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Holomorphic equivariant cohomology of Atiyah algebroids and localization

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Abstract. In a previous paper we developed an equivariant cohomology theory associated to a set of data formed by a differentiable manifold M carrying the action of a Lie group G, and a Lie algebroid A on M equipped with a compatible infinitesimal action of G. We also obtained a localization formula for a twisted version of this cohomology, and from it, a Bott-type formula. In this paper we slightly modify that theory to obtain a localization formula for an equivariant cohomology associated to an equivariant holomorphic vector bundle. This formula encompasses many residues formulas in complex geometry, in particular we shall show that it admits as a special case Carrell-Lieberman's residue formula [7, 6].

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1. Introduction

In [5] we developed an equivariant cohomology theory associated to a set of data formed by a differentiable manifold M carrying the action of a Lie group G, and a Lie algebroid A on M equipped with a compatible infinitesimal action of G. We also obtained a localization formula for a twisted version of this cohomology, and from it, a Bott-type formula.

Our aim in this paper is to slightly modify the proofs of [5] to obtain a localization formula for an equivariant cohomology associated to the Atiyah algebroid $\mathscr{D}_{\mathscr{E}}$ of a holomorphic vector bundle $\mathscr{E}$ on a complex manifold X and to a vector field on X that may be lifted to a section of $\mathscr{D}_{\mathscr{E}}$. This formula encompasses many residues formulas in complex geometry, in particular we shall show that it admits as a special case Carrell-Lieberman's residue formula [7, 6].

After recalling a few basic definitions in Section 2, we introduce the holomorphic Atiyah algebroids $\mathscr{D}_{\mathscr{E}}$ associated to a holomorphic vector bundle $\mathscr{E}$ on a complex manifold X, and assuming that there is a vector field on X that may be lifted to a section of $\mathscr{D}_{\mathscr{E}}$, we introduce a holomorphic equivariant complex and prove a related localization formula (Section 3). When $\mathscr{E} = 0$ this cohomology essentially reduces to K. Liu's holomorphic equivariant cohomology [11]. In Section 4 we show how our localization formula implies the Carrell-Lieberman residue formula [7, 6].

Further investigations along this line would naturally lead to consider generalizations of the holomorphic Lefschetz formulas and application to Courant algebroids, e.g., in connection to generalized complex geometry (cf. [10]).

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2. Preliminaries

Let M be a differentiable manifold M, and A Lie algebroid A with anchor map $a\colon A \to T_M$ [12]. We denote by $C^\bullet_A = \Gamma(\Lambda^\bullet A^*)$ the cohomology complex associated with A, with differential δ [12]. (The symbol Γ denotes global sections.) Let Q_A be the orientation line bundle $\wedge^r A \otimes \Omega^m_M$, where $r = \operatorname{rk} A$ and $m = \dim M$, and Ω^m_M is the bundle of differential m-forms on M [8] . For every $s \in \Gamma(A)$ one can define a map $L_s = \{s, \cdot\} =\colon \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^\bullet A)$ by letting

$$L_s(s_1 \wedge \cdots \wedge s_k) = \sum_{i=1}^{k} s_1 \wedge \cdots \wedge \{s, s_i\} \wedge \cdots \wedge s_k .$$

Moreover one defines a map

$$\begin{aligned} D\colon \Gamma(Q_A) &\to \Gamma(A^* \otimes Q_A) = \Gamma(A^*) \otimes_{C^\infty(M)} \Gamma(Q_A) \\ D\tau(s) &= L_s(X) \otimes \mu + X \otimes \mathscr{L}_{a(s)}\mu \end{aligned}$$

if $\tau = X \otimes \mu \in \Gamma(Q_A)$ and $s \in \Gamma(A)$. Here $\mathscr{L}$ is the usual Lie derivative with respect to a vector field. We consider the complex $\tilde{C}^\bullet_A = \Gamma(\wedge^\bullet A^* \otimes Q_A)$ with the differential $\tilde{\delta}$ defined by

$$\tilde{\delta}(\chi \otimes \tau) = \delta\chi \otimes \tau + (-1)^{\deg(\chi)} \chi \otimes D\tau$$

where $\xi \in C^\bullet_A$.

We define the map

$$\begin{aligned} p\colon \tilde{C}^\bullet_A &\to \Omega^{\bullet - r + m}(M) \\ p(\psi \otimes X \otimes \mu) &= (a(\psi \lrcorner X)) \lrcorner \mu \,. \end{aligned} \tag{1}$$

Proposition 2.1. *The morphism p is a chain map, in the sense that the diagram*

$$\begin{array}{ccc} \tilde{C}^k_A & \xrightarrow{p} & \Omega^{k-r+m}(M) \\ \downarrow \tilde{\delta} & & \downarrow d \\ \tilde{C}^{k+1}_A & \xrightarrow{p} & \Omega^{k-r+m+1}(M) \end{array}$$

commutes up to a sign, i.e., on C^k_A one has

$$p \circ \tilde{\delta} = (-1)^k \; d \circ p \,.$$

For a proof see [5]. We shall also use this map to integrate cochains in $\tilde{C}^\bullet_A$, by letting

$$\int_M \gamma = \int_M p(\gamma) \,.$$

3. Holomorphic equivariant cohomology of Atiyah algebroids

Let X be an n-dimensional compact complex manifold. We shall denote by Θ_X its holomorphic tangent bundle and by T_X its tangent bundle when X is regarded as a $2n$-dimensional smooth differentiable manifold. Ω^i_X will denote the bundle of holomorphic i-forms on X, while $\Omega^i_{X,\mathbb{C}}$ will denote the bundle of complex-valued smooth i-forms, and $\Omega^{p,q}_X$ the bundle of forms of type (p,q). (In general, we shall not use a different notation for a bundle and its sheaf of sections.)

3.1. **The Atiyah algebroid.** If $\mathscr{E}$ is a (rank r) holomorphic vector bundle on X we shall denote by $\mathscr{D}_\mathscr{E}$ the associated Atiyah algebroid, i.e., $\mathscr{D}_\mathscr{E}$ is the bundle of first order differential operators on $\mathscr{E}$ with scalar symbol, with bracket defined on its global sections by the usual commutator. $\mathscr{D}_\mathscr{E}$ sits inside an exact sequence

$$0 \to \mathit{End}(\mathscr{E}) \to \mathscr{D}_\mathscr{E} \xrightarrow{\sigma} \Theta_X \to 0 \tag{2}$$

(this should be regarded as an exact sequence of sheaves of $\mathscr{O}_X$-modules) where the anchor σ is the symbol map. If V is a global section of Θ_X, the obstruction to lift it to a global section of $\mathscr{D}_\mathscr{E}$ is a class $o(V)$ in $H^1(X, \mathit{End}(\mathscr{E})) \simeq \mathrm{Ext}^1(\mathscr{E},\mathscr{E})$. One can easily show that

$$o(V) = i_V(a(\mathscr{E}))$$

where

$$a(\mathscr{E}) \in H^1(X, \mathit{End}(\mathscr{E}) \otimes \Omega^1_X) \simeq \mathrm{Ext}^1(\Theta_X, \mathit{End}(\mathscr{E}))$$

is the Atiyah class of $\mathscr{E}$, i.e., the obstruction to the existence of holomorphic connections on $\mathscr{E}$ [1]. (One should notice that $a(\mathscr{E})$ is the class in $\mathrm{Ext}^1(\Theta_X, \mathit{End}(\mathscr{E}))$ defining the extension (2), and a holomorphic connection for $\mathscr{E}$ is exactly a splitting of the sequence (2).)

We shall assume throughout that a global holomorphic vector field V on X has been fixed, such that $o(V) = 0$, and will denote by $\tilde V$ a fixed lift of V to $\Gamma(\mathscr{D}_\mathscr{E})$. The pair $(\mathscr{E}, \tilde V)$ is called an *equivariant holomorphic vector bundle.*

3.2. **The cohomology complex.** We want to define a "holomorphic equivariant" cohomology complex associated with the Atiyah algebroid $\mathscr{D}_\mathscr{E}$. We set

$$D_\mathscr{E} = [\mathscr{C}^\infty_X \otimes \mathscr{D}_\mathscr{E}] \oplus T^{0,1}_X$$

$$Q^k_\mathscr{E} = \bigoplus_{p+q=k} \Gamma(\Lambda^p \mathscr{D}^*_\mathscr{E} \otimes \Omega^{0,q}_X)\,. \tag{3}$$

According to the theory developed in [9], the vector bundle $D_{\mathscr{E}}$ has the structure of a Lie algebroid, whose differential is the sum of the differential of $\mathscr{D}_{\mathscr{E}}$, and the $\bar\partial$ operator of X. An easy check shows that the orientation bundle of the algebroid $D_{\mathscr{E}}$ is trivial, so that the morphism p defined in (1) maps $Q^k_{\mathscr{E}} \to \Omega^{k-r^2}_{X,\mathbb{C}}$. This map is an isomorphism when $\mathscr{E}=0$. We define an "equivariant" complex

$$\mathfrak{Q}^\bullet_{\mathscr{E}} = \mathbb{C}[t]\otimes Q^\bullet_{\mathscr{E}}$$

with the usual equivariant grading

$$\deg(\mathscr{P}\otimes\beta) = 2\deg(\mathscr{P}) + \deg\beta$$

if $\mathscr{P}$ is a homogeneous polynomial in t and $\beta\in Q^\bullet_{\mathscr{E}}$. We also define a differential

$$\tilde\delta_V = \bar\partial_{\mathscr{D}_{\mathscr{E}}} + t\, i_{\tilde V}$$

where $\bar\partial_{\mathscr{D}_{\mathscr{E}}}$ is the Cauchy-Riemann operator of the holomorphic bundles $\Lambda^k\mathscr{D}^*_{\mathscr{E}}$. We have $\tilde\delta_V\colon \mathfrak{Q}^\bullet_{\mathscr{E}} \to \mathfrak{Q}^{\bullet+1}_{\mathscr{E}}$, and an easy computation shows that $\tilde\delta_V^2 = 0$, so that $(\mathfrak{Q}^\bullet_{\mathscr{E}},\tilde\delta_V)$ is a cohomology complex. We denote its cohomology by $H^\bullet_V(\mathscr{E})$.

There is a relation between the complex $\mathfrak{Q}^\bullet_0$ (that we obtain by setting $\mathscr{E}=0$ in $\mathfrak{Q}^\bullet_{\mathscr{E}}$) and Liu's holomorphic equivariant de Rham complex [11]. Liu's complex is defined by letting

$$A^{(k)} = \bigoplus_{q-p=k}\Omega^{p,q}(X)$$

with a differential

$$\delta_t = \bar\partial + t\, i_V$$

for some value of t. One has cohomology complexes $(A^{(k)},\delta_t)$, where the index k ranges from $-n$ to n. Liu shows that the corresponding cohomology groups $H^{(k)}_t(X)$ are independent of t, provided $t\neq 0$. We shall denote them by $H^\bullet_{\mathrm{Liu}}(X)$. An explicit computation shows the following relation. Let us denote by $H^\bullet_V(X)$ the cohomology groups $H^\bullet_V(\mathscr{E})$ corresponding to the case $\mathscr{E}=0$.

Proposition 3.1. *For every $k=-n,\dots,n$, the cohomology group $H^{(k)}_{\mathrm{Liu}}(X)$ is isomorphic to the subspace of $\oplus_j H^j_V(X)$ generated by classes that have a representative in the subspace $\oplus_p[\mathbb{C}[t]\otimes\Omega^{p,p+k}(X)]$ of $\oplus_j\mathfrak{Q}^j_0$.*

Proof. A class $[\omega] \in H^{(k)}_{\rm Liu}(X)$ is represented by an element

$$\omega = \sum_{p=0}^{n} \omega^{p,p+k}$$

(where $\omega^{p,p+k} \in \Omega^{p,p+k}(X)$) satisfying $(\bar\partial + t\, i_V)\omega = 0$, i.e.,

$$\bar\partial\omega^{p,p+k} + t\, i_V \omega^{p+1,p+k+1} = 0 \quad \text{for} \quad p = 0, \dots, n. \tag{4}$$

If we define the element in $\mathfrak{Q}^\bullet_0$

$$\xi = \sum_{p=0}^{n} t^{n-p}\omega^{p,p+k}$$

then the condition (4) is equivalent to $\tilde\delta_V \xi = 0$. This establishes the correspondence. □

In particular $H^{(0)}_{\rm Liu}(X)$ is isomorphic to the subspace of $\oplus_j H^j_V(X)$ generated over $\mathbb{C}[t]$ by the classes in $\oplus_p \Omega^{p,p}(X)$.

3.3. **The localization formula.** We may expect the following localization formula to hold.

Theorem 3.2. *Let X be an n-dimensional compact complex manifold, $\mathscr{E}$ a holomorphic vector bundle on X, and V a holomorphic vector field on X, which lifts to a section of $\mathscr{D}_\mathscr{E}$, and has isolated nondegenerate zeroes. If $\gamma \in \mathfrak{Q}^\bullet_\mathscr{E}$ is such that $\tilde\delta_V\gamma = 0$, we have*

$$\int_X \gamma(t) = (-2\pi i)^n \sum_j \frac{p(\gamma)_0(x_j)(t)}{\det \mathbb{L}_{V,j}} \tag{5}$$

where the points x_j are the zeroes of V, and for every j

$$\mathbb{L}_{V,j} \colon \Theta_{x_j}X \to \Theta_{x_j}X$$

is the endomorphism of the holomorphic tangent space $\Theta_{x_j}X$ induced by V.

Proof. The map $p\colon Q^k_\mathscr{E} \to \Omega^{k-r^2}_{X,\mathbb{C}}$ may be written — with respect to the decomposition (3) — as $p = \sum_{i+j=k} p_i \otimes I_j$, where $p_i\colon \mathscr{C}^\infty_X \otimes \Lambda^i\mathscr{D}^*_\mathscr{E} \to \Omega^{i-r^2,0}_X$, and $I_j\colon \Omega^{0,j}_X \to \Omega^{0,j}_X$ is the identity map. This implies the identity

$$p \circ \bar\partial_{D_\mathscr{E}} = (-1)^k \bar\partial \circ p.$$

This reduces the proof of the localization formula to the case $\mathscr{E} = 0$. Let us denote by $\delta_V = \bar\partial + t\, i_V$ the "holomorphic equivariant" differential for the complex $\mathfrak{Q}^\bullet_0 = \mathbb{C}[t] \otimes \Omega^\bullet_{X,\mathbb{C}}$.

We start by proving that if $\gamma \in \mathfrak{Q}_0^\bullet$ is equivariantly closed, i.e., $\delta_V\gamma = 0$, then $\gamma_{2n} = \bar\partial\alpha$ away from the zeroes of V, where γ_{2n} is the component of γ of degree $2n$ in the de Rham grading, and α is a $2n-1$-form. To prove this, choose an hermitian metric g on X, and denoting by $\tilde g\colon T_X^{0,1} \to \Omega_X^{1,0}$ the corresponding homomorphism, let $\theta = \tilde g(\bar V)$ (so θ is of type (1,0)). Note that $(\delta_V\theta)_0 = \|V\|^2$ so that $\delta_V\theta$ is invertible in the ring of differential forms away from the zeroes of V. We set

$$\tilde\alpha = \frac{\theta\wedge\gamma}{\delta_V\theta}$$

and define α as the $(n, n-1)$ component of $\tilde\alpha$. Then $\gamma_{2n} = \bar\partial\alpha = d\alpha$. By applying Stokes theorem, this reduces our goal to proving that

$$\lim_{\epsilon\to 0}\int_{\partial B_j(\epsilon)}\alpha = (-2\pi i)^n\frac{\gamma_0(x_j)}{\det \mathbb{L}_{V,j}}$$

where $B_j(\epsilon)$ is a ball of radius ϵ (in the hermitian metric we chose) with center x_j. This is proved exactly as in [4], Section 3. □

Remark 3.3. If we take $\mathscr{E} = 0$ and consider only classes γ in $\mathbb{C}[t]\otimes[\oplus_p\Omega^{p,p}(X)]$, this localization formula reduces to the one in Theorem 1.6 of [11]. As a consequence, our formula also has Bott's formula for the case of a holomorphic vector field as a special case. △

4. Carrell-Lieberman's localization formula

Eventually, we show how the localization formula (5) implies the Carrell-Lieberman localization formula [7, 6].

Let P be an Ad-invariant polynomial on the algebra $\mathrm{gl}(r,\mathbb{C})$ of $r\times r$ complex matrices, and let Φ be the polynomial expressing $P(a(\mathscr{E}))$ in terms of the Chern classes $c_1(\mathscr{E}),\dots,c_r(\mathscr{E})$, i.e.,

$$P(a(\mathscr{E})) = \Phi(c_1(\mathscr{E}),\dots,c_r(\mathscr{E}))\,.$$

Let K be the curvature of the Chern connection ∇ of the pair $(\mathscr{E}, h)$ (where h is an hermitian metric on $\mathscr{E}$), i.e., the unique connection on $\mathscr{E}$ which is compatible both with the complex structure of $\mathscr{E}$ and the metric h. Let V be a holomorphic vector field on X, with nondegenerate isolated zeroes. One knows that there exists a C^∞ endomorphism μ of $\mathscr{E}$ such that $\bar\partial_{\mathscr{E}}\mu = i_VK$, and $\mu(x_j) = \mathbb{L}_{V,j}$ for all zeroes x_j of V ([4], see also [3]). We define the equivariant curvature of the Chern connection as $\tilde{K} = K + t\,\mu$. By using the connection ∇ to split the exact sequence

$$0 \to End(\mathscr{E}) \otimes \mathscr{C}^\infty_X \to D_{\mathscr{E}} \to T_X \otimes \mathbb{C} \to 0$$

we may regard $\tilde{K}$ as an element in $\mathfrak{Q}^2_{\mathscr{E}}$, and one has $\tilde{\delta}_V \tilde{K} = 0$. Moreover we define the equivariant Chern forms of $\mathscr{E}$ as

$$\tilde{c}_k(\mathscr{E}) = P_k\left(\frac{i}{2\pi}\tilde{K}\right)$$

where P_k is the k-th elementary symmetric polynomial on the Lie algebra $\mathrm{gl}(r, \mathbb{C})$. Then $\Phi(\tilde{c}_1(\mathscr{E}), \ldots, \tilde{c}_r(\mathscr{E}))$ is a cocycle in $\mathfrak{Q}^\bullet_{\mathscr{E}}$, and

$$\int_X P(a(\mathscr{E})) = \int_X \Phi(\tilde{c}_1(\mathscr{E}), \ldots, \tilde{c}_r(\mathscr{E}))\,. \tag{6}$$

Finally, denote by $\nu_1^{(j)}, \ldots, \nu_r^{(j)}$ the Chern classes of the endomorphism

$$\mathbb{L}_{\tilde{V},j} \colon \mathscr{E}_{x_j} \to \mathscr{E}_{x_j}$$

given by a differential operator $\tilde{V}$ such that $\sigma(\tilde{V}) = V$ (note that at the zeroes x_j of V, the differential operator $\tilde{V}$ has degree 0). For the Chern classes of an endomorphism, see [4]. In view of equation (6), and proceeding as in [5] (Theorem 5.1), one can prove, under the same hypotheses on V and $\mathscr{E}$ as in Theorem 3.2, the following formula of Bott type.

Lemma 4.1.

$$\int_X P(a(\mathscr{E})) = \sum_j \frac{\Phi(\nu_1^{(j)}, \ldots, \nu_r^{(j)})}{\det(\mathbb{L}_{V,j})}. \tag{7}$$

Example 4.2. If $\mathscr{E} = \mathscr{O}_X$, we have $a(\mathscr{O}_X) = 0$ and the exact sequence defining the Atiyah algebroid $\mathscr{D}_{\mathscr{O}_X}$ splits. Every holomorphic vector field V on X lifts to a section $\tilde{V}$ of $\mathscr{D}_{\mathscr{O}_X}$ of the form $c + V$, with c a constant. Formula (7) yields the well-known identity [4]

$$\sum_j \frac{1}{\det(\mathbb{L}_{V,j})} = 0\,.$$

△

Equation (7) is Carrell-Lieberman's localization formula in disguise. Let Z be the 0-cycle in X defined by the zeroes of V, and let $x \in Z$. If around x the vector field V is written as

$$V = \sum_{i=1}^n a^i \frac{\partial}{\partial z^i}$$

and $f \in H^0(Z, \mathscr{O}_Z)$, the complex number

$$\mathrm{Res}_{V,x}(f) = \frac{f(x)}{J(a_1, \ldots, a_n)(x)}$$

is well defined. Here $J(a_1, \ldots, a_n)$ is the Jacobian determinant of the components $a_1, \ldots, a_n$ of V with respect to the local holomorphic coordinates $z^1, \ldots, z^n$; it is nonzero because the zeroes of V are assumed to be isolated. Therefore we have a morphism

$$\mathrm{Res}_V \colon H^0(Z, \mathscr{O}_Z) \to \mathbb{C}\,.$$

The endomorphisms $\mathbb{L}_{\tilde{V},x_j} \colon \mathscr{E}_{x_j} \to \mathscr{E}_{x_j}$ define an element $\mathbb{L}_{\tilde{V}} \in H^0(Z, End(\mathscr{E}))$, and for every j, and element $\mathbb{L}_{\tilde{V},x_j} \in H^0(Z_{|x_j}, End(\mathscr{E}))$. We have the identity

$$\frac{\Phi(\nu_1^{(j)}, \ldots, \nu_r^{(j)})}{\det(\mathbb{L}_{V,j})} = (2\pi i)^n \, \mathrm{Res}_V(P(\mathbb{L}_{\tilde{V},x_j}))$$

so that:

Corollary 4.3 (Carrell's localization formula)**.** *Under the same hypotheses of Theorem 3.2, we have*

$$\int_X P(a(\mathscr{E})) = (2\pi i)^n \, \mathrm{Res}_V(P(\mathbb{L}_{\tilde{V}}))\,.$$

Remark 4.4. The previous construction may be twisted by a line bundle $\mathscr{L}$, considering a section $V \in H^0(X, \Theta_X \otimes \mathscr{L})$ and defining a residue morphism $\mathrm{Res}_V \colon H^0(Z, \mathscr{L}^n) \to \mathbb{C}$. The advantage in making such a twist is that if X is projective, one can always choose $\mathscr{L}$ so that there exist sections of $H^0(X, \Theta_X \otimes \mathscr{L})$ that may be lifted to a morphism $\mathscr{D}_{\mathscr{E}} \to \mathscr{D}_{\mathscr{E}} \otimes \mathscr{L}$ [7, 6], i.e., any holomorphic vector bundle can be made into an equivariant one. The general localization formula (5) could be twisted in that way as well. This in particular means that we allow the vector field V to be meromorphic, and in this sense we are also generalizing the Baum-Bott localization formula [2]. △

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