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Comparison of two methods for the definition of the infinite time limit of the time evolution operator in scattering theory

Gábor Zsolt Tóth

International School for Advanced Studies (SISSA),
Via Beirut 2-4, 34014 Trieste, Italy

INFN, Sezione di Trieste, Italy

Research Institute for Particle and Nuclear Physics,
Hungarian Academy of Sciences, Pf. 49, 1525 Budapest, Hungary

e-mail: tgzs@cs.elte.hu

Abstract

A comparison of the methods of averaging and adiabatic switching for the definition of the infinite time limit of the time evolution operator in scattering theory is presented. The focus is on certain basic features which are relevant mainly in quantum field theory. Specifically, the differences between the unitarity properties of the operators obtained using the two methods, the difference between the necessary energy renormalizations and the existence of Lippmann-Schwinger equations are discussed. The manifestation of these differences in perturbation expansions is also described.

1 Introduction

The $T \to \pm\infty$ limits of the time evolution operator $U(0,-T)$ have an important role in scattering theory and also in some other areas of quantum physics. However, in general it is not completely obvious how to define these limits. In this short paper we discuss certain basic aspects of this problem; we describe and compare two methods for the definition of $\lim_{T\to\pm\infty} U(0,-T)$, the method of averaging and the method of adiabatic switching, which are often mentioned in the literature. Our aim is to emphasize certain features of the two methods and to highlight certain significant differences between them. The discussion is done in the context of scattering theory, where the time evolution operator is used to produce in and out states.

Section 2 contains a brief introduction of the in and out states, the S-matrix elements and the time evolution operator. In section 3 we describe the method of averaging. The main points that we emphasize are that in general a renormalization of the free particle Hamiltonian operator is necessary and that the $\lim_{T\to\pm\infty} U(0,-T)$ operators obtained by the averaging are in general not unitary, therefore normalization factors have to be included in the formulas which give the in and out states. We also write perturbation formulas for $\lim_{T\to\pm\infty} U(0,-T)$ and a Lippmann-Schwinger equation. The averaging described in section 3 is a representative of the kind of methods which give zero for the limit of oscillating functions.

In section 4 we describe the adiabatic switching method. The operators $\lim_{T\to\pm\infty} U(0,-T)$ obtained with this method are unitary, therefore no normalization factors are needed. The renormalization of the free particle Hamiltonian operator is still necessary. We write perturbation formulas for $\lim_{T\to\pm\infty} U(0,-T)$, which clearly differ, beyond first order, from the perturbation formulas for the method of averaging. The Lippmann-Schwinger equation that exists for the averaging method does not hold for the adiabatic switching method. In the framework of perturbation theory we also find that the renormalization needed for the free particle Hamiltonian operator differs, beyond first order, from the renormalization that is needed when the averaging is used.

The differences that we discuss are relevant mainly for quantum field theory, in quantum mechanical potential scattering the differences between the two methods usually vanish.

It seems to us that the nonunitary nature of $\lim_{T\to\pm\infty} U(0,-T)$ defined using the averaging method and the consequent necessity of the normalization factor is not always stated clearly in the literature, therefore we felt that this paper could be of interest. We were also motivated by recent development in scattering theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

We note that we dealt with the adiabatic switching method in a recent article [14]. In the present paper, as in [14], we treat scattering theory in a general way, without much reference to particular properties that various models can have.

2 The in and out states

Let H be the total Hamiltonian operator that describes a certain scattering process. We assume that H takes the form $H = H_0 + H_I$, where H_0 is a Hamiltonian operator that describes free particles and H_I is an interaction term. In some cases we use the form gH_{int} for H_I, where g is a coupling constant that can be used as a variable for power series expansion.

The S matrix elements are scalar products of in and out states:

$$S_{wv} = \langle w, out | v, in \rangle. \tag{1}$$

An important property of the in and out states is that they are eigenstates of H.

In quantum mechanics standard formulas (see [15, 16]) that are meant to give the in and out states in terms of H, H_0 and the eigenstates of H_0 are the following:

$$|v, in\rangle = \lim_{T\to\infty} U(0, -T)|v\rangle \tag{2}$$

$$|v, out\rangle = \lim_{T\to\infty} U(0, T)|v\rangle. \tag{3}$$

In these formulas $U(t_2, t_1)$ is the time evolution operator

$$U(t_2, t_1) = e^{\frac{i}{\hbar}H_0 t_2} e^{-\frac{i}{\hbar}H(t_2-t_1)} e^{-\frac{i}{\hbar}H_0 t_1}, \tag{4}$$

and $|v\rangle$ is an eigenstate of H_0. The in and out states $|v, in\rangle$ and $|v, out\rangle$ thus correspond to the eigenstate $|v\rangle$ of H_0. The energy of an eigenstate $|v\rangle$ will be denoted by E_v^0; i.e. $H_0|v\rangle = E_v^0|v\rangle$. The energy of $|v, in\rangle$ and $|v, out\rangle$ will be denoted by E_v: $H|v, in\rangle = E_v|v, in\rangle$, $H|v, out\rangle = E_v|v, out\rangle$ (it is natural to assume that the energies of $|v, in\rangle$ and $|v, out\rangle$ are equal. However, this is not essential for the subsequent discussion.)

In the following sections we aim to write (2) and (3) in a form that is valid generally, in particular also in quantum field theory. We will see that certain modifications of (2) and (3) are necessary for this, and these modifications depend on the way the $T \to \infty$ limit is defined. We will discuss two methods, which we mentioned in the introduction, for defining this limit.

3 The method of averaging

The components of $U(0, -T)|v\rangle$ or $U(0, T)|v\rangle$ are generally oscillatory functions of T and do not have a limit as $T \to \infty$, therefore one has to specify how the $T \to \infty$ limit is understood in (2) and (3). In order to properly define the $T \to \infty$ limit an averaging is often introduced in the following way (see e.g. [17]):

$$|v, in\rangle = \lim_{\epsilon\to 0^+} \lim_{T\to\infty} \hat{U}_\epsilon(0, -T)|v\rangle \tag{5}$$

$$|v, out\rangle = \lim_{\epsilon\to 0^+} \lim_{T\to\infty} \hat{U}_\epsilon(0, T)|v\rangle, \tag{6}$$

where

$$\hat{U}_\epsilon(0,-T) = \epsilon \int_0^T \mathrm{d}\tau\, e^{-\epsilon\tau} U(0,-\tau) \tag{7}$$

$$\hat{U}_\epsilon(0,T) = \epsilon \int_0^T \mathrm{d}\tau\, e^{-\epsilon\tau} U(0,\tau). \tag{8}$$

The above form of the definition of the in and out states is usually satisfactory in potential scattering (at least for sufficiently short range and regular potentials). However, in general (in quantum field theory, for example) one encounters divergences in the $\epsilon \to 0$ limit if one applies perturbation theory to calculate the components of the in and out states and the S-matrix elements (see e.g. [18] for a detailed explanation of the mentioned divergences). These divergences occur in theories in which the energy of $|v, in\rangle$ or $|v, out\rangle$ (which are eigenstates of H) is shifted with respect to the energy of $|v\rangle$ by any nonzero amount. (These shifts are called self-energy corrections. We stress that the mentioned divergences are distinct from the ultraviolet and infrared divergences, which we shall not discuss in this paper.)

The problem of self-energy corrections can also be illustrated in a nonperturbative framework; for this purpose let us consider the case when the Hilbert space is finite dimensional and the spectrum of H_0 is nondegenerate. Although the scattering is trivial in such a system, the self-energy corrections are nonzero. At $g = 0$ we have $|v, in\rangle = |v\rangle$, of course. However, for sufficiently small but nonzero values of g the eigenvalues of H are shifted from the values that they take at $g = 0$, and this has the consequence that $\langle W|v, in\rangle = 0$ for any eigenvector $|W\rangle$ of H. This can be seen in the following way. Let E_W be the eigenvalue for $|W\rangle$. Then we have

$$\langle W|U(0,-T)|v\rangle = \langle W|v\rangle \exp[\frac{i}{\hbar}(E_v^0 - E_W)T], \tag{9}$$

which is an oscillating function of T since $E_v^0 \neq E_W$. It is easy to verify that the averaging procedure yields zero for this function, thus $\langle W|v, in\rangle = 0$. This holds for all eigenvectors of H, and these eigenvectors form a basis in the Hilbert space, therefore $|v, in\rangle = 0$. In particular, $|v, in\rangle$ is discontinuous at $g = 0$ as a function of g.

The standard solution to the problem described above is the renormalization of H_0, which consists in replacing the initial H_0 operator in the definition of $U(t_2, t_1)$ by a new one, while H remains unchanged. This renormalized H_0 has to have the same eigenvalues as H. The vector $|v\rangle$ appearing in (5) and (6) should be an eigenvector of the renormalized H_0.

In [17, 19, 20], for instance, it is proposed that the eigenvalues of H_0 be shifted, without changing its eigenvectors. The renormalized H_0 will then take the form

$$H_0^{\mathrm{ren}} = H_0 + \int \mathrm{d}i\, \Delta E_i\, |v_i\rangle\langle v_i| = \int \mathrm{d}i\, E_i\, |v_i\rangle\langle v_i|, \tag{10}$$

where ΔE_i are the self-energy corrections, E_i are the eigenvalues of the H operator and the $|v_i\rangle$ vectors are eigenvectors of H_0 and constitute an orthonormal basis. The integration is understood in a general sense; it includes, for instance, summation over discrete states. The index i labels all eigenvectors of H_0.

In general the eigenspaces of H_0 are degenerate, therefore the choice of the basis vectors $|v_i\rangle$ in (10) is not completely fixed by the requirement that they should be eigenvectors of H_0. If a certain eigenspace of H_0 is degenerate, then this eigenspace can be further decomposed into subspaces the degeneracy of which remains unbroken by the interaction. In (10) the $|v_i\rangle$ vectors are supposed to belong to such subspaces.

The formulas (5) and (6) remain unsatisfactory even after the renormalization of H_0 because $\lim_{\epsilon\to 0^+}\lim_{T\to\infty}\hat{U}_\epsilon(0,-T)$ and $\lim_{\epsilon\to 0^+}\lim_{T\to\infty}\hat{U}_\epsilon(0,T)$ are generally not unitary (more precisely, not scalar product preserving), even though $U(t_2,t_1)$ is unitary. This is caused by the property of the averaging procedure that it assigns zero in the $T\to\infty$ limit to oscillating functions (i.e. to oscillating functions which oscillate around 0).

In order to illustrate the nonunitarity of $\lim_{\epsilon\to 0^+}\lim_{T\to\pm\infty}\hat{U}_\epsilon(0,-T)$ we consider again the case when the Hilbert space is finite dimensional and the spectrum of H_0 is nondegenerate. We assume that g is sufficiently small so that the spectrum of H is also nondegenerate, and that H_0 has been renormalized. Considering the definition of $U(0,-T)$ and the property that $|v,in\rangle$ is an eigenvector of H, we obviously have $\langle v,in|U(0,-T)|v\rangle=\langle v,in|v\rangle$. This implies that $\langle v,in|v,in\rangle=\langle v,in|\lim_{\epsilon\to 0^+}\lim_{T\to\infty}\hat{U}_\epsilon(0,-T)|v\rangle=\langle v,in|v\rangle$. However, $\langle v,in|v,in\rangle=\langle v,in|v\rangle$ cannot hold if $\langle v,in|v,in\rangle=\langle v|v\rangle$ unless $|v,in\rangle=|v\rangle$, which is not the case generally. Therefore $\langle v,in|v,in\rangle\neq\langle v|v\rangle$. In fact, the operator $\lim_{\epsilon\to 0^+}\lim_{T\to\infty}\hat{U}_\epsilon(0,-T)$ projects $|v\rangle$ onto the (one dimensional) space spanned by $|v,in\rangle$, implying in particular that $\langle v,in|v,in\rangle<\langle v|v\rangle$.

On the basis of the above considerations, we can say that the correct form of (5) and (6) should be

$$|v,in\rangle = \lim_{\epsilon\to 0^+}\lim_{T\to\infty}\frac{1}{\sqrt{Z_{v,\epsilon,in}(-T)}}\hat{U}_\epsilon(0,-T)|v\rangle \tag{11}$$

$$|v,out\rangle = \lim_{\epsilon\to 0^+}\lim_{T\to\infty}\frac{1}{\sqrt{Z_{v,\epsilon,out}(T)}}\hat{U}_\epsilon(0,T)|v\rangle, \tag{12}$$

where

$$Z_{v,\epsilon,in}(-T) = \frac{\langle v|\hat{U}_\epsilon(0,-T)^\dagger\hat{U}_\epsilon(0,-T)|v\rangle}{\langle v|v\rangle} \tag{13}$$

$$Z_{v,\epsilon,out}(T) = \frac{\langle v|\hat{U}_\epsilon(0,T)^\dagger\hat{U}_\epsilon(0,T)|v\rangle}{\langle v|v\rangle}. \tag{14}$$

The real numbers $Z_{v,\epsilon,in}$ and $Z_{v,\epsilon,out}$ can be called overlap factors, as they characterize the overlap between $|v\rangle$ and $|v,in\rangle$ or $|v,out\rangle$. $Z_{v,\epsilon,in}$ and $Z_{v,\epsilon,out}$ are included in (11) and (12)

in order to maintain the correct normalization of $|v, in\rangle$ and $|v, out\rangle$. In principle, it still remains to be checked whether the mappings $|v\rangle \to |v, in\rangle$ and $|v\rangle \to |v, out\rangle$ defined by (11) and (12) preserve orthogonality. However, we shall not discuss this problem here.

If an eigenspace of H_0 is degenerate, then this eigenspace can be further decomposed into subspaces the degeneracy of which remains unbroken by the interaction. In the formulas (11) and (12) it is assumed that $|v\rangle$ belongs to such a subspace.

The expressions (13) and (14) are somewhat formal since the denominator is usually not a finite number. If necessary, one can introduce a regularization (see e.g. the Appendix B of [14]) to properly define these quotients. The same applies to some of the formulas in section 4 as well.

The equations (11) and (12) are quite general; they can also be applied in quantum field theory. Nevertheless, they require a suitable choice of the renormalized H_0 operator, and, in particular, the knowledge of the self energy corrections. In perturbation theory these corrections can be calculated order by order from the condition that the $\epsilon \to 0$ limit in (11) and (12) should be convergent.

It is worth noting that in quantum mechanical potential scattering (assuming short range and sufficiently regular potentials) the self-energy corrections are usually 0 (thus the renormalization of H_0 is not necessary) and the overlap factors are usually equal to 1 in the sense that $\lim_{\epsilon\to 0^+} \lim_{T\to\infty} Z_{v,\epsilon,in}(-T) = 1$ and $\lim_{\epsilon\to 0^+} \lim_{T\to\infty} Z_{v,\epsilon,out}(T) = 1$. In quantum field theory, however, these quantities usually have nontrivial values. This can be seen in specific examples (e.g. in the Φ^4 theory) by perturbative calculations.

In the rest of this section we discuss some aspects of the application of perturbation theory to (11) and (12). The perturbation series for $\lim_{T\to\infty}\langle v_i|\hat{U}_\epsilon(0,-T)|v_j\rangle$ with unrenormalized H_0 reads

$$\lim_{T\to\infty}\langle v_i|\hat{U}_\epsilon(0,-T)|v_j\rangle = \langle v_i|v_j\rangle + \frac{-i}{\hbar}g\frac{\langle ij\rangle}{P(ij)+\epsilon}$$
$$+\sum_{k=2}^{\infty}\left(\frac{-i}{\hbar}\right)^k g^k \int \mathrm{d}m_1\mathrm{d}m_2\ldots\mathrm{d}m_{k-1}\frac{\langle im_{k-1}\rangle}{P(ij)+\epsilon}\frac{\langle m_{k-1}m_{k-2}\rangle}{P(m_{k-1}j)+\epsilon}\cdots\frac{\langle m_2m_1\rangle}{P(m_2j)+\epsilon}\frac{\langle m_1j\rangle}{P(m_1j)+\epsilon}, \tag{15}$$

where the notation

$$P(ij) = \frac{i}{\hbar}(E_i^0 - E_j^0), \qquad \langle ij\rangle = \langle v_i|H_{\mathrm{int}}|v_j\rangle \tag{16}$$

is used, and the integrations over the intermediate states are understood in a general sense, in particular they include summation over discrete states.

For $\lim_{T\to\infty}\langle v_i|\hat{U}_\epsilon(0,T)^\dagger|v_j\rangle$ we have

$$\lim_{T\to\infty}\langle v_i|\hat{U}_\epsilon(0,T)^\dagger|v_j\rangle = \langle v_i|v_j\rangle + \frac{-i}{\hbar}g\frac{\langle ij\rangle}{P(ji)+\epsilon}$$
$$+\sum_{k=2}^{\infty}\left(\frac{-i}{\hbar}\right)^k g^k\int \mathrm{d}m_1\mathrm{d}m_2\ldots\mathrm{d}m_{k-1}\frac{\langle im_1\rangle}{P(m_1i)+\epsilon}\frac{\langle m_1m_2\rangle}{P(m_2i)+\epsilon}\ldots\frac{\langle m_{k-2}m_{k-1}\rangle}{P(m_{k-1}i)+\epsilon}\frac{\langle m_{k-1}j\rangle}{P(ji)+\epsilon}. \tag{17}$$

These formulas can be used to derive perturbative formulas for the components of the in and out states and for the S-matrix elements.

It is worth mentioning that it follows from the above formulas that the vectors $\lim_{T\to\infty}\hat{U}_\epsilon(0,-T)|v_j\rangle$ and $\lim_{T\to\infty}\hat{U}_\epsilon(0,T)|v_j\rangle$ also satisfy the Lippmann-Schwinger equations

$$\lim_{T\to\infty}\hat{U}_\epsilon(0,-T)|v_j\rangle = |v_j\rangle + \frac{-i}{\hbar}g\frac{1}{\frac{i}{\hbar}(H_0-E_j^0)+\epsilon}H_{\mathrm{int}}\lim_{T\to\infty}\hat{U}_\epsilon(0,-T)|v_j\rangle \tag{18}$$

$$\lim_{T\to\infty}\hat{U}_\epsilon(0,T)|v_j\rangle = |v_j\rangle + \frac{-i}{\hbar}g\frac{1}{\frac{i}{\hbar}(H_0-E_j^0)-\epsilon}H_{\mathrm{int}}\lim_{T\to\infty}\hat{U}_\epsilon(0,T)|v_j\rangle, \tag{19}$$

where ϵ is positive.

In order to take into account the renormalization of H_0 one should replace in the formulas (15) and (17) the $|v_i\rangle$ vectors by the corresponding eigenvectors $|v_i,ren\rangle$ of H_0^{ren}, the $P(ij)$ given in (16) by $P(ij)=\frac{i}{\hbar}(E_i-E_j)$, and the $\langle ij\rangle$ given in (16) by $\langle ij\rangle=\langle v_i,ren|\frac{H-H_0^{\mathrm{ren}}}{g}|v_j,ren\rangle$. If one uses (10), then, of course, $|v_i,ren\rangle=|v_i\rangle$.

Since the renormalized H_0 depends on g, the series (15) and (17) will cease to be power series in g after the renormalization of H_0. In order to obtain power series again, one has to expand the terms of (15) and (17) and collect the terms having the same power of g.

We note that in the case when H_0 has discrete spectrum the Rayleigh-Schrödinger perturbation theory can be expected to be reproduced in effect.

As we mentioned earlier, the choice of the $|v_i\rangle$ vectors within degenerate eigenspaces of H_0 is not completely unrestricted; the $|v_i\rangle$ vectors should belong to subspaces the degeneracy of which is not broken by the interaction. The decomposition of a degenerate eigenspace into such subspaces depends, of course, on the interaction, and is not necessarily known in advance. Nevertheless, in perturbation theory a wrong choice of $|v_i\rangle$ vectors will result in divergences as the $\epsilon\to 0^+$ limit is taken. This can be used to find suitable $|v_i\rangle$ vectors. For example, in the case when the Hilbert space is finite dimensional, one finds at first order that $|v_i\rangle$ must be such that $\langle ij\rangle=0$ for all $j\neq i$ for which $E_j^0=E_i^0$. This condition is the same as the condition that one finds in Rayleigh-Schrödinger perturbation theory at first order.

4 The method of adiabatic switching

Another method that can be used to define the $T \to \infty$ limit in (2) and (3), instead of the averaging procedure described in section 3, is the adiabatic switching of the interaction. In this case the in and out states are given by

$$|v, in\rangle = \lim_{\epsilon \to 0^+} \lim_{T \to \infty} U_\epsilon(0, -T)|v\rangle \tag{20}$$

$$|v, out\rangle = \lim_{\epsilon \to 0^+} \lim_{T \to \infty} U_\epsilon(0, T)|v\rangle, \tag{21}$$

where

$$U_\epsilon(t_2, t_1) = T \exp\left[-\frac{i}{\hbar}\int_{t_1}^{t_2} H_{I,\epsilon}(t)\, dt\right] \tag{22}$$

and

$$H_{I,\epsilon}(t) = e^{-\epsilon|t|} e^{\frac{i}{\hbar}H_0 t} H_I e^{-\frac{i}{\hbar}H_0 t}. \tag{23}$$

In (22) T denotes time ordering. In potential scattering this definition, like (5) and (6), is usually satisfactory. In general, however, $\lim_{T\to\infty} U_\epsilon(0, -T)$ and $\lim_{T\to\infty} U_\epsilon(0, T)$ are not convergent as $\epsilon \to 0^+$, and this divergence can be related to the existence of nonzero self-energy corrections. One can try to handle this problem by the renormalization of H_0. In general the renormalized H_0 operators which are suitable for the adiabatic switching method are different from those which are suitable for the averaging method. In particular, it is not true that the renormalized H_0 operator has to have the same eigenvalues as H. To illustrate this point we consider the case when the Hilbert space is finite dimensional and both H_0 and H are diagonal and have nondegenerate spectra. The renormalized H_0 operator is also taken to be diagonal. It is not difficult to check, using the perturbation formulas (24) and (25) below, that if the adiabatic switching is applied, then the necessary shift of the eigenvalues of H_0 is not the full self-energy correction (i.e. the difference between the energy of $|v, in\rangle$ and the energy of $|v\rangle$). Specifically, if the self-energy correction is $gE_v^1 + g^2E_v^2 + \ldots$, then the required energy shift is $\Delta E_v = gE_v^1 + g^2\frac{E_v^2}{2} + \ldots$, i.e. at first order the two quantities agree, but at second order the required energy renormalization is only the half of the self-energy correction. Although the energy renormalizations are generally not equal to the self-energy corrections, the latter can of course be obtained from the eigenvalue equations for the in and out states.

Another significant difference from the averaging method is that $\lim_{T\to\infty} U_\epsilon(0, -T)$, $\lim_{T\to\infty} U_\epsilon(0, T)$ and their limit as $\epsilon \to 0^+$ are unitary, therefore it is not necessary to include any overlap factor in (20) and (21).

The perturbation series for $\lim_{T\to\infty}\langle v_i|U_\epsilon(0,-T)|v_j\rangle$ with unrenormalized H_0 reads

$$\lim_{T\to\infty}\langle v_i|U_\epsilon(0,-T)|v_j\rangle = \langle v_i|v_j\rangle + \frac{-i}{\hbar}g\frac{\langle ij\rangle}{P(ij)+\epsilon} + \sum_{k=2}^{\infty}\left(\frac{-i}{\hbar}\right)^k g^k \int \mathrm{d}m_1\mathrm{d}m_2\ldots\mathrm{d}m_{k-1} \frac{\langle im_{k-1}\rangle}{P(ij)+k\epsilon}\frac{\langle m_{k-1}m_{k-2}\rangle}{P(m_{k-1}j)+(k-1)\epsilon}\cdots\frac{\langle m_2m_1\rangle}{P(m_2j)+2\epsilon}\frac{\langle m_1j\rangle}{P(m_1j)+\epsilon}, \quad (24)$$

where the notation (16) is used. This formula differs from (15) in the coefficients of ϵ in the denominators. Up to first order, however, there is no difference between the adiabatic switching method and the averaging method. This implies, in particular, that the Z normalization constants in (11) and (12) can be taken to be 1 (i.e. are not needed) up to first order.

For $\lim_{T\to\infty}\langle v_i|U_\epsilon(T,0)|v_j\rangle$ we have

$$\lim_{T\to\infty}\langle v_i|U_\epsilon(T,0)|v_j\rangle = \langle v_i|v_j\rangle + \frac{-i}{\hbar}g\frac{\langle ij\rangle}{P(ji)+\epsilon} + \sum_{k=2}^{\infty}\left(\frac{-i}{\hbar}\right)^k g^k \int \mathrm{d}m_1\mathrm{d}m_2\ldots\mathrm{d}m_{k-1} \frac{\langle im_1\rangle}{P(m_1i)+\epsilon}\frac{\langle m_1m_2\rangle}{P(m_2i)+2\epsilon}\cdots\frac{\langle m_{k-2}m_{k-1}\rangle}{P(m_{k-1}i)+(k-1)\epsilon}\frac{\langle m_{k-1}j\rangle}{P(ji)+k\epsilon}. \quad (25)$$

These formulas can be used to derive perturbation series for the in and out states and for the S-matrix elements.

Due to the difference between the coefficients of ϵ in (24), (25) and in (15), (17) the vectors $\lim_{T\to\infty}U_\epsilon(0,-T)|v_j\rangle$ and $\lim_{T\to\infty}U_\epsilon(0,T)|v_j\rangle$ do not satisfy Lippmann-Schwinger equations similar to (18) and (19).

We note that in quantum field theory the use of the adiabatic switching method seems to be restricted mainly to the vacuum state, and in this case Gell-Mann and Low's formula [21, 22] is used, which is slightly different from (20) or (21) and does not require the renormalization of H_0.

In [14] we proposed the following formulas instead of (20) and (21):

$$|v,in\rangle = \lim_{\epsilon\to 0^+}\lim_{T\to\infty}\Pi_{v,\epsilon}(-T)U_\epsilon(0,-T)|v\rangle \quad (26)$$

$$|v,out\rangle = \lim_{\epsilon\to 0^+}\lim_{T\to\infty}\Pi_{v,\epsilon}(T)U_\epsilon(0,T)|v\rangle, \quad (27)$$

where $\Pi_{v,\epsilon}(T)$ is a complex number of absolute value 1, i.e. a phase factor, given by the formula

$$\Pi_{v,\epsilon}(T) = \frac{\sqrt{\langle v|U_\epsilon(T,0)|v\rangle\langle v|U_\epsilon(0,T)|v\rangle}}{\langle v|U_\epsilon(0,T)|v\rangle}. \quad (28)$$

The formulas (26) and (27) can be regarded as generalizations of Gell-Mann and Low's formula. The phase factors do not have any effect on transition probabilities. We argued

that the divergences in the $\epsilon \to 0$ limit are milder with these formulas than with (20) and (21), and the necessary renormalization of H_0 is less strictly determined. We also stated the conjecture that in most quantum field theories the divergence in the $\epsilon \to 0^+$ limit is entirely cancelled out by the phase factors, i.e. no renormalization of H_0 is needed at all.

5 Conclusion

In this paper we discussed the methods of averaging and adiabatic switching for the definition of the infinite time limit of the time evolution operator $U(0, \pm T)$ in scattering theory. We described two significant differences between the two methods. The first difference is that the operator obtained by the averaging method is not unitary in general, therefore normalization factors are needed in the definition of the in and out states in order to ensure the conservation of probabilities. In the case of the adiabatic switching method such a problem of unitarity does not arise. The other difference is that in the case of the averaging method the energy renormalizations needed in the free Hamiltonian operator are equal to the energy shifts caused by the interaction, whereas in the case of the adiabatic switching these quantities are generally different. A further difference between the two methods is that Lippmann-Schwinger equation exists for the averaging method, but similar equations do not hold for the adiabatic switching method. At the level of perturbation series these differences are encoded entirely in the (seemingly minor) difference between the coefficients of ϵ in the formulas (15), (17) and (24), (25). We also saw that the two methods are identical to first order, but generally differ at higher orders.

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