $\mathcal{L}^1$-a.e. in I. As done in [BB19, Section 3.1], we will consider the metric space Υ of curves γ: more precisely, let

$$\Upsilon = \Big\{(t_1, t_2, \gamma) : t_1 < t_2,\ \gamma \in C([t_1, t_2], \mathbb{R})\Big\}$$

with the distance

$$d\big((t_1, t_2, \gamma), (t_1', t_2', \gamma')\big) := |t_1 - t_1'| + |t_2 - t_2'| + \max\big\{|\gamma(s) - \gamma'(s)|, s \in [t_1, t_2] \cap [t_1', t_2']\big\},$$

and its subset made of characteristics

$$\Gamma = \Big\{(t_1, t_2, \gamma) \in \Upsilon : \gamma \text{ characteristic in } (t_1, t_2)\Big\}.$$

One can show that Γ is a Borel subset of Υ: indeed

$$\gamma \in \Gamma \quad \Longleftrightarrow \quad \sup_{t,s \in [t_1,t_2]\cap\mathbb{Q}} \left|\gamma(t) - \gamma(s) - \int_s^t \boldsymbol{b}(\tau, \gamma(\tau))d\tau\right| = 0,$$

i.e. it is the 0-level set of a Borel function.

Clearly, given $t_1 < t_2$, a function γ which is continuous in the closed interval $[t_1, t_2]$ can always be extended to the real line, so that Υ can be seen as a quotient of the space $\mathbb{R}^2 \times C(\mathbb{R}, \mathbb{R}^d)$ with the quotient topology. In what follows, to shorten the notation, instead of the triplet (t_1, t_2, γ) we will write only γ, and denote its interval of definition by $[t_\gamma^-, t_\gamma^+]$. We will sometimes consider γ as defined only in the open interval $I_\gamma := (t_\gamma^-, t_\gamma^+)$, i.e. $\gamma = \gamma \llcorner_{(t_\gamma^-, t_\gamma^+)}$: this is for convenience, since our results concern the intersection properties of families of curves in the open interval where they are characteristics.

We now recall the following important definition.

Definition 2.1 (Lagrangian representation of the vector field $\rho(1, \boldsymbol{b})$)**.** We say that a bounded, positive measure $\eta \in \mathcal{M}_b^+(\Upsilon)$ is a *Lagrangian representation of the vector-valued measure* $\rho(1, \boldsymbol{b})$ if the following conditions hold:

(1) η is concentrated on the set Γ of absolutely continuous solutions to the ODE
$$\dot\gamma(t) = \boldsymbol{b}(t, \gamma(t)), \tag{2.4}$$
which explicitly means for every $s, t \in I_\gamma$
$$\int_\Gamma \left| \gamma(t) - \gamma(s) - \int_s^t \boldsymbol{b}(\tau, \gamma(\tau))\, d\tau \right| \eta(d\gamma) = 0;$$
(2) if $(\mathrm{id}, \gamma)\colon I_\gamma \to I_\gamma \times \mathbb{R}^d$ denotes the map defined by $t \mapsto (t, \gamma(t))$, then it holds in the sense of measures
$$\rho(1, \boldsymbol{b}) = \int_\Gamma (\mathrm{id}, \gamma)_\sharp \big((1, \dot\gamma) \mathcal{L}^1 \llcorner_{I_\gamma}\big)\, \eta(d\gamma);$$
(3) we can decompose the divergence μ as local superposition of Dirac masses without cancellation, i.e.
$$\mu = \int_\Gamma \Big[\delta_{t_\gamma^-, \gamma(t_\gamma^-)}(dt, dx) - \delta_{t_\gamma^+, \gamma(t_\gamma^+)}(dt, dx) \Big]\, \eta(d\gamma),$$
$$|\mu| = \int \Big[\delta_{t_\gamma^-, \gamma(t_\gamma^-)}(dt, dx) + \delta_{t_\gamma^+, \gamma(t_\gamma^+)}(dt, dx) \Big]\, \eta(d\gamma),$$
where we recall that, for every γ, the interval in which it is a characteristic is denoted by $I_\gamma = (t_\gamma^-, t_\gamma^+)$.

The existence of such a measure η is ensured by the following

Theorem 2.2 ([Smi94])**.** *Let $\rho(1, \boldsymbol{b})$ be as in (2.1), i.e. satisfying (2.2) and (2.3). Then there exists a Lagrangian representation of $\rho(1, \boldsymbol{b})$ in the sense of Definition 2.1.*

For the proof (which reduces to a reparameterization of the curves), one can adapt the proof of [BB19, Theorem 3.2] (see also the discussion in [Bon17, Chapter 3]). Observe that for all γ the interval of definition is bounded (because we assume $\rho(1, \boldsymbol{b})$ with compact support), so that if $\mu^\pm$ is the positive/negative part of the divergence we can disintegrate η according to
$$\eta = \int_{\mathbb{R}^{d+1}} \eta_z\, \mu^-(dz) = \int_{\mathbb{R}^{d+1}} \eta_z\, \mu^+(dz), \qquad \mu^\pm = \big(t_\gamma^\pm, \gamma(t_\gamma^\pm)\big)_\sharp \eta. \tag{2.5}$$
We remark finally that, by the first and second points of Definition 2.1, it follows that
$$\int_\Gamma \bigg[\int_{I_\gamma} |\dot\gamma|\, \mathcal{L}^1 \bigg] \eta(d\gamma) = \int_\Gamma \bigg[\int_{I_\gamma} |\boldsymbol{b}(t, \gamma(t))|\, dt \bigg] \eta(d\gamma) = \int_{\mathbb{R}^{d+1}} \rho |\boldsymbol{b}| < +\infty$$
so that the total variation of η-a.e. curve is finite, and thus the limits $\gamma(t_\gamma^\pm) \in \mathbb{R}^d$ exist.

Remark 2.3. In the case $\mu = 0$ the existence of a Lagrangian representation can also be inferred from the so called Ambrosio's Superposition Principle [Amb04].

In the paper we will write, with a slight abuse of notation,
$$\mathrm{Graph}\, \gamma + B_r^d(0) := \big\{ (t, x) : t \in [t_\gamma^-, t_\gamma^+],\ x \in \gamma(t) + B_r^d(0) \big\}. \tag{2.6}$$

2.2. **Proper sets.** Proper sets were introduced in the paper [BB19], to whom we refer the reader for a complete treatment. Here we limit ourselves to recall the definition and the main properties we will use in the following sections. Since the measures we consider are not necessarily absolutely continuous w.r.t. $\mathcal{L}^{d+1}$, we use the definition of [BB19, Remark 4.3]; see also [BS] for the full discussion.

Let $f\colon \mathbb{R}^{d+1} \to \mathbb{R}$ be a bounded Lipschitz function with compact support.

Definition 2.4 (Inner Proper Sets)**.** The open, bounded set $\Omega = \{f > h\}$ is called $\rho(1, \boldsymbol{b})$-*inner proper* if there exists a sequence $\delta_n \searrow 0$ such that the measures

$$\nu^{\delta_n} := \frac{1}{\delta_n}(1, \boldsymbol{b}) \cdot \nabla f \rho \llcorner_{f^{-1}(h,h+\delta_n)}$$

satisfy

$$\nu^{\delta_n} \rightharpoonup \nu, \qquad |\nu^{\delta_n}| \rightharpoonup |\nu|,$$

for some measure $\nu \in \mathcal{M}(\mathbb{R}^{d+1})$.

It is fairly easy to see that ν is the distributional trace.

Definition 2.5 (Proper sets)**.** The set $\Omega_h = \{f > h\}$ is $\rho(1, \boldsymbol{b})$-*proper* if it is inner proper, $\{-f > h\}$ is inner proper, and the two traces coincides:

$$\mathsf{Tr}\left(\rho(1, \boldsymbol{b}), \Omega\right) = \mathsf{Tr}\left(\rho(1, \boldsymbol{b}), \mathbb{R}^{d+1} \setminus \Omega\right).$$

In the following we will write *proper* instead of $\rho(1, \boldsymbol{b})$-proper when there is no ambiguity about the vector field. Using [BB19, Lemma 4.4] or the results in [BS], we have the following proposition.

Proposition 2.6. *Let $f : \mathbb{R}^{d+1} \to \mathbb{R}$ is a Lipschitz function such that $E_h := \{f > h\}$ has compact closure for every $h \in \mathbb{R}$. Then E_h is proper for $\mathcal{L}^1$-a.e. $h \in \mathbb{R}$. In particular:*

(1) for every (t, x) the balls $\{B^{d+1}_r(t, x)\}_{r>0}$ are proper sets for $\mathcal{L}^1$-a.e. $r > 0$;

(2) for every fixed $(t, x) \in \mathbb{R}^{d+1}$ and $r, L > 0$, define the cylinder of center (t, x) and sizes r, L *as*

$$\mathrm{Cyl}^{r,L}_{t,x} := \Big\{(\tau, y) : |\tau - t| < Lr, \big|y - x - \boldsymbol{b}(t, x)(\tau - t)\big| < r\Big\}.$$

Then the cylinders $\{\mathrm{Cyl}^{r,L}_{t,x}\}_{r>0}$ (with $L > 0$ fixed) are proper sets for $\mathcal{L}^1$-a.e. $r > 0$.

Another useful property is expressed in the following proposition: under a transversality assumption of the boundaries, proper sets are closed under finite unions.

Proposition 2.7 ([BB19, Proposition 4.11])**.** *If Ω_1, Ω_2 are proper sets with*

$$\mathcal{H}^d\Big(\mathrm{Fr}\left(\partial\Omega_1 \cap \partial\Omega_2, \partial\Omega_1 \cup \partial\Omega_2\right)\Big) = 0, \tag{2.7}$$

then $\Omega := \Omega_1 \cup \Omega_2$ is proper.

In [BB19] a slightly different definition of proper sets is given, because in that paper it is assumed that $\rho(1, \boldsymbol{b})$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^{d+1}$: it is required that the trace is a measure a.c. w.r.t. $\mathcal{H}^d \llcorner_{\partial\Omega}$, and that $\mathcal{H}^d$-a.e. $x \in \partial\Omega$ is a Lebesgue point of $\rho(1, \boldsymbol{b})\mathcal{L}^{d+1}$. With that definition one can prove that proper sets can be suitably perturbed in order to adapt to the special time-space structure of the vector field $\rho(1, \boldsymbol{b})$. The perturbation is made in such a way that almost all the inflow and outflow of $\rho(1, \boldsymbol{b})$ occurs on open sets which are contained in countably many time-flat hyperplanes: due to the special space-time form of the vector field, many computations becomes simpler thanks to this modification of proper sets.

Theorem 2.8 ([BB19, Theorem 4.18])**.** *Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set. For every $\varepsilon > 0$ there exists a proper set Ω^ε such that*

(1) $\Omega \subset \Omega^\varepsilon \subset \Omega + B^{d+1}_\varepsilon(0)$;

(2) if

$$S^\varepsilon_1 = \Big\{(t, x) \in \partial\Omega^\varepsilon : \mathbf{n} = (1, 0) \text{ in a neigborhood of } (t, x)\Big\},$$

then S^ε_1 is made of Lebesgue points of $\rho(1, \boldsymbol{b})$ up to a $\mathcal{H}^d$-negligible set and

$$\left| \int_{S^\varepsilon_1} \rho\mathcal{H}^d - \int_{\partial\Omega} \rho[(1, \boldsymbol{b}) \cdot \mathbf{n}]^+ \, \mathcal{H}^d \right| < \varepsilon;$$

(3) if
$$S_2^\varepsilon = \Big\{(t,x) \in \partial\Omega^\varepsilon : \mathbf{n} = (-1,0) \text{ in a neigborhood of } (t,x)\Big\},$$
then S_2^ε is made of Lebesgue points of $\rho(1,\boldsymbol{b})$ up to a $\mathcal{H}^d$-negligible set and
$$\bigg| \int_{S_2^\varepsilon} \rho \mathcal{H}^d - \int_{\partial\Omega} \rho[(1,\boldsymbol{b}) \cdot \mathbf{n}]^- \mathcal{H}^d \bigg| < \varepsilon.$$

As observed in [BB19, Section 7], or directly from the proof of the above theorem, we can assume that the countably many sets $\{t > \text{const}\}$ whose boundaries contain $S_1^\varepsilon, S_2^\varepsilon$, are proper. In [BS] it is used a different approach, which does not need the above theorem. In this paper, however, we assume that the above theorem holds also for measure-valued vector fields as follows. We denote by
$$\mathsf{Tr}^\pm\big(\rho(1,\boldsymbol{b}),\Omega\big)$$
the positive/negative part of the measure $\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)$.

Theorem 2.9. *For every $\varepsilon > 0$ there exists a proper set Ω^ε such that*

(1) $\Omega \subset \Omega^\varepsilon \subset \Omega + B_\varepsilon^{d+1}(0)$;

(2) if
$$S_1^\varepsilon = \Big\{(t,x) \in \partial\Omega^\varepsilon : \Omega = \{t' > t\} \text{ in a neigborhood of } (t,x)\Big\},$$
then
$$\Big| \mathsf{Tr}^+\big(\rho(1,\boldsymbol{b}),\Omega^\varepsilon\big)(S_1^\varepsilon) - \mathsf{Tr}^+\big(\rho(1,\boldsymbol{b}),\Omega\big)(\partial\Omega)\Big| < \varepsilon;$$

(3) if
$$S_2^\varepsilon = \Big\{(t,x) \in \partial\Omega^\varepsilon : \Omega = \{t' < t\} \text{ in a neigborhood of } (t,x)\Big\},$$
then
$$\Big| \mathsf{Tr}^-\big(\rho(1,\boldsymbol{b}),\Omega^\varepsilon\big)(S_1^\varepsilon) - \mathsf{Tr}^-\big(\rho(1,\boldsymbol{b}),\Omega\big)(\partial\Omega)\Big| < \varepsilon.$$

The proof of the above theorem can be obtained by repeating the proof of Theorem 2.8 contained in [BB19].

2.2.1. *Restriction of Lagrangian representations to proper sets.* In addition to this perturbation, proper sets play an important role in connection to Lagrangian representations, as it is possible to restrict a Lagrangian representation to a proper set, in a suitable sense. Given $\rho(1,\boldsymbol{b})$ as in (2.1) and a proper set Ω, let $\{t_i\}_{i\in\mathbb{N}} \subset \mathbb{R}$ be a dense sequence, and label each open component of
$$\gamma^{-1}(\Omega) = \bigcup_{j\in\mathbb{N}} I^j, \qquad I^j = \big(t^{j,-}, t^{j,+}\big),$$
as follows:

(1) if $t^{j,-} = t_\gamma^-$, then denote $t^{j,+} = t_\gamma^{0,+}$;

(2) if $t^{j,+} = t_\gamma^+$, then denote $t^{j,-} = t_\gamma^{0,-}$;

(3) for the remaining open intervals I_γ^j, relabel I^j as
$$I_\gamma^i = \big(t_\gamma^{i,-}, t_\gamma^{i,+}\big), \quad \text{where} \quad i = \min\big\{i' : t_{i'} \in I_\gamma^j\big\},$$
i.e. the apex i of the interval I_γ^i refers to the fact that it contains the time t_i of the dense sequence and eventually some of the t_j, $j > i$.

Let $D_0^-, D_0^+, D_i \subset \varGamma$ be the domains of $t_\gamma^{0,-}$, $t_\gamma^{0,+}$, $t_\gamma^{i,\pm}$ respectively. This labeling is Borel (see [BB19, Lemma 5.5]) and we can now give the following

Definition 2.10. The *restriction operators* $\mathtt{R}_\Omega^{0,\pm}$, $\mathtt{R}_\Omega^i$ and $\mathtt{R}_\Omega\gamma$ are defined respectively as
$$\begin{gathered}\mathtt{R}_\Omega^{0,+}\gamma := \gamma\llcorner_{(t_\gamma^-, t_\gamma^{0,+})}, \qquad \mathtt{R}_\Omega^{0,-}\gamma := \gamma\llcorner_{(t_\gamma^{0,-}, t_\gamma^+)}, \qquad \mathtt{R}_\Omega^i\gamma := \gamma\llcorner_{(t_\gamma^{i,-}, t_\gamma^{i,+})}, \\ \mathtt{R}_\Omega\gamma := \mathtt{R}_\Omega^{0,+}\gamma \cup \mathtt{R}_\Omega^{0,-}\gamma \cup \bigcup_{i\in\mathbb{N}} \mathtt{R}_\Omega^i\gamma,\end{gathered} \tag{2.8}$$
and the measures η_Ω^i are defined as
$$\eta_\Omega^i := (\mathtt{R}_\Omega^i)_\sharp \eta. \tag{2.9}$$

Note that $\mathtt{R}_\Omega$ is multivalued and it is clear that if

$$\rho^i_\Omega(1,\boldsymbol{b}) := \int (\mathrm{id},\gamma)_\sharp\big((1,\dot\gamma)\,\mathcal{L}^1\big)\,\eta^i_\Omega(d\gamma), \tag{2.10}$$

then in Ω

$$\rho(1,\boldsymbol{b}) = \sum_i \rho^i_\Omega(1,\boldsymbol{b}).$$

Theorem 2.11 ([BB19, Theorem 6.8], [BS])**.** *If Ω is a proper set, the restriction operator $\mathtt{R}_\Omega$ maps a Lagrangian representation of $\rho(1,\boldsymbol{b})$ to a Lagrangian representation of $\rho(1,\boldsymbol{b})\llcorner_\Omega$.*

From the definition of $\mathtt{R}_\Omega$, one can deduce the following proposition.

Proposition 2.12 ([BB19, Proposition 6.10], [BS])**.** *Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set and $N \subset \Gamma$ a Borel set. It holds*

$$\eta\big(\big\{\gamma : \exists i \text{ s.t. } \mathtt{R}^i_\Omega\gamma \in N\big\}\big) \le (\mathtt{R}_\Omega)_\sharp\eta(N).$$

2.3. **Optimal transport and duality.** In this section we recall some results contained in the paper [Kel84]. They have already been exploited in the setting of Lagrangian representations in [BB19] and we recall here the main facts for the usefulness of the reader.

Given finitely many finite measures $\mu_i \ge 0$ over Polish (i.e. metrizable, complete and separable) spaces X_i, we define the set of *admissible transference plans* $\mathsf{Adm}(\{\mu_i\}_{i\in I})$ as

$$\mathsf{Adm}(\{\mu_i\}_{i\in I}) = \big\{\pi \ge 0 : (\mathtt{p}_i)_\sharp\pi \le \mu_i\big\} \subset \mathcal{M}^+\bigg(\prod_i X_i\bigg).$$

Given a positive Borel function $h \ge 0$, consider the following duality problem:

$$\sup_{\mathsf{Adm}(\{\mu_i\})} \int h\,\pi = \inf\bigg\{\sum_i \int h_i\,\mu_i, h_i \text{ Borel}, \sum_i h_i \ge h\bigg\}. \tag{2.11}$$

We recall the following deep duality result:

Theorem 2.13 ([Kel84, Theorems 2.14, 2.12])**.** *The equality (2.11) holds if h is a Borel function, and the infimum is actually a minimum.*

Moreover, in the case of two factors X_1, X_2 and when h is a characteristic function, the infimum can be restricted to (characteristic functions of) Borel sets.

Proposition 2.14 ([Kel84, Proposition 3.3.])**.** *If $n = 2$ and $h = \mathbb{1}_B$, then the r.h.s. of (2.11) can be replaced by*

$$\inf\Big\{\mu_1(B_1) + \mu_2(B_2) : B_1, B_2 \text{ Borel such that } \mathbb{1}_{B_1} + \mathbb{1}_{B_2} \ge \mathbb{1}_B\Big\},$$

and the infimum is attained, i.e. it is a minimum.

3. The local theory of forward untangling

Consider a vector field of the form $\rho(1,\boldsymbol{b})$ as in (2.1), satisfying (2.2) and (2.3).

Definition 3.1. A Lagrangian representation η of $\rho(1,\boldsymbol{b})$, with $\mathrm{div}(\rho(1,\boldsymbol{b})) = \mu$, is said to be *forward untangled* if the following condition holds true: η is concentrated on a set $\Delta \subset \Gamma$ made up of trajectories such that for every $(\gamma,\gamma') \in \Delta\times\Delta$ the following implication holds:

$$\text{if there exists } t \in \big(\max\{t^-_\gamma, t^-_{\gamma'}\}, \min\{t^+_\gamma, t^+_{\gamma'}\}\big] \text{ such that } \gamma(t) = \gamma'(t) \text{ then}$$
$$\mathrm{Graph}\gamma\llcorner_{[t,\min\{t^+_\gamma,t^+_{\gamma'}\}]} \text{ coincides with } \mathrm{Graph}\,\gamma'\llcorner_{[t,\min\{t^+_\gamma,t^+_{\gamma'}\}]}.$$

This means that the trajectories can bifurcate only in the "past".

Consider now a proper set $\Omega \subset \mathbb{R}^{d+1}$, and let Ω^ε be the perturbed set constructed in Theorem 2.9. For typographical convenience, in this section we will drop the index ε and refer to Ω^ε directly as Ω. Recall that the sets S_1, S_2 are defined in Theorem 2.9, so that essentially all inflow and outflow of $\rho(1,\boldsymbol{b})$ are occurring on open sets which are contained in finitely many time-flat hyperplanes $\{t = \text{const}\}$ and $\mathtt{p}_t(S_1) \subset \{\{t = \text{const}\}$ is locally proper$\}$. Define now

$$\eta^{\text{in}} := \eta\llcorner_{\{\mathrm{Graph}\,\gamma\cap S_1\neq\emptyset\}} = \int_{S_1} \eta^{\text{in}}_z\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz),$$

where the last formula is the disintegration of η^{in} w.r.t. its evaluation on S_1.

We begin by pointing out a necessary condition for a Lagrangian representation to be forward untangled. Let $\tilde{t}^-_\gamma$ be the entering time in Ω, i.e. $\gamma(\tilde{t}^-_\gamma) \in S_1$.

Proposition 3.2. *Let η be a forward untangled Lagrangian representation and let Ω be a perturbed proper set. Then, for every $\varpi, R > 0$ there exists $r > 0$ such that*

$$\int \frac{1}{\sigma(B^d_r(\gamma(\tilde{t}^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(\tilde{t}^-_{\gamma'}) \in \gamma(\tilde{t}^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[\tilde{t}^-_\gamma, \min\{t^+_\gamma, t^+_{\gamma'}\}]} \not\subseteq \mathrm{Graph}\, \gamma + B^d_R(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) \leq \varpi,$$

where

$$\sigma\big(B^d_r(\gamma(\tilde{t}^-_\gamma))\big) = \eta^{\mathrm{in}}\big(\big\{\gamma' : \gamma'(\tilde{t}^-_{\gamma'}) \in \gamma(\tilde{t}^-_\gamma) + B^d_r(0)\big\}\big) = \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)\big(\gamma(\tilde{t}^-_\gamma) + B^d_r(0)\big).$$

See (2.6) for the notation.

Proof. The assumption that Ω is proper, and thus the inner and outer distributional traces coincide, implies that $\eta(\{\gamma(t^-_\gamma) \in \partial\Omega\}) = 0$ so that η-a.e. γ crosses $\partial\Omega$ in an inner point $\tilde{t}^-_\gamma \in (t^-_\gamma, t^+_\gamma)$.

By the forward untangling, it follows that writing the disintegration

$$\eta^{\mathrm{in}} = \int_{S_1} \eta^{\mathrm{in}}_z \, \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz),$$

then for $\mathsf{Tr}^+\left(\rho(1, \boldsymbol{b}), \Omega\right)$-a.e. $z \in S_1$ there exists a curve γ_z such that

$$\eta^{\mathrm{in}}_z\big(\big\{\gamma' : \mathrm{Graph}\, \gamma' \subset \mathrm{Graph}\, \gamma_z\big\}\big) = 1,$$

i.e. only the curves which are restriction of γ_z enter in Ω from the point z. The map $z \mapsto \gamma_z$ can be taken Borel: indeed the crossing point $z \in S_1$ is a continuous function of γ, whose domain can be assumed to be a σ-compact subset $\Delta \subset \Gamma$ by inner regularity, and then its inverse $z \mapsto \{\mathrm{Graph}\, \gamma, \exists \tilde{t}^-_\gamma (\gamma(\tilde{t}^-_\gamma) = z)\}$ is a multifunction with σ-compact graph whose projection gives the graph of $z \mapsto \mathrm{Graph}\, \gamma_z$.

By Lusin's Theorem, for every $\delta > 0$, we can find a compact set $K_\delta \subset S_1$ with

$$\mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(S_1 \setminus K_\delta) < \delta \quad \text{and} \quad K_\delta \ni z \mapsto \gamma_z \text{ continuous w.r.t. } C^0\text{-topology.} \tag{3.1}$$

By the uniform continuity on compact sets, for every $R > 0$ there exists $r_R > 0$ such that

$$\forall z, z' \in K_\delta : \ \gamma_{z'}(\tilde{t}^-_{\gamma'}) \in \gamma_z(\tilde{t}^-_\gamma) + B^d_{r_R}(0) \quad \Rightarrow \quad \mathrm{Graph}\, \gamma_{z'} \llcorner_{[\tilde{t}^-_\gamma, \min\{t^+_\gamma, t^+_{\gamma'}\}]} \subset \mathrm{Graph}\, \gamma_z + B^d_R(0). \tag{3.2}$$

Since for $\mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)$-a.e. $z \in K_\delta$ it holds

$$\lim_{r \to 0} \frac{1}{\sigma(B^d_r(z))} \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)\big(B^d_r(z) \cap (S_1 \setminus K_\delta)\big) = 0$$

by definition of Lebesgue point, by Egorov's Theorem, we can further consider a compact set $K'_\delta \subset K_\delta$ such that

$$\mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(K_\delta \setminus K'_\delta) < \delta \tag{3.3}$$

the above convergence is uniform, i.e. for any $z \in K'_\delta$, $r < r'$ then

$$\frac{1}{\sigma(B^d_r(z))} \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)\big(B^d_r(z) \cap (S_1 \setminus K_\delta)\big) < \delta. \tag{3.4}$$

Set now

$$r = \min\{r_R, r'\}.$$

Observe that we can write

$$\begin{aligned} &\int \frac{1}{\sigma(B^d_r(\gamma(\tilde{t}^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(\tilde{t}^-_{\gamma'}) \in \gamma(\tilde{t}^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[\tilde{t}^-_\gamma, \min\{t^+_{\gamma'}, t^+_\gamma\}]} \not\subseteq \mathrm{Graph}\, \gamma + B^d_R(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) \\ &\leq \int \left\{ \frac{1}{\sigma(B^d_r(z))} \int \mathbb{1}_{\{z' : |z'-z| < r, \, \mathrm{Graph}\, \gamma_{z'} \llcorner_{[\tilde{t}^-_{\gamma_z}, \min\{t^+_{\gamma_{z'}}, t^+_{\gamma_z}\}]} \not\subseteq \mathrm{Graph}\, \gamma_z + B^d_R(0)\}} \, \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz') \right\} \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz), \end{aligned} \tag{3.5}$$

because

$$\left\{\gamma' : \begin{array}{l} \gamma'(\tilde t^-_{\gamma'}) \in \gamma(\tilde t^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\,\gamma' \llcorner_{[\tilde t^-_\gamma, \min\{t^+_{\gamma'}, t^+_\gamma\}]} \not\subseteq \mathrm{Graph}\,\gamma + B^d_R(0) \end{array} \right\}$$
$$\subset \Big\{\gamma' : \mathrm{Graph}\,\gamma' \subset \mathrm{Graph}\,\gamma_{z'}, |z'-z| < r, \mathrm{Graph}\,\gamma_{z'} \llcorner_{[\tilde t^-_{\gamma_z}, \min\{t^+_{\gamma_{z'}}, t^+_{\gamma_z}\}]} \not\subseteq \mathrm{Graph}\,\gamma_z + B^d_R(0)\Big\}.$$

Now we split the integral in z of the r.h.s. of (3.5) in two terms, one on the compact set K_δ and the other in the complement. For simplicity, denote by

$$\mathcal{A}_R := \Big\{(z,z') \in S_1 \times S_1 : \mathrm{Graph}\,\gamma_{z'} \llcorner_{[\max\{\tilde t^-_{\gamma_{z'}}, \tilde t^-_{\gamma_z}\}, \min\{t^+_{\gamma_{z'}}, t^+_{\gamma_z}\}]} \not\subseteq \mathrm{Graph}\,\gamma_z + B^d_R(0)\Big\}.$$

Then we have

$$\begin{aligned}
&\int \left\{ \frac{1}{\sigma(B^d_r(z))} \int \mathbb{1}_{\{z' : |z'-z|<r, \,\mathrm{Graph}\,\gamma_{z'} \llcorner_{[\tilde t^-_\gamma, \min\{t^+_{\gamma_{z'}}, t^+_{\gamma_z}\}]} \not\subseteq \mathrm{Graph}\,\gamma_z + B^d_R(0)\}} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&= \int \left\{ \frac{1}{\sigma(B^d_r(z))} \int \mathbb{1}_{B^d_r(z) \cap \mathcal{A}_R(z)}(z') \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&= \int \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R(z)}(z') \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&= \int_{(K'_\delta)^c} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \underbrace{\mathbb{1}_{\mathcal{A}_R(z)}(z')}_{\leq 1} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&\quad + \int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R(z)}(z') \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&\leq \int_{(K'_\delta)^c} \left\{ \cancel{\frac{1}{\sigma(B^d_r(z))} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(B^d_r(z))} \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&\quad + \int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R(z)}(z') \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&\overset{(3.3) \text{ and } (3.1)}{<} 2\delta + \int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R(z)}(z') \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz).
\end{aligned}$$

For the second integral we notice that the contribution of $z' \in B_r(z) \cap K_\delta$ is zero, in view of (3.2). Hence

$$\begin{aligned}
&\int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)} \mathbb{1}_{\mathcal{A}_R(z)}(z') \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&= \int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \int_{B^d_r(z)\setminus K_\delta} \mathbb{1}_{\mathcal{A}_R(z)}(z') \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz') \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&\leq \int_{K'_\delta} \left\{ \frac{1}{\sigma(B^d_r(z))} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)\big(B^d_r(z) \cap (S_1 \setminus K_\delta)\big) \right\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&\overset{(3.4)}{<} \delta.
\end{aligned}$$

The proof is concluded by taking δ so that $3\delta \leq \varpi$. □

Corollary 3.3. *Under the assumptions of Proposition 3.2 for every $R > 0$ it holds*

$$\lim_{r\to 0} \int \frac{1}{\sigma(B^d_r(\gamma(\tilde t^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(\tilde t^-_{\gamma'}) \in \gamma(\tilde t^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\,\gamma' \llcorner_{[\tilde t^-_\gamma, \min\{t^+_\gamma, t^+_{\gamma'}\}]} \not\subseteq \mathrm{Graph}\,\gamma + B^d_R(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) = 0.$$

We would like now to prove a converse implication, which is more delicate and thus we will split the proof in several lemmata. We will denote now by η a Lagrangian representation of $\mathrm{div}(\rho(1,\boldsymbol{b})) = \mu$ in Ω (which can be taken as the restriction of a Lagrangian representation in $\mathbb{R}^{d+1}$, in view of Theorem 2.11). Here $\tilde t^-_\gamma = t^-_\gamma$, and we will write for shortness

$$\mathrm{Graph}\,\gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} = \mathrm{Graph}\,\gamma' \llcorner_{[\max\{t^-_{\gamma'}, t^-_\gamma\}, \min\{t^+_{\gamma'}, t^-_\gamma\}]}.$$

As in Proposition 3.2 above, the proper set Ω is assumed to be perturbed, in the sense of Theorem 2.9.

Proposition 3.4. *Let η be a Lagrangian representation in a perturbed proper set $\Omega \subset \mathbb{R}^{d+1}$. Let $\varpi > 0$ so that for all $R > 0$ there exists $r = r(R) > 0$ such that*

$$\int_\Gamma \frac{1}{\sigma(B^d_r(\gamma(t^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' \in \Gamma : \begin{array}{l} \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_R(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) \le \varpi. \tag{3.6}$$

Then there exists a Borel function

$$S_1 \ni z \mapsto \bar\gamma_z \in \Gamma$$

such that if

$$U := \Big\{ \gamma \in \Gamma : \mathrm{Graph}\, \gamma \subset \mathrm{Graph}\, \bar\gamma_{\gamma(t^-_\gamma)} \Big\},$$

then it holds

$$\eta^{\mathrm{in}}(U^c) \le \inf_{C \ge 3/2} \left\{ 2C\varpi + \frac{\mu^-(\Omega)}{C} \right\}. \tag{3.7}$$

We begin by proving the following lemma, which shows how the piece of information contained in the hypothesis of Proposition 3.4 can be passed to the limit:

Lemma 3.5. *In the setting of Proposition 3.4, it holds*

$$\int \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z \left(\left\{ (\gamma, \gamma') : \begin{array}{l} \gamma'(t^-_{\gamma'}) = \gamma(t^-_\gamma), \\ \mathrm{Graph}\, \gamma \not\subseteq \mathrm{Graph}\, \gamma',\ \mathrm{Graph}\, \gamma' \not\subseteq \mathrm{Graph}\, \gamma \end{array} \right\} \right) \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz) \le \varpi. \tag{3.8}$$

Proof. For fixed $\bar R \ge R$, γ we have

$$\begin{aligned} &\eta^{\mathrm{in}}\Big(\Big\{ \gamma' : \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_R(0) \Big\} \Big) \\ &\ge \eta^{\mathrm{in}}\Big(\Big\{ \gamma' : \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_r(0), \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_{\bar R}(0) \Big\} \Big). \end{aligned}$$

By keeping $\bar R$ fixed and sending $R \searrow 0$, we obtain a family of $\{r_n\}_{n \in \mathbb{N}}$ such that

$$\int \frac{1}{\sigma(B^d_{r_n}(\gamma(t^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_{r_n}(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_{\bar R}(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) \le \varpi.$$

We now let $r_n \to 0$ and we make use of the following facts:

(1) the set

$$\Big\{ \gamma' : \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_R(0) \Big\}$$

is open in Γ;

(2) by the properties of the disintegration, for $\mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)$-a.e. $z \in S_1$ it holds

$$\fint_{B^d_r(z)} \eta_{z'}\, \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz) \rightharpoonup \eta_z, \qquad \text{as measures on } \Gamma. \tag{3.9}$$

At this point one uses Fatou's Lemma and the l.s.c. of the weak convergence on open sets to obtain

$$\begin{aligned} \varpi &\ge \liminf_n \int \frac{1}{\sigma(B^d_{r_n}(\gamma(t^-_\gamma)))} \eta^{\mathrm{in}} \left(\left\{ \gamma' : \begin{array}{l} \gamma'(t^-_{\gamma'}) \in \gamma(t^-_\gamma) + B^d_{r_n}(0), \\ \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_{\bar R}(0) \end{array} \right\} \right) \eta^{\mathrm{in}}(d\gamma) \\ \text{(Fatou)} \ &\ge \int \liminf_n \left\{ \fint_{B^d_{r_n}(\gamma(t^-_\gamma)))} \eta^{\mathrm{in}}_{z'} \Big(\Big\{ \gamma' : \mathrm{Graph}\, \gamma' \llcorner_{[t^-_\gamma, t^+_\gamma]} \not\subseteq \mathrm{Graph}\, \gamma + \mathrm{clos}\, B^d_{\bar R}(0) \Big\} \Big) \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz') \right\} \eta^{\mathrm{in}}(d\gamma) \\ &\overset{(3.9)}{\ge} \int \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z \Big(\Big\{ (\gamma, \gamma') : \max_{t \in [t^-_\gamma, \min\{t^+_{\gamma'}, t^+_\gamma\}]} \big\{ \mathrm{dist}\, \big(\gamma(t), \gamma'(t) \big) \big\} > \bar R \Big\} \Big) \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz). \end{aligned}$$

Finally, we send $\bar R \to 0$ and we use the Monotone Convergence Theorem, so that

$$\int \eta^{\mathrm{in}}_z \otimes \eta^{\mathrm{in}}_z \left(\left\{ (\gamma, \gamma') : \begin{array}{l} \gamma'(t^-_{\gamma'}) = \gamma(t^-_\gamma), \\ \mathrm{Graph}\, \gamma \not\subseteq \mathrm{Graph}\, \gamma', \\ \mathrm{Graph}\, \gamma' \not\subseteq \mathrm{Graph}\, \gamma \end{array} \right\} \right) \mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)(dz) \le \varpi,$$

which is what we wanted to prove. $\square$

We now state an elementary inequality which will be very useful to conclude the argument of the proof of Proposition 3.4.

Lemma 3.6. *If $D_0 \geq 3/2$ it holds*

$$1-\alpha \leq D_0(1-\alpha)\max\big\{1-\alpha, 2(\alpha-\beta)\big\} + \frac{\beta}{D_0}, \qquad \textit{for all} \quad 0 \leq \beta \leq \alpha \leq 1.$$

The proof is omitted. We are eventually ready to prove Proposition 3.4.

Proof (of Proposition 3.4). To begin, let us define a partial order relation on the set $\varGamma$ (note that we are just looking to curves contained in Ω, not in the whole $\mathbb{R}^{d+1}$). We consider the set

$$\mathcal{R} := \big\{(\gamma,\gamma') \in \varGamma^2 : \operatorname{Graph}\gamma \subset \operatorname{Graph}\gamma'\big\}.$$

It is immediate to check the relation $\mathcal{R}$ is a partial order on $\varGamma$. We will write $\gamma \preccurlyeq \gamma'$ for $(\gamma,\gamma') \in \mathcal{R}$, and $\gamma \prec \gamma'$ meaning $(\gamma,\gamma') \in \mathcal{R}$ and $\gamma \neq \gamma'$. Notice that, in this language, we can rephrase the conclusion of Lemma 3.5, namely Formula (3.8), by saying that

$$\begin{aligned} \varpi &\geq \int \eta_z^{\mathrm{in}} \otimes \eta_z^{\mathrm{in}} \left(\left\{ (\gamma,\gamma') : \begin{array}{l} \gamma'(t^-_{\gamma'}) = \gamma(t^-_{\gamma}), \\ \operatorname{Graph}\gamma \not\subseteq \operatorname{Graph}\gamma', \\ \operatorname{Graph}\gamma' \not\subseteq \operatorname{Graph}\gamma \end{array} \right\} \right) \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ &= \int \eta_z^{\mathrm{in}} \otimes \eta_z^{\mathrm{in}} \Big(\Big\{ (\gamma,\gamma') : \gamma'(t^-_{\gamma'}) = \gamma(t^-_{\gamma}),\, (\gamma,\gamma') \in \varGamma^2 \setminus \big(\mathcal{R} \cup \mathcal{R}^T\big) \Big\} \Big) \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz), \end{aligned} \tag{3.10}$$

where we have used the notation $\mathcal{R}^T$ to denote the set $\{(\gamma,\gamma') : (\gamma',\gamma) \in \mathcal{R}\}$.

Consider the function

$$z,\gamma \mapsto \eta_z^{\mathrm{in}}\big(\big\{\gamma' : \gamma' \preccurlyeq \gamma\big\}\big). \tag{3.11}$$

This function is u.s.c. in every compact set where $z \mapsto \eta_z^{\mathrm{in}}$ is weakly continuous: indeed, observe that if $\gamma_n \to \gamma$ then

$$\big\{\gamma' : \gamma' \preccurlyeq \gamma_n\big\} \quad \text{converges in Hausdorff distance to} \quad \big\{\gamma' : \gamma' \preccurlyeq \gamma\big\},$$

so that for every $\epsilon > 0$ it holds, for n large enough,

$$\big\{\gamma' : \gamma' \preccurlyeq \gamma_n\big\} \subset \big\{\gamma' : \operatorname{Graph}\gamma' \subset \operatorname{Graph}\gamma + \operatorname{clos} B^{d+1}_\epsilon(0)\big\}. \tag{3.12}$$

Then using the Monotone Convergence Theorem, the u.s.c. of measures of closed sets w.r.t. weak convergence and (3.12) we get for $z_n \to z$

$$\begin{aligned} \eta_z^{\mathrm{in}}\big(\big\{\gamma' : \gamma' \preccurlyeq \gamma\big\}\big) &= \lim_{\epsilon \searrow 0} \eta_z^{\mathrm{in}}\big(\big\{\gamma' : \operatorname{Graph}\gamma' \subset \operatorname{Graph}\gamma + \operatorname{clos} B^{d+1}_\epsilon(0)\big\}\big) \\ &\geq \lim_{\epsilon \searrow 0} \limsup_n \eta_{z_n}^{\mathrm{in}}\big(\big\{\gamma' : \operatorname{Graph}\gamma' \subset \operatorname{Graph}\gamma + \operatorname{clos} B^{d+1}_\epsilon(0)\big\}\big) \\ &\geq \lim_{\epsilon \searrow 0} \limsup_n \eta_{z_n}^{\mathrm{in}}\big(\big\{\gamma' : \gamma' \preccurlyeq \gamma_n\big\}\big) \\ &= \limsup_n \eta_{z_n}^{\mathrm{in}}\big(\big\{\gamma' : \gamma' \preccurlyeq \gamma_n\big\}\big), \end{aligned} \tag{3.13}$$

and the sought u.s.c. property is proven. For $z \in S_1$ let us now define

$$a_z := \sup_\gamma \eta_z^{\mathrm{in}}\big(\big\{\gamma' : \gamma' \preccurlyeq \gamma\big\}\big).$$

Being $z,\gamma \mapsto \eta_z^{\mathrm{in}}(\{\gamma',\gamma' \preccurlyeq \gamma\})$ u.s.c. by (3.13), it follows that for every a the set $\{z \in S_1 : a_z \geq a\}$ is a Souslin set (or analytic) by observing that

$$\{z : a_z \geq a\} = \mathtt{p}_z\big\{(z,\gamma) : \eta_z^{\mathrm{in}}\big(\big\{\gamma' : \gamma' \preccurlyeq \gamma\big\}\big) \geq a\big\},$$

i.e. a projection of a closed set [Sri98, Proposition 4.1.1]. In particular it is universally measurable. Thus, for $z \in S_1$, for every $\varepsilon > 0$, by definition of supremum, there exists γ_z such that, having set $A_z := \big\{\gamma' : \gamma' \preccurlyeq \gamma_z\big\}$, it holds

$$\eta_z^{\mathrm{in}}\big(A_z\big) \geq a_z - \varepsilon. \tag{3.14}$$

By [Sri98, Theorem 5.2.1], we can take γ_z to be $\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)_{\llcorner S_1}$-measurable (and hence also σ-continuous, i.e. its graph is a countable union of continuous graphs, by redefining γ_z on a $\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)$-negligible set). By prolonging the curve γ_z as

$$\gamma_z(t) \mapsto \begin{cases} \gamma_z(t) & t \in [t^-_\gamma, t^+_{\gamma_z}], \\ \gamma_z(t^+_\gamma) & t > t^+_{\gamma_z}, \end{cases}$$

and restricting it to the first exiting time, we can assume that its initial and final point belongs to S_1 and $\partial\Omega$, respectively: in particular (3.14) still holds and γ_z cannot be prolonged in Ω.

Set $B_z := \{\gamma' : \gamma' \prec \gamma_z\}$ and

$$b_z := \eta^{\mathrm{in}}_z(B_z).$$

Clearly, $b_z \leq a_z$ for $z \in S_1$; furthermore, we emphasize that B_z is the set of curves whose graph is contained in the graph of the almost-maximizer γ_z but are different from it: in view of this, these curves must have a final point inside the domain Ω, so that the following bound holds:

$$\begin{aligned} \mu^-(\Omega) &= \int_\Gamma \delta_{(t^+_\gamma,\gamma(t^+_\gamma))}(\Omega)\eta(d\gamma) \\ (\eta^{\mathrm{in}} \leq \mathtt{R}_\Omega\eta) \quad &\geq \int \bigg\{ \int_\Gamma \delta_{(t^+_\gamma,\gamma(t^+_\gamma))}(\Omega)\eta^{\mathrm{in}}_z(d\gamma) \bigg\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ \text{by above observation} \quad &\geq \int \bigg\{ \int_{B_z} \delta_{(t^+_\gamma,\gamma(t^+_\gamma))}(\Omega)\eta^{\mathrm{in}}_z(d\gamma) \bigg\} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ &= \int \eta^{\mathrm{in}}_z(B_z)\, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ &= \int b_z\, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz). \end{aligned} \tag{3.15}$$

Next, we observe that for every γ

$$\eta^{\mathrm{in}}_z\big(\{\gamma' : \gamma' \preccurlyeq \gamma\}\big) \leq a_z, \tag{3.16}$$

by the very definition of a_z. From the fact that γ_z cannot be prolonged and from the following implication

$$\gamma \notin A_z, \gamma \preccurlyeq \gamma' \quad \Longrightarrow \quad \gamma' \notin A_z, \tag{3.17}$$

we obtain the inclusions

$$\begin{aligned} \big\{(\gamma,\gamma') : \gamma \notin A_z, (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\big\} &= \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma \prec \gamma' \vee \gamma' \preccurlyeq \gamma\big\} \\ &= \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma \prec \gamma'\big\} \cup \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma' \preccurlyeq \gamma\big\} \\ &\underset{(3.17)}{\subset} \big\{(\gamma,\gamma') : \gamma' \notin A_z, \gamma \prec \gamma'\big\} \cup \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma' \preccurlyeq \gamma\big\} \\ &\subset \big\{(\gamma,\gamma') : \gamma' \notin A_z, \gamma \preccurlyeq \gamma'\big\} \cup \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma' \preccurlyeq \gamma\big\}. \end{aligned}$$

Being the last set symmetric in γ, γ', we obtain by symmetrizing the first one

$$\big\{(\gamma,\gamma') : (\gamma,\gamma') \notin A_z \times A_z, (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\big\} \subset \big\{(\gamma,\gamma') : \gamma' \notin A_z, \gamma \preccurlyeq \gamma'\big\} \cup \big\{(\gamma,\gamma') : \gamma \notin A_z, \gamma' \preccurlyeq \gamma\big\} \tag{3.18}$$

Integrating w.r.t. $\eta^{\mathrm{in}}_z \times \eta^{\mathrm{in}}_z$

$$\begin{aligned} \eta^{\mathrm{in}}_z \times \eta^{\mathrm{in}}_z\big(\big\{(\gamma,\gamma') : (\gamma,\gamma') \notin A_z \times A_z, (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\big\}\big) &\underset{(3.18)}{\leq} 2\int_{\Gamma\setminus A_z} \eta^{\mathrm{in}}_z\big(\{\gamma' : \gamma' \preccurlyeq \gamma\}\big)\eta^{\mathrm{in}}_z(d\gamma) \\ &\underset{(3.16)}{\leq} 2(1-\eta^{\mathrm{in}}_z(A_z))a_z. \end{aligned}$$

Hence, since

$$\eta^{\mathrm{in}}_z \times \eta^{\mathrm{in}}_z(A_z \times A_z) = \eta(A_z)^2,$$

we get

$$\begin{aligned} \eta^{\mathrm{in}}_z \times \eta^{\mathrm{in}}_z\big(\big\{(\gamma,\gamma') : (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\big\}\big) &\leq 2(1-\eta(A_z))a_z + \eta(A_z)^2 \\ &= 2a_z - a_z^2 + \big(a_z - \eta(A_z)\big)^2. \end{aligned} \tag{3.19}$$

Hence

$$\begin{aligned}\eta_z^{\rm in} \otimes \eta_z^{\rm in}\big(\big\{(\gamma,\gamma') : (\gamma,\gamma') \in \Gamma^2 \setminus (\mathcal{R} \cup \mathcal{R}^T)\big\}\big) &= 1 - \eta_z^{\rm in} \times \eta_z^{\rm in}\big(\big\{(\gamma,\gamma') : (\gamma,\gamma') \in \mathcal{R} \cup \mathcal{R}^T\big\}\big) \\ &\underset{(3.19)}{\geq} 1 - \Big(2a_z - a_z^2 + \big(a_z - \eta(A_z)\big)^2\Big) \\ &= (1-a_z)^2 - \big(a_z - \eta(A_z)\big)^2 \\ &\underset{(3.14)}{\geq} (1-a_z)^2 - \varepsilon^2.\end{aligned} \tag{3.20}$$

On the other hand, since γ_z cannot be prolonged,

$$\big\{(\gamma,\gamma') : (\gamma,\gamma') \in \Gamma^2 \setminus (\mathcal{R} \cup \mathcal{R}^T)\big\} \supset \big[\{\gamma_z\} \times (\Gamma \setminus A_z)\big] \cup \big[(\Gamma \setminus A_z) \times \{\gamma_z\}\big] \tag{3.21}$$

so that

$$\begin{aligned}\eta_z^{\rm in} \otimes \eta_z^{\rm in}\big(\big\{(\gamma,\gamma') : (\gamma,\gamma') \in \Gamma^2 \setminus (\mathcal{R} \cup \mathcal{R}^T)\big\}\big) &\underset{(3.21)}{\geq} 2\eta_z^{\rm in}(\{\gamma_z\})\big(1 - \eta_z^{\rm in}(A_z)\big) \\ &= 2\big(\eta_z^{\rm in}(A_z) - \eta_z^{\rm in}(B_z)\big)\big(1 - \eta_z^{\rm in}(A_z)\big) \\ &= 2\big(\eta_z^{\rm in}(A_z) - b_z\big)\big(1 - \eta_z^{\rm in}(A_z)\big) \\ &\underset{(3.14)}{\geq} 2\big(a_z - b_z - \varepsilon\big)(1-a_z).\end{aligned} \tag{3.22}$$

Hence we can continue (3.10) as

$$\begin{aligned}\varpi &\geq \int \eta_z^{\rm in} \otimes \eta_z^{\rm in}\big(\big\{(\gamma,\gamma') : (\gamma,\gamma') \in \Gamma^2 \setminus (\mathcal{R} \cup \mathcal{R}^T)\big\}\big)\, \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ \text{(3.20), (3.22)}\quad &\geq \int_{S_1} (1-a_z)\max\{1-a_z, 2(a_z-b_z)\}\ \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) - (2\varepsilon+\varepsilon^2)\int_{S_1} \mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega).\end{aligned} \tag{3.23}$$

On the other hand by Lemma 3.6 with $\alpha = a_z$ and $\beta = b_z$ and $D_0 \geq 3/2$ we have

$$\eta_z^{\rm in}(\Gamma \setminus A_z) \leq 1 - a_z + \varepsilon \leq D_0(1-a_z)\max\big\{1-a_z, 2(a_z-b_z)\big\} + \frac{b_z}{D_0} + \varepsilon,$$

so that,

$$\begin{aligned}&\int \eta_z^{\rm in}(\Gamma \setminus A_z)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\ &\quad\leq \int_{S_1}\left[D_0(1-a_z)\max\big\{1-a_z,2(a_z-b_z)\big\} + \frac{b_z}{D_0}\right]\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) + \varepsilon\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(S_1) \\ &\quad\underset{(3.23)}{\leq} D_0\varpi + \int \frac{b_z}{D_0}\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) + \varepsilon(1+2D_0+D_0\varepsilon)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(S_1).\end{aligned}$$

Since ε can be chosen arbitrary small,

$$\begin{aligned}\int \eta_z^{\rm in}(\Gamma \setminus A_z)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) &\underset{(3.15)}{\leq} D_0\varpi + \frac{\mu^-(\Omega)}{D_0} + \varepsilon(1+2D_0+D_0\varepsilon)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(S_1) \\ &\leq 2D_0\varpi + \frac{\mu^-(\Omega)}{D_0},\end{aligned} \tag{3.24}$$

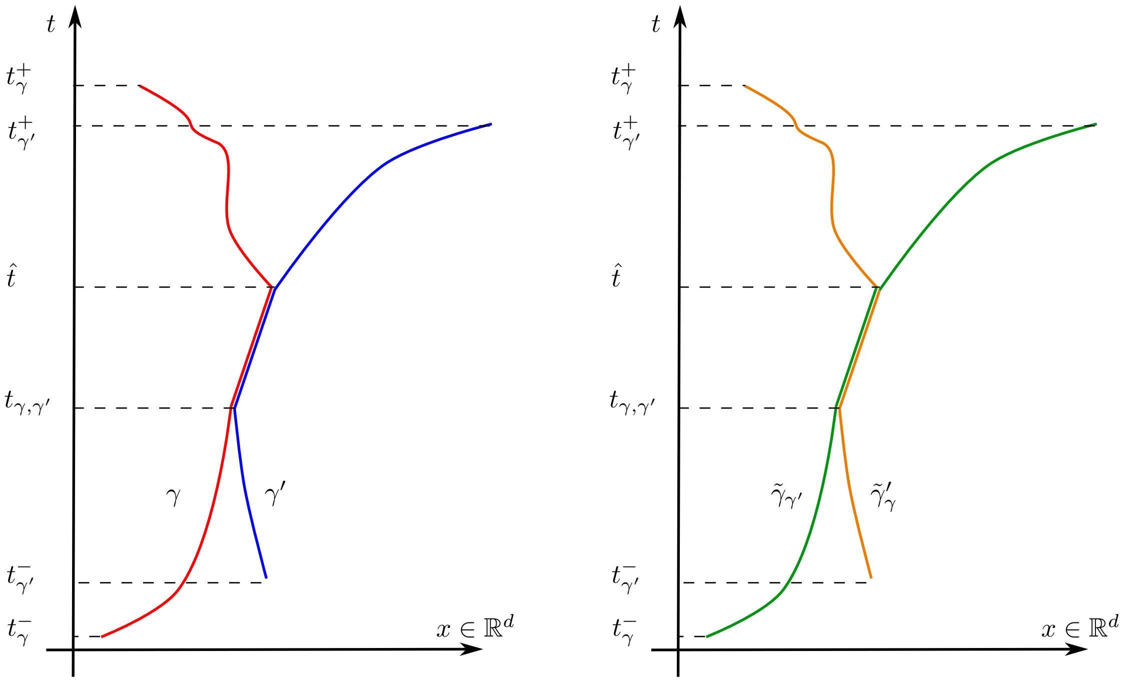
and this yields the desired conclusions: indeed, setting $U := \bigcup_{z\in S_1} A_z$, from (3.24) we have that

$$\eta^{\rm in}(U^c) \leq \inf_{C \geq 3/2}\left\{2C\varpi + \frac{\mu^-(\Omega)}{C}\right\}$$

and $\eta_z^{\rm in} \llcorner_U$ is supported on a set of curves which are restrictions of the curve γ_z by construction. □

In order to pass from uniqueness of the curves starting from the same initial point (Proposition 3.4) to *almost* forward untangling (i.e. forward untangling up to a set whose measure is controlled by ϖ and $\mu^-(\Omega)$), we assume that the condition (3.6) holds for every Lagrangian representation η of $\rho(1,\boldsymbol{b})\llcorner_\Omega$.

Proposition 3.7. *Assume that* (3.6) *holds for every Lagrangian representation* η *of* $\rho(1,\boldsymbol{b})$ *(in the perturbed proper set* Ω*). Then for every* η *there exists a set of* forward untangled *trajectories* U *such that*



(a) A pair of curves $(\gamma, \gamma') \in NF$, respectively in red and blue.

(b) The resulting curves $\tilde{\gamma}_{\gamma'}$ (green) and $\tilde{\gamma}'_{\gamma}$ (orange).

Figure 1. Two curves γ, γ' with $(\gamma, \gamma') \in NF$ and visual depiction of the exchanging map $\tilde{\gamma}_{\gamma'}$.

- *for* $\mathsf{Tr}(\rho(1, \boldsymbol{b}), \Omega)$*-a.e.* z *the measure* $\eta_z^{\mathrm{in}} \llcorner_U$ *is supported on a set of curves whose graphs are a subset of a given curve* $\gamma_z \in U$*;*
- *it holds*

$$\eta^{\mathrm{in}}(U^c) \leq \inf_{C \geq 1/2} \left\{ 18 C \varpi + \frac{\mu^-(\Omega)}{C} \right\}.$$

Proof. First, by Proposition 3.4 we know that for every η there exists a set U such that $\eta \llcorner_U$ has the property that η_z^{in} is supported on the set of curves which are subset of a given curve γ_z. We restrict η to U, removing a set of trajectories with η-measure bounded by (3.7).

Let us now define the set

$$NF := \big\{ (\gamma, \gamma') : \gamma(t_\gamma^-) \neq \gamma'(t_{\gamma'}^-),\ \gamma, \gamma' \text{ cross} \big\},$$

where we say that two trajectories cross if there exists $t \in (t_\gamma^-, t_\gamma^+) \cap (t_{\gamma'}^-, t_{\gamma'}^+)$ such that $\gamma(t) = \gamma'(t)$, but $(\gamma, \gamma') \notin \mathit{\Delta}$, i.e. they bifurcate at a time $\hat{t} > t$ (assuming this time to be the first time where the curves meet).

Consider the set of positive transference plans $\mathsf{Adm}\big(\eta^{\mathrm{in}}, \eta^{\mathrm{in}}\big)$, which are concentrated on the set NF, and let $\bar{\pi} \in \mathsf{Adm}\big(\eta^{\mathrm{in}}, \eta^{\mathrm{in}}\big)$ be a plan of positive mass: being NF symmetric, we can assume that also $\bar{\pi}$ is. We write the disintegration of $\bar{\pi}$ w.r.t. the marginals

$$\bar{\pi} = \int \bar{\pi}_\gamma \eta^{\mathrm{in}}(d\gamma). \tag{3.25}$$

For every $(\gamma, \gamma') \in NF$, if $t_{\gamma,\gamma'}$ is the first time of intersection so that γ, γ' are certainly splitting for some $\hat{t} > t_{\gamma,\gamma'}$ (see Figure 1a), define

$$\tilde{\gamma}_{\gamma'}(t) := \begin{cases} \gamma(t) & t_\gamma^- \leq t \leq t_{\gamma,\gamma'}, \\ \gamma'(t) & t_{\gamma,\gamma'} < t \leq t_{\gamma'}^+, \end{cases} \qquad \tilde{\gamma}'_{\gamma}(t) := \begin{cases} \gamma'(t) & t_{\gamma'}^- \leq t \leq t_{\gamma,\gamma'}, \\ \gamma(t) & t_{\gamma,\gamma'} < t \leq t_\gamma^+. \end{cases}$$

In other words, the map $\tilde\gamma_{\gamma'}$ is defined on $NF \subset \Gamma \times \Gamma$, takes values in Γ and encodes the operation of exchanging the trajectories of γ, γ' at the crossing time $t_{\gamma,\gamma'}$ (similar considerations hold for $\tilde\gamma'_\gamma$). See Figure 1b.

By the property $\bar\pi \in \mathsf{Adm}(\eta^{\rm in}, \eta^{\rm in})$, one deduces that

$$(\mathtt{p}_\gamma)_\sharp \bar\pi \le \eta^{\rm in}, \tag{3.26}$$

and thus the measure

$$\tilde\eta^{\rm in} := \eta^{\rm in} - \frac{1}{2}(\mathtt{p}_\gamma)_\sharp\bar\pi + \frac{1}{2}(\tilde\gamma_{\gamma'})_\sharp\bar\pi \tag{3.27}$$

is a non-negative measure. Moreover, since by construction

$$\delta_{\gamma(t)} + \delta_{\gamma'(t)} = \delta_{\tilde\gamma_{\gamma'}(t)} + \delta_{\tilde\gamma'_\gamma(t)}, \tag{3.28}$$

it follows that

$$\begin{aligned}
\int \delta_{\gamma(t)}\tilde\eta^{\rm in}(d\gamma) &\underset{(3.27)}{=} \int \delta_{\gamma(t)}\eta^{\rm in}(d\gamma) + \frac{1}{2}\int \big(\delta_{\tilde\gamma_{\gamma'}(t)} - \delta_{\gamma(t)}\big)\bar\pi_\gamma(d\gamma')\eta^{\rm in}(d\gamma) \\
&\underset{(3.28)}{=} \int \delta_{\gamma(t)}\eta^{\rm in}(d\gamma) + \frac{1}{4}\int \big(\delta_{\tilde\gamma_{\gamma'}(t)} - \delta_{\gamma(t)}\big)\bar\pi(d\gamma d\gamma') - \frac{1}{4}\int \big(\delta_{\tilde\gamma'_\gamma(t)} - \delta_{\gamma'(t)}\big)\bar\pi(d\gamma d\gamma') \\
\text{exchange } \gamma \to \gamma' \quad &= \int \delta_{\gamma(t)}\eta^{\rm in}(d\gamma) + \frac{1}{4}\int \big(\delta_{\tilde\gamma_{\gamma'}(t)} - \delta_{\gamma(t)}\big)\bar\pi(d\gamma d\gamma') - \frac{1}{4}\int \big(\delta_{\tilde\gamma_{\gamma'}(t)} - \delta_{\gamma(t)}\big)\bar\pi(d\gamma' d\gamma) \\
\bar\pi \text{ symmetric} \quad &= \int \delta_{\gamma(t)}\eta^{\rm in}(d\gamma).
\end{aligned}$$

This means that $\tilde\eta^{\rm in}$ is a representation of $(\mathtt{p}_{t,x})_\sharp\eta^{\rm in}$, where $\mathtt{p}_{t,x}(\gamma) = (t, \gamma(t))$ is the evaluation map (in time and space). In particular (the projections of) $\eta^{\rm in}$ and (respectively of) $\tilde\eta^{\rm in}$ have the same trace on $\partial\Omega$.

Write

$$\begin{aligned}
\tilde\eta^{\rm in} &= \int_{S_1} \tilde\eta^{\rm in}_z \,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \\
&= \int_{S_1}\left(\eta^{\rm in}_z - \frac{1}{2}\int \|\bar\pi_\gamma\|\eta^{\rm in}_z(d\gamma) + \frac{1}{2}\int (\tilde\gamma_{\gamma'})_\sharp\bar\pi_\gamma\eta^{\rm in}_z(d\gamma)\right)\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz).
\end{aligned}$$

The above formula follows from (3.25), and the fact that $\tilde\gamma_{\gamma'}$ has the same initial point of γ because of its definition. By $\bar\pi \in \mathsf{Adm}(\eta^{\rm in}, \eta^{\rm in})$ one deduces that $\|\bar\pi_\gamma\| \le 1$, and then

$$\tilde\eta^{\rm in}_z\big(\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\gamma_z\}\big) = \|\eta^{\rm in}_z\| - \frac{1}{2}\int \|\bar\pi_\gamma\|\eta^{\rm in}_z(d\gamma) \ge \frac{1}{2}. \tag{3.29}$$

We have used the fact that $\mathrm{Graph}\,\tilde\gamma_{\gamma'} \not\subseteq \mathrm{Graph}\,\gamma_z$ for $(\gamma,\gamma') \in NF$. The same argument implies that if $\mathrm{Graph}\,\bar\gamma \not\subseteq \mathrm{Graph}\,\gamma_z$ then

$$\begin{aligned}
\tilde\eta^{\rm in}_z\big(\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma\}\big) &= \eta^{\rm in}_z\big(\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma \cap \mathrm{Graph}\,\gamma_z\}\big) \\
&\quad - \frac{1}{2}\int_{\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma \cap \mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta^{\rm in}_z(d\gamma) \\
&\quad + \frac{1}{2}\int (\tilde\gamma_{\gamma'})_\sharp\bar\pi_\gamma\big(\{\gamma' : \mathrm{Graph}\,\gamma' \subset \mathrm{Graph}\,\bar\gamma, \mathrm{Graph}\,\gamma' \not\subseteq \mathrm{Graph}\,\gamma_z\}\big)\eta^{\rm in}_z(d\gamma) \\
&= \eta^{\rm in}_z\big(\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma \cap \mathrm{Graph}\,\gamma_z\}\big) \\
&\quad - \frac{1}{2}\int_{\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma \cap \mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta^{\rm in}_z(d\gamma) \\
&\quad + \frac{1}{2}\int \bar\pi_\gamma\big(\{\gamma' : \mathrm{Graph}\,\tilde\gamma_{\gamma'} \subset \mathrm{Graph}\,\bar\gamma, \mathrm{Graph}\,\tilde\gamma_{\gamma'} \not\subseteq \mathrm{Graph}\,\gamma_z\}\big)\eta^{\rm in}_z(d\gamma),
\end{aligned} \tag{3.30}$$

where we have used (3.27) and the very definition of push-forward of a measure. Observing that

$$\mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma \cap \mathrm{Graph}\,\gamma_z \quad \Longrightarrow \quad \mathrm{Graph}\,\tilde\gamma_{\gamma'} \not\subseteq \mathrm{Graph}\,\bar\gamma, \tag{3.31}$$

we get for the last integral of (3.30)

$$\begin{aligned}
&\int \bar\pi_\gamma\big(\big\{\gamma' : \mathrm{Graph}\,\tilde\gamma_{\gamma'} \subset \mathrm{Graph}\,\bar\gamma, \mathrm{Graph}\,\tilde\gamma_{\gamma'} \not\subseteq \mathrm{Graph}\,\gamma_z\big\}\big)\eta^{\mathrm{in}}_z(d\gamma) \\
&\underset{(3.31)}{=} \int_{\{\gamma:\mathrm{Graph}\,\gamma\not\subseteq\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \bar\pi_\gamma\big(\big\{\gamma' : \mathrm{Graph}\,\tilde\gamma_{\gamma'} \subset \mathrm{Graph}\,\bar\gamma, \mathrm{Graph}\,\tilde\gamma_{\gamma'} \not\subseteq \mathrm{Graph}\,\gamma_z\big\}\big)\eta^{\mathrm{in}}_z(d\gamma) \\
&\le \int_{\{\gamma:\mathrm{Graph}\,\gamma\not\subseteq\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta^{\mathrm{in}}_z(d\gamma).
\end{aligned} \tag{3.32}$$

Hence we can continue (3.30) as

$$\begin{aligned}
\tilde\eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma\big\}\big) &= \eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\big\}\big) \\
&\quad - \frac{1}{2}\int_{\{\gamma:\mathrm{Graph}\,\gamma\subset\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta^{\mathrm{in}}_z(d\gamma) \\
&\quad + \frac{1}{2}\int \bar\pi_\gamma\big(\big\{\gamma' : \mathrm{Graph}\,\tilde\gamma_{\gamma'} \subset \mathrm{Graph}\,\bar\gamma, \mathrm{Graph}\,\tilde\gamma_{\gamma'} \not\subseteq \mathrm{Graph}\,\gamma_z\big\}\big)\eta^{\mathrm{in}}_z(d\gamma) \\
&\underset{(3.32)}{\le} \eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\big\}\big) \\
&\quad - \frac{1}{2}\int_{\{\gamma:\mathrm{Graph}\,\gamma\subset\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta^{\mathrm{in}}_z(d\gamma) \\
&\quad + \frac{1}{2}\int_{\{\gamma:\mathrm{Graph}\,\gamma\not\subseteq\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta^{\mathrm{in}}_z(d\gamma).
\end{aligned} \tag{3.33}$$

Adding and subtracting in the RHS the term $\frac{1}{2}\int_{\{\gamma:\mathrm{Graph}\,\gamma\not\subseteq\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta^{\mathrm{in}}_z(d\gamma)$, we obtain

$$\begin{aligned}
\tilde\eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma\big\}\big) &\le \eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\big\}\big) \\
&\quad - \frac{1}{2}\int \|\bar\pi_\gamma\|\eta^{\mathrm{in}}_z(d\gamma) \\
&\quad + \int_{\{\gamma:\mathrm{Graph}\,\gamma\not\subseteq\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} \|\bar\pi_\gamma\|\eta^{\mathrm{in}}_z(d\gamma),
\end{aligned} \tag{3.34}$$

and doing the same with the term $\eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \not\subseteq \mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\big\}\big)$ we finally reach

$$\begin{aligned}
\tilde\eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma\big\}\big) &\le \|\eta^{\mathrm{in}}_z\| - \frac{1}{2}\int \|\bar\pi_\gamma\|\eta^{\mathrm{in}}_z(d\gamma) \\
&\quad + \int_{\{\gamma:\mathrm{Graph}\,\gamma\not\subseteq\mathrm{Graph}\,\bar\gamma\cap\mathrm{Graph}\,\gamma_z\}} (\|\bar\pi_\gamma\|-1)\eta^{\mathrm{in}}_z(d\gamma) \\
(\bar\pi_\gamma \text{ has mass less than or equal to } 1) &\le \|\eta^{\mathrm{in}}_z\| - \frac{1}{2}\int \|\bar\pi_\gamma\|\eta^{\mathrm{in}}_z(d\gamma) \\
(\text{by the first equality in } (3.29)) &= \tilde\eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\gamma_z\big\}\big).
\end{aligned} \tag{3.35}$$

All in all, we have thus shown that for $\mathrm{Graph}\,\bar\gamma \not\subseteq \mathrm{Graph}\,\gamma_z$ it holds

$$\tilde\eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\bar\gamma\big\}\big) \le \tilde\eta^{\mathrm{in}}_z\big(\big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\gamma_z\big\}\big),$$

which means that γ_z is a maximizer, i.e. a solution to the problem

$$\sup_\gamma \tilde\eta^{\mathrm{in}}_z\big(\big\{\gamma' : \mathrm{Graph}\,\gamma' \subset \mathrm{Graph}\,\gamma\big\}\big).$$

We are thus in position to apply the arguments of Proposition 3.4 with η replaced by $\tilde\eta$ and

$$A_z := \big\{\gamma : \mathrm{Graph}\,\gamma \subset \mathrm{Graph}\,\gamma_z\big\}.$$

By formula (3.24)

$$\|\bar\pi\| = \int \|\bar\pi_\gamma\|\eta^{\mathrm{in}}(d\gamma) = 2\int \tilde\eta^{\mathrm{in}}_z\big(\mathit{\Gamma}\setminus A_z\big)\,\mathsf{Tr}(\rho(1,\boldsymbol{b}),\Omega)(dz) \le 4D_0\varpi + \frac{2\mu^-(\Omega)}{D_0},$$

for all $D_0 \geq 3/2$, where we have used the observation that for a.e. z, η_z is concentrated on $\{\gamma \subset \gamma_z\}$, and then $\tilde{\gamma}_{\gamma'} \not\subseteq \gamma_z$ for $(\gamma, \gamma') \in NF$.

We now resort to Proposition 2.14 to deduce that there is a subset $M = M_1 \cup M_2 \subset U$ whose measure is at most

$$\eta^{\mathrm{in}}(M) \leq \inf_{D_0 \geq 3/2} \left\{ 4D_0 \varpi + \frac{2\mu^-(\Omega)}{D_0} \right\}$$

and such that in $U' = U \setminus M$ the trajectories γ_z, $z \in U'$, do not cross. Adding the measure of the set $\mathit{\Gamma} \setminus U$ estimated in Proposition 3.4 and calling U' with U, we obtain the statement. □

4. The global theory of forward untangling

4.1. **Subadditivity of untangling functional.** We now study how the local pieces of information contained in Propositions 3.7 can be glued in a global one. Roughly speaking, we will consider here the case in which the constant ϖ is replaced by a measure: we will show that a suitable functional (the *forward untangling functional*) is subadditive and thus it is natural to compare it with a measure.

We begin by giving the following

Definition 4.1. Let $\Omega \subset \mathbb{R}^{d+1}$ be a proper set. The *forward untangling functional for a Lagrangian representation* η is defined as

$$\mathfrak{f}^{\mathrm{for}}(\Omega) := \inf \Big\{ (\mathtt{R}_\Omega)_\sharp \eta^{\mathrm{in}}(N) : \mathit{\Gamma}(\Omega) \setminus N \subset \Delta^{\mathrm{for}} \Big\}. \tag{4.1}$$

In other words, the forward untangling functional computed on a proper set Ω gives the amount of curves we have to remove (from the ones seen by $(\mathtt{R}_\Omega)_\sharp \eta^{\mathrm{in}}$) so that the remaining ones are forward untangled. We remark that $\mathfrak{f}^{\mathrm{for}}$ depends on the representation η.

We now show the following property of the forward untangling functional:

Proposition 4.2. *The functional $\mathfrak{f}^{\mathrm{for}}$ defined in (4.1) is monotone w.r.t. inclusion and subadditive on the class of proper sets. More precisely, if $\mathrm{U}, \mathrm{V} \subset \mathbb{R}^{d+1}$ are proper sets whose union $\Omega := \mathrm{U} \cup \mathrm{V}$ is proper, then*

$$\mathfrak{f}^{\mathrm{for}}(\Omega) \leq \mathfrak{f}^{\mathrm{for}}(\mathrm{U}) + \mathfrak{f}^{\mathrm{for}}(\mathrm{V}).$$

Proof. The monotonicity follows by the elementary observation that a restriction of a forward untangled representation is forward untangled. We will thus concentrate on the subadditivity. By definition, for every $\varepsilon > 0$ there exists a set $N(\mathrm{U}) \subset \mathit{\Gamma}(\mathrm{U})$ such that

$$\mathfrak{f}^{\mathrm{for}}(\mathrm{U}) \geq (\mathtt{R}_\mathrm{U})_\sharp \eta^{\mathrm{in}}(N(\mathrm{U})) - \varepsilon$$

and

$$\mathit{\Gamma}(\mathrm{U}) \setminus N(\mathrm{U}) \subset \mathit{\Delta}.$$

Let $N(\mathrm{V})$ be an analogous set for V. Set

$$N := \big\{ \gamma \in \mathit{\Gamma}(\Omega) : \exists i \left(\mathtt{R}^i_\mathrm{U} \gamma \in N(\mathrm{U}) \right) \big\} \cup \big\{ \gamma \in \mathit{\Gamma}(\Omega) : \exists i \left(\mathtt{R}^i_\mathrm{V} \gamma \in N(\mathrm{V}) \right) \big\}.$$

By Proposition 2.12

$$\begin{aligned}
\eta^{\mathrm{in}}(N) &\leq \eta^{\mathrm{in}}\big(\big\{ \gamma \in \mathit{\Gamma}(\Omega) : \exists i \left(\mathtt{R}^i_\mathrm{U}(\gamma) \in N(\mathrm{U}) \right) \big\} \big) + \eta^{\mathrm{in}}\big(\big\{ \gamma \in \mathit{\Gamma}(\Omega) : \exists i \left(\mathtt{R}^i_\mathrm{V}(\gamma) \in N(\mathrm{V}) \right) \big\} \big) \\
&\leq (\mathtt{R}_\mathrm{U})_\sharp \eta^{\mathrm{in}}(N(\mathrm{U})) + (\mathtt{R}_\mathrm{V})_\sharp \eta^{\mathrm{in}}(N(\mathrm{V})) \\
&\leq \mathfrak{f}^{\mathrm{for}}(\mathrm{U}) + \mathfrak{f}^{\mathrm{for}}(\mathrm{V}) + 2\varepsilon.
\end{aligned}$$

Being ε arbitrary and $\mathfrak{f}^{\mathrm{for}}(\Omega) \leq \eta^{\mathrm{in}}(N)$, we thus obtain that $\mathfrak{f}^{\mathrm{for}}(\Omega) \leq \mathfrak{f}^{\mathrm{for}}(\mathrm{U}) + \mathfrak{f}^{\mathrm{for}}(\mathrm{V})$ if we show that $\mathit{\Gamma}(\Omega) \setminus N \subset \mathit{\Delta}$. To do this, observe that

$$\mathtt{R}_\mathrm{U}(\mathit{\Gamma}(\Omega)) \subset \mathit{\Gamma}(\mathrm{U}),$$

and the same for V. Hence, if $\mathrm{Graph}\, \gamma \llcorner_{\mathrm{clos}\, \Omega} \cap \mathrm{Graph}\, \gamma' \llcorner_{\mathrm{clos}\, \Omega} \neq \emptyset$ then

(1) if

$$\mathrm{Graph}\, \gamma \llcorner_{\mathrm{clos}\, \Omega} \cap \mathrm{Graph}\, \gamma' \llcorner_{\mathrm{clos}\, \Omega} \cap \mathrm{clos}\, \mathrm{U} \neq \emptyset,$$

and then they must coincide forward in time in $\mathrm{clos}\, \mathrm{U}$;

(2) or
$$\operatorname{Graph}\gamma\llcorner_{\operatorname{clos}\Omega}\cap\operatorname{Graph}\gamma'\llcorner_{\operatorname{clos}\Omega}\cap\operatorname{clos}\mathrm{V}\neq\emptyset,$$
and then they must coincide forward in time in $\operatorname{clos}\mathrm{V}$;

Hence, if $\gamma,\gamma'\in\Gamma(U\cup V)$ and if $\gamma(\bar t)=\gamma'(\bar t)$, then $\gamma=\gamma'$ in the interval of time such that $\bar t\in\gamma^{-1}(\operatorname{clos}U)$ or $\bar t\in\gamma^{-1}(\operatorname{clos}V)$. When γ exits one of the sets and enters in the other, a simple iterative argument gives they the property of remaining together is preserved forward in time, i.e. must coincide forward in time in $\operatorname{clos}\mathrm{U}\cup\operatorname{clos}\mathrm{V}=\operatorname{clos}\Omega$. □

We are thus led to consider the following

Assumption 4.3. There exist $\tau>0$ and a non-negative measure $\varpi^\tau\in\mathcal{M}^+(\mathbb{R}^{d+1})$ of mass τ such that for some $C\geq 1$, for all $(t,x)\in\Omega$ there exists a family of proper balls $\{B_r^{d+1}(t,x)\}_r$ such that it holds
$$\mathfrak{f}^{\,\mathrm{for}}(B_r^{d+1}(t,x))\leq 18C\varpi^\tau(B_r^{d+1}(t,x))+\frac{\mu^-(B_r^{d+1}(t,x))}{C}. \tag{4.2}$$

For future reference let us define the measure
$$\zeta_\tau^{C,\mathrm{for}}:=18C\varpi^\tau+\frac{\mu^-}{C}.$$

By means of a standard covering argument we have the following

Proposition 4.4. *If Assumption (4.3) holds in a proper set Ω with compact closure, then*
$$\mathfrak{f}^{\,\mathrm{for}}(\Omega)\leq C_d\zeta_C^{\tau,\mathrm{for}}(\operatorname{clos}\Omega), \tag{4.3}$$
where C_d is a dimensional constant.

Proof. By Besicovitch Covering Theorem [AFP00, Theorem 2.18], for any $\varepsilon>0$, we can cover the compact set $\operatorname{clos}\Omega$ with finitely many proper balls B_i such that (4.2) holds and
$$\sum_i\zeta_C^{\tau,\mathrm{for}}(B_i)\leq C_d\zeta_C^{\tau,\mathrm{for}}(\operatorname{clos}\Omega)+\varepsilon.$$

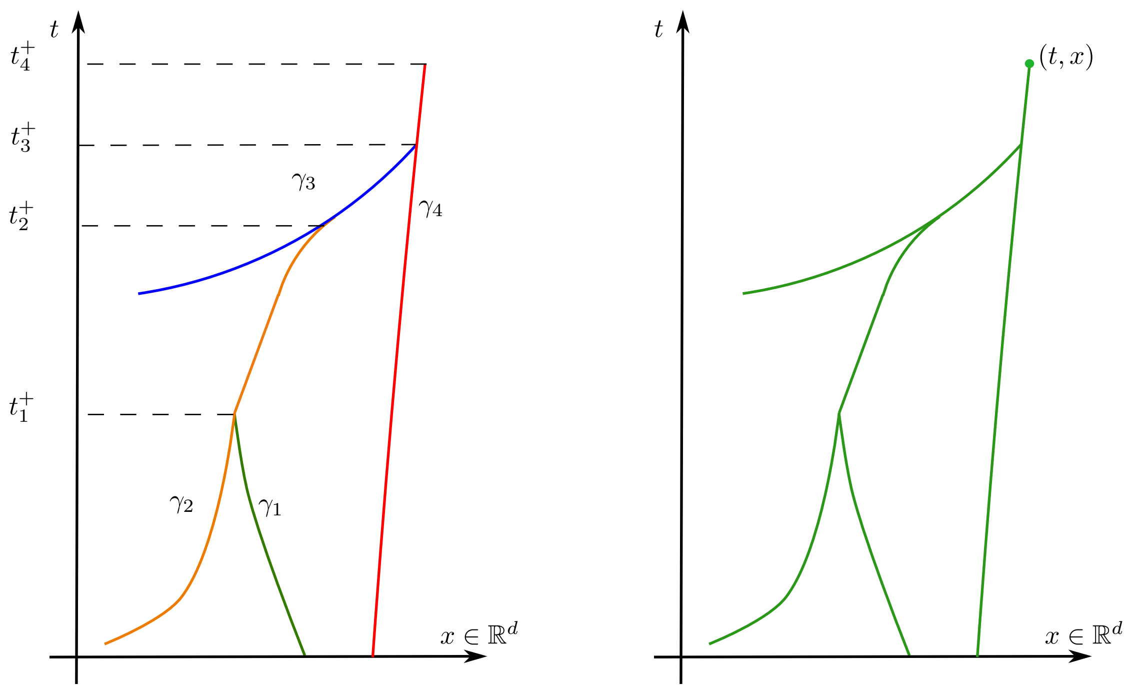
Thanks to the subadditivity (and the monotonicity) of $\mathfrak{f}^{\,\mathrm{for}}$ we can thus write
$$\mathfrak{f}^{\,\mathrm{for}}(\Omega)\leq\mathfrak{f}^{\,\mathrm{for}}\left(\bigcup_i B_i\right)\leq\sum_i\mathfrak{f}^{\,\mathrm{for}}(B_i)\leq C_d\zeta_C^{\tau,\mathrm{for}}(\operatorname{clos}\Omega)+\varepsilon$$
and sending $\varepsilon\to 0$ we obtain (4.3). □

We finally show that the validity of Assumption 4.3 is enough, thanks to the subadditivity proved in Proposition 4.2, to have that η is forward untangled.

Corollary 4.5. *Suppose there exist sequences $\tau_i\searrow 0$ and $C_i\nearrow+\infty$ such that Assumption 4.3 holds for τ_i,C_i and moreover*
$$C_i\tau_i\to 0.$$
Then η is forward untangled.

Proof. Observe that under the assumptions above $\|\zeta_{C_i}^{\tau_i,\mathrm{for}}\|\to 0$ which implies that η is concentrated on a set of trajectories such that for all (perturbed) proper set Ω the measure $(\mathtt{R}_\Omega\eta)^{\mathrm{in}}$ is forward untangled. By countable additivity, we can assume that η is concentrated on a set U of trajectories such that for countably many proper balls B_n generating the topology the entering trajectories are forward untangled. It is elementary to deduce that U is forward untangled. □

Using now the proof of Proposition 3.7, we conclude that, if every representation η satisfies the assumptions of the above corollary, then the set of forward untangled trajectories used by every Lagrangian representation η is made of subcurves of the same family of forward untangled curves γ_z. This observation will be made precise in Theorem 5.5.



(a) A concatenated family $\{\gamma_i\}_{i=1}^4$ of trajectories.

(b) We depicted in green the set F_x^t.

Figure 2. Concatenated families of trajectories and an example of set F_x^t.

Remark 4.6. A more careful analysis of the proof of the above proposition would allow to prove that the Lagrangian representation η is actually concentrated on a set of trajectories such that

$$\text{if there exists } t \in \big[\max\{t_\gamma^-, t_{\gamma'}^-\}, \min\{t_\gamma^+, t_{\gamma'}^+\}\big] \cap \{t > \min\{t_\gamma^-, t_{\gamma'}^-\}\} \text{ such that } \gamma(t) = \gamma'(t) \text{ then}$$
$$\operatorname{Graph}\gamma\llcorner_{[t,\min\{t_\gamma^+,t_{\gamma'}^+\}]} \text{ coincides with } \operatorname{Graph}\gamma'\llcorner_{[t,\min\{t_\gamma^+,t_{\gamma'}^+\}]}.$$

The difference is that no initial point of a trajectory belongs to the inner part of the graph of another trajectory. The proof is obtained by considering transference plans which move mass also among trajectories γ' which start from an inner point of $\gamma \subset \gamma_z$, with the notation of the proof of Proposition 3.7.

5. Uniqueness by forward untangling

In this technical section, we show how the forward untangling is related to the uniqueness problem of the continuity/transport equation.

5.1. **Decomposition and disintegration.** Let η be a forward untangled Lagrangian representation of $\rho(1, \boldsymbol{b})$ and let $\varDelta \subset \varGamma$ be a σ-compact set of forward untangled trajectories on which η is concentrated.

Definition 5.1. We say that a finite set of curves $\{\gamma_i\}_{i=0}^N \subset \varDelta$ is *concatenated* if the terminal times t_i^+ are increasing and $\gamma_i(t_i^+) = \gamma_{i+1}(t_i^+)$.

See Figure 2a. For every time t we define the following partition of $\varDelta$:

$$E_x^t := \bigg\{\gamma \in \varDelta : \exists N \in \mathbb{N}, \{\gamma_i\}_{i=0}^N \Big(\big\{\gamma_i\big\}_{i=0}^N \text{ concatenated } \wedge\ \gamma_0 = \gamma\ \wedge\ \gamma_N(t) = x\Big)\bigg\}. \tag{5.1}$$

By the forward untangling property of $\varDelta$, it is fairly easy to see that the above sets are a partition of $\varDelta$ into σ-compact sets of curves which can be concatenated to a point in $\{t\} \times \mathbb{R}^d$.

Define the σ-compact subsets of $\mathbb{R}^{d+1}$

$$F_x^t := \bigcup_{\gamma \in E_x^t} \operatorname{Graph}\big(\gamma\llcorner_{(t_\gamma^-,t]}\big). \tag{5.2}$$

Lemma 5.2. *The family of sets $\{E^t_x\}_x$ and $\{F^t_x\}_x$ are made of disjoint sets, and for every $s < t$ it holds*

$$(s, y) \in F^t_x \quad \Longrightarrow \quad E^s_y \subset E^t_x. \tag{5.3}$$

Moreover, for every $(s, y) \in F^t_x$, there is a unique curve connecting (s, y) to (t, x) (with $s \leq t$), and this curve is a characteristic. Finally,

$$\gamma \mapsto \{x : \gamma \in E^t_x\} \tag{5.4}$$

is a map with σ-compact graph.

Proof. The second property (5.3) holds by the definition of concatenation. Suppose now there is a curve $\gamma \in E^t_{x'} \cap E^t_{x''}$: by forward untangling, if two curves γ', γ'' intersect $\mathrm{Graph}\,\gamma$ at $t', t'' > t^-_\gamma$, then they must coincide for $t \geq \max\{t', t''\}$. In particular, if $\{\gamma'_{i'}\}_{i'=0,\dots,N'}$ is the sequence of curves for $E^t_{x'}$ concatenating γ to $\{x'\}$, and $\{\gamma''_{i''}\}_{i''=0,\dots,N''}$ is the sequence of curves for $E^t_{x''}$ concatenating γ to $\{x''\}$, one deduces by an iterative process that

$$\mathrm{Graph}\,\gamma'_{i'} \subset \bigcup_{i''} \mathrm{Graph}\,\gamma_{i''}.$$

Hence $x' = x$.

The same reasoning shows that there is a unique curve connecting $(s, y) \in F^t_x$ to the point (t, x) with $s \leq t$, and being the finite union of characteristics with the forward untangling property, it is fairly easy to see that it is a characteristic.

The last property follows as in the analysis of [BB19, Proposition 9.1]. Indeed, the graph of the function (5.4) is the set

$$\Big\{(\gamma, x) : \exists \gamma_i, i = 0, \dots, N \Big(\{\gamma_i\}^N_{i=0} \text{ concatenated } \wedge \ \gamma_0 = \gamma \ \wedge \ \gamma_N(t) = x\Big)\Big\},$$

which is σ-compact because

$$\Big\{(\gamma, \gamma') \in \mathit{\Delta} \times \mathit{\Delta} : \mathrm{Graph}\,\gamma \cap \mathrm{Graph}\,\gamma' \neq \emptyset\Big\}$$

is σ-compact. □

Corollary 5.3. *The set F^t_x is the union of characteristics $\{\tilde\gamma_{\mathfrak{b}}\}_{\mathfrak{b}\in\mathfrak{B}}$, being $\mathfrak{B}$ a family of indexes. Each $\tilde\gamma_{\mathfrak{b}}$ is defined in a time interval $(t^-_{\mathfrak{b}}, t]$ with the forward untangling property, i.e.*

$$\tilde\gamma_{\mathfrak{b}}(s) = \tilde\gamma_{\mathfrak{b}'}(s) \quad \Longrightarrow \quad \tilde\gamma_{\mathfrak{b}}(\tau) = \tilde\gamma_{\mathfrak{b}'}(\tau) \quad \forall \tau \in [s, t].$$

Moreover $\tilde\gamma_{\mathfrak{b}}(t) = x$.

Proof. Starting from any point $(s, y) \in F^t_x$, there is a unique characteristic $\tilde\gamma_{(s,y)}$ connecting it to the final point (t, x), which is obtained by piecing together the finite set of concatenated characteristic in $\mathit{\Delta}$ connecting (s, y) to (t, x). The set of indexes $\mathfrak{b}$ can be taken as the set of starting points of $\tilde\gamma$. □

We will now write the evolution of the PDE along each set $E^{\bar t}_x$, with $\bar t$ a fixed time. For this purpose, we will restrict η to the curves which belong to some $E^{\bar t}_x$ and consider only the part of these curves contained in $(-\infty, \bar t] \times \mathbb{R}^d$. Define the cutting map at time $\bar t$ as

$$r^{\bar t}(\gamma) := \gamma \llcorner_{(t^-_\gamma, \bar t]},$$

for those γ such that $t^-_\gamma < \bar t$. Let

$$\eta^{\bar t} := (r^{\bar t})_\sharp \big(\eta \llcorner_{E^{\bar t}}\big), \qquad \text{with} \quad E^{\bar t} := \bigcup_x E^{\bar t}_x,$$

and

$$f^{\bar t} : r^{\bar t}(E^{\bar t}) \to \mathbb{R}^d \quad \text{be the quotient Borel map, } f^{\bar t}(E^{\bar t}_x) = x: \tag{5.5}$$

this map has σ-compact graph by (5.4). In other words, we are considering the PDE

$$\mathrm{div}\,\big(\rho(1, \boldsymbol{b})\mathbb{1}_{(-\infty,\bar t)}\big) = \mu \llcorner_{(-\infty,\bar t)} - \rho(\bar t, dx) =: \mu^{\bar t},$$

and for definiteness we will consider $t \mapsto \rho(t)$ continuous from the left w.r.t. the weak convergence of measures. The existence of the measure $\rho(\bar t)$ follows from the standard trace computation on the set $(-\infty, \bar t) \times \mathbb{R}^d$: for every test function ϕ

$$\int \phi \rho(\bar t) = \int \phi(\gamma(\bar t)) \eta(d\gamma),$$

where we consider γ defined in the semiopen interval $(t^-_\gamma, \bar t]$ (because $\rho(t)$ is continuous from the left).

Define the image measure

$$m^{\bar t} = (f^{\bar t})_\sharp \eta^{\bar t}. \tag{5.6}$$

Since

$$\rho(\bar t) = (\mathtt{p}_{\bar t})_\sharp \eta,$$

where $\mathtt{p}_{\bar t}$ is the evaluation map $\mathtt{p}_{\bar t}(\gamma) := \gamma(\bar t)$, it follows that $\rho(\bar t) \ll m^{\bar t}$ and then

$$\rho(\bar t, dx) = \varrho(\bar t, x) m^{\bar t}(dx).$$

By disintegration theorem we can write

$$\eta^{\bar t} = \int \eta^{\bar t}_{\bar x} m^{\bar t}(d\bar x)$$

and hence

$$\begin{aligned}\mu^{\bar t} &= \int_\Gamma \Big[\delta_{(t^-_\gamma, \gamma(t^-_\gamma)} - \delta_{t^+_\gamma, \gamma(t^+_\gamma))}\Big] \eta^{\bar t}(d\gamma) \\ &= \int \bigg[\int_\Gamma \Big[\delta_{(t^-_\gamma, \gamma(t^-_\gamma)} - \delta_{t^+_\gamma, \gamma(t^+_\gamma))}\Big] \eta^{\bar t}_{\bar x}(d\gamma)\bigg] m^{\bar t}(d\bar x) =: \int \mu^{\bar t}_{\bar x} m^{\bar t}(d\bar x),\end{aligned}$$

where we have set

$$\mu^{\bar t}_{\bar x} := \int_\Gamma \Big[\delta_{(t^-_\gamma, \gamma(t^-_\gamma)} - \delta_{t^+_\gamma, \gamma(t^+_\gamma))}\Big] \eta^{\bar t}_{\bar x}(d\gamma). \tag{5.7}$$

Now we consider the projection operator (id, γ) and the measure $(\mathrm{id}, \gamma)_\sharp \eta^{\bar t}_{\bar x}$. Being η a Lagrangian representation, if we define the density

$$(1, \boldsymbol{b}) \rho^{\bar t}_{\bar x} = (\mathrm{id}, \gamma)_\sharp \bigg(\int \big[(1, \dot\gamma) \mathcal{L}^1 \llcorner_{(t^-_\gamma, t^+_\gamma]}\big] \eta^{\bar t}_{\bar x}\bigg),$$

then by direct computation

$$\begin{aligned}\int \big[(1, \boldsymbol{b}) \cdot \nabla_{t,x}\phi\big] \rho^{\bar t}_{\bar x}(t, dx) dt &= \int_\Gamma \bigg[\int_{t^-_\gamma}^{t^+_\gamma} (1, \gamma'(s)) \cdot \nabla_{t,x}\phi(s, \gamma(s))\, \mathcal{L}^1(ds)\bigg] \eta^{\bar t}_{\bar x}(d\gamma) \\ &= \int_\Gamma \bigg[\int_{t^-_\gamma}^{t^+_\gamma} \frac{d}{ds}\phi(s, \gamma(s)) \mathcal{L}^1(ds)\bigg] \eta^{\bar t}_{\bar x}(d\gamma) \\ &= \int_\Gamma \Big\langle \delta_{(t^-_\gamma, \gamma(t^-_\gamma)} - \delta_{t^+_\gamma, \gamma(t^+_\gamma))}, \phi \Big\rangle \eta^{\bar t}_{\bar x}(d\gamma),\end{aligned} \tag{5.8}$$

hence it holds in the sense of distributions

$$\mathrm{div}(\rho^{\bar t}_{\bar x}(1, \boldsymbol{b})) = \mu^{\bar t}_{\bar x}, \qquad \rho^{\bar t}_{\bar x}(\bar t, dx) = \varrho(\bar t, \bar x)\delta_{\bar x}, \tag{5.9}$$

for $m^{\bar t}$-a.e. $\bar x \in \mathbb{R}^d$.

The measures $\rho^{\bar t}_{\bar x}(t)$ can be obviously computed directly from the disintegration of $\rho(t)$ w.r.t. the partition $\{F^{\bar t}_{\bar x}(t)\}_{\bar x}$:

$$\rho(t, dx) \llcorner_{\cup_{\bar x}(F^{\bar t}_{\bar x})(t)} = \int \rho^{\bar t}_{\bar x}(t, dx) m^{\bar t}(d\bar x). \tag{5.10}$$

In the same way, one can recover $\mu^{\bar t}_{\bar x}$ by the disintegration

$$\mu \llcorner_{\cup_{\bar x} F^{\bar t}_{\bar x}} = \int \mu^{\bar t}_{\bar x} m^{\bar t}(d\bar x). \tag{5.11}$$

The balance (5.9) becomes

$$\varrho(\bar t, \bar x) = \rho^{\bar t}_{\bar x}(t, \mathbb{R}^d)(t, x) + \mu^{\bar t}_{\bar x}\big([t, \bar t) \times \mathbb{R}^d\big). \tag{5.12}$$

We collect these results in the following proposition.

Proposition 5.4. *If*

(1) $m^{\bar t}$ is the measure given by (5.6),
(2) $\rho^{\bar t}_x(t)$ is the family of probability measures given by (5.10),
(3) $\mu^{\bar t}_x$ is the disintegration of the measure $\mu \llcorner_{\{t \leq \bar t\}}$ on $F^{\bar t}_x$ as in (5.7),

then for $m^{\bar t}$*-a.e.* $\bar x$

$$\varrho(\bar t,\bar x) = \rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)(t,x) + \mu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big).$$

We now prove that the partition into the sets $F^{\bar t}_{\bar x}$ is essentially unique.

Theorem 5.5. *Assume that every Lagrangian representation of* $\rho(1,\boldsymbol{b})$ *is forward untangled. Then every* η' *is concentrated on a set of curves* $\varDelta'$ *such that if*

$$\gamma\in\varDelta',\qquad \gamma(t)\in F^{\bar t}_{\bar x}(t) \text{ for some } t^-_\gamma < t\le \bar t, \tag{5.13}$$

then

$$\operatorname{Graph}\gamma\llcorner_{(t,\bar t]}\subset F^{\bar t}_{\bar x}. \tag{5.14}$$

Proof. Let $\rho(t)$ be the left continuous density, and $\rho(t+)$ the trace of $\rho(1,\boldsymbol{b})$ in the proper open set $\{s>t\}$. Write the disintegration

$$\eta^{\mathrm{in}}_t = \eta\llcorner_{\{\gamma:t\in(t^-_\gamma,t^+_\gamma]\}} = \int \eta^{\mathrm{in}}_{(t,z)}\rho(t,dz).$$

By the forward untangling assumption, we have that for $\rho(t+)$-a.e. $z\in\mathbb{R}^d$ there exists a curve $\gamma_{(t,z)}(s)$ starting in (t,z) such that it holds

$$\eta^{\mathrm{in}}_{\bar t,z}\Big(\Big\{\gamma : \operatorname{Graph}\gamma\llcorner_{[\bar t,\infty)}\subset \operatorname{Graph}\gamma_{(t,z)}\Big\}\Big) = 1.$$

If we assume that every Lagrangian representation η' is forward untangled, one concludes that for every η' the curve $\gamma_{(t,z)}$ is the same (up to a prolongation) for $\rho(t)$-a.e. z. In particular, if $\gamma_{(t,z)}(s)\in F^{\bar t}_{\bar x}(s)$ for some $s\in[t,\bar t]$, then the curve $\gamma_{(\bar t,z)}\llcorner_{[s,\bar t]}$ can be assumed to satisfy (5.14).

Repeating the above observation for a dense sequence of times $\{t=t_i\}_{i\in\mathbb{N}}$, so that we pick every $\gamma\in\varDelta'$, we can conclude that η' is concentrated on a set of trajectories $\varDelta'$ such that if

$$\gamma\in\varDelta',\qquad \gamma(t)\in F^{\bar t}_{\bar x}(t) \text{ for } t\le\bar t, \tag{5.15}$$

then

$$\operatorname{Graph}\gamma\llcorner_{[t,\bar t]}\subset F^{\bar t}_{\bar x}, \tag{5.16}$$

It remains to prove that the measure of trajectories which are outside $E^{\bar t}_{\bar x}$ for an initial interval of times has η'-measure 0. Call A this set; by a partition argument, it is enough to consider the set $\tilde A\subset A$ of trajectories $\gamma\in A$ intersecting the set $\{t=\tilde t\}$ for some $\tilde t$, but such that the set $\{\gamma(\tilde t),\gamma\in A\}$ is not a subset of $\cup_x F^{\bar t}_x(\tilde t)$. The balance for η' as in (5.12) contains the additional terms $\eta'(\tilde A)$. Since the quantities in (5.12) can be computed without resorting to the Lagrangian representation, it follows that $\eta'(\tilde A)=0$ and hence also $\eta(A)=0$. $\square$

Remark 5.6 (Not strongly consistent disintegration)**.** One can define an equivalence relation as follows. Let $\{t_i\}_i$ be a dense countable sequence of times. For each t_i, let $F^{t_i}_x$ be the family of σ-compact sets constructed in (5.2). Define the equivalence relation E by

$$(t,x)E(t',x') \quad\Longrightarrow\quad \exists F^{t_i}_{x_i}\big((t,x),(t',x')\in F^{t_i}_{x_i}\big). \tag{5.17}$$

Since each F^t_x is σ-compact, it follows that E is made of σ-compact equivalence classes $F_{\mathfrak a}$, $\mathfrak a\in\mathfrak A$ for some index set.

However, one can make an example such that the disintegration w.r.t. E is not *strongly consistent*; we recall that a disintegration $\{\mu_\alpha\}_{\alpha\in A}$ of a measure μ on a space X is said to be *strongly consistent* with a partition $\{X_\alpha\}_{\alpha\in A}$ of X if it holds $\mu_a(X\setminus X_\alpha)=0$ for $\mathtt{p}_\sharp\mu$-a.e. a, being $\mathtt{p}$ the quotient map $\mathtt{p}\colon X\ni x\mapsto [x]\in X/\!\sim\, =: A$ (quotient taken w.r.t. the equivalence relation induced by the partition). Indeed, the problem is equivalent to ask if an equivalence relation whose graph is a countable union of graphs of equivalence relations which have a strongly consistent disintegration has a strongly consistent disintegration. This is easily seen to be false: for example, let us consider the equivalence relations on $[0,1]$ whose graph is

$$E_n := \big\{(x,y)\in[0,1]^2 : x-y=\mathbb{N}2^{-n}\mod 1\big\},\qquad n\in\mathbb{N}$$

and the equivalence relation $\sim$ on $[0,1]$ whose graph is the countable union of E_n, i.e.

$$E := \big\{(x,y) : \exists k,n\in\mathbb{N}, x-y=k2^{-n}\mod 1\big\}.$$

It is possible to show that the Lebesgue measure $\mathcal{L}^1 \llcorner [0,1]$ has *not* a strongly consistent disintegration w.r.t. the partition induced by $\sim$, as it is done in the standard construction of a Vitali set (see e.g. [RF10, Theorem 17, Chapter 2]).

5.2. **Composition rule.** We assume now that every Lagrangian representation of $\rho u(1,\boldsymbol{b})$ is forward untangled for every $u \in L^\infty(\rho)$ such that

$$\operatorname{div}_{t,x}\big(\rho u(1,\boldsymbol{b})\big) = \nu \in \mathcal{M}(\mathbb{R}^{d+1}). \tag{5.18}$$

By Theorem 5.5 the trees used to decompose $\rho(1,\boldsymbol{b})$ are suitable to decompose also $\rho u(1,\boldsymbol{b})$, and thus we obtain the formula

$$\varrho(\bar t,\bar x) u(\bar t,\bar x) = \int u(t,x)\rho^{\bar t}_{\bar x}(t,dx) + \nu^{\bar t}_{\bar x}\big([t,\bar t) \times \mathbb{R}^d\big), \tag{5.19}$$

where $\rho^{\bar t}_{\bar x}$ is given by (5.10) and $\nu^{\bar t}_{\bar x}$ is obtained by the disintegration analogous to (5.11)

$$\nu_{\llcorner \cup_{\bar x} E^{\bar t}_{\bar x}} = \int \nu^{\bar t}_{\bar x} m^{\bar t}(d\bar x). \tag{5.20}$$

For a given function β we now compute

$$\begin{aligned}
\varrho(\bar t,\bar x)\beta(u(\bar t,\bar x)) \underset{(5.19)}{=} {}& \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t,x)\rho^{\bar t}_{\bar x}(t,dx) + \frac{1}{\varrho(\bar t,\bar x)}\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\bigg) \\
= {}& \varrho(\bar t,\bar x)\bigg[\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t) + \frac{1}{\varrho(\bar t,\bar x)}\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\bigg) - \beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg)\bigg] \\
&+ \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) - \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) \\
&+ \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg)
\end{aligned}$$

In the case at t it holds $\rho^{\bar t}_{\bar x} = 0$, the above formula is right by just removing the indetermined terms.

If β is convex, by Jensen's inequality

$$\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) \leq \frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int \beta(u(t))\rho^{\bar t}_{\bar x}(t), \tag{5.21}$$

and then we obtain

$$\begin{aligned}
&\varrho(\bar t,\bar x)\beta(u(\bar t,\bar x)) - \int \beta(u(t))\rho^{\bar t}_{\bar x}(t) \\
&\quad\leq \varrho(\bar t,\bar x)\bigg[\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t) + \frac{1}{\varrho(\bar t,\bar x)}\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\bigg) - \beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg)\bigg] \\
&\qquad+ \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) - \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) \\
&\qquad+ \frac{\varrho(\bar t,\bar x)}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int \beta(u(t))\rho^{\bar t}_{\bar x}(t) - \int \beta(u(t))\rho^{\bar t}_{\bar x}(t).
\end{aligned}$$

Assume now β has at most linear growth and $\beta(0)=0$, in particular it is Lipschitz. From the chain rule for BV function we obtain that:

(1) being β Lipschitz,

$$\begin{aligned}
\varrho(\bar t,\bar x)\bigg[\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t) + \frac{1}{\varrho(\bar t,\bar x)}\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\bigg) - \beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg)\bigg]& \\
\leq \operatorname{Lip}(\beta)\big|\nu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\big|;&
\end{aligned}$$

(2) again from the Lipschitz assunption on β,

$$\begin{aligned}\varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\varrho(\bar t,\bar x)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) - \varrho(\bar t,\bar x)\beta\bigg(\frac{1}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg) \\ \leq \mathrm{Lip}(\beta)\bigg|\bigg(1-\frac{\varrho(\bar t,\bar x)}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\bigg)\int u(t)\rho^{\bar t}_{\bar x}(t)\bigg| \\ \leq \mathrm{Lip}(\beta)\|u\|_\infty\Big|\varrho(\bar t,\bar x)-\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)\Big| \\ \overset{(5.12)}{\leq} \mathrm{Lip}(\beta)\|u\|_\infty\big|\mu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\big|;\end{aligned}$$

(3) in the same fashion as above, since we are assuming $\beta(0)=0$,

$$\frac{\varrho(\bar t,\bar x)}{\rho^{\bar t}_{\bar x}(t,\mathbb{R}^d)}\int \beta(u(t))\rho^{\bar t}_{\bar x}(t) - \int \beta(u(t))\rho^{\bar t}_{\bar x}(t) \leq \mathrm{Lip}(\beta)\|u\|_\infty\big|\mu^{\bar t}_{\bar x}\big([t,\bar t)\times\mathbb{R}^d\big)\big|.$$

We thus obtain the following lemma.

Lemma 5.7. *For every Lipschitz convex function β it holds*

$$\varrho\beta(u)(\bar t,\bar x) \leq \int \beta(u(t))\rho^{\bar t}_{\bar x}(t) + C_{\beta,u}(2|\mu^{\bar t}_{\bar x}| + |\nu^{\bar t}_{\bar x}|)\big([t,\bar t)\times\mathbb{R}^d\big), \tag{5.22}$$

where $C_{\beta,u} \leq \mathrm{Lip}(\beta)(\|u\|_\infty + 1)$.

5.2.1. *Construction of the composition measure.* We restrict the computation of the composition measure to the set $F^{\bar t}_{\bar x}\cap[-\infty,\bar t)\times\mathbb{R}^d$, because integrating the result w.r.t. $m^{\bar t}(d\bar x)$ gives the measure in $\cup_{\bar x}F^{\bar t}_{\bar x}$, and then repeating the computation for a dense countable sequence of times t_i we obtain the composition measure in the whole $\mathbb{R}^{d+1}$.

A preliminary step which simplifies the computation of the composition measure is that we can write balances (5.19) and (5.22) on sets which are more general than time stripes.

Assume that the set A has compact closure, $(\bar t,\bar x)\in A$ and is such that for every $(t,x)\in F^{\bar t}_{\bar x}$ the unique characteristic in $F^{\bar t}_{\bar x}$ connecting (t,x) to $(\bar t,\bar x)$ has at most one intersection with ∂A. Then, if

$$\gamma \mapsto t_{\gamma,A}$$

is the unique intersection time, one can define the measure

$$\rho^{\bar t}_{\bar x,A} = (\mathtt{p}_{t_{\gamma,A},x})_\sharp \eta^{\bar t}_{\bar x}.$$

where

$$\mathtt{p}_{t_{\gamma_A},x}(\gamma) := \big(t_{\gamma,A},\gamma(t_{\gamma,A})\big)$$

is the evaluation map. The above measure is supported on ∂A and independent of the representation, because using the intersection property of A with $F^{\bar t}_{\bar x}$, for every $(t,x)\in\partial A\cap F^{\bar t}_{\bar x}$ the set F^t_x satisfies

$$F^t_x\setminus\{(t,x)\}\subset F^{\bar t}_{\bar x}\setminus \mathrm{clos}\,A,$$

and by writing the balances on each of these sets as in (5.8) we recover $\rho^{\bar t}_{\bar x,A}$: if $A'\subset\partial A\cap F^{\bar t}_{\bar x}$ and $\tilde t<\inf\{t:(t,x)\in A\}$, then

$$\rho^{\bar t}_{\bar x,A}(A') = \rho^{\bar t}_{\bar x}\bigg(\tilde t,\bigcup_{(t,x)\in A'}F^t_x(\tilde t)\bigg) + \mu^{\bar t}_{\bar x}\bigg(\bigcup_{(t,x)\in A'}F^t_x\cap(\tilde t,t)\times\mathbb{R}^d\bigg). \tag{5.23}$$

Using now formula (5.12), we obtain

$$\begin{aligned}\rho^{\bar t}_{\bar x,A}(\partial A) &\underset{(5.23)}{=} \rho^{\bar t}_{\bar t}(\tilde t,\mathbb{R}^d) + \mu^{\bar t}_{\bar x}\bigg(\bigcup_{(t,x)\in\partial A}F^t_x\cap(\tilde t,t)\times\mathbb{R}^d\bigg) \\ &\underset{(5.12)}{=} \varrho(\bar t,\bar x) - \mu^{\bar t}_{\bar x}(\mathrm{clos}\,A).\end{aligned}$$

Using the analogous formula (5.19) for the representation of $u\in L^\infty$ and the balance as above, one obtains

$$\varrho(\bar t,\bar x)u(\bar t,\bar x) = \int u\rho^{\bar t}_{\bar x,A} + \nu^{\bar t}_{\bar x}(\mathrm{clos}\,A), \tag{5.24}$$

The function $u(t,x)$ is the value computed on the top of each F^t_x. Using the same computations to prove Lemma 5.7, one can prove the following lemma.

Lemma 5.8. *If A is a set with compact closure such that $(\bar t,\bar x) \in A$ and each trajectory in $F^{\bar t}_{\bar x}$ has at most a single intersection with ∂A, then for every convex Lipschitz function β and $u \in L^\infty$ solving (5.18) it holds*

$$\varrho\beta(u)(\bar t,\bar x) \le \int \beta(u(t))\rho^{\bar t}_{\bar x,A}(t) + C_{\beta,u}\big(2|\mu^{\bar t}_{\bar x}| + |\nu^{\bar t}_{\bar x}|\big)\big(\bar A \cap [t,\bar t) \times \mathbb{R}^d\big). \tag{5.25}$$

As a corollary, we can apply it to proper sets, by means of the restriction operators $\mathtt{R}^i_\Omega$.

Corollary 5.9. *If Ω is a $\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})$-proper set, $(\bar t,\bar x) \in \operatorname{clos}\Omega$ and $F^{\bar t}_{\bar x,\Omega}$ is made by the connected components of $\gamma \in E^{\bar t}_{\bar x}$ which can be concatenated to $(\bar t,\bar x)$ in Ω, then (5.25) holds.*

The proof is based on the observation that for proper sets almost all trajectories are crossing transversally the boundary, and then one can patch the two estimates of Lemma 5.7 and Lemma 5.8 to move the point $(\bar t,\bar x)$ to the boundary and to replace the integrand in the r.h.s. (5.25) with the trace.

Proof. Since Ω is proper, for every $\epsilon > 0$ there exists $\delta > 0$ with the folowing property: the η-measure of the set of trajectories whose crossing times of $\partial\Omega$ are closer than δ is less than ϵ. It is thus enough to restrict η to the trajectories whose crossing times of $\partial\Omega$ have a distance greater than or equal to δ.

We can then apply Lemma 5.8 to these trajectories if the end point $(\bar t,\bar x)$ is inside Ω and the Lagrangian representation is the restriction $\eta'_\delta = (\mathtt{R}_{\Omega + B^{d+1}_{\delta/2}(0)})_\sharp \eta$ of η: indeed the trajectories used by η'_δ do not reenter Ω after leaving it (by the very definition of η'_δ). Being Ω proper the integral in the r.h.s. of (5.25) coincides with the trace.

If instead $(\bar t,\bar x) \in \partial\Omega$ we can apply (5.12) or (5.22) to the time slice $[\bar t - \delta/2, \bar t)$ (because again these trajectories remain inside Ω in the interval $[\bar t - \delta, \bar t)$) in order to obtain the balance as in (5.25). □

If we integrate (5.24) and (5.25) w.r.t. the exiting part of the trace, $\mathsf{Tr}^-(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega)$, and observe that the entering trace $\mathsf{Tr}^+(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega)$ is given by the projection of the measure $(\mathtt{R}_\Omega)_\sharp\eta$ on the initial points $(t^-_\gamma, \gamma(t^-_\gamma))$, we obtain the following proposition.

Proposition 5.10. *If Ω is a $\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})$-proper and $u\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})$-proper set and (5.18) holds, then*

$$\int u \,\mathsf{Tr}^-(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega) = \int u\,\mathsf{Tr}^+(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega) + \nu^{\bar t}_{\bar x}(\Omega).$$

Moreover for every Lipschitz convex function β it holds

$$\begin{aligned}\mathsf{j}^{\bar t}_{\bar x}(\Omega) :={}& C_{\beta,u}\big(2|\mu^{\bar t}_{\bar x}| + |\nu^{\bar t}_{\bar x}|\big)(\Omega) \\ &- \int \beta(u)\,\mathsf{Tr}^-(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega) + \int \beta(u)\,\mathsf{Tr}^+(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b}),\Omega) \ge 0.\end{aligned}$$

Proof. The only observation is that since $(|\mu| + |\nu|)(\partial\Omega) = 0$, we do not need the left continuity as in Lemma 5.8. □

We now extend the functional $\mathsf{j}^{\bar t}_{\bar x}$ to a general set E by setting

$$\mathsf{j}^{\bar t}_{\bar x}(E) := \inf\big\{\mathsf{j}^{\bar t}_{\bar x}(\Omega), E \subset \Omega \text{ proper}\big\}.$$

It is standard to verify that j is an outer measure, and it is additive on sets with positive distance. We then deduce from the Carathéodory criterion [AFP00, Theorem 1.49] that its restriction to the Borel sets is a positive measure:

Proposition 5.11. *For every Lipschitz, convex function β there exists a positive measure $\mathsf{j}^{\bar t}_{\bar x}$ such that if*

$$\operatorname{div}\big(\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})\big) = \mu^{\bar t}_{\bar x}, \qquad \operatorname{div}\big(u\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})\big) = \nu^{\bar t}_{\bar x},$$

then

$$\operatorname{div}\big(\beta(u)\rho^{\bar t}_{\bar x}(1,\boldsymbol{b})\big) = -\mathsf{j}^{\bar t}_{\bar x} + C_{\beta,u}(|\nu^{\bar t}_{\bar x}| + 2|\mu^{\bar t}_{\bar x}|). \tag{5.26}$$

Moreover

$$\mathsf{j}^{\bar t}_{\bar x}(\mathbb{R}^{d+1}) \le C_{\beta,u}(\|\nu^{\bar t}_{\bar x}\| + 2\|\mu^{\bar t}_{\bar x}\|). \tag{5.27}$$

Proof. The last estimate (5.27) follows by letting $\Omega \nearrow \mathbb{R}^d$, and recalling that $\rho(1, \boldsymbol{b})$ has compact support. □

The fact that we have the uniform bound (5.27) allows to obtain the estimate not only on a single $F^{\bar t}_{\bar x}$ but on the whole space: considering a dense sequence of times t_i and using a standard covering argument, we thus conclude that

Theorem 5.12. *If β is a Lipschitz convex function and u solves (5.18), then there is a positive measure $\mathfrak{j} = \mathfrak{j}_{\beta,u}$ such that*

$$\mathrm{div}_{t,x}\big(\beta(u)\rho(1,\boldsymbol{b})\big) = -\mathfrak{j}_{\beta,u} + C_{\beta,u}(|\nu| + 2|\mu|)$$

Moreover

$$\mathfrak{j}_{\beta,u}(\mathbb{R}^{d+1}) \leq C_{\beta,u}\big(\|\nu\| + 2\|\mu\|\big).$$

In order to get a representation of the measures $\rho(1, \boldsymbol{b})$ for every function β which is the difference of two convex functions (or equivalently whose second derivative is a bounded measure), it is enough to consider the family of 1-Lipschitz functions (Kruzkhov entropies)

$$\beta_{\bar u} = |u - \bar u|,$$

and assume that $\|u\|_{L^\infty(\rho)} = 1$. The formula (5.26) yields a function

$$[-1,1] \ni \bar u \;\mapsto\; \mathfrak{j}_{\beta_{\bar u},u} \in \mathcal{M}^+(\mathbb{R}^{d+1})$$

which is weakly continuous by its very definition (5.26). Using the representation

$$\beta(u) = a + bu + \frac{1}{2}\int_{-1}^{1} |u - \bar u| D^2\beta(d\bar u), \qquad a, b \in \mathbb{R},$$

one obtains

Proposition 5.13. *If $\mathfrak{j}_{\beta_{\bar u},u}$ are the composition measures for $\beta_{\bar u}(u) = |u - \bar u|$, then for every Lipschitz function whose second derivative is a bounded measure it holds*

$$\mathfrak{j}_{\beta,u} = a\mu + b\nu + \int_{-1}^{1} \mathfrak{j}_{\beta_{\bar u},u} D^2\beta(d\bar u). \tag{5.28}$$

Remark 5.14. The measure $\mathfrak{j}_{\beta,u}$ is related to the merging of curves due to (5.22), and if $|\beta''| \leq C$ one can verify by (5.22) that $\mathfrak{j}_{\beta,u} \leq C\mathfrak{j}_{(\cdot)^2/2,u}$. We however are not able to take a measure $\mathfrak{j}$ which does not depend on u and such that $\mathfrak{j}_{(\cdot)^2/2,u} \leq \mathfrak{j}$: indeed, one can image a binary tree, where every branch divides in half at time 2^{-i}. Now, taking $\mu \llcorner_{t>0} = 0$, the measures ρ are

$$\rho(t \in [2^{-i-1}, 2^{-i})) = \sum_{j=1}^{2^i} 2^{-i}\delta_{x_j - tv_j - c_j}, \quad x_j, v_j, c_j \text{ some suitable constants.}$$

One can take a solution $u_{\bar i}$ of $\mathrm{div}(\rho u(1,\boldsymbol{b})) = 0$ which is 0 for $t \geq 2^{-\bar i}$ and it bifurcates to $-1, 1$ at the branching at time $2^{-\bar i}$, then remaining constant for each previous trees. It is immediate to see that

$$\mathfrak{j}_{\bar i} = -\,\mathrm{div}\big(\rho u^2(1,\boldsymbol{b})\big) = \sum_{j=1}^{2^{\bar i}} 2^{-j}\delta_{x_j - 2^{-\bar i}v_j - c_j},$$

whose mass is 1. By varying the time $\bar t$, a measure $\mathfrak{j}$ such that $\mathfrak{j}_{\bar i} \leq \mathfrak{j}$ for all i should have mass $+\infty$, hence it is not a locally bounded measure.

Example 5.15. The measure $\mathfrak{j}_{\beta,u}$ does not seem to have any particular structure in more than one space dimension, where the flow is monotone: here we show that it can be a.c. w.r.t. the Lebesgue measure even in two space dimensions: for a related example see [BB18]. In [ABC13, Section 4] a Lipschitz function $H\colon [0,1]^2 \to [0,1]$ is constructed with the following properties:

(1) for every $x_1 \in [0,1]$ the function $x_2 \mapsto H(x_1, x_2)$ is increasing from $H(x_1, 0) = 0$ to $H(x_1, 1) = 1$;
(2) for every $h \in [0,1]$ the level set $H^{-1}(h)$ is a curve $x_2 = f_h(x_1)$;
(3) the set N for which $\nabla H = 0$ has positive Lebesgue measure and intersect $H^{-1}(h)$ is a single point $\bar x_h$ for $\mathcal{L}^1$-a.e. h.

If ℓ is the arc length on the rectifiable set

$$H^{-1}(h) = \Big\{ x(\ell,h), \ell \in [0,L_h] \Big\}$$

let $\bar{\ell}_h$ be the coordinate of this intersection point on the level set $H^{-1}(h)$, $\bar{x}_h = x(\bar{\ell}_h, h)$, and define the vector field on this level set as

$$\boldsymbol{b}(x(\ell,h)) = \begin{cases} -\nabla^\perp H(x(\ell,h)) & \ell < \bar{\ell}_h, \\ \nabla^\perp H(x(\ell,h)) & \ell \geq \bar{\ell}_h. \end{cases}$$

The vector field is Borel, since the set $\{x(\ell,h)\}_{\ell,h}$ can be taken σ-compact.

Now we follow the analysis of [ABC14]. Using the disintegration

$$\mathcal{L}^2 \llcorner_{[0,1]^2} = \int \frac{\mathcal{H}^1 \llcorner_{H^{-1}(h)}}{|\nabla H|} \mathcal{L}^1(dh) + \int \delta_{\bar{x}_h} \mathcal{L}^1(dh),$$

the reduction of the transport equation $\mathrm{div}(\rho(1,\boldsymbol{b})) = 0$, $\rho \ll \mathcal{L}^2$, on the level set $H^{-1}(h)$ is [ABC14, Lemma 3.7]

$$\partial_t \bigg(\frac{\rho(t,\ell)}{|\nabla H(x(\ell,h))|} \bigg) - \partial_\ell \big(\mathrm{sign}(\ell - \bar{\ell}_h) \rho(t,\ell) \big) + \partial_t \rho(t, \bar{\ell}_h) \delta_{\bar{\ell}_h}(d\ell) = 0.$$

If the initial data is $\rho = 1$, the solution is

$$\rho(\ell,h) = \begin{cases} 1 & \ell \neq \bar{\ell}_h, \\ 1+2t & \ell = \bar{\ell}_h. \end{cases} \tag{5.29}$$

Similarly another solution is given by

$$\rho(\ell,h) u(\ell,h) = \begin{cases} \mathrm{sign}(\ell - \bar{\ell}_h) & \ell \neq \bar{\ell}_h, \\ 0 & \ell = \bar{\ell}_h. \end{cases} \tag{5.30}$$

Thus for $\beta(u) = u^2$, by comparing (5.29) with (5.30) it follows that

$$\mathrm{j}_{(\cdot)^2,u} = 2\mathcal{L}^2 \llcorner_N = 2\mathcal{L}^2 \llcorner_{\{\nabla H = 0\}}.$$

6. Monotone vector fields

Let $A(t)$, $t \in \mathbb{R}$ be a maximal monotone operator on $\mathbb{R}^d$, i.e. assume

$$\forall y_1 \in A(t,x_1), y_2 \in A(t,x_2): \qquad (y_1 - y_2, x_1 - x_2) \geq 0$$

with

$$\forall R > 0: \qquad \int_{B^{d+1}_R(0)} |A(t,x)| \, dt \, dx < +\infty. \tag{6.1}$$

Since we will consider only locally defined solutions, the above integrability assumption can be clearly relaxed to some local condition and as [BG11, Section 2] we can also consider quasi-monotone operators.

Consider the ODE

$$\dot{x}(t) \in -A(t) \qquad \mathcal{L}^1\text{-a.e. } t > 0. \tag{6.2}$$

In [BG11] the following results are proved.

Proposition 6.1 ([BG11, Proposition 3.3])**.** *For any initial datum $x(0) = x_0 \in \mathbb{R}^d$ there exists a unique solution $x(t)$ to (6.2) with 1-Lipschitz dependence on the initial datum.*

Theorem 6.2 ([BG11, Theorem 1.2])**.** *If $A_n(t)$ is a family of monotone operators converging in $L^1(\mathbb{R}^{d+1})$ to A, then the flow $X_n(t)$ constructed in Proposition 6.1 for the operators $A_n(t)$ converges to the corresponding flow $X(t)$ for $A(t)$.*

The flow of Proposition 6.1 is defined for every initial point and forward untangled, because of the forward uniqueness. The last result we recall is that there is a universally measurable selection

$$\boldsymbol{a}(t,x) \in A(t,x)$$

such that it holds

$$\dot{X}(t,x_0) = \boldsymbol{a}(t, X(t,x_0)) \quad \mathcal{L}^1\text{-a.e. } t \in \mathbb{R}.$$

Define the set

$$\mathcal{N} = \left\{(t, x_0) \in \mathbb{R}^{d+1} : \nexists \frac{d}{dt} X(t, x_0)\right\}$$

In [BG11, Lemma 5.3] it is proved that this set is negligible for all measures of the form $\mathcal{L}^1 \times \mu$, $\mu \in \mathcal{M}(\mathbb{R}^d)$. Hence one can define

Definition 6.3 ([BG11, Definition 5.5])**.** For the time-dependent maximal monotone operator $A(t)$, define a single-valued, everywhere defined vector field $\boldsymbol{b} : \mathbb{R}^{d+1} \to \mathbb{R}^d$ by first setting

$$\boldsymbol{b}(t, X(t, x_0)) = \frac{d}{dt} X(t, x_0), \qquad (t, x) \notin \mathcal{N},$$

and then extending it arbitrarily on $X(\mathcal{N})$.

Consider now

$$\rho_t + \operatorname{div}(\rho \boldsymbol{b}) = 0, \qquad \rho(t = 0) = \rho_0. \tag{6.3}$$

The following proposition holds.

Proposition 6.4. *Let $A(t)$ be a maximal monotone operator, and ρ_0 be a non-negative measure. Then there is a unique non-negative measure-valued solution to the continuity equation (6.3) given by the formula*

$$\rho(t) = (X(t))_\sharp \rho_0.$$

The proof is a direct consequence of the forward uniqueness of the characteristics.

References

[ABC13] G. Alberti, S. Bianchini, and G. Crippa. Structure of level sets and Sard-type properties of Lipschitz maps. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 12(4):863–902, 2013.

[ABC14] G. Alberti, S. Bianchini, and G. Crippa. A uniqueness result for the continuity equation in two dimensions. *J. Eur. Math. Soc. (JEMS)*, 16(2):201–234, 2014.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Science Publications. Clarendon Press, 2000.

[Amb04] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Inventiones mathematicae*, 158(2):227–260, 2004.

[BB18] S. Bianchini and P. Bonicatto. Failure of the chain rule in the non steady two-dimensional setting. *Current Research in Nonlinear Analysis: In Honor of Haim Brezis and Louis Nirenberg*, pages 33–60, 2018.

[BB19] S. Bianchini and P. Bonicatto. A uniqueness result for the decomposition of vector fields in $\mathbb{R}^d$. *Inventiones mathematicae*, 220(1):255–393, 2019.

[BC13] F. Bouchut and G. Crippa. Lagrangian flows for vector fields with gradient given by a singular integral. *J. Hyperbolic Differ. Equ.*, 10(2):235–282, 2013.

[BG11] S. Bianchini and M. Gloyer. An estimate on the flow generated by monotone operators. *Comm. Partial Differential Equations*, 36(5):777–796, 2011.

[Bon17] P. Bonicatto. *Untangling of trajectories for non-smooth vector fields and Bressan's Compactness Conjecture*. PhD thesis, SISSA, 2017.

[BS] S. Bianchini and A. Stavitskiy. Forward untangling in metric measure spaces and applications.

[CNSS17] G. Crippa, C. Nobili, C. Seis, and S. Spirito. Eulerian and Lagrangian solutions to the continuity and Euler equations with L^1 vorticity. *SIAM Journal on Mathematical Analysis*, 49(5):3973–3998, 2017.

[Kel84] H. G. Kellerer. Duality theorems for marginal problems. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 67(4):399–432, 1984.

[RF10] H.L. Royden and P. Fitzpatrick. *Real Analysis*. Prentice Hall, 2010.

[Smi94] S. K. Smirnov. Decomposition of solenoidal vector charges into elementary solenoids and the structure of normal one-dimensional currents. *St. Petersburg Math. J.*, 5(4):841–867, 1994.

[Sri98] S.M. Srivastava. *A Course on Borel Sets*. Graduate Texts in Mathematics. Springer, 1998.

S. Bianchini: S.I.S.S.A., via Bonomea 265, 34136 Trieste, Italy
Email address: bianchin@sissa.it

P. Bonicatto: Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, CH-4051, Basel, Switzerland
Email address: paolo.bonicatto@unibas.ch