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On Dini derivatives of real functions

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Abstract. For a continuous function f, the set V_f made of those points where the lower left derivative is strictly less than the upper right derivative is totally disconnected. Besides continuity, alternative assumptions are proposed so to preserve this property. On the other hand, we construct a function f whose set V_f coincides with the entire domain, and nevertheless f is continuous on an infinite set, possibly having infinitely many cluster points. Some open problems are proposed.

1 Introduction and main result

Dini derivatives take their names after Ulisse Dini, who introduced them in 1878, cf. [4]; let us recall their standard notations

$$D_+f(x) = \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h}, \qquad D^+f(x) = \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h},$$
$$D_-f(x) = \liminf_{h\to 0^-} \frac{f(x+h)-f(x)}{h}, \qquad D^-f(x) = \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h}.$$

Here, and in the rest of the paper, we assume that $f : I \to \mathbb{R}$ is defined on some open interval $I \subseteq \mathbb{R}$. A fundamental step in the study of Dini derivatives was achieved in the first quarter of the twentieth century by Denjoy [3] for continuous functions, Young [8] for measurable functions, and Saks [7] for arbitrary ones. The Denjoy–Young–Saks theorem states that at each point x, except for a set of measure zero, one of the following four alternatives holds:

1. f has a finite derivative at x;
2. $D_-f(x) = D^+f(x) \in \mathbb{R}, \quad D^-f(x) = +\infty, \ D_+f(x) = -\infty;$
3. $D^-f(x) = D_+f(x) \in \mathbb{R}, \ D^+f(x) = +\infty, \ D_-f(x) = -\infty;$
4. $D^-f(x) = D^+f(x) = +\infty, \quad D_-f(x) = D_+f(x) = -\infty.$

Denjoy also explicitly constructed a continuous function realizing each of the previous four conditions on a perfect set of positive Lebesgue measure; a highly remarkable result, in consideration of the fact that continuous functions can exhibit very pathological behaviors (see, e.g., [5]). We refer to the book by Bruckner [1] for an extensive study of Dini derivatives and a more complete historical account.

In this paper, for any function $f : I \to \mathbb{R}$, we are interested in studying the set

$$V_f := \{x \in I : D_-f(x) < D^+f(x)\}.$$

It should be noticed that, in the above mentioned example by Denjoy, the set V_f is totally disconnected, i.e., it does not contain any nontrivial interval. The main question is: *how large can this set be?*

It is well known that there exist *non-continuous* functions $f : \mathbb{R} \to \mathbb{R}$ for which $V_f = \mathbb{R}$ (see for instance [2], where the function $f : \mathbb{R} \to \mathbb{R}$ has a dense graph in $\mathbb{R}^2$). On the contrary, we will prove that there are no *continuous* functions with such a property. To be more precise, let us introduce the following class of functions.

Definition 1. *We say that a function $f : I \to \mathbb{R}$ is* upper well behaved *if for every compact interval J contained in I there is a $x_J \in J$ such that $f(x_J) = \max f(J)$.*

Clearly, every continuous function is upper well behaved. On the other hand, one can easily find examples of upper well behaved functions which are nowhere continuous (e.g., the well known Dirichlet function).

Here is our first result.

Theorem 2. *If $f : I \to \mathbb{R}$ is upper well behaved, then the set V_f is totally disconnected.*

Our theorem complements Denjoy's example of a continuous function, for which $\mu(V_f) > 0$; it suggests that, if f is continuous, the set V_f should be "small", in some sense. Some questions then arise:

Q1. *If $f : I \to \mathbb{R}$ is continuous, or even upper well behaved, is the set V_f of first Baire category?*

Q2. *If $I = (a, b)$ and $f : I \to \mathbb{R}$ is continuous, can $\mu(V_f)$ be equal to $b - a$?*

Let us now investigate on the possibility for a function $f : \mathbb{R} \to \mathbb{R}$ to be such that $V_f = \mathbb{R}$ and, at the same time, to be continuous at some points of its domain. We will prove that such a function exists, and the set of its continuity points A can be infinite. However, we need to assume that the points of $\mathcal{D}(A)$, i.e., the cluster points of A, are all isolated. Here is the precise statement.

Theorem 3. *For any closed set $A \subseteq \mathbb{R}$ such that $\mathcal{D}(\mathcal{D}(A)) = \varnothing$, there exists a function $f : \mathbb{R} \to \mathbb{R}$, with $V_f = \mathbb{R}$, whose set of continuity points coincides with A.*

A further question then arises:

Q3. *If $V_f = \mathbb{R}$, can the function $f : \mathbb{R} \to \mathbb{R}$ be continuous on a dense set of points?*

The proofs of Theorem 2 and Theorem 3 are provided in the next section. They are based on the knowledge that every monotone function is differentiable almost everywhere, and on some simple properties of continued fractions.

2 Proofs

Proof of Theorem 2. By contradiction, let $[a, b] \subseteq V_f$, with $a < b$. Let $(x_n)_n$ be a sequence in $[a, b]$ such that $f(x_n) \to \inf f([a, b])$. Passing if necessary to a subsequence, we can assume that $x_n \to \check{x}$, for some $\check{x} \in [a, b]$. We have two cases.

Case 1: $\check{x} \in [a,b)$. We will prove that f is increasing in $(\check{x},b]$, hence almost everywhere differentiable there, a contradiction.

By contradiction, let α,β in $(\check{x},b]$ be such that $\alpha < \beta$ and $f(\alpha) > f(\beta)$. Being $\check{x} < \alpha$ and $f(\alpha) > \inf f([a,b])$, there is a n such that $x_n < \alpha$ and $f(x_n) < f(\alpha)$. Since f is upper well behaved, there is a $\hat{x} \in [x_n,\beta]$ such that $f(\hat{x}) = \max f([x_n,\beta])$. Being $f(\hat{x}) \geq f(\alpha) > \max\{f(x_n), f(\beta)\}$, it has to be $\hat{x} \in (x_n,\beta)$, whence $D_- f(\hat{x}) \geq 0 \geq D^+ f(\hat{x})$, a contradiction, since $\hat{x} \in V_f$.

Case 2: $\check{x} = b$. One proves in an analogous way that f is decreasing in $[a,b)$, hence almost everywhere differentiable there, a contradiction.

The proof is thus completed. □

Remark 4. *If we define a function $f : I \to \mathbb{R}$ to be* lower well behaved *when $(-f)$ is upper well behaved, then it can be proved that the set*

$$\Lambda_f := \{x \in I : D^- f(x) > D_+ f(x)\}$$

is totally disconnected.

In order to prove Theorem 3, we need a preliminary result.

Lemma 5. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function, and define*

$$f(x) = \begin{cases} \psi(x)\,, & \text{if } x = 0 \text{ or } x \in \mathbb{R}\setminus\mathbb{Q}\,, \\ \left(2 - \dfrac{1}{p}\right)\psi(x)\,, & \text{if } x \in \mathbb{Q}\setminus\{0\} \text{ and } |x| = \frac{p}{q} \text{ with } \gcd(p,q) = 1\,. \end{cases}$$

Then, the set of continuity points of f coincides with the set of zeros of ψ; moreover,

- *if $\psi(x) \neq 0$, then $D^+ f(x) = +\infty$ and $D_- f(x) = -\infty$;*
- *if $\psi(x) = 0$, then $D^+ f(x) = 2D^+\psi(x)$ and $D_- f(x) = 2D_-\psi(x)$.*

Proof. The result is proved by means of the theory of continued fractions, for which we refer to [6]. We fix $x \in \mathbb{R}$ and consider two cases.

Case 1: $\psi(x) \neq 0$. It is easy to prove that f is not continuous at these points.

If $x \in (0,+\infty)\setminus\mathbb{Q}$, let $(c_n(x))_{n\in\mathbb{N}}$ be the sequence of convergents of the continued fraction representing x. Define

$$x_n^+ = \frac{a_{2n}}{b_{2n}} = c_{2n}(x)\,, \qquad x_n^- = \frac{a_{2n+1}}{b_{2n+1}} = c_{2n+1}(x)\,.$$

The sequence $(x_n^+)_n$ converges to the right while $(x_n^-)_n$ converges to the left to x. Since the fractions $c_n(x)$ are in lowest terms, we have

$$\frac{f(x_n^+) - f(x)}{x_n^+ - x} = \frac{\left(2 - \dfrac{1}{a_{2n}}\right)\psi(c_{2n}(x)) - \psi(x)}{c_{2n}(x) - x} \to +\infty\,,$$

because the numerator tends to $\psi(x) > 0$ as $n \to +\infty$. Analogously,

$$\frac{f(x_n^-) - f(x)}{x_n^- - x} = \frac{\left(2 - \dfrac{1}{a_{2n+1}}\right)\psi(c_{2n+1}(x)) - \psi(x)}{c_{2n+1}(x) - x} \to -\infty\,,$$

Hence, $D^+ f(x) = +\infty$ and $D_- f(x) = -\infty$.

If $x \in (0, +\infty) \cap \mathbb{Q}$, let $x = \frac{a}{b}$ with $\gcd(a, b) = 1$, and define, for every $n \in \mathbb{N}$,

$$y_n^+ = \frac{a}{b} + \frac{1}{(2b)^n} = \frac{2^n ab^{n-1} + 1}{2^n b^n}, \qquad y_n^- = \frac{a}{b} - \frac{1}{(2b)^n} = \frac{2^n ab^{n-1} - 1}{2^n b^n}.$$

For every $n \geq 2$, the fractions are reduced to lowest terms, while their numerators tend to infinity as $n \to +\infty$. So,

$$\frac{f(y_n^+) - f(x)}{y_n^+ - x} = \frac{\left(2 - \dfrac{1}{2^n ab^{n-1} + 1}\right)\psi\,(y_n^+) - \left(2 - \dfrac{1}{a}\right)\psi\,(x)}{(2b)^{-n}} \to +\infty\,,$$

because the numerator tends to $\frac{1}{a}\psi(x) > 0$ as $n \to +\infty$. Analogously,

$$\frac{f(y_n^-) - f(x)}{y_n^- - x} = -\frac{\left(2 - \dfrac{1}{2^n ab^{n-1} - 1}\right)\psi\,(y_n^-) - \left(2 - \dfrac{1}{a}\right)\psi\,(x)}{(2b)^{-n}} \to -\infty\,.$$

Hence, $D^+ f(x) = +\infty$ and $D_- f(x) = -\infty$. We have thus proved the conclusion, in this case, for every $x > 0$.

A similar argument leads to the conclusion when $x < 0$. Finally, if $x = 0$, we define, for every $n \geq 1$,

$$z_n^+ = \frac{n+1}{n^2}\,, \qquad z_n^- = -\frac{n+1}{n^2}\,,$$

so that

$$\frac{f(z_n^\pm) - f(0)}{z_n^\pm - 0} = \frac{\left(2 - \dfrac{1}{n+1}\right)\psi\,(z_n^\pm) - \psi\,(0)}{z_n^\pm} \to \pm\infty\,,$$

since $\psi(0) > 0$, hence proving again that $D^+ f(0) = +\infty$ and $D_- f(0) = -\infty$.

<u>Case 2</u>: $\psi(x) = 0$. The continuity of f at x is trivial, since

$$\psi(y) \leq f(y) \leq 2\psi(y)\,, \qquad \text{for every } y \in \mathbb{R}\,. \tag{1}$$

The function

$$r_x(y) = \frac{\psi(y) - \psi(x)}{y - x} = \frac{\psi(y)}{y - x}$$

is continuous in its domain $\mathbb{R} \setminus \{x\}$, and

$$r_x(y)(y - x) \geq 0\,, \qquad \text{for every } y \in \mathbb{R} \setminus \{x\}\,. \tag{2}$$

Moreover,

$$D^+\psi(x) = \limsup_{y \to x^+} r_x(y)\,, \qquad D_-\psi(x) = \liminf_{y \to x^-} r_x(y)\,.$$

Correspondingly, we can find two sequences of irrational numbers $(\xi_n^-)_n$ in $(-\infty, x)$ and $(\xi_n^+)_n$ in $(x, +\infty)$ such that $\lim_n \xi_n^\pm = x$ and

$$\lim_n r_x(\xi_n^+) = D^+\psi(x)\,, \qquad \lim_n r_x(\xi_n^-) = D_-\psi(x)\,.$$

We now assume $x > 0$. Recalling the notation $(c_n(\zeta))_n$ for the sequence of the convergents of the continued fraction representing $\zeta \notin \mathbb{Q}$, we can find two sequences of positive rational numbers $(\zeta_n^\pm)_n$ such that

$$\zeta_n^- = c_{2\kappa(n)+1}(\xi_n^-) = \frac{\gamma_n^-}{\delta_n^-} \quad \text{and} \quad \zeta_n^+ = c_{2\kappa(n)}(\xi_n^+) = \frac{\gamma_n^+}{\delta_n^+},$$

where the choice $\kappa(n) > n$ is such that $|\xi_n^\pm - \zeta_n^\pm| < n^{-1}$, $|r_x(\xi_n^\pm) - r_x(\zeta_n^\pm)| < n^{-1}$, and $\gamma_n^\pm > n$. In particular, we can ensure that $\lim_n \zeta_n^\pm = x$ and

$$\lim_n r_x(\zeta_n^+) = D^+\psi(x), \qquad \lim_n r_x(\zeta_n^-) = D_-\psi(x).$$

Finally,

$$\frac{f(\zeta_n^+) - f(x)}{\zeta_n^+ - x} = \frac{f(\zeta_n^+)}{\zeta_n^+ - x} = \left(2 - \frac{1}{\gamma_n^+}\right) \frac{\psi(\zeta_n^+)}{\zeta_n^+ - x} \to 2D^+\psi(x),$$

$$\frac{f(\zeta_n^-) - f(x)}{\zeta_n^- - x} = \frac{f(\zeta_n^-)}{\zeta_n^- - x} = \left(2 - \frac{1}{\gamma_n^-}\right) \frac{\psi(\zeta_n^-)}{\zeta_n^- - x} \to 2D_-\psi(x).$$

Hence, $D^+f(x) = 2D^+\psi(x)$ and $D_-f(x) = 2D_-\psi(x)$, taking into account (1) and (2).

The cases when $x < 0$ or $x = 0$ can be carried out similarly. The proof is thus completed. □

Proof of Theorem 3. The aim is to construct a non-negative continuous function ψ whose set of zeros coincides with A, satisfying

$$D^+\psi(x) = +\infty \quad \text{and} \quad D_-\psi(x) = -\infty, \quad \text{for every } x \in A.$$

Let us first introduce some notations. We define the function

$$\sqrt[\sharp]{x} = \begin{cases} \sqrt{x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

and the following functions: for an interval $[c, d] \subseteq \mathbb{R} \cup \{\pm\infty\}$ and a constant $\eta > 0$,

$$g_{[c,d]}^\eta = \begin{cases} \eta \sqrt[\sharp]{1 - \left(\frac{2x - c - d}{d - c}\right)^2}, & \text{if } c, d \in \mathbb{R} \text{ and } c < d, \\ \sqrt[\sharp]{x - c}, & \text{if } c \in \mathbb{R} \text{ and } d = +\infty, \\ \sqrt[\sharp]{d - x}, & \text{if } c = -\infty \text{ and } d \in \mathbb{R}, \\ 1, & \text{if } c = -\infty \text{ and } d = +\infty, \\ 0, & \text{if } c = d. \end{cases}$$

Let us treat the case $\mathcal{D}(A) \neq \varnothing$, the other case being simpler. By hypothesis, $\mathcal{D}(A)$ is countable and made of isolated points, so we can list its elements as

$$\left.\begin{array}{r} -\infty = \cdots = \bar{\mathfrak{a}}_{-p-1} \\ \text{or} \\ -\infty < \cdots < \bar{\mathfrak{a}}_{-p-1} \end{array}\right\} < \bar{\mathfrak{a}}_{-p} < \cdots < \bar{\mathfrak{a}}_0 < \cdots < \bar{\mathfrak{a}}_q < \left\{\begin{array}{l} \bar{\mathfrak{a}}_{q+1} = \cdots = +\infty, \\ \text{or} \\ \bar{\mathfrak{a}}_{q+1} < \cdots < +\infty, \end{array}\right.$$

depending on the boundedness of the set. Fix $j \in \mathbb{Z}$ and consider the interval $[\bar{\mathfrak{a}}_j\,, \bar{\mathfrak{a}}_{j+1}]$.

$$\textit{If}\ \ (\bar{\mathfrak{a}}_j\,, \bar{\mathfrak{a}}_{j+1}) \cap A = \varnothing\,, \quad \textit{we set} \quad \psi_j(x) = g^1_{[\bar{\mathfrak{a}}_j\,, \bar{\mathfrak{a}}_{j+1}]}\,.$$

Otherwise, let $a_0^j \in (\bar{\mathfrak{a}}_j\,, \bar{\mathfrak{a}}_{j+1}) \cap A$. We are going to define a bilateral increasing sequence $(a_n^j)_{n\in\mathbb{Z}}$ in $[\bar{\mathfrak{a}}_j\,, \bar{\mathfrak{a}}_{j+1}] \cap A$ such that

$$\lim_{n\to-\infty} a_n^j = \bar{\mathfrak{a}}_j\,, \qquad \lim_{n\to+\infty} a_n^j = \bar{\mathfrak{a}}_{j+1}\,.$$

If $(\bar{\mathfrak{a}}_j\,, a_0^j) \cap A$ has infinitely-many elements, since $\mathcal{D}(A) \cap (\bar{\mathfrak{a}}_j\,, a_0^j) = \varnothing$, we can order them as

$$a_{-1}^j > a_{-2}^j > \cdots > a_{-n}^j > \cdots\,. \tag{3}$$

Otherwise, if $(\bar{\mathfrak{a}}_j\,, a_0^j) \cap A$ has finitely-many elements, say M, we can list them in order to have

$$a_{-1}^j > a_{-2}^j > \cdots > a_{-M}^j > a_{-(M+1)}^j = a_{-(M+2)}^j = \cdots = \bar{\mathfrak{a}}_j\,. \tag{4}$$

Similarly, if $(a_0^j\,,\, \bar{\mathfrak{a}}_{j+1}) \cap A$ has infinitely-many elements we can order them as

$$a_1^j < a_2^j < \cdots < a_n^j < \cdots\,.$$

Otherwise, if $(a_0^j\,,\, \bar{\mathfrak{a}}_{j+1}) \cap A$ has finitely-many elements, say N, we can list them in order to have

$$a_1^j < a_2^j < \cdots < a_N^j < a_{N+1}^j = a_{N+2}^j = \cdots = \bar{\mathfrak{a}}_{j+1}\,.$$

If a_n^j and a_{n+1}^j are real numbers, we set

$$s_n^j = \frac{a_{n+1}^j + a_n^j}{2}\,, \qquad \text{and} \qquad \eta_n^j = \min\{q_n^j, 1\}\,,$$

where

$$q_n^j = \begin{cases} \sqrt{s_n^j - \bar{\mathfrak{a}}_j}\,, & \text{if } n < 0\,, \\[2ex] \sqrt{\bar{\mathfrak{a}}_{j+1} - s_n^j}\,, & \text{if } n \geq 0\,. \end{cases}$$

Otherwise, if $a_n^j = -\infty$ or $a_{n+1}^j = +\infty$, we set $\eta_n^j = 1$. So, we can define

$$\psi_j(x) = \sum_{n\in\mathbb{Z}} g^{\eta_n^j}_{[a_n^j\,, a_{n+1}^j]}(x)\,.$$

Notice that, by definition, for every $j \in \mathbb{Z}$ the function ψ_j is null outside $(\bar{\mathfrak{a}}_j, \bar{\mathfrak{a}}_{j+1})$, while for every $x \in (\bar{\mathfrak{a}}_j, \bar{\mathfrak{a}}_{j+1})$, we have that $\psi_j(x) = 0$ if and only if $x \in \{a_n^j\ :\ n \in \mathbb{Z}\}$.

Finally, we set

$$\psi(x) = \sum_{j\in\mathbb{Z}} \psi_j(x)\,.$$

By the above construction, it is easy to verify that the set of zeros of ψ coincides with A. We now prove that ψ is continuous, with

$$D^+\psi(x) = +\infty \quad \text{and} \quad D_-\psi(x) = -\infty\,, \quad \text{for every } x \in A\,. \tag{5}$$

Let us fix $x \in \mathbb{R}$, and consider three different cases.

If $x \notin A$, we have that $x \in (a_n^j\,,\, a_{n+1}^j)$, for some j, n in $\mathbb{Z}$. The continuity at x is trivial since $\psi = g^{\eta_n^j}_{[a_n^j\,,\, a_{n+1}^j]}$ in a neighborhood of x.

If $x \in A \setminus \mathcal{D}(A)$, there exists $j \in \mathbb{Z}$ and $n \in \mathbb{Z}$ such that

$$a_{n-1}^j < x = a_n^j < a_{n+1}^j\,,$$

so we have

$$\psi_j(y) = g^{\eta_{n-1}^j}_{[a_{n-1}^j\,,\, a_n^j]}(y) + g^{\eta_n^j}_{[a_n^j\,,\, a_{n+1}^j]}(y)\,, \quad \text{for every } y \in [a_{n-1}^j\,,\, a_{n+1}^j]\,.$$

The continuity at x and property (5) follow by a simple calculation; in this case, the function ψ admits left and right derivatives at $x = a_n^j$.

Let us now consider the case when $x = \bar{\mathfrak{a}}_j \in \mathcal{D}(A)$. In order to prove that $\lim_{y \to \bar{\mathfrak{a}}_j^+} \psi(y) = 0$ and $D^+ f(\bar{\mathfrak{a}}_j) = +\infty$, we need to consider separately the two cases (3) or (4). Case (4) can be treated as above, since ψ coincides with $g^{\eta^j_{-(M+1)}}_{[a^j_{-(M+1)}\,,\, a^j_{-M}]}$ in a right neighborhood of $\bar{\mathfrak{a}}_j$.

Assume now that we are in case (3). Consider the sequence $(s_n^j)_{n\in\mathbb{Z}}$ defined above, so that $\lim_{n\to-\infty} s_n^j = \bar{\mathfrak{a}}_j$. Since, in a right neighborhood of $\bar{\mathfrak{a}}_j$,

$$\begin{aligned}
\psi(y) \, &= \psi_j(y) = \sum_{n<0} g^{\eta_n^j}_{[a_n^j\,,\, a_{n+1}^j]}(y) \\
&\le \sum_{n<0} \eta_n^j \, \chi_{[a_n^j\,,\, a_{n+1}^j]}(y) \le \sum_{n<0} \sqrt{s_n^j - \bar{\mathfrak{a}}_j}\; \chi_{[a_n^j\,,\, a_{n+1}^j]}(y)
\end{aligned}$$

(here χ_E denotes the characteristic function of the interval E), we have that $\lim_{y \to \bar{\mathfrak{a}}_j^+} \psi(y) = 0$.

On the other hand, in order to prove that $D^+ f(\bar{\mathfrak{a}}_j) = +\infty$, notice that, since $\lim_{n\to-\infty} q_n^j = 0$, for $n < 0$ with $|n|$ sufficiently large we have

$$\psi(s_n^j) = \psi_j(s_n^j) = g^{\eta_n^j}_{[a_n^j\,,\, a_{n+1}^j]}(s_n^j) = \eta_n^j = \sqrt{s_n^j - \bar{\mathfrak{a}}_j}\,,$$

thus giving us

$$\frac{\psi(s_n^j) - \psi(\bar{\mathfrak{a}}_j)}{s_n^j - \bar{\mathfrak{a}}_j} = \frac{\sqrt{s_n^j - \bar{\mathfrak{a}}_j} - 0}{s_n^j - \bar{\mathfrak{a}}_j} \to +\infty\,, \qquad \text{as } n \to -\infty\,.$$

A similar procedure shows that $\lim_{y \to \bar{\mathfrak{a}}_j^-} \psi(y) = 0$ and $D_- f(\bar{\mathfrak{a}}_j) = -\infty$.

We have thus proved that ψ is continuous and property (5) holds. Recalling Lemma 5, the proof is completed. □

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