

EXTENDED AFFINE WEYL GROUPS AND FROBENIUS MANIFOLDS – II

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ABSTRACT. For the root system of type B_l and C_l , we generalize the result of [5] by showing the existence of a Frobenius manifold structure on the orbit space of the extended affine Weyl group that corresponds to any vertex of the Dynkin diagram instead of a particular choice of [5].

1. INTRODUCTION

For an irreducible reduced root system R defined in l -dimensional Euclidean space V with Euclidean inner product (\cdot, \cdot) , we fix a basis of simple roots $\alpha_1, \dots, \alpha_l$ and denote by α_j^\vee , $j = 1, 2, \dots, l$ the corresponding coroots. The Weyl group W is generated by the reflections

$$\mathbf{x} \mapsto \mathbf{x} - (\alpha_j^\vee, \mathbf{x})\alpha_j, \quad \forall \mathbf{x} \in V, \quad j = 1, \dots, l. \quad (1.1)$$

The semi-direct product of W by the lattice of coroots yields the affine Weyl group W_a that acts on V by the affine transformations

$$\mathbf{x} \mapsto w(\mathbf{x}) + \sum_{j=1}^l m_j \alpha_j^\vee, \quad w \in W, \quad m_j \in \mathbb{Z}. \quad (1.2)$$

We denote by $\omega_1, \dots, \omega_l$ the fundamental weights that are defined by relations

$$(\omega_i, \alpha_j^\vee) = \delta_{ij}, \quad i, j = 1, \dots, l. \quad (1.3)$$

Fixing a simple root α_k , we define an *extended affine Weyl group* $\widetilde{W} = \widetilde{W}^{(k)}(R)$ as in [5]. It acts on the extended space

$$\widetilde{V} = V \oplus \mathbb{R}$$

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and is generated by the transformations

$$x = (\mathbf{x}, x_{l+1}) \mapsto (w(\mathbf{x}) + \sum_{j=1}^l m_j \alpha_j^\vee, x_{l+1}), \quad w \in W, m_j \in \mathbb{Z}, \quad (1.4)$$

and

$$x = (\mathbf{x}, x_{l+1}) \mapsto (\mathbf{x} + \omega_k, x_{l+1} - 1). \quad (1.5)$$

Let us introduce coordinates x_1, \dots, x_l on the space V by

$$\mathbf{x} = x_1 \alpha_1^\vee + \dots + x_l \alpha_l^\vee. \quad (1.6)$$

Denote by f the determinant of the Cartan matrix of the root system R .

Definition 1.1 ([5]). $\mathcal{A} = \mathcal{A}^{(k)}(R)$ is the ring of all \widetilde{W} -invariant Fourier polynomials of the form

$$\sum_{m_1, \dots, m_{l+1} \in \mathbb{Z}} a_{m_1, \dots, m_{l+1}} e^{2\pi i(m_1 x_1 + \dots + m_l x_l + \frac{1}{f} m_{l+1} x_{l+1})}$$

that are bounded in the limit

$$\mathbf{x} = \mathbf{x}^0 - i \omega_k \tau, \quad x_{l+1} = x_{l+1}^0 + i \tau, \quad \tau \rightarrow +\infty \quad (1.7)$$

for any $x^0 = (\mathbf{x}^0, x_{l+1}^0)$.

For the fixed simple root α_k , we introduce a set of numbers

$$d_j = (\omega_j, \omega_k), \quad j = 1, \dots, l \quad (1.8)$$

and define the following Fourier polynomials [5]

$$\tilde{y}_j(x) = e^{2\pi i d_j x_{l+1}} y_j(\mathbf{x}), \quad j = 1, \dots, l, \quad (1.9)$$

$$\tilde{y}_{l+1}(x) = e^{2\pi i x_{l+1}}. \quad (1.10)$$

Here $y_1(\mathbf{x}), \dots, y_l(\mathbf{x})$ are the W_a -invariant Fourier polynomials defined by

$$y_j(\mathbf{x}) = \frac{1}{n_j} \sum_{w \in W} e^{2\pi i(\omega_j, w(\mathbf{x}))}, \quad n_j = \#\{w \in W | e^{2\pi i(\omega_j, w(\mathbf{x}))} = e^{2\pi i(\omega_j, \mathbf{x})}\}. \quad (1.11)$$

It was shown in [5] that for some particular choices of the simple root α_k , a Chevalley type theorem holds true for the ring \mathcal{A} , i.e., it is generated by $\tilde{y}_1, \dots, \tilde{y}_{l+1}$, and thus the orbit space defined as $\mathcal{M} = \text{Spec} \mathcal{A}$ of the extended affine Weyl group \widetilde{W} is an affine algebraic variety of dimension $l + 1$. Furthermore, in [5] it was proved that on such an orbit space there exists a *Frobenius manifold structure* whose potential is a polynomial of $t^1, \dots, t^l, e^{t^{l+1}}$. Here t^1, \dots, t^{l+1} are the flat coordinates of the Frobenius manifold. For the root system of type A_l , there

is in fact no restrictions on the choice of α_k . However, for the root systems of type $B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2$ there is only one choice for each. Recall that the geometric structures on the orbit spaces \mathcal{M} generalize those that live on the orbit spaces of the finite Coxeter groups discovered in [9, 8] and [4].

In [10] P. Solodoway showed that the Chevalley type theorem of [5] can also be derived from the results of K. Wirthmüller [11], and in fact it holds true for any choice of the base element α_k , or equivalently, for any fixed vertex of the Dynkin diagram. So we have

Theorem 1.2 ([10, 11]). *The ring \mathcal{A} is isomorphic to the ring of polynomials of $\tilde{y}_1(x), \dots, \tilde{y}_{l+1}(x)$.*

A natural question, as it was pointed out in [5, 10], is whether the geometric structures revealed in [5] also exist on the orbit spaces of the extended affine Weyl groups for an arbitrary choice of α_k ? The purpose of the present paper is to give an affirmative answer to this question for the root systems of type B_l, C_l . We will show that on the corresponding orbit spaces there also exist Frobenius manifold structures with potentials that are polynomials in $t^1, \dots, t^{l-1}, t^l, \frac{1}{t}, e^{t^{l+1}}$. Here t^1, \dots, t^{l+1} are the flat coordinates of the resulting Frobenius manifold.

The paper is organized as follows: in Sec.2, we construct a flat pencil of metrics on each orbit space of the extended affine Weyl group of the root system of type C_l for any fixed vertex of the Dynkin diagram, then in Sec.3 we study properties of the flat coordinates of the flat metric (η^{ij}) . In Sec.4, we prove the existence of a Frobenius manifold structure on each orbit space. In Sec.5 we give some examples. In Sec.6 we show that to the root system of type B_l we can apply a similar construction as the one for the root system of type C_l . The resulting Frobenius manifolds are isomorphic to those that are obtained from C_l . Some concluding remarks are given in the last section.

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2. FLAT PENCILS OF METRICS ON THE ORBIT SPACES OF $\widetilde{W}^{(k)}(C_l)$

Let \mathcal{M} be the orbit space defined as $\text{Spec}\mathcal{A}$ of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$ for any fixed $1 \leq k \leq l$. We choose the standard base $\alpha_1, \dots, \alpha_l$ of simple roots for the root system C_l as given in [2]. As in [5] we define an indefinite metric $(\ , \)^\sim$ on $\widetilde{V} = V \oplus \mathbb{R}$ such that \widetilde{V} is the orthogonal direct sum of V and \mathbb{R} , V endowed with the W -invariant Euclidean metric

$$(dx_m, dx_n)^\sim = \frac{m}{4\pi^2}, \quad 1 \leq m \leq n \leq l \quad (2.1)$$

and \mathbb{R} endowed with the metric

$$(dx_{l+1}, dx_{l+1})^\sim = -\frac{1}{4\pi^2 d_k} = -\frac{1}{4k\pi^2}. \quad (2.2)$$

Here the numbers d_j are defined in (1.8) and take the values

$$d_1 = 1, \dots, d_{k-1} = k-1, \quad d_j = k, \quad k \leq j \leq l. \quad (2.3)$$

The W_a -invariant Fourier polynomials $y_1(\mathbf{x}), \dots, y_l(\mathbf{x})$ that are defined in (1.11) have the expressions

$$y_j(\mathbf{x}) = \sigma_j(\xi_1, \dots, \xi_l), \quad (2.4)$$

where

$$\xi_j = e^{2i\pi(x_j - x_{j-1})} + e^{-2i\pi(x_j - x_{j-1})}, \quad x_0 = 0, \quad j = 1, \dots, l \quad (2.5)$$

and $\sigma_j(\xi_1, \dots, \xi_l)$ is the j -th elementary symmetric polynomial of ξ_1, \dots, ξ_l . For the reason that will be clear later, we will use in what follows the following set of generators for the the ring of W_a -invariant Fourier polynomials which we still denote by y_1, \dots, y_l :

$$y_j(\mathbf{x}) = \sigma_j(\zeta_1, \dots, \zeta_l) \quad (2.6)$$

with

$$\zeta_j = \xi_j + 2, \quad j = 1, \dots, l. \quad (2.7)$$

Consequently we have a set of generators for the ring $\mathcal{A} = \mathcal{A}^{(k)}(C_l)$

$$\tilde{y}_j(x) = e^{2\pi i d_j x_{l+1}} y_j(\mathbf{x}), \quad j = 1, \dots, l, \quad (2.8)$$

$$\tilde{y}_{l+1}(x) = e^{2\pi i x_{l+1}}, \quad (2.9)$$

defined in the same way as in (1.9), (1.10). They form a global coordinates on \mathcal{M} . As in [5], we introduce the following local coordinates on \mathcal{M} :

$$y^1 = \tilde{y}_1, \dots, y^l = \tilde{y}_l, y^{l+1} = \log \tilde{y}_{l+1} = 2\pi i x_{l+1}. \quad (2.10)$$

They live on a covering $\widetilde{\mathcal{M}}$ of $\mathcal{M} \setminus \{\tilde{y}_{l+1} = 0\}$. The projection

$$P : \widetilde{V} \rightarrow \widetilde{\mathcal{M}} \quad (2.11)$$

induces a symmetric bilinear form on $T^*\widetilde{\mathcal{M}}$

$$(dy^i, dy^j)^\sim \equiv g^{ij}(y) := \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} (dx^a, dx^b)^\sim. \quad (2.12)$$

Denote

$$\Sigma = \{y \mid \det(g^{ij}(y)) = 0\}. \quad (2.13)$$

It turns out that Σ is an analogue of the discriminant. Namely, as it was shown in [5], Σ is the P -image of the union of hyperplanes

$$\{(\mathbf{x}, x_{l+1}) \mid (\beta, \mathbf{x}) = m \in \mathbb{Z}, x_{l+1} = \text{arbitrary}\}, \quad \beta \in \Phi^+, \quad (2.3)$$

where Φ^+ is the set of all positive roots.

Lemma 2.1. *The functions $g^{ij}(y)$ are weighted homogeneous polynomials in $y^1, \dots, y^l, e^{y^{l+1}}$ of the degree*

$$\deg g^{ij} = \deg y^i + \deg y^j, \quad (2.14)$$

where $\deg y^j = d_j$ and $\deg y^{l+1} = d_{l+1} = 0$.

Proof The assertion follows immediately from the Theorem 1.2 which says that the functions $\tilde{y}_1, \dots, \tilde{y}_{l+1}$ form a set of generators of the ring \mathcal{A} . \square

From this lemma we see that Σ is an algebraic subvariety in \mathcal{M} and the matrix (g^{ij}) is invertible in $\mathcal{M} \setminus \Sigma$. The inverse matrix $(g^{ij})^{-1}$ defines a flat metric on $\mathcal{M} \setminus \Sigma$.

Let us introduce the following new coordinates on \mathcal{M}

$$\theta^j = \begin{cases} e^{k y^{l+1}}, & j = 0, \\ y^j e^{(k-j)y^{l+1}}, & j = 1, \dots, k-1, \\ y^j, & j = k, \dots, l \end{cases} \quad (2.15)$$

Denote

$$\mu_j = 2\pi i(x_j - x_{j-1}), \quad \mu_{l+1} = y^{l+1}, \quad j = 1, \dots, l. \quad (2.16)$$

In the coordinates μ_1, \dots, μ_{l+1} the indefinite metric on \tilde{V} has the form

$$((d\mu_i, d\mu_j)^\sim) = \text{diag}(-1, \dots, -1, \frac{1}{k}). \quad (2.17)$$

Define

$$P(u) := \sum_{j=0}^l u^{l-j} \theta^j = e^{k\mu_{l+1}} \prod_{j=1}^l (u + \zeta_j). \quad (2.18)$$

We can easily verify that the function $P(u)$ satisfies

$$\frac{\partial P(u)}{\partial \mu_a} = \frac{1}{u + \zeta_a} P(u) (e^{\mu_a} - e^{-\mu_a}), \quad 1 \leq a \leq l; \quad (2.19)$$

$$\frac{\partial P(u)}{\partial \mu_{l+1}} = kP(u), \quad P'(u) := \frac{\partial P(u)}{\partial u} = P(u) \sum_{a=1}^l \frac{1}{u + \zeta_a}. \quad (2.20)$$

By using these identities, we have

Lemma 2.2. *The following formulae hold true for the generating functions of the metric (g^{ij}) and the contravariant components of its Levi-Civita connection in the coordinates $\theta^0, \dots, \theta^l$*

$$\begin{aligned} & \sum_{i,j=0}^l (d\theta^i, d\theta^j)^\sim u^{l-i} v^{l-j} = (dP(u), dP(v))^\sim \\ & = (k-l)P(u)P(v) + \frac{u^2 + 4u}{u-v} P'(u)P(v) - \frac{v^2 + 4v}{u-v} P(u)P'(v), \end{aligned}$$

$$\begin{aligned} & \sum_{i,j,m=0}^l \Gamma_m^{ij}(\theta) d\theta^m u^{l-i} v^{l-j} = \sum_{a,b,m=1}^{l+1} \frac{\partial P(u)}{\partial \mu_a} \frac{\partial^2 P(v)}{\partial \mu_b \partial \mu_m} d\mu_m (d\mu_a, d\mu_b) \\ & = (k-l)P(u)dP(v) + \frac{u^2 + 4u}{u-v} P'(u)dP(v) - \frac{v^2 + 4v}{u-v} P(u)dP'(v) \\ & \quad + \frac{2u + uv + 2v}{(u-v)^2} P(v)dP(u) - \frac{2u + uv + 2v}{(u-v)^2} P(u)dP(v). \end{aligned}$$

$$\text{Here } \Gamma_m^{ij}(\theta) = - \sum_{s=1}^{l+1} g^{is}(\theta) \Gamma_{sm}^j(\theta).$$

Proof By using (2.19), (2.20) we have

$$\begin{aligned}
 (dP(u), dP(v))^\sim &= \frac{1}{k} \frac{\partial P(u)}{\partial \mu_{l+1}} \frac{\partial P(v)}{\partial \mu_{l+1}} - \sum_{a=1}^l \frac{\partial P(u)}{\partial \mu_a} \frac{\partial P(v)}{\partial \mu_a} \\
 &= kP(u)P(v) - \sum_{a=1}^l P(u)P(v) \frac{\zeta_a^2 - 4\zeta_a}{(u + \zeta_a)(v + \zeta_a)} \\
 &= \sum_{s=1}^l P(u)P(v) \frac{v}{u-v} \left(1 - \frac{v}{v + \zeta_a}\right) - \sum_{a=1}^l P(u)P(v) \frac{u}{u-v} \left(1 - \frac{u}{u + \zeta_a}\right) \\
 &\quad + \sum_{a=1}^l P(u)P(v) \frac{1}{u + \zeta_a} \frac{4u}{u-v} - \sum_{a=1}^l P(u)P(v) \frac{1}{v + \zeta_a} \frac{4v}{u-v} + kP(u)P(v) \\
 &= (k-l)P(u)P(v) + \frac{u^2 + 4u}{u-v} P'(u)P(v) - \frac{v^2 + 4v}{u-v} P(u)P'(v).
 \end{aligned}$$

So we proved the first formula, the second formula can be proved in the same way. The lemma is proved. \square

The above lemma shows that in the coordinates $\theta^0, \dots, \theta^l$ the functions $g^{ij}(\theta)$ are quadratic polynomials, and the contravariant components Γ_m^{ij} are homogeneous linear functions¹. It reveals the following important properties of the flat metric:

Corollary 2.3. *In the coordinates y^1, \dots, y^{l+1} the functions $\Gamma_m^{ij}(y)$ are weighted homogeneous polynomials of degree*

$$\deg \Gamma_m^{ij}(y) = d_i + d_j - d_m. \quad (2.21)$$

Corollary 2.4. *In the coordinates y^1, \dots, y^{l+1} the polynomials $g^{ij}(y)$ and $\Gamma_m^{ij}(y)$ are at most linear in y^k .*

Now let us define a symmetric bilinear form on $T^*\mathcal{M}$ by

$$\langle dy^i, dy^j \rangle := \eta^{ij}(y) = \frac{\partial g^{ij}(y)}{\partial y^k}. \quad (2.22)$$

¹These metrics give rise to a quadratic Poisson structure on the space of ‘‘loops’’ $\{S^1 \rightarrow M\}$ (see [3] for the details):

$$\{\theta^i(s), \theta^j(s')\} = g^{ij}(\theta(s))\delta'(s-s') + \Gamma_m^{ij}(\theta(s))\theta_s^m \delta(s-s').$$

We plan to study such important class of quadratic metrics and Poisson structures in a separate publication.

Lemma 2.5. *The matrix $(\eta^{ij}(y))$ has the form*

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & k & P_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & k & R_1 & P_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & R_1 & R_2 & P_3 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k & R_1 & R_2 & \cdots & & R_{k-2} & P_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ P_1 & P_2 & P_3 & \cdots & P_{k-2} & P_{k-1} & P_k & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & Q_1 & Q_2 & \cdots & Q_{l-k} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & Q_2 & Q_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & Q_{l-k} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \quad (2.23)$$

Here

$$\begin{aligned} R_j &= 4(k-j+1)y^{j-1}e^{y^{l+1}} + (k-j)y^j, \\ P_j &= 4(k-j+1)y^{j-1}e^{y^{l+1}}, \\ Q_m &= 4my^{k+m} + (1 - \delta_{m,l-k})(m+1)y^{k+m+1}, \\ &1 \leq j \leq k, \quad 1 \leq m \leq l-k \end{aligned}$$

and we assume $y^0 = 1$.

It follows from the above lemma that

$$\det(\eta^{ij}) = (-1)^l k^{k-1} 4^{l-k} (l-k)^{l-k} (y^l)^{l-k},$$

so $(\eta^{ij})^{-1}$ defines a metric on $\mathcal{M} \setminus \{y \in \mathcal{M} | y^l = 0\}$.

Theorem 2.6. *The space \mathcal{M} carries a flat pencil of metrics (bilinear forms on T^*M)*

$$g^{ij}(y) \quad \text{and} \quad \eta^{ij}(y) = \frac{\partial g^{ij}(y)}{\partial y^k},$$

i.e., any linear combination $g^{ij} + \lambda \eta^{ij}$ defines a flat metric on certain open subset of \mathcal{M} and the contravariant components of the Levi-Civita connection of (η^{ij}) is given by

$$\gamma_m^{ij}(y) = \frac{\partial \Gamma_m^{ij}(y)}{\partial y^k}. \quad (2.24)$$

The metric $(g^{ij}(y))$ does not degenerate on $\mathcal{M} \setminus \Sigma$ and the metric $(\eta^{ij}(y))$ does not degenerate on $\mathcal{M} \setminus \{y \in \mathcal{M} | y^l = 0\}$.

Proof The result follows, applying Lemma D.1 of [4], from the fact that in the coordinates y^1, \dots, y^{l+1} the flat metric g^{ij} and the contravariant

components of its Levi-Civita connection depend at most linearly on y^k . The theorem is proved. \square

Remark 2.7. *Our particular choice of the basis of the $\widetilde{W}^{(k)}(C_l)$ -invariant Fourier polynomials (2.6)–(2.9) ensures that the components of the flat metric $(g^{ij}(y))$ are at most linear in y^k . This linearity is the most crucial step in the construction of the above flat pencil of metrics. If we choose the basis of the $\widetilde{W}^{(k)}(C_l)$ -invariant Fourier polynomials by using (2.4), (2.8) and (2.9), then we lose this linearity property of the functions $(g^{ij}(y))$ and the construction of the flat metric $(\eta^{ij}(y))$ becomes obscure.*

3. FLAT COORDINATES OF THE METRIC (η^{ij})

In this section, we will show that the flat coordinates of the metric (η^{ij}) are algebraic functions of $y^1, \dots, y^{l+1}, e^{y^{l+1}}$. To this end, we first perform changes of coordinates to simplify the matrix (η^{ij}) .

Lemma 3.1. *There exists a system of new coordinates z^1, \dots, z^{l+1} of the form*

$$z^j = y^j + p_j(y^1, \dots, y^{j-1}, e^{y^{l+1}}), \quad 1 \leq j \leq k, \quad z^{l+1} = y^{l+1}, \quad (3.1)$$

$$z^j = y^j + \sum_{m=j+1}^l c_m^j y^m, \quad k+1 \leq j \leq l, \quad (3.2)$$

where p_j are homogeneous polynomials of degree d_j and c_m^j are some constants such that in the new coordinates the metric (η^{ij}) still has the form (2.23) with the entries replaced by

$$R_j = 0, \quad P_j = 0, \quad Q_m = 4mz^{k+m}, \quad 1 \leq j \leq k, \quad 1 \leq m \leq l-k. \quad (3.3)$$

Proof Let us first note that the $(k+1) \times (k+1)$ matrix $(\tilde{\eta}^{ij})$ which has elements

$$\tilde{\eta}^{ij} = \eta^{ij}(y), \quad \tilde{\eta}^{k+1,m} = \tilde{\eta}^{m,k+1} = \delta_{j,k}, \quad 1 \leq i, j \leq k, \quad 1 \leq m \leq k+1$$

coincides, under renaming of the coordinate $y^{l+1} \mapsto y^{k+1}$, with the matrix $(\eta^{ij}(y))_{(k+1) \times (k+1)}$ that is constructed as in the last section with respect to the extended affine Weyl group $\widetilde{W}^{(k)}(C_k)$. Thus by using the results of [5] we can find homogeneous polynomials $p_j, 1 \leq j \leq k$ such that under the change of coordinates (3.1) and $z^j = y^j, k+1 \leq j \leq l$

the matrix $(\eta^{ij}(z))$ has the form (2.23) with entries

$$\begin{aligned} R_j = 0, \quad P_j = 0, \quad Q_m = 4mz^{k+m} + (1 - \delta_{m,l-k})(m+1)z^{k+m+1}, \\ 1 \leq j \leq k, \quad 1 \leq m \leq l-k. \end{aligned}$$

To finish the proof of the lemma, we need to perform a second change of coordinates. To this end, denote by Ψ a $n \times n$ matrix with entries as linear functions of a^1, \dots, a^n

$$\psi^{ij}(a) = 4(i+j-1)a^{i+j-1} + (i+j)a^{i+j}, \quad i, j \geq 1.$$

Here $a^m = 0$ for $m \geq n+1$. We are to find a linear transformation of the triangular form

$$a^j = \sum_{m=j}^n B_m^j b^m, \quad B_j^j = 1, \quad j \geq 1$$

such that

$$\begin{aligned} & \sum_{r,s=1}^n 4(r+s-1)b^{r+s-1} \frac{\partial a^i}{\partial b^r} \frac{\partial a^j}{\partial b^s} \\ &= 4(i+j-1) \sum_{m=i+j-1}^n B_m^{i+j-1} b^m + (i+j) \sum_{m=i+j}^n B_m^{i+j} b^m. \end{aligned}$$

Equivalently, the constants B_j^i must satisfy the relations

$$\begin{aligned} 4(i+j-1)B_m^{i+j-1} + (i+j)B_m^{i+j} = 4m \sum_{\alpha+\beta=m+1} B_\alpha^i B_\beta^j, \\ i+j \leq m \leq n. \end{aligned} \tag{3.4}$$

Introduce the generating functions

$$f^i(t) = \sum_{\alpha \geq 0} B_{i+\alpha}^i t^\alpha, \quad i = 1, 2, \dots \tag{3.5}$$

Then the relations in (3.4) can be encoded into the following equations of $f^i(t)$:

$$4(i+j-1)t^{i+j-2}f^{i+j-1} + (i+j)t^{i+j-1}f^{i+j} = 4 \frac{d}{dt} (t^{i+j-1}f^i f^j). \tag{3.6}$$

This system of equations has the following solution which was obtain by Si-Qi Liu

$$f^i(t) = \cosh \left(\frac{\sqrt{t}}{2} \right) \left(\frac{2 \sinh \left(\frac{\sqrt{t}}{2} \right)}{\sqrt{t}} \right)^{2i-1}. \tag{3.7}$$

From the above result we derive the existence of constants c_m^j , $k+1 \leq j \leq l$, $j+1 \leq m \leq l$ such that under the change of coordinates

$$z^i \mapsto z^i, \quad i = 1, \dots, k, l+1, \quad z^j \mapsto z^j + \sum_{m=j+1}^l c_m^j z^m, \quad k+1 \leq j \leq l,$$

the matrix $(\eta^{ij}(z))$ has the form (2.23) and with entries given by (3.3). The lemma is proved. \square

Lemma 3.2. *Under the change of coordinates*

$$w^i = z^i, \quad i = 1, \dots, k, l+1, \quad (3.8)$$

$$w^{k+1} = z^{k+1} (z^l)^{-\frac{1}{2(l-k)}}, \quad w^j = z^j (z^l)^{-\frac{j-k}{l-k}}, \quad w^l = (z^l)^{\frac{1}{2(l-k)}}, \quad (3.9)$$

$$j = k+2, \dots, l-1,$$

the components of the metric $(\eta^{ij}(z))$ are transformed to the form

$$\text{diag}(A_{(k-1) \times (k-1)}, B_{(l-k+2) \times (l-k+2)}),$$

where the matrix A has entries $A^{ij} = \delta_{i, k-i} k$ and the upper triangular matrix B has the form

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & \\ 0 & 0 & S_{k+3} & S_{k+4} & \cdots & S_{l-1} & S_l & & \\ 0 & 0 & S_{k+4} & S_{k+5} & \cdots & S_l & & & \\ \vdots & \vdots & \vdots & & & & & & \\ 0 & 0 & S_l & & & & & & \\ 0 & 2 & & & & & & & \\ 1 & & & & & & & & \end{pmatrix}. \quad (3.10)$$

with

$$S_{k+j} = 4j(w^l)^{-2} w^{k+j}, \quad S_l = 4(l-k)(w^l)^{-2}, \quad 3 \leq j \leq l-k-1. \quad (3.11)$$

Proof By a straightforward calculation. \square

Proposition 3.3. *In the coordinates w^1, \dots, w^{l+1} the Christoffel symbols of the metric (η^{ij}) have the following properties*

- (1) $\gamma_{ij}^m = 0$ for $m = 1, \dots, k, l, l+1$, $i, j = 1, \dots, l+1$;
- (2) $\gamma_{ij}^{k+1} = -\frac{\partial \eta_{ij}(w)}{\partial w^l}$ are weighted homogenous polynomials in w^{k+3}, \dots, w^l ;
- (3) γ_{ij}^m are weighted homogenous polynomials in w^{k+3}, \dots, w^l for $k+2 \leq m \leq l-1$ and $i, j = 1, \dots, l-1, l+1$;
- (4) $\gamma_{lj}^m = \frac{1}{w^l} \delta_j^m$, for $k+2 \leq m \leq l-1$, $1 \leq j \leq l+1$.

Proof The first three properties of γ_{ij}^m follow easily from the simple form of the matrix $(\eta^{ij}(w))$. To prove the last property, we only need to note that

$$\gamma_{lj}^m = \frac{1}{2} \sum_{s=1}^{l+1} \eta^{ms} \frac{\partial \eta_{sj}}{\partial w^l} = \frac{1}{2} \sum_{s=1}^{l+1} \frac{2}{w^l} \eta^{ms} \eta_{sj} = \frac{1}{w^l} \delta_j^m. \quad (3.12)$$

The proposition is proved. \square

Theorem 3.4. *We can choose the flat coordinates of the metric $(\eta^{ij}(w))$ in the form*

$$t^1 = w^1, \dots, t^k = w^k, \quad t^l = w^l, \quad t^{l+1} = w^{l+1}, \quad (3.13)$$

$$t^{k+1} = w^{k+1} + w^l h_{k+1}(w^{k+2}, \dots, w^{l-1}), \quad (3.14)$$

$$t^j = w^l (w^j + h_j(w^{j+1}, \dots, w^{l-1})). \quad (3.15)$$

Here h_j are weighted homogeneous polynomials of degree $\frac{k(l-j)}{l-k}$ for $j = k+1, \dots, l-2$ and $h_{l-1} = 0$.

Proof To find the flat coordinates $t = t(w)$, we need to solve the following system of PDEs

$$\frac{\partial^2 t}{\partial w^i \partial w^j} - \sum_{m=1}^{l+1} \gamma_{ij}^m \frac{\partial t}{\partial w^m} = 0, \quad i, j = 1, \dots, l+1. \quad (3.16)$$

From the above proposition we easily see that t^1, \dots, t^k, t^{l+1} are $k+1$ solutions of the above system. We still need to find $l-k$ independent solutions t^{k+1}, \dots, t^l . Introduce the $(l-k) \times (l-k)$ matrix

$$\Phi = (\phi_j^i), \quad \phi_j^i = \frac{\partial t^{k+i}}{\partial w^{k+j}}. \quad (3.17)$$

Then the system (3.16) is reduced to

$$\partial_m \Phi = \Phi A_m, \quad \partial_m = \frac{\partial}{\partial w^m}, \quad m = k+1, \dots, l. \quad (3.18)$$

They are regular at $\mathbf{w} = (w^{k+1}, \dots, w^l) = 0$ except the system with $m = l$. In this case the coefficient matrix has the simple form

$$A_l = \text{diag}(0, \frac{1}{w^l}, \dots, \frac{1}{w^l}, 0). \quad (3.19)$$

Now assume that Φ has the form

$$\Phi = \Psi \text{diag}(1, w^l, \dots, w^l, 1). \quad (3.20)$$

Then the systems in (3.18) are converted to

$$\partial_m \Psi = \Psi B_m, \quad \partial_l \Psi = 0, \quad m = k+1, \dots, l-1. \quad (3.21)$$

The entries of the coefficient matrices B_m are now weighted homogeneous polynomials of w^{k+1}, \dots, w^l . Thus we can find a unique analytic at $\mathbf{w} = 0$ solution Ψ of the above systems such that

$$\Psi|_{\mathbf{w}=0} = \text{diag}(1, \dots, 1). \quad (3.22)$$

From the weighted homogeneity of the coefficient matrices B_m it follows that the elements of Ψ are also weighted homogeneous. Since $\deg w^j > 0$ for $j = k+1, \dots, l$ we see that they are in fact polynomials of w^{k+1}, \dots, w^l . Thus the result of the theorem follows. The theorem is proved. \square

Due to the above construction, we can associate the following natural degrees to the flat coordinates

$$\tilde{d}_j = \deg t^j := \frac{j}{k}, \quad 1 \leq j \leq k, \quad (3.23)$$

$$\tilde{d}_m = \deg t^m := \frac{2l - 2m + 1}{2(l - k)}, \quad k + 1 \leq m \leq l, \quad (3.24)$$

$$\tilde{d}_{l+1} = \deg t^{l+1} := 0, \quad (3.25)$$

and we readily have the following corollary

Corollary 3.5. *In the flat coordinates t^1, \dots, t^{l+1} , the entries of the matrix (η^{ij}) has the form*

$$\eta^{ij} = \begin{cases} k, & j = k - i, & 1 \leq i \leq k - 1, \\ 4(l - k), & j = k + l + 1 - i, & k + 2 \leq i \leq l - 1, \\ 1, & i = l + 1, j = k & \text{or } i = k, j = l + 1, \\ 2, & i = l, j = k + 1 & \text{or } i = k + 1, j = l. \end{cases} \quad (3.26)$$

The entries of the matrix $(g^{ij}(t))$ and the Christoffel symbols $\Gamma_m^{ij}(t)$ are weighted homogeneous polynomials of $t^1, \dots, t^l, \frac{1}{t^l}, e^{t^{l+1}}$ of degrees $\tilde{d}_i + \tilde{d}_j$ and $\tilde{d}_i + \tilde{d}_j - \tilde{d}_m$ respectively. In particular,

$$\begin{aligned} g^{m, l+1} &= \tilde{d}_m t^m, & 1 \leq m \leq l, & \quad g^{l+1, l+1} = \frac{1}{k}, \\ \Gamma_j^{l+1, i} &= \tilde{d}_j \delta_{i, j}, & 1 \leq i, j \leq l + 1. \end{aligned} \quad (3.27)$$

Remark 3.6. *For the orbit spaces of finite reflection groups flat coordinates were constructed by Saito, Yano and Sekiguchi in [9] (see also [8]).*

The numbers $\tilde{d}_1, \tilde{d}_{l+1}$ satisfy a duality relation that is similar to that of [5]. To describe this duality relation, let us delete the k -th vertex of

the Dynkin diagram \mathcal{R} and obtain two components $\mathcal{R} \setminus \alpha_k = \mathcal{R}_1 \cup \mathcal{R}_2$. On each component we have an involution

$$i \mapsto i^*, \quad i = 1, \dots, k-1 \text{ and } i = k+1, \dots, l \quad (3.28)$$

given by the reflection with respect to the center of the component. We also define

$$k^* = l+1, \quad (l+1)^* = k, \quad (3.29)$$

then we have

$$\tilde{d}_i + \tilde{d}_{i^*} = 1, \quad i = 1, \dots, l+1, \quad (3.30)$$

and from the above corollary we see that η^{ij} is a nonzero constant iff $j = i^*$.

4. THE FROBENIUS MANIFOLD STRUCTURE ON THE ORBIT SPACE OF $\widetilde{W}^{(k)}(C_l)$

Now we are ready to describe the Frobenius manifold structure on the orbit space of the extended affine Weyl group $\widetilde{W}^{(k)}(C_l)$. Let us first recall the definition of Frobenius manifold, see [4] for details.

Definition 4.1. *A Frobenius algebra is a pair $(A, \langle \cdot, \cdot \rangle)$ where A is a commutative associative algebra with a unity e over a field \mathcal{K} (in our case $\mathcal{K} = \mathbb{C}$) and $\langle \cdot, \cdot \rangle$ is a \mathcal{K} -bilinear symmetric nondegenerate invariant form on A , i.e.,*

$$\langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle, \quad \forall x, y, z \in A.$$

Definition 4.2. *A Frobenius structure of charge d on an n -dimensional manifold M is a structure of Frobenius algebra on the tangent spaces $T_t M = (A_t, \langle \cdot, \cdot \rangle_t)$ depending (smoothly, analytically etc.) on the point t . This structure satisfies the following axioms:*

- FM1. *The metric $\langle \cdot, \cdot \rangle_t$ on M is flat, and the unity vector field e is covariantly constant, i.e., $\nabla e = 0$. Here we denote ∇ the Levi-Civita connection for this flat metric.*
- FM2. *Let c be the 3-tensor $c(x, y, z) := \langle x \cdot y, z \rangle$, $x, y, z \in T_t M$. Then the 4-tensor $(\nabla_w c)(x, y, z)$ is symmetric in $x, y, z, w \in T_t M$.*
- FM3. *The existence on M of a vector field E , called the Euler vector field, which satisfies the conditions $\nabla \nabla E = 0$ and*

$$[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot y,$$

$$E \langle x, y \rangle - \langle [E, x], y \rangle - \langle x, [E, y] \rangle = (2-d) \langle x, y \rangle$$

for any vector fields x, y on M .

A manifold M equipped with a Frobenius structure on it is called a Frobenius manifold.

Let us choose locally flat coordinates t^1, \dots, t^n for the invariant flat metric, then locally there exists a function $F(t^1, \dots, t^n)$, called the *potential* of the Frobenius manifold, such that

$$\langle u \cdot v, w \rangle = u^i v^j w^s \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^s} \quad (4.1)$$

for any three vector fields $u = u^i \frac{\partial}{\partial t^i}$, $v = v^j \frac{\partial}{\partial t^j}$, $w = w^s \frac{\partial}{\partial t^s}$. Here and in what follows summations over repeated indices are assumed. By definition, we can also choose the coordinates t^1 such that $e = \frac{\partial}{\partial t^1}$. Then in the flat coordinates the components of the flat metric can be expressed in the form

$$\frac{\partial^3 F}{\partial t^1 \partial t^i \partial t^j} = \eta_{ij} = \langle \frac{\partial}{\partial t^i}, \frac{\partial}{\partial t^j} \rangle, \quad i, j = 1, \dots, n. \quad (4.2)$$

The associativity of the Frobenius algebras is equivalent to the following overdetermined system of equations for the function F

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^k \partial t^m} = \frac{\partial^3 F}{\partial t^k \partial t^j \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^i \partial t^m} \quad (4.3)$$

for arbitrary indices i, j, k, m from 1 to n .

In the flat coordinates the Euler vector field E has the form

$$E = \sum_{i=1}^n (d_j^i t^j + r_i) \frac{\partial}{\partial t^i} \quad (4.4)$$

for some constants d_j^i, r_i , $i = 1, \dots, n$ which satisfy

$$d_1^i = \delta_1^i, \quad r_1 = 0.$$

From the axiom FM3 it follows that the potential F satisfies the quasi-homogeneity condition

$$\mathcal{L}_E F = (3 - d)F + \frac{1}{2} A_{ij} t^i t^j + B_i t^i + \text{constant}. \quad (4.5)$$

The system (4.2)–(4.5) is called the *WDVV equations of associativity* which is equivalent to the above definition of Frobenius manifold in the chosen system of local coordinates.

In our examples the constant matrix d_i^j is always diagonal, $d_i^j = \hat{d}_i \delta_i^j$.

Let us also recall an important geometrical structure on a Frobenius manifold M , the *intersection form* of M . This is a symmetric bilinear form $(,)^*$ on T^*M defined by the formula

$$(w_1, w_2)^* = i_E(w_1 \cdot w_2), \quad (4.6)$$

here the product of two 1-forms w_1, w_2 at a point $t \in M$ is defined by using the algebra structure on $T_t M$ and the isomorphism

$$T_t M \rightarrow T_t^* M \quad (4.7)$$

established by the invariant flat metric $\langle \cdot, \cdot \rangle$. In the flat coordinates t^1, \dots, t^n of the invariant metric, the intersection form can be represented by

$$(dt^i, dt^j)^* = \mathcal{L}_E F^{ij} = (d+1 - \hat{d}_i - \hat{d}_j) F^{ij} + A^{ij}, \quad (4.8)$$

where

$$A^{ij} = \eta^{ii'} \eta^{jj'} A_{i'j'}, \quad F^{ij} = \eta^{ii'} \eta^{jj'} \frac{\partial^2 F}{\partial t^{i'} \partial t^{j'}} \quad (4.9)$$

and $F(t)$ is the potential of the Frobenius manifold. Denote by $\Sigma \subset M$ the *discriminant* of M on which the intersection form degenerates, then an important property of the intersection form is that on $M \setminus \Sigma$ its inverse defines a new flat metric.

Theorem 4.3. *There exists a unique Frobenius structure of charge $d = 1$ on the orbit space $\mathcal{M} \setminus \{t^l = 0\}$ of $\widetilde{W}^{(k)}(C_l)$ polynomial in $t^1, t^2, \dots, t^l, \frac{1}{t^l}, e^{t^{l+1}}$ such that*

- (1) *The unity vector field e coincides with $\frac{\partial}{\partial y^k} = \frac{\partial}{\partial t^k}$;*
- (2) *The Euler vector field has the form*

$$E = \sum_{\alpha=1}^l \tilde{d}_\alpha t^\alpha \frac{\partial}{\partial t^\alpha} + \frac{\partial}{\partial t^{l+1}} \quad (4.10)$$

where $\tilde{d}_1, \dots, \tilde{d}_l$ are defined in (3.23)–(3.25).

- (3) *The invariant flat metric and the intersection form of the Frobenius structure coincide respectively with the metric (η^{ij}) and $(g^{ij}(t))$ on $\mathcal{M} \setminus \{t^l = 0\}$.*

Proof By following the lines of the proof of Lemma 2.6 that is given in [5] we can show the existence of a unique weighted homogeneous polynomial

$$G := G(t^1, \dots, t^{k-1}, t^{k+1}, \dots, t^l, \frac{1}{t^l}, e^{t^{l+1}})$$

of degree 2 such that the function

$$F = \frac{1}{2}(t^k)^2 t^{l+1} + \frac{1}{2} t^k \sum_{i,j \neq k} \eta_{ij} t^i t^j + G \quad (4.11)$$

satisfies the equations

$$g^{ij} = \mathcal{L}_E F^{ij}, \quad \Gamma_m^{ij} = \tilde{d}_j c_m^{ij}, \quad i, j, m = 1, \dots, l+1, \quad (4.12)$$

where $c_m^{ij} = \frac{\partial F^{ij}}{\partial t^m}$. Obviously, the function F satisfies the equations

$$\frac{\partial^3 F}{\partial t^k \partial t^i \partial t^j} = \eta_{ij}, \quad i, j = 1, \dots, l+1 \quad (4.13)$$

and the quasi-homogeneity condition

$$\mathcal{L}_E F = 2F. \quad (4.14)$$

From the properties of a flat pencil of metrics [4] it follows that F also satisfies the associativity equations

$$c_m^{ij} c_q^{mp} = c_m^{ip} c_q^{mj} \quad (4.15)$$

for any set of fixed indices i, j, p, q . Now the theorem follows from above properties of the function F and the simple identity $\mathcal{L}_E e = -e$. The theorem is proved. \square

5. SOME EXAMPLES

In this section we give some examples to illustrate the above construction of the Frobenius manifold structures. For the sake of simplicity of notations, instead of t^1, \dots, t^{l+1} we will redenote the flat coordinates of the metric η^{ij} by t_1, \dots, t_{l+1} , and we will also denote $\partial_i = \frac{\partial}{\partial t_i}$ in the the following examples.

Example 5.1. [$C_3, k = 1$] *Let R be the root system of type C_3 , take $k = 1$, then $d_1 = d_2 = d_3 = 1$, and*

$$\begin{aligned} y^1 &= e^{2i\pi x_4} (\zeta_1 + \zeta_2 + \zeta_3); \\ y^2 &= e^{2i\pi x_4} (\zeta_1 \zeta_2 + \zeta_1 \zeta_3 + \zeta_2 \zeta_3); \\ y^3 &= e^{2i\pi x_4} \zeta_1 \zeta_2 \zeta_3; \\ y^4 &= 2i\pi x_4, \end{aligned}$$

where $\zeta_j = e^{2i\pi(x_j - x_{j-1})} + e^{-2i\pi(x_j - x_{j-1})} + 2$ and $x_0 = 0$, $j = 1, 2, 3$. The metric $(,)^\sim$ has the form

$$((dx_i, dx_j)^\sim) = \frac{1}{4\pi^2} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The flat coordinates are

$$t_1 = y^1 - 2e^{y^4}, \quad t_2 = (y^2 - \frac{1}{6}y^3)(y^3)^{-\frac{1}{4}}, \quad t_3 = (y^3)^{\frac{1}{4}}, \quad t_4 = y^4$$

and the intersection form is given by

$$\begin{aligned}
g^{11} &= 2t_2t_3e^{t_4} + \frac{1}{3}t_3^4e^{t_4} + 4e^{2t_4}; \\
g^{12} &= \frac{7}{3}t_3^3e^{t_4} + \frac{7}{2}t_2e^{t_4}; \quad g^{13} = \frac{5}{2}t_3e^{t_4}; \quad g^{14} = t_1; \\
g^{22} &= 12t_3^2e^{t_4} - \frac{1}{4}t_2^2 + \frac{1}{12}t_3^3t_2 - \frac{1}{108}t_3^6 + \frac{1}{4}\frac{t_2^3}{t_3^3}; \\
g^{23} &= 2t_1 + 4e^{t_4} - \frac{1}{3}t_2t_3 + \frac{1}{72}t_3^4 - \frac{1}{4}\frac{t_2^2}{t_3^2}; \\
g^{24} &= \frac{3}{4}t_2; \quad g^{33} = \frac{1}{4}\frac{t_2}{t_3} - \frac{1}{12}t_3^2; \quad g^{34} = \frac{1}{4}t_3; \quad g^{44} = 1.
\end{aligned}$$

The potential has the expression

$$\begin{aligned}
F &= \frac{1}{2}t_1^2t_4 + \frac{1}{2}t_1t_2t_3 - \frac{1}{48}t_2^2t_3^2 + \frac{1}{1440}t_2t_3^5 - \frac{1}{36288}t_3^8 \\
&\quad + t_2t_3e^{t_4} + \frac{1}{6}t_3^4e^{t_4} + \frac{1}{2}e^{2t_4} + \frac{1}{48}\frac{t_2^3}{t_3}
\end{aligned}$$

and the Euler vector field is given by

$$E = t_1\partial_1 + \frac{3}{4}t_2\partial_2 + \frac{1}{4}t_3\partial_3 + \partial_4.$$

Remark 5.2. *If we take $k = 2$ or 3 for the C_3 root system, we obtain a Frobenius manifold structure that is isomorphic to the one given in Example 2.7 [$B_3, k = 2$] or Example 2.8 [$C_3, k = 3$] of [5].*

Example 5.3. [$C_4, k = 1$] *Let R be the root system of type C_4 . Take $k = 1$, then $d_1 = d_2 = d_3 = d_4 = 1$, and*

$$\begin{aligned}
y^1 &= e^{2i\pi x_5} (\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4); \\
y^2 &= e^{2i\pi x_5} \sum_{1 \leq a < b \leq 4} \zeta_a \zeta_b; \\
y^3 &= e^{2i\pi x_5} \sum_{1 \leq a < b < c \leq 4} \zeta_a \zeta_b \zeta_c; \\
y^4 &= e^{2i\pi x_5} \zeta_1 \zeta_2 \zeta_3 \zeta_4; \\
y^5 &= 2i\pi x_5,
\end{aligned}$$

where $\zeta_j = e^{2i\pi(x_j-x_{j-1})} + e^{-2i\pi(x_j-x_{j-1})} + 2$ and $x_0 = 0$, $j = 1, 2, 3, 4$.
The metric $(,)^\sim$ has the form

$$((dx_i, dx_j)^\sim) = \frac{1}{4\pi^2} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 & 0 \\ 1 & 2 & 3 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

To write down the flat coordinates, we first introduce the variables

$$\begin{aligned} w_1 &= y^1 - 2e^{y^5}, \quad w_2 = (y^2 - \frac{1}{6}y^3 + \frac{1}{30}y^4)(y^4)^{-\frac{1}{6}}, \\ w_3 &= (y^3 - \frac{1}{4}y^4)(y^4)^{-\frac{2}{3}}, \quad w_4 = (y^4)^{\frac{1}{6}}, \quad w_5 = y^5. \end{aligned}$$

Then we have

$$t_1 = w_1, \quad t_2 = w_2 - \frac{1}{12}w_3^2w_4, \quad t_3 = w_3w_4, \quad t_4 = w_4, \quad t_5 = w_5.$$

We omit the presentation of the long expression of the intersection form and only write down the potential F here

$$\begin{aligned} F &= \frac{1}{2}t_1^2t_5 + \frac{1}{2}t_1t_2t_4 - \frac{1}{6912}t_3^4 + \frac{1}{17280}t_3^3t_4^3 \\ &\quad - \frac{1}{288}t_2t_4t_3^2 - \frac{1}{34560}t_4^6t_3^2 + \frac{1}{24}t_1t_3^2 + \frac{1}{1440}t_3t_4^4t_2 \\ &\quad - \frac{1}{48}t_2^2t_4^2 - \frac{1}{60480}t_4^7t_2 + \frac{1}{345600}t_4^9t_3 - \frac{1}{7603200}t_4^{12} \\ &\quad + \frac{1}{12}e^{t_5}t_3^2 + \frac{1}{6}e^{t_5}t_3t_4^3 + \frac{1}{120}e^{t_5}t_4^6 + t_2t_4e^{t_5} + \frac{1}{2}e^{2t_5} \\ &\quad + \frac{1}{24}\frac{t_3t_2^2}{t_4} - \frac{1}{216}\frac{t_2t_3^3}{t_4^2} + \frac{1}{4320}\frac{t_3^5}{t_4^3}. \end{aligned}$$

The Euler vector field is given by

$$E = t_1\partial_1 + \frac{5}{6}t_2\partial_2 + \frac{1}{2}t_3\partial_3 + \frac{1}{6}t_4\partial_4 + \partial_5.$$

Example 5.4. $[C_4, k = 2]$ Let R be the root system of type C_4 . Take $k = 2$, then $d_1 = 1, d_2 = d_3 = d_4 = 2$, and

$$\begin{aligned} y^1 &= e^{2i\pi x_5} (\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4); \\ y^2 &= e^{4i\pi x_5} \sum_{1 \leq a < b \leq 4} \zeta_a \zeta_b; \\ y^3 &= e^{4i\pi x_5} \sum_{1 \leq a < b < c \leq 4} \zeta_a \zeta_b \zeta_c; \\ y^4 &= e^{4i\pi x_5} \zeta_1 \zeta_2 \zeta_3 \zeta_4; \\ y^5 &= 2i\pi x_5, \end{aligned}$$

where ζ_j are defined as in the above example. The metric $(,)^\sim$ has the form

$$((dx_i, dx_j)^\sim) = \frac{1}{4\pi^2} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 2 & 0 \\ 1 & 2 & 3 & 3 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

and the flat coordinates are given by

$$\begin{aligned} t_1 &= y^1 - 4e^{y^5}, \quad t_2 = y^2 - 2y^1 e^{y^5} + 6e^{2y^5}, \quad t_3 = (y^3 - \frac{1}{6}y^4)(y^4)^{-\frac{1}{4}}, \\ t_4 &= (y^4)^{\frac{1}{4}}, \quad t_5 = y^5. \end{aligned}$$

The intersection form has the following components

$$\begin{aligned} g^{11} &= 2t_2 - \frac{1}{2}t_1^2 + 4e^{2t_5}; \\ g^{12} &= 6t_1 e^{2t_5} + \frac{1}{2}t_4^4 e^{t_5} + \frac{1}{2}t_3 t_4 e^{t_5}; \\ g^{13} &= 5t_3 e^{t_5} + \frac{10}{3}t_4^3 e^{t_5}; \quad g^{14} = 3t_4 e^{t_5}; \quad g^{15} = \frac{1}{2}t_1; \\ g^{22} &= 2e^{t_5} t_1 t_3 t_4 + 8e^{4t_5} + 8t_3 t_4 e^{2t_5} + \frac{16}{3}e^{2t_5} t_4^4 + 4e^{2t_5} t_1^2 + \frac{1}{3}e^{t_5} t_4^4 t_1; \\ g^{23} &= \frac{56}{3}e^{2t_5} t_4^3 + 7e^{2t_5} t_3 + \frac{7}{3}t_1 t_4^3 e^{t_5} + \frac{7}{2}e^{t_5} t_1 t_3; \\ g^{24} &= 5e^{2t_5} t_4 + \frac{5}{2}t_4 e^{t_5} t_1; \quad g^{25} = t_2; \end{aligned}$$

$$\begin{aligned}
 g^{33} &= 12t_4^2e^{t_5}t_1 + 48t_4^2e^{2t_5} - \frac{1}{4}t_3^2 + \frac{1}{12}t_3t_4^3 - \frac{1}{108}t_4^6 + \frac{1}{4}\frac{t_3^3}{t_4^3}; \\
 g^{34} &= 2t_2 + 4e^{t_5}t_1 + 4e^{2t_5} - \frac{1}{3}t_4t_3 + \frac{1}{72}t_4^4 - \frac{1}{4}\frac{t_3^2}{t_4^2}; \\
 g^{35} &= \frac{3}{4}t_3; \quad g^{44} = \frac{1}{4}\frac{t_3}{t_4} - \frac{1}{12}t_4^2; \quad g^{45} = \frac{1}{4}t_4; \quad g^{55} = \frac{1}{2}.
 \end{aligned}$$

The Euler vector field is given by

$$E = \frac{1}{2}t_1\partial_1 + t_2\partial_2 + \frac{3}{4}t_3\partial_3 + \frac{1}{4}t_4\partial_4 + \frac{1}{2}\partial_5.$$

Finally, we have the potential

$$\begin{aligned}
 F &= \frac{1}{2}t_2^2t_5 + \frac{1}{4}t_1^2t_2 + \frac{1}{2}t_4t_3t_2 + \frac{1}{1440}t_4^5t_3 - \frac{1}{48}t_4^2t_3^2 \\
 &\quad - \frac{1}{36288}t_4^8 - \frac{1}{96}t_1^4 + \frac{1}{2}e^{2t_5}t_1^2 + \frac{1}{6}e^{t_5}t_1t_4^4 + \frac{2}{3}t_4^4e^{2t_5} \\
 &\quad + e^{t_5}t_1t_3t_4 + t_3t_4e^{2t_5} + \frac{1}{4}e^{4t_5} + \frac{1}{48}\frac{t_3^3}{t_4}.
 \end{aligned}$$

6. ON THE FROBENIUS MANIFOLD STRUCTURES RELATED TO THE ROOT SYSTEM OF TYPE B_l

For the root system R of type B_l , we also choose the standard base $\alpha_1, \dots, \alpha_l$ of simple roots as given in [2]. As in [5] we define an indefinite metric $(,)^\sim$ on $\tilde{V} = V \oplus \mathbb{R}$ such that \tilde{V} is the orthogonal direct sum of V and \mathbb{R} , V endowed with the W -invariant Euclidean metric

$$(dx_m, dx_n)^\sim = \frac{1}{4\pi^2}[(1 - \frac{1}{2}\delta_{n,l})m - \frac{l}{4}\delta_{n,l}\delta_{m,l}], \quad 1 \leq m \leq n \leq l \quad (6.1)$$

and \mathbb{R} endowed with the metric

$$(dx_{l+1}, dx_{l+1})^\sim = -\frac{1}{4\pi^2d_k}. \quad (6.2)$$

If the label k of the chosen simple root α_k is less than l , then the numbers d_j are given by

$$d_1 = 1, \dots, d_k = k, \quad d_{k+1} = \dots = d_{l-1} = k, \quad d_l = \frac{k}{2}. \quad (6.3)$$

In the case of $k = l$ we have

$$d_j = \frac{j}{2}, \quad j = 1, \dots, l-1, \quad d_l = \frac{l}{4}. \quad (6.4)$$

The basis of the W_a -invariant Fourier polynomials $y_1(\mathbf{x}), \dots, y_{l-1}(\mathbf{x}), y_l(\mathbf{x})$ are chosen in the same way as we did for the case of C_l

$$y_j(\mathbf{x}) = \sigma_j(\zeta_1, \dots, \zeta_l), \quad j = 1, \dots, l-1, \quad y_l(\mathbf{x}) = \zeta_1^{\frac{1}{2}} \dots \zeta_l^{\frac{1}{2}} \quad (6.5)$$

with $\zeta_1, \dots, \zeta_{l-1}$ defined by (2.7) and

$$\zeta_l = e^{2i\pi(x_{l-1}-2x_l)} + e^{-2i\pi(x_{l-1}-2x_l)} + 2. \quad (6.6)$$

So

$$\begin{aligned} \zeta_j^{\frac{1}{2}} &= 2 \cos \pi(x_j - x_{j-1}), \quad j \leq l-1 \\ \zeta_l^{\frac{1}{2}} &= 2 \cos \pi(2x_l - x_{l-1}). \end{aligned}$$

The generators of the ring $\widetilde{W}^{(k)}(B_l)$ have the same form as that of (2.9). It is easy to see that the components of the resulting metric $(g^{ij}(y))$ coincide with those corresponding to the root system of type C_l if we perform the change of coordinates

$$y^j \mapsto \bar{y}^j = y^j, \quad y_l(\mathbf{x}) \mapsto \bar{y}^l = (y^l)^2, \quad y^{l+1} \mapsto \bar{y}^{l+1} = y^{l+1}, \quad j = 1, \dots, l-1,$$

for $1 \leq k \leq l-1$ and

$$y^j \mapsto \bar{y}^j = y^j, \quad y_l(\mathbf{x}) \mapsto \bar{y}^l = (y^l)^2, \quad y^{l+1} \mapsto \bar{y}^{l+1} = \frac{1}{2}y^{l+1}, \quad j = 1, \dots, l-1,$$

for the case when $k = l$. Thus, the Frobenius manifold structure that we obtain in this way from B_l , by fixing the k -th vertex of the corresponding Dynkin diagram, is isomorphic to the one that we obtain from C_l by choosing the k -th vertex of the Dynkin diagram of C_l .

7. CONCLUDING REMARKS

It remains a challenging problem to understand whether the constructions of the present paper can be generalized to the root systems of the types $D_l, E_6, E_7, E_8, F_4, G_2$ with respect to the choice of an arbitrary vertex on the Dynkin diagram, as it was suggested in [10] motivating by the results of [11]. Another open problem is to obtain an explicit realization of the integrable hierarchies associated with the Frobenius manifolds of the type $W^{(k)}(R)$. So far this problem was solved only for $k = 1, R = A_1$, see [1, 6, 7] for details. We plan to study these problems in subsequent publications.

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