On deformations of multidimensional Poisson brackets of hydrodynamic type

M. Casati

Scuola Internazionale Superiore di Studi Avanzati
via Bonomea 265, 34136 Trieste (ITALY)

December 9, 2013

Abstract

The theory of Poisson Vertex Algebras (PVAs) [4] is a good framework to treat Hamiltonian partial differential equations. A PVA consists of a pair $(\mathcal{A}, \{\cdot_\lambda\cdot\})$ of a differential algebra $\mathcal{A}$ and a bilinear operation called the λ-bracket. We extend the definition to the class of algebras $\mathcal{A}$ endowed with $d \geq 1$ commuting derivations. We call this structure a *multidimensional PVA*: it is a suitable setting to study Hamiltonian PDEs with d spatial dimensions. We apply this theory to the study of deformations of the Poisson bracket of hydrodynamic type associated to the Euler's equations of motion of d-dimensional incompressible fluids.

Keywords Hamiltonian Operator, Hydrodynamic Poisson Bracket, Poisson Vertex Algebra

MSC 37K05 (primary), 37K25, 17B80

1 Introduction

The main goal of this paper is to extend the formalism of Poisson Vertex Algebras (PVAs) in order to study Hamiltonian evolutionary PDEs with several spatial dimensions. Such PVAs provide efficient techniques to characterize Hamiltonian operators of evolutionary PDEs. Let us first remind the basic definitions of the theory of PVAs.

1.1 Poisson Vertex Algebras

A Poisson Vertex Algebra [4] is a differential algebra $(\mathcal{A}, \partial)$ endowed with a bilinear operation $\mathcal{A} \times \mathcal{A} \to \mathbb{R}[\lambda] \otimes \mathcal{A}$ called a λ-bracket and satisfying the set of properties

1. $\{f_\lambda \partial g\} = (\lambda + \partial)\{f_\lambda g\}$
2. $\{\partial f_\lambda g\} = -\lambda\{f_\lambda g\}$
3. $\{f_\lambda gh\} = \{f_\lambda g\}h + \{f_\lambda h\}g$
4. $\{fg_\lambda h\} = \{f_{\lambda+\partial} h\}g + \{g_{\lambda+\partial} h\}f$
5. $\{g_\lambda f\} = -{}_{\to}\{f_{\lambda+\partial} g\}$
6. $\{f_\lambda\{g_\mu h\}\} - \{g_\mu\{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu} h\}$

Let us explain the notation used in 4. and 5. Expand $\{f_\lambda g\} = \sum C_n \lambda^n$ with $C_n \in \mathcal{A}$. Then in each term of the RHS of equation 4 one has

$$\{f_{\lambda+\partial} g\}h := \sum_n C_n \left(\lambda + \partial\right)^n h$$

Notice that using this convention $\{f_{\lambda+\partial} g\} = \{f_{\lambda+\partial} g\}1 = \{f_\lambda g\}$. The RHS of the fifth equation is defined by

$${}_{\to}\{f_{-\lambda-\partial} g\} := \sum (-\lambda - \partial)^n C_n$$

The notion of a PVA, that can be seen as the semiclassical limit of Vertex Algebras [18], has been introduced in order to deal with evolutionary Hamiltonian PDEs in which the unknown functions depend on one spatial variable. It provides a good framework for the study of integrability of such a class of equations, and also gives some insights into the study of nonlocal Poisson structures [7]. The main theorem, on which all the theory is based, is that from a λ-bracket of a PVA we can get the Poisson bracket between local functionals as

$$\left\{\int f, \int g\right\} = \int \{f_\lambda g\}\big|_{\lambda=0}$$

and, conversely, given a Poisson structure as a differential operator we can define a λ-bracket between the generators of the algebra as the symbol of the differential operator; its extension to the full algebra is directly achieved by using the so called *master formula*.

In order to apply the same framework to Hamiltonian system with d spatial variables we have extended the definition of a PVA introducing so-called *multidimensional Poisson Vertex Algebras*. For a suitable $\mathcal{A}$, modelled on the algebra of differential polynomials of several variables, we show that the same axioms of a standard PVA, conveniently rephrased, can be used to characterize Poisson structures on this more general space of maps. In the paper, we apply the formalism of multidimensional PVAs to the d-dimensional Poisson brackets of hydrodynamic type. As an illustrative example, we obtain a new derivation of the set of necessary and sufficient conditions for a homogeneous differential operator of order 1 to be an Hamiltonian structure described by Mokhov in the late '80s [21]. Moreover, we apply the technique of multidimensional PVAs to the study of deformations of such structures.

1.2 Poisson brackets of hydrodynamic type and their deformations

The notion of Poisson bracket of hydrodynamic type has been introduced in the early 1980s by Dubrovin and Novikov [11] in order to characterize the Hamiltonian structure of a class of equations that describe systems such as ideal fluids with internal degrees of freedom. Poisson brackets (also called Poisson structures or Hamiltonian operators by different authors) of this form can be used to describe, for instance, Euler's equation [22]. The Hamiltonian structure is, in these cases, given in term of a first order differential operator

$$P^{ij}(u(x)) = g^{ij}(u(x))\frac{\mathrm{d}}{\mathrm{d}x} + b^{ij}_k(u)u^k_x \tag{1.1}$$

for the maps $u^i(x) \in \mathcal{M}\colon S^1 \to M$. P is a well defined skewsymmetric operator if and only if the functions $g^{ij}(u)$ and $b^{ij}_k(u)$ are respectively the components of a (pseudo)Riemannian contravariant metric and the contravariant Christoffel symbols of a compatible connection. Finally, P is an Hamiltonian operator if and only if the connection has no torsion and the metric is flat. This means that there exists a coordinate system $\{u^i\}$ in which the Poisson structure is constant, the so-called *flat coordinates*. Hydrodynamic Poisson brackets for maps $u(\mathbf{x})$, $\mathbf{x} = \{x^1, \ldots, x^d\}$ have been studied for several years ([12], [21]), and it has been proved that there does not always exist a flat coordinate system, so they are essentially noncostant. Under certain nondegeneracy assumptions there exist coordinate systems in which they are at most linear [12]. The Lie–Poisson bracket of hydrodynamic type, namely the one associated to the Lie algebra of vector fields on a compact manifold [22], is linear. It has the form

$$P_{ij}(p(\mathbf{x})) = p_i(\mathbf{x})\frac{\partial}{\partial x^j} + p_j(\mathbf{x})\frac{\partial}{\partial x^i} + \frac{\partial p_j(\mathbf{x})}{\partial x^i} \tag{1.2}$$

for $p\colon M \to \Omega^1(M)$, $\dim M = d$.

There exist Hamiltonian operators on the space $\mathcal{M}$ which are of order greater than one: one of the first examples to be discovered and probably the most celebrated one is the second Hamiltonian structure of KdV equation [20]

$$P_2(u(x)) = \frac{\mathrm{d}^3}{\mathrm{d}x^3} + 4u\frac{\mathrm{d}}{\mathrm{d}x} + 2u_x$$

An analogue of the Darboux theorem for the Hamiltonian operators in the infinite dimensional manifold $\mathcal{M}$ has been proved independently by several authors ([17], [9], [13]). In [13] and [9], this problem takes the name of *triviality problem.*

Given a Hamiltonian operator P_0, $[P_0, P_0] = 0$, we define a deformation P_ϵ of this operator as a formal infinite sum

$$P_\epsilon = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \cdots$$

such that the Schouten relation is satisfied:

$$[P_\epsilon, P_\epsilon] = 0.$$

We say that such a deformation is trivial if there exists a general Miura type transform $\phi_\epsilon = \sum_{k=0}^{\infty} \epsilon^k \phi_k$ of the map space such that $\phi_{\epsilon *} P_0 = P_\epsilon$ and ϕ_k is homogeneous of degree k.

It must be stressed that this result holds for the one-dimensional case. To our knowledge, there do not exist analogous results even in dimension two. The research work by E. Ferapontov and collaborators has produced a big outcome in the direction of classifying the integrable Hamiltonian equations of hydrodynamic type with d spatial variables and their deformations (for instance, in [15], [14]); analogous results in the direction of deformations of the Poisson structure itself are not available. All the results stated in [4] are derived for a one dimensional Hamiltonian operator (in the original language, for a differential algebra with one derivation); since we want to deal with higher dimensional operators, in Section 2 we first extend the definitions and the theorems to the multidimensional Poisson Vertex Algebras, where the algebra $\mathcal{A}$ is endowed with d commuting derivations. It can model the space of maps in several variables we are interested in, which is called an *algebra of differential polynomials*. We give here the basic definition and the main theorem on which we rely for the applications. The details and all the proofs are in Section 2.3.

Definition 1 (Multidimensional PVAs)**.** A d-dimensional PVA is a differential algebra $\mathcal{A}$ endowed with d commuting derivation ∂_α, $\alpha = 1, \dots, d$ and with a bilinear operation $\{\cdot_{\boldsymbol{\lambda}}\cdot\}\colon \mathcal{A}\times\mathcal{A} \to \mathbb{R}[\lambda_1, \dots, \lambda_d]\otimes\mathcal{A}$ called a λ-bracket of rank d. The λ-bracket of a multidimensional PVA satisfies the following set of properties

1. $\{\partial_\alpha f_{\boldsymbol{\lambda}} g\} = -\lambda_\alpha \{f_{\boldsymbol{\lambda}} g\}$
2. $\{f_{\boldsymbol{\lambda}} \partial_\alpha g\} = (\lambda_\alpha + \partial_\alpha)\{f_{\boldsymbol{\lambda}} g\}$
3. $\{f_{\boldsymbol{\lambda}} gh\} = \{f_{\boldsymbol{\lambda}} g\}h + \{f_{\boldsymbol{\lambda}} h\}g$
4. $\{fg_{\boldsymbol{\lambda}} h\} = \{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}} h\}g + \{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}} g\}h$
5. $\{g_{\boldsymbol{\lambda}} f\} = -_{\to}\{f_{-\boldsymbol{\lambda}-\boldsymbol{\partial}} g\}$
6. $\{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}} h\}\} - \{g_{\boldsymbol{\mu}}\{f_{\boldsymbol{\lambda}} h\}\} = \{\{f_{\boldsymbol{\lambda}} g\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}} h\}$

The λ-bracket between two elements of $\mathcal{A}$ can be expanded as

$$\{f_{\boldsymbol{\lambda}} g\} = \sum_{n_1,\dots,n_d\in\mathbb{Z}_{\geq 0}} C_{n_1\dots n_d}\lambda_1^{n_1}\cdots\lambda_d^{n_d} := \sum_{N\in\mathbb{Z}^d_{\geq 0}} C_N \boldsymbol{\lambda}^N$$

with $C_N \in \mathcal{A}$. According to this decomposition, the RHS of the fourth property expands to

$$\{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}} g\}h = \sum_{N\in\mathbb{Z}^d_{\geq 0}} C_N(\boldsymbol{\lambda}+\boldsymbol{\partial})^N h := \sum_{n_1,\dots,n_d\in\mathbb{Z}_{\geq 0}} C_{n_1\dots n_d}\left(\prod_{\alpha=1}^{d}(\lambda_\alpha+\partial_\alpha)^{n_\alpha}\right) h.$$

The RHS of the fifth property is given by

$$_{\to}\{f_{-\boldsymbol{\lambda}-\boldsymbol{\partial}}g\} := \sum_{N\in\mathbb{Z}^d_{\geq 0}} (-\boldsymbol{\lambda}-\boldsymbol{\partial})^N C_N = \sum_{n_1,\dots,n_d\in\mathbb{Z}_{\geq 0}} \prod_{\alpha=1}^{d} (-\lambda_\alpha - \partial_\alpha)^{n_\alpha} C_{n_1\dots n_d}.$$

For each PVA with $\mathcal{A}$ an algebra of differential polynomials, we can define a Lie bracket among the local functionals with density $\mathcal{A}$ (Theorem 2). We have

$$\left\{\int f, \int g\right\} := \int \{f_{\boldsymbol{\lambda}}g\}|_{\boldsymbol{\lambda}=\mathbf{0}}. \tag{1.3}$$

Such a Lie bracket is exactly what is called a Poisson structure. We prove that there is a one-to-one correspondence between Poisson structures on local functionals and λ-brackets on their densities.

Arnold described the relation between the Lie algebra of vector fields on a manifold and the Euler's equations of motion for incompressible fluids [2], together with their Hamiltonian formulation. Novikov explicitly introduced a Poisson bracket for the system, which is the Lie–Poisson bracket of the algebra of vector fields (1.2) [22]. In Section 3 we consider the first order deformations of the Lie–Poisson structure (1.2), whose equivalent λ-bracket is

$$\{p_{i\boldsymbol{\lambda}}p_j\} = -\left(p_i\lambda_j + p_j\lambda_i + \partial_i p_j\right).$$

We give the set of necessary and sufficient conditions for a generic second order differential operator to be a deformation of the Lie–Poisson hydrodynamic bracket (Theorem 5). We consider in particular the case $d = 2$. The Lie–Poisson hydrodynamic bracket belongs to the class of Hamiltonian structures that Ferapontov calls of type III [15]; in fact, they are the normal forms of such structures. The main result in this paper is stated in Theorem 5. Any second order homogeneous differential operator compatible with the Lie–Poisson bracket of the vector fields on a 2-torus is a trivial deformation of the Lie–Poisson bracket itself.

2 Multidimensional Poisson Vertex Algebras

In this section we want to extend the notion of a Poisson Vertex Algebra [4] in the spirit of the extension of the more general structure of a Lie conformal algebra into the category of Lie pseudoalgebras [3]. Our ultimate aim is to identify Poisson brackets between local functionals on a space of maps with some λ-brackets, in the same way one can achieve this result for Poisson brackets on loop spaces.

2.1 Formal map space

Let M be a n-dimensional smooth manifold. We want to descrive a class of Poisson brackets on the space

$$\mathcal{M} = \mathrm{Maps}(\Sigma \to M)$$

where Σ is a compact d-dimensional smooth manifold. In order to avoid the problems arising from the integration, let us fix Σ to be $(S^1)^d = T^d$. Anyway, all the definitions and the theorems of this section are expressed and work at the formal level. It means that it is not important whether we are considering functions on the d-torus, or rapidly decaying functions on $\mathbb{R}^d$; in a way, we are just working in the general space in which integration by parts is allowed producing no boundary terms.

We describe such a space of maps according to the theory of formal variational calculus [16]. Our expositon is tightly related to the one of [13]. Let us define the formal map space $\mathcal{M}$ in terms of ring of functions on it.

Let $U \subset M$ be a chart on M with coordinates $(u^1, \dots, u^n)$ and denote $\mathcal{A} = \mathcal{A}(U)$ the space of polynomials in the independent variables u^i_I for $i = 1, \dots, n$ and $I \in \mathbb{Z}^d_+$ a multiindex (i.e., $I_\alpha = 1, 2, \dots$ with $\alpha = 1, \dots, d$)

$$f(x, u; u_I) := \sum_{m \geq 0} f_{i_1 I_1; \dots; i_m I_m}(x, u) u^{i_1}_{I_1} \dots u^{i_m}_{I_m} \tag{2.1}$$

with coefficients $f_{i_1 I_1; \dots; i_m I_m}(\mathbf{x}, u)$ smooth functions on $\Sigma \times M$. Such an expression is called a *differential polynomial*. The space $\mathcal{A}$, endowed with a family of operators

$$\begin{aligned} \partial_\alpha \colon \mathcal{A} &\to \mathcal{A} \\ f &\mapsto \frac{\partial f}{\partial x^\alpha} + u^i_{E_\alpha} \frac{\partial f}{\partial u^i} + u^i_{I+E_\alpha} \frac{\partial f}{\partial u^i_I} \end{aligned}$$

($\alpha = 1, \dots, d$ and $E_\alpha = (0, 0, \dots, \underbrace{1}_{\alpha}, 0, \dots, 0)$) satisfying the following commutation properties

$$[\partial_\alpha, \partial_\beta] = 0 \qquad \forall\, \alpha, \beta \tag{2.2a}$$

$$\left[\frac{\partial}{\partial u^i_I}, \partial_\alpha\right] = \frac{\partial}{\partial u^i_{I-E\alpha}} \qquad (= 0 \text{ if } I_\alpha = 0) \tag{2.2b}$$

$$\left[\frac{\partial}{\partial u^i_I}, \frac{\partial}{\partial u^j_J}\right] = 0 \qquad \forall\, (i, j, I, J) \tag{2.2c}$$

form what in [4] is called an *algebra of differential polynomials*.

Since we are interested in local (in the sense of [13]) structures on the space of maps, we do not have to take into account the explicit dependence on the points in Σ. This justifies the following definitions, where we will restrict ourselves to consider the space $\hat{\mathcal{A}} \subset \mathcal{A}$ of differential polynomials f that do not depend explicitly on x^α. The 'total derivatives' have thus the form

$$\partial_\alpha = \sum_{\substack{i=1,\dots,n \\ I \in \mathbb{Z}^d_{\geq 0}}} u^i_{I+E_\alpha} \frac{\partial}{\partial u^i_I} \tag{2.3}$$

and satisfy the same commutation relations as in (2.2).

Because of the lacking of dependence on the variables on Σ, we are allowed to identify the space of local functionals $\hat{\mathcal{F}}$ whose densities do not depend explicitly on x with the quotient $\hat{\mathcal{A}}/\sum_\alpha \partial_\alpha \hat{\mathcal{A}}$. The quotient operation is denoted $\int$.

Let us consider the space $\mathfrak{X}(\hat{\mathcal{A}})$ of vector fields on the formal space of maps. These are formal infinite sums

$$\xi = \sum_{I\in\mathbb{Z}^d_{\geq 0}} \xi^i_I(u_J)\frac{\partial}{\partial u^i_I} \tag{2.4}$$

with $\xi^i_I \in \hat{\mathcal{A}}$.

The derivative of a local functional $\int f \equiv \overline{f} \in \hat{\mathcal{F}}$ along a vector field ξ reads

$$\xi\overline{f} = \int \sum_{I\in\mathbb{Z}^d_{\geq 0}} \xi^i_I(u_J)\frac{\partial f}{\partial u^i_I}\mathrm{d}\mathbf{x} \tag{2.5}$$

while the Lie bracket between two of such vector fields is obtained in a straightforward way by composition of derivations $[\xi,\eta]f = \xi(\eta f) - \eta(\xi f)$. The total derivatives ∂_α can be regarded as vector fields with $\xi^i_I = u^i_{I+E_\alpha}$.

An evolutionary vector field is a derivation of $\hat{\mathcal{A}}$ which commutes with all the derivations ∂_α. A simple computation shows that the condition imposes $\xi^i_I = \partial_\alpha \xi^i_{I-E_\alpha}$. Applying this relation reculsively we get that an evolutionary vector field has form

$$\xi = \sum_{\substack{i=1,\dots,n\\ I\in\mathbb{Z}^d_{\geq 0}}} (\boldsymbol{\partial}^I X^i(u_J))\frac{\partial}{\partial u^i_I}. \tag{2.6}$$

We will adopt a multi-index notation and denote

$$\prod_{\alpha=1}^d (\partial_\alpha)^{I_\alpha} =: \boldsymbol{\partial}^I.$$

for $I \in \mathbb{Z}^d_{\geq 0}$. Analogue conventions will be adopted also for more general operators or expressions, always meaning that they must be regarded as “term by term” powers.

$\hat{\mathcal{A}}^n$, as a collection of n elements of $\hat{\mathcal{A}}$, can be regarded as a vector. We can introduce a symmetric bilinear pairing $\hat{\mathcal{A}}^n \times \hat{\mathcal{A}}^n \to \hat{\mathcal{F}}$ given by

$$(A,B) \mapsto \int \sum_{i=1}^n A_i \cdot B_i \tag{2.7}$$

and use it to identify $\hat{\mathcal{A}}^n$ with its dual space, namely the one-forms. The *variational derivative*, in this setting, is a map $\hat{\mathcal{F}} \to \hat{\mathcal{A}}^{n*}$. We can write $\delta\overline{f} = \frac{\delta\overline{f}}{\delta u^i}\delta u^i$, and each component is

$$\frac{\delta\overline{f}}{\delta u^i} := \sum_I (-\boldsymbol{\partial})^I \frac{\partial f}{\partial u^i_I}. \tag{2.8}$$

It is worthy noticing that we are giving as definition a formula which can actually be regarded as a proposition following by the rigorous construction of a *variational bicomplex*, which can be found for instance in [1].

Lemma 1. *In $\hat{\mathcal{F}}$, the variational derivative of a total derivative vanishes. Moreover, for any evolutionary vector field $\xi \in \mathfrak{X}(\mathcal{A})$ (2.6) and for any $f \in \hat{\mathcal{A}}$, we have*

$$\int \xi f = \int X^i \frac{\delta f}{\delta u^i}. \tag{2.9}$$

Proof. Applying the commutation rule (2.2b) to the definition we get

$$\frac{\delta}{\delta u^i}\partial_\alpha f = \sum_I \left((-1)^{|I|} \boldsymbol{\partial}^{I+E_\alpha} \frac{\partial f}{\partial u^i_I} + (-1)^{|I|} \boldsymbol{\partial}^I \frac{\partial f}{\partial u^i_{I-E_\alpha}} \right). \tag{2.10}$$

Now it is sufficient to impose $I' = I - E_\alpha$ in the second term to get the result

$$\sum_{I,I'} (-1)^{|I|} \boldsymbol{\partial}^{I+E_\alpha} \frac{\partial f}{\partial u^i_I} - (-1)^{|I'|} \boldsymbol{\partial}^{I'+E_\alpha} \frac{\partial f}{\partial u^i_{I'}} = 0. \tag{2.11}$$

In order to prove the second part of the lemma it is enough to recall the definition (2.6). Integrating by part one can discharge all the derivatives on $\frac{\partial f}{\partial u^i_I}$ with the correct sign and gets the definition of variational derivative (2.8). □

2.2 Poisson bivector and Poisson bracket

A bivector, in general, is an element of $\mathfrak{X}(\hat{\mathcal{A}})^{\wedge 2}$. In components, bivectors are written as infinite sums of expression of the form

$$\alpha = \frac{1}{2} \alpha^{i_1;i_2}_{I_1;I_2} \left(u(x_1), u(x_2); \dots\right) \frac{\partial}{\partial u^{i_1}_{I_1}(x_1)} \wedge \frac{\partial}{\partial u^{i_2}_{I_2}(x_2)} \tag{2.12}$$

antisymmetric with respect to simultaneous exchange of $i_1, I_1, x_1 \leftrightarrow i_2, I_2, x_2$. The Lie bracket between vector fields and the wedge product allow us to define the *Schouten-Nijenhuis* bracket between k-vectors. It is defined by extending the Lie bracket of vector fields with respect to the product, imposing the Leibniz rule $[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(k-1)l} \beta \wedge [\alpha, \gamma]$ if $\alpha \in \mathfrak{X}^k$ and $\beta \in \mathfrak{X}^l$. It turns out that the Schouten-Nijenhuis bracket is the unique extension of Lie bracket of vector fields which turns the graded algebra of k-vector fields into a Gerstenhaber algebra. The formulas for the Schouten-Nijenhuis bracket between generic k- and l- vectors are in general quite involved. We give the formula for the bracket of a bivector with a vector fields, which is equivalent to the Lie derivative. Given a vector field $\xi = \xi^i_I(x, u(x), \dots)\partial/\partial u^i_I$ and β a bivector of form (2.12) we get

$$\mathcal{L}_\xi \beta^{i_1,i_2}_{I_1,I_2} = \sum_{l,L} \xi^l_L(u_l) \frac{\partial}{\partial u^l_L(x_l)} \beta^{i_1,i_2}_{I_1 I_2} - \frac{\partial \xi^{i_1}_{I_1}}{\partial u^l_L(x_l)} \beta^{l,i_2}_{L,I_2} - \frac{\partial \xi^{i_2}_{I_2}}{\partial u^l_L(x_l)} \beta^{i_1,l}_{I_1,L}. \tag{2.13}$$

This formula is useful to define *translational invariant* k-vectors, which satisfy

$$\mathcal{L}_{\partial_\alpha}\beta = 0 \tag{2.14}$$

for all $\alpha = 1, \ldots, d$. The set of conditions (2.14) imposes relations among the components of the bivector analogue to the ones we have already given for evolutionary vector fields. Every translational invariant bivector β has coefficients

$$\beta^{i_1,i_2}_{I_1 I_2}(u_J(x_1), u_K(x_2)) = (\boldsymbol{\partial}_{x_1})^{I_1} (\boldsymbol{\partial}_{x_2})^{I_2} B^{i_1 i_2}(u_J(x_1), u_K(x_2)) \tag{2.15}$$

where the differential polynomials $B^{i_1 i_2}$ are antisymmetric in the simultaneous exchange of indices. We call such differential polynomials the *components* of the translational invariant bivector β.

A *local bivector* is a translational invariant bivector such that its dependence on x_1, x_2 is given by a distribution with the support on the diagonal $x_1 = x_2$, i.e.

$$B^{ij} = \sum_{P \in Z^d_{\geq 0}} C^{ij}_P(u_J(x)) \boldsymbol{\partial}^P_x \delta(x - y). \tag{2.16}$$

The delta functions and their derivatives are defined by the usual formulae

$$\int f(y)\delta(x-y)\mathrm{d}\mathbf{y} = f(x)$$

$$\int f(y)\boldsymbol{\partial}^I_x \delta(x-y)\mathrm{d}\mathbf{y} = \int f(y)(-\boldsymbol{\partial}_y)^I \delta(x-y)\mathrm{d}\mathbf{y} = \boldsymbol{\partial}^I f(x)$$

$$\int f(x_1, \ldots, x_k)\boldsymbol{\partial}^{I_2}_{x_1}\delta(x_1 - x_2)\ldots\boldsymbol{\partial}^{I_k}_{x_1}\delta(x_1 - x_k)\mathrm{d}\mathbf{x}_2\ldots\mathrm{d}\mathbf{x}_k =$$

$$= \boldsymbol{\partial}^{I_2}_{x_2}\ldots\boldsymbol{\partial}^{I_k}_{x_k} f(x_1, \ldots, x_k)\big|_{x_1 = x_2 = \cdots = x_k}.$$

Where we do not specify the variables on which the derivatives $\boldsymbol{\partial}$ act, it is meant that they act on the first ones. The value of a local bivector on two one-forms $\phi = \phi_i(x, \ldots)\delta u^i$ and $\psi = \psi_i(y, \ldots)\delta u^i$ is

$$\int \phi_i B^{ij}(x, \ldots)\boldsymbol{\partial}\psi_j \mathrm{d}\mathbf{x} \tag{2.17}$$

which easily follows from simply pairing the one-forms with the bivectors and integrating in such a way that the derivatives of Dirac's delta act on the second one form. The antisymmetry of the bivector, namely $(B\phi, \psi) = -(B\psi, \phi)$ imposes on the components

$$B^{ji}_S = \sum_{T \in \mathbb{Z}^d_{\geq 0}} (-1)^{|T|+1} \binom{T}{S} \boldsymbol{\partial}^{T-S} B^{ij}_T \tag{2.18}$$

where we denote $|T| = \sum_\alpha T_\alpha$ and use the binomial coefficient for multi-indices

$$\binom{A}{B} = \binom{a_1}{b_1} \cdots \binom{a_d}{b_d} \tag{2.19}$$

and

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!(a-b)!} & 0 \leq b \leq a \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

We will occasionally use also the multinomial coefficients

$$\binom{A}{B_1, \dots, B_n}, \qquad B_n = A - \textstyle\sum_{i=1}^{n-1} B_i$$

which definition is analogue to the one for binomial coefficients with multi-indices, given the usual multinomial coefficient

$$\binom{a}{b_1 \dots b_n} = \frac{a!}{b_1! b_2! \dots b_n!}. \tag{2.21}$$

From the componentwise expression (2.16) we see that each component can be interpreted as a differential operator acting on the Dirac's delta

$$B^{ij}(x, u(x), u(x)_I; \frac{\mathrm{d}}{\mathrm{d}x})\delta(x-y) \tag{2.22}$$

with

$$B^{ij}(x, u(x), u(x)_I; \frac{\mathrm{d}}{\mathrm{d}x}) = \sum B^{ij}_S \boldsymbol{\partial}^S.$$

A *local Poisson structure* is a local bivector $P \in \mathfrak{X}^{\wedge 2}(\mathcal{M})$ satisfying the Schouten relation $[P, P] = 0$.

Given a Poisson bivector it is possible to define a bilinear operation (that we will call a bracket) on the space of local densities $f \in \mathcal{A}$. It can be used to define a bracket on the space $\hat{\mathcal{F}}$ of local functionals, which is usually called the *Poisson bracket* of functionals. This name is somehow confusing since the Poisson bracket of functionals is not the bracket on a Poisson algebra; indeed, it fails to be a derivation, because of the lack of a product in the space of functionals.

Given a Poisson structure P, we first define the bracket on $\hat{\mathcal{A}}$ on the basis elements u^i; we will often refer to them as the *generators* of $\hat{\mathcal{A}}$. We have

$$\{u^i(x), u^j(y)\} = \sum_S P^{ij}_S \left(u(x), u_I(x)\right) \boldsymbol{\partial}^S \delta(x-y). \tag{2.23}$$

This definition extends to two generic densities $f, g \in \hat{\mathcal{A}}$ according to the formula

$$\{f(x), g(y)\} = \sum_{L,M} \frac{\partial f}{\partial u^i_L(x)} \frac{\partial g}{\partial u^j_M(y)} \boldsymbol{\partial}^L_x \boldsymbol{\partial}^M_y \{u^i(x), u^j(y)\}. \tag{2.24}$$

Such a bracket satisfies by definition the Leibniz rule, i.e. $\{f, gh\} = \{f, g\}h + g\{f, h\}$ and it is obviously bilinear. An important remark is that such a bracket does not satisfy neither the usual skewsymmetry property nor the Jacobi identity, thus it is not a Lie bracket and, *a fortiori*, not even a Poisson bracket. The reason why the two important properties do not hold is quite natural: we defined a Poisson bivector to be skewsymmetric in the sense (2.18), which means that the skewsymmetry makes sense only after the integration, i.e. on $\hat{\mathcal{F}}$. On $\hat{\mathcal{F}}$ Leibniz property does not hold, but we can give a genuine Lie bracket.

Definition 2 (Poisson bracket)**.** A Poisson bracket $\{,\}$ in $\hat{\mathcal{F}} = \hat{\mathcal{A}}/\mathrm{d}\hat{\mathcal{A}}$ is a bilinear operation

$$\begin{aligned} \{\cdot,\cdot\}\colon \hat{\mathcal{F}} \times \hat{\mathcal{F}} &\to \hat{\mathcal{F}} \\ \left(\int f, \int g\right) &\mapsto \left\{\int f, \int g\right\} \end{aligned}$$

satisfying the following two fundamental properties:

1. Skewsymmetry: $\{\int f, \int g\} = -\{\int g, \int f\}$
2. Jacobi identity: $\{\int f, \{\int g, \int h\}\} - \{\int g, \{\int f, \int h\}\} = \{\{\int f, \int g\}, \int h\}$

Applying the skewsymmetry property, Jacobi identity can also be written as the vanishing of the expression $\{\{\int f, \int g\}, \int h\} + \mathrm{cycl.}(f, g, h) = 0$ which is the usual way to write it.

Given a Poisson bivector P of form (2.16) satisfying $[P, P] = 0$, the Poisson bracket of two local functionals $\int f(u(x), u_I(x))\mathrm{d}\mathbf{x}$ and $\int g(u(y), u_I(y))\mathrm{d}\mathbf{y}$ is given by

$$\begin{aligned} \left\{\int f, \int g\right\} &= \int\int \{f(x), g(y)\}\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\ &= \int\int \sum_{L,M} \frac{\partial f}{\partial u^i_L(x)} \frac{\partial g}{\partial u^j_M(y)} \boldsymbol{\partial}^L_x \boldsymbol{\partial}^M_y \{u^i(x), u^j(y)\}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\ &= \int\int \frac{\delta f}{\delta u^i(x)} \frac{\delta g}{\delta u^j(y)} \{u^i(x), u^j(y)\}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\ &= \sum_{S\in\mathbb{Z}^d_{\geq 0}} \int \frac{\delta f}{\delta u^i(x)} P^{ij}_S(u(x), u_I(x))\boldsymbol{\partial}^S \frac{\delta g}{\delta u^j(x)}\mathrm{d}\mathbf{x} \end{aligned} \tag{2.25}$$

where the second equality is given by (2.24), the third one is obtained by integrating by parts and transferring the total derivatives $\boldsymbol{\partial}$ on the partial derivatives of f and g respectively and the fourth one by performing the integration with respect to $\mathbf{y}$ for the Dirac's delta.

We do not prove here that the Schouten condition for P is the crucial requirement for the bracket to be Lie, namely to satisfy Jacobi identity. Although it is possible to get this result with the Dirac's delta formalism – or even by regarding the Poisson bracket of densities as a distribution and evaluating it on test functions – we are going to shift our point of view and consider Barakat, De Sole and Kac's approach to Hamiltonian operators on a space of maps in terms of Poisson Vertex Algebras [4].

2.3 Poisson Vertex Algebras

Let $\hat{\mathcal{A}}$ be a differential algebra with d commuting derivations. Usually, we consider the algebra of differential polynomials or an extension thereof.

Definition 3 (λ-bracket)**.** A λ-bracket (of rank d) on $\hat{\mathcal{A}}$ is a $\mathbb{R}$-linear map

$$\begin{aligned}\{\cdot_{\boldsymbol{\lambda}}\cdot\}\colon \hat{\mathcal{A}}\times\hat{\mathcal{A}} &\to \mathbb{R}[\lambda_1,\dots,\lambda_d]\otimes\hat{\mathcal{A}}\\ (f,g)&\mapsto\{f_{\boldsymbol{\lambda}}g\}\end{aligned}$$

which is *sesquilinear*, namely

$$\{\partial_\alpha f_{\boldsymbol{\lambda}} g\} = -\lambda_\alpha\{f_{\boldsymbol{\lambda}}g\} \tag{2.26a}$$

$$\{f_{\boldsymbol{\lambda}}\partial_\alpha g\} = (\partial_\alpha+\lambda_\alpha)\{f_{\boldsymbol{\lambda}}g\} \tag{2.26b}$$

and obeys, respectively, the *right* and *left Leibniz rule*

$$\{f_{\boldsymbol{\lambda}}gh\} = \{f_{\boldsymbol{\lambda}}g\}h + \{f_{\boldsymbol{\lambda}}h\}g \tag{2.27a}$$

$$\{fg_{\boldsymbol{\lambda}}h\} = \{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}g + \{g_{\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}f \tag{2.27b}$$

The λ-bracket of two elements in $\hat{\mathcal{A}}$ is, in other words, a polynomial in $\lambda_1,\dots,\lambda_d$ (we will often refer to the collection of λ_α as $\boldsymbol{\lambda}$) with coefficients in $\hat{\mathcal{A}}$. In general, we can write $\{f_{\boldsymbol{\lambda}}g\} = A(f,g)_{i_1,\dots,i_d}\lambda_1^{i_1}\dots\lambda_d^{i_d}$ which, using the usual multiindex notation, is equivalent to writing $A(f,g)_I\boldsymbol{\lambda}^{\boldsymbol{I}}$. When, as in (2.27b), we write $\{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}}g\}$ it means that the λ product is $A(f,g)_I\,(\boldsymbol{\lambda}+\boldsymbol{\partial})^{\boldsymbol{I}}$, with the derivation acting on the right (if nothing is written on the right, it is equivalent to the derivatives acting on 1 and thus the only term not vanishing is $\boldsymbol{\lambda}^I$).

Definition 4 (Multidimensional Poisson Vertex Algebra)**.** A (d-dimensional) *Poisson Vertex Algebra* is a differential algebra $\hat{\mathcal{A}}$ endowed with a λ-bracket of rank d which is *skewsymmetric*

$$\{g_{\boldsymbol{\lambda}}f\} = -{}_{\to}\{f_{-\boldsymbol{\lambda}-\boldsymbol{\partial}}g\} \tag{2.28}$$

and satisfy the *PVA-Jacobi identity*

$$\{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}}h\}\} - \{g_{\boldsymbol{\mu}}\{f_{\boldsymbol{\lambda}}h\}\} = \{\{f_{\boldsymbol{\lambda}}g\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}}h\}. \tag{2.29}$$

The notation used in (2.28) means that the differential operators $(-\boldsymbol{\lambda}-\boldsymbol{\partial})$ must be regarded as acting on the coefficient of the bracket, too; namely ${}_{\to}\{f_{-\boldsymbol{\lambda}-\boldsymbol{\partial}}g\} = (-\boldsymbol{\lambda}-\boldsymbol{\partial})^I A(f,g)_I$.

Theorem 1 (Master formula)**.** *Let $\hat{\mathcal{A}}$ be the algebra of differential polynomials (or an extension thereof) as defined in Section 2.1. Given two elements $(f,g)\in\hat{\mathcal{A}}$, their λ bracket can be expressed in terms of the λ bracket between the so-called* generators *of $\hat{\mathcal{A}}$, $\{u^i\}_{i=1,\dots,n}$. We have*

$$\{f_{\boldsymbol{\lambda}}g\} = \sum_{\substack{i,j=1\dots,n\\ M,N\in\mathbb{Z}^d_{\geq 0}}} \frac{\partial g}{\partial u^j_N}(\boldsymbol{\lambda}+\boldsymbol{\partial})^N\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}}u^j\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\frac{\partial f}{\partial u^i_M}. \tag{2.30}$$

In particular, the skewsymmetry and the PVA-Jacobi property hold if and only if the same properties for the generators hold.

We give here only a sketch of the proof of the theorem. The complete – rather cumbersome – proof, which extends to the d-dimensional case the Theorem 1.15 of [4], will be presented in Section 2.4. Our aim is to prove that the master formula provides the unique bilinear opeation satisfying the properties of a PVA for any two elements of $\hat{\mathcal{A}}$. From sesquilinearity of the bracket between two generators we have that $\{u^i_M{}_{\boldsymbol{\lambda}} u^j_N\} = (\boldsymbol{\lambda}+\boldsymbol{\partial})^N(-\boldsymbol{\lambda})^M\{u^i{}_{\boldsymbol{\lambda}} u^j\}$. Moreover, from the right Leibniz property (2.27a) follows that $\{f_{\boldsymbol{\lambda}}\cdot\}$ is a derivation of $\hat{\mathcal{A}}$, thus it acts on g only by its derivatives $\frac{\partial g}{\partial u^j_N}$. We get

$$\{f_{\boldsymbol{\lambda}} g\} = \sum \{f_{\boldsymbol{\lambda}} u^j_N\}\frac{\partial g}{\partial u^j_N}. \tag{2.31}$$

Applying the sesquilinearity (2.26b) we thus obtain

$$\{f_{\boldsymbol{\lambda}} g\} = \sum \frac{\partial g}{\partial u^j_N}(\boldsymbol{\lambda}+\boldsymbol{\partial})^N\{f_{\boldsymbol{\lambda}} u^j\} \tag{2.32}$$

where the partial derivatives pf g have been put on the left to denote that the total derivatives in the parenthesis act only on the λ bracket itself, according to the right Leibniz rule (2.27a).

The way in which the derivatives of the first function f enter into the master formula, conversely, is dictated by the left Leibniz rule and the sesquilinearity for the first entry of the bracket. We have

$$\{f_{\boldsymbol{\lambda}} g\} = \sum \{u^i{}_{\boldsymbol{\lambda}+\boldsymbol{\partial}} g\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\frac{\partial f}{\partial u^i_M}. \tag{2.33}$$

Note that in (2.33) the total derivatives act also on the partial derivatives of f, as imposed by the left Leibniz rule (2.27b). One should then prove that the skewsymmetry and the PVA-Jacobi identity for the brackets between the generators are the only conditions needed for the corresponding properties between generic elements of $\hat{\mathcal{A}}$.

In Section 2.2 we have noticed that there is a remarkable difference between the bracket defined by the same Poisson bivector in the space of local densities and the one among local functionals. In short, while the former is not a Lie bracket but it is a derivation, the latter – despite being an actual Lie bracket – fails at being the bracket of a Poisson algebra. The main discovery which establishes a relation between the theory of Hamiltonian PDEs and Poisson Vertex Algebras has originally been proved in [4] for a PVA of rank 1, namely that the Poisson bracket (strictly speaking, the bracket defined by a Poisson bivector) among local densities is related to a λ bracket by the relation

$$\{f,g\} = \{f_{\boldsymbol{\lambda}} g\}\big|_{\boldsymbol{\lambda}=0} \qquad f,g\in\hat{\mathcal{A}}. \tag{2.34}$$

Its extension to the more general case we are dealing with is straightforward. This fact is summarized by the following

Theorem 2. *Let $\hat{\mathcal{A}}$ be an algebra of differential polynomials with a λ bracket and consider the bracket on $\hat{\mathcal{A}}$ defined in* (2.34)*. Then*

(a) The bracket (2.34) *induces a well-defined bracket on the quotient space $\hat{\mathcal{F}}$;*

(b) If the λ bracket satisfies the axioms of a PVA, then the induced bracket on $\hat{\mathcal{F}}$ is a Lie bracket.

Proof. Part (a). From the property of sesquilinearity we have that

$$\{f + \boldsymbol{\partial}^M h, g\} = \left(\{f_{\boldsymbol{\lambda}} g\} - \boldsymbol{\lambda}^M \{h_{\boldsymbol{\lambda}} g\}\right)\big|_{\boldsymbol{\lambda}=0} = \{f, g\} \tag{2.35}$$

$$\begin{aligned} \{f, g + \boldsymbol{\partial}^M h\} &= \left(\{f_{\boldsymbol{\lambda}} g\} + (\boldsymbol{\lambda} + \boldsymbol{\partial})^M \{f_{\boldsymbol{\lambda}} h\}\right)\big|_{\boldsymbol{\lambda}=0} \\ &= \{f, g\} + \boldsymbol{\partial}^M \{f, h\} \sim \{f, g\}. \end{aligned} \tag{2.36}$$

Part (b). The Jacobi property for the bracket follows immediately by setting $\boldsymbol{\lambda} = \boldsymbol{\mu} = 0$ in PVA-Jacobi, while the skewsymmetry is a consequence of the skewsymmetry for the λ-bracket. We have

$$\begin{aligned} \{g, f\} &= \{g_{\boldsymbol{\lambda}} f\}\big|_{\boldsymbol{\lambda}=0} \\ &= -_{\to}\{f_{-\boldsymbol{\lambda}-\boldsymbol{\partial}} g\}\big|_{\boldsymbol{\lambda}=0} \qquad \text{(skewsymmetry)} \\ &= -\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\{f_{-\boldsymbol{\lambda}} g\}\right)\big|_{\boldsymbol{\lambda}=0} \qquad \text{using (2.48)} \\ &= -\left(1 + \boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}} + \cdots\right)\{f_{-\boldsymbol{\lambda}} g\}\big|_{\boldsymbol{\lambda}=0} \\ &= -\{f, g\}. \end{aligned} \tag{2.37}$$

□

Conversely, given a Poisson bracket among local densities, the corresponding λ-bracket is its formal Fourier transform. The aim of this paragraph is to show that the Fourier transform of the bracket of local densities is indeed a λ-bracket, which satisfies the PVA axioms if and only if the bracket is defined by a local Poisson bivector. This result is very important because working with the λ-brackets we do not deal with differential operators on a quotient space, but with simple differential polynomials.

Definition 5 (Formal Fourier transform)**.** Given a $\hat{\mathcal{A}}$ valued formal distribution $D(\mathbf{x}, \mathbf{y})$ (with $\mathbf{x}, \mathbf{y} \in M$, $\dim M = d$), its *formal Fourier transform* is the linear map

$$D(\mathbf{x}, \mathbf{y}) \mapsto \int \mathrm{d}\mathbf{x} e^{\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})} D(\mathbf{x}, \mathbf{y}) =: \mathrm{F}D(\mathbf{y}, \boldsymbol{\lambda})$$

with values in $\hat{\mathcal{A}}[\lambda_1, \ldots, \lambda_d]$. It is equivalent to the one introduced, in a different context, by Kac and De Sole in [6]. The symbol of the integral $\int \mathrm{d}\mathbf{x}$ must be regarded as the quotient operator with respect to $\sum_\alpha \partial_{x^\alpha}$.

Lemma 2. *Let us consider a differential operator acting on a Dirac's delta*

$$P(u(\mathbf{x}), \boldsymbol{\partial}_x)\delta(\mathbf{x}-\mathbf{y}) = \sum_S P(u(\mathbf{x}))_S \boldsymbol{\partial}_x^S \delta(\mathbf{x}-\mathbf{y}).$$

Its formal Fourier transform is the symbol of the operator itself, namely

$$\sum_S P(u(\mathbf{x}))_S \boldsymbol{\lambda}^S. \tag{2.38}$$

Proof. Expanding the multiindex notation and keeping the sum implicit we have

$$\begin{aligned}
\mathrm{F}P(u(\mathbf{y}), \boldsymbol{\lambda}) &= \int e^{\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})} P_{s_1\ldots s_d}(u(\mathbf{y}))\partial_{y^1}^{s_1}\ldots\partial_{y^d}^{s_d}\delta(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{x} \\
&= \int e^{\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})} P_{s_1\ldots s_d}(u(\mathbf{y}))(-\partial_{x^1}^{s_1})\ldots(-\partial_{x^d}^{s_d})\delta(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{x}
\end{aligned}$$

integrating by parts

$$\begin{aligned}
&= \int \partial_{x^1}^{s_1}\ldots\partial_{x^d}^{s_d} e^{\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})} P_{s_1\ldots s_d}(u(\mathbf{y}))\delta(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{x} \\
&= \lambda_1^{s_1}\cdots\lambda_d^{s_d} P_{s_1\ldots s_d}(u(\mathbf{y}))
\end{aligned}$$

which, using the usual multiindex notation, is

$$= P_S(u(\mathbf{y}))\boldsymbol{\lambda}^S.$$

□

In order to prove our claim that the Fourier transform of a Poisson bracket of densities is a λ bracket, we proceed as follows: first, we will prove that the skewsymmetry and the Jacobi property of the bracket among the generators, i.e. the coordinate functions, imply the skewsymmetyry (2.28) and the PVA-Jacobi identity (2.29) for λ-bracket. Then we will compute the Fourier transform of the Poisson bracket between two generic densities and we will prove that it is expressed in terms of the Fourier transform of the bracket of generators by the master formula. Hence, the Fourier transform of the Poisson bracket is a λ-bracket.

The Poisson bracket of two coordinate functions $u^i(\mathbf{x})$ and $u^j(\mathbf{y})$ is given by

$$\{u^i(\mathbf{x}), u^j(\mathbf{y})\} = P^{ji}(u(\mathbf{y}))_S \boldsymbol{\partial}_y^S \delta(\mathbf{x}-\mathbf{y})$$

where $P_S^{ji}\boldsymbol{\partial}^S$ are the components of the Poisson bivector defining the bracket. From the lemma 2, its Fourier transform is

$$\{u^i_{\boldsymbol{\lambda}} u^j\}(\mathbf{y}) = P^{ji}(u(\mathbf{y}))_S \boldsymbol{\lambda}^S. \tag{2.39}$$

Lemma 3. *The Fourier transform* (2.39) *is skewsymmetric in the sense of* (2.28).

Proof. From the form of the Poisson brackets of generators we have that

$$\{u^i(\mathbf{x}), u^j(\mathbf{y})\} = P_S^{ji}(u(\mathbf{y}))\boldsymbol{\partial}_y^S\delta(\mathbf{x}-\mathbf{y})$$
$$\{u^j(\mathbf{y}), u^i(\mathbf{x})\} = P_S^{ij}(u(\mathbf{x}))\boldsymbol{\partial}_x^S\delta(\mathbf{y}-\mathbf{x}).$$

We recall the skewsymmetry relation of the Poisson bivector (2.18), which gives

$$P_S^{ji}(\mathbf{y}) = -\sum_T(-1)^{|T|}\binom{T}{S}\boldsymbol{\partial}^{T-S}P_S^{ij}(\mathbf{x}) \tag{2.40}$$

and apply it within the Fourier transform. We get

$$\begin{aligned}
\{u^i_{\boldsymbol{\lambda}}u^j\} &= \int e^{\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})}P_S^{ji}(u(\mathbf{y}))\boldsymbol{\partial}_y^S\delta(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{x}\\
&= -\int e^{\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})}(-1)^{|T|}\binom{T}{S}\boldsymbol{\partial}^{T-S}(P_S^{ij}(u(\mathbf{x})))(-\boldsymbol{\partial}_x^S)\delta(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{x}\\
&= -\int(-1)^{|T|}\binom{T}{S}\boldsymbol{\partial}_x^S\left[e^{\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})}\boldsymbol{\partial}_x^{T-S}P_T^{ij}(u(\mathbf{x}))\right]\delta(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{x}\\
&= -\int(-1)^{|T|}\binom{T}{S}\binom{S}{L}\boldsymbol{\lambda}^L\boldsymbol{\partial}^{S-L+T-S}P_T^{ij}(u(\mathbf{x}))\delta(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{x}\\
&= -(-\boldsymbol{\lambda}-\boldsymbol{\partial})^T P_T^{ij}(u(\mathbf{y}))\\
&= -_{\rightarrow}\{u^j_{-\boldsymbol{\lambda}-\boldsymbol{\partial}}u^i\}.
\end{aligned}$$

□

Lemma 4. *The Fourier transform* (2.39) *satisfies the PVA-Jacobi identity, namely*

$$\{u^i_{\boldsymbol{\lambda}}\{u^j_{\boldsymbol{\mu}}u^k\}\} - \{u^j_{\boldsymbol{\mu}}\{u^i_{\boldsymbol{\lambda}}u^k\}\} = \{\{u^i_{\boldsymbol{\lambda}}u^j\}_{\boldsymbol{\mu}+\boldsymbol{\lambda}}u^k\}.$$

Proof. Let us take the three generators $u^i(x)$, $u^j(y)$ and $u^k(z)$. For convenience, we drop the boldface typesetting to denote that the variables x, y, z are coordinates in $\mathbb{R}^d$. Let us consider the double Fourier-like transform

$$\int e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{u^i(x),\{u^j(y),u^k(z)\}\}\mathrm{d}x\mathrm{d}y \tag{2.41}$$

The first step is to expand the outer bracket , which gives

$$\begin{aligned}
&\int e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\left(\boldsymbol{\partial}_z^L\{u^i(x),u^l(z)\}\right)\frac{\partial\{u^j(y),u^k(z)\}}{\partial u^l_L}\mathrm{d}x\mathrm{d}y\\
&\qquad= \int e^{\boldsymbol{\lambda}\cdot(x-z)}\frac{\partial}{\partial u^l_L}\left(e^{\boldsymbol{\mu}\cdot(y-z)}\{u^j(y),u^k(z)\}\right)\left(\boldsymbol{\partial}_z^L\{u^i(x),u^l(z)\}\right)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

If we perform the integration with respect to y, which appears only in the first parenthesis, we get by definition the λ bracket (with parameter $\boldsymbol{\mu}$) of the two generators u^j and u^k. Thus, we have got the partial result

$$\int e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{u^i(x),\{u^j(y),u^k(z)\}\}\mathrm{d}x\mathrm{d}y \\ = \int e^{\boldsymbol{\lambda}\cdot(x-z)}\{u^i(x),\{u^j_{\boldsymbol{\mu}}u^k\}(z)\}\mathrm{d}x.$$

Let us for simplicity denote $\{u^j_{\boldsymbol{\mu}}u^k\}(z)=g(z)$. A step backwards in the computation brings us back to

$$\int e^{\boldsymbol{\lambda}\cdot(x-z)}\frac{\partial g(z)}{\partial u^l_L}\left(\boldsymbol{\partial}_z^L\{u^i(x),u^l(z)\}\right)\mathrm{d}x \\ = \binom{L}{T}\int \frac{\partial g(z)}{\partial u^l_L}e^{\boldsymbol{\lambda}\cdot(x-z)}\left(\boldsymbol{\partial}_z^T P^{li}_S(z)\right)\boldsymbol{\partial}_z^{L-T+S}\delta(x-z)\mathrm{d}x$$

where the second line is obtained by simply expanding the derivations of the bracket. By substituting as usual $\boldsymbol{\partial}_z\delta(x-z)$ with $(-\boldsymbol{\partial}_x)\delta(x-z)$ and integrating by parts we get

$$\binom{L}{T}\frac{\partial g(z)}{\partial u^l_L}\boldsymbol{\lambda}^{L-T+S}\left(\boldsymbol{\partial}_z^T P^{li}_S(z)\right) \\ = \frac{\partial g(z)}{\partial u^l_L}(\boldsymbol{\lambda}+\boldsymbol{\partial}_z)^T P^{li}_S(z)\boldsymbol{\lambda}^S \\ = \{u^i_{\boldsymbol{\lambda}}g\}$$

where the last equality is given by (2.32). Summarizing, we have

$$\int e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{u^i(x),\{u^j(y),u^k(z)\}\}\mathrm{d}x\mathrm{d}y = \{u^i_{\boldsymbol{\lambda}}\{u^j_{\boldsymbol{\mu}}u^k\}\}. \tag{2.42}$$

The second term for the Jacobi identity among three coordinate functions is the same with $u^i(x)$ replaced by $u^j(y)$. The same computations hold provided the switching, and this gives as second term of the Fourier transform of the Jacobi identity

$$\int e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{u^j(y),\{u^i(x),u^k(z)\}\}\mathrm{d}x\mathrm{d}y = \{u^j_{\boldsymbol{\mu}}\{u^j_{\boldsymbol{\lambda}}u^k\}\}. \tag{2.43}$$

The RHS term of the PVA-Jacobi identity is more complicated to achieve. As before, let us start from expanding the usual formula for the Poisson bracket

$$\int e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{\{u^i(x),u^j(y)\},u^k(z)\} \\ = \int e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\frac{\partial\{u^i(x),u^j(y)\}}{\partial u^l_L(y)}\boldsymbol{\partial}_y^L\{u^l(y),u^k(z)\}\mathrm{d}x\mathrm{d}y \\ = \int e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\frac{\partial\{u^i(x),u^j(y)\}}{\partial u^l_L(y)}\boldsymbol{\partial}_y^L\left(P^{kl}_M(z)\boldsymbol{\partial}_z^M\delta(y-z)\right)\mathrm{d}x\mathrm{d}y \\ = \int e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\frac{\partial\{u^i(x),u^j(y)\}}{\partial u^l_L(y)}P^{kl}_M(z)\boldsymbol{\partial}_y^L\boldsymbol{\partial}_z^M\delta(y-z)\mathrm{d}x\mathrm{d}y$$

The derivative with respect to y in the third does not act on $P^{kl}_M(z)$, so we could move it further. Moreover, for convenience we can trade $\boldsymbol{\partial}_y^L \boldsymbol{\partial}_z^M \delta(y-z)$ for $(-1)^{|L|}(-\boldsymbol{\partial}_y)^{M+L}\delta(y-z)$ exchanging two times the variables respect to which we derive the Dirac's delta. It allows us to integrate by parts the delta's derivatives, in order to get

$$
\begin{aligned}
&= (-1)^{|L|} \int \boldsymbol{\partial}_y^{L+M} \left(e^{\boldsymbol{\mu}\cdot(y-z)} \frac{\partial\{u^i(x), u^j(y)\}}{\partial u^l_L(y)} \right) e^{\boldsymbol{\lambda}\cdot(x-z)} P^{kl}_M(z)\delta(y-z)\mathrm{d}x\mathrm{d}y \\
&= (-1)^{|L|} \binom{L+M}{T} \int \boldsymbol{\partial}_y^T \left(\frac{\partial\{u^i(x), u^j(y)\}}{\partial u^l_L(y)} \right) \boldsymbol{\mu}^{L+M-T} e^{\boldsymbol{\mu}\cdot(y-z)} e^{\boldsymbol{\lambda}\cdot(x-z)} \cdot \\
&\qquad\qquad \cdot P^{kl}_M(z)\delta(y-z)\mathrm{d}x\mathrm{d}y \\
&= (-1)^{|L|} \binom{L+M}{T} \int \boldsymbol{\partial}_z^T \left(\frac{\partial\{u^i(x), u^j(z)\}}{\partial u^l_L(z)} \right) \boldsymbol{\mu}^{L+M-T} e^{\boldsymbol{\lambda}\cdot(x-z)} P^{kl}_M(z)\mathrm{d}x.
\end{aligned}
$$

From the form for $\{u^i(x), u^j(z)\}$ we see that the partial derivatives act only on the coefficients P^{ji}_N. So, we get that our expression is equal to

$$
\begin{aligned}
&(-1)^{|L|} \binom{L+M}{T} \boldsymbol{\mu}^{L+M-T} \cdot \\
&\qquad \cdot \int \boldsymbol{\partial}_z^T \left(\frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)} \boldsymbol{\partial}_z^N \delta(x-z) \right) e^{\boldsymbol{\lambda}\cdot(x-z)} P^{kl}_M(z)\mathrm{d}x
\end{aligned}
$$

Basically we repeat the computation applying the same rules for multiderivatives of product and the integration by parts of the Dirac's delta and we end, after the integration, with

$$
(-1)^{|L|} \binom{L+M}{T} \binom{T}{R} P^{kl}_M(z) \boldsymbol{\mu}^{L+M-T} \boldsymbol{\lambda}^{T-R+N} \boldsymbol{\partial}_z^R \frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)}.
$$

The rules for the product of binomials hold also in the multiindices case, since they are only a product of ordinary binomials. It means, by slightly abusing the notation, that

$$
\binom{A}{B}\binom{B}{C} = \frac{A!}{B!(A-B)!}\frac{B!}{C!(B-C)!} = \binom{A}{A-B, C}
$$

In our case, calling $L+M-T = Q$, we get

$$
\begin{aligned}
&(-1)^{|L|} \binom{L+M}{Q,R} P^{kl}_M(z) \boldsymbol{\mu}^Q \boldsymbol{\lambda}^{L+M-Q-R} \boldsymbol{\partial}^R \frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)} \boldsymbol{\lambda}^N \\
&\qquad\qquad = (-1)^{|L|} P^{kl}_M(z) (\boldsymbol{\lambda} + \boldsymbol{\mu} + \boldsymbol{\partial})^{L+M} \frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)} \boldsymbol{\lambda}^N.
\end{aligned}
$$

Finally, we can adsorb the sign in front of the expression, and we get

$$
P^{kl}_M(z)(\boldsymbol{\lambda} + \boldsymbol{\mu} + \boldsymbol{\partial})^M (-\boldsymbol{\lambda} - \boldsymbol{\mu} - \boldsymbol{\partial})^L \frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)} \boldsymbol{\lambda}^N
$$

which is clearly the expression in terms of (2.33) of $\{u^i_{\boldsymbol{\lambda}} u^j\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}} u^k\}$, the RHS of the PVA-Jacobi identity. We have finally proved that taking the double Fourier transform with respect to $e^{\boldsymbol{\lambda}\cdot(x-z)+\boldsymbol{\mu}\cdot(y-z)}$ of the Jacobi identity for the Poisson bracket of the generators gives the PVA-Jacobi identity among them. $\square$

To conclude this discussion, we want to show that taking the Fourier transform of the Poisson bracket between two densities gives a formula which coincides with the master formula for a λ-bracket. The computation is rather lengthy but in a sense straightforward. We want to compute

$$\int e^{\boldsymbol{\lambda}\cdot(x-y)}\{f(x), g(y)\}\mathrm{d}x.$$

We expand the Poisson bracket and get

$$\int e^{\boldsymbol{\lambda}\cdot(x-y)}\frac{\partial f(x)}{\partial u^i_M}\frac{\partial g(y)}{\partial u^j_N}\boldsymbol{\partial}_x^M\boldsymbol{\partial}_y^N\left(P^{ji}_S(y)\boldsymbol{\partial}_y^S\delta(x-y)\right)\mathrm{d}x.$$

The derivatives respect to x do not act on the coefficients P^{ji}_S because they depend on functions of y by definition. Inside the bracket, moreover, we can trade the derivatives respect to y with the ones respect to x exploiting the properties of Dirac's delta, thus obtaining

$$(-1)^{|M|}\int e^{\boldsymbol{\lambda}\cdot(x-y)}\frac{\partial f(x)}{\partial u^i_M}\frac{\partial g(y)}{\partial u^j_N}\boldsymbol{\partial}_y^N\left(P^{ji}_S(y)(-\boldsymbol{\partial}_x)^{M+S}\delta(x-y)\right)\mathrm{d}x.$$

Then we perform the derivatives $\boldsymbol{\partial}_y^N$ and use the same trick

$$(-1)^{|M|}\binom{N}{T}\int e^{\boldsymbol{\lambda}\cdot(x-y)}\frac{\partial f(x)}{\partial u^i_M}\frac{\partial g(y)}{\partial u^j_N}\boldsymbol{\partial}_y^T(P^{ji}_S(y)(-\boldsymbol{\partial}_x)^{N-T+M+S}\delta(x-y)\mathrm{d}x.$$

We integrate by parts and let the $\boldsymbol{\partial}_x$ act properly. Then, we can finally integrate the Dirac's delta and get

$$(-1)^{|M|}\frac{\partial g}{\partial u^j_N}\binom{N}{T}\binom{N-T+M+S}{R}\boldsymbol{\lambda}^{N-T+M+S-R}\boldsymbol{\partial}^T(P^{ji}_S)\boldsymbol{\partial}^R\frac{\partial f}{\partial u^i_M}.$$

By applying the Newton's binomial,

$$\begin{aligned}(-1)^{|M|}&\frac{\partial g}{\partial u^j_N}\binom{N}{T}\boldsymbol{\lambda}^{N-T}\boldsymbol{\partial}^T(P^{ji}_S)(\boldsymbol{\lambda}+\boldsymbol{\partial})^{M+S}\frac{\partial f}{\partial u^i_M}\\ &=(-1)^{|M|}\frac{\partial g}{\partial u^j_N}(\boldsymbol{\lambda}+\boldsymbol{\partial})^N(P^{ji}_S(\boldsymbol{\lambda}+\boldsymbol{\partial})^{M+S}\frac{\partial f}{\partial u^i_M})\\ &=\frac{\partial g}{\partial u^j_N}(\boldsymbol{\lambda}+\boldsymbol{\partial})^N(P^{ji}_S(\boldsymbol{\lambda}+\boldsymbol{\partial})^S(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\frac{\partial f}{\partial u^i_M}).\end{aligned}$$

Recalling the form of the λ-bracket between the generators, the last expression is

$$\frac{\partial g}{\partial u^j_N}(\boldsymbol{\lambda}+\boldsymbol{\partial})^N\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}}u^j\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\frac{\partial f}{\partial u^i_M}),$$

namely the master formula.

We have thus proved the following theorem

Theorem 3. *Given a local Poisson bivector P on the space of maps* $\mathrm{Map}(\Sigma, M) \cong \hat{\mathcal{A}}$*, the Fourier transform of the bracket induced by the bivector is the λ-bracket of a Poisson Vertex Algebra on $\hat{\mathcal{A}}$.*

$$\{f_{\boldsymbol{\lambda}}g\}(\mathbf{y}) := \int_\Sigma e^{\boldsymbol{\lambda}\cdot(\mathbf{x}-\mathbf{y})}\{f(\mathbf{x}), g(\mathbf{y})\}\mathrm{d}\mathbf{x}. \tag{2.44}$$

2.4 Proof of master formula (Theorem 1)

Proof of the sesquilinearity For the left sesquilinearity, we have

$$\begin{aligned}\{\partial_\alpha f_{\boldsymbol{\lambda}} g\} &= \sum\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}}g\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\left(\frac{\partial}{\partial u^i_M}\partial_\alpha f\right)\\ &= \sum\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}}g\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\left(\frac{\partial f}{\partial u^i_{M-E_\alpha}}+\partial_\alpha\frac{\partial f}{\partial u^i_M}\right)\\ &= \sum\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}}g\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^{M-E_\alpha}(-\lambda_\alpha-\partial_\alpha)\left(\frac{\partial f}{\partial u^i_{M-E_\alpha}}+\partial_\alpha\frac{\partial f}{\partial u^i_M}\right)\end{aligned}$$

splitting the sum into two pieces and setting $M-E_\alpha \rightsquigarrow M$ in the first one

$$\begin{aligned}&= \sum\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}}g\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\left((-\lambda_\alpha-\partial_\alpha)\frac{\partial f}{\partial u^i_M}+\partial_\alpha\frac{\partial f}{\partial u^i_M}\right)\\ &= -\lambda_\alpha\{f_{\boldsymbol{\lambda}}g\}.\end{aligned} \tag{2.45}$$

The proof for the right sesquilinearity is completely straightforward, namely

$$\begin{aligned}\{f_{\boldsymbol{\lambda}}\partial_\alpha g\} &= \sum\left(\frac{\partial}{\partial u^j_N}\partial_\alpha\ g\right)(\boldsymbol{\lambda}+\boldsymbol{\partial})^N\{f_{\boldsymbol{\lambda}}u^j\}\\ &= \sum\left(\frac{\partial g}{\partial u^j_{N-E_\alpha}}+\partial_\alpha\frac{\partial g}{\partial u^j_N}\right)(\boldsymbol{\lambda}+\boldsymbol{\partial})^N\{f_{\boldsymbol{\lambda}}u^j\}\end{aligned}$$

splitting the sum into two parts and with the same substitution as before

$$\begin{aligned}&= \sum\frac{\partial g}{\partial u^j_N}(\lambda_\alpha+\partial_\alpha)(\boldsymbol{\lambda}+\boldsymbol{\partial})^N\{f_{\boldsymbol{\lambda}}u^j\}+\left(\partial_\alpha\frac{\partial g}{\partial u^j_N}\right)(\boldsymbol{\lambda}+\boldsymbol{\partial})^N\{f_{\boldsymbol{\lambda}}u^j\}\\ &= (\lambda_\alpha+\partial_\alpha)\{f_{\boldsymbol{\lambda}}g\}.\end{aligned} \tag{2.46}$$

where the last equality holds since λ_α pops out from the first term and gathering the other two we end with ∂_α acting on the whole (2.32).

Proof of the Leibniz rules The master formula obviously satisfies the left and right Leibniz rule, since in a way we started from imposing them to the bracket. While the right Leibniz does not deserve any remark, we should notice that the derivations of (2.33) acting on the derivatives of the product give

$$\{fg_{\boldsymbol{\lambda}}h\} = \{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}(\boldsymbol{\lambda}+\boldsymbol{\partial})^M \left(\frac{\partial f}{\partial u^i_M}g + \frac{\partial g}{\partial u^i_M}f\right)$$

using the notation $\boldsymbol{\partial} = \boldsymbol{\partial}_f + \boldsymbol{\partial}_g$ to denote the arguments on which the derivative acts

$$\begin{aligned} &= \{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}_f+\boldsymbol{\partial}_g}h\}(\boldsymbol{\lambda}+\boldsymbol{\partial}_f+\boldsymbol{\partial}_g)^M \left(\frac{\partial f}{\partial u^i_M}g + \frac{\partial g}{\partial u^i_M}f\right) \\ &= \{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}_g}h\}g + \{g_{\boldsymbol{\lambda}+\boldsymbol{\partial}_f}h\}f \end{aligned} \tag{2.47}$$

Proof of the skewsymmetry We want to prove that skewsymmetry of the bracket, given the master formula, follows from the skewsymmetry of the bracket between the generators. First we need an useful formula for all the following computation. We will denote

$$\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}u\right) f(\boldsymbol{\lambda}) = f(\boldsymbol{\lambda}+\boldsymbol{\partial})u. \tag{2.48}$$

In words, we can say that the $\boldsymbol{\partial}$ in the exponent acts only on what is inside the parentheses while the shifting $\boldsymbol{\lambda} \rightsquigarrow \boldsymbol{\lambda}+\boldsymbol{\partial}$ is applied to all the $\boldsymbol{\lambda}$s which follow the expression. This notation is justified by the Taylor expansion of the exponential, which turns out to be equivalent to the RHS; the most important part is to always keep track of the terms on which the derivations are actually allowed to act.

We can prove a first lemma

$$\{u^j_{-\boldsymbol{\lambda}+\boldsymbol{\partial}}u^j\}f = -e^{-\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\left(\{u^i_{\boldsymbol{\lambda}}u^j\}f\right). \tag{2.49}$$

The derivative in the bracket of the LHS is meant to act only on f, so we can denote it as $\boldsymbol{\partial}_f$. Applying the skewsymmetry property for the generators we have

$$\{u^j_{\boldsymbol{\lambda}+\boldsymbol{\partial}_f}u^i\}f = -_{\rightarrow}\{u^i_{-(-\boldsymbol{\lambda}+\boldsymbol{\partial}_f)-\boldsymbol{\partial}_\lambda}u^j\}f \tag{2.50}$$

where we denoted with $\boldsymbol{\partial}_\lambda$ the derivative which acts on the bracket itself. Now, applying the (2.48) we get the result

$$\{u^j_{-\boldsymbol{\lambda}+\boldsymbol{\partial}_f}u^i\}f = -e^{-(\boldsymbol{\partial}_f+\boldsymbol{\partial}_\lambda)\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\left(\{u^i_{\boldsymbol{\lambda}}u^j\}f\right) = -e^{-\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\left(\{u^i_{\boldsymbol{\lambda}}u^j\}f\right) \tag{2.51}$$

where we dropped the subscripts from the derivative, because using the parentheses as prescribed there is no ambiguity. We can now prove the skewsymmetry of the full master formula. With the usual prescription (2.48) we have

$$
\begin{aligned}
{}_{\to}\{g_{-\boldsymbol{\lambda}-\boldsymbol{\partial}}f\} &= e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\lambda}}\left(\{g_{-\boldsymbol{\lambda}}f\}\right) \\
&= e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\left(\frac{\partial f}{\partial u^i_M}(-\boldsymbol{\lambda}+\boldsymbol{\partial})^M\{u^j_{-\boldsymbol{\lambda}+\boldsymbol{\partial}}u^i\}(\boldsymbol{\lambda}-\boldsymbol{\partial})^N\frac{\partial g}{\partial u^j_N}\right)
\end{aligned}
$$

carefully denoting on what the derivations act

$$
\begin{aligned}
&= e^{(\boldsymbol{\partial}_f+\boldsymbol{\partial}_\lambda+\boldsymbol{\partial}_g)\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\left(\frac{\partial f}{\partial u^i_M}(-\boldsymbol{\lambda}+\boldsymbol{\partial}_\lambda+\boldsymbol{\partial}_g)^M\{u^j_{-\boldsymbol{\lambda}+\boldsymbol{\partial}_g}u^i\}\cdot\right. \\
&\quad\left.(\boldsymbol{\lambda}-\boldsymbol{\partial}_g)^N\frac{\partial g}{\partial u^j_N}\right)
\end{aligned}
$$

and using the lemma

$$
\begin{aligned}
&= -e^{(\boldsymbol{\partial}_f+\boldsymbol{\partial}_\lambda+\boldsymbol{\partial}_g)\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\left(\frac{\partial f}{\partial u^i_M}(-\boldsymbol{\lambda}+\boldsymbol{\partial}_\lambda+\boldsymbol{\partial}_g)^M\cdot\right. \\
&\quad\left. e^{-(\boldsymbol{\partial}_\lambda+\boldsymbol{\partial}_g)\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\left(\{u^i_{\boldsymbol{\lambda}}u^j\}(\boldsymbol{\lambda}+\boldsymbol{\partial}_\lambda)^N\frac{\partial g}{\partial u^j_N}\right)\right).
\end{aligned}
\tag{2.52}
$$

Now we can simply apply the general rule and shift $\boldsymbol{\lambda}$ according to the exponentials. We get

$$
\begin{aligned}
&-\frac{\partial f}{\partial u^i_M}(-\boldsymbol{\lambda}-\boldsymbol{\partial}_f)^M e^{-(\boldsymbol{\partial}_\lambda+\boldsymbol{\partial}_g)\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\cdot \\
&\qquad\cdot\left(\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}_f+\boldsymbol{\partial}_\lambda+\boldsymbol{\partial}_g}u^j\}(\boldsymbol{\lambda}+2\boldsymbol{\partial}_\lambda+\boldsymbol{\partial}_f+\boldsymbol{\partial}_g)^N\frac{\partial g}{\partial u^j_N}\right) \\
&\qquad= -\frac{\partial f}{\partial u^i_M}(-\boldsymbol{\lambda}-\boldsymbol{\partial}_f)^M\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}_f}u^j\}(\boldsymbol{\lambda}+\boldsymbol{\partial}_\lambda+\boldsymbol{\partial}_f)^N\frac{\partial g}{\partial u^j_N}
\end{aligned}
\tag{2.53}
$$

where the derivatives acts on the terms denoted by their subscripts regardless the actual order. To avoid such a mess and get an easy to read formula, we can switch the ordering so that all the derivations acts towards right and we finally get

$$
\begin{aligned}
{}_{\to}\{g_{-\boldsymbol{\lambda}-\boldsymbol{\partial}}f\} &= -\frac{\partial g}{\partial u^j_N}(\boldsymbol{\lambda}+\boldsymbol{\partial})^N\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}}u^j\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\frac{\partial f}{\partial u^i_M} \\
&= -\{f_{\boldsymbol{\lambda}}g\}.
\end{aligned}
\tag{2.54}
$$

This result has been obtained by imposing since the beginning in (2.50) the skewsymmetry of the λ bracket between two generators and it basically relies on it. This is important because it allows us to use the master formula not only for the λ brackets which define a PVA, but also for the more generic ones which satisfy only the sesquilinearity and the Leibniz properties.

Proof of the PVA-Jacobi identity The final property to prove is the PVA-Jacobi identity for the bracket, provided that it holds for the generators. The first step we need to prove is that the property holds for generic u^i_I, u^j_J, u^k_K derivatives of the generators themselves. Let us consider

$$\begin{aligned}\{u^i_{I\boldsymbol{\lambda}}\{u^j_{J\boldsymbol{\mu}}u^k_K\}\} &= (-\boldsymbol{\lambda})^I\{u^i_{\boldsymbol{\lambda}}(-\boldsymbol{\mu})^J(\boldsymbol{\partial}+\boldsymbol{\mu})^K\{u^j_{\boldsymbol{\mu}}u^k\}\} \\ &= (-\boldsymbol{\lambda})^I(-\boldsymbol{\mu})^J\{u^i_{\boldsymbol{\lambda}}(\boldsymbol{\mu}+\boldsymbol{\partial})^K\{u^j_{\boldsymbol{\mu}}u^k\}\}\end{aligned} \tag{2.55}$$

The derivative on the second member of the bracket can be split by a straightforward generalization of binomial formula; then we apply at the same time the right sesquilinearity and the linearity of the bracket to get

$$\sum_L \binom{K}{L}(-\boldsymbol{\lambda})^I(-\boldsymbol{\mu})^J\boldsymbol{\mu}^L(\boldsymbol{\partial}+\boldsymbol{\lambda})^{K-L}\{u^i_{\boldsymbol{\lambda}}\{u^j_{\boldsymbol{\mu}}u^k\}\} = \\ (-\boldsymbol{\lambda})^I(-\boldsymbol{\mu})^J(\boldsymbol{\partial}+\boldsymbol{\lambda}+\boldsymbol{\mu})^K\{u^i_{\boldsymbol{\lambda}}\{u^j_{\boldsymbol{\mu}}u^k\}\} \tag{2.56}$$

We proceed in the same way for the second member of the PVA-Jacobi identity, since the only difference are the exchange of u^i_I with u^j_J and the simultaneous exchange of $\boldsymbol{\lambda}$ with $\boldsymbol{\mu}$. It gives the same coefficients in front of the second term of the PVA-Jacobi among the generators. The computation for the RHS member of the identity is slightly different, since we have

$$(\boldsymbol{\partial}+\boldsymbol{\lambda}+\boldsymbol{\mu})^K\{(-\boldsymbol{\lambda})^I(\boldsymbol{\partial}+\boldsymbol{\lambda})^J\{u^i_{\boldsymbol{\lambda}}u^j\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}}u^j\} = \\ \sum_L(-\boldsymbol{\lambda})^I(\boldsymbol{\partial}+\boldsymbol{\lambda}+\boldsymbol{\mu})^K\binom{J}{L}(\boldsymbol{\lambda})^L(-\boldsymbol{\lambda}-\boldsymbol{\mu})^{J-L}\{\{u^i_{\boldsymbol{\lambda}}u^j\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}}u^j\} = \\ (-\boldsymbol{\lambda})^I(\boldsymbol{\partial}+\boldsymbol{\lambda}+\boldsymbol{\mu})^K(-\boldsymbol{\mu})^J\{\{u^i_{\boldsymbol{\lambda}}u^j\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}}u^j\}. \tag{2.57}$$

The last equality is obtained by "reconstructing" backwards the Newton's binomial formula. Since all the three terms of the PVA-Jacobi identity for u^l_L can be expressed as the three terms of the identity for the generators on which the same differential operator acts, the fact that the property holds for the generators is sufficient to prove that it holds also on their derivatives. Thus, the final step in proving the master formula is checking that the PVA-Jacobi holds for any triple of differential functions (f, g, h).

We start from the first term

$$\{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}}h\}\}$$

We expand the inner bracket and apply the right Leibniz rule, getting

$$\sum_{k,K}\left\{f_{\boldsymbol{\lambda}}\frac{\partial h}{\partial u^k_K}\right\}\{g_{\boldsymbol{\mu}}u^k_K\} + \sum_{k,K}\frac{\partial h}{\partial u^k_K}\{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}}u^k_K\}\}.$$

Repeating the same computation for the first of the two terms we get

$$\sum_{k,l,K,L} \frac{\partial^2 h}{\partial u_K^k \partial u_L^l} \{f_{\boldsymbol{\lambda}} u_L^l\}\{g_{\boldsymbol{\mu}} u_K^k\}$$

which is the same, for the symmetry of the second derivative, as the first term we get computing $\{g_{\boldsymbol{\mu}}\{f_{\boldsymbol{\lambda}} h\}\}$. The computation for the second term is more cumbersome. First, we apply (2.33)

$$\sum_{k,K} \frac{\partial h}{\partial u_K^k} \{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}} u_K^k\}\} = \sum_{k,K,m,M} \frac{\partial h}{\partial u_K^k} \{u^m_{M\,\boldsymbol{\lambda}+\boldsymbol{\partial}}\{g_{\boldsymbol{\mu}} u_K^k\}\} \frac{\partial f}{\partial u_M^m}$$

and, using (2.48) twice, we end up with

$$\begin{aligned}
&= \sum_{\substack{k,m,n\\K,M,N}} \frac{\partial h}{\partial u_K^k} \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}} \frac{\partial f}{\partial u_M^m} \right) \left\{ u^m_{M\,\boldsymbol{\lambda}} \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}} \frac{\partial g}{\partial u_N^n} \right) \{u^n_{N\,\boldsymbol{\mu}} u_K^k\} \right\} \\
&= \sum_{\substack{k,m,n\\K,M,N}} \frac{\partial h}{\partial u_K^k} \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}} \frac{\partial f}{\partial u_M^m} \right) \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}} \frac{\partial g}{\partial u_N^n} \right) \{u^m_{M\,\boldsymbol{\lambda}}\{u^n_{N\,\boldsymbol{\mu}} u_K^k\}\} \\
&\quad + \frac{\partial h}{\partial u_K^k} \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}} \frac{\partial f}{\partial u_M^m} \right) \left\{ u^m_{M\,\boldsymbol{\lambda}} \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}} \frac{\partial g}{\partial u_N^n} \right) \right\} \{u^n_{N\,\boldsymbol{\mu}} u_K^k\}.
\end{aligned}$$

The expression in front of the first summand is apparently symmetric in the exchange $(f,\boldsymbol{\lambda}) \rightsquigarrow (g,\boldsymbol{\mu})$, so we expect that the computation of the second term for the PVA-Jacobi gives exactly

$$\begin{aligned}
&\sum_{\substack{k,m,n\\K,M,N}} \frac{\partial h}{\partial u_K^k} \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}} \frac{\partial f}{\partial u_M^m} \right) \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}} \frac{\partial g}{\partial u_N^n} \right) \cdot \\
&\qquad\qquad \cdot \left(\{u^m_{M\,\boldsymbol{\lambda}}\{u^n_{N\,\boldsymbol{\mu}} u_K^k\}\} - \{u^n_{N\,\boldsymbol{\mu}}\{u^m_{M\,\boldsymbol{\lambda}} u_K^k\}\} \right) \\
&= \sum_{\substack{k,m,n\\K,M,N}} \frac{\partial h}{\partial u_K^k} \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}} \frac{\partial f}{\partial u_M^m} \right) \left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}} \frac{\partial g}{\partial u_N^n} \right) \{\{u^m_{M\,\boldsymbol{\lambda}} u_N^n\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}} u_K^k\}
\end{aligned} \tag{2.58}$$

where we have taken into account that the PVA-Jacobi identity is satisfied by the derivatives of the generators, as we proved before. To expand the second summand we first consider the outer exponential and introduce as usual the notation $\boldsymbol{\partial}_f$ to stress that the derivation acts on $\partial f/\partial u$. We get

$$\frac{\partial h}{\partial u_K^k} \left\{ u^m_{M\,\boldsymbol{\lambda}+\boldsymbol{\partial}_f} \left(e^{\boldsymbol{\partial}_g \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}} \frac{\partial g}{\partial u_N^n} \right) \right\} \{u^n_{N\,\boldsymbol{\mu}} u_K^k\} \frac{\partial f}{\partial u_M^m}$$

where we can apply repeatedly the right sesquilinearity in the first bracket; formally, by expanding in series the second entry of the bracket, each $\boldsymbol{\partial}_g$ derivative with a certain power is replaced by $\boldsymbol{\lambda}+\boldsymbol{\partial}_f+\boldsymbol{\partial}_g$ with the same power outside the bracket, giving (collecting $\boldsymbol{\partial}_f$ and $\boldsymbol{\partial}_g$ into the only $\boldsymbol{\partial}$ and putting the bracket in the right order for the derivations to act)

$$\begin{aligned}\frac{\partial h}{\partial u_K^k} e^{(\boldsymbol{\lambda}+\boldsymbol{\partial})\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}} \{u^n_{N\,\boldsymbol{\mu}} u_K^k\} \left\{ u^m_{M\,\boldsymbol{\lambda}+\boldsymbol{\partial}} \frac{\partial g}{\partial u_N^n} \right\} \frac{\partial f}{\partial u_M^m} \\ = \frac{\partial h}{\partial u_K^k} \{u^n_{N\,\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}} u_K^k\} \left\{ u^m_{M\,\boldsymbol{\lambda}+\boldsymbol{\partial}} \frac{\partial g}{\partial u_N^n} \right\} \frac{\partial f}{\partial u_M^m}\end{aligned}$$

and, using (2.33) and (2.32) to rebuild a more compact form

$$= \{u^n_{N\,\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}} h\} \left\{ f_{\boldsymbol{\lambda}} \frac{\partial g}{\partial u_N^n} \right\}. \tag{2.59}$$

As we have already noticed, the second term of the PVA-Jacobi identity is obtained by the exchange of f with g and of $\boldsymbol{\lambda}$ with $\boldsymbol{\mu}$; so, the LHS of PVA-Jacobi identity for the master equation turns out to be

$$\begin{aligned}\sum_{\substack{m,n\\M,N}} \left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}} \frac{\partial f}{\partial u_M^m} \right) \left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}} \frac{\partial g}{\partial u_N^n} \right) \{\{u^m_{M\,\boldsymbol{\lambda}} u_N^n\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}} h\} \\ + \sum_{n,N} \{u^n_{N\,\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}} h\} \left\{ f_{\boldsymbol{\lambda}} \frac{\partial g}{\partial u_N^n} \right\} - \sum_{m,M} \{u^m_{M\,\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}} h\} \left\{ g_{\boldsymbol{\mu}} \frac{\partial f}{\partial u_M^m} \right\}.\end{aligned} \tag{2.60}$$

Now we consider the RHS. For convenience, we put $\boldsymbol{\lambda}+\boldsymbol{\mu}=\boldsymbol{\nu}$; we thus want to compute

$$\{\{f_{\boldsymbol{\lambda}} g\}_{\boldsymbol{\nu}} h\}.$$

Expanding the inner bracket and using the left Leibniz property we get

$$\sum_{n,N} \left\{ \frac{\partial g}{\partial u_N^n}{}_{\boldsymbol{\nu}+\boldsymbol{\partial}} h \right\} \{f_{\boldsymbol{\lambda}} u_N^n\} + \{\{f_{\boldsymbol{\lambda}} u_N^n\}_{\boldsymbol{\nu}+\boldsymbol{\partial}} h\} \frac{\partial g}{\partial u_N^n}$$

where for convenience the second term can be rewritten using the exponential convention and it gives

$$\sum_{n,N} \left\{ \frac{\partial g}{\partial u_N^n}{}_{\boldsymbol{\nu}+\boldsymbol{\partial}} h \right\} \{f_{\boldsymbol{\lambda}} u_N^n\} + \left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}} \frac{\partial g}{\partial u_N^n} \right) \{\{f_{\boldsymbol{\lambda}} u_N^n\}_{\boldsymbol{\nu}} h\}.$$

Again, the second term can be expanded giving as result

$$\sum_{n,N}\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}\frac{\partial g}{\partial u_N^n}\right)\{\{f_{\boldsymbol{\lambda}}u_N^n\}_{\boldsymbol{\nu}}h\}$$

$$=\sum_{\substack{n,m\\N,M}}\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}\frac{\partial g}{\partial u_N^n}\right)\left\{\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)\{u_M^m{}_{\boldsymbol{\lambda}}u_N^n\}_{\boldsymbol{\nu}}h\right\}$$

$$=\sum_{\substack{n,m\\N,M}}\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}\frac{\partial g}{\partial u_N^n}\right)\left\{\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)_{\boldsymbol{\nu}+\boldsymbol{\partial}}h\right\}\{u_M^m{}_{\boldsymbol{\lambda}}u_N^n\}$$
$$+\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}\frac{\partial g}{\partial u_N^n}\right)\{\{u_M^m{}_{\boldsymbol{\lambda}}u_N^n\}_{\boldsymbol{\nu}+\boldsymbol{\partial}}h\}\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right).$$

The second term can be written using once again the form (2.48), so that we have

$$=\sum_{\substack{n,m\\N,M}}\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}\frac{\partial g}{\partial u_N^n}\right)\left\{\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)_{\boldsymbol{\nu}+\boldsymbol{\partial}}h\right\}\{u_M^m{}_{\boldsymbol{\lambda}}u_N^n\}$$
$$+\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}\frac{\partial g}{\partial u_N^n}\right)\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)\{\{u_M^m{}_{\boldsymbol{\lambda}}u_N^n\}_{\boldsymbol{\nu}}h\}.$$

Concerning the first of the two terms, we recall that the derivation written inside of the bracket acts, as usual, only on the terms on the right, in particular it is not applied to $\partial g/\partial u$. With the usual convention for the first exponential, we must add $\boldsymbol{\partial}_g$ to each $\boldsymbol{\nu}$, thus giving

$$\sum_{\substack{n,m\\N,M}}\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}\frac{\partial g}{\partial u_N^n}\right)\left\{\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)_{\boldsymbol{\nu}+\boldsymbol{\partial}}h\right\}\{u_M^m{}_{\boldsymbol{\lambda}}u_N^n\}$$

$$=\sum_{\substack{n,m\\N,M}}\left\{\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)_{\boldsymbol{\nu}+\boldsymbol{\partial}+\boldsymbol{\partial}_g}h\right\}\frac{\partial g}{\partial u_N^n}\{u_M^m{}_{\boldsymbol{\lambda}}u_N^n\}$$

$$=\sum_{m,M}\left\{\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)_{\boldsymbol{\nu}+\boldsymbol{\partial}}h\right\}\{u_M^m{}_{\boldsymbol{\lambda}}g\}$$

where we dropped the explicit $\boldsymbol{\partial}_g$ in the last formula as usual. By sesquilinearity we can move the exponential outside the bracket and we get

$$\sum_{m,M}\left\{\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)_{\boldsymbol{\nu}+\boldsymbol{\partial}} h\right\}\{u_M^m{}_{\boldsymbol{\lambda}}g\}$$

$$= \sum_{m,M} e^{-(\boldsymbol{\nu}+\boldsymbol{\partial})\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\left\{\frac{\partial f}{\partial u_M^m}{}_{\boldsymbol{\nu}+\boldsymbol{\partial}}h\right\}\{u_M^m{}_{\boldsymbol{\lambda}}g\}$$

$$= \sum_{m,M}\left\{\frac{\partial f}{\partial u_M^m}{}_{\boldsymbol{\nu}+\boldsymbol{\partial}}h\right\}{}_{\rightarrow}\{u_M^m{}_{\boldsymbol{\lambda}-\boldsymbol{\nu}-\boldsymbol{\partial}}g\}$$

where we have put the arrow on the left of the last bracket because the derivation acts on the bracket, too (since it comes from a derivation which was put on the very left of the expression). Then, applying the skewsymmetry property to the second bracket we finally get

$$= -\sum_{m,M}\left\{\frac{\partial f}{\partial u_M^m}{}_{\boldsymbol{\nu}+\boldsymbol{\partial}}h\right\}\{g_{\boldsymbol{\nu}-\boldsymbol{\lambda}}u_M^m\}$$

$$= -\sum_{m,M}\left\{\frac{\partial f}{\partial u_M^m}{}_{\boldsymbol{\lambda}+\boldsymbol{\mu}+\boldsymbol{\partial}}h\right\}\{g_{\boldsymbol{\mu}}u_M^m\}.$$

Putting together the terms we have separately solved, the RHS of the identity is

$$\{\{f_{\boldsymbol{\lambda}}g\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}}h\} =$$

$$= \sum_{\substack{m,n\\M,N}}\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}\frac{\partial g}{\partial u_N^n}\right)\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\nu}}}e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)\{\{u_M^m{}_{\boldsymbol{\lambda}}u_N^n\}_{\boldsymbol{\nu}}\,h\}$$

$$+\sum_{n,N}\left\{\frac{\partial g}{\partial u_N^n}{}_{\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}}h\right\}\{f_{\boldsymbol{\lambda}}u_N^n\} - \sum_{m,M}\left\{\frac{\partial f}{\partial u_M^m}{}_{\boldsymbol{\lambda}+\boldsymbol{\mu}+\boldsymbol{\partial}}h\right\}\{g_{\boldsymbol{\mu}}u_M^m\}.$$

Recalling that $\boldsymbol{\mu} = \boldsymbol{\lambda} + \boldsymbol{\mu}$ in the first of the two summand it turns out that the RHS is equivalent to

$$\sum_{\substack{m,n\\M,N}}\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\mu}}}\frac{\partial g}{\partial u_N^n}\right)\left(e^{\boldsymbol{\partial}\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}}\frac{\partial f}{\partial u_M^m}\right)\left\{\{u_M^m{}_{\boldsymbol{\lambda}}u_N^n\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}}\,h\right\}$$

$$+\sum_{n,N}\left\{\frac{\partial g}{\partial u_N^n}{}_{\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}}h\right\}\{f_{\boldsymbol{\lambda}}u_N^n\} - \sum_{m,M}\left\{\frac{\partial f}{\partial u_M^m}{}_{\boldsymbol{\lambda}+\boldsymbol{\mu}+\boldsymbol{\partial}}h\right\}\{g_{\boldsymbol{\mu}}u_M^m\}. \quad (2.61)$$

We must compare (2.60) with (2.61). The first terms in the two expressions coincide; about the second and the third, we observe that the they can be rewritten. We get

$$
\begin{aligned}
\sum_{n,N}\{u^n_N{}_{\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}\left\{f_{\boldsymbol{\lambda}}\frac{\partial g}{\partial u^n_N}\right\} - \sum_{m,M}\{u^m_M{}_{\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}\left\{g_{\boldsymbol{\mu}}\frac{\partial f}{\partial u^m_M}\right\}& \\
= \sum_{\substack{m,n\\M,N}}\{u^n_N{}_{\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}\left(\frac{\partial^2 g}{\partial u^m_M\partial u^n_N}\{f_{\boldsymbol{\lambda}}u^m_M\}\right)& \\
- \{u^m_M{}_{\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}\left(\frac{\partial^2 f}{\partial u^n_N\partial u^m_M}\{g_{\boldsymbol{\lambda}}u^n_N\}\right)&
\end{aligned}
$$

for the ones in (2.60) and

$$
\begin{aligned}
\sum_{n,N}\left\{\frac{\partial g}{\partial u^n_N}{}_{\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}}h\right\}\{f_{\boldsymbol{\lambda}}u^n_N\} - \sum_{m,M}\left\{\frac{\partial f}{\partial u^m_M}{}_{\boldsymbol{\lambda}+\boldsymbol{\mu}+\boldsymbol{\partial}}h\right\}\{g_{\boldsymbol{\mu}}u^m_M\}& \\
= \sum_{\substack{m,n\\M,N}}\{u^m_M{}_{\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}\left(\frac{\partial^2 g}{\partial u^m_M\partial u^n_N}\{f_{\boldsymbol{\lambda}}u^n_N\}\right)& \\
- \{u^n_N{}_{\boldsymbol{\mu}+\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}\left(\frac{\partial^2 f}{\partial u^n_N\partial u^m_M}\{g_{\boldsymbol{\lambda}}u^m_M\}\right)&
\end{aligned}
$$

for the ones in (2.61); we get these two expansions using (2.33). Now, given the symmetry of the two expressions in (m, M) and (n, N) (the sums run for all the values of the four indices) it is apparent that all the terms in LHS and RHS are the same, and thus the PVA-Jacobi identity is proved.

This proves that the λ-bracket defined by the master formula is the λ-bracket of a PVA, provided that the bracket among the generators is the λ-bracket of a PVA itself.

2.5 Cohomology of Poisson Vertex Algebras

In Section 2.3 we have proved the correspondence between local Poisson bivectors and PVAs. It is well known [19] that from the Schouten relation $[P, P] = 0$ it follows that one can define a linear differential $\mathrm{d}_P = [P, \cdot]$ which is a coboundary operator, $\mathrm{d}_P^2 = 0$.

From the properties of the Schouten bracket it follows that $\mathrm{d}_P\colon \Lambda^k(M) \to \Lambda^{k+1}(M)$, both for the finite and the infinite dimensional setting (for the latter, see for instance [13]). One can define the cochain complex

$$
0 \to \Lambda^0(M) \xrightarrow{\mathrm{d}_P} \Lambda^1(M) \xrightarrow{\mathrm{d}_P} \Lambda^2(M) \xrightarrow{\mathrm{d}_P} \cdots
$$

and its cohomology, which is called the Poisson–Lichnerowicz cohomology of M.

It is quite natural to repeat the construction in the context of Poisson Vertex Algebras. All the details are exposed by De Sole and Kac for one-dimensional PVAs [8], but the definitions we are interested in are basically the same.

Definition 6 (Variational complex of a PVA)**.** Given an algebra of differential polynomials $\hat{\mathcal{A}}$, let us consider the free commutative superalgebra $\widetilde{\Omega}^\bullet(\hat{\mathcal{A}})$ over $\hat{\mathcal{A}}$ with odd generators δu^i_M, $i = 1, \dots, n$, $M \in \mathbb{Z}^d_{\geq 0}$. We define a grading on $\widetilde{\Omega}^\bullet(\hat{\mathcal{A}})$, imposing $\deg f = 0$ for $f \in \hat{\mathcal{A}}$ and $\deg \delta u^i_M = 1$, so that we can decompose $\widetilde{\Omega}^\bullet = \bigoplus_{k \in Z_{\geq 0}} \widetilde{\Omega}^k$. An odd derivation $\delta \colon \widetilde{\Omega}^k \to \widetilde{\Omega}^{k+1}$ is defined by

$$
\begin{aligned}
\delta f &:= \sum_{i,M} \frac{\partial f}{\partial u^i_M} \delta u^i_M & f \in \hat{\mathcal{A}} \\
\delta(\delta u^i_M) &= 0
\end{aligned}
$$

The derivation δ squares to 0, so it defines a complex $(\widetilde{\Omega}, \delta)$ which is called the *basic variational complex.* The total derivatives δ_α defined in (2.3) can be extended to $\widetilde{\Omega}$ by $\partial_\alpha \delta u^i_M = \delta u^i_{M+E_\alpha}$. One can easily prove that $[\partial_\alpha, \delta] = 0$ for any $\alpha = 1, \dots, d$. In such a way, we can define the reduced complex (Ω, δ), that is called the *variational complex.*

$$
\Omega = \oplus_{k \geq 0} \Omega^k, \qquad \Omega^k = \widetilde{\Omega}^k / \sum_\alpha \partial_\alpha \widetilde{\Omega}^k \tag{2.62}
$$

and δ is the induced differential between the quotient spaces.

The space $\Omega^0(\hat{\mathcal{A}})$ is immediately identified with $\hat{\mathcal{F}}$ the space of local functionals. The derivative δ is the proper variational derivative, since we can use $\delta u^i_M = \boldsymbol{\partial}^M \delta u^i$ and the quotient map to get exactly (2.8). esthe space of evolutionary vector fields and the space $\Omega^2(\hat{\mathcal{A}})$ is isomorphic to the space of local bivectors (see the proof in [4], which holds for all Ω^k and local k-vectors).

Let us consider an element $X \in \Omega^{k-1}(\hat{\mathcal{A}})$. According to the aforementioned result, we regard Ω^k as the space of local k-vector fields. Given a local bivector whose symbol is $\sum P_S \boldsymbol{\lambda}^S$, we extend to the multidimensional PVA the definition of [8] for the PVA differential and we get

$$
\begin{aligned}
(\mathrm{d}_P X)\big(F^0, \dots, F^k\big) = &\sum_{i=0}^k (-1)^{k+i} \int F^i \cdot P_S \boldsymbol{\partial}^S \frac{\delta}{\delta u} X(F^0, \overset{i}{\check{\dots}}, F^k) + \\
&+ \sum_{0 \leq i < j \leq k} (-1)^{k+i+j} X\Big(\frac{\delta}{\delta u} \int \big(F^i \cdot P_S \boldsymbol{\partial}^S F^j\big), F^0, \overset{i}{\check{\dots}} \overset{j}{\check{\dots}}, F^k\Big)
\end{aligned}
$$

The differential $\mathrm{d}_P \colon \Omega^{k-1} \to \Omega^k$ is in one-to-one correspondence with $[P, \cdot]$ the ordinary coboundary operator of the Poisson–Lichnerowicz cohomology, and it squares to 0 thanks to the PVA-Jacobi identity.

Exactly as in the classical case, one can then define the cohomology of the complex: the first cohomology group $H^1(\hat{\mathcal{A}}, \mathrm{d}_P)$ is identified with the symmetries of the Poisson bivector and the second cohomology group $H^2(\hat{\mathcal{A}}, \mathrm{d}_P)$ is identified with the Poisson structures compatible with the structure P. In the language of the λ-brackets, we say that $H^2(\hat{\mathcal{A}}, \mathrm{d}_{\{\cdot_{\boldsymbol{\lambda}}\cdot\}_0})$ is the space of the λ-brackets which are compatibles with $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_0$.

We introduce a grading on $\hat{\mathcal{A}}$. We simply define

$$\deg u^i_I = |I| := \sum_\alpha I_\alpha \qquad\qquad \deg f(u_i) = 0.$$

This means that $\deg \partial_\alpha f(u^i; u^i_I) = \deg f(u^i; u^i_I) + 1$. Moreover, we assign to $\boldsymbol{\lambda}^I$ the degree $|I|$. Any λ-bracket can be decomposed, according to this grading, into homogenous parts. A rigorous way to obtain such a decomposition is to introduce a formal indeterminate ϵ of degree -1 and rescale the bracket in such a way that $\sum_{k=-1}^{\infty} \epsilon^{k+1} \{u^i{}_{\boldsymbol{\lambda}} u^j\}^{[k]}$ is of degree 0; the reason why $\{u^i{}_{\boldsymbol{\lambda}} u^j\}^{[k]}$ is in fact of degree $k+1$ is that we want to keep consistency with the notation used in [13].

We can use this grading to decompose each cohomology group

$$H^k(\hat{\mathcal{A}}, \mathrm{d}_{\{\cdot_{\boldsymbol{\lambda}}\cdot\}_0}) = \bigoplus_n H^k_{[n]}(\hat{\mathcal{A}}, \mathrm{d}_{\{\cdot_{\boldsymbol{\lambda}}\cdot\}_0}).$$

We will call each $H^k_{[n]}$ the k-th cohomology group of n-th order.

3 Multidimensional Poisson brackets of hydrodynamic type

In this section we apply the formalism we have discussed in the previous one to the so-called multidimensional Poisson brackets of hydrodynamic type introduced by Dubrovin and Novikov in [12]. They are brackets on a space of maps $\Sigma \to M$ defined by a bivector whose components are differential operators of the first order linear with respect to the first derivatives of the maps; in terms of λ-bracket among the generators of $\hat{\mathcal{A}}$, they have the form

$$\{u^i{}_{\boldsymbol{\lambda}} u^j\} = \sum_{\alpha=1}^{d} g^{ij\alpha}(u)\lambda_\alpha + \sum_{\substack{\alpha=1\dots d\\ k=1\dots n}} b^{ij\alpha}_k(u)\partial_\alpha u^k. \tag{3.1}$$

Since the order of derivatives we will deal with is not very high, it is easier to switch back to a single-index notation, namely $\boldsymbol{\lambda}^{E_\alpha} = \lambda_\alpha$ and (for instance) $u_{E_\alpha+E_\beta} = \partial_{\alpha\beta} u$. Dubrovin and Novikov in [12] have found, generalizing the result for the one-dimensional case they discovered in [11], a set of necessary conditions for a differential operator of type (3.1) to define a Poisson bracket in the space of local functionals, provided g^α were nondegenerate. Some years later Mokhov [21] proved the complete set of axioms the collection of functions $(g^{ij\alpha}, b^{ij\alpha}_k)$ must fulfil. They are summarized in the following theorem

Theorem 4 ([21])**.** *Let P be a differential operator whose symbol is* (3.1)*. The bracket among local functionals of density f, g defined by*

$$\left\{\int f, \int g\right\} := \int \frac{\delta f}{\delta u^i} P^{ij} \frac{\delta g}{\delta u^j}$$

is a Poisson bracket – equivalently, (3.1) *is the λ-bracket of a Poisson Vertex Algebra – if and only if*

$$g^{ij\alpha} = g^{ji\alpha} \tag{3.2a}$$

$$\frac{\partial g^{ij\alpha}}{\partial u^k} = b_k^{ij\alpha} + b_k^{ji\alpha} \tag{3.2b}$$

$$\sum_{(\alpha,\beta)} \left(g^{ai\alpha} b_a^{jk\beta} - g^{aj\beta} b_a^{ik\alpha} \right) = 0 \tag{3.2c}$$

$$\sum_{(i,j,k)} \left(g^{ai\alpha} b_a^{jk\beta} - g^{aj\beta} b_a^{ik\alpha} \right) = 0 \tag{3.2d}$$

$$\sum_{(\alpha,\beta)} \left[g^{ai\alpha} \left(\frac{\partial b_a^{jk\beta}}{\partial u^r} - \frac{\partial b_r^{jk\beta}}{\partial u^a} \right) + b_a^{ij\alpha} b_r^{ak\beta} - b_a^{ik\alpha} b_r^{aj\beta} \right] = 0 \tag{3.2e}$$

$$g^{ai\beta} \frac{\partial b_r^{jk\alpha}}{\partial u^a} - b_a^{ij\beta} b_r^{ak\alpha} - b_a^{ik\beta} b_r^{ja\alpha} = g^{aj\alpha} \frac{\partial b_r^{ik\beta}}{\partial u^a} - b_a^{ja\alpha} b_r^{ak\beta} - b_a^{jk\alpha} b_r^{ia\beta} \tag{3.2f}$$

$$\begin{aligned} &\frac{\partial}{\partial u^s} \left[g^{ai\alpha} \left(\frac{\partial b_a^{jk\beta}}{\partial u^r} - \frac{\partial b_r^{jk\beta}}{\partial u^a} \right) + b_a^{ij\alpha} b_r^{ak\beta} - b_a^{ik\alpha} b_r^{aj\beta} \right] \\ &+ \frac{\partial}{\partial u^r} \left[g^{ai\beta} \left(\frac{\partial b_a^{jk\alpha}}{\partial u^s} - \frac{\partial b_s^{jk\alpha}}{\partial u^a} \right) + b_a^{ij\beta} b_s^{ak\alpha} - b_a^{ik\beta} b_s^{aj\alpha} \right] \\ &+ \sum_{(i,j,k)} \left[b_r^{ai\beta} \left(\frac{\partial b_s^{jk\alpha}}{\partial u^a} - \frac{\partial b_a^{jk\alpha}}{\partial u^s} \right) \right] + \sum_{(i,j,k)} \left[b_s^{ai\alpha} \left(\frac{\partial b_r^{jk\beta}}{\partial u^a} - \frac{\partial b_a^{jk\beta}}{\partial u^r} \right) \right] = 0 \end{aligned} \tag{3.2g}$$

The notation $\sum_{(a_1,a_2,\dots)}$ *used for instance in* (3.2c) *means the cyclic summation over the indices. Conditions* (3.2a) – (3.2b) *are equivalent to the skewsymmetry of the bracket, while the other ones are equivalent to the validity of the Jacobi identity.*

Proof. We explicitly impose the skewsymmetry condition (2.28) and the PVA-Jacobi identity (2.29) for the bracket (3.1) among three generators of $\hat{\mathcal{A}}$. The vanishing of the first degree terms in λ_α for (2.28) are the conditions (3.2a), while the vanishing of the coefficients of u_α^k are (3.2b). We then use the master formula to compute (2.29). It gives a degree 2 differential polynomial in the λ's and the μ's. The remaining conditions are the vanishing of the coefficients for, respectively, $\lambda_\alpha\lambda_\beta$, $\lambda_\alpha\mu_\beta$ (the coefficients for $\mu \leftrightarrow \lambda$ are equivalent, provided the skewsymmetry), $u_{\alpha\beta}^r$, $u_\alpha^r\lambda_\beta$, and $u_\alpha^r u_\beta^s$. □

3.1 Deformations of Lie–Poisson brackets of hydrodynamic type

We are interested in a particular class of Poisson brackets of hydrodynamic type, introduced by Novikov in [22].

We start from the Lie algebra $\mathfrak{g} = \mathfrak{X}(\Sigma)$ of the vector fields on a manifold, let us say a d-dimensional torus. It has been known for long time that this algebra is tightly related to the Euler's equation for ideal fluids ([2]). In some coordinates such vector fields can be written as $X(\mathbf{x}) = \sum X^i(\mathbf{x})\partial_i$, $i = 1 \ldots d$; the components of their commutator are $[X, Y]^i(\mathbf{x}) = \sum X^j(\mathbf{x})\partial_j Y^i(\mathbf{x}) - Y^j(\mathbf{x})\partial_j X^i(\mathbf{x})$. This implies that the structure functions of $\mathfrak{g}$ must have the form $C^i_{jk}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \delta^i_j \delta(\mathbf{z} - \mathbf{x})\partial_k \delta(\mathbf{y} - \mathbf{z}) - \delta^i_k \delta(\mathbf{y} - \mathbf{x})\partial_j \delta(\mathbf{z} - \mathbf{y})$. It is well known that, given a Lie algebra $\mathfrak{g}$, it is always possible to endow its dual space $\mathfrak{g}^*$ with a Poisson bracket called the *Lie–Poisson bracket*. In this setting, the coordinates on $\mathfrak{g}^*$ are a set of functions $p_i(\mathbf{x})$ such that

$$\int p_i(\mathbf{x}) v^i(\mathbf{x}) \mathrm{d}\mathbf{x}$$

behaves as a scalar under change of variables. Here, $v^i(\mathbf{x})$ are the components of a vector field. This means that $p_i(\mathbf{x})$ are densities of 1-forms. The Lie–Poisson bracket is linear in the coordinates and defined by the structure functions as

$$\{p_j(\mathbf{y}), p_k(\mathbf{z})\} = \int C^i_{jk}(\mathbf{x}, \mathbf{y}, \mathbf{z}) p_i(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{3.3}$$

In the same fashion as the general case we dealt with in the previous section, we define the λ-bracket for the generators of the differential polynomial algebra $C^\infty(p_i)[p_{iI}]$ as

$$\{p_{i\,\boldsymbol{\lambda}} p_j\} = -\left(p_i \lambda_j + p_j \lambda_i + \partial_i p_j\right). \tag{3.4}$$

We now regard the p's and their derivatives as independent variables, in the spirit of jet bundles and as we defined in Section 2.1, dropping their dependence on the points of Σ.

A general Miura type transformation is a change of coordinates in the space of generators of the PVA. Using the grading introduced in 2.5 we can define the *Miura group* as the group of transformations of form

$$\begin{aligned} & p_i \mapsto \tilde{p}_i = \sum_{k=0}^{\infty} \epsilon^k F_{i[k]}(p; p_I) \quad |I| \leq k \\ & F_{i[k]} \in \hat{\mathcal{A}}, \quad \deg F_{i[k]} = k \\ & \det\left(\frac{\partial F_{i[0]}(p)}{\partial p_j}\right) \neq 0. \end{aligned} \tag{3.5}$$

Definition 7. A *n-th order deformation* of a PVA $(\hat{\mathcal{A}}, \{\cdot_{\boldsymbol{\lambda}}\cdot\}_0)$ is a PVA defined by a deformed λ bracket

$$\{\cdot_{\boldsymbol{\lambda}}\cdot\} = \{\cdot_{\boldsymbol{\lambda}}\cdot\}_0 + \sum_{k=1}^{n} \epsilon^n \{\cdot_{\boldsymbol{\lambda}}\cdot\}_k \tag{3.6}$$

such that $\{\cdot_{\boldsymbol{\lambda}}\cdot\}$ is PVA-skewsymmetric and the PVA-Jacobi identity holds up to order n, namely

$$\{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}} h\}\} - \{g_{\boldsymbol{\mu}}\{f_{\boldsymbol{\lambda}} h\}\} - \{\{f_{\boldsymbol{\lambda}} g\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}} h\} = O(\epsilon^{n+1}).$$

Definition 8. A deformation is said to be *trivial* if there exists an element ϕ_ϵ of the group (3.5) which pulls back $\{\cdot_{\boldsymbol{\lambda}}\cdot\}$ to $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_0$,

$$\{\phi_\epsilon(a)_{\boldsymbol{\lambda}}\phi_\epsilon(b)\}_0 = \phi_\epsilon\left(\{a_{\boldsymbol{\lambda}}b\}\right), \qquad \forall a, b \in \hat{\mathcal{A}}.$$

In terms of PVA cohomology, a deformed bracket is trivial if it is a coboundary in $\Omega^2(\hat{\mathcal{A}}, \{\cdot_{\boldsymbol{\lambda}}\cdot\}_0)$.

In the rest of the paper, we investigate first order deformations of (3.4) — they are, in PVA formalism, the first order deformations of the Poisson bracket of hydrodynamic type giving the Hamiltonian structure of Euler's equation. We want to investigate whether deformations of the dispersionless Poisson brackets, different from the trivial ones, exist.

A first order deformation of (3.4) is a second degree homogeneous bracket. In general, such a bracket is of the form

$$\begin{aligned}\{p_i{}_{\boldsymbol{\lambda}}p_j\}_1 = A^{ab}_{ij}(p)\lambda_a\lambda_b + B^{a,bl}_{ij}(p)\partial_b p_l\lambda_a +& \\ + C^{al,bm}_{ij}(p)\partial_a p_l \partial_b p_m + D^{ab,l}(p)\partial_{ab}p_l& \end{aligned} \tag{3.7}$$

in which each index can take values between 1 and d and we adopt the Einstein convention for the sum over repeated indices; moreover, the commas in the upper indices are inserted just for the convenience of the reader, namely to distinguish the different symmetry properties of the indices. Here, A, B, C and D are arbitrary functions of the p's only. It should be apparent from the definition that A^{ab}_{ij} and $D^{ab,l}_{ij}$ are symmetric in the exchange of a and b while $C^{al,bm}_{ij}$ must be symmetric in the simultaneous exchange of (a, l) with (b, m). The formalism of the Poisson Vertex Algebras makes finding the conditions on A, B, C and D for the bracket $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_1$ to the first order deformation of (3.4) relatively simple, and anyhow straightforward. Applying it to the case $d = 2$, we proved the following

Theorem 5. *The first order second cohomology group for the Lie – Poisson Vertex Algebra associated with the algebra of the vector fields on a 2-torus, namely* (3.4)*, are trivial.*

In order to prove the theorem, we first need an important, albeit not very deep, lemma:

Lemma 5. *A homogeneous λ-bracket of degree 2 of form* (3.7) *is a first order deformation of the bracket* (3.4) *if and only if the following conditions hold:*

$$A^{ab}_{ij} = -A^{ab}_{ji} \tag{3.8a}$$

$$\frac{\partial A^{ab}_{ij}}{\partial p_l} = \frac{1}{2}\left(B^{a,bl}_{ij} - B^{a,bl}_{ji}\right) = \frac{1}{2}\left(B^{b,al}_{ij} - B^{b,al}_{ji}\right) \tag{3.8b}$$

$$B^{a,bl}_{ij} + B^{b,al}_{ji} = B^{b,al}_{ij} + B^{a,bl}_{ji} = 2D^{ab,l}_{ij} + 2D^{ab,l}_{ji} \tag{3.8c}$$

$$\frac{\partial B^{a,bm}_{ij}}{\partial p_l} + \frac{\partial B^{b,al}_{ji}}{\partial p_m} = \frac{\partial B^{b,al}_{ij}}{\partial p_m} + \frac{\partial B^{a,bm}_{ji}}{\partial p_l} = 2C^{al,bm}_{ij} + 2C^{al,bm}_{ji} \tag{3.8d}$$

This first set of equations gives the conditions for the skewsymmetry (in the sense of PVAs) of the bracket. Imposing the PVA-Jacobi identity up to the first order we get

$$D_{ji}^{ab,c}p_k + D_{jk}^{ab,c}p_i + \left(D_{ji}^{ab,l}\delta_k^c + D_{jk}^{ab,l}\delta_i^c\right)p_l + \circlearrowleft (a,b,c) = 0 \tag{3.8e}$$

$$\begin{aligned} &2\left(A_{ik}^{bc}\delta_j^a + A_{ik}^{ac}\delta_j^b\right) + 2\left(A_{ji}^{ab}\delta_k^c - A_{ik}^{ab}\delta_j^c + A_{kj}^{ab}\delta_i^c\right) + \\ &-\left(B_{ki}^{a,bc} - B_{ik}^{a,bc}\right)p_j - \left(B_{jk}^{c,ab} - B_{jk}^{c,ba}\right)p_i + \\ &-\left[\left(B_{ki}^{a,bl} - B_{ik}^{a,bl}\right)\delta_j^c + B_{jk}^{c,bl}\delta_i^a + B_{jk}^{c,al}\delta_i^b\right]p_l + \\ &-\left[\left(2D_{ji}^{bc,l} - B_{ji}^{c,bl}\right)\delta_k^a + \left(2D_{ji}^{ca,l} - B_{ji}^{c,al}\right)\delta_k^b + 2D_{ji}^{ab,l}\delta_k^c\right]p_l + \\ &-\left(2D_{ji}^{cb,a} - B_{ji}^{c,ba} + 2D_{ji}^{ca,b} - B_{ji}^{c,ab} + 2D_{ji}^{ab,c}\right)p_k = 0 \end{aligned} \tag{3.8f}$$

$$\begin{aligned} &\sum_{\sigma(a,b,c)}\left[\left(\frac{\partial D_{ij}^{ab,l}}{\partial p_m} + \frac{\partial D_{ij}^{ab,m}}{\partial p_l} - 2C_{ij}^{am,bl}\right)p_m\delta_k^c + \right. \\ &\left. + \left(\frac{\partial D_{ij}^{ab,l}}{\partial p_c} + \frac{\partial D_{ij}^{ab,c}}{\partial p_l} - 2C_{ij}^{ac,bl}\right)\right] = 0 \end{aligned} \tag{3.8g}$$

$$\begin{aligned} &\left(C_{ji}^{ba,cl} + C_{ji}^{ab,cl}\right)p_k - \left(C_{jk}^{ba,cl} + C_{jk}^{ab,cl}\right)p_i - \frac{\partial}{\partial p_l}\left(D_{ji}^{ab,c} + D_{ji}^{bc,a} + D_{ji}^{ca,b}\right)p_k + \\ &-\left(D_{jk}^{bc,a} + D_{jk}^{ac,b}\right)\delta_i^l - D_{ji}^{ab,c}\delta_k^l - \left(D_{jk}^{bc,l}\delta_i^a + D_{jk}^{ca,l}\delta_i^b + D_{jk}^{ab,l}\delta_i^c\right) + \\ &+\left[\left(C_{ji}^{bm,cl}\delta_k^a + C_{ji}^{am,cl}\delta_k^b\right) - \left(C_{jk}^{bm,cl}\delta_i^a + C_{jk}^{am,cl}\delta_i^b\right) + \right. \\ &\left. - \frac{\partial}{\partial p_l}\left(D_{ji}^{ab,m}\delta_k^c + D_{ji}^{bc,m}\delta_k^a + D_{ji}^{ca,m}\delta_k^b\right)\right]p_m = 0 \end{aligned} \tag{3.8h}$$

$$
\begin{aligned}
&B_{ik}^{b,cl}\delta_j^a - B_{jk}^{a,cl}\delta_i^b + B^{b,cl}\delta_k^a - B_{ij}^{a,cl}\delta_k^b + B_{jk}^{b,al}\delta_i^c + \\
&+B_{ik}^{a,cb}\delta_j^l - B_{jk}^{b,ca}\delta_i^l + \left(2D_{ij}^{ab,c} - B_{ij}^{a,bc}\right) + + \frac{\partial B_{ik}^{a,cl}}{\partial p_b} p_j - \frac{\partial B_{jk}^{b,cl}}{\partial p_a} p_i + \\
&+ \left[2C_{ji}^{ab,cl} - 2C_{ij}^{ba,cl} + 2\frac{\partial D_{ij}^{cb,a}}{\partial p_l} - 2\frac{\partial D_{ji}^{ca,b}}{\partial p_l} + \frac{\partial}{\partial p_l}\left(2D_{ij}^{ab,c} - B_{ij}^{a,bc}\right)\right] p_k + \\
&+ \left[\frac{\partial B_{ik}^{a,cl}}{\partial p_m}\delta_j^b - \frac{\partial B_{jk}^{b,cl}}{\partial p_m}\delta_i^a + 2\left(\frac{\partial D_{ij}^{cb,m}}{\partial p_l} - C_{ij}^{bm,cl}\right)\delta_k^a + \right. \\
&\left. -2\left(\frac{\partial D_{ji}^{ab,m}}{\partial p_l} - C_{ji}^{am,cl}\right)\delta_k^b + \frac{\partial}{\partial p_l}\left(2D_{ij}^{ab,m} - B_{ij}^{a,bm}\right)\delta_k^c\right] = 0
\end{aligned}
\tag{3.8i}
$$

$$
\begin{aligned}
&\left(C_{ji}^{ac,bl} + C_{ji}^{ab,cl} - \frac{\partial D_{ji}^{ab,c}}{\partial p_l} - \frac{\partial D_{ji}^{ac,b}}{\partial p_l} + \frac{\partial D_{ij}^{bc,a}}{\partial p_l} + \frac{\partial D_{ij}^{bc,l}}{\partial p_a} - C_{ij}^{ba,cl} - C_{ij}^{ca,bl}\right) p_k + \\
&-\frac{\partial D_{jk}^{bc,l}}{\partial p_a} p_i - D_{jk}^{bc,a}\delta_i^l - D_{jk}^{ab,l}\delta_i^c - D_{jk}^{ac,l}\delta_i^b + D_{ji}^{bc,l}\delta_k^a + \\
&+ \left[-\frac{\partial D_{jk}^{bc,l}}{p_m}\delta_i^a + \left(\frac{\partial D_{ij}^{bc,l}}{\partial p_m} - \frac{\partial D_{ij}^{bc,m}}{\partial p_l} - C_{ij}^{bm,cl} - C_{ij}^{cm,bl}\right)\delta_k^a + \right. \\
&\left. + \left(C_{ji}^{am,bl} - \frac{\partial D_{ji}^{ab,m}}{\partial p_l}\right)\delta_k^c + \left(C_{ji}^{am,cl} - \frac{\partial D_{ji}^{ac,m}}{\partial p_l}\right)\delta_k^b\right] p_m = 0
\end{aligned}
\tag{3.8j}
$$

$$
\begin{aligned}
&-2\frac{\partial}{\partial p_a}\left(C_{jk}^{bl,cm} p_i\right) + 2\frac{\partial C_{ij}^{bl,cm}}{\partial p_a} p_k + 2C_{ji}^{bl,cm}\delta_k^a \\
&+\frac{\partial}{\partial p_m}\left(2C_{ji}^{ac,bl} - 2C_{ij}^{ca,bl} + 2\frac{\partial D_{ij}^{bc,a}}{\partial p_l} - 2\frac{\partial D_{ji}^{ab,c}}{\partial p_l}\right) p_k + \\
&+\frac{\partial}{\partial p_l}\left(2C_{ji}^{ab,cm} - 2C_{ij}^{ba,cm} + 2\frac{\partial D_{ij}^{bc,a}}{\partial p_m} - 2\frac{\partial D_{ji}^{ac,b}}{\partial p_m}\right) p_k + \\
&-2C_{jk}^{bl,am}\delta_i^c - 2C_{jk}^{cm,al}\delta_i^b - 2C_{jk}^{bl,ca}\delta_i^m - 2C_{jk}^{cm,ba}\delta_i^l + \\
&+2\left(C_{ji}^{ac,bl} - \frac{\partial D_{ji}^{ab,c}}{\partial p_l}\right)\delta_k^m + 2\left(C_{ji}^{ab,cm} - \frac{\partial D_{ji}^{ac,b}}{\partial p_m}\right)\delta_k^l = 0
\end{aligned}
\tag{3.8k}
$$

$$
\begin{aligned}
&\sum_{\sigma(al,bm,cn)} \left[\frac{\partial}{\partial p_n}\left(\frac{\partial C_{ij}^{al,bm}}{\partial p_c} p_k - 2\frac{\partial C_{ij}^{ac,bm}}{\partial p_l} p_k + \frac{\partial^2 D_{ij}^{ab,c}}{\partial p_m \partial p_l} p_k\right) + \right. \\
&\left. +p_s\frac{\partial}{\partial p_n}\left(\frac{\partial C_{ij}^{al,bm}}{\partial p_s} - 2\frac{\partial C_{ij}^{as,bm}}{\partial p_l} + \frac{\partial^2 D_{ij}^{ab,s}}{\partial p_m \partial p_l}\right)\delta_k^c\right] = 0
\end{aligned}
\tag{3.8l}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial p_m}\left[\left(\frac{\partial D^{ab,l}_{ij}}{\partial p_c}+\frac{\partial D^{ab,c}_{ij}}{\partial p_l}-C^{ac,bl}_{ij}-C^{bc,al}_{ij}\right)p_k\right]+\\
&\quad+\left(\frac{\partial C^{al,cm}_{ij}}{\partial p_b}+\frac{\partial C^{bl,cm}_{ij}}{\partial p_a}-\frac{\partial C^{ab,cm}_{ij}}{\partial p_l}-\frac{\partial C^{ba,cm}_{ij}}{\partial p_l}+\right.\\
&\quad\left.-\frac{\partial C^{al,cb}_{ij}}{\partial p_m}-\frac{\partial C^{bl,ca}_{ij}}{\partial p_m}+\frac{\partial^2 D^{ac,b}_{ij}}{\partial p_l\partial p_m}+\frac{\partial^2 D^{bc,a}_{ij}}{\partial p_l\partial p_m}\right)p_k+\\
&\quad+p_n\left[\left(\frac{\partial^2 D^{ab,l}_{ij}}{\partial p_m\partial p_n}+\frac{\partial^2 D^{ab,n}_{ij}}{\partial p_l\partial p_m}-\frac{\partial C^{an,bl}_{ij}}{\partial p_m}-\frac{\partial C^{bn,al}_{ij}}{\partial p_m}\right)\delta^c_k+\right.\\
&\quad\left(\frac{\partial^2 D^{ac,n}_{ij}}{\partial p_l\partial p_m}+\frac{\partial C^{al,cm}_{ij}}{\partial p_n}-\frac{\partial C^{an,cm}_{ij}}{\partial p_l}-\frac{\partial C^{cn,al}_{ij}}{\partial p_m}\right)\delta^b_k+\\
&\quad\left(\frac{\partial^2 D^{ac,n}_{ij}}{\partial p_l\partial p_m}+\frac{\partial C^{al,cm}_{ij}}{\partial p_n}-\frac{\partial C^{an,cm}_{ij}}{\partial p_l}-\frac{\partial C^{cn,al}_{ij}}{\partial p_m}\right)\delta^b_k+\\
&\quad\left.\left(\frac{\partial^2 D^{bc,n}_{ij}}{\partial p_l\partial p_m}+\frac{\partial C^{bl,cm}_{ij}}{\partial p_n}-\frac{\partial C^{bn,cm}_{ij}}{\partial p_l}-\frac{\partial C^{cn,bl}_{ij}}{\partial p_m}\right)\delta^a_k\right]=0 \quad (3.8\text{m})
\end{aligned}
$$

Repeated indices are summated according to Einstein's rule; $\circlearrowleft (a,b,c)$ *means cyclic permutations of the indices* (a,b,c)*, while* $\sum_{\sigma(a,b,c)}$ *means the complete symmetrization with respect to the listed indices (or couples of indices).*

Proof. The set of equations (3.8a)–(3.8d) is obtained by computing (2.28) for the deformed bracket, taking the expression of first order in ϵ and splitting it into the terms which are homogeneous in λ's and derivatives of p. In particular, (3.8a), (3.8b), (3.8c) and (3.8d) are the coefficients for $\lambda_a\lambda_b$, $\lambda_a\partial_b p_l$, $\partial_{ab}p_l$ and $\partial_a p_l\partial_b p_m$ respectively.

When computing the PVA-Jacobi identity for $\{\cdot_{\boldsymbol\lambda}\cdot\}$, we end up with a degree 0 term in ϵ which is the PVA-Jacobi identity for the undeformed bracket, plus a degree 1 term which reads

$$
\begin{aligned}
\epsilon\Bigg(&\left\{p_{i\boldsymbol\lambda}\left\{p_{j\boldsymbol\mu}p_k\right\}_0\right\}_1+\left\{p_{i\boldsymbol\lambda}\left\{p_{j\boldsymbol\mu}p_k\right\}_1\right\}_0+\\
&-\left\{p_{j\boldsymbol\mu}\left\{p_{i\boldsymbol\lambda}p_k\right\}_0\right\}_1-\left\{p_{j\boldsymbol\mu}\left\{p_{i\boldsymbol\lambda}p_k\right\}_1\right\}_0+\\
&-\left\{\left\{p_{i\boldsymbol\lambda}p_j\right\}_{1\boldsymbol\lambda+\boldsymbol\mu}p_k\right\}_0-\left\{\left\{p_{i\boldsymbol\lambda}p_j\right\}_{0\boldsymbol\lambda+\boldsymbol\mu}p_k\right\}_1\Bigg) \quad (3.9)
\end{aligned}
$$

and terms of higher order that are discharged. The sets of equations (3.8e)–(3.8m) are then obtained, as in the previous case, collecting the homogeneous terms in λ, μ and derivatives of p, up to the third degree. □

Remark Let us consider the trivial case $d = 1$. The undeformed bracket reads $\{p_\lambda p\}_0 = -2p\lambda - p'$ (the prime means the only derivative of p, namely wrt x), which is the so-called *Virasoro-Magri PVA* with central charge 0 (see Ex. 1.18 in [4]). We easily get the well known result, shown for instance in [9], that such deformations do not exist in the scalar case. From the skewsymmetry conditions we get (now the indices have become useless) $A = 0$, $2D = B$, $2C = B'$; moreover, (3.8e) is enough to get $D = 0$, hence $B = C = 0$.

3.2 Proof of the Theorem 5

The set of conditions (3.8) selects, among all the second order homogeneous operators, the ones which are admissible deformations of (3.4). We focus on the case $d = 2$, i.e. we build deformations of the Lie–Poisson bracket for the vector fields on the 2-torus. We proceed along two paths which are ultimately going to meet. First, we specialize the conditions (3.8) to the present case, giving the most general form of compatible first order deformations. Then, we shift to consider the trivial deformations of (3.4), namely the ones which are obtained by a Miura transformation of the bracket itself. We will show that the compatibility conditions of the latter ones and the reduced system of the former coincide. That means that all the compatible deformations of order 1 are trivial. Since we are dealing with a two dimensional system, we denote (p_1, p_2) as (p, q). The parameters of the deformation are, by definition, 108. Imposing the conditions (3.8) we reduce their freedom until it is possible. The first conditions we want to impose to the functions A's, B's, C's, and D's are (3.8a)–(3.8d). A convenient way to do this is to split the original parameters in symmetric and antisymmetric part with respect to the indices (i, j). We get

$$A^{ab}_{ij(S)} = \frac{1}{2}\left(A^{ab}_{ij} + A^{ab}_{ji}\right) \overset{(3.8a)}{=} 0 \tag{3.10a}$$

$$A^{ab}_{12(A)} = \frac{1}{2}\left(A^{ab}_{12} - A^{ab}_{21}\right) = \tilde{A}^{ab} \tag{3.10b}$$

$$B^{a,bl}_{ij(S)} = \frac{1}{2}\left(B^{a,bl}_{ij} + B^{ab,l}_{ji}\right) = \tilde{B}^{a,bl}_{ij} \tag{3.10c}$$

$$B^{a,bl}_{12(A)} = \frac{1}{2}\left(B^{a,bl}_{12} - B^{ab,l}_{21}\right) \overset{(3.8b)}{=} \frac{\partial \tilde{A}^{ab}}{\partial p_l} \tag{3.10d}$$

$$\begin{aligned} C^{al,bm}_{ij(S)} &= \frac{1}{2}\left(C^{al,bm}_{ij} + C^{al,bm}_{ji}\right) \\ &\overset{(3.8d)}{=} \frac{1}{4}\left(\frac{\partial\left(\tilde{B}^{a,bm}_{ij} + B^{a,bm}_{ij(A)}\right)}{\partial p_l} + \frac{\partial\left(\tilde{B}^{b,al}_{ij} - B^{b,al}_{ij(A)}\right)}{\partial p_m}\right) \end{aligned} \tag{3.10e}$$

$$C^{al,bm}_{12(A)} = \frac{1}{2}\left(C^{al,bm}_{12} - C^{al,bm}_{21}\right) = \tilde{C}^{al,bm} \tag{3.10f}$$

$$D^{ab,l}_{ij(S)} = \frac{1}{2}\left(D^{ab,l}_{ij} + D^{ab,l}_{ji}\right) \tag{3.10g}$$

$$\overset{(3.8c)}{=} \frac{1}{4}\left(\tilde{B}^{a,bl}_{ij} + B^{a,bl}_{ij(A)} + \tilde{B}^{b,al}_{ij} - B^{b,al}_{ij(A)}\right)$$

$$D^{ab,l}_{12(A)} = \frac{1}{2}\left(D^{ab,l}_{12} - D^{ab,l}_{21}\right) = \tilde{D}^{ab,l}. \tag{3.10h}$$

Just imposing the skewsymmetry condition has reduced the number of free parameters (now they are the functions denoted with the tilde) to 43. There are anyway a lot of conditions among these 43 parameters, imposed by the equations (3.8e)–(3.8m). By inspection and the usage of the Mathematica package `SYM` [10] we can find all the algebraic relations between the functions, thus reducing the system to an overdetermined system of 45 equations for 9 unknown functions $\tilde{A}^{11}$, $\tilde{A}^{12}$, $\tilde{A}^{22}$, $\tilde{B}^{1,22}_{11}$,.$\tilde{B}^{2,11}_{11}$, $\tilde{B}^{1,21}_{11}$, $\tilde{B}^{2,11}_{22}$, $\tilde{B}^{1,22}_{22}$, and $\tilde{B}^{2,12}_{22}$. The expressions for the remaining ones in terms of these nine are left to the Appendix A. In order to further investigate the system we relied to a powerful computational tool which is called a *Janet basis* for the linear system of PDEs [23]. It has been implemented in the Maple package `Janet` [5]; we found that the conditions according which the deformed λ-bracket is a PVA up to the first order are the following two equations

$$\begin{aligned}
&-\frac{3}{4}p^2\tilde{A}^{11} + \frac{3}{4}q^2\tilde{A}^{22} - \frac{3}{4}q^3\tilde{B}^{1,22}_{11} - \frac{3}{4}p^3\tilde{B}^{2,11}_{22} - \frac{3}{8}q^3\frac{\partial\tilde{A}^{22}}{\partial q} + \frac{3}{8}p^3\frac{\partial\tilde{A}^{11}}{\partial p} + \\
&-\frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{2,11}_{11}}{\partial p} - \frac{1}{2}p^3q\frac{\partial\tilde{B}^{1,22}_{22}}{\partial p} - \frac{1}{2}pq^2\tilde{B}^{2,11}_{11} - \frac{1}{4}pq^2\tilde{B}^{1,21}_{11} - \frac{1}{4}p^2q\tilde{B}^{2,12}_{22} + \\
&-\frac{1}{2}p^2q\tilde{B}^{1,22}_{22} - p^2q^2\frac{\partial^2\tilde{A}^{11}}{\partial q^2} - \frac{1}{2}p^3q\frac{\partial^2\tilde{A}^{11}}{\partial p\partial q} + p^2q^2\frac{\partial^2\tilde{A}^{22}}{\partial p^2} + \frac{1}{2}pq^3\frac{\partial^2\tilde{A}^{22}}{\partial p\partial q} + \\
&+\frac{5}{8}p^2q\frac{\partial\tilde{A}^{11}}{\partial q} + \frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{2,12}_{22}}{\partial q} - \frac{1}{4}pq^2\frac{\partial\tilde{A}^{12}}{\partial q} - \frac{1}{2}pq^3\frac{\partial\tilde{B}^{2,11}_{11}}{\partial q} + \frac{1}{2}p^3q\frac{\partial\tilde{B}^{2,11}_{22}}{\partial q} + \\
&+\frac{1}{2}pq^3\frac{\partial\tilde{B}^{1,22}_{11}}{\partial p} + \frac{1}{4}p^2q\frac{\partial\tilde{A}^{12}}{\partial p} - \frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{1,22}_{22}}{\partial q} + \frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{1,21}_{11}}{\partial p} - \frac{5}{8}pq^2\frac{\partial\tilde{A}^{22}}{\partial p} = \\
&= 0
\end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
&-5p^2q^2\frac{\partial^2\tilde{A}^{12}}{\partial p\partial q}-2p^3q\frac{\partial^2\tilde{A}^{12}}{\partial p^2}+6p^2q^2\frac{\partial^2\tilde{A}^{11}}{\partial q^2}+5p^3q\frac{\partial^2\tilde{A}^{11}}{\partial p\partial q}+2pq^3\frac{\partial^2\tilde{A}^{22}}{\partial p\partial q}+\\
&+\frac{11}{4}p^2\tilde{A}^{11}-\frac{7}{4}q^2\tilde{A}^{22}+\frac{15}{4}q^3\tilde{B}^{1,22}_{11}+\frac{3}{4}p^3\tilde{B}^{2,11}_{22}-\frac{1}{8}q^3\frac{\partial\tilde{A}^{22}}{\partial q}-\frac{19}{8}p^3\frac{\partial\tilde{A}^{11}}{\partial p}+\\
&+p^4\frac{\partial^2\tilde{A}^{11}}{\partial p^2}+q^4\frac{\partial^2\tilde{A}^{22}}{\partial q^2}+\frac{5}{4}p^2q^2\frac{\partial\tilde{B}^{2,11}_{11}}{\partial p}+\frac{1}{2}p^3q\frac{\partial\tilde{B}^{1,22}_{22}}{\partial p}+\frac{5}{2}pq^2\tilde{B}^{2,11}_{11}+\frac{9}{4}pq^2\tilde{B}^{1,21}_{11}+\\
&-\frac{3}{4}p^2q\tilde{B}^{2,12}_{22}+\frac{1}{2}p^2q\tilde{B}^{1,22}_{22}+2q^4\frac{\partial\tilde{B}^{1,22}_{11}}{\partial q}+2pq^3\frac{\partial\tilde{B}^{1,21}_{11}}{\partial q}-2p^3q\frac{\partial\tilde{B}^{2,12}_{22}}{\partial p}-2p^4\frac{\partial\tilde{B}^{2,11}_{22}}{\partial p}+\\
&-2pq^3\frac{\partial^2\tilde{A}^{12}}{\partial q^2}-\frac{17}{8}p^2q\frac{\partial\tilde{A}^{11}}{\partial q}-\frac{9}{4}p^2q^2\frac{\partial\tilde{B}^{2,12}_{22}}{\partial q}+\frac{1}{4}pq^2\frac{\partial\tilde{A}^{12}}{\partial q}+\frac{5}{2}pq^3\frac{\partial\tilde{B}^{2,11}_{11}}{\partial q}+\\
&-\frac{7}{2}p^3q\frac{\partial\tilde{B}^{2,11}_{22}}{\partial q}+\frac{1}{2}pq^3\frac{\partial\tilde{B}^{1,22}_{11}}{\partial p}-\frac{5}{4}p^2q\frac{\partial\tilde{A}^{12}}{\partial p}+\frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{1,22}_{22}}{\partial q}+\frac{3}{4}p^2q^2\frac{\partial\tilde{B}^{1,21}_{11}}{\partial p}+\\
&+\frac{13}{8}pq^2\frac{\partial\tilde{A}^{22}}{\partial p}=0. \qquad (3.12)
\end{aligned}
$$

Now, let us consider the trivial deformations of (3.4), namely the deformed bracket given by performing a general Miura transformation (3.5) of the first order to the undeformed bracket. Such a change of coordinates will have the form

$$p_i \mapsto P_i = p_i + \sum_{j,k=1,2} \epsilon F_i^{jk}(p,q)\partial_j p_k$$

and thus depends on 8 arbitrary functions of ($p_1 \equiv p, p_2 \equiv q$). We compute $\{P_i(p)_{\boldsymbol{\lambda}} P_j(p)\}_0$, which is in our case very straightforward. We start by the expansion to the order ϵ,

$$\{P_{i\boldsymbol{\lambda}}P_j\}_0 = \{p_{i\boldsymbol{\lambda}}p_j\}_0 + \epsilon\left(\left\{F_i^{al}\partial_a p_{l\boldsymbol{\lambda}}p_j\right\}_0 + \left\{p_{i\boldsymbol{\lambda}}F_j^{al}\partial_a p_l\right\}_0\right) + O(\epsilon^2)$$

and then we use the master formula (2.30) for the two latter brackets. The expression we get is written in terms of the 'old' coordinates; up to the first order, we can invert the transformation by $p_i = P_i - \epsilon F_i^{al}(P)\partial_a P_l$, getting the formula for the deformed bracket. $\{P_i(p)_{\boldsymbol{\lambda}}P_j(p)\}_0 = \{P_{i\boldsymbol{\lambda}}P_j\}_0 + \epsilon\{P_{i\boldsymbol{\lambda}}P_j\}_1$, where the parameters of the first order deformed bracket are

$$
\begin{aligned}
A_{ij}^{ab} &= \frac{1}{2}\left(F_i^{ab}P_j + F_i^{ba}P_j - F_j^{ab}P_i - F_j^{ba}P_i + P_m F_i^{am}\delta_j^b\right. && \text{(3.13a)}\\
&\quad \left.+P_m F_i^{bm}\delta_j^a - P_m F_j^{am}\delta_i^b - P_m F_j^{bm}\delta_i^a\right)\\
B_{ij}^{a,bl} &= F_i^{bl}\delta_j^a - F_j^{al}\delta_i^b + F_i^{ab}\delta_j^l - F_j^{ba}\delta_i^l && \text{(3.13b)}\\
&\quad + \frac{\partial F_i^{ba}}{\partial P_l}P_j - \frac{\partial F_j^{bl}}{\partial P_a}P_i + \frac{\partial F_i^{ab}}{\partial P_l}P_j - \frac{\partial F_i^{bl}}{\partial P_a}P_j\\
&\quad + \frac{\partial F_i^{am}}{\partial P_l}P_m\delta_j^b - \frac{\partial F_i^{bl}}{\partial P_m}P_m\delta_j^a - \frac{\partial F_j^{bl}}{\partial P_m}P_m\delta_i^a + \frac{\partial F_i^{bm}}{\partial P_l}P_m\delta_j^a\\
C_{ij}^{al,bm} &= \frac{1}{2}\left(P_j\frac{\partial^2 F_i^{ab}}{\partial P_l\partial P_m} - P_j\frac{\partial^2 F_i^{ba}}{\partial P_l\partial P_m} - P_j\frac{\partial^2 F_i^{bm}}{\partial P_a\partial P_l} - P_j\frac{\partial^2 F_i^{al}}{\partial P_b\partial P_m}\right. && \text{(3.13c)}\\
&\quad + P_s\frac{\partial^2 F_i^{as}}{\partial P_l\partial P_m}\partial_j^b + P_s\frac{\partial^2 F_i^{bs}}{\partial P_l\partial P_m}\delta_j^a - P_s\frac{\partial^2 F_i^{al}}{\partial P_s\partial P_m}\delta_j^b - P_s\frac{\partial^2 F_i^{bm}}{\partial P_s\partial P_l}\delta_j^a\\
&\quad \left.+ \frac{\partial F_i^{ab}}{\partial P_l}\delta_j^m + \frac{\partial F_i^{ba}}{\partial P_m}\delta_j^l - \frac{\partial F_i^{bm}}{\partial P_a}\delta_j^l - \frac{\partial F_i^{al}}{\partial P_b}\delta_j^m\right)\\
D_i^{ab,l}j &= \frac{1}{2}\left(\frac{\partial F_i^{ab}}{\partial P_l}P_j + \frac{\partial F_i^{ba}}{\partial P_l}P_j - \frac{\partial F_i^{al}}{\partial P_b}P_j - \frac{\partial F_i^{bl}}{\partial P_a}P_j\right. && \text{(3.13d)}\\
&\quad \left.+P_m\frac{\partial F_i^{am}}{\partial P_l}\delta_j^b + P_m\frac{\partial F_i^{bm}}{\partial P_l}\delta_j^a - P_m\frac{\partial F_i^{al}}{\partial P_m}\delta_j^b - P_m\frac{\partial F_i^{bl}}{\partial P_m}\delta_j^a\right)
\end{aligned}
$$

Since the bracket defined by (3.13) is the Miura transformed of the undeformed one, it is a first order deformation of a PVA bracket; the set of coefficients satisfies, as it can be checked, the full set of conditions (3.8). For convenience, we go back to the old variables, namely to lower case p and q for P_1 and P_2. We give in terms of the eight parameters of the Miura transformations the nine unknown functions of (3.11) and (3.12), since we already know (Appendix A) how to compute all the remaining ones. They are, as one can easily check,

$$A^{11} = qF_1^{11} - 2pF_2^{11} - qF_2^{12} \tag{3.14a}$$

$$2A^{12} = pF_1^{11} + 2qF_1^{12} - pF_2^{12} + qF_2^{21} - 2pF_2^{21} - qF_2^{22} \tag{3.14b}$$

$$A^{22} = pF_1^{21} + 2qF_1^{22} - pF_2^{22} \tag{3.14c}$$

$$B_{11}^{1,21} = F_1^{12} - 2q\frac{\partial F_1^{21}}{\partial q} + p\frac{\partial F_1^{12}}{\partial p} - 2p\frac{\partial F_1^{21}}{\partial p} + q\frac{\partial F_1^{22}}{\partial p} \tag{3.14d}$$

$$B_{11}^{12,2} = F_1^{22} + p\frac{\partial F_1^{12}}{\partial q} + 2p\frac{\partial F_1^{21}}{\partial q} - q\frac{\partial F_1^{22}}{\partial q} - 4p\frac{\partial F_1^{22}}{\partial p} \tag{3.14e}$$

$$B_{11}^{2,11} = -F_1^{12} - 2p\frac{\partial F_1^{11}}{\partial q} + p\frac{\partial F_1^{12}}{\partial p} + 2p\frac{\partial F_1^{21}}{\partial p} + q\frac{\partial F_1^{22}}{\partial p} \tag{3.14f}$$

$$B_{22}^{2,12} = F_2^{21} + p\frac{\partial F_2^{11}}{\partial q} - 2q\frac{\partial F_2^{12}}{\partial q} + q\frac{\partial F_2^{21}}{\partial q} - 2p\frac{\partial F_2^{12}}{\partial p} \tag{3.14g}$$

$$B_{22}^{2,11} = F_2^{11} + q\frac{\partial F_2^{21}}{\partial p} + 2q\frac{\partial F_2^{12}}{\partial p} - p\frac{\partial F_2^{11}}{\partial p} - 4q\frac{\partial F_2^{11}}{\partial q} \tag{3.14h}$$

$$B_{22}^{2,12} = -F_2^{21} - 2q\frac{\partial F_2^{22}}{\partial p} + q\frac{\partial F_2^{21}}{\partial q} + 2q\frac{\partial F_2^{12}}{\partial q} + p\frac{\partial F_2^{11}}{\partial q} \tag{3.14i}$$

We can regard the set of equations (3.14) as an inhomogeneous linear system of 9 PDEs for the 8 unknown functions F's. A solution of the system, if there exists, is the set of the eight parameters of a Miura transformation which produces a given coboundary. The compatibility conditions among the parameters in the LHS of the system can be found using the tools of `Janet` package. We get a system of two second order differential equations, whose Janet basis is exactly (3.11) and (3.12).

A generic first order cocycle, i.e. a first order deformed bracket, can be written in terms of the nine parameters $(\tilde{A}^{11}, \dots, \tilde{B}_{22}^{2,12})$, provided that they satisfy (3.11) and (3.12). On the other hand, the same two conditions allow to find the eight parameters of a Miura transformations for which we get that cocycle. It follows that every cocycle in $\Omega^2_{[1]}(\hat{\mathcal{A}}, \{\cdot_{\boldsymbol{\lambda}}\cdot\}_0)$ is a coboundary, so that $H^2_{[1]}(\hat{\mathcal{A}}, \{\cdot_{\boldsymbol{\lambda}}\cdot\}_0) = 0$, for $\hat{\mathcal{A}}$ a 2-dimensional algebra of differential polynomials with 2 derivations and $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_0$ as in (3.4).

4 Concluding remarks

In this paper we have formulated the theory of multidimensional Poisson Vertex Algebras, showing how it can be applied to the study of evolutionary Hamiltonian PDEs. In particular, we have proved a first result in the theory of the deformation of multidimensional Poisson brackets of hydrodynamic type, namely the triviality of the first order second cohomology group for the Lie–Poisson bracket of vector fields on the 2-torus.

Further investigations will be devoted to higher order deformations of the Lie–Poisson λ-bracket, aiming to characterize the full second cohomology group, in the spirit of [17], [9] and [13].

A Components of the 1st order deformation of the λ bracket for $d = 2$

$$\tilde{B}^{1,11}_{11} = \frac{q^2}{p^2}\left(\frac{\partial \tilde{A}^{22}}{\partial q} + 2\tilde{B}^{1,22}_{11}\right) + \frac{q}{p}\left(-2\frac{\partial \tilde{A}^{12}}{\partial q} + \frac{5}{2}\frac{\partial \tilde{A}^{22}}{\partial p} + 2\tilde{B}^{1,21}_{11} + \tilde{B}^{2,11}_{11}\right) +$$
$$-\left(-\frac{\partial \tilde{A}^{11}}{\partial q} + \frac{\partial \tilde{A}^{12}}{\partial p} + \tilde{B}^{1,22}_{22} + \tilde{B}^{2,12}_{22}\right) + \frac{p\left(\frac{\partial \tilde{A}^{11}}{\partial p} - 2\tilde{B}^{2,11}_{22}\right)}{2q} +$$
$$+\frac{\tilde{A}^{12}}{p} - \frac{2q\tilde{A}^{22}}{p^2} - \frac{\tilde{A}^{11}}{q}$$

$$\tilde{B}^{2,22}_{22} = \frac{p^2}{q^2}\left(-\frac{\partial \tilde{A}^{11}}{\partial p} + 2\tilde{B}^{2,11}_{22}\right) + \frac{p}{q}\left(-\frac{5}{2}\frac{\partial \tilde{A}^{11}}{\partial q} + 2\frac{\partial \tilde{A}^{12}}{\partial p} + \tilde{B}^{1,22}_{22} + 2\tilde{B}^{2,12}_{22}\right) +$$
$$-\left(\frac{\partial \tilde{A}^{22}}{\partial p} - \frac{\partial \tilde{A}^{12}}{\partial q} + \tilde{B}^{1,21}_{11} + \tilde{B}^{2,11}_{11}\right) + \frac{q\left(-\frac{\partial \tilde{A}^{22}}{\partial q} - 2\tilde{B}^{1,22}_{11}\right)}{2p} +$$
$$+\frac{2p\tilde{A}^{11}}{q^2} - \frac{\tilde{A}^{12}}{q} + \frac{\tilde{A}^{22}}{p}$$

$$\tilde{B}^{1,11}_{22} = \tilde{B}^{2,22}_{11} = 0$$
$$\tilde{B}^{1,12}_{11} = -\frac{2p}{q}\tilde{B}^{1,11}_{11}$$
$$\tilde{B}^{2,21}_{22} = -\frac{2q}{p}\tilde{B}^{2,22}_{22}$$
$$\tilde{B}^{2,12}_{11} = 2\frac{p^2}{q^2}\tilde{B}^{1,11}_{11} - \frac{p}{q}\left(2\tilde{B}^{1,21}_{11} + 2\tilde{B}^{2,11}_{11}\right) - \tilde{B}^{1,22}_{11}$$
$$\tilde{B}^{1,21}_{22} = 2\frac{q^2}{p^2}\tilde{B}^{2,22}_{22} - \frac{q}{p}\left(2\tilde{B}^{1,22}_{22} + 2\tilde{B}^{2,12}_{22}\right) - \tilde{B}^{2,11}_{22}$$
$$\tilde{B}^{2,21}_{11} = \frac{p}{q}\left(\tilde{B}^{1,21}_{11} + \tilde{B}^{2,11}_{11}\right) - \frac{p^2}{q^2}\tilde{B}^{1,11}_{11}$$
$$\tilde{B}^{1,12}_{22} = \frac{q}{p}\left(\tilde{B}^{1,22}_{22} + \tilde{B}^{2,12}_{22}\right) - \frac{q^2}{p^2}\tilde{B}^{2,22}_{22}$$
$$\tilde{B}^{1,11}_{12} = -\frac{1}{2}\left(\frac{q}{p}\left(\tilde{B}^{1,22}_{22} + \tilde{B}^{2,12}_{22}\right) - \frac{q^2}{p^2}\tilde{B}^{2,22}_{22}\right)$$
$$\tilde{B}^{2,22}_{12} = -\frac{1}{2}\left(\frac{p}{q}(\tilde{B}^{1,21}_{11} + \tilde{B}^{2,11}_{11}) - \frac{p^2}{q^2}\tilde{B}^{1,11}_{11}\right)$$

$$\tilde{B}^{1,12}_{12} = \frac{1}{2}\left(-\frac{2q}{p}\tilde{B}^{2,22}_{22} - \tilde{B}^{1,11}_{11} + 2\tilde{B}^{1,22}_{22} + 2\tilde{B}^{2,12}_{22}\right)$$
$$\tilde{B}^{2,21}_{12} = \frac{1}{2}\left(-\frac{2p}{q}\tilde{B}^{1,11}_{11} + 2\tilde{B}^{1,21}_{11} + 2\tilde{B}^{2,11}_{11} - \tilde{B}^{2,22}_{22}\right)$$
$$\tilde{B}^{1,22}_{12} = -\frac{\partial \tilde{A}^{12}}{\partial q} + 2\frac{\partial \tilde{A}^{22}}{\partial p} + \frac{q}{p}\frac{\partial \tilde{A}^{22}}{\partial q} - \frac{2}{p}\tilde{A}^{22} + (2\tilde{B}^{1,21}_{11} + \tilde{B}^{2,11}_{11}) - \frac{3p}{2q}\tilde{B}^{1,11}_{11} + \frac{2q}{p}\tilde{B}^{1,22}_{11}$$
$$\tilde{B}^{2,11}_{12} = -2\frac{\partial \tilde{A}^{11}}{\partial q} + \frac{\partial \tilde{A}^{12}}{\partial p} - \frac{p}{q}\frac{\partial \tilde{A}^{11}}{\partial p} + \frac{2}{q}\tilde{A}^{11} + (\tilde{B}^{1,22}_{22} + 2\tilde{B}^{2,12}_{22}) + \frac{2p}{q}\tilde{B}^{2,11}_{22} - \frac{3q}{2p}\tilde{B}^{2,22}_{22}$$
$$\tilde{B}^{1,21}_{12} = \frac{1}{2}\left(-\frac{4\tilde{A}^{11}}{q} + 4\frac{\partial \tilde{A}^{11}}{\partial q} - 2\frac{\partial \tilde{A}^{12}}{\partial p} + \frac{2p\frac{\partial \tilde{A}^{11}}{\partial p}}{q} - \frac{4p\tilde{B}^{2,11}_{22}}{q} + \frac{3q\tilde{B}^{2,22}_{22}}{p} + 2\tilde{B}^{1,11}_{11} - 3\tilde{B}^{1,22}_{22} - 5\tilde{B}^{2,12}_{22}\right)$$
$$\tilde{B}^{2,12}_{12} = \frac{1}{2}\left(\frac{4\tilde{A}^{22}}{p} + 2\frac{\partial \tilde{A}^{12}}{\partial q} - 4\frac{\partial \tilde{A}^{22}}{\partial p} - \frac{2q\frac{\partial \tilde{A}^{22}}{\partial q}}{p} + \frac{3p\tilde{B}^{1,11}_{11}}{q} - \frac{4q\tilde{B}^{1,22}_{11}}{p} - 5\tilde{B}^{1,21}_{11} - 3\tilde{B}^{2,11}_{11} + 2\tilde{B}^{2,22}_{22}\right)$$

$$
\begin{aligned}
\tilde{C}_{12}^{11,11} &= \frac{1}{4}\frac{\partial}{\partial p}\left(\frac{q(\tilde{B}_{22}^{1,22}+B_{22}^{2,12})}{p}-\left(\frac{q}{p}\right)^2\tilde{B}_{22}^{2,22}\right)\\
\tilde{C}_{12}^{22,22} &= -\frac{1}{4}\frac{\partial}{\partial q}\left(\frac{p(\tilde{B}_{11}^{1,21}+\tilde{B}_{11}^{2,11})}{q}-\left(\frac{p}{q}\right)^2\tilde{B}_{11}^{1,11}\right)\\
\tilde{C}_{12}^{11,12} &= -\frac{1}{8}\left(\frac{\partial}{\partial q}\left(\left(\frac{q}{p}\right)^2\tilde{B}_{22}^{2,22}-\frac{q(\tilde{B}_{22}^{1,22}+B_{22}^{2,12})}{p}\right)+\frac{\partial}{\partial p}\left(-\frac{2q\tilde{B}_{22}^{2,22}}{p}+\tilde{B}_{11}^{1,11}+2\tilde{B}_{22}^{1,22}+2B_{22}^{2,12}\right)\right)\\
\tilde{C}_{12}^{21,22} &= \frac{1}{8}\left(\frac{\partial}{\partial q}\left(-\frac{2p\tilde{B}_{11}^{1,11}}{q}+2\tilde{B}_{11}^{1,21}+2\tilde{B}_{11}^{2,11}+\tilde{B}_{22}^{2,22}\right)+\frac{\partial}{\partial p}\left(\left(\frac{p}{q}\right)^2\tilde{B}_{11}^{1,11}-\frac{p(\tilde{B}_{11}^{1,21}+\tilde{B}_{11}^{2,11})}{q}\right)\right)\\
\tilde{C}_{12}^{12,12} &= -\frac{1}{4}\frac{\partial}{\partial q}\left(-\frac{2q\tilde{B}_{22}^{2,22}}{p}+\tilde{B}_{11}^{1,11}+2\tilde{B}_{22}^{1,22}+2B_{22}^{2,12}\right)\\
\tilde{C}_{12}^{21,21} &= \frac{1}{4}\frac{\partial}{\partial p}\left(-\frac{2p\tilde{B}_{11}^{1,11}}{q}+2\tilde{B}_{11}^{1,21}+2\tilde{B}_{11}^{2,11}+\tilde{B}_{22}^{2,22}\right)\\
\tilde{C}_{12}^{12,22} &= -\frac{1}{8}\frac{\partial(\tilde{B}_{11}^{1,21}+\tilde{B}_{11}^{2,11}+2\tilde{B}_{22}^{2,22})}{\partial q}\\
\tilde{C}_{12}^{11,21} &= \frac{1}{8}\frac{\partial(2\tilde{B}_{11}^{1,11}+\tilde{B}_{22}^{1,22}+B_{22}^{2,12})}{\partial p}\\
\tilde{C}_{12}^{11,22} &= \frac{1}{4}\left(2\left(\frac{\partial^2\tilde{A}^{11}}{\partial q^2}-\frac{\partial^2\tilde{A}^{22}}{\partial p^2}\right)-\frac{\partial}{\partial p}\left(\frac{q\left(\frac{\partial\tilde{A}^{22}}{\partial q}+2\tilde{B}_{11}^{1,22}\right)}{p}-\frac{2\tilde{A}^{22}}{p}-\frac{3p\tilde{B}_{11}^{1,11}}{2q}+3\tilde{B}_{11}^{1,21}+2\tilde{B}_{11}^{2,11}\right)\right.\\
&\quad\left.+\frac{\partial}{\partial q}\left(\frac{p\left(\frac{\partial\tilde{A}^{11}}{\partial p}-2\tilde{B}_{22}^{2,11}\right)}{q}-\frac{2\tilde{A}^{11}}{q}+\frac{3q\tilde{B}_{22}^{2,22}}{2p}+2\tilde{B}_{11}^{1,11}-\tilde{B}_{22}^{1,22}-2B_{22}^{2,12}\right)+\frac{p}{q}\frac{\partial\tilde{B}_{11}^{1,11}}{\partial p}\right)\\
\tilde{C}_{12}^{12,21} &= \frac{1}{8}\left(4\frac{\partial^2\tilde{A}^{11}}{\partial q^2}-4\frac{\partial^2\tilde{A}^{22}}{\partial p^2}+\right.\\
&\quad+\frac{\partial}{\partial p}\left(-\frac{2q\left(\frac{\partial\tilde{A}^{22}}{\partial q}+2\tilde{B}_{11}^{1,22}\right)}{p}+\frac{4\tilde{A}^{22}}{p}+\frac{3p\tilde{B}_{11}^{1,11}}{q}-5\tilde{B}_{11}^{1,21}-3\tilde{B}_{11}^{2,11}+2\tilde{B}_{22}^{2,22}\right)\\
&\quad+\frac{\partial}{\partial q}\left(\frac{2p\left(\frac{\partial\tilde{A}^{11}}{\partial p}-2\tilde{B}_{22}^{2,11}\right)}{q}-\frac{4\tilde{A}^{11}}{q}+\frac{3q\tilde{B}_{22}^{2,22}}{p}+2\tilde{B}_{11}^{1,11}-3\tilde{B}_{22}^{1,22}-5B_{22}^{2,12}\right)\\
&\quad\left.+\frac{2q\frac{\partial\tilde{B}_{22}^{2,22}}{\partial q}}{p}\right)
\end{aligned}
$$

$$\tilde{D}^{11,1}_{12} = \frac{1}{4}\left(\frac{q(\tilde{B}^{1,22}_{22} + B^{2,12}_{22})}{q} - \frac{q^2\tilde{B}^{2,22}_{22}}{q^2}\right)$$

$$\tilde{D}^{22,2}_{12} = -\frac{1}{4}\left(\frac{q(\tilde{B}^{1,21}_{11} + \tilde{B}^{2,11}_{11})}{q} - \frac{q^2\tilde{B}^{1,11}_{11}}{q^2}\right)$$

$$\tilde{D}^{11,2}_{12} = -\frac{1}{4}\left(-\frac{2q\tilde{B}^{2,22}_{22}}{q} + \tilde{B}^{1,11}_{11} + 2\tilde{B}^{1,22}_{22} + 2B^{2,12}_{22}\right)$$

$$\tilde{D}^{22,1}_{12} = \frac{1}{4}\left(-\frac{2q\tilde{B}^{1,11}_{11}}{q} + 2\tilde{B}^{1,21}_{11} + 2\tilde{B}^{2,11}_{11} + \tilde{B}^{2,22}_{22}\right)$$

$$\tilde{D}^{12,1}_{12} = \frac{1}{8}(2\tilde{B}^{1,11}_{11} + \tilde{B}^{1,22}_{22} + B^{2,12}_{22})$$

$$\tilde{D}^{12,2}_{12} = -\frac{1}{8}(\tilde{B}^{1,21}_{11} + \tilde{B}^{2,11}_{11} + 2\tilde{B}^{2,22}_{22})$$

Acknowledgements

I want to thank in particular prof. B. Dubrovin for introducing me to the subject of the deformation of Poisson brackets of hydrodynamic type, and for the accurate supervision of my research. I'm very grateful for the discussions with D. Valeri, A. De Sole and F. Magri about Poisson Vertex Algebras during the the series of seminars in Febraury 2013 at Università degli Studi di Milano–Bicocca and with P. Lorenzoni for his useful advice about multidimensional Poisson brackets.

This work is partially supported by the European Research Council Advanced Grant FroM–PDE and by PRIN 2010–11 Grant Geometric and analytic theory of Hamiltonian systems in finite and infinite dimensions of Italian Ministry of Universities and Researches.

References

[1] Ian M. Anderson. Introduction to the variational bicomplex. In *Mathematical aspects of classical field theory (Seattle, WA, 1991)*, volume 132 of *Contemp. Math.*, pages 51–73. Amer. Math. Soc., Providence, RI, 1992.

[2] V. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier (Grenoble)*, 16(fasc. 1):319–361, 1966.

[3] Bojko Bakalov, Alessandro D'Andrea, and Victor G. Kac. Theory of finite pseudoalgebras. *Adv. Math.*, 162(1):1–140, 2001.

[4] Aliaa Barakat, Alberto De Sole, and Victor G. Kac. Poisson vertex algebras in the theory of Hamiltonian equations. *Jpn. J. Math.*, 4(2):141–252, 2009.

[5] Y. A. Blinkov, C. F. Cid, V. P. Gerdt, W. Plesken, and D. Robertz. The MAPLE Package "Janet": II. Linear Partial Differential Equations. In V. G. Ganzha, E. W. Mayr, and E. V. Vorozhtsov, editors, *Proceedings of the 6th International Workshop on Computer Algebra in Scientific Computing, Sep. 20-26, 2003, Passau (Germany)*, pages 41–54, 2003.

[6] Alberto De Sole and Victor G. Kac. Finite vs affine W-algebras. *Jpn. J. Math.*, 1(1):137–261, 2006.

[7] Alberto De Sole and Victor G. Kac. Non-local Poisson structures and applications to the theory of integrable systems. *Jpn. J. Math.*, 8(2):233–347, 2013.

[8] Alberto De Sole and Victor G. Kac. The variational Poisson cohomology. *Jpn. J. Math.*, 8(1):1–145, 2013.

[9] Luca Degiovanni, Franco Magri, and Vincenzo Sciacca. On deformation of Poisson manifolds of hydrodynamic type. *Comm. Math. Phys.*, 253(1):1–24, 2005.

[10] Stelios Dimas and Dimitri Tsoubelis. Sym: A new symmetry – finding package for *Mathematica*. In *Prooceding of 10th International Conference in MOdern GRoup ANalysis*, pages 64–70, 2005.

[11] B. A. Dubrovin and S. P. Novikov. Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method. *Dokl. Akad. Nauk SSSR*, 270(4):781–785, 1983.

[12] B. A. Dubrovin and S. P. Novikov. Poisson brackets of hydrodynamic type. *Dokl. Akad. Nauk SSSR*, 279(2):294–297, 1984.

[13] Boris A. Dubrovin and Y. Zhang. Normal forms of hierarchies of integrable pdes, frobenius manifolds and gromov-witten invariants.

[14] E. V. Ferapontov, V. S. Novikov, and N. M. Stoilov. Dispersive deformations of Hamiltonian systems of hydrodynamic type in $2+1$ dimensions. *Phys. D*, 241(23-24):2138–2144, 2012.

[15] E. V. Ferapontov, A. V. Odesskii, and N. M. Stoilov. Classification of integrable two-component Hamiltonian systems of hydrodynamic type in $2+1$ dimensions. *J. Math. Phys.*, 52(7):073505, 28, 2011.

[16] I. M. Gel′fand and L. A. Dikiĭ. Asymptotic properties of the resolvent of Sturm-Liouville equations, and the algebra of Korteweg-de Vries equations. *Uspehi Mat. Nauk*, 30(5(185)):67–100, 1975.

[17] Ezra Getzler. A Darboux theorem for Hamiltonian operators in the formal calculus of variations. *Duke Math. J.*, 111(3):535–560, 2002.

[18] Victor Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 1998.

[19] André Lichnerowicz. Les variétés de Poisson et leurs algèbres de Lie associées. *J. Differential Geometry*, 12(2):253–300, 1977.

[20] Franco Magri. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.*, 19(5):1156–1162, 1978.

[21] O. I. Mokhov. Poisson brackets of Dubrovin-Novikov type (DN-brackets). *Funktsional. Anal. i Prilozhen.*, 22(4):92–93, 1988.

[22] S. P. Novikov. The Hamiltonian formalism and a multivalued analogue of Morse theory. *Uspekhi Mat. Nauk*, 37(5(227)):3–49, 248, 1982.

[23] W. Plesken and D. Robertz. Janet's approach to presentations and resolutions for polynomials and linear PDEs. *Arch. Math. (Basel)*, 84(1):22–37, 2005.