# A DYNAMIC MODEL FOR VISCOELASTICITY IN DOMAINS WITH TIME–DEPENDENT CRACKS

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Abstract. In this paper, we prove the existence of solutions for a class of viscoelastic dynamic systems on time–dependent cracking domains. Finally we exhibit an example which shows that our model overcomes the viscoelastic crack paradox; indeed, the energy-dissipation balance is satisfied provided we take into account the additional dissipative term due to crack growth.

**Keywords**: linear second order hyperbolic systems, dynamic fracture mechanics, elastodynamics, viscoelasticity, cracking domains.

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## 1. Introduction

In this paper we study the dynamic crack growth in viscoelastic materials with long memory. The purpose is to introduce a model which is not affected by the viscoelastic crack paradox, which states that cracks cannot grow in a viscoelastic material of Kelvin-Voigt's type (see the comment after (1.11)).

To describe our model we start with a short description of the standard approach to dynamic fracture in the case of linearly elastic materials with no viscosity. In this situation, the deformation of the elastic part of the material evolves according to elastodynamics, while the evolution of the crack follows Griffith's dynamic criterion, see [13]. This principle, originally formulated in [11] for the quasi–static setting, states that there is an exact balance between the energy released by the elastic part and the energy used to increase the crack, which is postulated to be proportional to the area increment of the crack itself.

For an antiplane displacement, elastodynamics together with the stress-strain relation $\sigma(t,x) = \nabla u(t,x)$, leads to the following wave equation with some prescribed boundary and initial conditions

$$\ddot{u}(t,x) - \operatorname{div} \sigma(t,x) = f(t,x) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma_t. \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^2$ is an open bounded set, which represents the cross–section of the body in the reference configuration, $\Gamma_t \subset \overline{\Omega}$ models the cross–section of the crack at time $t$, $u(t,\cdot)\colon \Omega \setminus \Gamma_t \to \mathbb{R}$ is the antiplane displacement, and $f$ is a forcing term. In this case, Griffith's dynamic criterion reads

$$\mathcal{E}(t) + \mathcal{H}^1(\Gamma_t \setminus \Gamma_0) = \mathcal{E}(0) + \text{work of external forces}, \tag{1.2}$$

where $\mathcal{E}(t)$ is the total energy at time $t$, given by the sum of kinetic and elastic energy, and $\mathcal{H}^1$ represents the one–dimensional Hausdorff measure. From the mathematical point of view, a first step to study the evolution of the fracture is to solve the wave equation (1.1) when the time evolution of the crack is assigned, see for example [3, 6, 8, 14].

When we want to take into account the viscoelastic properties of the material, we can consider three main different rheological models: Kelvin–Voigt, Maxwell, and Standard Viscoelastic Material (see, for example, [16, Chapter 7, Section 1]). These models are based on the behaviour of simple rheological units obtained by some combinations of elastic and viscous elements, shown in Fig.1.

Actually, Kelvin–Voigt and Maxwell units can be considered as particular cases of Standard Material units.

To describe these models in more detail, it is convenient to start with the one–dimensional situation, where now the displacement $u(t,x)$ is defined for $(t,x) \in (0,T) \times (0,L)$, and the strain is given by $u_x(t,x)$. If the spring stiffnesses are denoted by $k_1, k_2$ and the viscous resistance by $c$, such that the force at the viscous element is the product of $c$ and the strain rate $\dot{u}_x(t,x)$, the Kelvin–Voigt unit is characterized by the stress–strain relation

$$\sigma_V(t,x) = k_1 u_x(t,x) + c\dot{u}_x(t,x), \tag{1.3}$$

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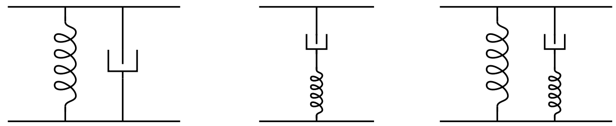


FIGURE 1. Viscoelastic mechanical models:
(a) Kelvin–Voigt unit, (b) Maxwell unit, (c) The Standard Material unit

the Maxwell unit yields

$$\dot{u}_x(t,x) = \frac{1}{k_2}\dot{\sigma}_M(t,x) + \frac{1}{c}\sigma_M(t,x) \quad \text{that is} \quad \sigma_M(t,x) = \mathrm{e}^{-\frac{k_2}{c}t}\sigma_M(0,x) + k_2\int_0^t \mathrm{e}^{-\frac{k_2}{c}(t-s)}\dot{u}_x(s,x)\mathrm{d}s, \tag{1.4}$$

and the Standard–Material unit gives

$$\sigma_S(t,x) + \frac{c}{k_2}\dot{\sigma}_S(t,x) = k_1 u_x(t,x) + c\left(1+\frac{k_1}{k_2}\right)\dot{u}_x(t,x), \tag{1.5}$$

which has the following explicit form

$$\sigma_S(t,x) = \mathrm{e}^{-\frac{k_2}{c}t}\sigma_S(0,x) + \frac{k_1k_2}{c}\int_0^t \mathrm{e}^{-\frac{k_2}{c}(t-s)}u_x(s,x)\mathrm{d}s + (k_1+k_2)\int_0^t \mathrm{e}^{-\frac{k_2}{c}(t-s)}\dot{u}_x(s,x)\mathrm{d}s. \tag{1.6}$$

For $k_2 \to \infty$ in (1.5) we obtain the Kelvin–Voigt unit because $\sigma_S \to \sigma_V$, while the Maxwell unit corresponds to $k_1 = 0$. The relations (1.4) and (1.6), after integrating by parts with respect to time, lead to the following common stress–strain relation

$$\sigma(t,x) = \mathrm{e}^{-\frac{k_2}{c}t}\tilde{\sigma}(0,x) + Cu_x(t,x) - B\int_0^t \mathrm{e}^{-\frac{k_2}{c}(t-s)}u_x(s,x)\mathrm{d}s, \tag{1.7}$$

where $C$ and $B$ are two positive constants depending on $k_1$,$k_2$, and $c$, and $\tilde{\sigma}(0,x)$ also depends on $u_x(0,x)$.

The main difference among the three models is that the Kelvin-Voigt unit provides a local model (the medium is said to have "short memory"), since the state of stress at the instant $t$ only depends on the strain at that instant. The other two models, instead, are non–local in time (the medium is said to have "long memory"), which means that the state of stress at the instant $t$ depends also on the history of the strain between 0 and $t$.

In the case of dimension $d \geq 1$, in the generalized antiplane case, formula (1.3) becomes

$$\sigma_V(t,x) = C\,\nabla u(t,x) + B\,\nabla\dot{u}(t,x), \tag{1.8}$$

while formula (1.7) takes the form

$$\sigma(t,x) = \mathrm{e}^{-\frac{k_2}{c}t}\tilde{\sigma}(0,x) + C\,\nabla u(t,x) - B\int_0^t \mathrm{e}^{-\frac{k_2}{c}(t-s)}\nabla u(s,x)\mathrm{d}s. \tag{1.9}$$

In the framework of dynamic fracture in viscoelastic bodies, when the crack evolution $t \mapsto \Gamma_t$ is prescribed, formula (1.8) for Kelvin-Voigt's model, with a suitable choice of the constants, leads to the following equation

$$\ddot{u}(t,x) - \Delta u(t,x) - \Delta\dot{u}(t,x) = f(t,x) \qquad t\in[0,T],\ x\in\Omega\setminus\Gamma_t.$$

This problem was already considered in [6], which shows that the solutions satisfy the following energy–dissipation balance

$$\mathcal{E}(t) + \mathcal{D}(t) = \mathcal{E}(0) + \text{work of external forces}, \tag{1.10}$$

where the viscous dissipation is given by

$$\mathcal{D}(t) = \int_0^t\int_\Omega |\nabla\dot{u}|^2\,\mathrm{d}x\,\mathrm{d}s.$$

In this case Griffith's dynamic criterion, which is given by (1.2) when no viscosity is present, becomes

$$\mathcal{E}(t) + \mathcal{D}(t) + \mathcal{H}^{d-1}(\Gamma_t\setminus\Gamma_0) = \mathcal{E}(0) + \text{work of external forces}, \tag{1.11}$$

which is incompatible with (1.10), unless $\Gamma_t = \Gamma_0$ for every $t$. In other words cracks cannot move in this model. This phenomenon was already well known in mechanical literature as the viscoelastic paradox, see for instance [16, Chapter 7].

To overcome this difficulty, a modified Kelvin–Voigt's model was considered in [4], with possibly degenerate viscosity coefficients. Under a suitable choice of these coefficients, it is shown that one can obtain a solution which satisfies Griffith's dynamic criterion (1.11).

In this paper we deal, in a unified way, with Maxwell's model and with the Standard Material model, when the crack evolution $t \mapsto \Gamma_t$ is prescribed. Formula (1.9), with a suitable choice of the coefficients, leads to the following equation

$$\ddot{u}(t,x) - \Delta u(t,x) + \int_0^t \mathrm{e}^{s-t} \Delta u(s,x) \mathrm{d}s = f(t,x) \quad t \in [0,T],\, x \in \Omega \setminus \Gamma_t. \tag{1.12}$$

The main results of this paper are Theorem 4.9 and Theorem 4.15, in which we prove, by two different methods, the existence of a weak solution to (1.12). This is done not only in the antiplane case, but also in the more general case of linear elasticity in dimension $d$, that is when the displacement is vector–valued and the elastic energy depends on the symmetrized gradient of the displacement.

The first method, considered in Theorem 4.9, is based on a particular generalization of Lax–Milgram's Theorem ([12, Chapter 3, Theorem 1.1]). We follow the lines of the proof of Theorem 2.1 in [5]. The main difficulty is the fact that in our situation the set $\Omega \setminus \Gamma_t$, where the equation is given, depends on time. This requires some changes in the choice of the functional spaces used to adapt the proof in [5].

The second method, provided by Theorem 4.15, uses a time discretization scheme that yields a solution which, in addition, satisfies the energy–dissipation inequality (4.80). This procedure, adopted in [6] for wave equation (1.1) in a time–dependent domain, consists of the following steps: time discretization, construction of approximate solution, discrete energy estimates, and passage to the limit.

The main difficulty to apply this procedure, in the same way it was done in [6], is given by the identification of the term in the energy-dissipation balance which corresponds to the non–local in time viscous term $\int_0^t e^{s-t} \Delta u(s,x) \mathrm{d}s$ in (1.12).

To solve this issue, we introduce an auxiliary variable $w$ and we transform our equation (1.12) into an equivalent system (see (4.37)) of two equations in the two variables $u$ and $w$, without long memory terms, which have to be solved on the time–dependent domain $\Omega \setminus \Gamma_t$. The advantage of this strategy lies in the fact that we transform a non–local model (the equation) into a local one (the system).

We discretize the time interval $[0,T]$ by using the time step $\tau_n := \frac{T}{n}$. To define the approximate solution $(u_n, w_n)$ at time $(k+1)\tau_n$, we solve an incremental problem (see (4.49)) depending on the values of $(u_n, w_n)$ at times $(k-1)\tau_n$ and $k\tau_n$. Since system (4.37) has a natural notion of energy, we can obtain a discrete energy estimate for $(u_n, w_n)$. Then, we extend $(u_n, w_n)$ to the whole interval $[0,T]$ by a suitable interpolation, and the energy estimates allow us to apply a compactness result and pass to the limit, along a subsequence of $(u_n, w_n)$. It is now possible to prove that the limit of $(u_n, w_n)$ satifies system (4.37), which is equivalent to our equation (1.12). As a byproduct, from the discrete energy estimates we obtain the energy–dissipation inequality (4.80).

We complete the paper by providing an example in $d = 2$ of a weak solution to (1.12) for which the crack can grow while balancing the energy including the contribution of the crack. More precisely, when $\Gamma_t$ moves with constant speed along the $x_1$–axis, we construct a function $u$ which solves (1.12) with a suitable forcing term and satisfies

$$\mathcal{E}(t) + \mathcal{D}(t) + \mathcal{H}^1(\Gamma_t \setminus \Gamma_0) = \mathcal{E}(0) + \text{work of external forces}, \tag{1.13}$$

where $\mathcal{E}$ and $\mathcal{D}$ are the elastic energy and the viscous dissipation corresponding to (1.12) (see (4.101) and (4.102) in Remark 4.22).

The paper is organized as follows. In Section 2 we fix the notation adopted throughout the paper. Afterwards, in Section 3 we list the standard assumptions on the family of cracks $\{\Gamma_t\}_{t\in[0,T]}$, we state the evolution problem in the general case, and we specify the notion of weak solution to the problem. In Section 4 we deal with the existence of a weak solution to the viscoelastic dynamic model; in particular in Subsection 4.1, we provide a weak solution by means of a special variant of Projections Theorem. After that, in Subsection 4.2, as previously anticipated, we define a vector-valued system equivalent to the equation and in Subsection 4.3 we implement the time discretization method on such a system. We conclude the section by showing the validity of the energy–dissipation inequality. The last part of the work is Section 5, where in dimension $d = 2$ we provide an example of a moving crack that satisfies Griffith's dynamic energy–dissipation balance (1.13).

## 2. Notation

The space of $m \times d$ matrices with real entries is denoted by $\mathbb{R}^{m\times d}$; in case $m = d$, the subspace of symmetric matrices is denoted by $\mathbb{R}^{d\times d}_{sym}$. Given two vectors $v_1, v_2 \in \mathbb{R}^d$, their Euclidean scalar product is denoted by $v_1 \cdot v_2 \in \mathbb{R}$. Given $A \in \mathbb{R}^{m\times d}$, we use $A^T$ to denote its transpose; we use $A_1 \cdot A_2 \in \mathbb{R}$ to denote the Euclidean scalar product of two matrices $A_1, A_2 \in \mathbb{R}^{d\times d}$.

Given a function $f\colon \mathbb{R}^d \to \mathbb{R}^m$, we denote its Jacobian matrix by the symbol $\nabla f$, whose components are $(\nabla f)_{ij} := \partial_j f^i$, $i = 1,\dots,m$, $j = 1,\dots,d$, where $\partial_j$ denotes the partial derivative with respect to the variable $x_j$. For a tensor field $F\colon \mathbb{R}^d \to \mathbb{R}^{m\times d}$, by $\operatorname{div} F$ we mean the divergence of $F$ with respect to rows, namely $(\operatorname{div} F)_i := \sum_{j=1}^d \partial_j F_{ij}$, for $i = 1,\dots,m$.

The $d$–dimensional Lebesgue measure is denoted by $\mathcal{L}^d$ and the $(d-1)$–dimensional Hausdorff measure by $\mathcal{H}^{d-1}$. We adopt standard notations for Lebesgue and Sobolev spaces on open subsets of $\mathbb{R}^d$; given an open set $\Omega \subseteq \mathbb{R}^d$ we use $\|\cdot\|_\infty$ to denote the norm of $L^\infty(\Omega;\mathbb{R}^m)$. The boundary values of a Sobolev function are always intended in the sense of traces. Given a bounded open set $\Omega$ with Lipschitz boundary, we denote by $\nu$ the outer unit normal vector to $\partial\Omega$, which is defined $\mathcal{H}^{d-1}$–a.e. on the boundary. Given a Banach space $X$, its norm is denoted by $\|\cdot\|_X$; if $X$ is a Hilbert space, we use $(\cdot,\cdot)_X$ to denote its scalar product. The dual space of $X$ is denoted by $X'$, and we use $\langle\cdot,\cdot\rangle_{X',X}$ to denote the duality product between $X'$ and $X$. Given two Banach spaces $X_1$ and $X_2$, the space of linear and continuous maps from $X_1$ to $X_2$ is denoted by $\mathscr{L}(X_1;X_2)$; given $\mathbb{A} \in \mathscr{L}(X_1;X_2)$ and $u \in X_1$, we write $\mathbb{A}u \in X_2$ to denote the image of $u$ under $\mathbb{A}$. Given an open interval $(a,b) \subseteq \mathbb{R}$, $L^p(a,b;X)$ is the space of $L^p$ functions from $(a,b)$ to $X$. Given $u \in L^p(a,b;X)$, we denote by $\dot u \in \mathcal{D}'(a,b;X)$ its distributional derivative. The set of continuous functions from $[a,b]$ to $X$ is denoted by $C^0([a,b];X)$. Given a reflexive Banach space $X$, $C^0_w([a,b];X)$ is the set of weakly continuous functions from $[a,b]$ to $X$, namely

$$C^0_w([a,b];X) := \{u\colon [a,b] \to X : t \mapsto \langle x', u(t)\rangle_{X',X} \text{ is continuous from } [a,b] \text{ to } \mathbb{R} \text{ for every } x' \in X'\}.$$

## 3. Formulation of the evolution problem, notion of solution

Let $T$ be a positive real number and let $\Omega \subset \mathbb{R}^d$ be a bounded open set (which represents the reference configuration of the body) with Lipschitz boundary. Let $\partial_D\Omega$ be a (possibly empty) Borel subset of $\partial\Omega$, on which we prescribe Dirichlet condition, and let $\partial_N\Omega$ be its complement, on which we give Neumann condition. Let $\Gamma \subset \overline{\Omega}$ be the prescribed crack path. We assume the following hypotheses on the geometry of the cracks:

- (E1) $\Gamma$ is a closed set with $\mathcal{L}^d(\Gamma) = 0$ and $\mathcal{H}^{d-1}(\Gamma \cap \partial\Omega) = 0$;
- (E2) for every $x \in \Gamma$ there exists an open neighborhood $U$ of $x$ in $\mathbb{R}^d$ such that $(U \cap \Omega) \setminus \Gamma$ is the union of two disjoint open sets $U^+$ and $U^-$ with Lipschitz boundary;
- (E3) $\{\Gamma_t\}_{t\in[0,T]}$ is a family of closed subsets of $\Gamma$ satisfying $\Gamma_s \subset \Gamma_t$ for every $0 \le s \le t \le T$.

Notice that the set $\Gamma_t$ represents the crack at time $t \in [0,T]$. Fixed $d \in \mathbb{N}$, thanks to (E1)–(E3) the space $L^2(\Omega \setminus \Gamma_t;\mathbb{R}^d)$ coincides with $L^2(\Omega;\mathbb{R}^d)$ for every $t \in [0,T]$. In particular, we can extend a function $u \in L^2(\Omega\setminus\Gamma_t;\mathbb{R}^d)$ to a function in $L^2(\Omega;\mathbb{R}^d)$ by setting $u = 0$ on $\Gamma_t$. Moreover, the trace of $u \in H^1(\Omega\setminus\Gamma;\mathbb{R}^d)$ is well defined on $\partial\Omega$. Indeed, we may find a finite number of open sets with Lipschitz boundary $U_j \subset \Omega\setminus\Gamma$, $j = 1,\dots k$, such that $\partial\Omega \setminus (\Gamma \cap \partial\Omega) \subset \cup_{j=1}^k \partial U_j$. Since $\mathcal{H}^{d-1}(\Gamma\cap\partial\Omega) = 0$, there exists a constant $C > 0$, depending only on $\Omega$ and $\Gamma$, such that

$$\|u\|_{L^2(\partial\Omega;\mathbb{R}^d)} \le C\|u\|_{H^1(\Omega\setminus\Gamma;\mathbb{R}^d)} \quad \text{for every } u \in H^1(\Omega\setminus\Gamma;\mathbb{R}^d). \tag{3.1}$$

Similarly, we can find a finite number of open sets $V_j \subset \Omega\setminus\Gamma$, $j = 1,\dots l$, with Lipschitz boundary, such that $\Omega\setminus\Gamma = \cup_{j=1}^l V_j$. By using second Korn's inequality in each $V_j$ (see, e.g., [15, Theorem 2.4]) and taking the sum over $j$ we can find a constant $C_K$, depending only on $\Omega$ and $\Gamma$, such that

$$\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})} \le C_K\left(\|u\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|eu\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{sym})}\right) \quad \text{for every } u \in H^1(\Omega\setminus\Gamma;\mathbb{R}^d), \tag{3.2}$$

where $eu$ is the symmetric part of $\nabla u$, i.e., $eu := \frac{1}{2}(\nabla u + \nabla u^T)$.

For every $t \in [0,T]$ we define

$$U_t := H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d).$$

Notice that in the definition of $U_t$ we are considering only the distributional gradient of $u$ in $\Omega \setminus \Gamma_t$ and not the one in $\Omega$. Thanks to (3.2), we can use on the space $U_t$ the equivalent norm

$$\|u\|_{U_t} := (\|u\|_H^2 + \|eu\|_H^2)^{\frac{1}{2}} \quad \text{for every } u \in U_t.$$

To simplify our exposition, for every $d \in \mathbb{N}$ we define the spaces $H := L^2(\Omega;\mathbb{R}^d)$, $H_N := L^2(\partial_N\Omega;\mathbb{R}^d)$ and $H_D := L^2(\partial_D\Omega;\mathbb{R}^d)$; we always identify the dual of $H$ by $H$ itself, and $L^2(0,T;H)$ by the space $L^2((0,T)\times\Omega;\mathbb{R}^d)$.

Thanks to (3.2), the space $U_t$ coincides with the usual Sobolev space $H^1(\Omega\setminus\Gamma_t;\mathbb{R}^d)$. Furthermore, by (3.1), we can consider for every $t \in [0,T]$ the set

$$U_t^D := \{u \in U_t \,:\, u = 0 \text{ on } \partial_D\Omega\},$$

which is a Hilbert space with respect to $\|\cdot\|_{U_t}$. Moreover, by combining (3.2) with (3.1), we derive also the existence of a positive constant $C_{tr}$ such that

$$\|u\|_{H_N} \le C_{tr}\|u\|_{U_T} \quad \text{for every } u \in U_T. \tag{3.3}$$

Let $\mathbb{C},\mathbb{V}\colon \Omega \to \mathscr{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym})$ be the elasticity and viscosity tensors. They are two fourth-order tensors satisfying:

$$\mathbb{C}_{ijhk}, \mathbb{V}_{ijhk} \in L^\infty(\Omega) \quad \text{for every } i,j,h,k = 1,\dots,d, \tag{3.4}$$

$$\mathbb{C}(x)\eta_1\cdot\eta_2 = \eta_1\cdot\mathbb{C}(x)\eta_2, \quad \mathbb{V}(x)\eta_1\cdot\eta_2 = \eta_1\cdot\mathbb{V}(x)\eta_2 \quad \text{for a.e. } x\in\Omega \text{ and for every } \eta_1,\eta_2\in\mathbb{R}^{d\times d}_{sym}, \tag{3.5}$$

$$\mathbb{C}(x)\eta\cdot\eta \ge C_{\mathbb{C}}|\eta|^2, \quad \mathbb{V}(x)\eta\cdot\eta \ge C_{\mathbb{V}}|\eta|^2 \quad \text{for a.e. } x\in\Omega \text{ and for every } \eta\in\mathbb{R}^{d\times d}_{sym}, \tag{3.6}$$

for two positive constants $C_{\mathbb{C}}, C_{\mathbb{V}}$ independent of $x$.

Let us now specify the hypothesis on the forcing term, boundary, and initial conditions. Let $f$ be a function such that $f = f_1 + f_2$ with $f_1 \in L^2(0,T;H)$ and $f_2 \in H^1(0,T;U_T')$. Fixed $\beta > 0$ and given some functions $N \in H^1(0,T;H_N)$, $z \in H^2(0,T;H)\cap H^1(0,T;U_0)$, $u^0 \in U_0$ with $u^0 - z(0) \in U_0^D$, and $u^1 \in H$, we want to find a solution to the following viscoelastic dynamic system

$$\ddot u(t) - \operatorname{div}(\mathbb{C}eu(t)) - \frac{1}{\beta}\operatorname{div}(\mathbb{V}eu(t)) + \operatorname{div}\left(\int_0^t \frac{1}{\beta^2}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{V}eu(\tau)\mathrm{d}\tau\right) = f(t) \quad \text{in } \Omega\setminus\Gamma_t,\ t\in(0,T), \tag{3.7}$$

satisfying the boundary and initial conditions

$$u(t) = z(t) \qquad \text{on } \partial_D\Omega, \quad t\in(0,T), \tag{3.8}$$

$$\left[\left(\mathbb{C} + \frac{\mathbb{V}}{\beta}\right)eu(t) - \int_0^t \frac{1}{\beta^2}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{V}eu(\tau)\mathrm{d}\tau\right]\nu = N(t) \qquad \text{on } \partial_N\Omega, \quad t\in(0,T), \tag{3.9}$$

$$\left[\left(\mathbb{C} + \frac{\mathbb{V}}{\beta}\right)eu(t) - \int_0^t \frac{1}{\beta^2}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{V}eu(\tau)\mathrm{d}\tau\right]\nu = 0 \qquad \text{on } \Gamma_t, \quad t\in(0,T), \tag{3.10}$$

$$u(0) = u^0, \quad \dot u(0) = u^1. \tag{3.11}$$

As usual, the Neumann boundary conditions are only formal, and their meaning will be specified in Definition 3.3.

**Remark 3.1.** Notice that the stress–strain relation implied in (3.7), that is

$$\sigma_\beta(t) = \left(\mathbb{C} + \frac{\mathbb{V}}{\beta}\right)eu(t) - \int_0^t \frac{1}{\beta^2}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{V}eu(\tau)\mathrm{d}\tau,$$

after integrating by parts in time, can be written in the following form

$$\sigma_\beta(t) = \mathbb{C}eu(t) + \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{V}e\dot u(\tau)\mathrm{d}\tau + \frac{1}{\beta}\mathrm{e}^{-\frac{t}{\beta}}\mathbb{V}eu(0). \tag{3.12}$$

Passing to the limit as $\beta \to 0^+$ in (3.12), for every $t > 0$ we obtain Kelvin-Voigt's material stress–strain relation

$$\sigma_0(t) = \mathbb{C}eu(t) + \mathbb{V}e\dot u(t).$$

For convenience of notations we set $\mathbb{B} := \frac{\mathbb{V}}{\beta}$, and its $\beta$ dependence is not explicit because it is irrelevant to our study. Throughout the paper we always assume that the family $\{\Gamma_t\}_{t\in[0,T]}$ satisfies (E1)–(E3), as well as $\mathbb{C}$, $\mathbb{B}$, $f$, $z$, $u^0$, and $u^1$ the previous hypotheses. Let us define the following functional spaces:

$$\mathcal{U} := \{\varphi \in L^2(0,T;U_T) \cap H^1(0,T;H) : \varphi(t) \in U_t \text{ for a.e. } t \in (0,T)\}.$$

$$\mathcal{U}^D := \{\varphi \in \mathcal{U} : \varphi(t) \in U_t^D \text{ for a.e. } t \in (0,T)\},$$

**Lemma 3.2.** *The space $\mathcal{U}$ is a Hilbert space with respect to the following norm:*

$$\|\varphi\|_{\mathcal{U}} := \left(\|\varphi\|^2_{L^2(0,T;U_T)} + \|\dot\varphi\|^2_{L^2(0,T;H)}\right)^{\frac12} \quad \forall \varphi \in \mathcal{U}.$$

*Moreover, $\mathcal{U}^D$ is a closed subspace of $\mathcal{U}$.*

*Proof.* It is clear that $\|\cdot\|_{\mathcal{U}}$ is a norm induced by a scalar product on the set $\mathcal{U}$ . We just have to check the completeness of this space with respect to this norm. Let $\{\varphi_k\}_k \subseteq \mathcal{U}$ be a Cauchy sequence. Then, $\{\varphi_k\}_k$ and $\{\dot\varphi_k\}_k$ are Cauchy sequences, respectively, in $L^2(0,T;U_T)$ and $L^2(0,T;H)$, which are complete Hilbert spaces. Thus there exists $\varphi \in L^2(0,T;U_T)$ with $\dot\varphi \in L^2(0,T;H)$ such that $\varphi_k \to \varphi$ in $L^2(0,T;U_T)$ and $\dot\varphi_k \to \dot\varphi$ in $L^2(0,T;H)$. In particular there exists a subsequence $\{\varphi_{k_j}\}_j$ such that $\varphi_{k_j}(t) \to \varphi(t)$ in $U_T$ for a.e. $t \in (0,T)$. Since $\varphi_{k_j}(t) \in U_t$ for a.e. $t \in (0,T)$ we deduce that $\varphi(t) \in U_t$ for a.e. $t \in (0,T)$. Hence $\varphi \in \mathcal{U}$ and $\varphi_k \to \varphi$ in $\mathcal{U}$. With a similar argument, we can prove that $\mathcal{U}^D \subseteq \mathcal{U}$ is a closed subspace. □

With this in mind, now we are in a position to express in which sense a function $u$ is a solution to the system (3.7)–(3.11). To this purpose, we give the definition of weak solution for the system and the meaning of initial conditions. For convenience, we will use the following notation

$$\langle\langle \cdot,\cdot\rangle\rangle := \langle\cdot,\cdot\rangle_{U_T',U_T}.$$

**Definition 3.3** (Weak solution)**.** We say that $u \in \mathcal{U}$ is a *weak solution* to system (3.7) with boundary conditions (3.8)–(3.10) if $u - z \in \mathcal{U}^D$ and

$$-\int_0^T (\dot u(t), \dot v(t))_H \,\mathrm{d}t + \int_0^T ((\mathbb{C}+\mathbb{B})eu(t), ev(t))_H \,\mathrm{d}t - \int_0^T\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), ev(t))_H \,\mathrm{d}\tau\mathrm{d}t$$
$$= \int_0^T (N(t), v(t))_{H_N}\,\mathrm{d}t + \int_0^T (f_1(t), v(t))_H\,\mathrm{d}t + \int_0^T \langle\langle f_2(t), v(t)\rangle\rangle\,\mathrm{d}t$$

for every $v \in \mathcal{U}^D$ such that $v(0) = v(T) = 0$.

**Definition 3.4** (Initial conditions)**.** We say that $u \in \mathcal{U}$ weak solution to (3.7) satisfies the initial conditions (3.11) if

$$\lim_{t\to 0^+}\|u(t) - u^0\|_H = 0, \quad \lim_{t\to 0^+}\|\dot u(t) - u^1\|_{(U_0^D)'} = 0.$$

**Remark 3.5.** Without loss of generality we may assume $z(t) = 0$ and $u^0 = 0$. Indeed, if $\tilde u \in \mathcal{U}^D$ verifies the following equalities:

- for some $\bar f_1 \in L^2(0,T;H)$, $\bar f_2 \in H^1(0,T;U_T')$

$$-\int_0^T (\dot{\tilde u}(t), \dot v(t))_H \,\mathrm{d}t + \int_0^T ((\mathbb{C}+\mathbb{B})e\tilde u(t), ev(t))_H \,\mathrm{d}t - \int_0^T\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}e\tilde u(\tau), ev(t))_H \,\mathrm{d}\tau\mathrm{d}t$$
$$= \int_0^T (N(t), v(t))_{H_N}\,\mathrm{d}t + \int_0^T (\bar f_1(t), v(t))_H\,\mathrm{d}t + \int_0^T \langle\langle \bar f_2(t), v(t)\rangle\rangle\,\mathrm{d}t, \tag{3.13}$$

  for every $v \in \mathcal{U}^D$ such that $v(0) = v(T) = 0$;
- for some $\tilde u^1 \in H$

$$\lim_{t\to 0^+}\|\tilde u(t)\|_H = 0, \quad \lim_{t\to 0^+}\|\dot{\tilde u}(t) - \tilde u^1\|_{(U_0^D)'} = 0; \tag{3.14}$$

then by taking

$$\bar f_1 := f_1 - \ddot z, \qquad \tilde u^1 := u^1 - \dot z(0),$$

and defining for a.e. $t \in [0,T]$ the action of $\bar f_2(t)$ on the space $U_T$ in the following way

$$\langle\langle \bar f_2(t), \varphi\rangle\rangle := \langle\langle f_2(t), \varphi\rangle\rangle + \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}ez(\tau), e\varphi)_H\,\mathrm{d}\tau - ((\mathbb{C}+\mathbb{B})ez(t), e\varphi)_H$$

$$+((1-\mathrm{e}^{-\frac{t}{\beta}})\mathbb{B}eu^0-(\mathbb{C}+\mathbb{B})eu^0,e\varphi)_H-((1-\mathrm{e}^{-\frac{t}{\beta}})\mathbb{B}ez(0)-(\mathbb{C}+\mathbb{B})ez(0),e\varphi)_H,$$

we easily see that the function $u(t):=\tilde{u}(t)+u^0-z(0)+z(t)$ satisfies Definition 3.3 and 3.4.

## 4. Existence Results

In this section we will provide two different methods to prove the existence of solutions to the viscoelastic dynamic system (3.7)–(3.11). The second method that we will show, is based on a minimizing movement deriving from the theory of gradient flow. It is the usual tool used to prove the existence of solutions in these kind of problems as you can find in [4], [6], [9]. Thanks to that, we will be also able to give a particular energy–dissipation inequality, which is verified by the solution previously found. Consequently, by this inequality, we will prove that the solution will satisfy the initial conditions (3.11) in a stronger sense than Definition 3.4. On the contrary, the first method is to be considered in the framework of functional analysis; it derives from an idea of C. Dafermos as you can find in [5], based on a generalization of Lax–Milgram's Theorem, which you can read in [12].

4.1. **A special variant of Projections Theorem.** As previously anticipated, in this subsection we will prove an existence result for a particular functional equation, which we will use to show the existence of a weak solution to our viscoelastic dynamic problem in the sense of identity (3.13).

Let $X$ be a Hilbert space and $V\subseteq X$ be a linear subspace, endowed with the scalar product $(\cdot,\cdot)_V$ which makes it a Pre-Hilbert space. Suppose that the inclusion of $V$ in $X$ is a continuous map, which means that for some $c>0$

$$\|\varphi\|_X\leq c\|\varphi\|_V \qquad \forall\varphi\in V. \tag{4.1}$$

Now let us consider a bilinear form $B:X\times V\to\mathbb{R}$ such that

$$B(\cdot,\varphi):X\to\mathbb{R} \qquad \text{is a linear continuous function on } X \text{ for every } \varphi\in V, \tag{4.2}$$

$$B(\varphi,\varphi)\geq\alpha\|\varphi\|_V^2 \quad \text{for every } \varphi\in V, \text{for some } \alpha>0. \tag{4.3}$$

**Theorem 4.1.** *Suppose that hypotheses (4.1)–(4.3) are satisfied. Let $L:V\to\mathbb{R}$ be a linear continuous map, then there exists $u\in X$ such that*

$$B(u,\varphi)=L(\varphi) \qquad \forall\varphi\in V.$$

For the proof see [12, Chapter 3, Theorem 1.1]

**Lemma 4.2.** *Let us consider the following spaces:*

$$\mathcal{E}_c:=\{v\in C_c^\infty(0,T;U_T)|\ \ v(t)\in U_t^D, \quad \forall t\in(0,T)\},$$
$$\mathcal{E}:=\{\varphi\in C^\infty([0,T];U_T)|\ \ \varphi(0)=0,\ \varphi(t)\in U_t^D, \quad \forall t\in(0,T)\}.$$

*For every $v\in\mathcal{E}_c$ the function*

$$\varphi_v(t)=\int_0^t\frac{v(s)}{s-T}\mathrm{d}s$$

*is well defined and $\varphi_v\in\mathcal{E}$.*

*Proof.* In the first instance, we can notice that $\varphi_v$ is well defined because $v$ is a function with compact support, hence it vanishes in a neighbourhood of $T$. Moreover, $\varphi_v(0)=0$ by definition of the integral and $\varphi_v\in C^\infty([0,T];U_T)$ because it is a primitive of a function with the same regularity. Now, we can observe that $v(s)\in U_s^D\subseteq U_t^D$ for all $s\leq t$, therefore we have $\frac{v(s)}{s-T}\in U_t^D$ for $s\leq t$, and by the properties of Bochner's integral we get $\varphi_v(t)\in U_t^D$. □

**Lemma 4.3.** *Given $v\in\mathcal{U}^D$ with $v(0)=v(T)=0$, there exists a sequence of functions $\{v_k\}_k\subseteq\mathcal{E}_c$, such that*

$$v_k\xrightarrow[k\to\infty]{\mathcal{U}}v.$$

For the proof see [9, Lemma 2.8].

**Remark 4.4.** The last lemma shows that the space $\mathcal{E}_c$ is dense in the space of functions belonging to $\mathcal{U}^D$ which vanish on the boundary of the time interval. Then it will be enough to prove the validity of relation (3.13) only for test functions in the space $\mathcal{E}_c$.

**Proposition 4.5.** *Let $u \in \mathcal{U}^D$ be a weak solution according to the formulation (3.13). Then there exists a function $\dot{u}_d \in L^2(0,T;(U_0^D)')$ which is the distributional derivative of $\dot{u}$.*

*Proof.* We define the action of $\dot{u}_d$ in the following way: for a.e. $t \in (0,T)$

$$\langle \dot{u}_d(t), v\rangle := -((\mathbb{C}+\mathbb{B})eu(t), ev)_H + \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), ev)_H \,\mathrm{d}\tau + (N(t), v)_{H_N} + (\bar{f}_1(t), v)_H + \langle\langle \bar{f}_2(t), v\rangle\rangle, \tag{4.4}$$

for every $v \in U_0^D$, where $\langle \cdot, \cdot \rangle$ represents the duality product between $(U_0^D)'$ and $U_0^D$.
Let us consider a test function $\varphi \in C_c^\infty(0,T)$, then $\psi(t) := \varphi(t)v \in C_c^\infty(0,T;U_0)$ for every $v \in U_0^D$, and consequently $\psi \in \mathcal{E}_c$. If we multiply both members of (4.4) by $\varphi(t)$, after integrating on $(0,T)$ and thanks to the weak formulation (3.13), we obtain

$$\begin{aligned}\int_0^T \langle \dot{u}_d(t), v\rangle \varphi(t)\mathrm{d}t &= -\int_0^T ((\mathbb{C}+\mathbb{B})eu(t), e\psi(t))_H \,\mathrm{d}t + \int_0^T\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\psi(t))_H \,\mathrm{d}\tau\mathrm{d}t \\ &+ \int_0^T (N(t), \psi(t))_{H_N}\,\mathrm{d}t + \int_0^T (\bar{f}_1(t), \psi(t))_H\,\mathrm{d}t + \int_0^T \langle\langle \bar{f}_2(t), \psi(t)\rangle\rangle\,\mathrm{d}t = -\int_0^T (\dot{u}(t), v)_H \dot{\varphi}(t)\mathrm{d}t,\end{aligned}$$

which implies

$$\langle \int_0^T \dot{u}_d(t)\varphi(t)\mathrm{d}t, v\rangle = \langle -\int_0^T \dot{u}(t)\dot{\varphi}(t)\mathrm{d}t, v\rangle \qquad \forall v \in U_0^D.$$

Hence, we get

$$\int_0^T \dot{u}_d(t)\varphi(t)\mathrm{d}t = -\int_0^T \dot{u}(t)\dot{\varphi}(t)\mathrm{d}t \qquad \forall \varphi \in C_c^\infty(0,T)$$

as elements of $(U_0^D)'$, that concludes the proof. □

**Remark 4.6.** Proposition 4.5 implies that $\dot{u} \in H^1(0,T;(U_0^D)')$, then it admits a continuous representative. Therefore, we can say that there exists $\dot{u}(0) \in (U_0^D)'$ such that

$$\lim_{t\to 0^+} \|\dot{u}(t) - \dot{u}(0)\|_{(U_0^D)'} = 0.$$

**Proposition 4.7.** *Let $u \in \mathcal{U}^D$ be a function which verifies relation (3.13) for every function $\psi$ which belongs to the following space*

$$Lip_T^0 := \{\psi \in Lip([0,T];U_T) : \psi(0) = 0, \ \ \psi(t) = 0 \quad \forall t \in I(T), \ \ \psi(t) \in U_t^D \quad \forall t \in [0,T]\}$$

*for some $I(T)$ open neighbourhood of $T$. Then $u$ satisfies the identity*

$$\begin{aligned}-\int_0^T (\dot{u}(t), \dot{\Psi}(t))_H\,\mathrm{d}t + \int_0^T ((\mathbb{C}+\mathbb{B})eu(t), e\Psi(t))_H\,\mathrm{d}t - \int_0^T\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\Psi(t))_H\,\mathrm{d}\tau\mathrm{d}t \\ = \int_0^T (N(t), \Psi(t))_{H_N}\,\mathrm{d}t + \int_0^T (\bar{f}_1(t), \Psi(t))_H\,\mathrm{d}t + \int_0^T \langle\langle \bar{f}_2(t), \Psi(t)\rangle\rangle\,\mathrm{d}t + \langle \dot{u}(0), \Psi(0)\rangle\end{aligned} \tag{4.5}$$

*for every function $\Psi$ which belongs to the space*

$$Lip_T := \{\Psi \in Lip([0,T];U_T) : \Psi(T) = 0 \quad \Psi(t) \in U_t^D \quad \forall t \in [0,T]\}.$$

*Proof.* We proceed in two steps.

*Step 1.* We first show that $u$ satisfies (4.5) for every $\Psi \in Lip_T^* \subseteq Lip_T$ where

$$Lip_T^* := \{\Psi \in Lip([0,T];U_T) : \Psi(t) = 0 \quad \forall t \in I(T), \ \ \Psi(t) \in U_t^D \quad \forall t \in [0,T]\}.$$

Let us consider $\Psi \in Lip_T^*$ and define for every $\epsilon \in (0,T)$ the function

$$\psi_\epsilon(t) := \begin{cases} \frac{t}{\epsilon}\Psi(0) & t \in [0,\epsilon] \\ \Psi(t-\epsilon) & t \in [\epsilon, T]. \end{cases}$$

Then for $\epsilon$ small enough we have $\psi_\epsilon \in Lip_T^0$, and by assumptions we can use it as a test function in (3.13) to get the equality $I_\epsilon + I_\epsilon^m = 0$, where the two terms $I_\epsilon$ and $I_\epsilon^m$ are defined in the following way

$$I_\epsilon := -\int_\epsilon^T (\dot{u}(t), \dot{\Psi}(t-\epsilon))_H\,\mathrm{d}t + \int_\epsilon^T ((\mathbb{C}+\mathbb{B})eu(t), e\Psi(t-\epsilon))_H\,\mathrm{d}t - \int_\epsilon^T (N(t), \Psi(t-\epsilon))_{H_N}\,\mathrm{d}t$$

$$-\int_\epsilon^T\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau),e\Psi(t-\epsilon))_H\,\mathrm{d}\tau\mathrm{d}t-\int_\epsilon^T(\bar f_1(t),\Psi(t-\epsilon))_H\,\mathrm{d}t-\int_\epsilon^T\langle\langle \bar f_2(t),\Psi(t-\epsilon)\rangle\rangle\,\mathrm{d}t,$$

and

$$I_\epsilon^m:=-\fint_0^\epsilon(\dot u(t),\Psi(0))_H\,\mathrm{d}t+\fint_0^\epsilon((\mathbb{C}+\mathbb{B})eu(t),te\Psi(0))_H\,\mathrm{d}t-\fint_0^\epsilon\int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau),te\Psi(0))_H\,\mathrm{d}\tau\mathrm{d}t$$
$$-\fint_0^\epsilon(N(t),t\Psi(0))_{H_N}\,\mathrm{d}t-\fint_0^\epsilon(\bar f_1(t),t\Psi(0))_H\,\mathrm{d}t-\fint_0^\epsilon\langle\langle\bar f_2(t),t\Psi(0)\rangle\rangle\,\mathrm{d}t.$$

Let us study the convergence of $I_\epsilon$ and $I_\epsilon^m$ as $\epsilon\to0^+$. First of all, we notice that

$$\begin{aligned}\|\psi_\epsilon-\Psi\|^2_{L^2(0,T;U_T)}&=\int_0^T\|\psi_\epsilon(t)-\Psi(t)\|^2_{U_T}\mathrm{d}t=\int_0^\epsilon\left\|\frac{t}{\epsilon}\Psi(0)-\Psi(t)\right\|^2_{U_T}\mathrm{d}t+\int_\epsilon^T\|\Psi(t-\epsilon)-\Psi(t)\|^2_{U_T}\mathrm{d}t\\&\le2\|\Psi(0)\|^2_{U_T}\int_0^\epsilon\frac{t^2}{\epsilon^2}\mathrm{d}t+2\int_0^\epsilon\|\Psi(t)\|^2_{U_T}\mathrm{d}t+\int_\epsilon^T L_\Psi^2|t-\epsilon-t|^2\mathrm{d}t\\&=\frac{2}{3}\epsilon\|\Psi(0)\|^2_{U_T}+2\int_0^\epsilon\|\Psi(t)\|^2_{U_T}\mathrm{d}t+L^2_\Psi\epsilon^2(T-\epsilon)\xrightarrow[\epsilon\to0^+]{}0.\end{aligned}\tag{4.6}$$

Thanks to (3.4), (4.6), and to the absolute continuity of Lebesgue's integral, we have

$$\begin{aligned}&\left|\int_\epsilon^T((\mathbb{C}+\mathbb{B})eu(t),e\Psi(t-\epsilon))_H\,\mathrm{d}t-\int_0^T((\mathbb{C}+\mathbb{B})eu(t),e\Psi(t))_H\,\mathrm{d}t\right|\\&\le\left|\int_\epsilon^T((\mathbb{C}+\mathbb{B})eu(t),e\Psi(t-\epsilon)-e\Psi(t))_H\,\mathrm{d}t\right|+\left|\int_0^\epsilon((\mathbb{C}+\mathbb{B})eu(t),e\Psi(t))_H\,\mathrm{d}t\right|\\&\le\|\mathbb{C}+\mathbb{B}\|_\infty\int_0^T\|eu(t)\|_H\|\psi_\epsilon(t)-\Psi(t)\|_{U_T}\mathrm{d}t+\int_0^\epsilon|((\mathbb{C}+\mathbb{B})eu(t),e\Psi(t))_H|\,\mathrm{d}t\\&\le\|\mathbb{C}+\mathbb{B}\|_\infty\|u\|_{L^2(0,T;U_T)}\|\psi_\epsilon-\Psi\|_{L^2(0,T;U_T)}+\int_0^\epsilon|((\mathbb{C}+\mathbb{B})eu(t),e\Psi(t))_H|\,\mathrm{d}t\xrightarrow[\epsilon\to0^+]{}0.\end{aligned}\tag{4.7}$$

In the same way we can prove

$$\int_\epsilon^T\int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau),e\Psi(t-\epsilon))_H\,\mathrm{d}\tau\mathrm{d}t\xrightarrow[\epsilon\to0^+]{}\int_0^T\int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau),e\Psi(t))_H\,\mathrm{d}\tau\mathrm{d}t,\tag{4.8}$$

$$\int_\epsilon^T(N(t),\Psi(t-\epsilon))_{H_N}\,\mathrm{d}t\xrightarrow[\epsilon\to0^+]{}\int_0^T(N(t),\Psi(t))_{H_N}\,\mathrm{d}t,\tag{4.9}$$

$$\int_\epsilon^T(\bar f_1(t),\Psi(t-\epsilon))_H\,\mathrm{d}t\xrightarrow[\epsilon\to0^+]{}\int_0^T(\bar f_1(t),\Psi(t))_H\,\mathrm{d}t,\tag{4.10}$$

$$\int_\epsilon^T\langle\langle\bar f_2(t),\Psi(t-\epsilon)\rangle\rangle\,\mathrm{d}t\xrightarrow[\epsilon\to0^+]{}\int_0^T\langle\langle\bar f_2(t),\Psi(t)\rangle\rangle\,\mathrm{d}t.\tag{4.11}$$

Notice that, thanks to the continuity of the translation operator in $L^2$, and again by the absolute continuity of Lebesgue's integral, we get

$$\begin{aligned}&\left|\int_\epsilon^T(\dot u(t),\dot\Psi(t-\epsilon))_H\,\mathrm{d}t-\int_0^T(\dot u(t),\dot\Psi(t))_H\,\mathrm{d}t\right|\\&\le\left|\int_\epsilon^T(\dot u(t),\dot\Psi(t-\epsilon)-\dot\Psi(t))_H\,\mathrm{d}t\right|+\left|\int_0^\epsilon(\dot u(t),\dot\Psi(t))_H\,\mathrm{d}t\right|\\&\le\int_0^T\|\dot u(t)\|_H\|\dot\Psi(t-\epsilon)-\dot\Psi(t)\|_H\,\mathrm{d}t+\int_0^\epsilon|(\dot u(t),\dot\Psi(t))_H|\,\mathrm{d}t\\&\le\|\dot u\|_{L^2(0,T;H)}\|\dot\Psi(\cdot-\epsilon)-\dot\Psi(\cdot)\|_{L^2(0,T;H)}+\int_0^\epsilon|(\dot u(t),\dot\Psi(t))_H|\,\mathrm{d}t\xrightarrow[\epsilon\to0^+]{}0.\end{aligned}\tag{4.12}$$

Thanks to (4.7)–(4.12) we can say

$$I_\epsilon \xrightarrow[\epsilon\to 0^+]{} -\int_0^T (\dot u(t),\dot\Psi(t))_H\,\mathrm{d}t + \int_0^T ((\mathbb{C}+\mathbb{B})eu(t), e\Psi(t))_H\,\mathrm{d}t - \int_0^T\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\Psi(t))_H\,\mathrm{d}\tau\mathrm{d}t$$
$$- \int_0^T (N(t),\Psi(t))_{H_N}\,\mathrm{d}t - \int_0^T (\bar f_1(t),\Psi(t))_H\,\mathrm{d}t - \int_0^T \langle\langle \bar f_2(t),\Psi(t)\rangle\rangle\,\mathrm{d}t.$$

Now we analyze the limit of $I^m_\epsilon$ as $\epsilon\to 0^+$. For the first term we have

$$⨍_0^\epsilon (\dot u(t),\Psi(0))_H\,\mathrm{d}t = \left(⨍_0^\epsilon \dot u(t)\mathrm{d}t, \Psi(0)\right)_H = \langle ⨍_0^\epsilon \dot u(t)\mathrm{d}t,\Psi(0)\rangle \xrightarrow[\epsilon\to 0^+]{} \langle \dot u(0),\Psi(0)\rangle \tag{4.13}$$

because thanks to Remark 4.6 it holds

$$\lim_{t\to 0^+}\|\dot u(t)-\dot u(0)\|_{(U_0^D)'} = 0.$$

Moreover

$$\left|⨍_0^\epsilon ((\mathbb{C}+\mathbb{B})eu(t), te\Psi(0))_H\,\mathrm{d}t\right| \le \|\mathbb{C}+\mathbb{B}\|_\infty \|e\Psi(0)\|_H ⨍_0^\epsilon t\|u(t)\|_{U_T}\mathrm{d}t \le C\epsilon^{\frac12}\left(\int_0^\epsilon \|u(t)\|^2_{U_T}\mathrm{d}t\right)^{\frac12} \xrightarrow[\epsilon\to 0^+]{} 0. \tag{4.14}$$

In the same way, we can prove

$$⨍_0^\epsilon \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), te\Psi(0))_H\,\mathrm{d}\tau\mathrm{d}t \xrightarrow[\epsilon\to 0^+]{} 0, \tag{4.15}$$

$$⨍_0^\epsilon (N(t), t\Psi(0))_{H_N}\,\mathrm{d}t \xrightarrow[\epsilon\to 0^+]{} 0, \tag{4.16}$$

$$⨍_0^\epsilon (\bar f_1(t), t\Psi(0))_H\,\mathrm{d}t \xrightarrow[\epsilon\to 0^+]{} 0, \tag{4.17}$$

$$⨍_0^\epsilon \langle\langle \bar f_2(t), t\Psi(0))_H\,\mathrm{d}t \xrightarrow[\epsilon\to 0^+]{} 0, \tag{4.18}$$

hence, by (4.13)–(4.18) we obtain $I^m_\epsilon \xrightarrow[\epsilon\to 0^+]{} -\langle \dot u(0),\Psi(0)\rangle$. Finally, thanks to the arbitrariness of $\Psi$ we have the thesis of the Step 1.

*Step 2.* Now we show that $u$ satisfies (4.5) for every $\Psi\in Lip_T$. Let us consider $\Psi\in Lip_T$ and define for every $n\in\mathbb{N}$ a sequence of functions in the following way

$$\theta_n(t) := \begin{cases} 1 & t\in[0,T-\frac{2}{n}] \\ -nt+nT-1 & t\in[T-\frac{2}{n}, T-\frac{1}{n}] \\ 0 & t\in[T-\frac1n, T]. \end{cases}$$

By setting $\Psi_n := \theta_n\Psi$ we have $\Psi_n\in Lip^*_T$ for every $n\in\mathbb{N}$, because $\Psi_n(t)=0$ for every $t\in[T-\frac1n,T]$, then thanks to Step 1 we can say

$$-\int_0^T (\dot u(t),\dot\Psi_n(t))_H\,\mathrm{d}t + \int_0^T ((\mathbb{C}+\mathbb{B})eu(t), e\Psi_n(t))_H\,\mathrm{d}t - \int_0^T\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\Psi_n(t))_H\,\mathrm{d}\tau\mathrm{d}t$$
$$= \int_0^T (N(t),\Psi_n(t))_{H_N}\,\mathrm{d}t + \int_0^T (\bar f_1(t),\Psi_n(t))_H\,\mathrm{d}t + \int_0^T \langle\langle \bar f_2(t),\Psi_n(t)\rangle\rangle\,\mathrm{d}t + \langle \dot u(0),\Psi(0)\rangle. \tag{4.19}$$

Notice that

$$\|\Psi_n-\Psi\|^2_{L^2(0,T;U_T)} = \int_0^T |\theta_n(t)-1|^2\|\Psi(t)\|^2_{U_T}\,\mathrm{d}t \le \|\Psi\|^2_{L^\infty(0,T;U_T)}\int_{T-\frac2n}^T |\theta_n(t)-1|^2\mathrm{d}t$$
$$\le \frac{C}{n} + C\int_{T-\frac2n}^{T-\frac1n} |\theta_n(t)-1|^2\mathrm{d}t = \frac{C}{n} + C\left|\frac{(2-n(T-t))^3}{3n}\right|_{T-\frac2n}^{T-\frac1n} = \frac{4C}{3n} \xrightarrow[n\to\infty]{} 0. \tag{4.20}$$

Moreover, given that for every $t\in[0,T]$ we have

$$0 = \Psi(T) = \Psi(t) + \int_t^T \dot\Psi(s)\mathrm{d}s,$$

for the derivative in time, we can write

$$\begin{aligned}\|\dot\Psi_n-\dot\Psi\|^2_{L^2(0,T;H)}&\le\int_0^T|\theta_n(t)-1|^2\|\dot\Psi(t)\|^2_H\,\mathrm{d}t+\int_0^T|\dot\theta_n(t)|^2\|\Psi(t)\|^2_H\,\mathrm{d}t\\&\le\frac{4C'}{3n}+n^2\int_{T-\frac{2}{n}}^{T-\frac1n}\|\Psi(t)\|^2_H\,\mathrm{d}t\le\frac{4C'}{3n}+n^2\int_{T-\frac{2}{n}}^{T-\frac1n}(T-t)\int_t^T\|\dot\Psi(s)\|^2_H\,\mathrm{d}s\mathrm{d}t\\&\le\frac{4C'}{3n}+2n\int_{T-\frac{2}{n}}^{T-\frac1n}\int_t^T\|\dot\Psi(s)\|^2_H\,\mathrm{d}s\mathrm{d}t\le\frac{4C'}{3n}+2\int_{T-\frac{2}{n}}^T\|\dot\Psi(s)\|^2_H\,\mathrm{d}s\le\frac{16C'}{3n}\xrightarrow[n\to\infty]{}0.\end{aligned}\tag{4.21}$$

By (4.20) and (4.21) we can pass to the limit in (4.19) as $n\to\infty$ to obtain the thesis. □

By following a C. Dafermos' idea found in [5] we can state the following result:

**Proposition 4.8.** *Suppose that there exists $u\in\mathcal{U}^D$ which satisfies the initial condition $u(0)=0$ in the sense of (3.14), and such that for every $\varphi\in\mathcal{E}$ the following identity holds:*

$$\begin{aligned}&\int_0^T(\dot u(t),\dot\varphi(t))_H\,\mathrm{d}t+\int_0^T(t-T)\left[(\dot u(t),\ddot\varphi(t))_H-\left((\mathbb{C}+\mathbb{B})eu(t)-\int_0^t\frac1\beta\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau,e\dot\varphi(t)\right)_H\right]\mathrm{d}t\\&\qquad=T(\tilde u^1,\dot\varphi(0))_H-\int_0^T(t-T)\left[(N(t),\dot\varphi(t))_{H_N}+(\bar f_1(t),\dot\varphi(t))_H+\langle\langle\bar f_2(t),\dot\varphi(t)\rangle\rangle\right]\mathrm{d}t,\end{aligned}\tag{4.22}$$

*where $\tilde u^1:=u^1-\dot z(0)$. Then $u$ is a weak solution in the sense of formulation (3.13), $u(0)=0$ and $\dot u(0)$ coincides with $\tilde u^1$ in $(U_0^D)'$. Moreover, if $u\in\mathcal{U}^D$ is a weak solution according to (3.13) and (3.14), then it satisfies (4.22) for every $\varphi\in\mathcal{E}$.*

*Proof.* Let us consider a function $v\in\mathcal{E}_c$, hence by Lemma 4.2 the function defined by

$$\varphi_v(t)=\int_0^t\frac{v(s)}{s-T}\mathrm{d}s\tag{4.23}$$

is well defined and belongs to the space $\mathcal{E}$. After using it as a test function in (4.22) we obtain

$$\begin{aligned}&-\int_0^T(\dot u(t),\dot\varphi_v(t)+(t-T)\ddot\varphi_v(t))_H\,\mathrm{d}t+\int_0^T\left((\mathbb{C}+\mathbb{B})eu(t)-\int_0^t\frac1\beta\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau,e((t-T)\dot\varphi_v(t))\right)_H\mathrm{d}t\\&\qquad=\int_0^T(N(t),(t-T)\dot\varphi_v(t))_{H_N}\,\mathrm{d}t+\int_0^T(\bar f_1(t),(t-T)\dot\varphi_v(t))_H\,\mathrm{d}t+\int_0^T\langle\langle\bar f_2(t),(t-T)\dot\varphi_v(t)\rangle\rangle\,\mathrm{d}t,\end{aligned}\tag{4.24}$$

since $\dot\varphi_v(0)=\frac{v(0)}{-T}=0$. Notice that $v(t)=(t-T)\dot\varphi_v(t)$ and consequently $\dot v(t)=\dot\varphi_v(t)+(t-T)\ddot\varphi_v(t)$, by the definition of $\varphi_v$ itself. Thanks to this remark and by (4.24), we conclude that $u\in\mathcal{U}^D$, solution to (4.22), satisfies (3.13) for every $v\in\mathcal{E}_c$, and so we get that $u$ is a weak solution, in the sense of relation (3.13), by Lemma 4.3.

Now we will prove that $\tilde u^1$ coincides with $\dot u(0)$. Since the function $u$ satisfies (3.13) for every $v\in\mathcal{U}^D$ such that $v(0)=v(T)=0$, then in particular it verifies the same identity for every $v\in Lip_T^0$. Thanks to Proposition 4.7 it verifies (4.5) for every $v\in Lip_T$ and therefore for every function in the following space

$$\mathcal{E}^*:=\{v\in C^\infty([0,T];U_T):\ \ v(t)=0\quad\forall t\in I(T),\ \ v(t)\in U_t^D\quad\forall t\in(0,T)\}.$$

Moreover, if we define $\varphi_v$ as in (4.23) we have $\varphi_v\in\mathcal{E}$ and we can use it as a test function in (4.22) to deduce

$$\begin{aligned}&-\int_0^T(\dot u(t),\dot v(t))_H\,\mathrm{d}t+\int_0^T((\mathbb{C}+\mathbb{B})eu(t),ev(t))_H\,\mathrm{d}t-\int_0^T\int_0^t\frac1\beta\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau),ev(t))_H\,\mathrm{d}\tau\mathrm{d}t\\&\qquad=\int_0^T(N(t),v(t))_H\,\mathrm{d}t+\int_0^T(\bar f_1(t),v(t))_H\,\mathrm{d}t+\int_0^T\langle\langle\bar f_2(t),v(t)\rangle\rangle\,\mathrm{d}t+(\tilde u^1,v(0))_H.\end{aligned}\tag{4.25}$$

By difference between (4.5) and (4.25) we get $\langle \tilde{u}^1 - \dot{u}(0), v(0)\rangle = 0$ for every $v \in \mathcal{E}^*$. Since for every $p \in U_0^D$ there exists a function $v \in \mathcal{E}^*$ such that $v(0) = p$, we can say that $\langle \tilde{u}^1 - \dot{u}(0), p\rangle = 0$ for every $p \in U_0^D$ and so $\tilde{u}^1 - \dot{u}(0) = 0$ as element of $(U_0^D)'$. This proves the first part of the proposition.

Vice versa, let $u \in \mathcal{U}^D$ be a weak solution in the sense of (3.13) and (3.14). Then $u$ satisfies equality (3.13) for every $v \in \mathcal{U}^D$ such that $v(0) = v(T) = 0$, then for every $v \in Lip_T^0$. Again by Proposition 4.7, $u$ verifies (4.5), with $\tilde{u}^1$ in place of $\dot{u}(0)$, for every function $v \in Lip_T$. Let us consider a function $\varphi \in \mathcal{E}$ then $v_\varphi(t) = (t-T)\dot{\varphi}(t) \in Lip_T$, and so it can be used as a test function in (4.5). By noticing that $\dot{v}_\varphi(t) = \dot{\varphi}(t) + (t-T)\ddot{\varphi}(t)$ and $v_\varphi(0) = -T\dot{\varphi}(0)$ we obtain the thesis. □

Thanks to the previous proposition, we obtain the equivalence between the viscoelastic dynamic problem (3.7)–(3.11) (in the sense already explained) and the identity (4.22). Therefore, to get our weak solution, it will be enough to prove the existence of a solution to Dafermos' equation (4.22). At this level, we use the special variant of Projections Theorem, that is Theorem 4.1, before mentioned. About that, after defining for every $t \in (0,T]$ these functional spaces

$$\mathcal{U}^t := \{u \in L^2(0,t;U_T) \cap H^1(0,t;H) : u(0) = 0,\ u(s) \in U_s^D \text{ for a.e. } s \in (0,t)\},$$
$$\mathcal{E}^t := \{\varphi \in C^\infty([0,t];U_T) : \varphi(0) = 0,\ \varphi(s) \in U_s^D, \quad \forall s \in (0,t)\},$$

we can state the following theorem:

**Theorem 4.9.** *There exists $t_0 \in (0,T]$ and a function $u \in \mathcal{U}^{t_0}$ which verifies the equation (4.22) on the interval $[0,t_0]$ for every $\varphi \in \mathcal{E}^{t_0}$.*

*Proof.* We fix $t_0 \in (0,T]$ such that

$$\begin{cases} t_0 < \frac{1}{2C_{\mathbb{C}}} & \text{if } \frac{1}{2C_{\mathbb{C}}} < T \\ t_0 = T & \text{otherwise.} \end{cases} \tag{4.26}$$

On the space $\mathcal{U}^{t_0}$ we take the usual scalar product, instead on the space $\mathcal{E}^{t_0}$ we consider the following one

$$(\phi,\varphi)_{\mathcal{E}^{t_0}} := \int_0^{t_0} [(\dot{\phi}(t),\dot{\varphi}(t))_H + (\phi(t),\varphi(t))_{U_T}]\mathrm{d}t + t_0(\dot{\phi}(0),\dot{\varphi}(0))_H \qquad \forall \phi,\varphi \in \mathcal{E}^{t_0}.$$

Let us define the bilinear form $B : \mathcal{U}^{t_0} \times \mathcal{E}^{t_0} \to \mathbb{R}$ in the following way

$$B(u,\varphi) := \int_0^{t_0} (\dot{u}(t),\dot{\varphi}(t))_H + (t-t_0)\left[(\dot{u}(t),\ddot{\varphi}(t))_H - \left((\mathbb{C}+\mathbb{B})eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau, e\dot{\varphi}(t)\right)_H\right]\mathrm{d}t,$$

and a linear operator $D : \mathcal{E}^{t_0} \to \mathbb{R}$ represented by

$$D(\varphi) := t_0(\tilde{u}^1,\dot{\varphi}(0))_H + \int_0^{t_0}(t-t_0)\langle\langle \dot{\bar{f}}_2(t),\varphi(t)\rangle\rangle\mathrm{d}t + \int_0^{t_0}\langle\langle \bar{f}_2(t),\varphi(t)\rangle\rangle\mathrm{d}t$$
$$+ \int_0^{t_0}(t-t_0)(\dot{N}(t),\varphi(t))_{H_N}\mathrm{d}t + \int_0^{t_0}(N(t),\varphi(t))_{H_N}\mathrm{d}t - \int_0^{t_0}(t-t_0)(\bar{f}_1(t),\dot{\varphi}(t))_H\,\mathrm{d}t.$$

Notice that, thanks to these definitions, the formulation (4.22) can be rephrased as follows

$$B(u,\varphi) = D(\varphi) \qquad \text{for every } \varphi \in \mathcal{E}^{t_0}.$$

Now we are in the framework of Theorem 4.1, and we want to show that hypothesis (4.1)–(4.3) are satisfied. Foremost, we will prove the existence of a constant $\alpha > 0$ such that

$$B(\varphi,\varphi) \geq \alpha\|\varphi\|^2_{\mathcal{E}^{t_0}} \qquad \text{for every } \varphi \in \mathcal{E}^{t_0}.$$

By definition we have

$$B(\varphi,\varphi) = \int_0^{t_0}\|\dot{\varphi}(t)\|_H^2 + (t-t_0)\Big[(\dot{\varphi}(t),\ddot{\varphi}(t))_H - ((\mathbb{C}+\mathbb{B})e\varphi(t),e\dot{\varphi}(t))_H + \int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}e\varphi(\tau),e\dot{\varphi}(t))_H\,\mathrm{d}\tau\Big]\mathrm{d}t. \tag{4.27}$$

Now we define

$$\psi(t) := \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}e\varphi(\tau)\mathrm{d}\tau \quad \text{and consequently we have} \quad \dot{\psi}(t) = \frac{1}{\beta}e\varphi(t) - \int_0^t\frac{1}{\beta^2}\mathrm{e}^{-\frac{t-\tau}{\beta}}e\varphi(\tau)\mathrm{d}\tau,$$

then (4.27) can be reworded as

$$B(\varphi,\varphi)=\int_0^{t_0}\|\dot\varphi(t)\|_H^2+(t-t_0)[(\dot\varphi(t),\ddot\varphi(t))_H-((\mathbb{C}+\mathbb{B})e\varphi(t),e\dot\varphi(t))_H+(\mathbb{B}\psi(t),e\dot\varphi(t))]\mathrm{d}t. \tag{4.28}$$

Thanks to the chain rule and to the symmetry property (3.5), we can write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\varphi(t)\|_H^2=(\dot\varphi(t),\ddot\varphi(t))_H,\qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((\mathbb{C}+\mathbb{B})e\varphi(t),e\varphi(t))_H=((\mathbb{C}+\mathbb{B})e\varphi(t),e\dot\varphi(t))_H,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}\psi(t),e\varphi(t))_H=(\mathbb{B}\dot\psi(t),e\varphi(t))_H+(\mathbb{B}\psi(t),e\dot\varphi(t))_H,$$

therefore, if we substitute these informations in the identity (4.28), we get

$$\begin{aligned}B(\varphi,\varphi)&=\int_0^{t_0}\|\dot\varphi(t)\|_H^2\,\mathrm{d}t+\frac{1}{2}\int_0^{t_0}(t-t_0)\frac{\mathrm{d}}{\mathrm{d}t}\|\dot\varphi(t)\|_H^2\,\mathrm{d}t-\frac{1}{2}\int_0^{t_0}(t-t_0)\frac{\mathrm{d}}{\mathrm{d}t}((\mathbb{C}+\mathbb{B})e\varphi(t),e\varphi(t))_H\,\mathrm{d}t\\
&\qquad+\int_0^{t_0}(t-t_0)\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{B}\psi(t),e\varphi(t))_H\mathrm{d}t-\int_0^{t_0}(t-t_0)(\mathbb{B}\dot\psi(t),e\varphi(t))_H\,\mathrm{d}t\\
&=\frac{t_0}{2}\|\dot\varphi(0)\|_H^2+\frac{1}{2}\int_0^{t_0}\|\dot\varphi(t)\|_H^2\,\mathrm{d}t+\frac{1}{2}\int_0^{t_0}((\mathbb{C}+\mathbb{B})e\varphi(t),e\varphi(t))_H\,\mathrm{d}t\\
&\qquad-\int_0^{t_0}(\mathbb{B}\psi(t),e\varphi(t))_H\,\mathrm{d}t-\int_0^{t_0}(t-t_0)(\mathbb{B}\dot\psi(t),e\varphi(t))_H\,\mathrm{d}t\\
&=\frac{t_0}{2}\|\dot\varphi(0)\|_H^2+\frac{1}{2}\int_0^{t_0}\|\dot\varphi(t)\|_H^2\,\mathrm{d}t+\frac{1}{2}\int_0^{t_0}((\mathbb{C}+\mathbb{B})e\varphi(t),e\varphi(t))_H\,\mathrm{d}t\\
&\qquad-\int_0^{t_0}(t-t_0)(\beta\mathbb{B}\dot\psi(t),\dot\psi(t))_H\,\mathrm{d}t-\int_0^{t_0}(t-t_0)(\mathbb{B}\dot\psi(t),\psi(t))-\int_0^{t_0}(\mathbb{B}\psi(t),e\varphi(t))_H\,\mathrm{d}t\\
&=\frac{t_0}{2}\|\dot\varphi(0)\|_H^2+\frac{1}{2}\int_0^{t_0}\|\dot\varphi(t)\|_H^2\,\mathrm{d}t+\frac{1}{2}\int_0^{t_0}(\mathbb{C}e\varphi(t),e\varphi(t))_H\,\mathrm{d}t\\
&\qquad+\frac{1}{2}\int_0^{t_0}(\mathbb{B}(e\varphi(t)-\psi(t)),e\varphi(t)-\psi(t))_H\,\mathrm{d}t+\int_0^{t_0}(t_0-t)(\beta\mathbb{B}\dot\psi(t),\dot\psi(t))_H\,\mathrm{d}t.\end{aligned} \tag{4.29}$$

Thanks to the coerciveness in (3.6) and to the definition of the $U_T$–norm, we have

$$(\mathbb{C}e\varphi(t),e\varphi(t))_H\ge C_{\mathbb{C}}\|\varphi(t)\|_{U_T}^2-C_{\mathbb{C}}\|\varphi(t)\|_H^2\qquad\forall t\in(0,T). \tag{4.30}$$

Moreover, since

$$\varphi(t)=\varphi(0)+\int_0^t\dot\varphi(s)\mathrm{d}s=\int_0^t\dot\varphi(s)\mathrm{d}s,$$

inequality (4.30) implies

$$\frac{1}{2}\int_0^{t_0}(\mathbb{C}e\varphi(t),e\varphi(t))_H\,\mathrm{d}t\ge\frac{C_{\mathbb{C}}}{2}\int_0^{t_0}\|\varphi(t)\|_{U_T}^2\,\mathrm{d}t-\frac{C_{\mathbb{C}}t_0}{2}\int_0^{t_0}\|\dot\varphi(t)\|_H^2\,\mathrm{d}t,$$

therefore, by (4.29) and thanks to the choice done in (4.26), we can deduce

$$B(\varphi,\varphi)\ge\frac{t_0}{2}\|\dot\varphi(0)\|_H^2+\frac{1-C_{\mathbb{C}}t_0}{2}\int_0^{t_0}\|\dot\varphi(t)\|_H^2\,\mathrm{d}t+\frac{C_{\mathbb{C}}}{2}\int_0^{t_0}\|\varphi(t)\|_{U_T}^2\,\mathrm{d}t\ge\min\left(\frac{1}{4},\frac{C_{\mathbb{C}}}{2},\frac{t_0}{2}\right)\|\varphi\|_{\mathcal{E}^{t_0}}^2,$$

that is the condition (4.3).

In the second step, we show that $B(\cdot,\varphi)$ is a continuous operator on $\mathcal{U}^{t_0}$ for every $\varphi\in\mathcal{E}^{t_0}$, and that $D:\mathcal{E}^{t_0}\to\mathbb{R}$ has same property on the space $\mathcal{E}^{t_0}$. To this end, let us consider $\{u_k\}_k\subseteq\mathcal{U}^{t_0}$ such that

$$u_k\xrightarrow[k\to\infty]{\mathcal{U}^{t_0}}u.$$

Accordingly

$$U_k:=u_k-u\xrightarrow[k\to\infty]{L^2(0,t_0;U_T)}0\quad\text{and}\quad\dot U_k:=\dot u_k-\dot u\xrightarrow[k\to\infty]{L^2(0,t_0;H)}0.$$

Then, by using Cauchy-Schwarz's inequality we get

$$\begin{aligned}|B(U_k,\varphi)| &\le \int_0^{t_0} |(\dot U_k,\dot\varphi)_H|\,\mathrm{d}t + t_0\int_0^{t_0} |(\dot U_k,\ddot\varphi)_H|\,\mathrm{d}t + t_0\int_0^{t_0} |((\mathbb{C}+\mathbb{B})eU_k, e\dot\varphi)_H|\,\mathrm{d}t \\ &\qquad + t_0\int_0^{t_0}\left|\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eU_k(\tau), e\dot\varphi(t))_H\,\mathrm{d}\tau\right|\mathrm{d}t \\ &\le \|\dot U_k\|_{L^2(0,t_0;H)}\|\dot\varphi\|_{L^2(0,t_0;H)} + t_0\|\dot U_k\|_{L^2(0,t_0;H)}\|\ddot\varphi\|_{L^2(0,t_0;H)} \\ &\qquad + t_0\|\mathbb{C}+\mathbb{B}\|_\infty\|U_k\|_{L^2(0,t_0;U_T)}\|\dot\varphi\|_{L^2(0,t_0;U_T)} + \frac{t_0}{\beta}\|\mathbb{B}\|_\infty\int_0^{t_0}\left|\left(\int_0^t eU_k(\tau)\mathrm{d}\tau, e\dot\varphi(t)\right)_H\right|\mathrm{d}t.\end{aligned} \tag{4.31}$$

Notice that

$$\begin{aligned}\int_0^{t_0}\left|\left(\int_0^t eU_k(\tau)\mathrm{d}\tau, e\dot\varphi(t)\right)_H\right|\mathrm{d}t &\le \|e\dot\varphi\|_{L^2(0,t_0;H)}\left\|\int_0^t eU_k(\tau)\mathrm{d}\tau\right\|_{L^2(0,t_0;H)} \\ &\le \|e\dot\varphi\|_{L^2(0,t_0;H)}\left(\int_0^{t_0}\left(\int_0^t \|eU_k(\tau)\|_H\mathrm{d}\tau\right)^2\mathrm{d}t\right)^{\frac12} \\ &\le \|e\dot\varphi\|_{L^2(0,t_0;H)}\left(\int_0^{t_0}\left(\int_0^{t_0} \|eU_k(\tau)\|_H\mathrm{d}\tau\right)^2\mathrm{d}t\right)^{\frac12} \\ &\le \|e\dot\varphi\|_{L^2(0,t_0;H)}\left(\int_0^{t_0} t_0\int_0^{t_0} \|eU_k(\tau)\|_H^2\mathrm{d}\tau\mathrm{d}t\right)^{\frac12} \\ &= \|e\dot\varphi\|_{L^2(0,t_0;H)}\left(t_0^2\|eU_k\|^2_{L^2(0,t_0;H)}\right)^{\frac12} \le t_0\|e\dot\varphi\|_{L^2(0,t_0;H)}\|U_k\|_{L^2(0,t_0;U_T)},\end{aligned}$$

and thanks to this, by considering (4.31), we have

$$|B(U_k,\varphi)| \le C_1\|\dot U_k\|_{L^2(0,t_0;H)} + C_2\|U_k\|_{L^2(0,t_0,U_T)} \xrightarrow[k\to\infty]{} 0.$$

Now it remains to show that $D$ is a continuous operator on $\mathcal{E}^{t_0}$, and since it is linear we will show its boundedness. Let $\varphi\in\mathcal{E}^{t_0}$, then

$$\begin{aligned}|D(\varphi)| \le &\left|\int_0^{t_0}\left[(t-t_0)(\bar f_1(t),\dot\varphi(t))_H - (t-t_0)\langle\langle\dot{\bar f}_2(t),\varphi(t)\rangle\rangle - \langle\langle\bar f_2(t),\varphi(t)\rangle\rangle\right]\mathrm{d}t\right| \\ &+\left|\int_0^{t_0}(t-t_0)(\dot N(t),\varphi(t))_{H_N}\,\mathrm{d}t + \int_0^{t_0}(N(t),\varphi(t))_{H_N}\,\mathrm{d}t\right| + t_0\|\tilde u^1\|_H\|\dot\varphi(0)\|_H.\end{aligned}$$

In particular we have

$$\begin{aligned}&\int_0^{t_0}\left|(t-t_0)(\bar f_1(t),\dot\varphi(t))_H - \langle\langle\bar f_2(t),\varphi(t)\rangle\rangle - (t-t_0)\langle\langle\dot{\bar f}_2(t),\varphi(t)\rangle\rangle\right|\mathrm{d}t \\ &\le t_0\left(\int_0^{t_0}\|\bar f_1(t)\|_H^2\mathrm{d}t\right)^{\frac12}\left(\int_0^{t_0}\|\dot\varphi(t)\|_H^2\,\mathrm{d}t\right)^{\frac12} + \left(\int_0^{t_0}\|(t-t_0)\dot{\bar f}_2(t)+\bar f_2(t)\|^2_{U_T'}\,\mathrm{d}t\right)^{\frac12}\left(\int_0^{t_0}\|\varphi(t)\|^2_{U_T}\,\mathrm{d}t\right)^{\frac12} \\ &\le t_0\|\bar f_1\|_{L^2(0,T;H)}\|\varphi\|_{\mathcal{E}^{t_0}} + 2\sqrt2\max(t_0,1)\|\bar f_2\|_{H^1(0,T;U_T')}\|\varphi\|_{\mathcal{E}^{t_0}} \le C\|\varphi\|_{\mathcal{E}^{t_0}},\end{aligned}$$

and thanks to trace inequality (3.3) we deduce

$$\begin{aligned}&\left|\int_0^{t_0}(t-t_0)(\dot N(t),\varphi(t))_{H_N}\,\mathrm{d}t + \int_0^{t_0}(N(t),\varphi(t))_{H_N}\,\mathrm{d}t\right| \\ &\quad\le\left(\int_0^{t_0}\|(t-t_0)\dot N(t)+N(t)\|^2_{H_N}\,\mathrm{d}t\right)^{\frac12}\left(\int_0^{t_0}\|\varphi(t)\|^2_{H_N}\mathrm{d}t\right)^{\frac12} \le 2\sqrt2 C_{tr}\max(t_0,1)\|N\|_{H^1(0,T;H_N)}\|\varphi\|_{\mathcal{E}^{t_0}}\end{aligned}$$

Moreover we can write

$$t_0\|\tilde{u}^1\|_H\|\dot{\varphi}(0)\|_H \le t_0\|\tilde{u}^1\|_H t_0^{-\frac{1}{2}}\|\varphi\|_{\mathcal{E}^{t_0}} = t_0^{\frac{1}{2}}\|\tilde{u}^1\|_H\|\varphi\|_{\mathcal{E}^{t_0}}.$$

By applying Theorem 4.1 we have the existence of a solution to (4.22) on the interval $[0,t_0]$. □

This proves the existence of a weak solution in the sense of relations (3.13) and (3.14) on the interval $[0,t_0]$. Actually, thanks to Remark 3.5, we can find a weak solution, on the interval $[0,t_0]$, to the viscoelastic dynamic system (3.7)–(3.11) in the sense of Definition 3.3 and Definition 3.4.

Now we want to show that is possible to find a weak solution on the whole interval $[0,T]$. To this purpose, thanks to Proposition 4.8 and Theorem 4.9, we can find a weak solution $u_1$, on the interval $[0,t_0]$, to the viscoelastic dynamic system (3.7) with boundary and initial conditions (3.8)–(3.11). Let us consider $\hat{t}_0 \in [\frac{3}{4}t_0, t_0]$ such that $u_1(\hat{t}_0) \in U_T$ and $\hat{t}_0$ is a Lebesgue's point for $\dot{u}_1$, that is

$$\lim_{\epsilon\to 0^+} \fint_{\hat{t}_0-\epsilon}^{\hat{t}_0} \|\dot{u}_1(t) - \dot{u}_1(\hat{t}_0)\|_H = 0. \tag{4.32}$$

By repeating the same argument, we can also find a weak solution $u_2$, on the interval $[\hat{t}_0, 2\hat{t}_0]$, to (3.7) with boundary conditions as before, and the initial conditions given by

$$\lim_{t\to\hat{t}_0^+} \|u_2(t) - u_1(\hat{t}_0)\|_H = 0, \qquad \lim_{t\to\hat{t}_0^+} \|\dot{u}_2(t) - \dot{u}_1(\hat{t}_0)\|_{(U_0^D)'} = 0.$$

For convenience of notation, we set for every $a,b \in \mathbb{R}$ the following space

$$\mathcal{U}(a,b) := \{v \in L^2(a,b;U_T) \cap H^1(a,b;H) : \ v(t) \in U_t \ \text{ for a.e. } t \in (a,b)\},$$
$$\mathcal{U}^D(a,b) := \{v \in \mathcal{U}(a,b) : \ v(t) \in U_t^D \ \text{ for a.e. } t \in (a,b)\}.$$

**Lemma 4.10.** *Let $u \in \mathcal{U}(0,\hat{t}_0)$ be a weak solution to the viscoelastic dynamic system (3.7)–(3.10) on the interval $[0,\hat{t}_0]$, then for every $\theta \in \mathcal{U}^D(0,\hat{t}_0)$ such that $\theta(0)=0$, the following identity holds*

$$(\dot{u}(\hat{t}_0),\theta(\hat{t}_0))_H - \int_0^{\hat{t}_0} (\dot{u}(t),\dot{\theta}(t))_H \mathrm{d}t + \int_0^{\hat{t}_0} \left( (\mathbb{C}+\mathbb{B})eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau, e\theta(t) \right)_H \mathrm{d}t$$
$$= \int_0^{\hat{t}_0} (N(t),\theta(t))_{H_N}\mathrm{d}t + \int_0^{\hat{t}_0} (f_1(t),\theta(t))_H\,\mathrm{d}t + \int_0^{\hat{t}_0} \langle\langle f_2(t),\theta(t)\rangle\rangle \mathrm{d}t. \tag{4.33}$$

*Moreover, by considering $u \in \mathcal{U}(\hat{t}_0, 2\hat{t}_0)$ weak solution to the viscoelastic dynamic system (3.7)–(3.10) on the interval $[\hat{t}_0, 2\hat{t}_0]$, then for every $\Theta \in \mathcal{U}^D(\hat{t}_0, 2\hat{t}_0)$ such that $\Theta(2\hat{t}_0)=0$, the following identity holds*

$$-(\dot{u}(\hat{t}_0),\Theta(\hat{t}_0))_H - \int_{\hat{t}_0}^{2\hat{t}_0} (\dot{u}(t),\dot{\Theta}(t))_H \mathrm{d}t + \int_{\hat{t}_0}^{2\hat{t}_0} \left( (\mathbb{C}+\mathbb{B})eu(t) - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau, e\Theta(t) \right)_H \mathrm{d}t$$
$$= \int_{\hat{t}_0}^{2\hat{t}_0} (N(t),\Theta(t))_{H_N}\mathrm{d}t + \int_{\hat{t}_0}^{2\hat{t}_0} (f_1(t),\Theta(t))_H\,\mathrm{d}t + \int_{\hat{t}_0}^{2\hat{t}_0} \langle\langle f_2(t),\Theta(t)\rangle\rangle \mathrm{d}t. \tag{4.34}$$

*Proof.* To prove identity (4.33), let us consider $\theta \in \mathcal{U}^D(0,\hat{t}_0)$ such that $\theta(0)=0$ and define for $\epsilon \in (0,\hat{t}_0)$ the function

$$\theta_\epsilon(t) = \begin{cases} \theta(t) & t \in [0,\hat{t}_0-\epsilon] \\ \frac{\hat{t}_0-t}{\epsilon}\theta(t) & t \in [\hat{t}_0-\epsilon,\hat{t}_0]. \end{cases}$$

Since $\theta_\epsilon \in \mathcal{U}^D(0,\hat{t}_0)$, and also $\theta_\epsilon(0)=\theta(0)=0$ and $\theta_\epsilon(\hat{t}_0) = \frac{\hat{t}_0-\hat{t}_0}{\epsilon}\theta(\hat{t}_0)=0$, we can use it as a test function in (3.13) to obtain $I_\epsilon + J_\epsilon = K_\epsilon$, where we define

$$I_\epsilon := -\int_0^{\hat{t}_0-\epsilon} (\dot{u}(t),\dot{\theta}(t))_H\,\mathrm{d}t + \fint_{\hat{t}_0-\epsilon}^{\hat{t}_0} (\dot{u}(t),\theta(t))_H \mathrm{d}t$$
$$+ \int_0^{\hat{t}_0-\epsilon} ((\mathbb{C}+\mathbb{B})eu(t), e\theta(t))_H\,\mathrm{d}t - \int_0^{\hat{t}_0-\epsilon}\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-\tau}{\beta}}(\mathbb{B}eu(\tau), e\theta(t))_H\,\mathrm{d}\tau\mathrm{d}t,$$

$$J_\epsilon := -\int_{\hat{t}_0-\epsilon}^{\hat{t}_0} (\hat{t}_0 - t)(\dot{u}(t), \dot{\theta}(t))_H \mathrm{d}t + \int_{\hat{t}_0-\epsilon}^{\hat{t}_0} (\hat{t}_0 - t) \left( (\mathbb{C} + \mathbb{B})eu(t) - \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} \mathbb{B}eu(\tau)\mathrm{d}\tau, e\theta(t) \right)_H \mathrm{d}t,$$

$$K_\epsilon := \int_0^{\hat{t}_0-\epsilon} (N(t), \theta(t))_{H_N}\, \mathrm{d}t + \int_0^{\hat{t}_0-\epsilon} (f_1(t), \theta(t))_H\, \mathrm{d}t + \int_0^{\hat{t}_0-\epsilon} \langle\langle f_2(t), \theta(t)\rangle\rangle \mathrm{d}t$$
$$+ \int_{\hat{t}_0-\epsilon}^{\hat{t}_0} (\hat{t}_0 - t)(N(t), \theta(t))_{H_N}\, \mathrm{d}t + \int_{\hat{t}_0-\epsilon}^{\hat{t}_0} (\hat{t}_0 - t)(f_1(t), \theta(t))_H\, \mathrm{d}t + \int_{\hat{t}_0-\epsilon}^{\hat{t}_0} (\hat{t}_0 - t)\langle\langle f_2(t), \theta(t)\rangle\rangle \mathrm{d}t.$$

Thanks to the absolute continuity of the integral and to (4.32), we can say

$$I_\epsilon \xrightarrow[\epsilon\to 0^+]{} -\int_0^{\hat{t}_0} (\dot{u}(t), \dot{\theta}(t))_H\, \mathrm{d}t + \int_0^{\hat{t}_0} \left( (\mathbb{C} + \mathbb{B})eu(t) - \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} \mathbb{B}eu(\tau)\mathrm{d}\tau, e\theta(t) \right)_H \mathrm{d}t + (\dot{u}(\hat{t}_0), \theta(\hat{t}_0))_H,$$

$$J_\epsilon \xrightarrow[\epsilon\to 0^+]{} 0, \qquad K_\epsilon \xrightarrow[\epsilon\to 0^+]{} \int_0^{\hat{t}_0} (N(t), \theta(t))_{H_N}\, \mathrm{d}t + \int_0^{\hat{t}_0} (f_1(t), \theta(t))_H\, \mathrm{d}t + \int_0^{\hat{t}_0} \langle\langle f_2(t), \theta(t)\rangle\rangle \mathrm{d}t,$$

which conclude the proof of (4.33).

To prove (4.34), it is enough to consider, for every $\Theta \in \mathcal{U}^D(\hat{t}_0, 2\hat{t}_0)$ such that $\Theta(2\hat{t}_0) = 0$, the function

$$\Theta_\epsilon(t) = \begin{cases} \frac{t-\hat{t}_0}{\epsilon}\Theta(t) & t \in [\hat{t}_0, \hat{t}_0 + \epsilon] \\ \Theta(t) & t \in [\hat{t}_0 + \epsilon, 2\hat{t}_0] \end{cases}$$

and to repeat similar argument before performed. □

**Theorem 4.11.** *Let $u_1 \in \mathcal{U}(0, \hat{t}_0)$ be a weak solution to the viscoelastic dynamic system (3.7)–(3.11) on the interval $[0, \hat{t}_0]$. If $u_2 \in \mathcal{U}(\hat{t}_0, 2\hat{t}_0)$ is a weak solution on the interval $[\hat{t}_0, 2\hat{t}_0]$, which satisfies the initial conditions in the sense of continuity, that is*

$$\lim_{t\to\hat{t}_0^+} \|u_2(t) - u_1(\hat{t}_0)\|_H = 0, \qquad \lim_{t\to\hat{t}_0^+} \|\dot{u}_2(t) - \dot{u}_1(\hat{t}_0)\|_{(U_0^D)'} = 0,$$

*then there exists a weak solution $u \in \mathcal{U}(0, 2\hat{t}_0)$ to the viscoelastic dynamic system (3.7)–(3.11) on the whole interval $[0, 2\hat{t}_0]$.*

*Proof.* Notice that the initial data $u_1(\hat{t}_0)$ and $\dot{u}_1(\hat{t}_0)$ are well defined because we have $u_1 \in C^0([0, \hat{t}_0]; H)$ and $\dot{u}_1 \in C^0([0, \hat{t}_0]; (U_0^D)')$. Since $u_1$ is a weak solution to (3.7)–(3.11) on the interval $[0, \hat{t}_0]$, then for every $v \in \mathcal{U}^D(0, \hat{t}_0)$ such that $v(0) = v(\hat{t}_0) = 0$ we have

$$-\int_0^{\hat{t}_0} (\dot{u}_1(t), \dot{v}(t))_H\, \mathrm{d}t + \int_0^{\hat{t}_0} ((\mathbb{C} + \mathbb{B})eu_1(t), ev(t))_H\, \mathrm{d}t - \int_0^{\hat{t}_0}\int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu_1(\tau), ev(t))_H\, \mathrm{d}\tau\mathrm{d}t$$
$$= \int_0^{\hat{t}_0} (N(t), v(t))_{H_N}\, \mathrm{d}t + \int_0^{\hat{t}_0} (f_1(t), v(t))_H\, \mathrm{d}t + \int_0^{\hat{t}_0} \langle\langle f_2(t), v(t)\rangle\rangle \mathrm{d}t.$$

In a completely similar way, given that $u_2$ is a weak solution on interval $[\hat{t}_0, 2\hat{t}_0]$, for every $v \in \mathcal{U}^D(\hat{t}_0, 2\hat{t}_0)$ such that $v(\hat{t}_0) = v(2\hat{t}_0) = 0$ we have

$$-\int_{\hat{t}_0}^{2\hat{t}_0} (\dot{u}_2(t), \dot{v}(t))_H\, \mathrm{d}t + \int_{\hat{t}_0}^{2\hat{t}_0} ((\mathbb{C} + \mathbb{B})eu_2(t), ev(t))_H\, \mathrm{d}t - \int_{\hat{t}_0}^{2\hat{t}_0}\int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu_2(\tau), ev(t))_H\, \mathrm{d}\tau\mathrm{d}t$$
$$= \int_{\hat{t}_0}^{2\hat{t}_0} (N(t), v(t))_{H_N}\, \mathrm{d}t + \int_{\hat{t}_0}^{2\hat{t}_0} (f_1(t), v(t))_H\, \mathrm{d}t + \int_{\hat{t}_0}^{2\hat{t}_0} \langle\langle f_2(t), v(t)\rangle\rangle \mathrm{d}t.$$

Now we define

$$u(t) := \begin{cases} u_1(t) & t \in [0, \hat{t}_0] \\ u_2(t) & t \in [\hat{t}_0, 2\hat{t}_0]. \end{cases}$$

Let us fix $v \in \mathcal{U}^D(0, 2\hat{t}_0)$ such that $v(0) = v(2\hat{t}_0) = 0$, then $v \in \mathcal{U}^D(0, \hat{t}_0)$ and $v(0) = 0$, therefore thanks to (4.33) we get

$$(\dot{u}_1(\hat{t}_0), v(\hat{t}_0))_H - \int_0^{\hat{t}_0} (\dot{u}_1(t), \dot{v}(t))_H \mathrm{d}t + \int_0^{\hat{t}_0} \left( (\mathbb{C} + \mathbb{B})eu_1(t) - \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} \mathbb{B}eu_1(\tau)\mathrm{d}\tau, ev(t) \right)_H \mathrm{d}t$$

$$= \int_0^{\hat{t}_0} (N(t), v(t))_{H_N}\,\mathrm{d}t + \int_0^{\hat{t}_0} (f_1(t), v(t))_H\,\mathrm{d}t + \int_0^{\hat{t}_0} \langle\langle f_2(t), v(t)\rangle\rangle \mathrm{d}t. \tag{4.35}$$

In the same way, since $v \in \mathcal{U}^D(\hat{t}_0, 2\hat{t}_0)$ and $v(2\hat{t}_0) = 0$, by (4.34) we obtain

$$-(\dot{u}_2(\hat{t}_0), v(\hat{t}_0))_H - \int_{\hat{t}_0}^{2\hat{t}_0} (\dot{u}_2(t), \dot{v}(t))_H \mathrm{d}t + \int_{\hat{t}_0}^{2\hat{t}_0} \left( (\mathbb{C}+\mathbb{B})eu_2(t) - \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} \mathbb{B}eu_2(\tau)\mathrm{d}\tau, ev(t) \right)_H \mathrm{d}t$$
$$= \int_{\hat{t}_0}^{2\hat{t}_0} (N(t), v(t))_{H_N}\,\mathrm{d}t + \int_{\hat{t}_0}^{2\hat{t}_0} (f_1(t), v(t))_H\,\mathrm{d}t + \int_{\hat{t}_0}^{2\hat{t}_0} \langle\langle f_2(t), v(t)\rangle\rangle \mathrm{d}t. \tag{4.36}$$

By summing (4.35) and (4.36) we can write the following identity

$$- \int_0^{2\hat{t}_0} (\dot{u}(t), \dot{v}(t))_H\,\mathrm{d}t + \int_0^{2\hat{t}_0} ((\mathbb{C}+\mathbb{B})eu(t), ev(t))_H\,\mathrm{d}t - \int_0^{2\hat{t}_0} \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau), ev(t))_H\,\mathrm{d}\tau\mathrm{d}t$$
$$= \int_0^{2\hat{t}_0} (N(t), v(t))_{H_N}\,\mathrm{d}t + \int_0^{2\hat{t}_0} (f_1(t), v(t))_H\,\mathrm{d}t + \int_0^{2\hat{t}_0} \langle\langle f_2(t), v(t)\rangle\rangle \mathrm{d}t.$$

By the arbitrariness of $v$ we have that $u$ satisfies (3.13) and this concludes the proof. □

This proves the existence of a weak solution on the interval $[0, 2\hat{t}_0] \supseteq [0, t_0]$. By iterating a finite number of times the same arguments, we can find a weak solution on the whole interval $[0, T]$.

4.2. **Equivalent system to the equation.** As already anticipated, in this subsection, we present a new system of two vector-valued equations, which is equivalent to our viscoelastic dynamic system (3.7) in a precise sense. In particular, as we have already done for the viscoelastic dynamic system, we will give the definition of weak solution to the vector-valued system, and we will prove its equivalence to Definition 3.3. To this purpose, let us consider the vector-valued system

$$\begin{cases} \ddot{u}(t) - \operatorname{div}(\mathbb{C}eu(t)) - \operatorname{div}(\mathbb{B}(eu(t) - w(t))) = g(t) & \text{in } \Omega \setminus \Gamma_t, \quad t \in (0,T), \\ \beta\dot{w}(t) + w(t) - eu(t) = 0 \end{cases} \tag{4.37}$$

with the following boundary and initial conditions

$$u(t) = z(t) \qquad \text{on } \partial_D\Omega, \quad t \in (0,T), \tag{4.38}$$
$$[(\mathbb{C}+\mathbb{B})eu(t) - \mathbb{B}w(t)]\nu = -\mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}w^0\nu + N(t) \qquad \text{on } \partial_N\Omega, \quad t \in (0,T), \tag{4.39}$$
$$[(\mathbb{C}+\mathbb{B})eu(t) - \mathbb{B}w(t)]\nu = -\mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}w^0\nu \qquad \text{on } \Gamma_t, \quad t \in (0,T), \tag{4.40}$$
$$u(0) = u^0, \quad w(0) = w^0, \quad \dot{u}(0) = u^1, \tag{4.41}$$

where $w^0 \in H$ and by definition we set $g(t) := f_1(t) + f_2(t) + \mathrm{e}^{-\frac{t}{\beta}} \operatorname{div}(\mathbb{B}w^0)$ for a.e. $t \in (0,T)$. As we have already specified for the viscoelastic dynamic system (3.7)–(3.11), also in this case, the Neumann boundary conditions are only formal, and their meaning will be specified in Definition 4.12.

Now we define in which sense a couple of functions $(u, w)$ is to be understood as a solution of the previous system (4.37)–(4.41). To this aim, for convenience of notation, we pose $\mathcal{AC} := AC([0,T]; H)$.

**Definition 4.12** (Weak solution)**.** We say that $(u, w) \in \mathcal{U} \times \mathcal{AC}$ is a *weak solution* to system (4.37), with boundary conditions (4.38)–(4.40), if the following conditions hold

- the function $u - z \in \mathcal{U}^D$ and for every $\varphi \in \mathcal{U}^D$ such that $\varphi(0) = \varphi(T) = 0$ it is verified

$$- \int_0^T (\dot{u}(t), \dot{\varphi}(t))_H\,\mathrm{d}t + \int_0^T (\mathbb{C}eu(t), e\varphi(t))_H\,\mathrm{d}t + \int_0^T (\mathbb{B}(eu(t) - w(t)), e\varphi(t))_H\,\mathrm{d}t$$
$$= \int_0^T (N(t), \varphi(t))_{H_N}\,\mathrm{d}t + \int_0^T (f_1(t), \varphi(t))_H\,\mathrm{d}t + \int_0^T \langle\langle f_2(t), \varphi(t)\rangle\rangle \mathrm{d}t - \int_0^T \mathrm{e}^{-\frac{t}{\beta}} (\mathbb{B}w^0, e\varphi(t))_H\,\mathrm{d}t \tag{4.42}$$

- for a.e. $t \in (0,T)$ we have the validity of

$$\begin{cases} \beta \dot{w}(t) + w(t) = eu(t) \\ w(0) = w^0 \end{cases} \tag{4.43}$$

where the equality is to be understood in the sense of the Hilbert space $H$.

**Definition 4.13** (Initial conditions)**.** We say that $(u,w) \in \mathcal{U} \times \mathcal{AC}$ weak solution to (4.37) satisfies the initial conditions (4.41) if

$$\lim_{t\to 0^+} \|u(t) - u^0\|_H = 0, \quad \lim_{t\to 0^+} \|w(t) - w^0\|_H = 0, \quad \lim_{t\to 0^+} \|\dot{u}(t) - u^1\|_{(U_0^D)'} = 0. \tag{4.44}$$

Let us now prove that the new problem is equivalent to the first one:

**Theorem 4.14.** *The viscoelastic dynamic system* (3.7) *with boundary and initial conditions* (3.8)–(3.11) *is equivalent to the vector-valued system* (4.37) *with boundary and initial conditions* (4.38)–(4.41)*.*

*Proof.* Let us consider $(u,w) \in \mathcal{U} \times \mathcal{AC}$ solution to the system (4.37)–(4.41), then by the theory of ordinary differential equations valued in Hilbert spaces, the second equation gives us for a.e. $t \in (0,T)$

$$w(t) = w^0 \mathrm{e}^{-\frac{t}{\beta}} + \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} eu(\tau)\mathrm{d}\tau. \tag{4.45}$$

Moreover, $u - z \in \mathcal{U}^D$ and identity (4.42) holds for every $\varphi \in \mathcal{U}^D$ such that $\varphi(0) = \varphi(T) = 0$. Now, if we substitute (4.45) in (4.42) we obtain

$$\begin{aligned} -\int_0^T (\dot{u}(t), \dot{\varphi}(t))_H \,\mathrm{d}t + \int_0^T \left( (\mathbb{C} + \mathbb{B})eu(t) - \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} \mathbb{B}eu(\tau)\mathrm{d}\tau, e\varphi(t) \right)_H \mathrm{d}t - \int_0^T \mathrm{e}^{-\frac{t}{\beta}} (\mathbb{B}w^0, e\varphi(t))_H \,\mathrm{d}t \\ = \int_0^T (N(t), \varphi(t))_{H_N} \,\mathrm{d}t + \int_0^T (f(t), \varphi(t))_H \,\mathrm{d}t + \int_0^T \langle\langle f_2(t), \varphi(t) \rangle\rangle \mathrm{d}t - \int_0^T \mathrm{e}^{-\frac{t}{\beta}} (\mathbb{B}w^0, e\varphi(t))_H \,\mathrm{d}t, \end{aligned}$$

from which we deduce that $u$ satisfies (3.7)–(3.11) in the weak sense of Definition 3.3 and 3.4.

On the contrary, if we consider a solution $u \in \mathcal{U}$ to (3.7)–(3.11) then $u - z \in \mathcal{U}^{\mathcal{D}}$ and we have the validity of the following equality

$$\begin{aligned} -\int_0^T (\dot{u}(t), \dot{\varphi}(t))_H \,\mathrm{d}t + \int_0^T ((\mathbb{C} + \mathbb{B})eu(t), e\varphi(t))_H \,\mathrm{d}t - \int_0^T \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\mathbb{B}eu(\tau), e\varphi(t))_H \,\mathrm{d}\tau\mathrm{d}t \\ = \int_0^T (N(t), \varphi(t))_{H_N} \,\mathrm{d}t + \int_0^T (f_1(t), \varphi(t))_H \,\mathrm{d}t + \int_0^T \langle\langle f_2(t), \varphi(t) \rangle\rangle \mathrm{d}t, \end{aligned} \tag{4.46}$$

for every $\varphi \in \mathcal{U}^{\mathcal{D}}$ such that $\varphi(0) = \varphi(T) = 0$. If we choose $w^0 \in H$ and we define for a.e. $t \in (0,T)$ the function $w$ in the following way

$$w(t) := w^0 \mathrm{e}^{-\frac{t}{\beta}} + \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} eu(\tau)\mathrm{d}\tau, \tag{4.47}$$

then $w$ satisfies by definition the second equation of the system (4.37) in the sense of Definition 4.12. Moreover, by summing in both hand sides of (4.46) the term

$$-\int_0^T \mathrm{e}^{-\frac{t}{\beta}} (\mathbb{B}w^0, e\varphi(t))_H \,\mathrm{d}t,$$

we get the identity (4.42).

Now we show that $w$, defined in (4.47), belongs to the space $\mathcal{AC}$. For every $s, t \in [0,T]$ such that $s < t$ the following inequalities hold

$$\begin{aligned} \|w(t) - w(s)\|_H &\le \left\| \mathrm{e}^{-\frac{t}{\beta}} \int_0^t \frac{1}{\beta} \mathrm{e}^{\frac{\tau}{\beta}} eu(\tau)\mathrm{d}\tau - \mathrm{e}^{-\frac{s}{\beta}} \int_0^s \frac{1}{\beta} \mathrm{e}^{\frac{\tau}{\beta}} eu(\tau)\mathrm{d}\tau \right\|_H + |\mathrm{e}^{-\frac{t}{\beta}} - \mathrm{e}^{-\frac{s}{\beta}}| \|w^0\|_H \\ &\le L(t-s)\|w^0\|_H + \frac{\mathrm{e}^{\frac{T}{\beta}}}{\beta} |\mathrm{e}^{-\frac{t}{\beta}} - \mathrm{e}^{-\frac{s}{\beta}}| \int_0^s \|eu(\tau)\|_H \,\mathrm{d}\tau + \mathrm{e}^{-\frac{t}{\beta}} \frac{\mathrm{e}^{\frac{T}{\beta}}}{\beta} \int_s^t \|eu(\tau)\|_H \,\mathrm{d}\tau. \end{aligned}$$

By setting

$$F(t) := \int_0^t \|eu(\tau)\|_H \, \mathrm{d}\tau$$

we have that $F \in H^1(0,T)$ because $\|eu(\cdot)\|_H \in L^2(0,T)$. Then $F$ admits a continuous representative. Thanks to this we can write

$$\|w(t) - w(s)\|_H \le \int_s^t \left[ L\|w^0\|_H + L\frac{\mathrm{e}^{\frac{T}{\beta}}}{\beta}\|F\|_{C^0([0,T])} + \frac{\mathrm{e}^{\frac{T}{\beta}}}{\beta}\|eu(\tau)\|_H \right] \mathrm{d}\tau := \int_s^t \ell(\tau)\mathrm{d}\tau.$$

Since $\ell \in L^1(0,T)$ we can say that $w \in \mathcal{AC}$, so that $(u,w) \in \mathcal{U} \times \mathcal{AC}$ is a weak solution to (4.37)–(4.41) in the sense of Definition 4.12 and 4.13. □

4.3. **Discretization in Time and Energy Estimate.** In this subsection we prove the existence of a weak solution to the vector-valued system (4.37)–(4.41), by means of a time discretization scheme in the same spirit of [6]. Moreover, we provide an energy estimate satisfied by this solution.

**Theorem 4.15.** *There exists a weak solution $(u,w) \in \mathcal{U} \times \mathcal{AC}$ to the vector-valued system (4.37)–(4.41) satisfying the initial conditions $u(0) = u^0$, $\dot{u}(0) = u^1$ and $w(0) = w^0$ in the sense of Definition 4.13. Moreover $u \in C^0_w([0,T];U_T)$, $\dot{u} \in C^0_w([0,T];H) \cap H^1(0,T;(U_0^D)')$, and*

$$\lim_{t\to 0^+} u(t) = u^0 \text{ in } U_T, \qquad \lim_{t\to 0^+} \dot{u}(t) = u^1 \text{ in } H.$$

Let us fix $n \in \mathbb{N}$ and set

$$\tau_n := \frac{T}{n}, \quad u_n^0 := u^0, \quad u_n^{-1} := u^0 - \tau_n u^1, \quad w_n^0 := w^0, \quad z_n^0 := z(0).$$

We define

$$\begin{aligned} &U_n^k := U_{k\tau_n}^D, \quad z_n^k := z(k\tau_n) && \text{for } k = 0,\dots,n, \\ &\delta z_n^0 := \dot{z}(0), \quad \delta z_n^k := \frac{z_n^k - z_n^{k-1}}{\tau_n}, \quad \delta^2 z_n^k := \frac{\delta z_n^k - \delta z_n^{k-1}}{\tau_n} && \text{for } k = 1,\dots,n. \end{aligned} \tag{4.48}$$

Regarding the part related to forcing term, for $k = 1,\dots,n$ we pose

$$(f_1)_n^k := \frac{1}{\tau_n}\int_{(k-1)\tau_n}^{k\tau_n} f_1(s)\,\mathrm{d}s,$$

and for $k = 0,\dots,n$ we set

$$N_n^k := N(k\tau_n), \quad (f_2)_n^k := f_2(k\tau_n), \quad h_n^k := \mathrm{e}^{-\frac{k\tau_n}{\beta}}\mathbb{B}w^0.$$

Moreover, we define $\delta N_n^k$, $\delta(f_2)_n^k$ and $\delta h_n^k$ for $k = 1,\dots,n$ in the same way we have already done in (4.48).

For every $k = 1,\dots,n$ let $(u_n^k, w_n^k) \in U_T \times H$, with $u_n^k - z_n^k \in U_n^k$, be the solution to the following discrete equation

$$\begin{aligned} &(\delta^2 u_n^k, \varphi)_H + (\mathbb{C}eu_n^k, e\varphi)_H + (\mathbb{B}(eu_n^k - w_n^k), e\varphi - \psi)_H + \beta(\mathbb{B}\delta w_n^k, \psi)_H \\ &\qquad = (N_n^k, \varphi)_{H_N} + ((f_1)_n^k, \varphi)_H + \langle\langle (f_2)_n^k, \varphi\rangle\rangle - (h_n^k, e\varphi)_H, \qquad \forall (\varphi,\psi) \in U_n^k \times H \end{aligned} \tag{4.49}$$

where by definition we have

$$\begin{aligned} &\delta u_n^k := \frac{u_n^k - u_n^{k-1}}{\tau_n}, && \text{for } k = 0,\dots,n, \\ &\delta^2 u_n^k := \frac{\delta u_n^k - \delta u_n^{k-1}}{\tau_n}, \quad \delta w_n^k := \frac{w_n^k - w_n^{k-1}}{\tau_n} && \text{for } k = 1,\dots,n. \end{aligned}$$

Notice that if we choose as a test function the pair $(\varphi, 0)$ with $\varphi \in U_n^k$, we get

$$(\delta^2 u_n^k, \varphi)_H + ((\mathbb{C}+\mathbb{B})eu_n^k - \mathbb{B}w_n^k, e\varphi)_H = (N_n^k, \varphi)_{H_N} + ((f_1)_n^k, \varphi)_H + \langle\langle (f_2)_n^k, \varphi\rangle\rangle - (h_n^k, e\varphi)_H,$$

which is an approximation in time of the first equation of system (4.37); conversely if we use as a test function $(0,\psi)$ such that $\psi \in H$, we have

$$(\beta\delta w_n^k + w_n^k - eu_n^k, \psi)_H = 0,$$

thus $\beta\delta w_n^k + w_n^k - eu_n^k \overset{H}{=} 0$, which is an approximation in time of the second equation of system (4.37). For $n$ large enough, Lax–Milgram's theorem gives us the existence of a unique solution $(u_n^k, w_n^k)$ to (4.49).

In the next lemma we show an estimate for the family $\{(u_n^k, w_n^k)\}_{k=1}^n$, uniform with respect to $n \in \mathbb{N}$, which will be used later to pass to the limit in the discrete equation (4.49).

**Lemma 4.16.** *There exists a constant $C > 0$, independent of $n \in \mathbb{N}$, such that*

$$\max_{i=1,..,n} \|\delta u_n^i\|_H + \max_{i=1,..,n} \|eu_n^i\|_H + \max_{i=1,..,n} \|w_n^i\|_H + \sum_{i=1}^n \tau_n \|\delta w_n^i\|_H^2 \le C. \tag{4.50}$$

*Proof.* To simplify our computations, we define the following two different bilinear symmetric forms:

$$a : (U_T \times H) \times (U_T \times H) \to \mathbb{R} \qquad b : H \times H \to \mathbb{R}$$
$$a((u, w), (\varphi, \psi)) := (\mathbb{C}eu, e\varphi)_H + (\mathbb{B}(eu - w), e\varphi - \psi)_H, \qquad b(w, \psi) := \beta(\mathbb{B}w, \psi)_H.$$

If we call $\omega_n^k := (u_n^k, w_n^k)$, by taking as a test function in (4.49) the pair $(\varphi, \psi) = \tau_n(\delta u_n^k - \delta z_n^k, \delta w_n^k) \in U_n^k \times H$, for $k = 1, \dots, n$ we obtain

$$\begin{aligned}\|\delta u_n^k\|_H^2 - (\delta u_n^{k-1}, \delta u_n^k)_H - \tau_n(\delta^2 u_n^k, \delta z_n^k)_H + a(\omega_n^k, \omega_n^k) - a(\omega_n^k, \omega_n^{k-1}) - \tau_n a(\omega_n^k, (\delta z_n^k, 0)) + \tau_n b(\delta w_n^k, \delta w_n^k) \\ = \tau_n(N_n^k, \delta u_n^k - \delta z_n^k)_{H_N} + \tau_n((f_1)_n^k, \delta u_n^k - \delta z_n^k)_H + \tau_n\langle\langle (f_2)_n^k, \delta u_n^k - \delta z_n^k\rangle\rangle - \tau_n(h_n^k, e\delta u_n^k - e\delta z_n^k)_H.\end{aligned} \tag{4.51}$$

Thanks to the following identities

$$\|\delta u_n^k\|_H^2 - (\delta u_n^{k-1}, \delta u_n^k)_H = \frac{1}{2}\|\delta u_n^k\|_H^2 - \frac{1}{2}\|\delta u_n^{k-1}\|_H^2 + \frac{\tau_n^2}{2}\|\delta^2 u_n^k\|_H^2,$$
$$a(\omega_n^k, \omega_n^k) - a(\omega_n^k, \omega_n^{k-1}) = \frac{1}{2}a(\omega_n^k, \omega_n^k) - \frac{1}{2}a(\omega_n^{k-1}, \omega_n^{k-1}) + \frac{\tau_n^2}{2}a(\delta\omega_n^k, \delta\omega_n^k),$$

and by omitting the terms with $\tau_n^2$, which are non negative, from (4.51) we derive

$$\frac{1}{2}\|\delta u_n^k\|_H^2 - \frac{1}{2}\|\delta u_n^{k-1}\|_H^2 + \frac{1}{2}a(\omega_n^k, \omega_n^k) - \frac{1}{2}a(\omega_n^{k-1}, \omega_n^{k-1}) + \tau_n b(\delta w_n^k, \delta w_n^k) \le \tau_n L_n^k,$$

where

$$\begin{aligned}L_n^k := (N_n^k, \delta u_n^k - \delta z_n^k)_{H_N} + ((f_1)_n^k, \delta u_n^k - \delta z_n^k)_H + \langle\langle (f_2)_n^k, \delta u_n^k - \delta z_n^k\rangle\rangle \\ - (h_n^k, e\delta u_n^k - e\delta z_n^k)_H + (\delta^2 u_n^k, \delta z_n^k)_H + a(\omega_n^k, (\delta z_n^k, 0)).\end{aligned}$$

We fix $i \in \{1, \dots, n\}$ and sum over $k = 1, \dots, i$ to obtain the following discrete energy inequality

$$\frac{1}{2}\|\delta u_n^i\|_H^2 + \frac{1}{2}a(\omega_n^i, \omega_n^i) + \sum_{k=1}^i \tau_n b(\delta w_n^k, \delta w_n^k) \le \mathcal{E}_0 + \sum_{k=1}^i \tau_n L_n^k, \tag{4.52}$$

where by definition

$$\mathcal{E}_0 := \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}(\mathbb{C}eu^0, eu^0)_H + \frac{1}{2}(\mathbb{B}(eu^0 - w^0), eu^0 - w^0)_H.$$

Let us now estimate the right–hand side in (4.52) from above. We can write

$$\left|\sum_{k=1}^i \tau_n((f_1)_n^k, \delta u_n^k - \delta z_n^k)_H\right| \le \|f_1\|_{L^2(0,T;H)}^2 + \frac{1}{2}\|\dot z\|_{L^2(0,T;H)}^2 + \frac{1}{2}\sum_{k=1}^i \tau_n\|\delta u_n^k\|_H^2, \tag{4.53}$$

$$\left|\sum_{k=1}^i \tau_n(h_n^k, \delta z_n^k)_H\right| \le \frac{1}{2}\sum_{k=1}^i \tau_n \mathrm{e}^{-2\frac{k\tau_n}{\beta}}\|\mathbb{B}w^0\|_H^2 + \frac{1}{2}\sum_{k=1}^i \tau_n\|\delta z_n^k\|_H^2 \le \frac{T}{2}\|\mathbb{B}w^0\|_H^2 + \frac{1}{2}\|\dot z\|_{L^2(0,T;H)}^2, \tag{4.54}$$

$$\begin{aligned}\left|\sum_{k=1}^i \tau_n(N_n^k, \delta z_n^k)_H\right| &\le \frac{1}{2}\sum_{k=1}^i \tau_n\|N_n^k\|_{H_N}^2 + \frac{1}{2}\sum_{k=1}^i \tau_n\|\delta z_n^k\|_{H_N}^2 \\ &\le T\|N(0)\|_{H_N}^2 + T^2\sum_{k=1}^i \tau_n\|\delta N_n^k\|_{H_N}^2 + \frac{C_{tr}^2}{2}\|\dot z\|_{L^2(0,T;U_T)}^2\end{aligned}$$

$$\leq C(N(0)) + T^2\|\dot N\|^2_{L^2(0,T;H_N)} + \frac{C_{tr}^2}{2}\|\dot z\|^2_{L^2(0,T;U_T)}, \tag{4.55}$$

$$\left|\sum_{k=1}^i \tau_n a(\omega_n^k,(\delta z_n^k,0))\right| \leq \frac{1}{2}\|\mathbb{C}\|^2_\infty \sum_{k=1}^i \tau_n\|eu_n^k\|^2_H + \frac{1}{2}\|\mathbb{B}\|^2_\infty\sum_{k=1}^i\tau_n\|eu_n^k - w_n^k\|^2_H + \sum_{k=1}^i \tau_n\|e\delta z_n^k\|^2_H$$

$$\leq K\left[\sum_{k=1}^i\tau_n\|eu_n^k\|^2_H + \sum_{k=1}^i\tau_n\|eu_n^k - w_n^k\|^2_H\right] + \|e\dot z\|^2_{L^2(0,T;H)}. \tag{4.56}$$

Notice that the following discrete integrations by parts hold

$$\sum_{k=1}^i \tau_n(\delta^2 u_n^k,\delta z_n^k)_H = (\delta u_n^i,\delta z_n^i)_H - (\delta u_n^0,\delta z_n^0)_H - \sum_{k=1}^i\tau_n(\delta u_n^{k-1},\delta^2 z_n^k)_H, \tag{4.57}$$

$$\sum_{k=1}^i \tau_n(h_n^k, e\delta u_n^k)_H = (eu_n^i, h_n^i)_H - (eu_n^0,h_n^0)_H - \sum_{k=1}^i\tau_n(\delta h_n^k, eu_n^{k-1})_H, \tag{4.58}$$

$$\sum_{k=1}^i \tau_n\langle\langle (f_2)_n^k, \delta u_n^k - \delta z_n^k\rangle\rangle = \langle\langle (f_2)_n^i, u_n^i - z_n^i\rangle\rangle - \langle\langle (f_2)_n^0, u_n^0 - z_n^0\rangle\rangle - \sum_{k=1}^i\tau_n\langle\langle \delta(f_2)_n^k, u_n^{k-1} - z_n^{k-1}\rangle\rangle, \tag{4.59}$$

$$\sum_{k=1}^i \tau_n(N_n^k,\delta u_n^k)_{H_N} = (N_n^i, u_n^i)_{H_N} - (N_n^0,u^0)_{H_N} - \sum_{k=1}^i\tau_n(\delta N_n^k, u_n^{k-1})_{H_N}. \tag{4.60}$$

By (4.57) we can write

$$\left|\sum_{k=1}^i(\delta^2 u_n^k,\delta z_n^k)_H\right| \leq \frac{1}{2\epsilon_1}\|\delta z_n^i\|^2_H + \frac{\epsilon_1}{2}\|\delta u_n^i\|^2_H + \|u^1\|_H\|\dot z(0)\|_H + \sum_{k=1}^i\tau_n\|\delta u_n^{k-1}\|_H\|\delta^2 z_n^k\|_H$$

$$\leq C_{\epsilon_1} + \frac{\epsilon_1}{2}\|\delta u_n^i\|^2_H + \frac{1}{2}\sum_{k=1}^i\tau_n\|\delta u_n^k\|^2_H + \frac{1}{2}\sum_{k=1}^i\tau_n\|\delta^2 z_n^k\|^2_H$$

$$\leq C_{\epsilon_1} + \|\ddot z\|^2_{L^2(0,T;H)} + \frac{\epsilon_1}{2}\|\delta u_n^i\|^2_H + \frac{1}{2}\sum_{k=1}^i\tau_n\|\delta u_n^k\|^2_H. \tag{4.61}$$

The second line of (4.61) is valid because

$$\sum_{k=1}^i\tau_n\|\delta u_n^{k-1}\|^2_H = \sum_{k=0}^{i-1}\tau_n\|\delta u_n^k\|_H \leq T\|u^1\|^2_H + \sum_{k=1}^i\tau_n\|\delta u_n^k\|_H. \tag{4.62}$$

Thanks to (4.58) and to (4.62) (applied to $eu_n^{k-1}$ in place of $\delta u_n^{k-1}$) we have

$$\left|\sum_{k=1}^i\tau_n(h_n^k, e\delta u_n^k)_H\right| \leq \frac{1}{2\epsilon_2}\|h_n^i\|^2_H + \frac{\epsilon_2}{2}\|eu_n^i\|^2_H + \|eu^0\|_H\|h(0)\|_H + \sum_{k=1}^i\tau_n\|\delta h_n^k\|_H\|eu_n^{k-1}\|_H$$

$$\leq C_{\epsilon_2} + \frac{\epsilon_2}{2}\|eu_n^i\|^2_H + \frac{1}{2}\sum_{k=1}^i\tau_n\|eu_n^k\|^2_H + \frac{1}{2}\sum_{k=1}^i\tau_n\|\delta h_n^k\|^2_H$$

$$\leq C_{\epsilon_2} + \|\dot h\|^2_{L^2(0,T;H)} + \frac{\epsilon_2}{2}\|eu_n^i\|^2_H + \frac{1}{2}\sum_{k=1}^i\tau_n\|eu_n^k\|^2_H. \tag{4.63}$$

Moreover, notice that

$$u_n^i = \sum_{k=1}^i\tau_n\delta u_n^k + u^0,$$

then

$$\|u_n^i\|_H \leq \sum_{k=1}^i\tau_n\|\delta u_n^k\|_H + \|u^0\|_H \leq \sqrt{T}\left(\sum_{k=1}^i\tau_n\|\delta u_n^k\|^2_H\right)^{\frac{1}{2}} + \|u^0\|_H. \tag{4.64}$$

Therefore, thanks to (4.59) and to (4.62) (applied to $u_n^{k-1}$ in the space $U_T$ in place of $\delta u_n^{k-1}$ in the space $H$) we obtain

$$\begin{aligned}
\left|\sum_{k=1}^{i}\tau_n\langle\langle (f_2)_n^k,\delta z_n^k\rangle\rangle\right| &\le \frac{1}{2}\sum_{k=1}^{i}\tau_n\|(f_2)_n^k\|_{U_T'}^2+\frac{1}{2}\sum_{k=1}^{i}\tau_n\|\delta z_n^k\|_{U_T}^2\\
&\le T\sum_{k=1}^{i}\tau_n\sum_{s=1}^{k}\tau_n\|\delta(f_2)_n^s\|_{U_T'}^2+\sum_{k=1}^{i}\tau_n\|f_2(0)\|_{U_T'}^2+\frac{1}{2}\|\dot z\|_{L^2(0,T;U_T)}^2\\
&\le T^2\sum_{k=1}^{i}\tau_n\|\delta(f_2)_n^k\|_{U_T'}^2+T\|f_2\|_{C^0([0,T];U_T')}^2+\frac{1}{2}\|\dot z\|_{L^2(0,T;U_0)}^2\\
&\le T\|f_2\|_{C^0([0,T];U_T')}^2+\frac{1}{2}\|\dot z\|_{L^2(0,T;U_0)}^2+T^2\|\dot f_2\|_{L^2(0,T;U_T')}^2
\end{aligned}\tag{4.65}$$

and

$$\begin{aligned}
\left|\sum_{k=1}^{i}\tau_n\langle\langle (f_2)_n^k,\delta u_n^k\rangle\rangle\right| &\le \frac{1}{2\eta_1}\|(f_2)_n^i\|_{U_T'}^2+\frac{\eta_1}{2}\|u_n^i\|_{U_T}^2+\|f_2(0)\|_{U_T'}\|u^0\|_{U_T}+\sum_{k=1}^{i}\tau_n\|\delta(f_2)_n^k\|_{U_T'}\|u_n^{k-1}\|_{U_T}\\
&\le K_1+\frac{1}{2\eta_1}\|f_2\|_{C^0([0,T];U_T')}^2+\frac{\eta_1}{2}\|eu_n^i\|_H^2+\eta_1 T\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|_H^2+\frac{1}{2}\sum_{k=1}^{i}\tau_n\|\delta(f_2)_n^k\|_{U_T'}^2\\
&\qquad+\frac{1}{2}\sum_{k=1}^{i}\tau_n\|eu_n^k\|_H^2+T^2\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|_H^2\\
&\le K_{1,\eta_1}+\frac{\eta_1}{2}\|eu_n^i\|_H^2+(\eta_1 T+T^2)\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|_H^2+\frac{1}{2}\|\dot f_2\|_{L^2(0,T;U_T')}^2+\frac{1}{2}\sum_{k=1}^{i}\tau_n\|eu_n^k\|_H^2\\
&\le C_{\eta_1}+\frac{\eta_1}{2}\|eu_n^i\|_H^2+C_{\eta_1}\left(\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|_H^2+\sum_{k=1}^{i}\tau_n\|eu_n^k\|_H^2\right).
\end{aligned}\tag{4.66}$$

By (4.60), (4.62) (applied again to $u_n^{k-1}$ in the space $U_T$ in place of $\delta u_n^{k-1}$ in the space $H$) and (4.64) we can write

$$\begin{aligned}
\left|\sum_{k=1}^{i}\tau_n(N_n^k,\delta u_n^k)_{H_N}\right| &\le \frac{1}{2\eta_2}\|N_n^i\|_{H_N}^2+\frac{\eta_2}{2}\|u_n^i\|_{H_N}^2+|(N(0),u^0)_{H_N}|+\sum_{k=1}^{i}\tau_n\|\delta N_n^k\|_{H_N}\|u_n^{k-1}\|_{H_N}\\
&\le K_2+\frac{1}{2\eta_2}\|N\|_{C^0(0,T;H_N)}^2+C_{tr}^2\frac{\eta_2}{2}\|u_n^i\|_{U_T}^2+\frac{1}{2}\sum_{k=1}^{i}\tau_n\|\delta N_n^k\|_{H_N}^2+\frac{C_{tr}^2}{2}\sum_{k=1}^{i}\tau_n\|u_n^k\|_{U_T}^2\\
&\le K_{2,\eta_2}+C_{tr}^2\frac{\eta_2}{2}\|eu_n^i\|_H^2+C_{tr}^2T\eta_2\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|_H^2+\frac{1}{2}\|\dot N\|_{L^2(0,T;H_N)}^2\\
&\qquad+\frac{C_{tr}^2}{2}\sum_{k=1}^{i}\tau_n\|eu_n^k\|_H^2+C_{tr}^2T^2\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|_H^2\\
&\le C_{\eta_2}+C_{tr}^2\frac{\eta_2}{2}\|eu_n^i\|_H^2+C_{\eta_2}\left(\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|_H^2+\sum_{k=1}^{i}\tau_n\|eu_n^k\|_H^2\right).
\end{aligned}\tag{4.67}$$

By considering (4.52)–(4.67) and using (3.5)–(3.6) we obtain

$$\begin{aligned}
&\frac{1-\epsilon_1}{2}\|\delta u_n^i\|_H^2+\frac{C_{\mathbb{C}}-\epsilon_2-\eta_1-C_{tr}^2\eta_2}{2}\|eu_n^i\|_H^2+\frac{C_{\mathbb{B}}}{2}\|eu_n^i-w_n^i\|_H^2+\beta C_{\mathbb{B}}\sum_{k=1}^{i}\tau_n\|\delta w_n^k\|_H^2\\
&\quad\le C_1+(1+C_{\eta_1}+C_{\eta_2})\sum_{k=1}^{i}\tau_n\|\delta u_n^k\|_H^2+\left(K+\frac{1}{2}+C_{\eta_1}+C_{\eta_2}\right)\sum_{k=1}^{i}\tau_n\|eu_n^k\|_H^2+K\sum_{k=1}^{i}\tau_n\|eu_n^k-w_n^k\|_H^2,
\end{aligned}$$

therefore, if we choose $\epsilon_1=\frac{1}{2}$ and $\epsilon_2=2\eta_1=2C_{tr}^2\eta_2=\frac{C_{\mathbb{C}}}{4}$ we get

$$\frac{1}{4}\|\delta u_n^i\|_H^2+\frac{C_{\mathbb{C}}}{4}\|eu_n^i\|_H^2+\frac{C_{\mathbb{B}}}{2}\|eu_n^i-w_n^i\|_H^2+\beta C_{\mathbb{B}}\sum_{k=1}^i\tau_n\|\delta w_n^k\|_H^2$$
$$\leq C_1+C_2\sum_{k=1}^i\tau_n\left[\|\delta u_n^k\|_H^2+\|eu_n^k\|_H^2+\|eu_n^k-w_n^k\|_H^2+\sum_{l=1}^k\tau_n\|\delta w_n^l\|_H^2\right]. \tag{4.68}$$

If we define

$$a_n^i:=\|\delta u_n^i\|_H^2+\|eu_n^i\|_H^2+\|eu_n^i-w_n^i\|_H^2+\sum_{k=1}^i\tau_n\|\delta w_n^k\|_H^2,$$

from (4.68) we can derive

$$a_n^i\leq\tilde{C}_1+\tilde{C}_2\sum_{k=1}^i\tau_n a_n^k.$$

Thanks to a discrete version of Gronwall's lemma (see, [1, Lemma 3.2.4]) we deduce that $a_n^i$ is bounded by a constant $C^*$ independent of $i$ and $n$, i.e.

$$\|\delta u_n^i\|_H^2+\|eu_n^i\|_H^2+\|eu_n^i-w_n^i\|_H^2+\sum_{k=1}^i\tau_n\|\delta w_n^k\|_H^2\leq C^* \qquad \text{for every } i=1,\dots,n \text{ and for every } n\in\mathbb{N},$$

therefore

$$\|\delta u_n^i\|_H^2+\|eu_n^i\|_H^2+\|w_n^i\|_H^2+\sum_{k=1}^i\tau_n\|\delta w_n^k\|_H^2\leq 2C^* \qquad \text{for every } i=1,\dots,n \text{ and for every } n\in\mathbb{N}.$$

□

We now want to pass to the limit into the discrete equation (4.49) to obtain a weak solution to the system (4.37)–(4.41). We start by defining the following approximating sequences of our limit solution

$$\begin{aligned}
&u_n(t):=u_n^k+(t-k\tau_n)\delta u_n^k, && \tilde{u}_n(t):=\delta u_n^k+(t-k\tau_n)\delta^2u_n^k && t\in[(k-1)\tau_n,k\tau_n],\ k=1,\dots,n,\\
&u_n^+(t):=u_n^k, && \tilde{u}_n^+(t):=\delta u_n^k && t\in((k-1)\tau_n,k\tau_n],\ k=1,\dots,n,\\
&u_n^-(t):=u_n^{k-1}, && \tilde{u}_n^-(t):=\delta u_n^{k-1} && t\in[(k-1)\tau_n,k\tau_n),\ k=1,\dots,n,
\end{aligned}$$

and the same approximations $w_n,w_n^+,w_n^-$ for the function $w$.

Now, we have the following lemma:

**Lemma 4.17.** *There exists $(u,w)\in\mathcal{U}\times\mathcal{AC}$, with $u-z\in\mathcal{U}^D$, such that, up to a not relabeled subsequence*

$$u_n\xrightharpoonup[n\to\infty]{H^1(0,T;H)}u,\quad u_n^\pm\xrightharpoonup[n\to\infty]{L^2(0,T;U_T)}u,\quad \tilde{u}_n^\pm\xrightharpoonup[n\to\infty]{L^2(0,T;H)}\dot{u}, \tag{4.69}$$
$$w_n\xrightharpoonup[n\to\infty]{H^1(0,T;H)}w,\quad w_n^\pm\xrightharpoonup[n\to\infty]{L^2(0,T;H)}w, \tag{4.70}$$

*Proof.* Thanks to Lemma 4.16 the sequences

$$\begin{aligned}
&\{u_n\}_n\subseteq H^1(0,T;H)\cap L^\infty(0,T;U_T), && \{w_n\}_n\subseteq H^1(0,T;H)\cap L^\infty(0,T;H),\\
&\{u_n^\pm\}_n\subseteq L^\infty(0,T;U_T), && \{w_n^\pm\}_n\subseteq L^\infty(0,T;H),\\
&\{\tilde{u}_n^\pm\}_n\subseteq L^\infty(0,T;H),
\end{aligned}$$

are uniformly bounded. Indeed we have $\|eu_n^i\|_H\leq C$ for every $n\in\mathbb{N}$ and $i=1,..,n$, and therefore thanks to (4.64) we have $\|u_n^i\|_{U_T}\leq\bar{C}$. Thanks to this,

$$\|u_n\|_{L^\infty(0,T;U_T)}\leq\max_{k=1,..,n}\sup_{t\in[(k-1)\tau_n,k\tau_n]}\|\left(1-k+t\tau_n^{-1}\right)u_n^k+\left(k-t\tau_n^{-1}\right)u_n^{k-1}\|_{U_T}\leq 2\bar{C}.$$

By Banach-Alaoglu's theorem there exist some functions $u\in H^1(0,T;H)$, $v_1\in L^2(0,T;U_T)$, $w\in H^1(0,T;H)$ and $v_2\in L^2(0,T;H)$ such that, up to a not relabeled subsequence

$$u_n\xrightharpoonup[n\to\infty]{L^2(0,T;U_T)}u,\qquad \dot{u}_n\xrightharpoonup[n\to\infty]{L^2(0,T;H)}\dot{u},\qquad u_n^+\xrightharpoonup[n\to\infty]{L^2(0,T;U_T)}v_1, \tag{4.71}$$

$$w_n \xrightarrow[n\to\infty]{L^2(0,T;H)} w, \qquad \dot w_n \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot w, \qquad w_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} v_2. \tag{4.72}$$

Since there exists a constant $C>0$ such that

$$\|u_n - u_n^+\|_{L^\infty(0,T;H)} \le C\tau_n \xrightarrow[n\to\infty]{} 0, \quad \|w_n - w_n^+\|_{L^\infty(0,T;H)} \le C\tau_n \xrightarrow[n\to\infty]{} 0,$$

by using (4.71), (4.72) and triangle inequality, we can conclude that $u = v_1$ and $w = v_2$. Moreover, given that

$$\begin{aligned} &u_n^-(t) = u_n^+(t-\tau_n), \qquad w_n^-(t) = w_n^+(t-\tau_n) && \text{for } t\in(\tau_n,T),\\ &\tilde u_n^-(t) = \tilde u_n^+(t-\tau_n), && \text{for } t\in(\tau_n,T),\\ &\tilde u_n^+(t) = \dot u_n(t), && \text{for a.e. } t\in(0,T), \end{aligned}$$

thanks to (4.71), (4.72) and the continuity of the translations in $L^2$ we deduce

$$u_n^- \xrightarrow[n\to\infty]{L^2(0,T;U_T)} u, \quad \tilde u_n^\pm \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot u, \quad w_n^- \xrightarrow[n\to\infty]{L^2(0,T;H)} w.$$

Now let us check that $(u,w)\in\mathcal{U}\times\mathcal{AC}$. To this aim, we define the following sets

$$\begin{aligned} \tilde{\mathcal{U}} &:= \{u\in L^2(0,T;U_T) : u(t)\in U_t \text{ for a.e. } t\in(0,T)\} \subseteq L^2(0,T;U_T),\\ \tilde{\mathcal{U}}^D &:= \{u\in \tilde{\mathcal{U}} : u(t)\in U_t^D \text{ for a.e. } t\in(0,T)\} \subseteq L^2(0,T;U_T). \end{aligned}$$

We have that $\tilde{\mathcal{U}}$ is a (strong) closed convex subset of $L^2(0,T;U_T)$, and so by Hahn-Banach's theorem the set $\tilde{\mathcal{U}}$ is weakly closed. In the same way we can prove that $\tilde{\mathcal{U}}^D$ is also a weakly closed set. Notice that $\{u_n^-\}_n\subseteq\tilde{\mathcal{U}}$, indeed

$$u_n^-(t) = u_n^{k-1}\in U_n^{k-1}\subseteq U_t \quad \text{for } t\in[(k-1)\tau_n, k\tau_n),\ k=1,\dots,n.$$

Since $u_n^- \xrightarrow[n\to\infty]{L^2(0,T;U_T)} u$, we conclude that $u\in\tilde{\mathcal{U}}$. Moreover $\tilde u_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot u$ and so $\dot u\in L^2(0,T;H)$, from which we have $u\in\mathcal{U}$. Finally, to show that $u-z\in\mathcal{U}^D$ we observe

$$u_n^-(t) - z_n^-(t) = u_n^{k-1} - z_n^{k-1} \in U_n^{k-1} \subseteq U_t^D \quad \text{for } t\in[(k-1)\tau_n, k\tau_n),\ k=1,\dots,n,$$

therefore $\{u_n^- - z_n^-\}_n\subseteq\tilde{\mathcal{U}}^D$. Since

$$u_n^- \xrightarrow[n\to\infty]{L^2(0,T;U_T)} u, \quad z_n^- \xrightarrow[n\to\infty]{L^2(0,T;U_0)} z,$$

we get $u-z\in\mathcal{U}^D$.

It remains to prove that $w\in\mathcal{AC}$. Given that $w\in H^1(0,T;H)$, it is well known (see for example [2, Corollary A.2]) that it admits a continuous representative, and in particular such representative is an absolute continuous function. □

**Lemma 4.18.** *The limit function $(u,w)\in\mathcal{U}\times\mathcal{AC}$ of Lemma 4.17 is a weak solution to system (4.37).*

*Proof.* We only need to prove that $(u,w)\in\mathcal{U}\times\mathcal{AC}$ satisfies (4.42) and (4.43). We fix $n\in\mathbb{N}$ and the functions $\varphi\in C_c^1(0,T;U_T)$ such that $\varphi(t)\in U_t^D$ for every $t\in(0,T)$, and $\psi\in C_c^1(0,T;H)$. We consider

$$\begin{aligned} &\varphi_n^k := \varphi(k\tau_n) && \psi_n^k := \psi(k\tau_n) && \text{for } k=0,\dots,n,\\ &\delta\varphi_n^k := \frac{\varphi_n^k - \varphi_n^{k-1}}{\tau_n} && \delta\psi_n^k := \frac{\psi_n^k - \psi_n^{k-1}}{\tau_n} && \text{for } k=1,\dots,n, \end{aligned}$$

and the approximating sequences

$$\begin{aligned} &\varphi_n^+(t) := \varphi_n^k, && \tilde\varphi_n^+(t) := \delta\varphi_n^k && t\in((k-1)\tau_n, k\tau_n],\ k=1,\dots,n,\\ &\psi_n^+(t) := \psi_n^k, && \tilde\psi_n^+(t) := \delta\psi_n^k && t\in((k-1)\tau_n, k\tau_n],\ k=1,\dots,n. \end{aligned}$$

If we use $\tau_n(\varphi_n^k,0)\in U_n^k\times H$ as a test function in (4.49), after summing over $k=1,\dots,n$, we get

$$\sum_{k=1}^n \tau_n(\delta^2 u_n^k, \varphi_n^k)_H + \sum_{k=1}^n \tau_n((\mathbb{C}+\mathbb{B})eu_n^k - \mathbb{B}w_n^k, e\varphi_n^k)_H$$

$$= \sum_{k=1}^{n} \tau_n (N_n^k, \varphi_n^k)_{H_N} + \sum_{k=1}^{n} \tau_n ((f_1)_n^k, \varphi_n^k)_H + \sum_{k=1}^{n} \tau_n \langle\langle (f_2)_n^k, \varphi_n^k \rangle\rangle - \sum_{k=1}^{n} \tau_n (h_n^k, e\varphi_n^k)_H. \tag{4.73}$$

By means of a time discrete integration by parts we obtain

$$\sum_{k=1}^{n} \tau_n (\delta^2 u_n^k, \varphi_n^k)_H = - \sum_{k=1}^{n} \tau_n (\delta u_n^{k-1}, \delta\varphi_n^k)_H = - \int_0^T (\tilde{u}_n^-(t), \tilde{\varphi}_n^+(t))_H \,\mathrm{d}t,$$

and thanks to (4.73) we deduce

$$\begin{aligned} - \int_0^T (\tilde{u}_n^-, \tilde{\varphi}_n^+)_H \,\mathrm{d}t &+ \int_0^T ((\mathbb{C}+\mathbb{B})eu_n^+ - \mathbb{B}w_n^+, e\varphi_n^+)_H \,\mathrm{d}t \\ &= \int_0^T (N_n^+, \varphi_n^+)_{H_N} \,\mathrm{d}t + \int_0^T ((f_1)_n^+, \varphi_n^+)_H \,\mathrm{d}t + \int_0^T \langle\langle (f_2)_n^+, \varphi_n^+ \rangle\rangle \mathrm{d}t - \int_0^T (h_n^+, e\varphi_n^+)_H \,\mathrm{d}t. \end{aligned} \tag{4.74}$$

Thanks to (4.69), (4.70), and to the following convergences

$$\varphi_n^+ \xrightarrow[n\to\infty]{L^2(0,T;U_T)} \varphi, \quad \tilde{\varphi}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot{\varphi}$$

we can pass to the limit in (4.74), and we get that $u \in \mathcal{U}$ satisfies (4.42) for every function $\varphi \in C_c^1(0,T;U_T)$ such that $\varphi(t) \in U_t^D$ for every $t \in (0,T)$. By using Lemma 4.3, we have that $u \in \mathcal{U}$ satisfies identity (4.42) for every $\varphi \in \mathcal{U}^D$ such that $\varphi(0) = \varphi(T) = 0$.

On the contrary, if we use $\tau_n(0, \psi_n^k) \in U_n^k \times H$ as a test function in (4.49), we have

$$(\beta\delta w_n^k + w_n^k - eu_n^k, \psi_n^k)_H = 0,$$

which corresponds to

$$(\beta\dot{w}_n(t) + w_n^+(t) - eu_n^+(t), \psi_n^+(t))_H = 0.$$

Therefore, for every $(a,b) \subset (0,T)$, thanks to (4.71) and (4.72), we can write

$$0 = \lim_{n\to\infty} \fint_a^b (\beta\dot{w}_n(t) + w_n^+(t) - eu_n^+(t), \psi_n^+(t))_H \,\mathrm{d}t = \fint_a^b (\beta\dot{w}(t) + w(t) - eu(t), \psi(t))_H \,\mathrm{d}t. \tag{4.75}$$

Now we pass to the limit in (4.75) as $a \to b$ and we obtain

$$(\beta\dot{w}(b) + w(b) - eu(b), \psi(b))_H = 0 \qquad \forall b \in (0,T).$$

Given that, fixed $t^* \in (0,T)$ for every $p \in H$ there exists $\psi_p(t) := p(t+1-t^*) \in \mathcal{AC}$ such that $\psi_p(t^*) = p$, we can say that for a.e. $t \in (0,T)$ we have $\beta\dot{w}(t) + w(t) - eu(t) = 0$ in $H$.

Finally, we conclude that $(u,w) \in \mathcal{U} \times \mathcal{AC}$ is a weak solution to system (4.37). □

Now, if we consider the discrete equation (4.49), for every $(\varphi,\psi) \in U_0^D \times H \subseteq U_n^k \times H$, with the condition $\|(\varphi,\psi)\|_{U_0\times H} \le 1$, we have

$$\begin{aligned} |(\delta^2 u_n^k, \varphi)_H| \le \|\mathbb{C}\|_\infty \|eu_n^k\|_H + \|\mathbb{B}\|_\infty \|eu_n^k - w_n^k\|_H + \beta\|\mathbb{B}\|_\infty \|\delta w_n^k\|_H \\ + \|N_n^k\|_{H_N} + \|(f_1)_n^k\|_H + \|(f_2)_n^k\|_{U_T'} + \|h_n^k\|_H. \end{aligned}$$

Therefore, taking the supremum over $(\varphi,\psi) \in U_0^D \times H$ with $\|(\varphi,\psi)\|_{U_0\times H} \le 1$, we obtain the existence of $C' > 0$ such that

$$\|\delta^2 u_n^k\|_{(U_0^D)'}^2 \le C'(\|eu_n^k\|_H^2 + \|eu_n^k - w_n^k\|_H^2 + \|\delta w_n^k\|_H^2 + \|N_n^k\|_{H_N} + \|(f_1)_n^k\|_H^2 + \|(f_2)_n^k\|_{U_T'}^2 + \|h_n^k\|_H^2).$$

By multiplying this inequality by $\tau_n$ and then by summing over $k = 1, \dots, n$, we get

$$\sum_{k=1}^{n} \tau_n \|\delta^2 u_n^k\|_{(U_0^D)'}^2 \le C' \left( \sum_{k=1}^{n} \tau_n \|eu_n^k\|_H^2 + \sum_{k=1}^{n} \tau_n \|eu_n^k - w_n^k\|_H^2 + \sum_{k=1}^{n} \tau_n \|\delta w_n^k\|_H^2 + C'' \right), \tag{4.76}$$

where

$$C'' := \|f_1\|_{L^2(0,T;H)}^2 + 2T(\|f_2\|_{C^0([0,T];U_T')}^2 + \|N\|_{C^0([0,T];H_N)}^2) + 2T^2(\|\dot{f}_2\|_{L^2(0,T,U_T')}^2 + \|\dot{N}\|_{L^2(0,T,H_N)}^2) + T\|\mathbb{B}w^0\|_H^2.$$

Thanks to (4.76) and Lemma 4.16 we conclude that there exists a positive constant $\tilde{C}$ independent on $n \in \mathbb{N}$ such that

$$\sum_{k=1}^{n} \tau_n \|\delta^2 u_n^k\|^2_{(U_0^D)'} \le \tilde{C}. \tag{4.77}$$

In particular $\{\tilde{u}_n\}_n \subseteq H^1(0,T;(U_0^D)')$ is uniformly bounded (notice that $\dot{\tilde{u}}_n(t) = \delta^2 u_n^k$ for $t \in ((k-1)\tau_n, k\tau_n)$ and $k = 1,\dots,n$). Hence, up to extracting a further (not relabeled) subsequence from the one of Lemma 4.17, we get

$$\tilde{u}_n \xrightharpoonup[n\to\infty]{H^1(0,T;(U_0^D)')} z, \tag{4.78}$$

and by using the following estimate

$$\|\tilde{u}_n - \tilde{u}_n^+\|_{L^2(0,T;(U_0^D)')} \le \sqrt{\tilde{C}}\tau_n \xrightarrow[n\to\infty]{} 0$$

we conclude that $z = \dot{u}$. Let us recall the following result, whose proof can be found for example in [10].

**Lemma 4.19.** *Let $X, Y$ be two reflexive Banach spaces such that $X \hookrightarrow Y$ continuously. Then*

$$L^\infty(0,T;X) \cap C^0_w([0,T];Y) = C^0_w([0,T];X).$$

Since $H^1(0,T;(U_0^D)') \hookrightarrow C^0([0,T],(U_0^D)')$, by using Lemma 4.17 and Lemma 4.19 we deduce that our weak solution $(u,w) \in \mathcal{U} \times \mathcal{AC}$ satisfies

$$u \in C^0_w([0,T];U_T), \quad \dot{u} \in C^0_w([0,T];H), \quad w \in C^0([0,T];H).$$

By (4.69), (4.70) and (4.78) we hence obtain

$$u_n(t) \xrightharpoonup[n\to\infty]{H} u(t), \quad w_n(t) \xrightharpoonup[n\to\infty]{H} w(t), \quad \tilde{u}_n(t) \xrightharpoonup[n\to\infty]{(U_0^D)'} \dot{u}(t) \qquad \forall t \in [0,T] \tag{4.79}$$

so that $u(0) = u^0$ and $\dot{u}(0) = u^1$, since $u_n(0) = u^0$ and $\tilde{u}_n(0) = u^1$. The validity of initial conditions (4.41) in the sense of Definition 4.13, is a consequence of an energy–dissipation inequality which holds for the weak solution $(u,w) \in \mathcal{U} \times \mathcal{AC}$ of Lemma 4.17. Let us define the total energy as

$$\mathcal{E}_{u,w}(t) := \frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}(\mathbb{C}eu(t), eu(t))_H + \frac{1}{2}(\mathbb{B}(eu(t)-w(t)), eu(t)-w(t))_H \quad t \in [0,T].$$

Notice that $\mathcal{E}_{u,w}(t)$ is well defined for every time $t \in [0,T]$ since $u \in C^0_w([0,T];U_T)$, $\dot{u} \in C^0_w([0,T];H)$ and $w \in C^0(0,T;H)$, and that

$$\mathcal{E}_{u,w}(0) = \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}(\mathbb{C}eu^0, eu^0)_H + \frac{1}{2}(\mathbb{B}(eu^0 - w^0), eu^0 - w^0)_H.$$

Now we are in a position to prove the energy-dissipation inequality before mentioned. For convenience of notation we pose $h(t) := \mathrm{e}^{-\frac{t}{\beta}}\mathbb{B}w^0$.

**Theorem 4.20.** *The weak solution $(u,w) \in \mathcal{U} \times \mathcal{AC}$ to the vector-valued system (4.37), given by Lemma 4.17, satisfies for every $t \in [0,T]$ the following energy–dissipation inequality*

$$\mathcal{E}_{u,w}(t) + \beta\int_0^t (\mathbb{B}\dot{w}(s), \dot{w}(s))_H \,\mathrm{d}s \le \mathcal{E}_{u,w}(0) + \mathcal{W}_{tot}(t) \tag{4.80}$$

*where the total work is defined as*

$$\begin{aligned}\mathcal{W}_{tot}(t) :=& \int_0^t [(f_1(s), \dot{u}(s) - \dot{z}(s))_H - \langle\langle \dot{f}_2(s), u(s) - z(s)\rangle\rangle]\mathrm{d}s + \langle\langle f_2(t), u(t) - z(t)\rangle\rangle - \langle\langle f_2(0), u^0 - z(0)\rangle\rangle \\ &- \int_0^t (\dot{N}(s), u(s) - z(s))_{H_N}\,\mathrm{d}s + (N(t), u(t) - z(t))_{H_N} - (N(0), u^0 - z(0))_{H_N} \\ &+ \int_0^t [((\mathbb{C}+\mathbb{B})eu(s) - \mathbb{B}w(s), e\dot{z}(s))_H - (\dot{u}(s), \ddot{z}(s))_H]\mathrm{d}s + (\dot{u}(t), \dot{z}(t))_H - (u^1, \dot{z}(0))_H \\ &+ \int_0^t [(\dot{h}(s), eu(s))_H + (h(s), e\dot{z}(s))_H]\mathrm{d}s - (h(t), eu(t))_H + (h(0), eu^0)_H \end{aligned} \tag{4.81}$$

**Remark 4.21.** From the classical point of view, the total work on the solution $u$ at time $t \in [0,T]$ is given by

$$\mathcal{W}^C_{tot}(t) := \mathcal{W}_{load}(t) + \mathcal{W}_{bdry}(t), \tag{4.82}$$

where $\mathcal{W}_{load}(t)$ is the work on the solution $u$ at time $t \in [0,T]$ due to the loading term, which is defined as

$$\mathcal{W}_{load}(t) := \int_0^t \langle\langle g(s), \dot u(s)\rangle\rangle \mathrm{d}s = \int_0^t (f_1(s), \dot u(s))_H \,\mathrm{d}s + \int_0^t \langle\langle f_2(s), \dot u(s)\rangle\rangle \,\mathrm{d}s + \int_0^t \mathrm{e}^{-\frac{s}{\beta}} \langle\langle \mathrm{div}(\mathbb{B}w^0), \dot u(s)\rangle\rangle \mathrm{d}s,$$

and $\mathcal{W}_{bdry}(t)$ is the work on the solution $u$ at time $t \in [0,T]$ due to the varying boundary conditions, which one expects to be equal to

$$\mathcal{W}_{bdry}(t) := \int_0^t (N(s), \dot u(s))_{H_N} \,\mathrm{d}s - \int_0^t \mathrm{e}^{-\frac{s}{\beta}} (\mathbb{B}w^0\nu, \dot u(s))_{H_N} \,\mathrm{d}s + \int_0^t (((\mathbb{C}+\mathbb{B})eu(s) - \mathbb{B}w(s))\nu, \dot z(s))_{H_D} \,\mathrm{d}s.$$

Unfortunately, $\mathcal{W}_{load}(t)$ is not well defined under our assumptions on $u$, because $\dot u \notin L^2(0,T;U_T)$. Hence, in our context, $\mathcal{W}_{load}(t)$ makes sense if we consider, in place of it, the following term

$$\begin{aligned} \tilde{\mathcal{W}}_{load}(t) := &\int_0^t \Big[ (f_1(s), \dot u(s))_H - \langle\langle \dot f_2(s), u(s)\rangle\rangle + \frac{1}{\beta}\mathrm{e}^{-\frac{s}{\beta}} \langle\langle \mathrm{div}(\mathbb{B}w^0), u(s)\rangle\rangle \Big] \mathrm{d}s \\ &+ \langle\langle f_2(t), u(t)\rangle\rangle - \langle\langle f_2(0), u^0\rangle\rangle + \mathrm{e}^{-\frac{t}{\beta}} \langle\langle \mathrm{div}(\mathbb{B}w^0), u(t)\rangle\rangle - \langle\langle \mathrm{div}(\mathbb{B}w^0), u^0\rangle\rangle. \end{aligned} \tag{4.83}$$

Of course, if $u$ is regular enough we have $\tilde{\mathcal{W}}_{load} = \mathcal{W}_{load}$. Also $\mathcal{W}_{bdry}(t)$ is not well defined under our assumptions on $u$. The term involving the Dirichlet datum $z$ is more difficult to handle since the trace of the function $((\mathbb{C}+\mathbb{B})eu - w)\nu$ on $\partial_D\Omega$ is not well defined. Therefore, if we assume that $w^0 \in U_0$, $u \in L^2(0,T;H^2(\Omega\setminus\Gamma;\mathbb{R}^d))$, $w \in L^2(0,T;H^1(\Omega\setminus\Gamma;\mathbb{R}^d))$, and that $\Gamma$ is a smooth manifold, then we can integrate by part the relation (4.42) to deduce that $u$ satisfies the first equation of (4.37). In this case, $((\mathbb{C}+\mathbb{B})eu - w)\nu \in L^2(0,T;H_D)$ and by using (4.37), together with the divergence theorem and the integration by parts formula, we deduce

$$\begin{aligned} &\int_0^t (((\mathbb{C}+\mathbb{B})eu(s) - \mathbb{B}w(s))\nu, \dot z(s))_{H_D} \,\mathrm{d}s \\ &= \int_0^t [(\mathrm{div}((\mathbb{C}+\mathbb{B})eu(s) - \mathbb{B}w(s)), \dot z(s))_H + ((\mathbb{C}+\mathbb{B})eu(s) - \mathbb{B}w(s), e\dot z(s))_H] \,\mathrm{d}s \\ &\qquad + \int_0^t \Big[ \mathrm{e}^{-\frac{s}{\beta}} (\mathbb{B}w^0\nu, \dot z(s))_{H_N} - (N(s), \dot z(s))_{H_N} \Big] \mathrm{d}s \\ &= \int_0^t \Big[ (\ddot u(s), \dot z(s))_H - (N(s), \dot z(s))_{H_N} - (f_1(s), \dot z(s))_H - \langle\langle f_2(s), \dot z(s)\rangle\rangle - \mathrm{e}^{-\frac{s}{\beta}} (\mathrm{div}(\mathbb{B}w^0), \dot z(s))_H \Big] \mathrm{d}s \\ &\qquad + \int_0^t \Big[ ((\mathbb{C}+\mathbb{B})eu(s) - \mathbb{B}w(s), e\dot z(s))_H + \mathrm{e}^{-\frac{s}{\beta}} (\mathbb{B}w^0\nu, \dot z(s))_{H_N} \Big] \mathrm{d}s \\ &= \int_0^t \Big[ ((\mathbb{C}+\mathbb{B})eu(s) - \mathbb{B}w(s), e\dot z(s))_H - (f_1(s), \dot z(s))_H + \mathrm{e}^{-\frac{s}{\beta}} (\mathbb{B}w^0, e\dot z(s))_H - \mathrm{e}^{-\frac{s}{\beta}} (\mathbb{B}w^0\nu, \dot z(s))_{H_D} \Big] \mathrm{d}s \\ &\qquad + \int_0^t [\langle\langle \dot f_2(s), z(s)\rangle\rangle - (\dot u(s), \ddot z(s))_H - (N(s), \dot z(s))_{H_N}] \mathrm{d}s \\ &\qquad + (\dot u(t), \dot z(t))_H - (u^1, \dot z(0))_H - \langle\langle f_2(t), z(t)\rangle\rangle + \langle\langle f_2(0), z(0)\rangle\rangle. \end{aligned} \tag{4.84}$$

By setting $H_{\partial\Omega} := L^2(\partial\Omega;\mathbb{R}^k)$ for every $k \in \mathbb{N}$, thanks to (4.84) and to the definition of $\mathcal{W}_{bdry}$, we have

$$\begin{aligned} \mathcal{W}_{bdry}(t) = &\int_0^t \Big[ ((\mathbb{C}+\mathbb{B})eu - \mathbb{B}w, e\dot z)_H - (f_1, \dot z)_H + \mathrm{e}^{-\frac{s}{\beta}} (\mathbb{B}w^0, e\dot z)_H - \mathrm{e}^{-\frac{s}{\beta}} (\mathbb{B}w^0\nu, \dot u)_{H_{\partial\Omega}} + (N, \dot u - \dot z)_{H_N} \Big] \mathrm{d}s \\ &+ \int_0^t [\langle\langle \dot f_2, z\rangle\rangle - (\dot u, \ddot z)_H] \mathrm{d}s + (\dot u(t), \dot z(t))_H - (u^1, \dot z(0))_H - \langle\langle f_2(t), z(t)\rangle\rangle + \langle\langle f_2(0), z(0)\rangle\rangle. \end{aligned}$$

Thanks to (4.83) and (4.84), by the definition of classical work, we get

$$\mathcal{W}^C_{tot}(t) = \int_0^t [(f_1, \dot u - \dot z)_H - \langle\langle \dot f_2, u - z\rangle\rangle] \mathrm{d}s + \langle\langle f_2(t), u(t) - z(t)\rangle\rangle - \langle\langle f_2(0), u^0 - z(0)\rangle\rangle$$

$$
\begin{aligned}
&+\int_0^t[((\mathbb{C}+\mathbb{B})eu-\mathbb{B}w,e\dot z)_H-(\dot u,\ddot z)_H]\mathrm{d}s+(\dot u(t),\dot z(t))_H-(u^1,\dot z(0))_H-\int_0^t \mathrm{e}^{-\frac{s}{\beta}}(\mathbb{B}w^0\nu,\dot u)_{H_{\partial\Omega}}\,\mathrm{d}s\\
&+\mathrm{e}^{-\frac{t}{\beta}}(\mathrm{div}(\mathbb{B}w^0),u(t))_H-(\mathrm{div}(\mathbb{B}w^0),u^0)_H+\int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{s}{\beta}}(\mathrm{div}(\mathbb{B}w^0),u(s))_H\,\mathrm{d}s+\int_0^t(N,\dot u-\dot z)_{H_N}\,\mathrm{d}s\\
=&\int_0^t[(f_1,\dot u-\dot z)_H-\langle\langle \dot f_2,u-z\rangle\rangle]\mathrm{d}s+\langle\langle f_2(t),u(t)-z(t)\rangle\rangle-\langle\langle f_2(0),u^0-z(0)\rangle\rangle\\
&-\int_0^t(\dot N,u-z)_{H_N}\,\mathrm{d}s+(N(t),u(t)-z(t))_{H_N}-(N(0),u^0-z(0))_{H_N}\\
&+\int_0^t[((\mathbb{C}+\mathbb{B})eu-\mathbb{B}w,e\dot z)_H-(\dot u,\ddot z)_H]\mathrm{d}s+(\dot u(t),\dot z(t))_H-(u^1,\dot z(0))_H\\
&-\mathrm{e}^{-\frac{t}{\beta}}(\mathbb{B}w^0,eu(t))_H+(\mathbb{B}w^0,eu^0)_H-\int_0^t\frac{1}{\beta}\mathrm{e}^{-\frac{s}{\beta}}(\mathbb{B}w^0,eu)_H\,\mathrm{d}s=\mathcal{W}_{tot}(t)
\end{aligned}
$$

Therefore, the definition of total work given in (4.81) is coherent with the classical one (4.82).

*Proof of Theorem 4.20.* Fixed $t\in(0,T]$, for every $n\in\mathbb{N}$ there exists a unique $j\in\{1,\dots,n\}$ such that $t\in((j-1)\tau_n,j\tau_n]$. In particular, called $\lceil x\rceil$ the superior integer part of the number $x$, it reads as

$$
j(n)=\left\lceil\frac{t}{\tau_n}\right\rceil.
$$

After setting $t_n:=j\tau_n$, we can rewrite (4.52) as

$$
\begin{aligned}
\frac12\|\tilde u_n^+(t)\|_H^2+\frac12(\mathbb{C}eu_n^+(t),eu_n^+(t))_H+\frac12(\mathbb{B}(eu_n^+(t)-w_n^+(t)),eu_n^+(t)-w_n^+(t))_H&\\
+\beta\int_0^{t_n}(\mathbb{B}\dot w_n(\tau),\dot w_n(\tau))_H\,\mathrm{d}\tau\le\mathcal{E}_0+\mathcal{W}_n^+(t),&\qquad(4.85)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{W}_n^+(t):=&\int_0^{t_n}[(N_n^+,\tilde u_n^+-\tilde z_n^+)_{H_N}+((f_1)_n^+,\tilde u_n^+-\tilde z_n^+)_H+\langle\langle (f_2)_n^+,\tilde u_n^+-\tilde z_n^+\rangle\rangle-(h_n^+,e\tilde u_n^+-e\tilde z_n^+)_H]\mathrm{d}\tau\\
&+\int_0^{t_n}\left[((\mathbb{C}+\mathbb{B})eu_n^+-\mathbb{B}w_n^+,e\tilde z_n^+)_H+(\dot{\tilde u}_n,\tilde z_n^+)_H\right]\mathrm{d}\tau
\end{aligned}
$$

Thanks to (4.50) and (4.77), we have

$$
\begin{aligned}
&\|w_n(t)-w_n^+(t)\|_H^2=\|w_n^j+(t-j\tau_n)\delta w_n^j-w_n^j\|_H^2\le\tau_n^2\|\delta w_n^j\|_H^2\le C\tau_n\xrightarrow[n\to\infty]{}0,\\
&\|u_n(t)-u_n^+(t)\|_H=\|u_n^j+(t-j\tau_n)\delta u_n^j-u_n^j\|_H\le\tau_n\|\delta u_n^j\|_H\le C\tau_n\xrightarrow[n\to\infty]{}0,\\
&\|\tilde u_n(t)-\tilde u_n^+(t)\|_{(U_0^D)'}^2=\|\delta u_n^j+(t-j\tau_n)\delta^2u_n^j-\delta u_n^j\|_{(U_0^D)'}^2\le\tau_n^2\|\delta^2u_n^j\|_{(U_0^D)'}^2\le\tilde C\tau_n\xrightarrow[n\to\infty]{}0.
\end{aligned}
$$

The last convergences and (4.79) imply

$$
u_n^+(t)\xrightarrow[n\to\infty]{H}u(t),\qquad w_n^+(t)\xrightarrow[n\to\infty]{H}w(t),\qquad \tilde u_n^+(t)\xrightarrow[n\to\infty]{(U_0^D)'}\dot u(t),
$$

and since $\|u_n^+(t)\|_{U_T}+\|\tilde u_n^+(t)\|_H\le C$ for every $n\in\mathbb{N}$, we get

$$
u_n^+(t)\xrightarrow[n\to\infty]{U_T}u(t),\qquad w_n^+(t)\xrightarrow[n\to\infty]{H}w(t),\qquad \tilde u_n^+(t)\xrightarrow[n\to\infty]{H}\dot u(t).\tag{4.86}
$$

By (4.86) and the lower semicontinuity property of $v\mapsto\|v\|_H^2$, $v\mapsto(\mathbb{C}v,v)_H$, and $v\mapsto(\mathbb{B}v,v)_H$, we conclude

$$
\|\dot u(t)\|_H^2\le\liminf_{n\to\infty}\|\tilde u_n^+(t)\|_H^2,\tag{4.87}
$$

$$
(\mathbb{C}eu(t),eu(t))_H\le\liminf_{n\to\infty}(\mathbb{C}eu_n^+(t),eu_n^+(t))_H,\tag{4.88}
$$

$$
(\mathbb{B}(eu(t)-w(t)),eu(t)-w(t))_H\le\liminf_{n\to\infty}(\mathbb{B}(eu_n^+(t)-w_n^+(t)),eu_n^+(t)-w_n^+(t))_H.\tag{4.89}
$$

Moreover, thanks to Lemma 4.17, and in particular by (4.70) we get

$$\int_0^t (\mathbb{B}\dot w(s), \dot w(s))_H \,\mathrm{d}s \le \liminf_{n\to\infty} \int_0^t (\mathbb{B}\dot w_n(s), \dot w_n(s))_H \,\mathrm{d}s \le \liminf_{n\to\infty} \int_0^{t_n} (\mathbb{B}\dot w_n(s), \dot w_n(s))_H \,\mathrm{d}s, \tag{4.90}$$

since $t \le t_n$ and $v \mapsto \int_0^t (\mathbb{B}v, v)_H \,\mathrm{d}s$ is a non negative quadratic form on $L^2(0,T;H)$.

Let us study the right–hand side of (4.85). Given that we have

$$\chi_{[0,t_n]}(f_1)_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} f_1, \qquad \tilde u_n^+ - \tilde z_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \dot u - \dot z,$$

we can deduce

$$\int_0^{t_n} ((f_1)_n^+, \tilde u_n^+ - \tilde z_n^+)_H \,\mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^t (f_1, \dot u - \dot z)_H \,\mathrm{d}\tau. \tag{4.91}$$

In a similar way, we can prove

$$\int_0^{t_n} (h_n^+, e\tilde z_n^+) \mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^t (h, e\dot z)_H \,\mathrm{d}\tau \tag{4.92}$$

$$\int_0^{t_n} ((\mathbb{C}+\mathbb{B})eu_n^+ - \mathbb{B}w_n^+, e\tilde z_n^+)_H \,\mathrm{d}\tau \xrightarrow[n\to\infty]{} \int_0^t ((\mathbb{C}+\mathbb{B})eu - \mathbb{B}w, e\dot z)_H \,\mathrm{d}\tau, \tag{4.93}$$

since the following convergences hold

$$\chi_{[0,t_n]} e\tilde z_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]} e\dot z, \qquad h_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} h, \qquad (\mathbb{C}+\mathbb{B})eu_n^+ - \mathbb{B}w_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} (\mathbb{C}+\mathbb{B})eu - \mathbb{B}w.$$

Thanks to the discrete integration by parts formulas (4.57)–(4.60) we have

$$\int_0^{t_n} (\dot{\tilde u}_n, \tilde z_n^+)_H \,\mathrm{d}\tau = (\tilde u_n^+(t), \tilde z_n^+(t))_H - (u^1, \dot z(0))_H - \int_0^{t_n} (\tilde u_n^-, \dot{\tilde z}_n)_H \,\mathrm{d}\tau,$$

$$\int_0^{t_n} (h_n^+, e\tilde u_n^+)_H \,\mathrm{d}\tau = (eu_n^+(t), h_n^+(t))_H - (eu^0, h(0))_H - \int_0^{t_n} (eu_n^-, \tilde h_n^+)_H \,\mathrm{d}\tau,$$

$$\int_0^{t_n} \langle\langle (f_2)_n^+, \tilde u_n^+ - \tilde z_n^+\rangle\rangle \mathrm{d}\tau = \langle\langle (f_2)_n^+(t), u_n^+(t) - z_n^+(t)\rangle\rangle - \langle\langle f_2(0), u^0 - z(0)\rangle\rangle - \int_0^{t_n} \langle\langle (\widetilde{f_2})_n^+, u_n^- - z_n^-\rangle\rangle \mathrm{d}\tau,$$

$$\int_0^{t_n} (N_n^+, \tilde u_n^+ - \tilde z_n^+)_{H_N} \,\mathrm{d}\tau = (N_n^+(t), u_n^+(t) - z_n^+(t))_{H_N} - (N(0), u^0 - z(0))_{H_N} - \int_0^{t_n} (\tilde N_n^+, u_n^- - z_n^-)_{H_N} \,\mathrm{d}\tau.$$

By arguing as before we can deduce

$$\int_0^{t_n} (\dot{\tilde u}_n, \tilde z_n^+)_H \,\mathrm{d}\tau \xrightarrow[n\to\infty]{} (\dot u(t), \dot z(t))_H - (u^1, \dot z(0))_H - \int_0^t (\dot u, \ddot z)_H \,\mathrm{d}\tau, \tag{4.94}$$

$$\int_0^{t_n} (h_n^+, e\tilde u_n^+)_H \,\mathrm{d}\tau \xrightarrow[n\to\infty]{} (h(t), eu(t))_H - (h(0), eu^0)_H - \int_0^t (\dot h, eu)_H \,\mathrm{d}\tau, \tag{4.95}$$

$$\int_0^{t_n} \langle\langle (f_2)_n^+, \tilde u_n^+ - \tilde z_n^+\rangle\rangle \mathrm{d}\tau \xrightarrow[n\to\infty]{} \langle\langle f_2(t), u(t) - z(t)\rangle\rangle - \langle\langle f_2(0), u^0 - z(0)\rangle\rangle - \int_0^t \langle\langle \dot f_2, u - z\rangle\rangle \mathrm{d}\tau, \tag{4.96}$$

$$\int_0^{t_n} (N_n^+, \tilde u_n^+ - \tilde z_n^+)_{H_N} \,\mathrm{d}\tau \xrightarrow[n\to\infty]{} (N(t), u(t) - z(t))_{H_N} - (N(0), u^0 - z(0))_{H_N} - \int_0^t (\dot N, u - z)_{H_N} \,\mathrm{d}\tau, \tag{4.97}$$

thanks to Lemma 4.17, to (4.86), and to the following convergences

$$\|\tilde z_n^+(t) - \dot z(t)\|_H = \left\| \frac{z(j\tau_n) - z((j-1)\tau_n)}{\tau_n} - \dot z(t) \right\|_H \le \int_{(j-1)\tau_n}^{j\tau_n} \|\dot z(\tau) - \dot z(t)\|_H \,\mathrm{d}\tau \xrightarrow[n\to\infty]{} 0,$$

$$\|h_n^+(t) - h(t)\|_H = \|\mathbb{B}w^0\|_H |\mathrm{e}^{-\frac{j\tau_n}{\beta}} - \mathrm{e}^{-\frac{t}{\beta}}| \le \frac{L}{\beta} \|\mathbb{B}w^0\|_H |t - j\tau_n| \le C\tau_n \xrightarrow[n\to\infty]{} 0,$$

$$\|(f_2)_n^+(t) - f_2(t)\|_{U_T'} = \|f_2(j\tau_n) - f_2(t)\|_{U_T'} \le (j\tau_n - t)^{\frac12} \|\dot f_2\|_{L^2(0,T;U_T')} \le C\tau_n^{\frac12} \xrightarrow[n\to\infty]{} 0,$$

$$\|z_n^+(t) - z(t)\|_{H_N} \le C_{tr} \|z_n^+(t) - z(t)\|_{U_T} = C_{tr} \|z(j\tau_n) - z(t)\|_{U_T} \le C_{tr} (j\tau_n - t)^{\frac12} \|\dot z\|_{L^2(0,T;U_T)} \xrightarrow[n\to\infty]{} 0,$$

$$\|N_n^+(t) - N(t)\|_{H_N} = \|N(j\tau_n) - N(t)\|_{H_N} \le (j\tau_n - t)^{\frac{1}{2}} \|\dot{N}\|_{L^2(0,T;H_N)} \xrightarrow[n\to\infty]{} 0,$$

$$\chi_{[0,t_n]}\dot{\tilde{z}}_n \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]}\ddot{z}, \quad \chi_{[0,t_n]}\tilde{h}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H)} \chi_{[0,t]}\dot{h}, \quad \chi_{[0,t_n]}(\widetilde{f_2})_n^+ \xrightarrow[n\to\infty]{L^2(0,T;U_T')} \chi_{[0,t]}\dot{f}_2,$$

$$z_n^- \xrightarrow[n\to\infty]{L^2(0,T;U_T)} z, \quad \chi_{[0,t_n]}\tilde{N}_n^+ \xrightarrow[n\to\infty]{L^2(0,T;H_N)} \chi_{[0,t]}\dot{N}.$$

By combining (4.85) and (4.87)–(4.97) we deduce

$$\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}(\mathbb{C}eu(t), eu(t))_H + \frac{1}{2}(\mathbb{B}(eu(t)-w(t)), eu(t)-w(t))_H + \beta\int_0^t (\mathbb{B}\dot{w}(s), \dot{w}(s))_H\,\mathrm{d}s \le \mathcal{E}_0 + \mathcal{W}_{tot}(t)$$

which is the energy–dissipation inequality (4.80) for $t \in (0,T]$. Finally, for $t = 0$ the inequality trivially holds since $u(0) = u^0$ and $\dot{u}(0) = u^1$. □

**Remark 4.22.** Thanks to the last theorem and to the equivalence between the viscoelastic dynamic system (3.7)–(3.11) and the vector-valued system (4.37)–(4.41), we can derive an energy–dissipation inequality for a solution to our viscoelastic dynamic system. As can be seen from (4.42) and the proof of Theorem 4.14 it is not restrictive to assume $w^0 = 0$.

Let $(u, w)$ be the solution to (4.37) provided by Lemma 4.17, then it satisfies the energy–dissipation inequality (4.80). Moreover, thanks to Theorem 4.14 the function $u$ is a solution to (3.7). Therefore if we substitute (4.45) in (4.80) we get for the conservative part

$$\begin{aligned}\mathcal{E}_{u,w}(t) &= \frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}(\mathbb{C}eu(t), eu(t))_H + \frac{1}{2}(\mathbb{B}(eu(t)-w(t)), eu(t)-w(t))_H \\ &= \frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}((\mathbb{C}+\mathbb{B})eu(t), eu(t))_H - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-s}{\beta}}(\mathbb{B}eu(s), eu(t))_H\,\mathrm{d}s \\ &\quad + \frac{1}{2\beta^2}\int_0^t\int_0^t \mathrm{e}^{-\frac{2t-s-\tau}{\beta}}(\mathbb{B}eu(\tau), eu(s))_H\,\mathrm{d}\tau\mathrm{d}s \end{aligned} \tag{4.98}$$

and for the dissipation

$$\begin{aligned}\beta\int_0^t (\mathbb{B}\dot{w}(s), \dot{w}(s))_H\,\mathrm{d}s &= \int_0^t (\mathbb{B}\dot{w}(s), eu(s)-w(s))_H\,\mathrm{d}s = \int_0^t (\mathbb{B}\dot{w}(s), eu(s))_H\,\mathrm{d}s - \int_0^t (\mathbb{B}\dot{w}(s), w(s))_H\,\mathrm{d}s \\ &= \frac{1}{\beta}\int_0^t \left(\mathbb{B}eu(s) - \int_0^s \frac{1}{\beta}\mathrm{e}^{-\frac{s-\tau}{\beta}}\mathbb{B}eu(\tau)\mathrm{d}\tau, eu(s)\right)_H \mathrm{d}s - \frac{1}{2}(\mathbb{B}w(t), w(t))_H \\ &= \frac{1}{\beta}\int_0^t (\mathbb{B}eu(s), eu(s))_H\,\mathrm{d}s - \frac{1}{\beta^2}\int_0^t\int_0^s \mathrm{e}^{-\frac{s-\tau}{\beta}}(\mathbb{B}eu(\tau), eu(s))_H\,\mathrm{d}\tau\mathrm{d}s \\ &\quad - \frac{1}{2\beta^2}\int_0^t\int_0^t \mathrm{e}^{-\frac{2t-s-\tau}{\beta}}(\mathbb{B}eu(\tau), eu(s))_H\,\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{4.99}$$

By substituting the same informations in the total work, we obtain

$$\begin{aligned}\mathcal{W}_{tot}(t) &= \int_0^t \left[(f_1, \dot{u}-\dot{z})_H + ((\mathbb{C}+\mathbb{B})eu, e\dot{z})_H - \int_0^s \frac{1}{\beta}\mathrm{e}^{-\frac{s-\tau}{\beta}}(\mathbb{B}eu(\tau), e\dot{z}(s))_H\,\mathrm{d}\tau\right]\mathrm{d}s \\ &\quad - \int_0^t [(\dot{u}, \ddot{z})_H + \langle\langle \dot{f}_2, u-z\rangle\rangle + (\dot{N}, u-z)_{H_N}]\mathrm{d}s + (\dot{u}(t), \dot{z}(t))_H - (u^1, \dot{z}(0))_H \\ &\quad + \langle\langle f_2(t), u(t)-z(t)\rangle\rangle - \langle\langle f_2(0), u^0 - z(0)\rangle\rangle + (N(t), u(t)-z(t))_{H_N} - (N(0), u^0 - z(0))_{H_N}. \end{aligned} \tag{4.100}$$

After defining the following energy

$$\begin{aligned}\mathcal{E}(t) &:= \frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}((\mathbb{C}+\mathbb{B})eu(t), eu(t))_H \\ &\quad - \int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-s}{\beta}}(\mathbb{B}eu(s), eu(t))_H\,\mathrm{d}s + \frac{1}{2\beta^2}\int_0^t\int_0^t \mathrm{e}^{-\frac{2t-s-\tau}{\beta}}(\mathbb{B}eu(\tau), eu(s))_H\,\mathrm{d}\tau\mathrm{d}s, \end{aligned} \tag{4.101}$$

and the dissipative term

$$\mathcal{D}(t) := \frac{1}{\beta}\int_0^t (\mathbb{B}eu(s), eu(s))_H\,\mathrm{d}s - \frac{1}{\beta^2}\int_0^t\int_0^s \mathrm{e}^{-\frac{s-\tau}{\beta}}(\mathbb{B}eu(\tau), eu(s))_H\,\mathrm{d}\tau\mathrm{d}s$$

$$- \frac{1}{2\beta^2} \int_0^t \int_0^t \mathrm{e}^{-\frac{2t-s-\tau}{\beta}} (\mathbb{B}eu(\tau), eu(s))_H \, \mathrm{d}\tau \mathrm{d}s, \tag{4.102}$$

thanks to (4.98), (4.99), and (4.100) we can rephrase the energy–dissipation inequality (4.80) as follows

$$\mathcal{E}(t) + \mathcal{D}(t) \leq \mathcal{E}(0) + \mathcal{W}(t),$$

where the total work $\mathcal{W}$, depending this time just on the function $u$, is expressed in (4.100).

Finally, thanks to Theorem 4.20 we are ready to show that our solution satisfies the initial conditions in a stronger sense than the ones stated in Definition 4.13. To this purpose, we can state the following lemma:

**Lemma 4.23.** *The weak solution $(u, w) \in \mathcal{U} \times \mathcal{AC}$ to the vector-valued system (4.37) of Lemma 4.17, satisfies*

$$\lim_{t \to 0^+} u(t) = u^0 \ \text{in } U_T, \quad \lim_{t \to 0^+} w(t) = w^0 \ \text{in } H, \quad \lim_{t \to 0^+} \dot{u}(t) = u^1 \ \text{in } H. \tag{4.103}$$

*Proof.* Firstly notice that the second condition in (4.103) is already satisfied because $w \in C^0([0,T];H)$. Moreover, by sending $t \to 0^+$ into the energy–dissipation inequality (4.80) and using that $u \in C^0_w([0,T];U_T)$ and $\dot{u} \in C^0_w([0,T];H)$ we deduce

$$\begin{aligned}
\mathcal{E}_{u,w}(0) &= \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}(\mathbb{C}eu^0, eu^0)_H + \frac{1}{2}(\mathbb{B}(eu^0 - w^0), eu^0 - w^0)_H \\
&\leq \frac{1}{2}\left[\liminf_{t \to 0^+}\|\dot{u}(t)\|_H^2 + \liminf_{t \to 0^+}(\mathbb{C}eu(t), eu(t))_H + \lim_{t \to 0^+}(\mathbb{B}(eu(t) - w(t)), eu(t) - w(t))_H\right] \\
&\leq \liminf_{t \to 0^+}\left[\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}(\mathbb{C}eu(t), eu(t))_H + \frac{1}{2}(\mathbb{B}(eu(t) - w(t)), eu(t) - w(t))_H\right] \\
&= \liminf_{t \to 0^+} \mathcal{E}_{u,w}(t) \leq \limsup_{t \to 0^+} \mathcal{E}_{u,w}(t) \leq \mathcal{E}_{u,w}(0),
\end{aligned}$$

since the right–hand side of (4.80) is continuous in $t$, and $u(0) = u^0$ and $\dot{u}(0) = u^1$. Therefore, there exists $\lim_{t \to 0^+} \mathcal{E}_{u,w}(t) = \mathcal{E}_{u,w}(0)$. Thanks to the lower semicontinuity of the real functions

$$t \mapsto \|\dot{u}(t)\|_H^2, \quad t \mapsto (\mathbb{C}eu(t), eu(t))_H, \quad t \mapsto (\mathbb{B}(eu(t) - w(t)), eu(t) - w(t))_H$$

we can say

$$\begin{aligned}
\mathcal{E}_{u,w}(0) &\leq \frac{1}{2}\liminf_{t \to 0^+}\|\dot{u}(t)\|_H^2 + \liminf_{t \to 0^+}\left[\frac{1}{2}(\mathbb{C}eu(t), eu(t))_H + \frac{1}{2}(\mathbb{B}(eu(t) - w(t)), eu(t) - w(t))_H\right] \\
&\leq \frac{1}{2}\limsup_{t \to 0^+}\|\dot{u}(t)\|_H^2 + \liminf_{t \to 0^+}\left[\frac{1}{2}(\mathbb{C}eu(t), eu(t))_H + \frac{1}{2}(\mathbb{B}(eu(t) - w(t)), eu(t) - w(t))_H\right] \\
&\leq \limsup_{t \to 0^+}\left[\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}(\mathbb{C}eu(t), eu(t))_H + \frac{1}{2}(\mathbb{B}(eu(t) - w(t)), eu(t) - w(t))_H\right] = \mathcal{E}_{u,w}(0).
\end{aligned} \tag{4.104}$$

By (4.104) we deduce

$$\lim_{t \to 0^+}\|\dot{u}(t)\|_H^2 = \|u^1\|_H^2.$$

If we argue in the same way of (4.104), we can also show the validity of the following limit

$$\lim_{t \to 0^+}(\mathbb{C}eu(t), eu(t))_H = (\mathbb{C}eu^0, eu^0)_H.$$

Finally, since we have

$$\dot{u}(t) \xrightarrow[t \to 0^+]{H} u^1, \quad eu(t) \xrightarrow[t \to 0^+]{H} eu^0$$

and $u \in C^0([0,T];H)$, we deduce (4.103). In particular the functions $u \colon [0,T] \to U_T$ and $\dot{u} \colon [0,T] \to H$ are continuous at $t = 0$, which means (4.44). ☐

We can finally prove Theorem 4.15.

*Proof of Theorem 4.15.* It is enough to combine Lemma 4.18 and Lemma 4.23. ☐

**Remark 4.24.** We have proved Theorem 4.15 for the $d$-dimensional linear elastic case, namely when the displacement $u$ is a vector–valued function. The same result is true with identical proofs in the antiplane case, that is when the displacement $u$ is a scalar function and satisfies (1.12).

## 5. Example of a moving crack

We conclude this paper with an example of a moving crack $\{\Gamma_t\}_{t\in[0,T]}$ and a weak solution to the viscoelastic dynamic system (3.7)–(3.11), which satisfies the energy–dissipation balance of Griffith's dynamic criterion, as it happens in [7] for the purely elastic case. In dimension $d = 2$ we consider an antiplane evolution, which means that the displacement $u$ is scalar, and we take $\Omega := \{x \in \mathbb{R}^2 : |x| < R\}$, with $R > 0$. We fix a constant $0 < c < 1$ such that $cT < R$, and we set

$$\Gamma_t := \{(\sigma, 0) \in \overline{\Omega} : \sigma \le ct\}.$$

Let us define the following function

$$S(x_1, x_2) := Im(\sqrt{x_1 + ix_2}) = \frac{1}{\sqrt{2}} \frac{x_2}{\sqrt{|x| + x_1}} \quad x \in \mathbb{R}^2 \setminus \{(\sigma, 0) : \sigma \le 0\},$$

where $Im$ denotes the imaginary part of a complex number. Notice that $S \in U_0 \setminus H^2(\Omega \setminus \Gamma_0)$, and it is a weak solution to

$$\begin{cases} \Delta S = 0 & \text{in } \Omega \setminus \Gamma_0, \\ \nabla S \cdot \nu = \partial_2 S = 0 & \text{on } \Gamma_0. \end{cases}$$

Let us consider the function

$$u(t, x) := \frac{2}{\sqrt{\pi}} S\left(\frac{x_1 - ct}{\sqrt{1 - c^2}}, x_2\right) \quad t \in [0, T],\, x \in \Omega \setminus \Gamma_t$$

and let $z(t)$ be its restriction to $\partial\Omega$. Since $u(t)$ has a singularity only at the crack tip $(ct, 0)$, the function $z(t)$ can be seen as the trace on $\partial\Omega$ of a function belonging to $H^2(0, T; H) \cap H^1(0, T; U_0)$, still denoted by $z(t)$. It is easy to see that $u$ solves the wave equation

$$\ddot{u}(t) - \Delta u(t) = 0 \quad \text{in } \Omega \setminus \Gamma_t,\, t \in (0, T),$$

with boundary conditions

$$\begin{aligned} u(t) &= z(t) && \text{on } \partial\Omega, \quad t \in (0, T), \\ \frac{\partial u}{\partial \nu}(t) &= \nabla u(t) \cdot \nu = 0 && \text{on } \Gamma_t, \quad t \in (0, T), \end{aligned}$$

and initial data

$$\begin{aligned} u^0(x_1, x_2) &:= \frac{2}{\sqrt{\pi}} S\left(\frac{x_1}{\sqrt{1 - c^2}}, x_2\right) \in U_0, \\ u^1(x_1, x_2) &:= -\frac{2}{\sqrt{\pi}} \frac{c}{\sqrt{1 - c^2}} \partial_1 S\left(\frac{x_1}{\sqrt{1 - c^2}}, x_2\right) \in H. \end{aligned}$$

Notice that $u \in \mathcal{U}$; moreover $u - z \in \mathcal{U}^D$, where now $\partial_D\Omega = \partial\Omega$. In this case $u$ verifies for every $v \in \mathcal{U}^D$ such that $v(0) = v(T) = 0$ the following identity

$$-\int_0^T (\dot{u}(t), \dot{v}(t))_H \,\mathrm{d}t + \int_0^T (\nabla u(t), \nabla v(t))_H \,\mathrm{d}t - \frac{1}{2}\int_0^T \int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\nabla u(\tau), \nabla v(t))_H \,\mathrm{d}\tau\mathrm{d}t = \int_0^T \langle\langle f(t), v(t)\rangle\rangle \mathrm{d}t,$$

where for every $t \in [0, T]$ the functional $f(t) \in U_T'$ is defined by

$$\langle\langle f(t), \varphi\rangle\rangle := -\frac{1}{2}\int_0^t \frac{1}{\beta} \mathrm{e}^{-\frac{t-\tau}{\beta}} (\nabla u(\tau), \nabla\varphi)_H \,\mathrm{d}\tau \qquad \forall \varphi \in U_T. \tag{5.1}$$

By noticing that for every $t \in [0, T]$ and for every $\varphi \in U_T$ we have

$$\langle\langle \dot{f}(t), \varphi\rangle\rangle := -\frac{1}{2\beta}(\nabla u(t), \nabla\varphi)_H + \frac{1}{2\beta^2}\int_0^t \mathrm{e}^{-\frac{t-\tau}{\beta}} (\nabla u(\tau), \nabla\varphi)_H \,\mathrm{d}\tau, \tag{5.2}$$

we can say that $f \in H^1(0, T; U_T')$, therefore $u$ is a weak solution to the viscoelastic dynamic system (3.7) with forcing term given by $f$, in the antiplane case, according to Definition 3.3.

Thanks to the computations done in [7, Section 4], we know that $u$ satisfies for every $t \in [0, T]$ the following energy–dissipation balance for the undamped equation, where $ct$ coincides with the length of $\Gamma_t \setminus \Gamma_0$

$$\frac{1}{2}\|\dot{u}(t)\|_H^2 + \frac{1}{2}\|\nabla u(t)\|_H^2 + ct = \frac{1}{2}\|\dot{u}(0)\|_H^2 + \frac{1}{2}\|\nabla u(0)\|_H^2 + \int_0^t (\frac{\partial u}{\partial \nu}(s), \dot{z}(s))_{H_{\partial\Omega}} \,\mathrm{d}s, \tag{5.3}$$

where

$$\int_0^t (\frac{\partial u}{\partial \nu}(s), \dot z(s))_{H_{\partial\Omega}}\,\mathrm{d}s = \int_0^t (\nabla u(s), \nabla \dot z(s))_H\,\mathrm{d}s - \int_0^t (\dot u(s), \ddot z(s))_H\,\mathrm{d}s + (\dot u(t), \dot z(t))_H - (\dot u(0), \dot z(0))_H.$$

Thanks to (5.1) and (5.2), for every $t \in [0,T]$ we can write

$$\langle\langle f(t), u(t) - z(t)\rangle\rangle - \int_0^t \langle\langle \dot f(s), u(s) - z(s)\rangle\rangle \mathrm{d}s = -\frac{1}{2}\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-s}{\beta}}(\nabla u(s), \nabla u(t))_H \mathrm{d}s + \frac{1}{2\beta}\int_0^t \|\nabla u(s)\|_H^2 \mathrm{d}s$$
$$- \frac{1}{2\beta}\int_0^t\int_0^s \frac{1}{\beta}\mathrm{e}^{-\frac{s-\tau}{\beta}}(\nabla u(\tau), \nabla u(s))_H \mathrm{d}\tau\mathrm{d}s + \frac{1}{2}\int_0^t\int_0^s \frac{1}{\beta}\mathrm{e}^{-\frac{s-\tau}{\beta}}(\nabla u(\tau), \nabla \dot z(s))_H \mathrm{d}\tau\mathrm{d}s. \tag{5.4}$$

By adding (5.4) to both hand sides of (5.3), we deduce that $u$ satisfies for every $t \in [0,T]$ the following Griffith's energy–dissipation balance

$$\frac{1}{2}\|\dot u(t)\|_H^2 + \frac{1}{2}\|\nabla u(t)\|_H^2 - \frac{1}{2}\int_0^t \frac{1}{\beta}\mathrm{e}^{-\frac{t-s}{\beta}}(\nabla u(s), \nabla u(t))_H \mathrm{d}s + \frac{1}{2\beta}\int_0^t \|\nabla u(s)\|_H^2 \mathrm{d}s$$
$$- \frac{1}{2\beta}\int_0^t\int_0^s \frac{1}{\beta}\mathrm{e}^{-\frac{s-\tau}{\beta}}(\nabla u(\tau), \nabla u(s))_H \mathrm{d}\tau\mathrm{d}s + ct = \frac{1}{2}\|u^1\|_H^2 + \frac{1}{2}\|\nabla u^0\|_H^2 + \mathcal{W}(t), \tag{5.5}$$

where in this case the total work takes the form

$$\mathcal{W}(t) := \int_0^t \Big[(\nabla u(s), \nabla \dot z(s))_H - \langle\langle \dot f(s), u(s) - z(s)\rangle\rangle - (\dot u(s), \ddot z(s))_H - \int_0^s \frac{1}{2\beta}\mathrm{e}^{-\frac{s-\tau}{\beta}}(\nabla u(\tau), \nabla \dot z(s))_H\,\mathrm{d}\tau\Big]\mathrm{d}s$$
$$+ \langle\langle f(t), u(t) - z(t)\rangle\rangle + (\dot u(t), \dot z(t))_H - (\dot u(0), \dot z(0))_H.$$

Thanks to (5.5), by using the same notation in (4.101) and (4.102) of Remark 4.22, we can write

$$\mathcal{E}(t) + \mathcal{D}(t) + \mathcal{H}^1(\Gamma_t \setminus \Gamma_0) = \mathcal{E}(0) + \mathcal{W}(t),$$

and then we can conclude that in this model Griffith's dynamic energy–dissipation balance can be satisfied by a moving crack.

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