Fractional powers and singular perturbations of quantum differential Hamiltonians

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We consider the fractional powers of singular (point-like) perturbations of the Laplacian and the singular perturbations of fractional powers of the Laplacian, and we compare two such constructions focusing on their perturbative structure for resolvents and on the local singularity structure of their domains. In application to the linear and non-linear Schrödinger equations for the corresponding operators, we outline a programme of relevant questions that deserve being investigated.

I. BACKGROUND: AT THE EDGE OF FRACTIONAL QUANTUM MECHANICS AND ZERO-RANGE INTERACTIONS

At the edge of the theory of quantum Hamiltonians with zero-range interactions, the theory of partial differential operators, and the recent mainstream of fractional quantum mechanics, there are two constructions, each of which is classical in nature and the combination of which has been receiving an increasing attention in the recent times.

The first is the construction of fractional powers of a differential operator with the non-negative symbol and more generally the fractional power of a non-negative self-adjoint operator on $L^2(\mathbb{R}^d)$. For concreteness, let us focus on the negative Laplacian

$$-\Delta = -\sum_{j=1}^{d}\frac{\partial^2}{\partial x_j^2}, \qquad x \equiv (x_1,\dots,x_d)\in\mathbb{R}^d,$$

which we shall simply call the Laplacian. In this case, the definition of $(-\Delta)^{s/2}$ for $s\in\mathbb{R}$ is obvious in terms of the corresponding power of the Fourier multiplier for $-\Delta$,

$$((-\Delta)^{s/2}f)(x) = (|p|^s\widehat{f})^{\vee}(x),$$

where $(\mathcal{F}f)(p)\equiv\widehat{f}(p)=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\mathrm{i}xp}f(x)\mathrm{d}x$ is the present convention for the Fourier transform. In fact, $\mathcal{F}(-\Delta)\mathcal{F}^*$ gives an explicit multiplication form and hence an explicit spectral decomposition for $-\Delta$ as a non-negative self-adjoint operator on $L^2(\mathbb{R}^d)$, and thus $(-\Delta)^{s/2}$ given by the identity above coincides with the construction with functional calculus.

The second construction is that the kind of perturbation of a given pseudo-differential operator that heuristically amounts to add to it a potential supported at one point only, whence the jargon of singular perturbation.[2,3] This is a typical restriction-extension construction, where the first one restricts the initial operator to sufficiently smooth functions vanishing in neighbourhoods of a given point $x_0\in\mathbb{R}^d$ and then one builds an operator extension of such restriction that is self-adjoint on $L^2(\mathbb{R}^d)$. This procedure, when the initial operator is $-\Delta$, is known to be equivalent to the somewhat

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more concrete scheme of taking the limit $\varepsilon \downarrow 0$ in Schrödinger operators of the form $-\Delta + V_\varepsilon$, where V_ε is a regular potential essentially supported at a scale ε^{-1} around x_0 and magnitude diverging with ε, which shrink to a delta-like profile centred at x_0.[2,3]

Both constructions are well known and relevant in several contexts: Sobolev spaces, fractional Sobolev norms, high and low regularity theory for non-linear partial differential equations, etc., concerning the former, and solvable yet realistic models with computable spectral features (eigenvalues, scattering matrix,...) in quantum mechanics, chemistry, biology, acoustics, concerning the latter.

Recently, especially for the solution theory of non-linear Schrödinger equations whose linear part is governed by singular Hamiltonians of point interactions,[8,12] as well as for linear Schrödinger-like equations with singular perturbations of fractional powers of the Laplacian,[4,10,11,14–18,20] the interest has increased around various ways of *combining the two constructions above*.

The goal of this work is to set up the problem in a systematic way for two operations that in a sense do not commute, to make the rigorous constructions of the operators of interest and to make qualitative and quantitative comparisons.

We shall consider, in the appropriate sense that we are going to specify, the class of singular perturbations of the d-dimensional Laplacian, supported at a point $x_0 \in \mathbb{R}^d$, in the homogeneous and inhomogeneous case, namely (informally speaking),

$$\begin{aligned} \mathfrak{h}_\tau &= -\Delta + \text{ singular perturbation at } x_0, \\ \mathfrak{h}_\tau + \mathbb{1} &= -\Delta + \mathbb{1} + \text{ singular perturbation at } x_0, \end{aligned} \tag{1}$$

and then we shall combine this construction with that of fractional powers, thus considering on one hand the operators

$$\begin{aligned} \mathfrak{h}_\tau^{s/2} &= (-\Delta + \text{ singular perturbation at } x_0)^{s/2}, \\ (\mathfrak{h}_\tau + \mathbb{1})^{s/2} &= (-\Delta + \mathbb{1} + \text{ singular perturbation at } x_0)^{s/2} \end{aligned} \tag{2}$$

and on the other hand, the operators

$$\begin{aligned} \mathsf{k}_\tau^{(s/2)} &= (-\Delta)^{s/2} + \text{ singular perturbation at } x_0, \\ \mathfrak{d}_\tau^{(s/2)} &= (-\Delta + \mathbb{1})^{s/2} + \text{ singular perturbation at } x_0. \end{aligned} \tag{3}$$

All operators in (1)–(3) above are meant to be self-adjoint.

Let us remark that $\mathfrak{h}_\tau^{s/2}$ and $(\mathfrak{h}_\tau + \mathbb{1})^{s/2}$ are going to be genuine fractional powers of a non-negative self-adjoint operator on $L^2(\mathbb{R}^3)$ (to this aim, one has to consider non-restrictively only those singular perturbations that produce non-negative operators), whereas the different notation for the superscript $s/2$ in $\mathsf{k}_\tau^{(s/2)}$ and in $\mathfrak{d}_\tau^{(s/2)}$ is to indicate that the latter operators are instead singular perturbations of $s/2$-th powers (and not fractional powers of singular perturbations).

In all cases, $\tau \in (\mathbb{R} \cup \{\infty\})^{\mathcal{J}}$, for some $\mathcal{J} \in \mathbb{N}$, is going to be a parameter that qualifies one element in the infinite family of self-adjoint realisations of the considered singular perturbation, with the customary convention that "$\tau = \infty$" denotes the absence of perturbation.

As for the choice of the point x_0, there is no loss of generality in choosing $x_0 = 0$, which we shall do henceforth.

The knowledge of singular quantum Hamiltonians of the type (1) is well established in the literature, and we review them in Sec. II. The study of their fractional powers, hence of the operators of the type (2), has only started recently, and in the second part of Sec. II, we give an account of the main known facts about them.

In comparison to (1) and (2), we shall then discuss the rigorous construction of operators of the type (3) (Secs. III–VII). Our presentation will have two main focuses, which reflect into the formulation of our following results:

1. to qualify the nature of the perturbation in the resolvent sense (*finite rank* vs *infinite-rank* perturbations);
2. to qualify the natural decomposition of the domain of the considered operators into a *regular component* and a *singular component* and to determine the boundary condition constraining such two components.

The first issue is central for deducing an amount of properties from the unperturbed to the perturbed operators. The second issue also arises naturally, as one can see heuristically that the considered operators must act in an ordinary way on those functions supported away from the perturbation centre, and therefore their domains must contain a subspace of H^s-regular functions, where s is the considered power, next to a more singular component, that is, the signature of the perturbation.

In this respect, we are going to highlight profound differences between two constructions, say,

$$(-\Delta + \mathbb{1} + \text{ singular perturbation at } x_0)^{s/2},$$
$$(-\Delta + \mathbb{1})^{s/2} + \text{ singular perturbation at } x_0,$$

which are therefore "non-commutative."

We organized the material as follows:

- the analysis of operators of fractional power of point interaction Hamiltonians is presented in Sec. II;
- on the other hand, the construction of singular perturbations of fractional Laplacians is presented in Secs. III–VII, with the general setup in Sec. III, the detailed discussion of the paradigmatic scenario of rank-one perturbations is presented in Secs. IV and V (homogeneous case) and in Sec. VI (inhomogeneous case), and an outlook on high-rank perturbations is presented in Sec. VII;
- finally, in Sec. VIII, we outline an amount of relevant questions that deserve being investigated in application to the linear and non-linear Schrödinger equations governed by the operators constructed in this work.

II. SINGULAR PERTURBATIONS OF THE LAPLACIAN AND THEIR FRACTIONAL POWERS

In this section, we qualify the operators $\mathfrak{h}_\tau$, $\mathfrak{h}_\tau^{s/2}$, and $(\mathfrak{h}_\tau + \mathbb{1})^{s/2}$ of (1) and (2). Whereas the former is well known in all dimensions in which it is not trivial, namely, $d = 1, 2, 3$, the latter two operators have been studied mainly in three dimensions, and thus in this section, we choose $d = 3$ for our presentation.

For chosen $\lambda > 0$, we set

$$G_\lambda(x) := \frac{e^{-\sqrt{\lambda}|x|}}{4\pi|x|} = \frac{1}{(2\pi)^{\frac{3}{2}}}\Big(\frac{1}{p^2+\lambda}\Big)^{\vee}(x), \qquad x, p \in \mathbb{R}^3, \tag{4}$$

whence

$$(-\Delta + \lambda)G_\lambda = \delta(x) \tag{5}$$

as a distributional identity on $\mathbb{R}^3$. We also set

$$\mathring{\mathfrak{h}} := \overline{-\Delta \upharpoonright C_0^\infty(\mathbb{R}^3 \setminus \{0\})} \tag{6}$$

as an operator closure with respect to the Hilbert space $L^2(\mathbb{R}^3)$.

As well known,[13] $\mathring{\mathfrak{h}}$ is densely defined, closed, and symmetric on $L^2(\mathbb{R}^3)$, with

$$\mathcal{D}(\mathring{\mathfrak{h}}) = \Big\{ f \in H^2(\mathbb{R}^3) \,\Big|\, \int_{\mathbb{R}^3} \widehat{f}(p)\,\mathrm{d}p = 0 \Big\}, \qquad \mathring{\mathfrak{h}} f = -\Delta f. \tag{7}$$

Its Friedrichs extension is given by the self-adjoint Laplacian on $L^2(\mathbb{R}^3)$ with domain $H^2(\mathbb{R}^3)$, and its adjoint is the operator

$$\begin{aligned} \mathcal{D}(\mathring{\mathfrak{h}}^*) &= \left\{ g \in L^2(\mathbb{R}^3) \,\middle|\, \begin{array}{c} \widehat{g}(p) = \widehat{f^\lambda}(p) + \dfrac{\eta}{(p^2+\lambda)^2} + \dfrac{\xi}{p^2+\lambda} \\ f^\lambda \in \mathcal{D}(\mathring{\mathfrak{h}}), \quad \eta, \xi \in \mathbb{C} \end{array} \right\} \\ &= H^2(\mathbb{R}^3) \dotplus \mathrm{span}\{G_\lambda\}, \\ \big((\widehat{\mathring{\mathfrak{h}}^* + \lambda)g}\big)(p) &= (p^2+\lambda)\Big(\widehat{f^\lambda}(p) + \frac{\eta}{(p^2+\lambda)^2}\Big). \end{aligned} \tag{8}$$

The structure in Eq. (8) is typical of a well-known decomposition [see, e.g., Eq. (2.5) and (2.10) in Ref. 7]. We observe that $\mathcal{D}(\mathring{\mathfrak{h}}^*)$ does *not* depend on λ, only the splitting of $g \in \mathcal{D}(\mathring{\mathfrak{h}}^*)$ into a H^2-function plus a less regular component does. The last identity in (8) may be re-interpreted distributionally as

$$\mathring{\mathfrak{h}}^* g = -\Delta g + (2\pi)^{\frac{3}{2}} \xi\, \delta(x) g,$$

where neither $-\Delta g$ nor $(2\pi)^{\frac{3}{2}} \xi\, \delta(x) g$ belongs to $L^2(\mathbb{R}^3)$, but their sum does.

The fact that $\ker(\mathring{\mathfrak{h}}^* + \lambda\mathbb{1}) = \operatorname{span}\{G_\lambda\}$ indicates that $\mathring{\mathfrak{h}}$ has deficiency index 1 and hence admits a one-(real-)parameter family of self-adjoint extensions. They can be classified in terms of the standard parameterization of the Kreĭn-Višik-Birman self-adjoint extension theory (see, e.g., Ref. 7, Sec. 3), identifying each of them as a restriction of $\mathring{\mathfrak{h}}^*$ by imposing in (8) Birman's self-adjointness condition $\eta = \tau\xi$ for some $\tau \in \mathbb{R} \cup \{\infty\}$ (see, e.g., Ref. 13, Theorem 1, and Corollary 1), whence the following theorem.

Theorem II.1. *Let $\lambda > 0$.*

(i) *The self-adjoint extensions in $L^2(\mathbb{R}^3)$ of the operator $\mathring{\mathfrak{h}}$ form the family $(\mathfrak{h}_\tau)_{\tau\in\mathbb{R}\cup\{\infty\}}$, where $\mathfrak{h}_\infty = \mathring{\mathfrak{h}}_F$, the Friedrichs extension, and all other (proper) extensions are given by $\mathfrak{h}_\tau = \mathring{\mathfrak{h}}^* \upharpoonright \mathcal{D}(\mathfrak{h}_\tau)$, where*

$$\begin{aligned}\mathcal{D}(\mathfrak{h}_\tau) &:= \left\{ g \in L^2(\mathbb{R}^3) \,\middle|\, \begin{array}{c} \widehat{g}(p) = \widehat{f^\lambda}(p) + \dfrac{\tau\,\xi}{(p^2+\lambda)^2} + \dfrac{\xi}{p^2+\lambda} \\ \xi \in \mathbb{C}, \quad f^\lambda \in \mathcal{D}(\mathring{\mathfrak{h}}) \end{array} \right\} \\ &= \Big\{ g = F^\lambda + \tfrac{8\pi\sqrt{\lambda}}{\tau} F^\lambda(0)\, G_\lambda \,\Big|\, F^\lambda \in H^2(\mathbb{R}^3) \Big\}.\end{aligned} \tag{9}$$

(ii) *Each extension is semi-bounded from below and*

$$\begin{aligned} \inf \sigma(\mathfrak{h}_\tau + \lambda\mathbb{1}) \geqslant 0 \quad &\Leftrightarrow \quad \tau \geqslant 0, \\ \inf \sigma(\mathfrak{h}_\tau + \lambda\mathbb{1}) > 0 \quad &\Leftrightarrow \quad \tau > 0, \\ (\mathfrak{h}_\tau + \lambda\mathbb{1}) \textit{ is invertible} \quad &\Leftrightarrow \quad \tau \neq 0. \end{aligned} \tag{10}$$

(iii) *For each $\tau \in \mathbb{R}$, the quadratic form of the extension $\mathfrak{h}_\tau$ is given by*

$$\mathcal{D}[\mathfrak{h}_\tau] = H^1(\mathbb{R}^3) \dotplus \operatorname{span}\{G_\lambda\}, \tag{11}$$

$$\begin{aligned}\mathfrak{h}_\tau[F^\lambda + \kappa_\lambda G_\lambda] = \|\nabla F^\lambda\|^2_{L^2(\mathbb{R}^3)} - \lambda \|F^\lambda + \kappa_\lambda G_\lambda\|^2_{L^2(\mathbb{R}^3)} + \lambda \|F^\lambda\|^2_{L^2(\mathbb{R}^3)} \\ + \frac{\tau}{8\sqrt{\lambda}} |\kappa_\lambda|^2\end{aligned} \tag{12}$$

for any $F^\lambda \in H^1(\mathbb{R}^3)$ and $\kappa_\lambda \in \mathbb{C}$.

Remark II.2. The τ-parameterization depends on the initial choice of the λ-shift, and thus the same extension is described by infinitely many different pairs (λ, τ)—of course with a unique τ once λ is chosen. This is clear by inspecting the boundary condition at $x = 0$ between the regular and singular component of a generic $g \in \mathcal{D}(\mathfrak{h}_\tau)$: for any two pairs (λ, τ) and (λ', τ'), identifying the same extension $\mathfrak{h}_\tau$, owing to (9), one has

$$\widehat{g} = \widehat{F^\lambda} + \frac{\xi}{p^2+\lambda} = \widehat{F^{\lambda'}} + \frac{\xi'}{p^2+\lambda'},$$

$$\textit{with} \quad \xi' := \tfrac{8\pi\sqrt{\lambda}}{\tau'\,(2\pi)^3} \Big(\int_{\mathbb{R}^3} \widehat{F^\lambda} \mathrm{d}p \Big), \quad \widehat{F^{\lambda'}} := \widehat{F^\lambda} + \tfrac{\xi'(\lambda'-\lambda)}{(p^2+\lambda)(p^2+\lambda')},$$

and also

$$\begin{aligned} \int_{\mathbb{R}^3} \widehat{F^\lambda} \mathrm{d}p &= \tfrac{\tau\,(2\pi)^3}{8\pi\sqrt{\lambda}}\, \xi, \\ \int_{\mathbb{R}^3} \widehat{F^{\lambda'}} \mathrm{d}p &= \tfrac{\tau'\,(2\pi)^3}{8\pi\sqrt{\lambda'}}\, \xi' \cdot \tfrac{\tau\sqrt{\lambda'}}{\tau'\sqrt{\lambda}} \big(1 + \tfrac{2\sqrt{\lambda}(\sqrt{\lambda'}-\sqrt{\lambda})}{\tau}\big), \end{aligned}$$

whence necessarily $1=\frac{\tau\sqrt{\lambda'}}{\tau'\sqrt{\lambda}}(1+\frac{2\sqrt{\lambda}(\sqrt{\lambda'}-\sqrt{\lambda})}{\tau})$ *or equivalently*

$$\frac{\tau-2\lambda}{\sqrt{\lambda}}=\frac{\tau'-2\lambda'}{\sqrt{\lambda'}}. \tag{13}$$

Thus, when referring to the extension $\mathfrak{h}_\tau$, *we shall always implicitly declare the choice of* λ *and any other* λ' *and* τ' *satisfying (13) actually identify the same extension.*

In the literature, it is customary to find the second expression of (9) with the alternative extension parameter α defined by

$$\alpha:=\frac{\tau-2\lambda}{8\pi\sqrt{\lambda}}, \tag{14}$$

thus re-writing $(\mathfrak{h}_\tau)_{\tau\in\mathbb{R}\cup\{\infty\}}$ as $(\mathfrak{h}_\alpha)_{\alpha\in\mathbb{R}\cup\{\infty\}}$ with $\mathfrak{h}_\alpha:=\mathring{\mathfrak{h}}^*\upharpoonright\mathcal{D}(\mathfrak{h}_\alpha)$ [see Remark 3 in Ref. 13 and Eq. (I.1.1.26) in Ref. 2. Of course, owing to Remark II.2, in particular, to formula (13), the parameter α identifies unambiguously an extension irrespectively of the shift λ chosen for the explicit domain decomposition. From this and a bit of further spectral analysis (Ref. 2, Sec. I.1.1), one then deduces the following.

Theorem II.3.

(i) *The self-adjoint extensions in* $L^2(\mathbb{R}^3)$ *of the operator* $\mathring{\mathfrak{h}}$ *form the family* $(\mathfrak{h}_\alpha)_{\alpha\in\mathbb{R}\cup\{\infty\}}$, *where* $\mathfrak{h}_\infty=\mathring{\mathfrak{h}}_F$, *the Friedrichs extension, and all other (proper) extensions are given by* $\mathfrak{h}_\alpha:=\mathring{\mathfrak{h}}^*\upharpoonright\mathcal{D}(\mathfrak{h}_\alpha)$, *where, for arbitrary* $\lambda>0$,

$$\begin{aligned}\mathcal{D}(\mathfrak{h}_\alpha)&=\Big\{g=F^\lambda+(\alpha+\tfrac{\sqrt{\lambda}}{4\pi})^{-1}F^\lambda(0)\,G_\lambda\,\Big|\,F^\lambda\in H^2(\mathbb{R}^3)\Big\},\\(\mathfrak{h}_\alpha+\lambda)\,g&=(-\Delta+\lambda)\,F^\lambda.\end{aligned} \tag{15}$$

(ii) *The spectrum of* $\mathfrak{h}_\alpha$ *is given by*

$$\begin{aligned}\sigma_{\mathrm{ess}}(\mathfrak{h}_\alpha)=\sigma_{\mathrm{ac}}(\mathfrak{h}_\alpha)&=[0,+\infty),\qquad \sigma_{\mathrm{sc}}(\mathfrak{h}_\alpha)=\emptyset,\\ \sigma_{\mathrm{p}}(\mathfrak{h}_\alpha)&=\begin{cases}\emptyset & \textit{if } \alpha\in[0,+\infty],\\ \{-(4\pi\alpha)^2\} & \textit{if } \alpha\in(-\infty,0).\end{cases}\end{aligned} \tag{16}$$

The negative eigenvalue $-(4\pi\alpha)$,[2] *when it exists, is non-degenerate and the corresponding eigenfunction is* $|x|^{-1}e^{-4\pi|\alpha|\,|x|}$. *Thus,* $\alpha\geqslant0$ *corresponds to a non-confining 'repulsive' contact interaction.*

(iii) *For each* $\alpha\in\mathbb{R}$, *the quadratic form of the extension* $\mathfrak{h}_\alpha$ *is given by*

$$\begin{aligned}\mathcal{D}[\mathfrak{h}_\alpha]&=H^1(\mathbb{R}^3)\dotplus\mathrm{span}\{G_\lambda\},\\ \mathfrak{h}_\alpha[F^\lambda+\kappa_\lambda G_\lambda]&=-\lambda\|F^\lambda+\kappa_\lambda G_\lambda\|^2_{L^2(\mathbb{R}^3)}\\&\qquad+\|\nabla F^\lambda\|^2_{L^2(\mathbb{R}^3)}+\lambda\|F^\lambda\|^2_{L^2(\mathbb{R}^3)}+(\alpha+\tfrac{\sqrt{\lambda}}{4\pi})\,|\kappa_\lambda|^2\end{aligned} \tag{17}$$

for arbitrary $\lambda>0$.

(iv) *The resolvent of* $\mathfrak{h}_\alpha$ *is given by*

$$(\mathfrak{h}_\alpha+\lambda\mathbb{1})^{-1}=(-\Delta++\lambda\mathbb{1})^{-1}+(\alpha+\tfrac{\sqrt{\lambda}}{4\pi})^{-1}|G_\lambda\rangle\langle G_\lambda| \tag{18}$$

for arbitrary $\lambda>0$.

For the operators $\mathfrak{h}_\alpha+\lambda\mathbb{1}$ with λ sufficiently large and the operators $\mathfrak{h}_\alpha$ with $\alpha\geqslant0$, the self-adjoint functional calculus defines the powers $(\mathfrak{h}_\alpha+\lambda\mathbb{1})^{s/2}$ or $\mathfrak{h}_\alpha^{s/2}$ for $s\in\mathbb{R}$.

A surely relevant regime is surely $s\in[0,2]$: the integer powers $s=2,1,0$ correspond, respectively, to the considered operator, its square root whose domain is then the form domain of the considered operator, and the identity; the fractional powers in between are of interest when one needs to discuss the corresponding linear or non-linear dynamics in spaces of convenient fractional regularity.

From the thorough analysis of such fractional powers made in Ref. 8, one has the following.

Theorem II.4. *Let $\alpha \geqslant 0$, $\lambda > 0$, and $s \in [0,2]$, and set*

$$\begin{aligned} \widetilde{H}^s_\alpha(\mathbb{R}^3) &:= \mathcal{D}(\mathfrak{h}_\alpha^{s/2}) = \mathcal{D}(\mathfrak{h}_\alpha + \lambda\mathbb{1})^{s/2}, \\ \|g\|_{\widetilde{H}^s_\alpha(\mathbb{R}^3)} &:= \|(\mathfrak{h}_\alpha + \lambda\mathbb{1})^{s/2} g\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{19}$$

(i) *One has*

$$\left.\begin{aligned} \widetilde{H}^s_\alpha(\mathbb{R}^3) &= H^s(\mathbb{R}^3) \\ \|\psi\|_{\widetilde{H}^s_\alpha(\mathbb{R}^3)} &\approx \|\psi\|_{H^s(\mathbb{R}^3)} \end{aligned}\right\} \quad \text{if } s \in [0, \tfrac{1}{2}), \tag{20}$$

$$\left.\begin{aligned} \widetilde{H}^s_\alpha(\mathbb{R}^3) &= H^s(\mathbb{R}^3) \dotplus \operatorname{span}\{G_\lambda\} \\ \|F^\lambda + \kappa_\lambda G_\lambda\|_{\widetilde{H}^s_\alpha(\mathbb{R}^3)} &\approx \|F^\lambda\|_{H^s(\mathbb{R}^3)} + (1+\alpha)|\kappa_\lambda| \end{aligned}\right\} \quad \text{if } s \in (\frac{1}{2}, \frac{3}{2}), \tag{21}$$

$$\left.\begin{aligned} \widetilde{H}^s_\alpha(\mathbb{R}^3) &= \Big\{ g = F^\lambda + (\alpha + \tfrac{\sqrt{\lambda}}{4\pi})^{-1} F^\lambda(0)\, G_\lambda \,\Big|\, F^\lambda \in H^s(\mathbb{R}^3) \Big\} \\ \|F^\lambda + \kappa_\lambda G_\lambda\|_{\widetilde{H}^s_\alpha(\mathbb{R}^3)} &\approx \|F^\lambda\|_{H^s(\mathbb{R}^3)} \end{aligned}\right\} \quad \text{if } s \in (\frac{3}{2}, 2), \tag{22}$$

where "$\approx$" denotes the equivalence of norms.

(ii) *For $g \in \mathcal{D}(\mathfrak{h}_\alpha^{s/2})$,*

$$\begin{aligned} (\mathfrak{h}_\alpha + \lambda\mathbb{1})^{s/2} g &= (-\Delta + \lambda\mathbb{1})^{s/2} g \\ &\quad - 4 \sin \tfrac{s\pi}{2} \int_0^{+\infty} \mathrm{d}t\, \frac{t^{s/2}\, \langle G_{\lambda+t}, g\rangle_{L^2(\mathbb{R}^3)}}{4\pi\alpha + \sqrt{\lambda+t}}\, G_{\lambda+t}. \end{aligned} \tag{23}$$

(iii) *One has the resolvent identity*

$$(\mathfrak{h}_\alpha + \lambda\mathbb{1})^{-s/2} = (-\Delta + \lambda\mathbb{1})^{-s/2} - \int_0^{+\infty} \mathrm{d}t\, \frac{4 t^{-\frac{s}{2}} \sin\frac{s\pi}{2}}{4\pi\alpha + \sqrt{\lambda+t}}\, |G_{\lambda+t}\rangle\langle G_{\lambda+t}|. \tag{24}$$

The transition cases $s = \frac{1}{2}$ and $s = \frac{3}{2}$ too can be qualified, however with less explicit formulas—see Propositions 8.1 and 8.2 in Ref. 8.

Thus, as described in Theorem II.4, for $s > \frac{1}{2}$, $\widetilde{H}^s_\alpha(\mathbb{R}^3)$ decomposes into a regular H^s-component and a singular component with local singularity $|x|^{-1}$, *precisely as the domain of* $\mathfrak{h}_\alpha$ *itself*, and for $s > \frac{3}{2}$, regular and singular parts are constrained by a local boundary condition among them *of the same type as in* (15); low powers $s < \frac{1}{2}$, instead, only produce domains where no leading singularity can be singled out.

It is remarkable that one has such an explicit and clean knowledge of the fractional powers of the singular perturbations of the Laplacian: the singular perturbation yields an operator $\mathfrak{h}_\alpha$ that is not a (pseudo-)differential operator any longer and its powers are not simply recovered as Fourier multipliers, nor is it *a priori* obvious how the fractional power affects the local boundary condition between regular and singular components of elements in $\mathcal{D}(\mathfrak{h}_\alpha)$.

It is also worth remarking that whereas $\mathfrak{h}_\alpha$, $\mathfrak{h}_\alpha^{s/2}$, and $(\mathfrak{h}_\alpha + \lambda\mathbb{1})^{s/2}$ for $s \in (\frac{1}{2}, 2)$ share the same singular behavior $|x|^{-1}$ of the functions in their domain, one noticeable difference is given by Eqs. (18) and (24): indeed, $(\mathfrak{h}_\alpha + \lambda\mathbb{1})^{-1}$ is a rank-one perturbation of $(-\Delta + \lambda\mathbb{1})^{-1}$, while $(\mathfrak{h}_\alpha + \lambda\mathbb{1})^{-s/2}$ is not a finite rank perturbation of $(-\Delta + \lambda\mathbb{1})^{-s/2}$.

III. SINGULAR PERTURBATIONS OF THE FRACTIONAL LAPLACIAN: GENERAL SETTING FOR THE HOMOGENEOUS CASE

In comparison to $\mathfrak{h}_\tau^{s/2}$, the fractional power of a singular perturbation of the Laplacian, we start discussing in this section the rigorous construction of operators of the type $\mathsf{k}_\tau^{(s/2)}$, as introduced informally in (3), namely, the self-adjoint singular perturbations of the fractional power of the Laplacian. Then, in Sec. IV, we shall consider the concrete cases of dimension $d = 3$, and in Sec. V, the case $d = 1$.

For chosen $d \in \mathbb{N}$, $\lambda > 0$, and $s \in \mathbb{R}$, we set

$$\mathsf{G}_{s,\lambda}(x) := \frac{1}{(2\pi)^{\frac{d}{2}}}\Big(\frac{1}{|p|^s + \lambda}\Big)^{\vee}(x), \qquad x, p \in \mathbb{R}^d, \tag{25}$$

whence

$$((-\Delta)^{s/2} + \lambda)\,\mathsf{G}_{s,\lambda} = \delta(x) \tag{26}$$

as a distributional identity on $\mathbb{R}^d$. We also set

$$\mathring{\mathsf{k}}^{(s/2)} := \overline{(-\Delta)^{s/2} \upharpoonright C^\infty_0(\mathbb{R}^d \setminus \{0\})} \tag{27}$$

as an operator closure with respect to the Hilbert space $L^2(\mathbb{R}^d)$. Thus, in comparison to Sec. II, when $d = 3$, one has $\mathsf{G}_{2,\lambda} = G_\lambda$ and $\mathring{\mathsf{k}}^{(1)} = \mathring{\mathfrak{h}}$.

The domain of $\mathring{\mathsf{k}}^{(s/2)}$, as a consequence of the operator closure in (27), is a space of functions with H^s-regularity and vanishing conditions at $x = 0$ for each function and its partial derivatives. The amount of vanishing conditions depends on d and s; in order to classify such conditions we introduce the intervals

$$I^{(d)}_n := \begin{cases} (0, \frac{d}{2}) & n = 0 \\ (\frac{d}{2} + n - 1, \frac{d}{2} + n) & n = 1, 2, \dots \end{cases} \tag{28}$$

For our purposes, it is convenient to use momentum coordinates to express the vanishing conditions that qualify the domain of $\mathring{\mathsf{k}}^{(s/2)}$: thus, with the notation $p \equiv (p_1, \dots, p_d) \in \mathbb{R}^d$, by means of an approximation argument (see the Appendix), we find

$$\begin{aligned}\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)}) = H^s_0(\mathbb{R}^d \setminus \{0\}) &= \overline{C^\infty_0(\mathbb{R}^d \setminus \{0\})}^{\|\,\|_{H^s}} \\ &= \begin{cases} H^s(\mathbb{R}^d) & \text{if } s \in I^{(d)}_0 \\ \left\{ \begin{array}{c} f \in H^s(\mathbb{R}^d) \text{such that} \\ \displaystyle\int_{\mathbb{R}^d} p_1^{\gamma_1} \cdots p_d^{\gamma_d} \widehat{f}(p)\, \mathrm{d}p = 0 \\ \gamma_1, \dots, \gamma_d \in \mathbb{N}_0,\ \sum_{j=1}^d \gamma_j \leqslant n - 1 \end{array} \right\} & \text{if } s \in I^{(d)}_n,\ n = 1, 2, \dots \end{cases}\end{aligned} \tag{29}$$

Clearly, $\int_{\mathbb{R}^d} p_1^{\gamma_1} \cdots p_d^{\gamma_d} \widehat{f}(p)\, \mathrm{d}p = 0$ is the same as $(\frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_d}}{\partial x_d^{\gamma_d}} f)(0) = 0$, with the notation $x \equiv (x_1, \dots, x_d) \in \mathbb{R}^d$.

The expression of $\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)})$ for the endpoint values $s = \frac{d}{2} + n$ requires an amount of extra analysis besides the arguments proof of (29): we do not discuss it here, an omission that does not affect the conceptual structure of our presentation.

Being densely defined, closed, and positive, either the symmetric operator $\mathring{\mathsf{k}}^{(s/2)}$ is already self-adjoint on $L^2(\mathbb{R}^d)$ or it admits infinitely many self-adjoint extensions. As well known, the infinite multiplicity of such extensions is quantified by the *deficiency index* of $\mathring{\mathsf{k}}^{(s/2)}$, which is the quantity

$$\mathcal{J}(s, d) := \dim \ker\, ((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda \mathbb{1}) \tag{30}$$

for one and hence for all $\lambda > 0$. The self-adjointness of $\mathring{\mathsf{k}}^{(s/2)}$ is equivalent to $\mathcal{J}(s, d) = 0$.

It is not difficult to compute $\mathcal{J}(s, d)$ for generic values of d and s and to identify a natural basis of the $\mathcal{J}(s, d)$-dimensional space $\ker\, ((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda \mathbb{1})$.

Lemma III.1. For given $d \in \mathbb{N}$ and $s > 0$,

$$s \in I^{(d)}_n \quad \Rightarrow \quad \mathcal{J}(s, d) = \binom{d + n - 1}{d}. \tag{31}$$

In particular, when $s \in I^{(d)}_n$ for some $n \in \mathbb{N}$, then

$$\ker\, ((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda \mathbb{1}) = \mathrm{span}\Big\{ u^\lambda_{\gamma_1, \dots, \gamma_d} \,\Big|\, \gamma_1, \dots, \gamma_d \in \mathbb{N}_0,\ \sum_{j=1}^d \gamma_j \leqslant n - 1 \Big\}, \tag{32}$$

where

$$\widehat{u^\lambda}_{\gamma_1,\dots,\gamma_d}(p) := \frac{p_1^{\gamma_1}\cdots p_d^{\gamma_d}}{|p|^s+\lambda}. \tag{33}$$

It is worth noticing, comparing (25) and (33), that

$$u^\lambda_{0,\dots,0} = (2\pi)^{\frac{d}{2}}\,\mathsf{G}_{s,\lambda}. \tag{34}$$

Proof of Lemma III.1. When $s\in I_0^{(d)}$, we see from (29) that $\mathring{\mathsf{k}}^{(s/2)}$ is self-adjoint; then $\ker\,((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1})$ is trivial and $\mathcal{J}(s,d)=0$, consistently with (31). When $s\in I_n^{(d)}$, $n=1,2,\dots$, then $u\in\ker\,((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1})=\mathrm{ran}\,(\mathring{\mathsf{k}}^{(s/2)}+\lambda\mathbb{1})^\perp$ is equivalent to

$$0=\int_{\mathbb{R}^3}\widehat{u}(p)(|p|^s+\lambda)\widehat{f}(p)\mathrm{d}p \qquad \forall f\in\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)})$$

and one argues from (29) that $\ker\,((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1})$ is spanned by linearly independent functions of the form $u^\lambda_{\gamma_1,\dots,\gamma_d}$, with $\sum_{j=1}^d\gamma_j\leqslant n-1$. Such functions are as many as the linearly independent monomials in d variables with degree at most equal to $n-1$, and therefore their number equals $\begin{pmatrix} d+n-1\\ d\end{pmatrix}$. □

The knowledge of $\ker\,((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1})$ and of the inverse of the Friedrichs extension of $\mathring{\mathsf{k}}^{(s/2)}$ is the two inputs for the Kreĭ-Višik-Birman extension theory (see, e.g., Ref. 7, Sec. 3), by means of which we can produce the whole family of self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$.

Such a construction is particularly clean in the case, relevant in applications, of deficiency index one: the comprehension of this case is instructive to understand the case of the higher deficiency index. Moreover, as we shall see, in this case, the self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$ turns out to be rank-one perturbations, in the resolvent sense: we will use the jargon $\mathcal{J}=1$ or "rank one" interchangeably.

Therefore, in this work, we make the presentational choice to discuss in detail the $\mathcal{J}(s,d)=1$ scenario when $s\in I_1^{(d)}$, deferring to Sec. VII an outlook on the high-$\mathcal{J}$ scenario. This corresponds to analyzing the regimes $s\in(\frac{1}{2},\frac{3}{2})$ when $d=1$, $s\in(1,2)$ when $d=2$, $s\in(\frac{3}{2},\frac{5}{2})$, when $d=3$, etc.

The construction of the self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$ in any such regimes is technically the very same, irrespectively of d, except for a noticeable peculiarity when $d=1$, as opposite to $d=2$, $3,\dots$.

Indeed, when $s\in I_1^{(d)}$ and hence $\mathcal{J}(s,d)=1$, we know from Lemma III.1 and (34) that $\ker\,((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1})=\mathrm{span}\{\mathsf{G}_{s,\lambda}\}$, and the function $\mathsf{G}_{s,\lambda}$ may or may not have a *local singularity* as $x\to 0$. As follows from the d-dimensional distributional identity

$$2^{\frac{s}{2}}\Gamma(\tfrac{s}{2})\,\frac{1}{|p|^s} = 2^{\frac{d-s}{2}}\Gamma(\tfrac{d-s}{2})\,\widehat{\Big(\frac{1}{|x|^{d-s}}\Big)}, \qquad s\in(0,d),$$

$\mathsf{G}_{s,0}$ has a singularity $\sim|x|^{-(d-s)}$ when $s<d$, it has a logarithmic singularity when $s=d$, and it is continuous at $x=0$ when $s>d$, with asymptotics

$$\begin{aligned} |x|^{d-s}\,\mathsf{G}_{s,\lambda}(x) &\xrightarrow{x\to 0} \Lambda_s^{(d)} := \frac{\Gamma(\frac{d-s}{2})}{(2\pi)^{\frac{d}{2}}\,2^{s-\frac{d}{2}}\,\Gamma(\frac{d}{2})}, && s\in(0,d),\\ \mathsf{G}_{s,\lambda}(x) &\xrightarrow{x\to 0} \mathsf{G}_{s,\lambda}(0) = \big(2^{d-1}\pi^{\frac{d}{2}-1}\Gamma(\tfrac{d}{2})\,\lambda^{\frac{s-d}{s}}\,s\sin\tfrac{\pi d}{s}\big)^{-1}, && s>d. \end{aligned} \tag{35}$$

Now, all the considered regimes $s\in(1,2)$ when $d=2$, $s\in(\frac{3}{2},\frac{5}{2})$ when $d=3$, etc. lie *below* the transition value $s=d$ between the local singular and the local regular behavior of $\mathsf{G}_{s,\lambda}$, whereas the regime $s\in(\frac{1}{2},\frac{3}{2})$ when $d=1$ lies *across* the transition value $s=1$.

The same type of distinction clearly occurs for the spanning functions (32) and (33) of $\ker\,((\mathring{\mathsf{k}}^{(s/2)})^{*}+\lambda\mathbb{1})$ for higher deficiency index $\mathcal{J}(s,d)$.

In the present context, the peculiarity described above when $d=1$ results in certain different steps of the construction of the self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$ and ultimately in the type of parameterization of such extensions, as we shall see.

Therefore, we articulate our discussion on the extensions of $\mathring{\mathsf{k}}^{(s/2)}$ when the deficiency index is one discussing first the three-dimensional case (Sec. IV) and then the one-dimensional case (Sec. V). As commented already, for generic $d\geqslant 2$, the discussion and the final results are completely analogous to $d=3$.

IV. RANK-ONE SINGULAR PERTURBATIONS OF THE FRACTIONAL LAPLACIAN: HOMOGENEOUS CASE IN DIMENSION THREE

In terms of the general discussion of Sec. III, we consider here the operator $\mathring{\mathsf{k}}^{(s/2)}$ on $L^2(\mathbb{R}^3)$ when $s\in(\frac{3}{2},\frac{5}{2})$. $\mathring{\mathsf{k}}^{(s/2)}$ acts as the fractional Laplacian $(-\Delta)^{s/2}$ on the domain

$$\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)}) = \Big\{f\in H^s(\mathbb{R}^3)\,\Big|\,\int_{\mathbb{R}^3}\widehat{f}(p)\,\mathrm{d}p=0\Big\}, \tag{36}$$

and its deficiency index is 1.

One has the following construction.

Theorem IV.1. *Let $s\in(\frac{3}{2},\frac{5}{2})$ and $\lambda>0$.*

(i) *The self-adjoint extensions in $L^2(\mathbb{R}^3)$ of the operator $\mathring{\mathsf{k}}^{(s/2)}$ form the family $(\mathsf{k}_\tau^{(s/2)})_{\tau\in\mathbb{R}\cup\{\infty\}}$, where $\mathsf{k}_\infty^{(s/2)}$ is its Friedrichs extension, namely, the self-adjoint fractional Laplacian $(-\Delta)^{s/2}$, and all other extensions are given by*

$$\begin{aligned}\mathcal{D}(\mathsf{k}_\tau^{(s/2)}) &:= \left\{ g\in L^2(\mathbb{R}^3)\,\left|\, \begin{array}{l} \widehat{g}(p)=\widehat{f^\lambda}(p)+\dfrac{\tau\,\xi}{(|p|^s+\lambda)^2}+\dfrac{\xi}{|p|^s+\lambda} \\ \xi\in\mathbb{C},\ f^\lambda\in H^s(\mathbb{R}^3),\ \displaystyle\int_{\mathbb{R}^3}\widehat{f^\lambda}(p)\,\mathrm{d}p=0 \end{array}\right.\right\} \\ &= \Big\{ g=F^\lambda+\tfrac{2\pi s^2\sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}}{\tau(s-3)}\,F^\lambda(0)\,\mathsf{G}_{s,\lambda}\,\Big|\,F^\lambda\in H^s(\mathbb{R}^3)\Big\}\end{aligned} \tag{37}$$

and

$$(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1})g := \mathcal{F}^{-1}\Big((|p|^s+\lambda)\Big(\widehat{f^\lambda}+\frac{\tau\,\xi}{(|p|^s+\lambda)^2}\Big)\Big). \tag{38}$$

(ii) *Each extension is semi-bounded from below and*

$$\begin{aligned} \inf\sigma(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1})\geqslant 0 \quad&\Leftrightarrow\quad \tau\geqslant 0,\\ \inf\sigma(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1})> 0 \quad&\Leftrightarrow\quad \tau> 0,\\ (\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1}) \textit{ is invertible} \quad&\Leftrightarrow\quad \tau\neq 0. \end{aligned} \tag{39}$$

(iii) *For each $\tau\in\mathbb{R}$, the quadratic form of the extension $\mathsf{k}_\tau^{(s/2)}$ is given by*

$$\mathcal{D}[\mathsf{k}_\tau^{(s/2)}] = H^{\frac{s}{2}}(\mathbb{R}^3)\dotplus\mathrm{span}\{\mathsf{G}_{s,\lambda}\}, \tag{40}$$

$$\begin{aligned}\mathsf{k}_\tau^{(s/2)}[F^\lambda+\kappa_\lambda\mathsf{G}_{s,\lambda}] &= \||\nabla|^{\frac{s}{2}}F^\lambda\|^2_{L^2(\mathbb{R}^3)}-\lambda\|F^\lambda+\kappa_\lambda\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R}^3)}\\ &\quad+\lambda\|F^\lambda\|^2_{L^2(\mathbb{R}^3)}+\frac{\tau(s-3)}{2\pi s^2\lambda^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}|\kappa_\lambda|^2\end{aligned} \tag{41}$$

for any $F^\lambda\in H^{s/2}(\mathbb{R}^3)$ and $\kappa_\lambda\in\mathbb{C}$.

(iv) *For $\tau \neq 0$, one has the resolvent identity*

$$(\mathsf{k}_\tau^{(s/2)} + \lambda\mathbb{1})^{-1} = ((-\Delta)^{s/2} + \lambda\mathbb{1})^{-1} + \frac{2\pi s^2 \sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}}{\tau(s-3)} |\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}|. \tag{42}$$

Proof. The whole construction is based upon the Kreĭ-Višik-Birman self-adjoint extension scheme. Since $\ker((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1}) = \mathrm{span}\{\mathsf{G}_{s,\lambda}\}$ and the Friedrichs extension of $\mathring{\mathsf{k}}^{(s/2)} + \lambda\mathbb{1}$ is the Fourier multiplier $(|p|^s + \lambda)$, one has the following formula for the adjoint (see, e.g., Ref. 7, Theorem 2.2):

$$\mathcal{D}((\mathring{\mathsf{k}}^{(s/2)})^*) = \left\{ g \in L^2(\mathbb{R}^3) \,\middle|\, \begin{array}{l} \widehat{g}(p) = \widehat{f^\lambda}(p) + \dfrac{\eta}{(|p|^s+\lambda)^2} + \dfrac{\xi}{|p|^s+\lambda} \\ \eta,\xi \in \mathbb{C},\ f^\lambda \in H^s(\mathbb{R}^3),\ \displaystyle\int_{\mathbb{R}^3} \widehat{f^\lambda}(p)\,\mathrm{d}p = 0 \end{array} \right\},$$

$$((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1})g = \mathcal{F}^{-1}\Big((|p|^s+\lambda)\Big(\widehat{f^\lambda} + \frac{\eta}{(|p|^s+\lambda)^2}\Big)\Big).$$

Each element of the one-parameter family of self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$ is identified (see, e.g., Ref. 7, Theorem 3.4) by the *Birman self-adjointness condition* $\eta = \tau\xi$ for some $\tau \in \mathbb{R} \cup \{\infty\}$. This establishes the first line of (37). Setting $\widehat{F^\lambda} := \widehat{f^\lambda} + (|p|^s + \lambda)^{-2}\tau\xi$, the boundary condition between F^λ and ξ in Fourier transform reads

$$\int_{\mathbb{R}^3} \widehat{F^\lambda}(p)\,\mathrm{d}p = \xi\, \frac{4\pi^2\tau(s-3)}{s^2\lambda^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}. \tag{43}$$

Then, from $F^\lambda(0) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \widehat{F^\lambda}\mathrm{d}p$ and using (25) with $d = 3$, the second line of (37) follows. Since $\mathsf{k}_\tau^{(s/2)}$ is a restriction of $(\mathring{\mathsf{k}}^{(s/2)})^*$, from the above action of the adjoint, one deduces (38). This completes the proof of part (i).

Part (ii) lists standard facts of the Kreĭ-Višik-Birman theory—see Ref. 7, Theorems 3.5, and 5.1.

The quadratic form is characterised in the extension theory (Ref. 7, Theorem 3.6) by the formulas $\mathcal{D}[\mathsf{k}_\tau^{(s/2)}] = \mathcal{D}[\mathsf{k}_F^{(s/2)}] \dotplus \ker((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1})$ (where "F" stands for Friedrichs), whence (40), and $(\mathsf{k}_\tau^{(s/2)} + \lambda\mathbb{1})[F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}] = ((-\Delta)^{s/2} + \lambda\mathbb{1})[F^\lambda] + \tau|\kappa_\lambda|^2\|\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R}^3)}$, whence (41). The proof of part (iii) is completed.

The Kreĭn's resolvent formula for deficiency index 1 (Ref. 7, Theorem 6.6) prescribes

$$f(\mathsf{k}_\tau^{(s/2)} + \lambda\mathbb{1})^{-1} = ((-\Delta)^{s/2} + \lambda\mathbb{1})^{-1} + \beta_{\lambda,\tau}\,|\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}|$$

for some scalar $\beta_{\lambda,\tau}$ to be determined, whenever $(\mathsf{k}_\tau^{(s/2)} + \lambda\mathbb{1})$ is invertible, hence for $\tau \neq 0$. Thus, for a generic $h \in L^2(\mathbb{R}^3)$, the element $g := (\mathsf{k}_\tau^{(s/2)} + \lambda\mathbb{1})^{-1}h \in \mathcal{D}(\mathsf{k}_\tau^{(s/2)})$ reads, in the view of (37) and of the resolvent formula above,

$$\widehat{g}(p) = \widehat{F^\lambda}(p) + \frac{\xi_\lambda}{|p|^s + \lambda},$$

$$\widehat{F^\lambda}(p) := \frac{\widehat{h}(p)}{|p|^s+\lambda}, \qquad \xi_\lambda := \frac{\beta_{\lambda,\tau}}{(2\pi)^3}\int_{\mathbb{R}^3}\frac{\widehat{h}(q)}{|q|^s+\lambda}\,\mathrm{d}q.$$

The boundary condition (43) for F^λ and ξ_λ then implies $1 = \beta_{\lambda,\tau}\,\frac{\tau(s-3)}{2\pi s^2\sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}}$, which determines $\beta_{\lambda,\tau}$ and proves (42), thus completing also the proof of (iv). □

In analogy to what is argued in Remark II.2, the τ-parameterization of the family $(\mathsf{k}_\tau^{(s/2)})_{\tau\in\mathbb{R}\cup\{\infty\}}$ depends on the initially chosen shift $\lambda > 0$, meaning that with a different choice $\lambda' > 0$, the same self-adjoint realisation previously identified by τ with shift λ is now selected by a different extension parameter τ'. In certain contexts, it is more convenient to switch onto a natural parameterization that identifies one extension irrespectively of the infinitely many different pairs (λ, τ) attached to it by the parameterization of Theorem IV.1. We shall do it in the next theorem: observe that indeed, as compared to Theorem IV.1, here below, $\lambda > 0$ is arbitrary.

Theorem IV.2. *Let* $s\in(\frac{3}{2},\frac{5}{2})$.

(i) *The self-adjoint extensions in* $L^2(\mathbb{R}^3)$ *of the operator* $\mathring{\mathsf{k}}^{(s/2)}$ *form the family* $(\mathsf{k}_\alpha^{(s/2)})_{\alpha\in\mathbb{R}\cup\{\infty\}}$, *where* $\mathsf{k}_\infty^{(s/2)}$ *is its Friedrichs extension, namely, the self-adjoint fractional Laplacian* $(-\Delta)^{s/2}$, *and all other extensions are given, for arbitrary* $\lambda>0$, *by*

$$\begin{aligned}\mathcal{D}(\mathsf{k}_\alpha^{(s/2)})&=\left\{g=F^\lambda+\frac{F^\lambda(0)}{\alpha-\frac{\lambda^{\frac{3}{s}-1}}{2\pi s\sin(\frac{3\pi}{s})}}\,\mathsf{G}_{s,\lambda}\,\middle|\,F^\lambda\in H^s(\mathbb{R}^3)\right\},\\ (\mathsf{k}_\alpha^{(s/2)}+\lambda)\,g&=((-\Delta)^{s/2}+\lambda)\,F^\lambda.\end{aligned}\tag{44}$$

(ii) *For each* $\alpha\in\mathbb{R}$, *the quadratic form of the extension* $\mathsf{k}_\alpha^{(s/2)}$ *is given by*

$$\mathcal{D}[\mathsf{k}_\alpha^{(s/2)}]=H^{\frac{s}{2}}(\mathbb{R}^3)\dotplus\mathrm{span}\{\mathsf{G}_{s,\lambda}\},\tag{45}$$

$$\begin{aligned}\mathsf{k}_\alpha^{(s/2)}[F^\lambda+\kappa_\lambda\mathsf{G}_{s,\lambda}]&=\||\nabla|^{\frac{s}{2}}F^\lambda\|^2_{L^2(\mathbb{R}^3)}-\lambda\|F^\lambda+\kappa_\lambda\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R}^3)}\\ &\quad+\lambda\|F^\lambda\|^2_{L^2(\mathbb{R}^3)}+\Big(\alpha-\frac{\lambda^{\frac{3}{s}-1}}{2\pi s\sin(\frac{3\pi}{s})}\Big)|\kappa_\lambda|^2\end{aligned}\tag{46}$$

for arbitrary $\lambda>0$.

(iii) *The resolvent of* $\mathsf{k}_\alpha^{(s/2)}$ *is given by*

$$\begin{aligned}(\mathsf{k}_\alpha^{(s/2)}+\lambda\mathbb{1})^{-1}&=((-\Delta)^{s/2}+\lambda\mathbb{1})^{-1}\\ &\quad+\Big(\alpha-\frac{\lambda^{\frac{3}{s}-1}}{2\pi s\sin(\frac{3\pi}{s})}\Big)^{-1}|\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}|\end{aligned}\tag{47}$$

for arbitrary $\lambda>0$.

(iv) *Each extension is semi-bounded from below, and*

$$\begin{aligned}&\sigma_{\mathrm{ess}}(\mathsf{k}_\alpha^{(s/2)})=\sigma_{\mathrm{ac}}(\mathsf{k}_\alpha^{(s/2)})=[0,+\infty),\qquad\sigma_{\mathrm{sc}}(\mathsf{k}_\alpha^{(s/2)})=\emptyset,\\ &\sigma_{\mathrm{disc}}(\mathsf{k}_\alpha^{(s/2)})=\begin{cases}\emptyset & \textit{if }\alpha\geqslant0,\\ \{E_\alpha^{(s)}\} & \textit{if }\alpha<0,\end{cases}\end{aligned}\tag{48}$$

where the eigenvalue $E_\alpha^{(s)}$ *is non-degenerate and is given by*

$$E_\alpha^{(s)}=-(2\pi|\alpha|\,s\,\sin(-\tfrac{3\pi}{s}))^{\frac{s}{3-s}},\tag{49}$$

with the (non-normalised) eigenfunction being $\mathsf{G}_{s,\lambda=|E_\alpha^{(s)}|}$.

Proof. Reasoning as in Remark II.2, we seek for the relation $\tau=\tau(\lambda)$ that ensures that all the pairs $(\lambda,\tau(\lambda))$, with $\lambda>0$, preserve the decomposition (37) and (38) and thus label the same element of the family of extensions.

For chosen λ and τ, a function $g\in\mathcal{D}(\mathsf{k}_\tau^{(s/2)})$ decomposes uniquely as

$$\widehat{g}=\widehat{F^\lambda}+\frac{\xi}{|p|^s+\lambda},\qquad\begin{matrix}F^\lambda\in H^s(\mathbb{R}^3)\\ \xi\in\mathbb{C}\end{matrix},\qquad\int_{\mathbb{R}^3}\widehat{F^\lambda}\mathrm{d}p=\xi\,\frac{4\pi^2\tau(s-3)}{s^2\lambda^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}.$$

Let now $\lambda'>0$ and $\tau'\in\mathbb{R}$ be such that for the *same* function g in the domain of the *same* self-adjoint realisation $\mathsf{k}_\tau^{(s/2)}$, one also has

$$\widehat{g}=\widehat{F^{\lambda'}}+\frac{\xi'}{|p|^s+\lambda'},\qquad\begin{matrix}F^{\lambda'}\in H^s(\mathbb{R}^3)\\ \xi'\in\mathbb{C}\end{matrix},\qquad\int_{\mathbb{R}^3}\widehat{F^{\lambda'}}\mathrm{d}p=\xi'\,\frac{4\pi^2\tau'(s-3)}{s^2\lambda'^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}.$$

The new splitting of g is equivalent to

$$\xi' = \xi, \qquad F^{\lambda'} = F^{\lambda} + \frac{\xi}{|p|^s + \lambda} - \frac{\xi'}{|p|^s + \lambda'},$$

and the boundary condition for $F^{\lambda'}$ and ξ' is equivalent to

$$\xi\,\frac{4\pi^2\tau(s-3)}{s^2\lambda^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})} + \int_{\mathbb{R}^3}\Big(\frac{\xi'}{|p|^s+\lambda} - \frac{\xi'}{|p|^s+\lambda'}\Big)\mathrm{d}p = \xi'\,\frac{4\pi^2\tau'(s-3)}{s^2\lambda'^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}. \tag{50}$$

Let us analyze the integral in (50). Both summands in the integrand diverge, with two identical divergences that cancel out. Thus, by means of the identity $r^2(r^s+\lambda)^{-1} = r^{2-s} - \lambda r^{2-s}(r^s+\lambda)^{-1}$, one has

$$\begin{aligned}\int_{\mathbb{R}^3}\Big(\frac{1}{|p|^s+\lambda} - \frac{1}{|p|^s+\lambda'}\Big)\mathrm{d}p &= 4\pi\lim_{R\to+\infty}\Big(\int_0^R \frac{r^2}{r^s+\lambda}\,\mathrm{d}r - \int_0^R \frac{r^2}{r^s+\lambda'}\,\mathrm{d}r\Big)\\ &= 4\pi\lim_{R\to+\infty}\Big(\int_0^R \frac{\lambda'\,\mathrm{d}r}{r^{s-2}(r^s+\lambda')} - \int_0^R \frac{\lambda\,\mathrm{d}r}{r^{s-2}(r^s+\lambda)}\Big)\\ &= \frac{4\pi^2}{\lambda^{1-\frac{3}{s}}s\sin(\frac{3\pi}{s})} - \frac{4\pi^2}{\lambda'^{1-\frac{3}{s}}s\sin(\frac{3\pi}{s})}.\end{aligned}$$

Plugging the result of the above computation into (50) yields

$$\frac{\tau(3-s) - s\lambda}{\lambda^{2-\frac{3}{s}}} = \frac{\tau'(3-s) - s\lambda'}{\lambda'^{2-\frac{3}{s}}},$$

which shows, in complete analogy to (13) when $s = 2$, that all pairs (λ, τ) such that

$$-\frac{\tau(3-s) - s\lambda}{2\pi s^2\sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}} =: \alpha \tag{51}$$

indeed label the same extension [the pre-factor $-2\pi s^2\sin(\frac{3\pi}{s})$ having being added for convenience]. Thus, $\alpha\in\mathbb{R}\cup\{\infty\}$ defined in (51) is the natural parameterization we were aiming for (and the Friedrichs case $\tau\to+\infty$ corresponds to $\alpha\to+\infty$).

Upon replacing $\frac{2\pi s^2\sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}}{\tau(s-3)} = \big(\alpha - \frac{1}{2\pi s\sin(\frac{3\pi}{s})\lambda^{1-\frac{3}{s}}}\big)^{-1}$ in the formulas of Theorem IV.1, we deduce at once all formulas of parts (i), (ii), and (iii), together of course with the certainty, proved above, that the decompositions are now λ-independent.

Since the deficiency index is 1 and hence all extensions are a rank-one perturbation, in the resolvent sense, of the self-adjoint fractional Laplacian, then all extensions have the same essential spectrum $[0,+\infty)$ of the latter and additionally may have *at most* one negative non-degenerate eigenvalue, and in any case, all extensions are semi-bounded from below—all these being general facts of the extension theory; see, e.g., Ref. 7, Theorem 5.9 and Corollary 5.10. This proves, in particular, the first line in (48).

The occurrence of a negative eigenvalue $E_\alpha = -\lambda$ of an extension $\mathsf{k}_\alpha^{(s/2)}$, for some $\lambda > 0$, can be read out from the resolvent formula (47) as the pole of $(\mathsf{k}_\alpha^{(s/2)} + \lambda\mathbb{1})^{-1}$, that is, imposing

$$\alpha - \frac{1}{2\pi s\sin(\frac{3\pi}{s})\lambda^{1-\frac{3}{s}}} = 0,$$

i.e.,

$$\alpha = -\lambda^{\frac{3-s}{s}}\big(2\pi s\sin(-\tfrac{3\pi}{s})\big)^{-1}.$$

The identity above can be only satisfied by some $\lambda > 0$ when $\alpha < 0$ because $\sin(-\frac{3\pi}{s}) > 0$, in which case

$$\lambda = \big(2\pi|\alpha|\,s\,\sin(-\tfrac{3\pi}{s})\big)^{\frac{s}{3-s}}.$$

Alternatively, one can argue from (37) and (38) that the eigenvalue $-\lambda$ must correspond to the eigenfunction $(\frac{1}{|p|^s+\lambda})^\vee$, that is, an element of the domain with only singular component, and to the parameter $\tau = 0$, hence with $f^\lambda \equiv 0$ in the notation therein. Then, setting $\tau = 0$ in (51) yields the same condition above on α and λ. This proves (49) and the second line in (48) when $\alpha < 0$, and it also qualifies the eigenfunction.

When such a bound state is absent and therefore when $\alpha \geqslant 0$, for what is argued before, one has $\sigma(\mathsf{k}_\alpha^{(s/2)}) = \sigma_{\mathrm{ess}}(\mathsf{k}_\alpha^{(s/2)}) = [0, +\infty)$. This proves the second line in (48) when $\alpha \geqslant 0$ and completes the proof of part (iv). □

Mirroring the observations made in the conclusions of Sec. II, we see that the elements of the domains of both the operators $\mathfrak{h}_\alpha^{s/2}$ [the fractional power of the singular perturbation of $(-\Delta)$] and $\mathsf{k}_\alpha^{(s/2)}$ [the singular perturbation of the fractional power $(-\Delta)^{s/2}$] split into a regular H^s-component plus a singular component; however, in the former case, the local singularity is $|x|^{-1}$ for all considered powers $s \in (\frac{1}{2}, 2]$, whereas in the latter, it is the singularity of $\mathsf{G}_{s,\lambda}$, namely, $|x|^{-(3-s)}$ for all powers $s \in (\frac{3}{2}, \frac{5}{2})$.

In either case, a local boundary condition constrains regular and singular components: working out the asymptotics as $x \to 0$ in (15) and (44) by means of (35), we find

$$\begin{aligned} g(x) &\sim (\alpha + (4\pi|x|)^{-1}) \qquad \text{as } x \to 0, \quad g \in \mathcal{D}(\mathfrak{h}_\alpha^{s/2}), \quad s \in (\tfrac{3}{2}, 2], \\ g(x) &\sim (\alpha + \Lambda_s |x|^{-(3-s)}) \qquad \text{as } x \to 0, \quad g \in \mathcal{D}(\mathsf{k}_\alpha^{(s/2)}), \quad s \in (\tfrac{3}{2}, \tfrac{5}{2}), \end{aligned} \tag{52}$$

where Λ_s is defined in (35). [Observe that $\Lambda_{s=2} = (4\pi)^{-1}$, consistently.]

Furthermore, whereas $(\mathfrak{h}_\alpha + \lambda\mathbb{1})^{-s/2}$ is not a finite rank perturbation of $(-\Delta + \lambda\mathbb{1})^{-s/2}$, $(\mathsf{k}_\alpha^{(s/2)} + \lambda\mathbb{1})^{-1}$ is indeed a rank-one perturbation of $((-\Delta)^{s/2} + \lambda\mathbb{1})^{-1}$.

V. RANK-ONE SINGULAR PERTURBATIONS OF THE FRACTIONAL LAPLACIAN: HOMOGENEOUS CASE IN DIMENSION ONE

In terms of the general discussion of Sec. III, we now consider here the operator $\mathring{\mathsf{k}}^{(s/2)}$ on $L^2(\mathbb{R})$ when $s \in (\frac{1}{2}, \frac{3}{2}) \setminus \{1\}$.

We start with identifying the Friedrichs extension $\mathsf{k}_F^{(s/2)}$ of $\mathring{\mathsf{k}}^{(s/2)}$. Unlike the three-dimensional case, the structure of $\mathsf{k}_F^{(s/2)}$ depends on whether $s < 1$ or $s > 1$.

Proposition V.1. Let $s \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$.

(i) *The quadratic form of the Friedrichs extension* $\mathsf{k}_F^{(s/2)}$ *of* $\mathring{\mathsf{k}}^{(s/2)}$ *is*

$$\begin{aligned} \mathcal{D}[\mathsf{k}_F^{(s/2)}] &= \begin{cases} H^{s/2}(\mathbb{R}) & \text{if } s \in (\frac{1}{2}, 1) \\ H_0^{s/2}(\mathbb{R} \setminus \{0\}) & \text{if } s \in (1, \frac{3}{2}), \end{cases} \\ \mathsf{k}_F^{(s/2)}[f, g] &= \langle\, |\nabla|^{\frac{s}{2}} f, \nabla|^{\frac{s}{2}} g \rangle. \end{aligned} \tag{53}$$

(ii) *When* $s \in (\frac{1}{2}, 1)$, *one has*

$$\begin{aligned} \mathcal{D}(\mathsf{k}_F^{(s/2)}) &= H^s(\mathbb{R}), \\ \mathsf{k}_F^{(s/2)} f &= (-\Delta)^{\frac{s}{2}} f. \end{aligned} \tag{54}$$

(iii) *When* $s \in (1, \frac{3}{2})$, *for every* $\lambda > 0$, *one has*

$$\begin{aligned} \mathcal{D}(\mathsf{k}_F^{(s/2)}) &= \left\{ f = \phi - \frac{\phi(0)}{\mathsf{G}_{s,\lambda}(0)}\, \mathsf{G}_{s,\lambda} \,\middle|\, \phi \in H^s(\mathbb{R}) \right\}, \\ (\mathsf{k}_F^{(s/2)} + \lambda\mathbb{1}) f &= ((-\Delta)^{\frac{s}{2}} + \lambda)\phi. \end{aligned} \tag{55}$$

In particular, $\mathcal{D}(\mathsf{k}_F^{(s/2)}) \subset H^s(\mathbb{R}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\}$. *In this regime of* s, $(\mathsf{k}_F^{(s/2)} + \lambda\mathbb{1})$ *has an everywhere defined and bounded inverse on* $L^2(\mathbb{R})$ *with*

$$(\mathsf{k}_F^{(s/2)} + \lambda\mathbb{1})^{-1} = ((-\Delta)^{\frac{s}{2}} + \lambda\mathbb{1})^{-1} - \frac{1}{G_{s,\lambda}(0)}\,|G_{s,\lambda}\rangle\langle G_{s,\lambda}|. \tag{56}$$

Proof. Following the standard form construction of the Friedrichs extension (see, e.g., Ref. 7, Theorem A.2), the Friedrichs form domain is the completion of $\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)})$ with respect to the $H^{\frac{s}{2}}$-norm, and therefore

$$\mathcal{D}[\mathsf{k}_F^{(s/2)}] = \overline{H_0^s(\mathbb{R}\setminus\{0\})}^{\|\,\|_{H^{s/2}(\mathbb{R})}} = \begin{cases} H^{s/2}(\mathbb{R}) & \text{if } s\in(\frac{1}{2},1),\\ H_0^{s/2}(\mathbb{R}\setminus\{0\}) & \text{if } s\in(1,\frac{3}{2}),\end{cases}$$

the last identity being proved precisely as (29). Moreover, $\mathsf{k}_F^{(s/2)}[f,g] = \lim_{n\to+\infty}\langle\,|\nabla|^{\frac{s}{2}}f_n, \nabla|^{\frac{s}{2}}g_n\rangle$ for all the sequences $(f_n)_n, (g_n)_n$ of $H^{\frac{s}{2}}$-approximants of f and g, respectively, whence $\mathsf{k}_F^{(s/2)}[f,g] = \langle\,|\nabla|^{\frac{s}{2}}f, \nabla|^{\frac{s}{2}}g\rangle$. This completes the proof of part (i).

The self-adjoint operator associated with the form (53) is qualified by the formulas

$$\mathcal{D}(\mathsf{k}_F^{(s/2)} + \lambda\mathbb{1}) = \left\{ f\in\mathcal{D}[\mathsf{k}_F^{(s/2)}] \;\middle|\; \begin{array}{c} \exists\, u_f\in L^2(\mathbb{R}) \text{ such that} \\ \langle\,|\nabla|^{\frac{s}{2}}f, \nabla|^{\frac{s}{2}}g\rangle + \lambda\langle f,g\rangle = \langle u_f, g\rangle \\ \forall\, g\in\mathcal{D}[\mathsf{k}_F^{(s/2)}] \end{array}\right\},$$

$$(\mathsf{k}_F^{(s/2)} + \lambda\mathbb{1})f = u_f$$

valid for any $\lambda > 0$. [By the density of $H_0^{s/2}(\mathbb{R}\setminus\{0\})$ in $L^2(\mathbb{R})$, the above u_f is unique.] In particular, $\mathcal{D}(\mathsf{k}_F^{(s/2)})$ is independent of λ, only its internal decomposition is. When $s\in(\frac{1}{2},1)$, the condition identifying f and u_f is clearly equivalent to $f\in H^s(\mathbb{R})$ and $u_f = (-\Delta)^{\frac{s}{2}}f$, which yields part (ii) of the thesis.

When instead $s\in(1,\frac{3}{2})$, the above condition reads

$$\begin{gathered} \int_{\mathbb{R}} \overline{|p|^{\frac{s}{2}}\widehat{f}}\;|p|^{\frac{s}{2}}\widehat{g}\,\mathrm{d}p + \lambda\int_{\mathbb{R}}\overline{\widehat{f}}\,\widehat{g}\,\mathrm{d}p = \int_{\mathbb{R}}\overline{\widehat{u_f}}\,\widehat{g}\,\mathrm{d}p \\ \forall g \text{ such that } \int_{\mathbb{R}} |(|p|^{\frac{s}{2}}+1)\,\widehat{g}(p)|^2\mathrm{d}p < +\infty \text{ and } \int_{\mathbb{R}}\widehat{g}\,\mathrm{d}p =: 0, \end{gathered} \tag{57}$$

that is,

$$\int_{\mathbb{R}} \overline{((|p|^s+\lambda)\widehat{f} - \widehat{u_f})}\,\widehat{g}\,\mathrm{d}p = 0$$

for all the g's indicated in (57) and for some $u_f\in L^2(\mathbb{R})$. It is easy to see that this is the same as

$$\widehat{f}(p) = \frac{\widehat{u_f}(p)+c}{|p|^s+\lambda}, \qquad \widehat{u_f}(p) = (|p|^s+\lambda)\widehat{f}(p) - c$$

for some $c\in\mathbb{C}$. Now, the condition for f to vanish at $x=0$ and belong to $H^{s/2}(\mathbb{R})$ is equivalent to

$$0 = \int_{\mathbb{R}}\frac{\widehat{u_f}(p)+c}{|p|^s+\lambda}\,\mathrm{d}p \qquad\text{and}\qquad +\infty > \int_{\mathbb{R}}\Big|(|p|^{\frac{s}{2}}+1)\,\frac{\widehat{u_f}(p)+c}{|p|^s+\lambda}\Big|^2\,\mathrm{d}p.$$

The finiteness of the second integral above is guaranteed by $s>1$ and $u_f\in L^2(\mathbb{R})$, whereas the vanishing of the first integral is the same as

$$\int_{\mathbb{R}}\frac{\widehat{u_f}(p)}{|p|^s+\lambda}\,\mathrm{d}p = -c\int_{\mathbb{R}}\frac{\mathrm{d}p}{|p|^s+\lambda} = -2\pi c\,\mathsf{G}_{s,\lambda}(0),$$

i.e.,

$$\sqrt{2\pi}\,c = -\frac{1}{\mathsf{G}_{s,\lambda}(0)}\int_{\mathbb{R}}((-\Delta)^{\frac{s}{2}}+\lambda)^{-1}u_f\,\mathrm{d}x.$$

Therefore, as a distributional identity,

$$u_f = ((-\Delta)^{\frac{s}{2}}+\lambda)f - \sqrt{2\pi}\,c\,\delta = ((-\Delta)^{\frac{s}{2}}+\lambda)\Big(f + \frac{\int_{\mathbb{R}}((-\Delta)^{\frac{s}{2}}+\lambda)^{-1}u_f}{\mathsf{G}_{s,\lambda}(0)}\,\mathsf{G}_{s,\lambda}\Big).$$

In turn, the condition $u_f \in L^2(\mathbb{R})$ is equivalent to $\phi := ((-\Delta)^{\frac{s}{2}} + \lambda)^{-1} u_f \in H^s(\mathbb{R})$: in terms of such ϕ, the previous identity reads

$$f = \phi - \frac{\int_{\mathbb{R}} \phi \, \mathrm{d}x}{\mathsf{G}_{s,\lambda}(0)} \mathsf{G}_{s,\lambda},$$

i.e., the condition $f(0) = 0$ reads $\int_{\mathbb{R}} \phi \, \mathrm{d}x = \phi(0)$. We have thus found

$$f = \phi - \frac{\phi(0)}{\mathsf{G}_{s,\lambda}(0)} \mathsf{G}_{s,\lambda}, \qquad u_f = ((-\Delta)^{\frac{s}{2}} + \lambda)\phi,$$

which proves (55). The following inversion formula is then a straightforward consequence,

$$\begin{aligned} (\mathsf{k}_F^{(s/2)} + \lambda \mathbb{1})^{-1} u =& \phi_u - \frac{\phi_u(0)}{\mathsf{G}_{s,\lambda}(0)} \mathsf{G}_{s,\lambda}, \\ \phi_u :=& ((-\Delta)^{\frac{s}{2}} + \lambda)^{-1} u \in H^s(\mathbb{R}). \end{aligned}$$

On the other hand,

$$\phi_u(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\widehat{u}(p)}{|p|^s + \lambda} \, \mathrm{d}p = \langle G_{s,\lambda}, u \rangle,$$

whence (56) and the conclusion of the proof of part (iii). □

As a special case of (56) above,

$$\begin{aligned} \mathsf{h}_{s,\lambda} :=& \sqrt{2\pi}\, (\mathring{\mathsf{k}}_F^{(s/2)} + \lambda \mathbb{1})^{-1} \, \mathsf{G}_{s,\lambda} \\ =& \begin{cases} \mathcal{F}^{-1}\Big(\dfrac{1}{(|p|^s + \lambda)^2}\Big) & \text{if } s \in (\frac{1}{2}, 1), \\ \mathcal{F}^{-1}\Big(\dfrac{1}{(|p|^s + \lambda)^2} - \dfrac{s-1}{\lambda\, s} \dfrac{1}{|p|^s + \lambda}\Big) & \text{if } s \in (1, \frac{3}{2}). \end{cases} \end{aligned} \tag{58}$$

Indeed, $\langle \mathsf{G}_{s,\lambda}, \mathsf{G}_{s,\lambda} \rangle = \frac{s-1}{\lambda^{2-\frac{1}{s}} s^2 \sin \frac{\pi}{s}}$ and [as follows from (35)] $\mathsf{G}_{s,\lambda}(0) = (\lambda^{1-\frac{1}{s}} s \sin \frac{\pi}{s})^{-1}$, whence $\|G_{s,\lambda}\|^2_{L^2(\mathbb{R})} / \mathsf{G}_{s,\lambda}(0) = \frac{s-1}{\lambda\, s}$.

We can now establish the following construction.

Theorem V.2. *Let $s \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$ and $\lambda > 0$. Set*

$$\omega(s) := \frac{s^2 \sin(\frac{\pi}{s})}{s-1}, \qquad \theta_s := \begin{cases} 0 & \text{if } s < 1, \\ 1 & \text{if } s > 1. \end{cases} \tag{59}$$

(i) *The self-adjoint extensions in $L^2(\mathbb{R})$ of the operator $\mathring{\mathsf{k}}^{(s/2)}$ form the family $(\mathsf{k}_\tau^{(s/2)})_{\tau \in \mathbb{R} \cup \{\infty\}}$, where $\mathsf{k}_\infty^{(s/2)}$ is its Friedrichs extension, already qualified in Proposition V.1, and all other extensions are given by*

$$\mathcal{D}(\mathsf{k}_\tau^{(s/2)}) := \left\{ g = F^\lambda + \Big(\frac{\omega(s) \lambda^{2-\frac{1}{s}}}{\tau} - \frac{\theta_s}{\mathsf{G}_{s,\lambda}(0)} \Big) F^\lambda(0)\, \mathsf{G}_{s,\lambda} \,\Big|\, F^\lambda \in H^s(\mathbb{R}) \right\}, \tag{60}$$

$$(\mathsf{k}_\tau^{(s/2)} + \lambda \mathbb{1}) g := ((-\Delta)^{\frac{s}{2}} + \lambda) F^\lambda. \tag{61}$$

(ii) *Each extension is semi-bounded from below and*

$$\begin{aligned} \inf \sigma(\mathsf{k}_\tau^{(s/2)} + \lambda \mathbb{1}) \geqslant 0 \quad &\Leftrightarrow \quad \tau \geqslant 0, \\ \inf \sigma(\mathsf{k}_\tau^{(s/2)} + \lambda \mathbb{1}) > 0 \quad &\Leftrightarrow \quad \tau > 0, \\ (\mathsf{k}_\tau^{(s/2)} + \lambda \mathbb{1}) \text{ is invertible} \quad &\Leftrightarrow \quad \tau \neq 0. \end{aligned} \tag{62}$$

(iii) *For each $\tau \in \mathbb{R}$, the quadratic form of the extension $\mathsf{k}_\tau^{(s/2)}$ is given by*

$$\mathcal{D}[\mathsf{k}_\tau^{(s/2)}] = \begin{cases} H^{s/2}(\mathbb{R}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\} & \text{if } s \in (\frac{1}{2}, 1) \\ H_0^{s/2}(\mathbb{R}\setminus\{0\}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\} & \text{if } s \in (1, \frac{3}{2}) \end{cases}, \tag{63}$$

$$\begin{aligned} \mathsf{k}_\tau^{(s/2)}[F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}] = \| |\nabla|^{\frac{s}{2}} F^\lambda \|_{L^2(\mathbb{R})}^2 - \lambda \| F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda} \|_{L^2(\mathbb{R})}^2 \\ + \lambda \| F^\lambda \|_{L^2(\mathbb{R})}^2 + \frac{\tau}{\omega(s) \lambda^{2-\frac{1}{s}}} |\kappa_\lambda|^2 \end{aligned} \tag{64}$$

for any $F^\lambda \in \mathcal{D}[\mathsf{k}_F^{(s/2)}]$ and $\kappa_\lambda \in \mathbb{C}$.

(iv) *For $\tau \neq 0$, one has the resolvent identity*

$$(\mathsf{k}_\tau^{(s/2)} + \lambda \mathbb{1})^{-1} = ((-\Delta)^{s/2} + \lambda \mathbb{1})^{-1} + \Big(\frac{\omega(s) \lambda^{2-\frac{1}{s}}}{\tau} - \frac{\theta_s}{\mathsf{G}_{s,\lambda}(0)} \Big) |\mathsf{G}_{s,\lambda}\rangle\langle \mathsf{G}_{s,\lambda}|. \tag{65}$$

Proof. We proceed along the line of the Proof of Theorem IV.1, based upon the Kreĭn-Višik-Birman self-adjoint extension scheme and Proposition V.1.

The adjoint of $\mathring{\mathsf{k}}^{(s/2)}$ is qualified by

$$\begin{aligned} \mathcal{D}((\mathring{\mathsf{k}}^{(s/2)})^*) &= \left\{ g \in L^2(\mathbb{R}) \,\middle|\, \begin{array}{c} \widehat{g}(p) = \widehat{f^\lambda}(p) + \eta\, \widehat{\mathsf{h}_{s,\lambda}} + \dfrac{\xi}{|p|^s + \lambda} \\ \eta, \xi \in \mathbb{C},\ f^\lambda \in H^s(\mathbb{R}),\ \displaystyle\int_{\mathbb{R}} \widehat{f^\lambda}(p)\, \mathrm{d}p = 0 \end{array} \right\}, \\ ((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda \mathbb{1}) g &= (\mathsf{k}_F^{(s/2)} + \lambda \mathbb{1})(f^\lambda + \eta\, \mathsf{h}_{s,\lambda}), \end{aligned} \tag{66}$$

where $\mathsf{h}_{s,\lambda}$ is the function (58), and the self-adjoint restrictions of $(\mathring{\mathsf{k}}^{(s/2)})^*$ are qualified by the self-adjointness condition $\eta = \tau\xi$ for some $\tau \in \mathbb{R} \cup \{\infty\}$. For clarity of presentation, let us split the discussion into the two regimes $s < 1$ and $s > 1$.

A. First case: $s \in (\frac{1}{2}, 1)$

Let $F^\lambda := f^\lambda + \tau\, \xi\, \mathsf{h}_{s,\lambda}$. When f^λ and ξ run over their possible domains, F^λ spans the whole Friedrichs domain $H^s(\mathbb{R})$. Moreover,

$$\int_{\mathbb{R}} \widehat{F^\lambda}(p)\, \mathrm{d}p = \xi\, \frac{2\pi\tau}{\lambda^{2-\frac{1}{s}} \omega(s)}. \tag{67}$$

Thus, the first line in (66) and (67) yields (60). Owing to (54) and to the fact that $\mathsf{k}_\tau^{(s/2)}$ is a restriction of $(\mathring{\mathsf{k}}^{(s/2)})^*$, one deduces (61) from (66). Thus, part (i) is proved.

Parts (ii) and (iii) are proved as in Theorem IV.1: in particular, $\mathcal{D}[\mathsf{k}_\tau^{(s/2)}] = \mathcal{D}[\mathsf{k}_F^{(s/2)}] \dotplus \ker((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda \mathbb{1})$ and (53) implies (63), whereas $(\mathsf{k}_\tau^{(s/2)} + \lambda \mathbb{1})[F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}] = ((-\Delta)^{s/2} + \lambda \mathbb{1})[F^\lambda] + \tau |\kappa_\lambda|^2 \| \mathsf{G}_{s,\lambda} \|_{L^2(\mathbb{R})}^2$ and (53) implies (64).

The Kreĭn's resolvent formula for deficiency index 1 and (54) prescribe

$$(\mathsf{k}_\tau^{(s/2)} + \lambda \mathbb{1})^{-1} = ((-\Delta)^{s/2} + \lambda \mathbb{1})^{-1} + \beta_{\lambda,\tau}\, |\mathsf{G}_{s,\lambda}\rangle\langle \mathsf{G}_{s,\lambda}|$$

for some scalar $\beta_{\lambda,\tau}$ to be determined, whenever $(\mathsf{k}_\tau^{(s/2)} + \lambda \mathbb{1})$ is invertible, hence for $\tau \neq 0$. Thus, for a generic $h \in L^2(\mathbb{R})$, the element $g := (\mathsf{k}_\tau^{(s/2)} + \lambda \mathbb{1})^{-1} h \in \mathcal{D}(\mathsf{k}_\tau^{(s/2)})$ reads, in the view of (37) and of the resolvent formula above,

$$\widehat{g}(p) = \widehat{F^\lambda}(p) + \frac{\xi_\lambda}{|p|^s + \lambda}, \qquad \widehat{F^\lambda}(p) := \frac{\widehat{h}(p)}{|p|^s + \lambda}, \qquad \xi_\lambda := \frac{\beta_{\lambda,\tau}}{2\pi} \int_{\mathbb{R}} \frac{\widehat{h}(q)}{|q|^s + \lambda}\, \mathrm{d}q.$$

The boundary condition (66) for F^λ and ξ_λ then implies $1=\beta_{\lambda,\tau}\,\frac{\tau}{\omega(s)\lambda^{2-\frac{1}{s}}}$, which determines $\beta_{\lambda,\tau}$ and proves (65), thus completing also the Proof of (iv).

B. Second case: $s\in(1,\frac{3}{2})$

Let $F^\lambda := f^\lambda+\tau\,\xi\,\mathsf{h}_{s,\lambda}$. (This is for consistency with the first case, but such F^λ is not going to be the same as the F^λ of the thesis: functions will be renamed later.) When f^λ and ξ run over their possible domains, F^λ spans the whole Friedrichs domain (55). In particular, $F^\lambda(0)=0$, which shows that the boundary condition in g between F^λ and $g-F^\lambda$ cannot have the form (67) any longer.

Owing to (55), we can re-write

$$\mathcal{D}(\mathsf{k}_F^{(s/2)})\ni F^\lambda=f^\lambda+\tau\,\xi\,\mathsf{h}_{s,\lambda}=\phi^\lambda_{\tau,\xi}-\frac{\phi^\lambda_{\tau,\xi}(0)}{\mathsf{G}_{s,\lambda}(0)}\,\mathsf{G}_{s,\lambda},\tag{68}$$

with $\phi^\lambda_{\tau,\xi}$ running over the whole $H^s(\mathbb{R})$ when F^λ runs over the whole $\mathcal{D}(\mathsf{k}_F^{(s/2)})$. Using (58), this is the same as

$$\widehat{f^\lambda}+\frac{\tau\,\xi}{(|p|^s+\lambda)^2}-\tau\,\xi\,\frac{s-1}{\lambda\,s}\,\frac{1}{|p|^s+\lambda}=\widehat{\phi^\lambda_{\tau,\xi}}-\frac{\phi^\lambda_{\tau,\xi}(0)}{\sqrt{2\pi}\,\mathsf{G}_{s,\lambda}(0)}\,\frac{1}{|p|^s+\lambda},$$

whence the identification

$$\widehat{\phi^\lambda_{\tau,\xi}}=\widehat{f^\lambda}+\frac{\tau\,\xi}{(|p|^s+\lambda)^2},\qquad \frac{\phi^\lambda_{\tau,\xi}(0)}{\sqrt{2\pi}\,\mathsf{G}_{s,\lambda}(0)}=\tau\,\xi\,\frac{s-1}{\lambda\,s}.\tag{69}$$

[In fact, a straightforward computation confirms that assuming the first of (69), the second follows.]

From (66) (with $\eta=\tau\xi$), we then see that a generic $g\in\mathcal{D}(\mathsf{k}_\tau^{(s/2)})$ has the form

$$\begin{aligned}\widehat{g}=\widehat{F^\lambda}+\sqrt{2\pi}\,\xi\,\widehat{\mathsf{G}}_{s,\lambda}&=\widehat{\phi^\lambda_{\tau,\xi}}+\Big(\sqrt{2\pi}\,\xi-\frac{\phi^\lambda_{\tau,\xi}(0)}{\mathsf{G}_{s,\lambda}(0)}\Big)\,\widehat{\mathsf{G}}_{s,\lambda}\\&=\widehat{\phi^\lambda_{\tau,\xi}}+\Big(\frac{1}{\tau}\,\frac{\lambda\,s}{s-1}-1\Big)\frac{\phi^\lambda_{\tau,\xi}(0)}{\mathsf{G}_{s,\lambda}(0)}\,\widehat{\mathsf{G}}_{s,\lambda},\end{aligned}\tag{70}$$

where we used (68) in the second step and the second identity of (69) in the third step. Since

$$\frac{\lambda\,s}{s-1}\,\frac{1}{\mathsf{G}_{s,\lambda}(0)}=\omega(s)\,\lambda^{2-\frac{1}{s}},$$

as follows from (35) and (59), then (70) becomes

$$g=\phi^\lambda_{\tau,\xi}+\Big(\frac{\omega(s)\,\lambda^{2-\frac{1}{s}}}{\tau}-\frac{1}{\mathsf{G}_{s,\lambda}(0)}\Big)\,\phi^\lambda_{\tau,\xi}(0)\,\mathsf{G}_{s,\lambda}.\tag{71}$$

Moreover,

$$(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1})g=(\mathsf{k}_F^{(s/2)}+\lambda\mathbb{1})F^\lambda=((-\Delta)^{\frac{s}{2}}+\lambda)\phi^\lambda_{\tau,\xi},\tag{72}$$

having used the second line of (66) in the first identity and (55) in the second identity.

Renaming $\phi^\lambda_{\tau,\xi}$ into F^λ, now F^λ runs over the whole $H^s(\mathbb{R})$ when g runs over the whole $\mathcal{D}(\mathsf{k}_\tau^{(s/2)})$: this fact and (71) then yield (60), whereas (72) yields (61). Part (i) is proved.

The proof of parts (ii) and (iii) follows the same scheme as in the case $s\in(\frac{1}{2},1)$: thus, $\mathcal{D}[\mathsf{k}_\tau^{(s/2)}]=\mathcal{D}[\mathsf{k}_F^{(s/2)}]\dotplus\ker((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1})$ and (53) implies (63), whereas $(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1})[F^\lambda+\kappa_\lambda\mathsf{G}_{s,\lambda}]=((-\Delta)^{s/2}+\lambda\mathbb{1})[F^\lambda]+\tau|\kappa_\lambda|^2\|\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R})}$ and (53) implies (64).

Concerning part (iv), The Kreĭn's resolvent formula and (56) prescribe for all $\tau\neq0$,

$$\begin{aligned}(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1})^{-1}&=(\mathsf{k}_F^{(s/2)}+\lambda\mathbb{1})^{-1}+\beta_{\lambda,\tau}\,|\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}|\\&=((-\Delta)^{\frac{s}{2}}+\lambda\mathbb{1})^{-1}+\Big(\beta_{\lambda,\tau}-\frac{1}{\mathsf{G}_{s,\lambda}(0)}\Big)\,|\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}|\end{aligned}\tag{73}$$

for some scalar $\beta_{\lambda,\tau}$ to be determined. Owing to (73) and to (60), in order for

$$g:=(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1})^{-1}h=((-\Delta)^{\frac{s}{2}}+\lambda\mathbb{1})^{-1}h+\Big(\beta_{\lambda,\tau}-\frac{1}{\mathsf{G}_{s,\lambda}(0)}\Big)\langle\mathsf{G}_{s,\lambda},h\rangle\,\mathsf{G}_{s,\lambda}$$

to belong to $\mathcal{D}(\mathsf{k}_\tau^{(s/2)})$ for a generic $h \in L^2(\mathbb{R})$, keeping into account that $F^\lambda := ((-\Delta)^{\frac{s}{2}} + \lambda\mathbb{1})^{-1}h$ is a generic function in $H^s(\mathbb{R})$ and that $\langle \mathsf{G}_{s,\lambda}, h\rangle = \langle \mathsf{G}_{s,\lambda}, ((-\Delta)^{\frac{s}{2}} + \lambda\mathbb{1})F^\lambda\rangle = F^\lambda(0)$, one must necessarily have $\beta_{\lambda,\tau} = \omega(s)\lambda^{2-\frac{1}{s}}/\tau$. Then (73) yields (65). $\square$

Analogously to the change of parameterization from Theorem IV.1 to Theorem IV.2, we deduce from Theorem V.2 the following version.

Theorem V.3. *Let $s \in (\frac{1}{2}, 1) \cup (1, \frac{3}{2})$ and*

$$\Theta(s,\lambda) := \left(\lambda^{1-\frac{1}{s}} s \sin\left(\tfrac{\pi}{s}\right)\right)^{-1}, \qquad \lambda > 0. \tag{74}$$

(i) *The self-adjoint extensions in $L^2(\mathbb{R})$ of the operator $\mathring{\mathsf{k}}^{(s/2)}$ form the family $(\mathsf{k}_\alpha^{(s/2)})_{\alpha \in \mathbb{R}\cup\{\infty\}}$, where for arbitrary $\lambda > 0$,*

$$\begin{aligned} \mathcal{D}(\mathsf{k}_\alpha^{(s/2)}) &= \left\{ \begin{array}{c} g = F^\lambda + \dfrac{F^\lambda(0)}{\alpha - \Theta(s,\lambda)}\,\mathsf{G}_{s,\lambda} \\ F^\lambda \in H^s(\mathbb{R}) \end{array} \right\}, \\ (\mathsf{k}_\alpha^{(s/2)} + \lambda)\,g &= ((-\Delta)^{s/2} + \lambda)\,F^\lambda. \end{aligned} \tag{75}$$

The Friedrichs extension $\mathsf{k}_F^{(s/2)}$, already qualified in Proposition V.1, corresponds to $\alpha = \infty$ when $s \in (\frac{1}{2}, 1)$ and to $\alpha = 0$ when $s \in (1, \frac{3}{2})$. For generic s, the extension with $\alpha = \infty$ is the ordinary self-adjoint fractional Laplacian $(-\Delta)^{s/2}$ on $L^2(\mathbb{R})$.

(ii) *For each $\alpha \in \mathbb{R}$, the quadratic form of the extension $\mathsf{k}_\alpha^{(s/2)}$ is given by*

$$\mathcal{D}[\mathsf{k}_\alpha^{(s/2)}] = \begin{cases} H^{s/2}(\mathbb{R}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\} & \text{if } s \in (\frac{1}{2}, 1) \\ H_0^{s/2}(\mathbb{R}\setminus\{0\}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\} & \text{if } s \in (1, \frac{3}{2}) \text{ and } \alpha \neq 0, \end{cases} \tag{76}$$

$$\begin{aligned} \mathsf{k}_\alpha^{(s/2)}[F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}] &= \| |\nabla|^{\frac{s}{2}} F^\lambda \|^2_{L^2(\mathbb{R})} - \lambda \| F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda} \|^2_{L^2(\mathbb{R})} \\ &\quad + \lambda \| F^\lambda \|^2_{L^2(\mathbb{R})} + \left(\frac{\theta_s}{\Theta(s,\lambda)} + \frac{1}{\alpha - \Theta(s,\lambda)} \right)^{-1} |\kappa_\lambda|^2 \end{aligned} \tag{77}$$

for arbitrary $\lambda > 0$.

(iii) *The resolvent of $\mathsf{k}_\alpha^{(s/2)}$ is given by*

$$\begin{aligned} (\mathsf{k}_\alpha^{(s/2)} + \lambda\mathbb{1})^{-1} &= ((-\Delta)^{s/2} + \lambda\mathbb{1})^{-1} \\ &\quad + (\alpha - \Theta(s,\lambda))^{-1}\, |\mathsf{G}_{s,\lambda}\rangle\langle \mathsf{G}_{s,\lambda}| \end{aligned} \tag{78}$$

for arbitrary $\lambda > 0$.

(iv) *For each $\alpha \in \mathbb{R}$, the extension $\mathsf{k}_\alpha^{(s/2)}$ is semi-bounded from below, and*

$$\sigma_{\mathrm{ess}}(\mathsf{k}_\alpha^{(s/2)}) = \sigma_{\mathrm{ac}}(\mathsf{k}_\alpha^{(s/2)}) = [0, +\infty), \qquad \sigma_{\mathrm{sc}}(\mathsf{k}_\alpha^{(s/2)}) = \emptyset, \tag{79}$$

$$\sigma_{\mathrm{disc}}(\mathsf{k}_\alpha^{(s/2)}) = \begin{cases} \emptyset & \text{if } (s-1)\,\alpha \leqslant 0, \\ \{-E_\alpha^{(s)}\} & \text{if } (s-1)\,\alpha > 0, \end{cases} \tag{80}$$

where the eigenvalue $-E_\alpha^{(s)}$ is non-degenerate and is given by

$$E_\alpha^{(s)} = \left(\alpha s \sin\left(\tfrac{\pi}{s}\right)\right)^{\frac{s}{1-s}}, \tag{81}$$

with the (non-normalised) eigenfunction being $\mathsf{G}_{s,\lambda = |E_\alpha^{(s)}|}$.

Proof. For any two pairs (λ,τ) and (λ',τ'), identifying the *same* self-adjoint realisation $\mathsf{k}_\tau^{(s/2)}$, a function $g\in\mathcal{D}(\mathsf{k}_\tau^{(s/2)})$ decomposes as

$$g = F^\lambda + A(\lambda,\tau)F^\lambda(0)\mathsf{G}_{s,\lambda} = F^{\lambda'} + A(\lambda',\tau')F^{\lambda'}(0)\mathsf{G}_{s,\lambda'}$$

$$A(\lambda,\tau) := \frac{\omega(s)\lambda^{2-\frac{1}{s}}}{\tau} - \frac{\theta_s}{G_{s,\lambda}(0)},$$

and the uniqueness of the decomposition implies

$$F^{\lambda'} = F^\lambda + A(\lambda,\tau)F^\lambda(0)\mathsf{G}_{s,\lambda} - A(\lambda',\tau')F^{\lambda'}(0)\mathsf{G}_{s,\lambda'}. \tag{82}$$

In order for $F^{\lambda'}$ to belong to $H^s(\mathbb{R})$, the non-H^s singularities at $x=0$ of $\mathsf{G}_{s,\lambda}$ and $\mathsf{G}_{s,\lambda'}$ must cancel out, that is,

$$A(\lambda,\tau)F^\lambda(0) = A(\lambda',\tau')F^{\lambda'}(0). \tag{83}$$

Plugging (83) into (82) and evaluating (82) at $x=0$ yields

$$\frac{A(\lambda,\tau)F^\lambda(0)}{F^{\lambda'}(0)} = F^{\lambda'}(0) = F^\lambda(0) + A(\lambda,\tau)F^\lambda(0)\,\frac{1}{2\pi}\int_{\mathbb{R}}\Big(\frac{1}{|p|^s+\lambda} - \frac{1}{|p|^s+\lambda'}\Big)\,\mathrm{d}p. \tag{84}$$

A straightforward analysis of the integral above [exploiting the compensation of singularities when $s\in(\frac{1}{2},1)$] shows that

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}}\Big(\frac{1}{|p|^s+\lambda} - \frac{1}{|p|^s+\lambda'}\Big)\,\mathrm{d}p &= \frac{1}{s\sin\frac{\pi}{s}}\,(\lambda^{\frac{1-s}{s}} - \lambda'^{\frac{1-s}{s}}) \\ &= \Theta(s,\lambda) - \Theta(s,\lambda').\end{aligned} \tag{85}$$

Combining (84) and (85) together implies that (λ,τ) and (λ',τ') are linked by the relation

$$\frac{1}{A(\lambda,\tau)} + \Theta(s,\lambda) = \frac{1}{A(\lambda',\tau')} + \Theta(s,\lambda') =: \alpha,$$

i.e.,

$$\alpha - \Theta(s,\lambda) = \Big(\frac{\omega(s)\lambda^{2-\frac{1}{s}}}{\tau} - \frac{\theta_s}{\Theta(s,\lambda)}\Big)^{-1}, \tag{86}$$

which gives the natural extension parameter α. It is immediate from (86) that the Friedrichs case $\tau\to+\infty$ corresponds to $\alpha\to+\infty$ when $s\in(\frac{1}{2},1)$ and to $\alpha=0$ when $s\in(1,\frac{3}{2})$.

Upon replacing (86) in the formulas of Theorem IV.1, we deduce parts (i), (ii), and (iii). Moreover, arguing as in the analogous point of the proof of Theorem IV.2, formulas (79) follow, and one also concludes that each $\mathsf{k}_\alpha^{(s/2)}$ may have *at most* one negative non-degenerate eigenvalue $E_\alpha=-\lambda$, $\lambda>0$.

The occurrence of E_α is read out from the resolvent formula (78) as the pole of $(\mathsf{k}_\alpha^{(s/2)}+\lambda\mathbb{1})^{-1}$, that is, imposing $\alpha-\Theta(s,\lambda)=0$ and hence

$$\alpha = (\lambda^{\frac{s-1}{s}}\, s\,\sin(\tfrac{\pi}{s}))^{-1}. \tag{87}$$

When $s<1$, (87) can be only satisfied by some $\lambda>0$ when $\alpha<0$, in which case

$$\lambda = (\alpha s\sin(\tfrac{\pi}{s}))^{\frac{s}{1-s}} \qquad (s<1,\ \alpha<0).$$

When instead $s>1$, a solution $\lambda>0$ to (87) exists only when $\alpha>0$ and is given by

$$\lambda = (\alpha s\sin(\tfrac{\pi}{s}))^{\frac{s}{1-s}} \qquad (s>1,\ \alpha>0).$$

Hence we proved also (80), which completes the proof of part (iv). □

In the regime $s\in(1,\frac{3}{2})$, Theorem V.3(ii) can be re-phrased in the following even more natural formulation, which shows that $\mathsf{k}_\alpha^{(s/2)}$ can be equivalently qualified as a *form perturbation* of $(-\Delta)^{s/2}$.

Proposition V.4. Let $s\in(1,\frac{3}{2})$. The self-adjoint extensions in $L^2(\mathbb{R})$ of $\mathring{\mathsf{k}}^{(s/2)}$ form the family $(\mathsf{k}_\alpha^{(s/2)})_{\alpha\in\mathbb{R}\cup\{\infty\}}$, where $\alpha=0$ labels the Friedrichs extension given by (60), $\alpha=\infty$ labels the ordinary self-adjoint fractional Laplacian $(-\Delta)^{s/2}$, and for $\alpha\in\mathbb{R}\setminus\{0\}$, one has

$$\begin{aligned}\mathcal{D}[\mathsf{k}_\alpha^{(s/2)}] &= H_0^{s/2}(\mathbb{R}\setminus\{0\}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\} = H^{s/2}(\mathbb{R}),\\ \mathsf{k}_\alpha^{(s/2)}[g] &= \||\nabla|^{\frac{s}{2}} g\|^2_{L^2(\mathbb{R})} - \frac{1}{\alpha}\,|g(0)|^2\end{aligned} \tag{88}$$

for every $\lambda > 0$.

Proof. In the view of Theorem V.3(ii), we only need to prove the second line of (88) for a generic $g \in \mathcal{D}[\mathsf{k}_\alpha^{(s/2)}]$. We set $\Sigma := (\frac{1}{\Theta(s,\lambda)} - \frac{1}{\alpha-\Theta(s,\lambda)})^{-1}$ for short and decompose $g = F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}$ for some $F^\lambda \in H_0^{s/2}(\mathbb{R}\setminus\{0\})$ and $\kappa_\lambda = g(0)/\mathsf{G}_{s,\lambda}(0)$. Applying (77), we find

$$\begin{aligned}\mathsf{k}_\alpha^{(s/2)}[g] &= -\lambda\|g\|^2_{L^2(\mathbb{R})} + \|\,|\nabla|^{\frac{s}{2}}(g - \kappa_\lambda \mathsf{G}_{s,\lambda})\|^2_{L^2(\mathbb{R})} + \lambda\|g - \kappa_\lambda\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R})} + \Sigma|\kappa_\lambda|^2\\ &= \|\,|\nabla|^{\frac{s}{2}} g\|^2_{L^2(\mathbb{R})} + |\kappa_\lambda|^2(\|\,|\nabla|^{\frac{s}{2}}\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R})} + \lambda\|\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R})} + \Sigma)\\ &\quad - 2\,\mathfrak{Re}\,\kappa_\lambda(\langle|\nabla|^{\frac{s}{2}} g, |\nabla|^{\frac{s}{2}}\mathsf{G}_{s,\lambda}\rangle + \lambda\langle g, \mathsf{G}_{s,\lambda}\rangle).\end{aligned}$$

Since $\|\,|\nabla|^{\frac{s}{2}}\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R})} + \lambda\|\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R})} = \langle \mathsf{G}_{s,\lambda}, ((-\Delta)^{\frac{s}{2}} + \lambda)\mathsf{G}_{s,\lambda}\rangle = \mathsf{G}_{s,\lambda}(0) = \Theta(s,\lambda)$ and analogously $\langle|\nabla|^{\frac{s}{2}} g, |\nabla|^{\frac{s}{2}}\mathsf{G}_{s,\lambda}\rangle + \lambda\langle g, \mathsf{G}_{s,\lambda}\rangle = \overline{g(0)}$, then

$$\mathsf{k}_\alpha^{(s/2)}[g] = \|\,|\nabla|^{\frac{s}{2}} g\|^2_{L^2(\mathbb{R})} + \frac{|g(0)|^2}{\Theta(s,\lambda)^2}\,(\Theta(s,\lambda) + \Sigma) - 2\,\frac{|g(0)|^2}{\Theta(s,\lambda)}.$$

The coefficient of the $|g(0)|^2$-term above amounts to

$$\frac{1}{\Theta(s,\lambda)^2}\Big(\Theta(s,\lambda) + \frac{1}{\frac{1}{\Theta(s,\lambda)} - \frac{1}{\alpha-\Theta(s,\lambda)}}\Big) - \frac{2}{\Theta(s,\lambda)} = -\frac{1}{\alpha},$$

whence indeed $\mathsf{k}_\alpha^{(s/2)}[g] = \||\nabla|^{\frac{s}{2}} g\|^2_{L^2(\mathbb{R})} - \alpha^{-1}|g(0)|^2$. □

VI. RANK-ONE SINGULAR PERTURBATIONS OF THE FRACTIONAL LAPLACIAN: INHOMOGENEOUS CASE

For completeness of presentation, in this section, we work out the *inhomogeneous* version of the operator $\mathsf{k}_\tau^{(s/2)}$ of Sec. IV, for concreteness in dimension $d = 3$. That is, instead of constructing a singular perturbation of $(-\Delta)^{s/2}$, we consider the singular perturbation of the fractional power $(-\Delta + \mathbb{1})^{s/2}$. This is going to be the operator $\mathfrak{d}_\tau^{(s/2)}$ introduced informally in (3).

The conceptual scheme is the very same as in Secs. III and IV, and only certain explicit computations are modified in an easy way. Thus, we content ourselves to state the main results without proofs.

For chosen $\lambda > 0$ and $s \in \mathbb{R}$, we set

$$\mathcal{G}_{s,\lambda}(x) := \frac{1}{(2\pi)^{\frac{3}{2}}}\Big(\frac{1}{(p^2+\lambda)^{s/2}}\Big)^{\vee}(x), \qquad x, p \in \mathbb{R}^3, \tag{89}$$

whence $(-\Delta + \lambda\mathbb{1})^{s/2}\,\mathcal{G}_{s,\lambda} = \delta(x)$ distributionally. We also set

$$\mathring{\mathfrak{d}}_\lambda^{(s/2)} := \overline{(-\Delta + \lambda\mathbb{1})^{s/2} \upharpoonright C_0^\infty(\mathbb{R}^3\setminus\{0\})} \tag{90}$$

as an operator closure with respect to the Hilbert space $L^2(\mathbb{R}^3)$. Thus, in comparison to Sec. II, $\mathcal{G}_{2,\lambda} = G_\lambda$ and $\mathring{\mathfrak{d}}_\lambda^{(1)} = \mathring{\mathfrak{h}} + \lambda\mathbb{1}$. Moreover,

$$\begin{aligned}|x|^{3-s}\,\mathcal{G}_{s,\lambda}(x) &\xrightarrow{x\to 0} \Lambda_s = \frac{\Gamma(\frac{3-s}{2})}{(2\pi)^{\frac{3}{2}}\,2^{s-\frac{3}{2}}\,\Gamma(\frac{s}{2})}, \qquad s \in (0,3),\\ \mathcal{G}_{s,\lambda}(x) &\xrightarrow{x\to 0} \mathcal{G}_{s,\lambda}(0) = \frac{\Gamma(\frac{s-3}{2})}{8\pi^{\frac{3}{2}}\,\lambda^{\frac{s-3}{2}}\,\Gamma(\frac{s}{2})}, \qquad s > 3.\end{aligned} \tag{91}$$

Reasoning as in (29) and in Lemma III.1, we see that when $s \in (\frac{3}{2}, \frac{5}{2})$, the deficiency index of $\mathring{\mathfrak{d}}_\lambda^{(s/2)}$ equals 1.

One has the following construction.

Theorem VI.1. *Let $s \in (\frac{3}{2}, \frac{5}{2})$ and $\lambda > 0$.*

(i) *The self-adjoint extensions in $L^2(\mathbb{R}^3)$ of the operator $\mathring{\mathfrak{d}}_\lambda^{(s/2)}$ form the family $(\mathfrak{d}_{\lambda,\tau}^{(s/2)})_{\tau \in \mathbb{R}\cup\{\infty\}}$, where $\mathfrak{d}_{\lambda,\infty}^{(s/2)}$ is its Friedrichs extension, namely, the self-adjoint fractional shifted Laplacian $(-\Delta + \lambda\mathbb{1})^{s/2}$, and all other extensions are given by*

$$\mathcal{D}(\mathfrak{d}_{\lambda,\tau}^{(s/2)}) := \left\{ g \in L^2(\mathbb{R}^3) \,\middle|\, \begin{array}{l} \widehat{g}(p) = \widehat{f^\lambda}(p) + \dfrac{\tau\,\xi}{(p^2+\lambda)^s} + \dfrac{\xi}{(p^2+\lambda)^{s/2}} \\ \xi \in \mathbb{C},\ f^\lambda \in H^s(\mathbb{R}^3),\ \displaystyle\int_{\mathbb{R}^3} \widehat{f^\lambda}(p)\,\mathrm{d}p = 0 \end{array} \right\}$$
$$= \left\{ g = F^\lambda + \tfrac{8\pi^{\frac{3}{2}}\lambda^{s-\frac{3}{2}}\Gamma(s)}{\tau\Gamma(s-\frac{3}{2})}\, F^\lambda(0)\,\mathcal{G}_{s,\lambda} \,\middle|\, F^\lambda \in H^s(\mathbb{R}^3) \right\}, \tag{92}$$

where $\widehat{F^\lambda} = \widehat{f^\lambda}(p) + (p^2+\lambda)^{-s}\tau\,\xi$ and

$$\mathfrak{d}_{\lambda,\tau}^{(s/2)} g := (-\Delta + \lambda\mathbb{1})^{s/2} F^\lambda. \tag{93}$$

(ii) *Each extension is semi-bounded from below and*

$$\begin{aligned} \inf\sigma(\mathfrak{d}_{\lambda,\tau}^{(s/2)}) \geqslant 0 \quad &\Leftrightarrow \quad \tau \geqslant 0, \\ \inf\sigma(\mathfrak{d}_{\lambda,\tau}^{(s/2)}) > 0 \quad &\Leftrightarrow \quad \tau > 0, \\ \mathfrak{d}_{\lambda,\tau}^{(s/2)} \text{is invertible} \quad &\Leftrightarrow \quad \tau \neq 0. \end{aligned} \tag{94}$$

(iii) *The eigenvalue zero of the extension $\mathfrak{d}_{\lambda,\tau=0}^{(s/2)}$ is non-degenerate and the (non-normalised) eigenfunction is $\mathcal{G}_{s,\lambda}$. When $\tau < 0$, the operator $\mathfrak{d}_{\lambda,\tau}^{(s/2)}$ admits one non-degenerate negative eigenvalue $E_\tau < \tau$.*

(iv) *For each $\tau \in \mathbb{R}$, the quadratic form of the extension $\mathfrak{d}_{\lambda,\tau}^{(s/2)}$ is given by*

$$\mathcal{D}[\mathfrak{d}_{\lambda,\tau}^{(s/2)}] = H^{\frac{s}{2}}(\mathbb{R}^3) \dotplus \mathrm{span}\{\mathcal{G}_{s,\lambda}\}, \tag{95}$$

$$\mathfrak{d}_{\lambda,\tau}^{(s/2)}[F^\lambda + \kappa_\lambda \mathcal{G}_{s,\lambda}] = \|(-\Delta + \lambda\mathbb{1})^{s/4} F^\lambda\|^2_{L^2(\mathbb{R}^3)} + \frac{\tau\Gamma(s-\frac{3}{2})}{8\pi^{\frac{3}{2}}\lambda^{s-\frac{3}{2}}\Gamma(s)}|\kappa_\lambda|^2 \tag{96}$$

for any $F^\lambda \in H^{\frac{s}{2}}(\mathbb{R}^3)$ and $\kappa_\lambda \in \mathbb{C}$.

(v) *For $\tau \neq 0$, one has the resolvent identity*

$$(\mathfrak{d}_{\lambda,\tau}^{(s/2)})^{-1} = (-\Delta + \lambda)^{-s/2} + \tfrac{8\pi^{\frac{3}{2}}\lambda^{s-\frac{3}{2}}\Gamma(s)}{\tau\Gamma(s-\frac{3}{2})}\, |\mathcal{G}_{s,\lambda}\rangle\langle\mathcal{G}_{s,\lambda}|. \tag{97}$$

It is clear that, as opposite to $\mathfrak{h}_\tau^{s/2}$ or $\mathsf{k}_\tau^{(s/2)}$, the shift $\lambda > 0$ is inherent the very construction of the operator $\mathfrak{d}_{\lambda,\tau}^{(s/2)}$: in fact, the domain of $\mathring{\mathfrak{d}}_\lambda^{(s/2)}$ is independent of $\lambda > 0$, but its action is not (the difference $\mathring{\mathfrak{d}}_\lambda^{(s/2)} - \mathring{\mathfrak{d}}_{\lambda'}^{(s/2)}$ is a bounded, yet a non-trivial operator); thus also the adjoint $(\mathring{\mathfrak{d}}_\lambda^{(s/2)})^*$ and its self-adjoint restrictions are λ-dependent [the adjoints $(\mathring{\mathfrak{h}}^{s/2})^*$ and $(\mathring{\mathsf{k}}^{(s/2)})^*$ are λ-independent, instead].

Let us also elaborate further on part (iii) of the theorem. As in Secs. I–IV, both statements are classical facts in the Kreĭn-Višik-Birman extension scheme. Chosen $\lambda > 0$ and $s \in (\frac{3}{2}, \frac{5}{2})$, the negative eigenvalue E_τ of the extension $\mathfrak{d}_{\lambda,\tau}^{(s/2)}$ for $\tau < 0$ is obtained as follows. Let g be the corresponding eigenfunction and decompose $\widehat{g} = \widehat{f^\lambda} + (p^2+\lambda)^{-s}\tau + (p^2+\lambda)^{-s/2}$ according to (92) (without loss of generality, we re-absorbed ξ in f^λ). Then the condition $\mathfrak{d}_{\lambda,\tau}^{(s/2)} g = E_\tau g$, owing to the property (93), reads

$$(p^2+\lambda)^{\frac{s}{2}}\widehat{f^\lambda} + (p^2+\lambda)^{-\frac{s}{2}}\tau = E_\tau\widehat{f^\lambda} + (p^2+\lambda)^{-s}\tau E_\tau + (p^2+\lambda)^{-\frac{s}{2}}E_\tau,$$

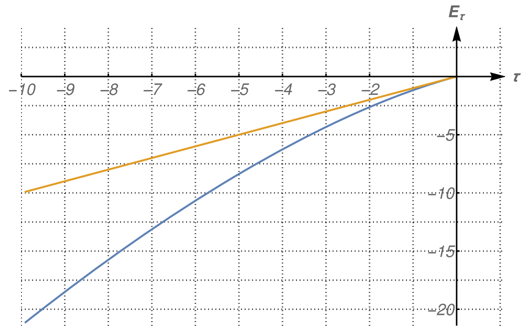

FIG. 1. Plot of the negative eigenvalue E_τ of the extension $\mathfrak{d}^{(s/2)}_{\lambda,\tau}$ vs τ for $\lambda = 1$ and $s = 1.8$ (blue curve). The reference orange curve gives the corresponding value of τ. Indeed, $E_\tau < \tau$ and $E_{\tau\to 0} = 0$.

whence

$$\widehat{f}(p) = \frac{1}{(p^2+\lambda)^{\frac{s}{2}} - E_\tau}\Big(\frac{\tau E_\tau}{(p^2+\lambda)^s} - \frac{\tau - E_\tau}{(p^2+\lambda)^{\frac{s}{2}}}\Big).$$

The fact that $f^\lambda \in H^s(\mathbb{R}^3)$ is then obvious, whereas the condition $\int_{\mathbb{R}^3} \widehat{f^\lambda}\,\mathrm{d}p = 0$ selects the value of E_τ in terms of τ (and of λ)—a numerical example is provided in Fig. 1.

Let us conclude by remarking that two relevant features of the homogeneous case are present in the inhomogeneous case too.

First and most importantly, the operator $\mathfrak{d}^{(s/2)}_{\lambda,\tau}$ is a rank-one perturbation, in the resolvent sense, of $(-\Delta+\lambda)^{s/2}$, precisely as $\mathsf{k}^{(s/2)}_\tau$ is a rank-one perturbation of $(-\Delta)^{s/2}$.

Second, the elements of the domain of $\mathfrak{d}^{(s/2)}_{\lambda,\tau}$ decompose into a regular H^s-part and a singular part, constrained to the former by a local boundary condition, where the local singularity when $s \in (\frac{3}{2}, \frac{5}{2})$ is of the form $|x|^{3-s}$ as $x \to 0$.

VII. HIGH DEFICIENCY INDEX (HIGH FRACTIONAL POWER) SCENARIO

Let us outline in this section how the previous constructions of the self-adjoint extensions of the operators $\mathring{\mathsf{k}}^{(s/2)}$ or $\mathring{\mathfrak{d}}^{(s/2)}_\lambda$ get modified when $s > \frac{d}{2} + 1$.

We recall from Lemma III.1 that when $s \in I^{(d)}_n(\frac{d}{2} + n - 1, \frac{d}{2} + n)$, $n \in \mathbb{N}$, one has

$$\ker\,((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1}) \;=\; \mathrm{span}\Big\{u^\lambda_{\gamma_1,\dots,\gamma_d}\,\Big|\,\gamma_1,\dots,\gamma_d \in \mathbb{N}_0,\ \sum_{j=1}^d \gamma_j \leqslant n-1\Big\}, \tag{98}$$

and analogously

$$\ker\,((\mathring{\mathfrak{d}}^{(s/2)}_\lambda)^* + \lambda\mathbb{1}) \;=\; \mathrm{span}\Big\{v^\lambda_{\gamma_1,\dots,\gamma_d}\,\Big|\,\gamma_1,\dots,\gamma_d \in \mathbb{N}_0,\ \sum_{j=1}^d \gamma_j \leqslant n-1\Big\}, \tag{99}$$

having defined

$$\widehat{v^\lambda}_{\gamma_1,\dots,\gamma_d}(p) := \frac{p_1^{\gamma_1}\cdots p_d^{\gamma_d}}{(p^2+\lambda)^{\frac{s}{2}}}. \tag{100}$$

The same extension scheme applied in Sec. IV provides an analogous classification of all the self-adjoint extensions in the case of generic deficiency index $\mathcal{J}(s,d)$, where now *each* extension of $\mathring{\mathsf{k}}^{(s/2)}$ is an operator $\mathsf{k}^{(s/2)}_T$ labeled by a self-adjoint operator T in some subspace $\mathcal{D}(T)$ of $\ker\,((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1}) \cong \mathbb{C}^{\mathcal{J}(s,d)}$, hence labeled by some $N\times N$ hermitian matrix, $1 \leqslant N \leqslant \mathcal{J}(s,d)$.

Explicitly (see, e.g., Theorem 3.4 in Ref. 7),

$$\mathcal{D}(\mathsf{k}_T^{(s/2)}) = \left\{ g \in L^2(\mathbb{R}^3) \,\middle|\, \begin{array}{l} g = f + (\mathring{\mathsf{k}}_F^{(s/2)} + \lambda\mathbb{1})^{-1}(Tu + w) + u \\ \text{where } f \in H_0^s(\mathbb{R}^3 \setminus \{0\}),\ u \in \mathcal{D}(T), \\ w \in \ker\,((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1}) \cap \mathcal{D}(T)^\perp \end{array} \right\}, \tag{101}$$

$$(\mathsf{k}_T^{(s/2)} + \lambda\mathbb{1})g = (\mathsf{k}_F^{(s/2)} + \lambda\mathbb{1})F_\lambda$$

$$F_\lambda := f + (\mathring{\mathsf{k}}_F^{(s/2)} + \lambda\mathbb{1})^{-1}(Tu + w) \in H^s(\mathbb{R}^3),$$

with $\mathsf{k}_F^{(s/2)}$ denoting the Friedrichs extension.

Analogously, the self-adjoint extensions of $\mathring{\mathfrak{d}}_\lambda^{(s/2)}$ form a family of operators $\mathfrak{d}_{\lambda,T}^{(s/2)}$ with

$$\mathcal{D}(\mathfrak{d}_{\lambda,T}^{(s/2)}) = \left\{ g \in L^2(\mathbb{R}^3) \,\middle|\, \begin{array}{c} g = f + (\mathring{\mathfrak{d}}_{\lambda,F}^{(s/2)})^{-1}(Tv + w) + v \\ \text{where } f \in H_0^s(\mathbb{R}^3 \setminus \{0\}),\ v \in \mathcal{D}(T), \\ w \in \ker(\mathring{\mathfrak{d}}_\lambda^{(s/2)})^* \cap \mathcal{D}(T)^\perp \end{array} \right\}, \tag{102}$$

$$\mathfrak{d}_{\lambda,T}^{(s/2)} g = \mathring{\mathfrak{d}}_{\lambda,F}^{(s/2)} F_\lambda$$

$$F_\lambda := f + (\mathring{\mathfrak{d}}_{\lambda,F}^{(s/2)})^{-1}(Tv + w) \in H^s(\mathbb{R}^3).$$

The theory provides also a counterpart classification of the the quadratic forms of the extensions (see Ref. 7, Theorem 3.6).

The above formulas show that for high powers s, the operators $\mathring{\mathsf{k}}^{(s/2)}$ and $\mathring{\mathfrak{d}}_\lambda^{(s/2)}$ have a richer variety [a $\mathcal{J}(s,d)^2$-parameter family] of self-adjoint extensions. The parameterizing matrix T determines a more complicated set of "boundary conditions" between the "Friedrichs" part of a generic element of the extension domain and the remaining part: the resulting constraint involves the evaluation at $x = 0$ of some number of partial derivatives of the former component.

This construction produces finite-rank perturbations in the resolvent sense; hence extensions are all semi-bounded from below and may admit a (finite) number of negative eigenvalues, up to $\mathcal{J}(s,d)$, counting the multiplicity.

Unlike the case of deficiency index 1, depending on the extension parameter T, the large-p vanishing behavior in momentum space of the singular component may be milder than that of the Green function, and therefore the local singularity of g in position space may be more severe than the behavior of the Green function as $x \to 0$.

Let us comment on how the *worst leading singularity* at $x = 0$ of a generic function $g \in \mathcal{D}(\mathsf{k}_T^{(s/2)})$ depends on s and d—the discussion for $g \in \mathcal{D}(\mathfrak{d}_{\lambda,T}^{(s/2)})$ is identical.

As expressed by (101), such a singularity is due to those functions of type $u^\lambda_{\gamma_1,\dots,\gamma_d}$ that span $\mathcal{D}(T)$. When $s \in I_n^{(d)}$, the worst local singularity occurs when such functions decrease at infinity in momentum coordinates with the slowest possible vanishing rate compatible with s and d, that is, when $\gamma_1 + \cdots + \gamma_d = n - 1$.

Let u be any such most singular function, which then behaves as $|\widehat{u}(p)| \approx |p|^{-(s+1-n)}$ as $|p| \to +\infty$. Then $|u(x)| \approx |x|^{-(d-1+n-s)}$ as $x \to 0$. Since the map

$$I_n^{(d)} \ni s \mapsto d - 1 + n - s$$

is monotone decreasing and takes values in $(\frac{d}{2} - 1, \frac{d}{2})$, if the extension $\mathsf{k}_T^{(s/2)}$ is such that $\mathcal{D}(T) \ni u$, then the functions in $\mathcal{D}(\mathsf{k}_T^{(s/2)})$ display a local singularity that ranges from $|x|^{-\frac{d}{2}}$ to $|x|^{-\frac{d-1}{2}}$ as long as s increases in $I_n^{(d)}$, precisely as (52) when s increases in $I_1^{(3)}$.

Noticeably, at the transition values $s \in \mathbb{N} + \frac{1}{2}$, the above picture undergoes a discontinuity in s due to the further control of one more derivative in $\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)})$, as a consequence of Sobolev's lemma and consequently to emergence in $\ker\,((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1})$ of elements that in momentum coordinates vanish more slowly at infinity.

VIII. APPLICATIONS AND PERSPECTIVES

Besides the operator-theoretic and functional-analytic interest *per se* of the constructions of the operators (3), our discussion is profoundly inspired to an amount of natural applications.

Singular perturbations model point-like impurities and more generally point-like interactions. In the realm of the evolutive equations of relevance for quantum mechanics, they naturally govern the evolution of systems subject to such "singular potentials."

This includes the linear Schrödinger evolution

$$\mathrm{i}\partial_t u = \mathfrak{h}_\alpha u$$

as well as the class of semi-linear Schrödinger equations

$$\mathrm{i}\partial_t u = \mathfrak{h}_\alpha u + \mathcal{N}(u)$$

of the free Laplacian plus a point-like perturbation, with physically relevant non-linearities such as the power-law local non-linearity $\mathcal{N}(u) = |u|^{\gamma-1}u$ or the Hartree type non-local non-linearity $\mathcal{N}(u) = (w * |u|^2)u$. The reconstruction of the *linear* propagator from the resolvent of $\mathfrak{h}_\alpha$ is already known in the literature,[1,19] as well as the dispersive properties and space-time estimates of such a propagator,[5,9] and the existence, completeness, and L^p-boundedness of the wave operators for the pairs $(\mathfrak{h}_\alpha, -\Delta)$.[6]

In addition, for the study of the *non-linear* problem in a suitable space (the energy space in the first place, as well as other spaces of lower or higher regularity), the knowledge is needed of the corresponding singular norms, namely, the norms $\|u\|_{\widetilde{H}^s_\alpha(\mathbb{R}^3)} = \|(\mathfrak{h}_\alpha + \mathbb{1})^{s/2}u\|_{L^2(\mathbb{R}^3)}$ considered in Theorem II.4 above and Ref. 8. In this respect and with such tools, the study of certain non-linear Schrödinger equations with singular potentials has already started.[12]

An analogous *systematic* knowledge for $\mathsf{k}_\tau^{(s/2)}$ and $\mathfrak{d}_{\lambda,\tau}^{(s/2)}$ is by know lacking.

This is even more needed due to the relevance of various models of singular perturbations of fractional differential operators. A relevant example is the *powers* of the quantum-mechanical semi-relativistic energy operator $\sqrt{-\Delta + m^2}$, the singular perturbation of which yields precisely operators of the type $\mathfrak{d}_{m^2,\tau}^{(s/2)}$ considered in Sec. VI or, in the case of zero rest energy, of the type $\mathsf{k}_\tau^{(s/2)}$ as in Sec. IV.

What one finds in the literature is an increasing amount of recent studies[4,10,11,14–16,18,20] where the singular perturbation of the fractional Laplacian is approached through Green's function methods (together with Wick-like rotations to obtain the propagator from the resolvent) that have the virtue of highlighting the singular structure carried over by what we denoted with $\mathsf{G}_{s,\lambda}$ and $\mathcal{G}_{s,\lambda}$, however with no specific concern to the multiplicity of self-adjoint realisations and the associated local boundary conditions or to the increase of the deficiency index with the power s.

As above, for each extension, one would like to qualify the linear propagator, its space-time estimates, and the fractional norms and to use these tools for the associated non-linear problems.

We trust that the research programme emerging from the above considerations may be successfully addressed over the next future!

APPENDIX: CHARACTERISATION OF $H^s_0(\mathbb{R}^d \setminus \{0\})$

We show in this Appendix how to prove the characterisation (29) of the space $H^s_0(\mathbb{R}^d \setminus \{0\})$.

It is not restrictive to fix $d = 3$ and to discuss and compare the first two regimes $s \in I_0^{(3)}$ and $s \in I_1^{(3)}$. The argument for $s \in I_n^{(3)}$, $n = 2, 3, \dots$, is completely analogous.

Thus, let us prove the following property:

Lemma A.1.

$$H^s_0(\mathbb{R}^3 \setminus \{0\}) = \begin{cases} H^s(\mathbb{R}^3) & \text{if } s \in [0, \frac{3}{2}), \\ \{f \in H^s(\mathbb{R}^3) \,|\, \int_{\mathbb{R}^3} \widehat{f}(p)\, \mathrm{d}p = 0\} & \text{if } s \in (\frac{3}{2}, \frac{5}{2}). \end{cases}$$

Proof. We consider first the case $s \in [0, \frac{3}{2})$. The inclusion

$$H_0^s(\mathbb{R}^3 \setminus \{0\}) \subset H^s(\mathbb{R}^3)$$

is obvious. For the other inclusion, for any $f \in H^s(\mathbb{R}^3)$ and for arbitrary $\varepsilon > 0$, we want to find $f_\varepsilon \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ such that $\|f_\varepsilon - f\|_{H^s} \leqslant \varepsilon$, and by means of a standard density argument, it is not restrictive to assume further that $f \in \mathcal{S}(\mathbb{R}^3)$ and $\widehat{f}$ is compactly supported. Let $\phi \in C^\infty(\mathbb{R}^3)$ be such that

$$\phi(x) = 0 \quad \text{for } |x| \leqslant 1, \qquad \phi(x) = 1 \quad \text{for } |x| \geqslant 2,$$

and let $\psi := \phi - 1$, $\phi_n(x) := \phi(n|x|)$, and $\psi_n(x) := \psi(n|x|)$, for $n \in \mathbb{N}$. Thus, $\psi \in C_0^\infty(\mathbb{R}^3)$ and $\phi_n f \in C^\infty(\mathbb{R}^3)$, with $\phi_n f$ vanishing in a neighbourhood of $x = 0$. Moreover,

$$\|\phi_n f - f\|_{H^s} = \|D^s(\phi_n f - f)\|_{L^2} = \|D^s(\psi_n f)\|_{L^2} = (2\pi)^{\frac{3}{2}} \|\langle p\rangle^s (\widehat{\psi}_n * \widehat{f})\|_{L^2},$$

where $D^s := (1-\Delta)^{s/2}$ and $\langle p\rangle = \sqrt{1+|p|^2}$. Clearly, $\widehat{\psi}_n(p) = \frac{1}{n^3}\widehat{\psi}(\frac{p}{n})$. Therefore,

$$\begin{aligned}
\|\phi_n f - f\|_{H^s}^2 &= (2\pi)^3 \int_{\mathbb{R}^3} \mathrm{d}p\, \langle p\rangle^{2s} \Big| \int_{\operatorname{supp}\widehat{f}} \mathrm{d}q\, \widehat{f}(q)\, \widehat{\psi}_n(p-q)\Big|^2 \\
&\leqslant \frac{(2\pi)^3 \|\widehat{f}\|_{L^2}^2}{n^6} \int_{\mathbb{R}^3} \mathrm{d}p \int_{\operatorname{supp}\widehat{f}} \mathrm{d}q\, \langle p\rangle^{2s} |\widehat{\psi}(\tfrac{p-q}{n})|^2 \\
&\lesssim n^{2s} \int_{\frac{\operatorname{supp}\widehat{f}}{n}} \mathrm{d}q \int_{\mathbb{R}^3} \mathrm{d}p\, \langle p\rangle^{2s} |\widehat{\psi}(p-q)|^2 \\
&\lesssim n^{2s-3} \xrightarrow{n\to+\infty} 0.
\end{aligned}$$

The last step above follows from the continuity of $q \mapsto \int_{\mathbb{R}^3} \mathrm{d}p\, \langle p\rangle^{2s} |\widehat{\psi}(p-q)|^2$. In particular, we can choose $n := n(\varepsilon)$ sufficiently large such that

$$\|\phi_n f - f\|_{H^s} \leqslant \frac{\varepsilon}{2}.$$

For such n, we can find a smooth function $x \mapsto \chi(x)$ that produces a slow cutoff at infinity so that

$$\|\chi\phi_n f - \phi_n f\|_{H^s} \leqslant \frac{\varepsilon}{2}.$$

We have thus identified a function $f_\varepsilon := \chi\phi_n f \in C_0^\infty(\mathbb{R}\setminus\{0\})$ satisfying

$$\|f_\varepsilon - f\|_{H^s} \leqslant \|\chi\phi_n f - \phi_n f\|_{H^s} + \|\phi_n f - f\|_{H^s} \leqslant \varepsilon.$$

Let us discuss now the case $s \in (\frac{3}{2}, \frac{5}{2})$. Owing to Sobolev's lemma, one has the continuous embedding $H^s(\mathbb{R}^3) \hookrightarrow C^{0,s-\frac{3}{2}}(\mathbb{R}^3)$ and hence any limit in the H^s-norm of elements in $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ must vanish at the origin. Therefore, we only need to prove the inclusion

$$H_0^s(\mathbb{R}^3 \setminus \{0\}) \supset \Big\{ f \in H^s(\mathbb{R}^3) \,\Big|\, \int \widehat{f}(p)\,\mathrm{d}p = 0 \Big\},$$

that is, for any $f \in H^s(\mathbb{R}^3)$ with $f(0) = 0$ and for arbitrary $\varepsilon > 0$, we want to find $f_\varepsilon \in C_0^\infty(\mathbb{R}^3\setminus\{0\})$ such that $\|f_\varepsilon - f\|_{H^s} \leqslant \varepsilon$. Since the function f_R defined by

$$\widehat{f_R} := \widehat{f} \cdot \mathbf{1}_{\{|p|\leqslant R\}} - \Big(\int_{|p|\leqslant R} \widehat{f}(p)\mathrm{d}p \Big) \frac{\mathbf{1}_{\{|p|\leqslant 1\}}}{|\{|p| \leqslant 1\}|}, \qquad R > 0,$$

has the obvious properties $f_R \in \mathcal{S}(\mathbb{R}^3)$, $\int_{\mathbb{R}^3} \widehat{f_R}(p)\,\mathrm{d}p = 0$, and $\|f - f_R\|_{H^s} \to 0$ as $R \to +\infty$, it is not restrictive to assume from the beginning that $f \in \mathcal{S}(\mathbb{R}^3)$ with $f(0) = 0$ and with compactly supported

$\widehat{f}$. With the same notation as in the first part of the proof,

$$\begin{aligned}
\|\phi_n f - f\|_{H^s}^2 &= (2\pi)^3 \int_{\mathbb{R}^3} \mathrm{d}p\, \langle p\rangle^{2s} \Big| \int_{\mathrm{supp}\widehat{f}} \mathrm{d}q\, \widehat{f}(q)\widehat{\psi}_n(p-q)\Big|^2 \\
&= (2\pi)^3 \int_{\mathbb{R}^3} \mathrm{d}p\, \langle p\rangle^{2s} \Big| \int_{\mathrm{supp}\widehat{f}} \mathrm{d}q\, \widehat{f}(q)(\widehat{\psi}_n(p-q) - \widehat{\psi}_n(p))\Big|^2 \\
&\leqslant \frac{(2\pi)^3 \|\widehat{f}\|^2_{L^2(\mathbb{R}^3)}}{n^6} \int_{\mathbb{R}^3} \mathrm{d}p \int_{\mathrm{supp}\widehat{f}} \mathrm{d}q\, \langle p\rangle^{2s} |\widehat{\psi}(\tfrac{p-q}{n}) - \widehat{\psi}(\tfrac{p}{n})|^2 \\
&\lesssim n^{2s} \int_{\mathbb{R}^3} \mathrm{d}p \int_{\frac{\mathrm{supp}\widehat{f}}{n}} \mathrm{d}q\, \langle p\rangle^{2s} \Big| \int_0^1 \mathrm{d}t\, (\nabla\widehat{\psi})(p-tq)\cdot q\Big|^2 \\
&\lesssim n^{2s-2} \int_{\frac{\mathrm{supp}\widehat{f}}{n}} \mathrm{d}q \int_{\mathbb{R}^3} \mathrm{d}p\, \langle p\rangle^{2s} \int_0^1 \mathrm{d}t\, |(\nabla\widehat{\psi})(p-tq)|^2 \\
&\lesssim n^{2s-5} \xrightarrow{n\to+\infty} 0,
\end{aligned}$$

where we used the condition $\int_{\mathbb{R}^3} \widehat{f}(q)\,\mathrm{d}q = 0$ in the second step, the bound $|q| \lesssim n^{-1}$ for $q \in \frac{\mathrm{supp}\widehat{f}}{n}$ in the penultimate step, and the continuity of the function $q \mapsto \int_{\mathbb{R}^3} \mathrm{d}p \int_0^1 \mathrm{d}t\, \langle p\rangle^{2s} |\widehat{(\nabla)}\psi(p-tq)|^2$ in the last step. In particular, we can choose $n := n(\varepsilon)$ sufficiently large such that

$$\|\phi_n f - f\|_{H^s} \leqslant \frac{\varepsilon}{2}.$$

For such n, we can find a smooth function $x \mapsto \chi(x)$ that produces a slow cutoff at infinity so that

$$\|\chi\phi_n f - \phi_n f\|_{H^s} \leqslant \frac{\varepsilon}{2}.$$

We have thus identified a function $f_\varepsilon := \chi\phi_n f \in C_0^\infty(\mathbb{R}\setminus\{0\})$ satisfying

$$\|f_\varepsilon - f\|_{H^s} \leqslant \|\chi\phi_n f - \phi_n f\|_{H^s} + \|\phi_n f - f\|_{H^s} \leqslant \varepsilon,$$

which concludes the proof. □

When $n = 2, 3, \dots$ and $s \in I_n = (n+\frac{1}{2}, n+\frac{3}{2})$, Sobolev's lemma guarantees that the closure in the H^s-norm of $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ comes with the vanishing at $x = 0$ of the the function and its first partial derivatives up to order n. Then one can complete the characterisation of $H_0^s(\mathbb{R}^3 \setminus \{0\})$ by a similar argument as above, now replacing f with its partial derivatives. This yields the formula

$$H_0^s(\mathbb{R}^3\setminus\{0\}) = \left\{ \begin{array}{c} f \in H^s(\mathbb{R}^3) \text{ such that} \\ \displaystyle\int_{\mathbb{R}^3} p_1^{\gamma_1} p_2^{\gamma_2} p_3^{\gamma_3} \widehat{f}(p)\,\mathrm{d}p = 0 \\ \gamma_1, \gamma_2, \gamma_3 \in \mathbb{N}_0,\ \gamma_1+\gamma_2+\gamma_3 \leqslant n-1 \end{array} \right\}, \qquad s \in (n+\tfrac{1}{2}, n+\tfrac{3}{2}).$$

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