Multi-bump solitons to linearly coupled systems of Nonlinear Schrödinger equations

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Abstract

This paper is devoted to study a class of systems of nonlinear Schrödinger equations:

$$\begin{cases} -\Delta u + u - u^3 = \epsilon v, \\ -\Delta v + v - v^3 = \epsilon u, \end{cases}$$

in $\mathbb{R}^n$ with dimension $n = 1, 2, 3$. Our main result is concerned with the existence of solitary waves with a feature that is novel for autonomous equations: one component is a multi-bump, while the other one has a negative peak. More precisely, denote by $\mathcal{P}$ a regular polytope centered at the origin of $\mathbb{R}^n$ such that its side is greater than the radius. Then, there exists a solution with a component having bumps located near the vertices of $\xi\mathcal{P}$, where $\xi \sim \log(1/\epsilon)$.

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Contents

1 Introduction

Nonlinear Schrödinger Equations (NLS) have been broadly investigated in many aspects, like existence of solitary waves (see e.g. [6, 23] which contain several further references), concentration and multi-bump phenomena for semiclassical states (see e.g. [7, 8, 10, 12, 15, 19]), to cite a few. In particular, the existence of multi-bump solutions requires the presence of a suitable potential depending upon x which breaks the symmetry invariance of the autonomous NLS.

On the other hand, the study of systems of NLS equations begun quite recently. Of course, the results one can achieve depend on the way the system is coupled. The case in which the coupling is nonlinear has been studied in [5, 18, 21, 22], see also [4, 16].

Here, motivated by applications to nonlinear optics, see [2], we deal with a class of Nonlinear Schrödinger Equations (NLS) which are linearly coupled, namely

$$\begin{cases} -u_1'' + u_1 - u_1^3 = \lambda u_2, \\ -u_2'' + u_2 - u_2^3 = \lambda u_1. \end{cases} \tag{1.1}$$

Systems like (1.1) arise in nonlinear optics. For example, the propagation of optical pulses in nonlinear dual-core fiber can be described by two linearly coupled NLS equations like

$$\begin{cases} i\phi_z + \phi_{xx} + |\phi|^2\phi + \kappa\psi &= 0, \\ \\ i\psi_z + \psi_{xx} + |\psi|^2\psi + \kappa\phi &= 0, \end{cases} \tag{1.2}$$

where $\phi(z, x)$ and $\psi(z, x)$ are the complex valued envelope functions, and κ is the (normalized) coupling coefficient between the two cores. We will look for standing waves of the form

$$\phi(z,x)=\widehat{u}_1(x)e^{i\omega z},\quad \psi(z,x)=\widehat{u}_2(x)e^{i\omega z}, \tag{1.3}$$

where $\widehat{u}_1(x)$, $\widehat{u}_2(x)$ are real valued functions and $\omega>0$. Substituting (1.3) into (1.2), and setting

$$u_1(x)=\frac{1}{\sqrt{\omega}}\,\widehat{u}_1\left(\frac{x}{\sqrt{\omega}}\right),\quad u_2(x)=\frac{1}{\sqrt{\omega}}\,\widehat{u}_2\left(\frac{x}{\sqrt{\omega}}\right),\quad \lambda=\frac{\kappa}{\omega},$$

we find exactly (1.1).

More precisely, we are interested in the existence of *bounded* states, namely solutions (u,v) such that $u,u',v,v'\in L^2(\mathbb{R})$ and for this reason we will search solutions (u,v) of (1.1) such that $u,v\in W^{1,2}(\mathbb{R})$.

Let

$$U_\alpha(x)=\frac{\sqrt{2\alpha}}{\cosh(\sqrt{\alpha}\,x)} \tag{1.4}$$

denote the even positive solution of $-u''+\alpha u=u^3,\ u\in W^{1,2}(\mathbb{R})$. We will also set $U=U_1$. For $\epsilon=0$ system (1.1) is decoupled and has the *trivial* solutions $(0,0)$, $(\pm U,0)$ and $(0,\pm U)$. Furthermore, for all $\lambda>0$, (1.1) has two families of soliton like solutions, given by

$$\begin{aligned}&(U_{1-\lambda},U_{1-\lambda}),\ (-U_{1-\lambda},-U_{1-\lambda}),\quad \text{for } 0\le\lambda\le 1,\ \text{(symmetric states)}\\ &(U_{1+\lambda},-U_{1+\lambda}),\ (-U_{1+\lambda},U_{1+\lambda}),\quad \text{for } \lambda\ge 0,\ \text{(anti-symmetric states)}.\end{aligned}$$

We are mainly interested in the properties of the system (1.1) that are not shared by a single NLS equation. For this reason, the main purpose of this paper is to prove that, for $\lambda>0$ sufficiently small, (1.1) possesses new solutions, different from the previous ones, with the feature that one component is a multi-bump soliton, in a sense that is made precise in Theorem 1.1. Up to our knowledge, there is no similar result in the literature. More precisely, in [2] a rigorous analysis on the bifurcation from the family of symmetric states is carried out, providing the existence of solutions having both the component positive and with a single peak. Let us also mention that in such paper there is an indication, based on numerical computations, that solutions whose first component has two bumps, should bifurcate from the antisymmetric states at $\lambda=1$.

Our main result dealing with the ODE case is concerned with (1.1) with $\lambda=\epsilon$, namely

$$\begin{cases}-u_1''+u_1-u_1^3=\epsilon u_2,\\ -u_2''+u_2-u_2^3=\epsilon u_1.\end{cases} \tag{1.5}$$

Theorem 1.1 *There exists $\epsilon_0>0$ such that if $0<\epsilon<\epsilon_0$, system (1.5) has a solution $(u_{1,\epsilon},u_{2,\epsilon})\in W^{1,2}(\mathbb{R})\times W^{1,2}(\mathbb{R})$ such that $u_{1,\epsilon}\sim U(x+\xi_\epsilon)+U(x-\xi_\epsilon)$, $u_{2,\epsilon}\sim -U(x)$, where*

$$\frac{\log(1/\epsilon)}{1+\delta}<\xi_\epsilon<\log(1/\epsilon), \tag{1.6}$$

for a fixed $\delta\in(0,1)$.

Remarks 1.2 (i) We suspect that the solutions $(u_{1,\epsilon}, u_{2,\epsilon})$ could be continued for $\epsilon = \lambda \in (0, 1)$. For $\lambda = 1$, this solution should coincide with the antisymmetric state $(U_2, -U_2)$. This is confirmed by the numerical results carried out in [2].

(ii) In addition to the solutions of Theorem 1.1, (1.5) possesses other solutions for ϵ small bifurcating from $(\pm U, 0)$, $\sim (0, \pm U)$, $(\pm U, \pm U)$, $(\pm U, \mp U)$ (the latter two correspond to symmetric, respectively anti-symmetric, states). See Remark 3.2 in the next section.

(iii) From the point of view of Dynamical Systems, the origin $(0, 0)$ is a saddle-saddle equilibrium of (1.5). This kind of problems have been studied for example in [9], where the existence of a chaotic behavior under suitable assumptions is proved. These assumptions are not satisfied by (1.5). Actually, we believe that, in the case studied in the present paper, there are no solutions with more than two bumps, (see also Remark 3.8 later on).

The proof of Theorem 1.1 is carried out in Section 3 and relies on an abstract perturbation result discussed in Section 2. Roughly, we look for solutions $(u_{1,\epsilon}, u_{2,\epsilon})$ bifurcating from a manifold of functions like $\mathbf{z}_\xi(x) = (U(x + \xi) + U(x - \xi), -U(x))$, where the bifurcation parameter ξ satisfies (1.6). The first component of $\mathbf{z}_\xi$ has a repulsive effect, while the second one, being a negative soliton, has an attractive effect. This attraction, resp. repulsion, depends on the distance between the peaks, and an equilibrium is achieved for a suitable value of $\xi \sim \log(1/\epsilon)$. This heuristic arguments also explain why we cannot find multibump solutions close to $(U(x+\xi)+U(x-\xi), U(x))$. From the mathematical point of view, we use a finite dimensional reduction, which is introduced in Section 2. This procedure leads to the problem of finding critical points of a one-dimensional reduced auxiliary functional.

Section 3 is completed by a final subsection 3.4 where we deal with systems of 3 (or more) NLS equations, showing the existence of several types of solutions, see Theorem 3.9.

The second part of this paper is devoted to the PDE counterpart of (1.5) in dimension $n = 2, 3$, that is:

$$\begin{cases} -\Delta u_1 + u_1 = u_1^3 + \epsilon u_2, \\ -\Delta u_2 + u_2 = u_2^3 + \epsilon u_1, \end{cases} \tag{1.7}$$

In addition to solutions like the ones found in Theorem 1.1, we will show that there exists new solutions with many maxima. Precisely, let $\mathcal{P}$ be a regular polytope centered at the origin (i.e. a regular polygon in $\mathbb{R}^2$ or a platonic solid in $\mathbb{R}^3$). Let $\{p_1, \ldots p_m\}$ be the vertices of $\mathcal{P}$, with sides s and radius r, and assume that

$$s = min\{|p_i - p_j| : i \neq j\} > r = |p_1|. \tag{1.8}$$

Let us explicitly point out that (1.8) is satisfied by the regular polygons in $\mathbb{R}^2$ with less than 6 sides and by all the regular polyhedra in $\mathbb{R}^3$ (Platonic solids) with the exception of the the dodecahedron.

Theorem 1.3 *Let the regular polytope* $\mathcal{P}$ *satisfy* (1.8). *There exists* $\epsilon_0 > 0$ *such that if* $0 < \epsilon < \epsilon_0$, *system* (1.7) *has a solution* $(u_{1,\epsilon}, u_{2,\epsilon}) \in W^{1,2}(\mathbb{R}^n) \times W^{1,2}(\mathbb{R}^n)$

such that $u_{2,\epsilon} \sim -U(x)$*, while* $u_{1,\epsilon}$ *is a multibump with maxima located near* ξp_i*, with* ξ *satisfying:*

$$\xi \in T_\epsilon := \left(\frac{\log(1/\epsilon)}{s-r+\delta} \, , \, \frac{\log(1/\epsilon)}{s-r} \right), \tag{1.9}$$

where s, r *are defined in* (1.8) *and* $\delta \in (0,1)$ *is a fixed constant.*

As we will see more precisely in Section 4, we cannot repeat the arguments used in the ODE case, because sharper estimates are needed. We overcome this difficulty by using a new manifold which is closer to the solutions we are looking for. Roughly, we first find a pair (V_ϵ, v_ϵ) bifurcating from $(U, 0)$, see Subsection 4.1. Next, we define the manifold of approximate solutions as the set of pairs (z_1, z_2), with $z_1 = \sum V_\epsilon(x - \xi p_i) - v_\epsilon \sim \sum U(\cdot + \xi p_i)$, while $z_2 = \sum v_\epsilon(x - \xi p_i) - V_\epsilon \sim -U$. It is possible to show that the abstract setting applies, see Subsections 4.2 and 4.3. Indeed, starting from the new manifold, we can obtain rather precise estimates that allows us to control again the behavior of the reduced functional, see Subsection 4.4, yielding the conclusion.

As anticipated before, both the ODE and PDE results are in striking contrast with the case of a single NLS equation. Actually, multi-bump solutions arise as solutions of NLS equations like

$$-\epsilon^2 \Delta u + u + V(x) u = u^3,$$

as $\epsilon \to 0$, under the assumption that V has several critical points, see for example [10], [15], [19]. Moreover, semiclassical states with several peaks near a point of maximum of V, have been found in [13]. These peaks converge, as $\epsilon \to 0$, to the critical points, resp. maximum, of V. On the other hand, the bumps of our solutions spread out at infinity and do not depend upon the presence of any potential. Indeed, it is a specific remarkable feature of the systems we are dealing with, that multi-bump solutions exist in the autonomous case as well.

The last section of the paper is an Appendix in which we prove some technical estimates needed in our proofs.

2 Abstract Setting

This section is devoted to outline the abstract procedure we will use to find critical points of a perturbed functional. Though the general ideas are related to those in [6], we cannot give a precise reference and thus we prefer to carry out the arguments in some details.

Consider a Hilbert space H and let $\Phi_\epsilon \in C^2(H, \mathbb{R})$ be a functional. We assume that there exists a smooth manifold $Z \subset H$ such that

$$\|\Phi'_\epsilon(z)\| \to 0, \qquad \text{as} \quad \epsilon \to 0, \quad \forall\, z \in Z. \tag{A1}$$

Our aim is to show that, for $\epsilon \ll 1$ the functional Φ_ϵ has a critical point near a suitable point $z^* \in Z$. For this, let us consider the tangent space $T_z Z$ to Z at z and let W be the orthogonal complement of $T_z Z$, namely

$$W = (T_z Z)^\perp .$$

Looking for critical points of the form $z+w$, where $w \in W$, we first consider the *Auxiliary equation*

$$P\Phi'_\epsilon(z+w)=0, \tag{2.1}$$

where P denotes the projection onto W. In order to solve the Auxiliary equation, we assume that there exists $\gamma>0$ such that $P\Phi''_\epsilon(z)$ is invertible for all $z \in Z$ with inverse satisfying

$$\|[P\Phi''_\epsilon(z)]^{-1}\| \le \gamma, \qquad \forall\, z \in Z. \tag{A2}$$

Setting

$$B_{\epsilon,\gamma}=\{w \in W:\ \|w\| \le 2\gamma\|\Phi'_\epsilon(z)\|\},$$

we further assume that

$$\|\Phi''_\epsilon(z+w)-\Phi''_\epsilon(z)\| \to 0, \quad \text{as} \quad \epsilon \to 0, \quad \forall\, z \in Z,\ \forall w \in B_{\epsilon,\gamma}. \tag{A3}$$

Remark 2.1 The previous assumption is quite natural (take into account that $(A1)$ and $w \in B_{\epsilon,\gamma}$ imply that $\|w\| \to 0$ as $\epsilon \to 0$). In particular, $(A3)$ is always satisfied if Φ_ϵ is of class C^3 and $D^3\Phi_\epsilon$ is bounded on bounded sets.

Let us define the map $S_\epsilon : B_{\epsilon,\gamma} \mapsto W$ by setting

$$S_\epsilon(w)=w-[P\Phi''_\epsilon(z)]^{-1}(P\Phi'_\epsilon(z+w)). \tag{2.2}$$

Clearly, any fixed point of S_ϵ gives rise to a solution of (2.1).

Lemma 2.2 *For ϵ small enough the map S_ϵ maps the ball $B_{\epsilon,\gamma}$ into itself and is a contraction. As a consequence, there exists a unique $w_{\epsilon,z} \in B_{\epsilon,\gamma}$ such that $S_\epsilon(w_{\epsilon,z})=w_{\epsilon,z}$.*

Proof. First, let us show that S_ϵ is a contraction in $B_{\epsilon,\gamma}$ for ϵ small. For $v, w \in B_{\epsilon,\gamma}$ one has

$$\begin{aligned} S'_\epsilon(w)[v] &= v-[P\Phi''_\epsilon(z)]^{-1}(P\Phi''_\epsilon(z+w)[v]) \\ &= [P\Phi''_\epsilon(z)]^{-1}(P\Phi''_\epsilon(z)[v]-P\Phi''_\epsilon(z+w)[v]). \end{aligned}$$

Thus we find

$$\|S'_\epsilon(w)[v]\| \le \gamma\|\Phi''_\epsilon(z)[v]-\Phi''_\epsilon(z+w)[v])\|. \tag{2.3}$$

Then, using (A3), we infer that given $\kappa \in (0,\frac{1}{2})$ there exists $\epsilon_0>0$ such that for all $|\epsilon|<\epsilon_0$ there holds

$$\|S'_\epsilon(w)[\mathbf{v}]\| \le \kappa\,\|v\|. \tag{2.4}$$

This obviously implies that S_ϵ is a contraction, provided $\epsilon \ll 1$.

Next, we show that S_ϵ maps $B_{\epsilon,\gamma}$ into itself. One has

$$\|S_\epsilon(0)\|=\|[P\Phi''_\epsilon(z)]^{-1}(P\Phi'_\epsilon(z))\| \le \gamma\|\Phi'_\epsilon(z)\|.$$

From (2.4), we get:

$$\|S_\epsilon(w) - S_\epsilon(0)\| \le \kappa\, \|w\|,$$

and this yields

$$\begin{aligned}\|S_\epsilon(w)\| &\le \|S_\epsilon(0)\| + \|S_\epsilon(w) - S_\epsilon(0)\| \le \gamma\|\Phi'_\epsilon(z)\| + \kappa\, \|w\| \\ &\le \gamma\|\Phi'_\epsilon(z)\| + 2\kappa\gamma\|\Phi'_\epsilon(z)\| < 2\gamma\|\Phi'_\epsilon(z)\|.\end{aligned}$$

This shows that $S_\epsilon(B_{\epsilon,\gamma}) \subset B_{\epsilon,\gamma}$ and completes the proof of the lemma. ■

Remark 2.3 In particular, one has that

$$\|w_{\epsilon,z}\| \le 2\gamma\|\Phi'_\epsilon(z)\|. \tag{2.5}$$

Moreover, (A1), and the fact that Φ''_ϵ is bounded, imply that $\|w_{\epsilon,z}\| \to 0$ and that $\|D_z w_{\epsilon,z}\| \to 0$ as $\epsilon \to 0$.

Let us now consider the *reduced* functional

$$\widetilde{\Phi}_\epsilon(z) = \Phi_\epsilon(z + w_{\epsilon,z}), \qquad z \in Z.$$

Lemma 2.4 *If $z_\epsilon \in Z$ is a critical point of $\widetilde{\Phi}_\epsilon$, then $u_\epsilon := z_\epsilon + w_{\epsilon,z_\epsilon}$ is a critical point of Φ_ϵ.*

Proof. To simplify the notation, we will assume that Z is one dimensional. One has that

$$(\Phi'_\epsilon(z_\epsilon + w_{\epsilon,z_\epsilon}) \mid z'_\epsilon + D_z w_{\epsilon,z_\epsilon}) = 0, \quad z'_\epsilon \in T_{z_\epsilon}Z,\ \|z'_\epsilon\| = 1.$$

From (2.1) it follows that $\Phi'_\epsilon(z_\epsilon + w_{\epsilon,z_\epsilon}) = \alpha_\epsilon z'_\epsilon$, for some $\alpha_\epsilon \in \mathbb{R}$. Thus we get $\alpha_\epsilon\|z'_\epsilon\|^2 + \alpha_\epsilon(z'_\epsilon \mid D_z w_{\epsilon,z_\epsilon}) = 0$. Using Remark 2.3, we infer that $(z'_\epsilon \mid D_z w_{\epsilon,z_\epsilon}) \to 0$ as $\epsilon \to 0$ and thus $\alpha_\epsilon = 0$. ■

Remark 2.5 The particular case in which $\Phi_\epsilon(u) = \Phi_0(u) + \epsilon\Psi(u)$, and Z is a critical manifold of Φ_0, namely $\Phi'_0(z) = 0$ for every $z \in Z$, has been broadly discussed in [6, Chapter 2]. Assumptions $(A1)$ and $(A3)$ are obviously satisfied, while $(A2)$ follows from the assumption that Z is non-degenerate, in the sense that $\Phi''_0(z)$ is Fredholm operator with index zero such that $\mathrm{Ker}[\Phi''_0(z)] = T_z Z$, for all $z \in Z$. In this way the preceding results cover the abstract setting of [6]. Let us also remind that, in such a specific case, one has that $\widetilde{\Phi}_\epsilon(z) = c + \epsilon\Psi(z) + o(\epsilon)$, and hence any maximum or minimum of $\Psi(z)$ gives rise to a critical point of Φ_ϵ, for $\epsilon \sim 0$.

3 The ODE case

In this section we carry out the proof of Theorem 1.1. Though one could obtain such a result as a particular case of Theorem 1.3, we prefer to give a separate proof because the arguments are more direct and require much less technicalities.

3.1 Preliminaries

We will search solutions of (1.5) in the Hilbert space $H := X \times X$, where $X = \{u \in W^{1,2}(\mathbb{R}) : u(x) = u(-x)\}$, endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}} |u'|^2 dx + \int_{\mathbb{R}} u^2 dx. \tag{3.1}$$

These solutions are the critical points of the functional $\Phi_\epsilon : H \to \mathbb{R}$ defined by setting

$$\Phi_\epsilon(\mathbf{u}) = I(u_1) + I(u_2) - \epsilon \int_{\mathbb{R}} u_1 u_2 dx, \qquad \mathbf{u} = (u_1, u_2) \tag{3.2}$$

where

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_{\mathbb{R}} u^4 dx. \tag{3.3}$$

This fits into the abstract setting discussed in the previous Section, with

$$\Phi_0(\mathbf{u}) = I(u_1) + I(u_2), \qquad \Psi(\mathbf{u}) = \int_{\mathbb{R}} u_1 u_2 dx.$$

As anticipated before, Φ_0 has pairs of trivial critical points given by $(\pm U, 0)$, $(0, \pm U)$ and $(\pm U, \pm U)$. Since we will use frequently in the sequel the properties of U, let us collect them in a lemma which deals with general n dimensional case.

Lemma 3.1 *[19, pag. 226-227] The equation $-\Delta u + u = u^3$ has a unique positive radial solution $U \in W^{1,2}(\mathbb{R}^n)$, $n = 1, 2, 3$. Such an U satisfies* [1]

$$U(x) \sim |x|^{(1-n)/2} e^{-|x|}, \qquad (|x| \to \infty). \tag{3.4}$$

Moreover, the solutions of the linearized equation $-\Delta\phi + \phi - 3U^2\phi = 0$, $\phi \in W^{1,2}(\mathbb{R}^n)$ satisfy $\phi \in \mathcal{T} = \langle \frac{\partial U}{\partial x_i} \rangle_{i=1}^n$. Finally, the expression

$$\int_{\mathbb{R}^n} \left[|\nabla v|^2 + v^2 - 3U^2 v^2 \right] dx$$

is negative for $v = U$, zero for $v \in \mathcal{T}$, and $\geq c\|v\|^2$, $c > 0$, for $v \perp U$, $v \perp \mathcal{T}$.

In particular, as an immediate consequence of the previous lemma, one has:

Remark 3.2 The pairs $(\pm U, 0)$, $(0, \pm U)$ and $(\pm U, \pm U)$ are non-degenerate in H. For example, the equation $\Phi_0''(\pm U, \pm U)[\phi, \psi] = 0$ is the linear decoupled system

$$\begin{cases} -\phi'' + \phi - 3U^2\phi = 0, \\ -\psi'' + \psi - 3U^2\psi = 0, \end{cases}$$

whose non-trivial solutions are spanned, according to the previous lemma, by $(U', 0)$, $(0, U')$. Since U' is an odd function, the claim follows. Therefore, a straight

[1] when $n = 1$, U is given exactly by equation (1.4).

application of the Implicit Function Theorem yields the existence of a family of solutions emanating from $(\pm U, 0)$, $(0, \pm U)$. There are also solutions bifurcating from $(\pm U, \pm U)$ and $(\pm U, \mp U)$, which are the already known symmetric and anti-symmetric states.

The existence of multi-bump solitons cannot be proved in a direct way as the ones found in Remark 3.2, but it will be obtained by using the abstract results discussed in Section 2. For this, let us start by introducing a family in H near which we will find the multi-bump solitons we are looking for. We set

$$Z = \{\mathbf{z}_\xi := (U(x+\xi) + U(x-\xi), -U) \,:\, \xi \in T_\epsilon\}. \tag{3.5}$$

where

$$T_\epsilon = \left(\frac{\log(1/\epsilon)}{1+\delta}, \log(1/\epsilon) \right).$$

Here and throughout the paper $0 < \delta < 1$ is a fixed constant.

For brevity, we will use the following notation

$$U_\xi(x) = U(x+\xi), \quad U_{-\xi}(x) = U(x-\xi), \quad z_\xi(x) = U_\xi(x) + U_{-\xi}(x).$$

In particular, the elements of Z have the form

$$\mathbf{z}_\xi := (z_\xi, -U) \,:\, \xi \in T_\epsilon.$$

3.2 Study of the auxiliary equation

In this section we will show that the abstract setting discussed in Section 2 applies. First of all, $(A3)$ is trivially satisfied. It suffices to point out that Φ_ϵ has the third derivative bounded on bounded sets and use Remark 2.1. In the sequel we are going to prove that assumptions $(A1)$ and $(A2)$ hold true.

3.2.1 Some preliminary estimates and proof of $(A1)$

Let us first prove some estimates which will be also useful in the sequel.

Lemma 3.3 *For ϵ sufficiently small, $\xi \in T_\epsilon$ and any m, $p \in \mathbb{N}$, one has that*

$$\begin{cases} \displaystyle\int_{\mathbb{R}} U_\xi^m U_{-\xi}^p dx \sim e^{-2(m\wedge p)\xi}, & \text{if } m \neq p, \\ \displaystyle\int_{\mathbb{R}} U_\xi^m U_{-\xi}^p dx \sim \xi\, e^{-2m\xi}, & \text{if } m = p, \end{cases} \tag{3.6}$$

$$\begin{cases} \displaystyle\int_{\mathbb{R}} U_\xi^m U^p dx \sim e^{-(m\wedge p)\xi}, & \text{if } m \neq p, \\ \displaystyle\int_{\mathbb{R}} U_\xi^m U^p dx \sim \xi\, e^{-m\xi}, & \text{if } m = p. \end{cases} \tag{3.7}$$

Proof. First we observe that

$$e^{-|x|} \leq \frac{1}{\cosh(x)} \leq 2e^{-|x|}, \quad \forall x \in \mathbb{R},$$

immediately yields:

$$\int_{\mathbb{R}} U_\xi^m U_{-\xi}^p \sim \int_{\mathbb{R}} e^{-m|x+\xi|} e^{-p|x-\xi|}.$$

Assume for example that $m > p$. Then

$$\int_{\mathbb{R}} U_\xi^m U_{-\xi}^p dx \sim \int_{\mathbb{R}} e^{-m|x|} e^{-p|x-2\xi|} dx$$

$$= e^{-2p\xi} \left(\int_{-\infty}^{0} e^{(m+p)x} dx + \int_{0}^{2\xi} e^{(p-m)x} dx + \int_{2\xi}^{+\infty} e^{-(m+p)x+4n\xi} dx \right)$$

$$= e^{-2p\xi} \left(\frac{1}{m+p} + 1 + \frac{1}{p-m} + e^{-2(m-p)\xi} \right).$$

Since $\xi \in T_\epsilon$, then for ϵ small enough, we get

$$\epsilon^{2(m-p)} < e^{-2(m-p)\xi} < \epsilon^{2(m-p)/(1+\delta)},$$

proving the first of (3.6).

Suppose now that $m = p$ then, for ϵ small enough,

$$\int_{\mathbb{R}} U^m(x+\xi) U^m(x-\xi) dx \sim \int_{\mathbb{R}} e^{-m|x|} e^{-m|x-2\xi|} dx$$

$$= \int_{-\infty}^{0} e^{2m(x-\xi)} dx + \int_{0}^{2\xi} e^{-2m\xi} dx + \int_{2\xi}^{+\infty} e^{2m(\xi-x)} dx$$

$$= e^{-2m\xi} \left(\frac{1}{2m} + 2\xi + \frac{1}{2m} \right) = O(\xi e^{-2m\xi}),$$

uniformly with respect to $\xi \in T_\epsilon$. This proves the second of (3.6).

Moreover, since $U_\xi^m U^p \sim e^{-m|x+\xi|} e^{-p|x|}$, the preceding arguments readily yield (3.7). ■

Now we are ready to show:

Lemma 3.4 *Assumption (A1) is satisifed. Precisely, there exist $\epsilon_0 > 0$ and $C > 0$ such that for $\epsilon < \epsilon_0$ we get*

$$\|\Phi'_\epsilon(\mathbf{z}_\xi)\| < C\epsilon, \qquad \forall\, \xi \in T_\epsilon.$$

Proof. Let $(\varphi, \psi) \in H$ with $\|(\varphi, \psi)\| = 1$, then

$$\Phi_\epsilon'(\mathbf{z}_\xi)[\varphi, \psi] = I'(z_\xi)[\varphi] + I'(-U)[\psi] - \epsilon \int_{\mathbb{R}} (-U\varphi + z_\xi \psi).$$

Since U_ξ, $U_{-\xi}$ and $-U$ are all solutions to the $-u'' + u = u^3$, we deduce that $I'(-U)[\psi] = 0$ as well as

$$I'(z_\xi)[\varphi] = (z_\xi|\varphi) - \int_{\mathbb{R}} z_\xi^3 \varphi = -3 \int_{\mathbb{R}} (U_\xi^2 U_{-\xi} + U_\xi U_{-\xi}^2)\varphi.$$

Then we get

$$\begin{aligned}\Phi_\epsilon'(\mathbf{z}_\xi)[\varphi, \psi] &= -3 \int_{\mathbb{R}} (U_\xi^2 U_{-\xi} + U_\xi U_{-\xi}^2)\varphi - \epsilon \int_{\mathbb{R}} (-U\varphi + (U_\xi + U_{-\xi})\psi) \\ &\leq c_1 \epsilon + c_2 \|U_\xi^2 U_{-\xi}\|_{L^2}.\end{aligned}$$

Hereafter, c_i denote positive constants. Using (3.6) and since $\xi \in T_\epsilon$, we infer

$$\Phi_\epsilon'(\mathbf{z}_\xi)[\varphi, \psi] \leq c_1 \epsilon + c_3 e^{-2\xi} < c_4 \epsilon,$$

proving the lemma. ■

3.2.2 Invertibility of $\Phi_\epsilon''(\mathbf{z}_\xi)$ on W and proof of $(A2)$

First let us observe that $T_{\mathbf{z}_\xi} Z = span \left\{ \dfrac{\partial \mathbf{z}_\xi}{\partial \xi} \right\} = (U_\xi' - U_{-\xi}', 0)$. We shall use the following orthogonal decomposition of the space W: $W = E \oplus F$, $E = span\{(z_\xi, 0), (0, -U)\}$, $F = E^\perp \cap W$.

We first test the bilinear form $\Phi_\epsilon''(\mathbf{z}_\xi)$ on E. Letting $a, b \in \mathbb{R}$ we find

$$\Phi_\epsilon''(\mathbf{z}_\xi)[(a z_\xi, -bU)]^2 = a^2 I''(z_\xi)[z_\xi]^2 + b^2 I''(-U)[-U]^2 - 2\epsilon ab \int_{\mathbb{R}} z_\xi U.$$

Let us evaluate separately the three terms in the preceding equation.

$$\begin{aligned}I''(z_\xi)[z_\xi]^2 &= \|z_\xi\|^2 - 3 \int_{\mathbb{R}} z_\xi^4 \\ &= \|U_\xi\|^2 + \|U_{-\xi}\|^2 + 2(U_\xi|U_{-\xi}) - 3 \int_{\mathbb{R}} (U_\xi + U_{-\xi})^4.\end{aligned}$$

Using (3.6) we infer that $(U_\xi|U_{-\xi}) = o(1)$ and

$$\int_{\mathbb{R}} (U_\xi + U_{-\xi})^4 = \int_{\mathbb{R}} U_\xi^4 + \int_{\mathbb{R}} U_{-\xi}^4 + o(1),$$

provided $\epsilon \ll 1$ and $\xi \in T_\epsilon$. This and $\|U_{\pm\xi}\|^2 = \int_{\mathbb{R}} U_{\pm\xi}^4$ imply

$$I''(z_\xi)[z_\xi]^2 = -2 \int_{\mathbb{R}} \left(U_\xi^4 + U_{-\xi}^4\right) + o(1), \quad (\epsilon \ll 1, \ \xi \in T_\epsilon). \tag{3.8}$$

Moreover, one has that $I''(-U)[-U]^2 = -2\int_{\mathbb{R}} U^4$, while (3.7) implies that

$$\int_{\mathbb{R}} z_\xi U = o(1).$$

This and (3.8) yield

$$\Phi''_\epsilon(\mathbf{z}_\xi)[(az_\xi, -bU)]^2 = -2b^2 \int U^4 - 2a^2 \int U_\xi^4 + U_{-\xi}^4 + o(1), \quad (\epsilon \ll 1,\ \xi \in T_\epsilon),$$

and thus there exists $C_1 > 0$ such that for all $\epsilon \ll 1$ one has

$$\Phi''_\epsilon(\mathbf{z}_\xi)[v]^2 \le -C_1\|v\|^2, \qquad \forall\, v \in E. \tag{3.9}$$

We are now going to prove that $\Phi''_\epsilon(\mathbf{z}_\xi)$ is positive definite on F. Choose an arbitrary $\mathbf{w} \in F$, $\|\mathbf{w}\| = 1$. Then, $\mathbf{w} = (w_1, w_2)$, where $w_1 \perp z_\xi$, $w_2 \perp U$. Since $\mathbf{w}$ is orthogonal to $T_{\mathbf{z}_\xi} Z$, we also have that $w_1 \perp (U'_\xi - U'_{-\xi})$. Since the functions w_1, w_2 and U are even, we can deduce that:

$$w_1 \quad \perp \quad span\{U_\xi, U_{-\xi}, U'_\xi, U'_{-\xi}\}, \tag{3.10}$$
$$w_2 \quad \perp \quad span\{U, U'\}. \tag{3.11}$$

The same calculations as before yield

$$\Phi''_\epsilon(\mathbf{z}_\xi)[\mathbf{w}]^2 = \|w_1\|^2 + \|w_2\|^2 - 3\int_{\mathbb{R}} \left[z_\xi^2 w_1^2 + U^2 w_2^2\right] - 2\epsilon \int_{\mathbb{R}} w_1 w_2.$$

It is well known that, since w_2 satisfies (3.11) we have that $\|w_2\|^2 - 3\int U^2 w_2^2 \ge c_1\|w_2\|^2$, and hence

$$\Phi''_\epsilon(\mathbf{z}_\xi)[\mathbf{w}]^2 \ge \|w_1\|^2 - 3\int_{\mathbb{R}} (U_\xi^2 w_1^2 + U_{-\xi}^2 w_1^2) + c_1\|w_2\|^2 + o(1). \tag{3.12}$$

Let us now estimate the remaining term, namely:

$$\|w_1\|^2 - 3\int U_\xi^2 w_1^2 + U_{-\xi}^2 w_1^2. \tag{3.13}$$

We first claim that there exists $R \in (\xi^{1/4}, \xi^{1/2})$ such that

$$\int_{R<|x\pm\xi|<R+1} [w_i'(x)^2 + w_i(x)^2]dx < 2\xi^{-1/2}, \quad i = 1,\ 2. \tag{3.14}$$

To see this, we remark that from $\|\mathbf{w}\| = 1$ it follows

$$\sum_{R\in\mathbb{N}}^{\xi^{\frac14}<R<\xi^{\frac12}} \int_{R<|x\pm\xi|<R+1} [w_i'(x)^2 + w_i(x)^2]dx \le 1.$$

Since the above sum has more than $\frac{\xi^{1/2}}{2}$ summands (for ξ large), then, it is always possible to choose $R \in \mathbb{N}$, $R \in (\xi^{1/4}, \xi^{1/2})$ in such a way that (3.14) holds.

Let us fix an R such that (3.14) holds and define the smooth cut-off functions $\chi_\xi, \chi_{-\xi}, \chi_0 : \mathbb{R} \to [0,1]$ by setting

$$\chi_\xi(x) = \begin{cases} 1 & x > \xi - R, \\ 0 & x < \xi - R - 1, \\ |\chi'_\xi(x)| \le 2 & x \in \mathbb{R}, \end{cases}$$

$$\chi_{-\xi}(x) = \chi_\xi(-x), \qquad \chi_0 = 1 - \chi_{-\xi} - \chi_\xi.$$

Let us remark that χ_0 satisfies:

$$\chi_0(x) = \begin{cases} 1 & |x| < \xi - R - 1, \\ 0 & |x| > \xi - R, \\ |\chi'_0(x)| \le 2. & x \in \mathbb{R} \end{cases}$$

Let us decompose: $w_1 = v_\xi + v_0 + v_{-\xi}$, where

$$v_{\pm\xi} = w_1 \chi_{\pm\xi}, \qquad v_0 = w_1 \chi_0.$$

From (3.14) it follows $(v_\xi|v_0) = o(1)$, $(v_{-\xi}|v_0) = o(1)$, $(v_\xi|v_{-\xi}) = o(1)$, and thus

$$\|w_1\|^2 = \|v_\xi\|^2 + \|v_0\|^2 + \|v_{-\xi}\|^2 + o(1).$$

Using again (3.14), we obtain

$$(U_\xi|v_0) = o(1), \quad (U_\xi|v_{-\xi}) = o(1).$$

Since $w_1 \perp U_\xi$, one gets that $(U_\xi|v_\xi) = o(1)$. Analogously,

$$\begin{aligned} (U_{-\xi}|v_0) = o(1), \quad (U_{-\xi}|v_\xi) = o(1) \quad &\Rightarrow \quad (U_{-\xi}|v_{-\xi}) = o(1), \\ (U'_\xi|v_0) = o(1), \quad (U'_\xi|v_{-\xi}) = o(1) \quad &\Rightarrow \quad (U'_\xi|v_\xi) = o(1), \\ (U'_{-\xi}|v_0) = o(1), \quad (U'_{-\xi}|v_\xi) = o(1) \quad &\Rightarrow \quad (U'_{-\xi}|v_{-\xi}) = o(1). \end{aligned}$$

This allows us to find a first estimate of (3.13):

$$\|w_1\|^2 - 3\int_{\mathbb{R}} U_\xi^2 w_1^2 + U_{-\xi}^2 w_1^2 = \|v_\xi\|^2 + \|v_0\|^2 + \|v_{-\xi}\|^2 - 3\int_{\mathbb{R}} (U_\xi^2 v_\xi^2 + U_{-\xi}^2 v_{-\xi}^2) + o(1).$$

It is now convenient to make a further decomposition, by setting $v_\xi = \bar{v}_\xi + \hat{v}_\xi$, where $\bar{v}_\xi \in span\{U_\xi, U'_\xi\}$ and $\hat{v}_\xi \in span\{U_\xi, U'_\xi\}^\perp$. Since $(U_\xi|v_\xi) = o(1)$ and $(U'_\xi|v_\xi) = o(1)$, see before, we infer that $\|\bar{v}_\xi\| = o(1)$, yielding

$$\|v_\xi\|^2 - 3\int U_\xi^2 v_\xi^2 = \|\hat{v}_\xi\|^2 - 3\int U_\xi^2 \hat{v}_\xi^2 + o(1).$$

Moreover, since $\hat{v}_\xi \perp span\{U_\xi, U'_\xi\}$, it follows (see Lemma 3.1)

$$\|\hat{v}_\xi\|^2 - 3\int U_\xi^2 \hat{v}_\xi^2 \ge c_2 \|\hat{v}_\xi\|^2 \ge c_3 \|v_\xi\|^2,$$

and therefore we get

$$\|v_\xi\|^2 - 3\int U_\xi^2 v_\xi^2 \geq c_3\|v_\xi\|^2 + o(1). \tag{3.15}$$

Similarly, we find

$$\|v_{-\xi}\|^2 - 3\int U_{-\xi}^2 v_{-\xi}^2 \geq c_4\|v_{-\xi}\|^2 + o(1). \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.13) we finally infer

$$\|w_1\|^2 - 3\int U_\xi^2 w_1^2 + U_{-\xi}^2 w_1^2 \geq c_3\|v_\xi\|^2 + c_4\|v_{-\xi}\|^2 + \|v_0\|^2 + o(1) \geq c_5\|w_1\|^2 + o(1).$$

From this and (3.12) we deduce there exists $c_6 > 0$ such that for all $\epsilon \ll 1$ and all $\xi \in T_\epsilon$ one has:

$$\Phi_\epsilon''(\mathbf{z}_\xi)[\mathbf{w}]^2 \geq c_6\|\mathbf{w}\|^2.$$

In conclusion we have proved

Lemma 3.5 *There exists a positive constant γ such that for ϵ small enough and $\xi \in T_\epsilon$, $P\Phi_\epsilon''(\mathbf{z}_\xi)$ is uniformly invertible and there results $\|[P\Phi_\epsilon''(\mathbf{z}_\xi)]^{-1}\| \leq \gamma$. In other words, (A2) holds.*

3.3 Proof of Theorem 1.1

The previous arguments allow us to use Lemma 2.2 which yields a solution $\mathbf{w}_{\epsilon,\xi}$ of the auxiliary equation. It remains to find a critical point of the reduced functional. For this it is useful the following lemma which provides an expansion of $\widetilde{\Phi}_\epsilon(\xi)$.

Lemma 3.6 *There exist positive constants $C, \underline{C}, \overline{C} > 0$ such that*

$$\widetilde{\Phi}_\epsilon(\xi) = C - a_{\epsilon,\xi}e^{-2\xi} + b_{\epsilon,\xi}\,\epsilon\,\xi\,e^{-\xi} + O(\epsilon^2), \quad \forall \xi \in T_\epsilon,\ \epsilon \sim 0,$$

where $a_{\epsilon,\xi},\ b_{\epsilon,\xi} \in [\underline{C}, \overline{C}]$.

Proof. Since the norm of Φ_ϵ'' is uniformly bounded, we get

$$\Phi_\epsilon(\mathbf{z} + \mathbf{w}) = \Phi_\epsilon(\mathbf{z}_\xi) + \Phi_\epsilon'(\mathbf{z})[\mathbf{w}_{\epsilon,\xi}] + \mathrm{O}(\|\mathbf{w}_{\epsilon,\xi}\|^2).$$

First, we are going to neglect the terms of order ϵ^2. In this way, we are going to prove that the influence of $\mathbf{w}_{\epsilon,\xi}$ in the reduced functional, $\widetilde{\Phi}_\epsilon$, is not relevant in order to find critical points. Observe that by Lemma 3.4 and (2.5) we have that, for $\epsilon \ll 1$,

$$\|\Phi_\epsilon'(\mathbf{z}_\xi)[\mathbf{w}_{\epsilon,\xi}]\| \leq c_1\epsilon^2, \quad \forall\, \xi \in T_\epsilon.$$

On the other hand,

$$\begin{aligned}\Phi_\epsilon(\mathbf{z}_\xi) &= I(z_\xi) + I(-U) + \epsilon \int_{\mathbb{R}} z_\xi U \\ &= \frac{3}{4}\int_{\mathbb{R}} U^4 - \int_{\mathbb{R}} U_\xi^3 U_{-\xi} - \frac{3}{2}\int_{\mathbb{R}} U_\xi^2 U_{-\xi}^2 + 2\epsilon \int_{\mathbb{R}} U_\xi U.\end{aligned}$$

Now we study the behavior of each term in the above decomposition. The first term is constant, i.e.,

$$c = \tfrac{3}{4}\int U^4, \tag{3.17}$$

while Lemma 3.3 yields

$$\int_{\mathbb{R}} U_\xi^2 U_{-\xi}^2 \sim \xi e^{-4\xi}.$$

Since $\xi \in T_\epsilon$ it follows that there exists $c_1 > 0$ and $\theta > 0$ such that for $\epsilon \ll 1$ one has

$$\left| \int U_\xi^2 U_{-\xi}^2 \right| \le c_1 \epsilon^{-(2+\theta)\xi}, \qquad \forall\, \xi \in T_\epsilon.$$

On the other hand, using once again Lemma 3.3 we get

$$-\int_{\mathbb{R}} U_\xi^3 U_{-\xi} \sim -e^{-2\xi}, \qquad 2\epsilon \int_{\mathbb{R}} U_\xi U \sim \epsilon \xi e^{-\xi}, \tag{3.18}$$

and the lemma follows. ■

Proof of Theorem 1.1 completed. According to the previous discussion, to complete the proof of Theorem 1.1 it remains to show

Lemma 3.7 *For ϵ sufficiently small, the reduced functional $\widetilde{\Phi}_\epsilon$ possesses a critical point $\xi^*(\epsilon) \in T_\epsilon$.*

Proof. Let $f(\xi) = -c_1 e^{-2\xi} + c_2 \epsilon\, \xi\, e^{-\xi}$, $c_1, c_2 > 0$. By an elementary calculation one finds that f has a strict maximum at ξ_ϵ such that

$$\frac{e^{-\xi_\epsilon}}{\xi_\epsilon - 1} = \frac{c_2}{2c_1}\epsilon. \tag{3.19}$$

Observe that for all $\xi \gg 1$ there holds

$$e^{-(1+\delta)\xi} \le \frac{2c_1}{c_2}\frac{e^{-\xi}}{\xi - 1} \le e^{-\xi}. \tag{3.20}$$

Since from (3.19) it follows that $\xi_\epsilon \to +\infty$ as $\epsilon \to 0$, then ξ_ϵ satisfies (3.20) provided $\epsilon \ll 1$ and hence

$$-\frac{1}{1+\delta}\log\epsilon < \xi_\epsilon < -\log\epsilon.$$

Let us evaluate f at ξ_ϵ and at the extrema of T_ϵ. One has

$$f(\xi_\epsilon) = \frac{c_2^2}{4c_1}\epsilon^2(\xi_\epsilon^2 - 1) > c_3\,\epsilon^2 \log^2(1/\epsilon), \qquad (\epsilon \ll 1).$$

On the other hand, taking $\zeta_\epsilon = -\frac{1}{1+\delta}\log\epsilon$ and $\zeta'_\epsilon = -\log\epsilon$, one has

$$f(\zeta_\epsilon) = -c_1\epsilon^{\frac{2}{1+\delta}} + c_2\zeta_\epsilon\epsilon^{\frac{2+\delta}{1+\delta}} \sim -\epsilon^{\frac{2}{1+\delta}} < 0, \quad \text{as } \epsilon \to 0,$$

as well as

$$f(\zeta'_\epsilon) = -c_1\epsilon^2 + c_2\epsilon^2\zeta'_\epsilon < c_2\,\epsilon^2\log(1/\epsilon), \qquad (\epsilon \ll 1).$$

Lemma 3.6 together with the above estimates imply the existence of a maximum for $\widetilde{\Phi}_\epsilon$ at a certain $\xi^*(\epsilon) \in (\zeta_\epsilon, \zeta'_\epsilon) = T_\epsilon$, provided ϵ is sufficiently small. This completes the proof of Theorem 1.1. ■

Remark 3.8 One can try to repeat the preceding arguments in order to find solutions whose first component has k bumps, $k > 2$, while the second component is negative and has h bump(s), $h \geq 1$. Unfortunately, the procedure fails because the reduced functional has no critical point. We suspect that there are no solution at all having the first component with more than two bumps. This is confirmed by the numerical computation in [2].

The preceding observation also highlights the difference between our results and those of [9]. In that paper, solutions with infinitely many bumps are found. Their idea is to use two given homoclinics satisfying certain geometrical properties, to construct multi-bump solutions by means of a gluing procedure. Observe that, in a sense, we are also gluing the homoclinics $(U_\xi, 0)$, $(0, -U)$ and $(U_{-\xi}, 0)$, but they not verify the conditions assumed in [9].

3.4 Systems with more than two equations

In this section the results found above will be extended to systems with more than two NLS equations. They arise when one considers the propagation of pulses in a m-core couplers with circular symmetry. To simplify notation, we will consider the case $m = 3$, only. The general case is similar. Setting $\mathbf{u} = (u_1, u_1, u_3) \in H := X \times X \times X$, we consider a system like

$$\left\{\begin{array}{rcl} -u_1'' + u_1 - u_1^3 &=& \epsilon(u_2 + u_3), \\ -u_2'' + u_2 - u_2^3 &=& \epsilon(u_1 + u_3), \\ -u_3'' + u_3 - u_3^3 &=& \epsilon(u_1 + u_2), \end{array}\right. \tag{3.21}$$

whose corresponding functional is given by

$$\Phi_\epsilon(\mathbf{u}) = I(u_1) + I(u_2) + I(u_3) - \epsilon\int_{\mathbb{R}} [u_1u_2 + u_1u_3 + u_2u_3]\,dx.$$

Of course, any triple in which one component is zero and the other two satisfy (1.5), is a solution of (3.21). In addition, dealing with solutions having all three components different from zero, we shall distinguish various other solutions. According to the notation introduced in [3], we consider:

- *A-type solutions*, branching off from solutions with one zero component while the remaining two components are equal, like $(U, U, 0)$.

Focusing, for example, on the family emanating from $(U, U, 0)$, the corresponding linearized system is

$$\begin{cases} -\phi_1'' + \phi_1 - 3U^2\phi_1 &= 0, \\ -\phi_2'' + \phi_2 - 3U^2\phi_2 &= 0, \\ -\phi_3'' + \phi_3 &= 0. \end{cases} \tag{3.22}$$

The same arguments carried out in Remark 3.2 show that $(U, U, 0)$ is non-degenerate and hence there exists a family of solutions $\mathbf{u}_\epsilon \in H$, whose components satisfy

$$u_{j,\epsilon} \sim U,\ (j = 1, 2), \qquad u_{3,\epsilon} \sim 0, \qquad (\epsilon \to 0). \tag{3.23}$$

Another kind of solutions is given by the

- *B-type solutions*, which have one multi-bump component, a second component near $-U$, a third near 0.

The existence of this type of solutions can be proved by the same arguments used in Theorem 1.1. We will focus on the particular case (the other ones, obtained by permutation of the indices $1, 2, 3$, are quite similar) in which one has:

$$u_{1,\epsilon} \sim U_\xi + U_{-\xi}, \quad u_{2,\epsilon} \sim -U, \quad u_{3,\epsilon} \sim 0, \qquad (\epsilon \to 0). \tag{3.24}$$

These solutions bifurcate from the manifold

$$Z_B = \{\mathbf{z}_\xi := (U_\xi + U_{-\xi}, -U, 0) \,:\, \xi \in T_\epsilon\}$$

which has exactly the same properties of Z defined in (3.5). Therefore, repeating the same arguments used for Theorem 1.1, one can prove the existence of the B-type solutions.

Next, we are going to consider solutions which have all the three components far from zero. The first family is the

- *C_1-type solutions*, which have the components satisfying, for example,

$$u_{j,\epsilon} \sim U,\ (j = 1, 2), \qquad u_{3,\epsilon} \sim -U, \qquad (\epsilon \to 0). \tag{3.25}$$

The existence of these solutions follows immediately applying the Implicit function Theorem, as for the A-type solutions. Actually, $(U, U, -U)$ is non-degenerate in the space H. A second family consists of the

- *C_2-type solutions*, which have two multi-bump components, while the third one is near $-U$.

As for the B-type solutions, the existence of the C_2-type solutions can be proved by the same arguments used in Theorem 1.1. Let us focus, for example, on solutions such that

$$u_{j,\epsilon} \sim U_\xi + U_{-\xi},\ (j=1,2), \qquad u_{3,\epsilon} \sim -U, \qquad (\epsilon \to 0). \tag{3.26}$$

The corresponding manifold is

$$Z_C = \{\mathbf{z}_\xi := (U_\xi + U_{-\xi}, U_\xi + U_{-\xi}, -U) \,:\, \xi \in T_\epsilon\}.$$

Here we work on the space

$$\widehat{H} = \{\mathbf{u} \in X \times X \times X : u_1 = u_2\}.$$

According to the *Symmetric Criticality Principle*, see [20] or [23, Theorem 1.28], the critical points of Φ_ϵ on $\widehat{H}$ are critical points of Φ_ϵ and hence solutions of (3.21). Carrying out similar arguments as in the proof of Theorem 1.1, we are led to study the reduced functional, whose leading part is, as before $\Phi_\epsilon(\mathbf{z}_\xi)$. In the present case one finds

$$\Phi(\mathbf{z}_\xi) = 2I(U_\xi + U_{-\xi}) + I(-U) - \epsilon \int_{\mathbb{R}} \left[-2U(U_\xi + U_{-\xi}) + (U_\xi + U_{-\xi})^2\right] dx,$$

One has:

$$2I(U_\xi + U_{-\xi}) + I(-U) \sim c' - 2\, e^{-2\xi},$$

as well as

$$\begin{aligned} &\epsilon \int_{\mathbb{R}} 2U(U_\xi + U_{-\xi}) dx - \epsilon \int_{\mathbb{R}} (U_\xi + U_{-\xi})^2 dx \\ &\quad = 4\epsilon \int_{\mathbb{R}} U_\xi U dx - 2\epsilon \int_{\mathbb{R}} U_\xi^2 dx - 2 \int_{\mathbb{R}} U_\xi U_{-\xi} dx \\ &\quad \sim 4\epsilon\, \xi\, e^{-\xi} - c''\, \epsilon - 2\epsilon\, \xi\, e^{-2\xi}. \end{aligned}$$

Thus, setting $c^* = c' - c''\epsilon$ we get

$$\widetilde{\Phi}_\epsilon(\xi) \sim c^* - 2\, e^{-2\xi} + 4\epsilon\, \xi\, e^{-\xi} + O(\epsilon^2), \qquad \forall\, \xi \in T_\epsilon,$$

which is nothing but Lemma 3.6. It follows that we can repeat once more the arguments of Lemma 3.7 to infer the existence of C_2-type solutions.

Finally, there exists a last family, different from all the previous ones:

- *E-type solutions*, such that $u_{1,\epsilon}(x) = u_{2,\epsilon}(-x)$ and $u_{3,\epsilon}(x) = u_{3,\epsilon}(-x)$.

More precisely, we will find solutions of the type $u_{1,\epsilon} \sim U_\xi$, $u_{2,\epsilon} \sim U_{-\xi}$ and $u_{3,\epsilon} \sim -U(x)$. Let us remark explicitly that the first two components are asymmetric with respect to x. To prove the existence of E-type solutions we will use again the perturbation method, but in a way different than in the previous cases. First of all, we shall work on the space

$$\widetilde{H} = \{\mathbf{u} \in E \times E \times E : u_1(x) = u_2(-x), u_3(x) = u_3(-x)\}, \quad E = W^{1,2}(\mathbb{R}).$$

Using again the *Symmetric Criticality Principle*, the critical points of Φ_ϵ on $\widetilde{H}$ solutions of (3.21). Let us consider the manifold

$$Z_E = \{\widetilde{\mathbf{z}}_\xi = (U_\xi, U_{-\xi}, -U) \in \widetilde{H} : \xi \in \mathbb{R}\}.$$

Clearly any $\widetilde{\mathbf{z}}_\xi \in Z_E$ is a critical point of Φ_0. Moreover, as for (3.22), one has that $(\phi_1, \phi_2, \phi_3) \in Ker\Phi_0''(\widetilde{\mathbf{z}}_\xi)$ if and only if $\phi_1 = a\,U'_\xi$, $\phi_2 = b\,U'_{-\xi}$, and $\phi_3 = c\,U'$ for some $a, b, c \in \mathbb{R}$. Then the requirement that $(\phi_1, \phi_2, \phi_3) \in \widetilde{H}$ implies $a = -b$ and $c = 0$. Therefore, $Ker[\Phi_0''(\widetilde{\mathbf{z}}_\xi)] = T_{\widetilde{\mathbf{z}}_\xi} Z_E$, namely Z_E is a non-degenerate critical manifold, see [6, Section 2.2]. Set

$$\Gamma(\xi) = -\int_{\mathbb{R}} G(\widetilde{\mathbf{z}}_\xi)dx = -\int_{\mathbb{R}} UU_\xi dx - \int_{\mathbb{R}} UU_{-\xi} dx + \int_{\mathbb{R}} U_\xi U_{-\xi} dx.$$

According to [6, Thm. 2.16], see also Remark 2.5, any maximum or minimum $\widetilde{\xi}$ of Γ gives rise to a critical point $\widetilde{\mathbf{u}}_\epsilon$ of Φ_ϵ such that $\widetilde{\mathbf{u}}_\epsilon \to \widetilde{\mathbf{z}}_{\widetilde{\xi}}$ as $\epsilon \to 0$. By a direct computation it follows that

$$\Gamma(0) < 0, \quad \Gamma'(0) = 0, \quad \Gamma''(0) < 0.$$

Therefore there exists $\widetilde{\xi} > 0$ such that Γ has a minimum at $\pm\widetilde{\xi}$. Thus we infer that (3.21) possesses, for ϵ sufficiently small, a family of solutions $\widetilde{\mathbf{u}}_\epsilon$ of type E. Precisely, the components of $\widetilde{\mathbf{u}}_\epsilon$ satisfy:

$$\widetilde{u}_{1,\epsilon} \to U_{\widetilde{\xi}}, \quad \widetilde{u}_{2,\epsilon} \to U_{-\widetilde{\xi}}, \quad \widetilde{u}_{3,\epsilon} \to -U, \quad (\epsilon \to 0). \tag{3.27}$$

Summarizing, we can state the following result:

Theorem 3.9 *For ϵ sufficiently small, the system* (3.21) *has the following solutions:*

(a) *A-type solutions, whose components satisfy* (3.23) *(and its permutations);*

(b) *B-type solutions, whose components satisfy* (3.24) *(and its permutations);*

(c_1) *C_1-type solutions, whose components satisfy* (3.25) *(and its permutations);*

(c_2) *C_2-type solutions, whose components satisfy* (3.26) *(and its permutations);*

(e) *E-type solutions, whose components satisfy* (3.27) *(and its permutations).*

Remark 3.10 The system considered in [3] is slightly different than (3.21), because the left hand side is $-u_j'' + qu_j - u_j^3$, q being a real parameter. This makes their results not directly comparable with ours. For example, dealing with the E-type solutions, there is a numerical evidence (see [3, Figure 9.3]) that the peaks of the two positive components should tend to zero when the bifurcation parameter $q/\epsilon \approx 2.747$, while we have shown that they converge to $\pm\widetilde{\xi}$ as $\epsilon \to 0$, see point (e) before.

4 The PDE case

In this part we are concerned with the proof of Theorem 1.3. To cover also the case of two bumps solutions, it is understood that the polytope $\mathcal{P}$ may also denote the degenerate polygon with two vertices, which are symmetric with respect to the origin (in such case $s = 2$, $r = 1$).

Let $\mathcal{G}$ be the *polyedral group* associated to $\mathcal{P}$, namely the group of all isometries that leave invariant $\mathcal{P}$, see [11, pag. 46]. The Sobolev space of functions which are invariant under $\mathcal{G}$, will be denoted by

$$\mathcal{X} = \{u \in W^{1,2}(\mathbb{R}^n) : u \circ g = u \ a.e. \text{ in } \mathbb{R}^n \, , \ \forall g \in \mathcal{G}\}$$

We will work in $\mathcal{H} = \mathcal{X} \times \mathcal{X}$ and consider the functional $\Phi_\epsilon : \mathcal{H} \to \mathbb{R}$,

$$\Phi_\epsilon(\mathbf{u}) = I(u_1) + I(u_2) - \epsilon\Psi(\mathbf{u}), \qquad \mathbf{u} = (u_1, u_2),$$

where

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int u^4 dx, \quad \Psi(\mathbf{u}) = \int u_1 u_2 dx,$$

and

$$\|u\|^2 = \int (|\nabla u|^2 + u^2) dx.$$

We also denote by $(u \mid v) = \int (\nabla u \cdot \nabla v + uv) dx$ the scalar product in $W^{1,2}(\mathbb{R}^n)$, corresponding to the preceding norm.

Since $\mathcal{X} = \text{Fix}(\mathcal{G})$, the Palais Symmetric Criticality Principle implies that the critical points of Φ_ϵ on $\mathcal{H}$ correspond to solutions of (1.7).

Let us remark that when $\mathcal{P}$ is the degenerate polytope with two symmetric points, the preceding theorem is nothing but the extension of Theorem 1.1 in $\mathbb{R}^n$, $n = 2, 3$.

In the proof of Theorem 1.3 we follow the same abstract approach as in the ODE case. However, as anticipated in the Introduction, the proof of this theorem requires some new ingredients. Since here the distance of the peaks to the origin is larger, the interaction among the bumps is weaker, and then we need better estimates for the reduced functional. To overcome this problem, we will consider a manifold Z which approximates the solutions we are looking for more accurately. This yields sharper estimates and allows us to obtain the conclusion. For this purpose, in the next Subsection, we will first construct a pair $(V_\epsilon, v_\epsilon) \in \mathcal{H}$ bifurcating from $(U, 0)$ where U denotes the unique positive radial solution of $-\Delta u + u = u^3$, in $W^{1,2}(\mathbb{R}^n)$, see Lemma 3.1.

4.1 Bifurcation from $(U, 0)$

Consider the problem

$$\begin{cases} -\Delta u + u - u^3 = \epsilon v, \\ -\Delta v + v - v^3 = \epsilon u, \end{cases} \tag{4.1}$$

where we assume $u, v \in H^1_r(\mathbb{R}^n)$, $n = 2, 3$. For $\epsilon = 0$ there is a solution of the form $(U, 0)$, where U denotes the unique positive radial solution of $-\Delta u + u = u^3$. In next proposition we obtain bifurcation on the parameter ϵ from this solution.

Proposition 4.1 *There exists $\epsilon_0 > 0$ and a C^∞ curve*

$$(-\epsilon_0, \epsilon_0) \to (V_\epsilon, v_\epsilon) \in H^2_r(\mathbb{R}^n) \times H^2_r(\mathbb{R}^n),$$

of solutions of (4.1). *Moreover,* $(V_0, v_0) = (U, 0)$.

Proof. Define $F : H^2_r(\mathbb{R}^n) \times H^2_r(\mathbb{R}^n) \times \mathbb{R} \to L^2_r(\mathbb{R}^n) \times L^2_r(\mathbb{R}^n)$,

$$F(u, v, \epsilon) = \begin{pmatrix} -\Delta u + u - u^3 - \epsilon v \\ -\Delta v + v - v^3 - \epsilon u \end{pmatrix}.$$

Obviously, the solutions of (4.1) correspond to the zeroes of F. We plan to obtain local bifurcation from $(U, 0)$ by means of the Implicit Function Theorem. Clearly one has:

- $F \in \mathcal{C}^\infty$,
- $F(U, 0, 0) = (0, 0)$,
- $\partial_{u,v} F(U, 0, 0)[\varphi, \psi] = (-\Delta\varphi + \varphi - 3U^2\varphi, -\Delta\psi + \psi)$.

We now verify that this operator is invertible, that is, given $(f, g) \in L^2_r(\mathbb{R}^n) \times L^2_r(\mathbb{R}^n)$, the problem

$$\left\{ \begin{array}{rcl} -\Delta\varphi + \varphi - 3U^2\varphi & = & f \\ -\Delta\psi + \psi & = & g \end{array} \right. \tag{4.2}$$

has a solution in $H^2_r(\mathbb{R}^n) \times H^2_r(\mathbb{R}^n)$.

Clearly, the second equation of (4.2) is uniquely solvable. Moreover, the Fredholm Alternative and Lemma 3.1 imply that also the first equation is uniquely solvable in $H^1_r(\mathbb{R}^n)$. Therefore, there exist $(\varphi, \psi) \in H^1_r(\mathbb{R}^n) \times H^1_r(\mathbb{R}^n)$ solving (4.2). Finally, they are in $H^2(\mathbb{R}^n)$ because of the regularity result of Agmon-Douglis-Nirenberg [1]. Hence, the Implicit Function Theorem implies the existence of a C^∞ curve $\epsilon \mapsto (V_\epsilon, v_\epsilon) \in H^2_r(\mathbb{R}^n) \times H^2_r(\mathbb{R}^n)$ such that $F(V_\epsilon, v_\epsilon, \epsilon) = 0$.
■

Recall for further purposes that:

$$\|V_\epsilon - U\|_{H^2} \to 0, \ \|v_\epsilon\|_{H^2} \to 0 \ \text{ as } \epsilon \to 0. \tag{4.3}$$

As we shall see, we will use the functions V_ϵ, v_ϵ to construct the manifold of approximated solutions. In order to deal with these functions, we will need to use their Taylor expansion in ϵ:

$$\begin{array}{l} V_\epsilon = U + \epsilon w_1 + \frac{\epsilon^2}{2} w_2 + \ldots \\ v_\epsilon = \epsilon \tilde{w}_1 + \frac{\epsilon^2}{2} \tilde{w}_2 + \ldots \end{array}$$

In order to compute w_k, $\tilde{w}_k$, $k = 1, 2, \ldots$, we just make the derivative of:

$$\left\{ \begin{array}{rcl} -\Delta V_\epsilon + V_\epsilon - V_\epsilon^3 & = & \epsilon v_\epsilon, \\ -\Delta v_\epsilon + v_\epsilon - v_\epsilon^3 & = & \epsilon V_\epsilon, \end{array} \right.$$

with respect to ϵ. Then, denoting by a "dot" the derivative with respect to ϵ, we find

$$\begin{cases} -\Delta \dot{V}_\epsilon + \dot{V}_\epsilon - 3V_\epsilon^2 \dot{V}_\epsilon &= v_\epsilon + \epsilon \dot{v}_\epsilon, \\ -\Delta \dot{v}_\epsilon + \dot{v}_\epsilon - 3v_\epsilon^2 \dot{v}_\epsilon &= V_\epsilon + \epsilon \dot{V}_\epsilon. \end{cases}$$

For $\epsilon = 0$ we get

$$\begin{cases} -\Delta w_1 + w_1 - 3U^2 w_1 &= 0, \\ -\Delta \tilde{w}_1 + \tilde{w}_1 &= U. \end{cases}$$

In particular $w_1 = 0$. Taking the second derivative with respect to ϵ we obtain that

$$\begin{cases} -\Delta \ddot{V}_\epsilon + \ddot{V}_\epsilon - 3[2V_\epsilon \dot{V}_\epsilon^2 + V_\epsilon^2 \ddot{V}_\epsilon] &= 2\dot{v}_\epsilon + \epsilon \ddot{v}_\epsilon, \\ -\Delta \ddot{v}_\epsilon + \ddot{v}_\epsilon - 3[2v_\epsilon \dot{v}_\epsilon^2 + v_\epsilon^2 \ddot{v}_\epsilon] &= 2\dot{V}_\epsilon + \epsilon \ddot{V}_\epsilon, \end{cases}$$

hence for $\epsilon = 0$ we find that

$$\begin{cases} -\Delta w_2 + w_2 - 3U^2 w_2 &= 2\tilde{w}_1, \\ -\Delta \tilde{w}_2 + \tilde{w}_2 &= 2w_1 = 0. \end{cases}$$

In particular $\tilde{w}_2 = 0$. In general,

$$\begin{cases} -\Delta w_k + w_k - \dfrac{d^k}{d\epsilon^k}\Big|_{\epsilon=0} (V_\epsilon^3) = k\tilde{w}_{k-1}, \\[2ex] -\Delta \tilde{w}_k + \tilde{w}_k - \dfrac{d^k}{d\epsilon^k}\Big|_{\epsilon=0} (v_\epsilon^3) = k w_{k-1}. \end{cases} \tag{4.4}$$

In (4.4) one has that for l, m, i, j, s nonnegative integers:

$$\frac{d^k}{d\epsilon^k}\Big|_{\epsilon=0} (V_\epsilon^3) = k! \left[\sum_{\substack{l \neq m \\ 2l+m=k}} \frac{3w_l^2 w_m}{(l!)^2 m!} + \sum_{\substack{i\neq j\neq s\neq i \\ i+j+s=k}} \frac{6w_i w_j w_s}{i!j!s!} + f(k)\frac{w_{k/3}^3}{(k/3)!}, \right] \tag{4.5}$$

where $f(k) = 1$ if $k/3 \in \mathbb{N}$, $f(k) = 0$ on the contrary. An analogous expression holds for $\dfrac{d^k}{d\epsilon^k}\Big|_{\epsilon=0} (v_\epsilon^3)$.

Lemma 4.2 *$w_k = 0$ if k is odd and $\tilde{w}_k = 0$ if k is even.*

Proof. We argue by induction. For $k = 0, 1, 2$ the result holds. Suppose that it holds for every $t < k$.

1. Assume k is odd. In such case, observe that in (4.5) any summand has a factor w_p with p odd. Moreover, all these p's are smaller than k, except the term with $l = 0$, $m = k$. Then the expression (4.5) will be equal to $3U^2 w_k$. Inserting this into the first equation of (4.4) we have:

   $$-\Delta w_k + w_k - 3U^2 w_k = k\tilde{w}_{k-1}.$$

   By induction, $\tilde{w}_{k-1} = 0$ which implies that $w_k = 0$.

2. Assume that k is even. Arguing in the same way as above, we obtain that $\tilde{w}_k$ verifies:
$$-\Delta\tilde{w}_k + \tilde{w}_k = kw_{k-1}.$$
By induction, $w_{k-1} = 0$ and hence we infer that $\tilde{w}_k = 0$.

■

Now we establish the decay of the functions w_k and $\tilde{w}_k$. The following lemma will be useful.

Lemma 4.3 *Assume that $f(x) = f(|x|)$ is a radial function verifying that $|f(r)| \leq Cr^{\alpha}r^{\frac{1-n}{2}}e^{-r}$, $\forall\, r > 1$, where $\alpha > -1$ and $C > 0$. Let $u \in H^1_r(\mathbb{R}^n)$ be the solution of the equation $-\Delta u + u = f$. Then $|u(r)| \leq C' r^{\alpha+1}r^{\frac{1-n}{2}}e^{-r}$.*

Proof. Writing the equation $-\Delta u + u = f$ in radial coordinates we obtain

$$\begin{cases} -u'' - (n-1)\frac{u'}{r} + u &= f(r) \\ \lim_{r\to\infty} u(r) = 0. \end{cases}$$

First, consider the homogeneous equation $-u'' - \frac{n-1}{r}u' + u = 0$. The set of solutions is a two dimensional vector space generated by u_1, u_2, which are defined as:

$$u_1(r) = r^{\frac{2-n}{2}} B_K\left(\frac{n-2}{2}, r\right), \; u_2(r) = r^{\frac{2-n}{2}} B_I\left(\frac{n-2}{2}, r\right).$$

In the above expressions B_K, B_I are the modified Bessel functions of the first and second type respectively, see [14]. From the properties of the Bessel functions we have $u_1(r) \sim r^{\frac{1-n}{2}}e^{-r}$, $u_2(r) \sim r^{\frac{1-n}{2}}e^{r}$.

Given R, A fixed constants and f a fixed function, the solution of the problem

$$\begin{cases} -u'' - (n-1)\frac{u'}{r} + u &= f \\ u(R) = A, \quad \lim_{r\to\infty} u(r) &= 0 \end{cases}$$

can be obtained by the variation of constants formula:

$$u(r) = u_1(r)\int_R^r u_2(s)f(s)s^{n-1}ds + u_2(r)\int_R^r u_1(s)f(s)s^{n-1}ds + au_1(r) + bu_2(r)$$

for some real constants a, b. Let us study the first summand in the above expression:

$$\left|u_1(r)\int_R^r u_2(s)f(s)s^{n-1}ds\right| \leq u_1(r)\int_R^r u_2(s)|f(s)|s^{n-1}ds$$

$$\leq \int_R^r u_1(s)u_2(s)|f(s)|s^{n-1}ds \sim \int_R^r |f(s)|ds < \int_R^{+\infty} |f(s)|ds < \infty.$$

Therefore, by using that $\lim_{r\to\infty} u(r) = 0$, we conclude that

$$b = -\int_R^{+\infty} u_1(s)f(s)s^{n-1}ds.$$

Hence, we can rewrite u as follows:

$$u(r) = u_1(r)\int_R^r u_2(s)f(s)s^{n-1}ds + u_2(r)\int_r^{+\infty} u_1(s)f(s)s^{n-1}ds + au_1(r),$$

where the constant a is given by the boundary condition $u(R) = A$. We prove now the decay of u:

$$\begin{aligned}\left|\frac{u(r)}{r^{\alpha+1}u_1(r)}\right| &\leq \frac{1}{r^{\alpha+1}}\int_R^r u_2(s)|f(s)|s^{n-1} \\ &+\frac{u_2(r)}{r^{\alpha+1}u_1(r)}\int_r^{+\infty} u_1(s)|f(s)|s^{n-1}ds + \frac{a}{r^{\alpha+1}}.\end{aligned}$$

We study each term separately:

$$\frac{1}{r^{\alpha+1}}\int_R^r u_2(s)|f(s)|s^{n-1}ds \leq \frac{C}{r^{\alpha+1}}\int_R^r s^\alpha ds = \frac{C}{r^{\alpha+1}}\frac{r^{\alpha+1}-R^{\alpha+1}}{\alpha+1} \leq \frac{C}{\alpha+1}.$$

$$\frac{u_2(r)}{r^{\alpha+1}u_1(r)}\int_r^{+\infty} u_1(s)|f(s)|s^{n-1}ds \leq C\frac{e^{2r}}{r^{\alpha+1}}\int_r^{+\infty} s^\alpha e^{-2s}ds.$$

Using that $\dfrac{e^r}{r^\alpha}$ is increasing for $r \geq r_0$ we get:

$$\begin{aligned}\frac{e^{2r}}{r^{\alpha+1}}\int_r^{+\infty} s^\alpha e^{-2s}ds &= \frac{e^r}{r}\frac{e^r}{r^\alpha}\int_r^{+\infty} s^\alpha e^{-2s}ds \\ &\leq \frac{e^r}{r}\int_r^{+\infty}\frac{e^s}{s^\alpha}s^\alpha e^{-2s}ds = \frac{e^r}{r}\int_r^{+\infty} e^{-s}ds = \frac{1}{r}.\end{aligned}$$

As a consequence,

$$\frac{u_2(r)}{r^{\alpha+1}u_1(r)}\int_r^{+\infty} u_1(s)|f(s)|s^{n-1}ds \to 0 \quad \text{as} \quad r \to +\infty.$$

So the result follows. ∎

Remark 4.4 The same arguments used before imply that:

- If $\alpha = -1$ then $u(r) \leq C'\log(r)r^{\frac{1-n}{2}}e^{-r}$.
- If $\alpha < -1$ then $u(r) \leq C'r^{\frac{1-n}{2}}e^{-r}$.

The above estimates are sharp, as can be easily checked in the one dimensional case ($n = 1$).

Corollary 4.1 *The functions w_k, $\tilde{w}_k$ satisfy the estimate:*

$$|w_k(r)|,\, |\tilde{w}_k(r)| \le C_k r^k r^{\frac{1-n}{2}} e^{-r} \quad \textit{for } r > 1,\ C_k > 0. \tag{4.6}$$

Proof. We argue by induction. Clearly for $k = 0$ (4.6) holds. Suppose (4.6) holds for any $m < k$.

First, consider the function w_k which satisfies (see the first equation in (4.4))

$$-\Delta w_k + w_k = k\tilde{w}_{k-1} + Y_k,$$

where Y_k denotes the right hand side of (4.5). Now let us remark that the decay of $k\tilde{w}_{k-1}$ is at most like $r^{k-1} r^{\frac{1-n}{2}} e^{-r}$, while the one of Y_k is stronger. Actually, $3U^2 w_k$, corresponding to the first term in (4.5) with $l = 0, m = k$, decays like $r^{1-n} e^{-2r} w_k$, while all the remaining terms contain three factors w_j, $j < k$. Therefore, w_k solves an equation of type $-\Delta w_k + w_k = f$ where $|f(r)| \le C'_{k-1} r^{k-1} r^{\frac{1-n}{2}} e^{-r}$. By Lemma 4.3 the conclusion follows.

In a similar way we can argue with the function $\tilde{w}_k$. ■

4.2 The approximate solutions: proof of (A1)

Given $i \in \{1 \dots m\}$, $\epsilon > 0$ small and $\xi \in \mathbb{R}$, define:

$$V_i(x) = V^i_{\epsilon,\xi}(x) = V_\epsilon(x - \xi p_i)\ , \quad v_i(x) = v^i_{\epsilon,\xi}(x) = v_\epsilon(x - \xi p_i)$$

where V_ϵ, v_ϵ are the functions found in Subsection 4.1. Sometimes we will write V_i, v_i with a double purpose: for short, and not to confuse with powers.

In the sequel, we will always assume that ξ is a parameter satisfying (1.9), where $\delta > 0$ is a small positive constant to be chosen.

Let

$$z_1 = z_1(\epsilon, \xi) := V^1_{\epsilon,\xi} + \cdots + V^m_{\epsilon,\xi} - v_\epsilon$$

and

$$z_2 = z_2(\epsilon, \xi) := v^1_{\epsilon,\xi} + \cdots + v^m_{\epsilon,\xi} - V_\epsilon.$$

Since V_ϵ and v_ϵ are radial, it follows that $z_i \in \mathcal{X}$, $i = 1, 2$. Let us define the following manifold of approximate solutions:

$$Z = \{\mathbf{z}_{\epsilon,\xi} = (z_1(\epsilon, \xi), z_2(\epsilon, \xi)) :\ \xi \in T_\epsilon\}.$$

Remark 4.5 The solutions $(u_{1,\epsilon}, u_{2,\epsilon})$ of Theorem 1.3 will bifurcate from suitable points of Z and thus have the following asymptotic behavior:

- $u_{1,\epsilon} \sim z_1(\epsilon, \xi) = V^1_{\epsilon,\xi} + \cdots + V^m_{\epsilon,\xi} - v_\epsilon \sim \sum U(\cdot + \xi p_i)$,

- $u_{2,\epsilon} \sim z_2(\epsilon, \xi) = v^1_{\epsilon,\xi} + \cdots + v^m_{\epsilon,\xi} - V_\epsilon \sim -U$.

We need the counterpart of the estimates proved in Lemma 3.3. Actually, it is convenient to prove some slightly more general estimates. Given a function $u : \mathbb{R}^n \to \mathbb{R}$ and $\xi \in \mathbb{R}^n$, we define $u_\xi(x) = u(x - \xi)$.

Lemma 4.6 *Given $u,\ u' : \mathbb{R}^n \to \mathbb{R}$ two positive continuous radial functions such that:*

$$u(x) \sim |x|^a e^{-b|x|},\ u'(x) \sim |x|^{a'} e^{-b'|x|} \quad (x \to \infty)$$

where $a,\ a' \in \mathbb{R}$, $b > 0$, $b' > 0$. Let $\xi \in \mathbb{R}^n$ tend to infinity. Then, the following asymptotic estimates hold:

(i) If $b < b'$,

$$\int_{\mathbb{R}^n} u_\xi u' \sim e^{-b|\xi|} |\xi|^a.$$

Clearly, if $b > b'$, a similar expression holds, by replacing a and b with a' and b'.

(ii) If $b = b'$, suppose, for simplicity, that $a \geq a'$. Then:

$$\int_{\mathbb{R}^n} u_\xi u' \sim \begin{cases} e^{-b|\xi|} |\xi|^{a+a'+\frac{n+1}{2}} & \text{if } a' > -\frac{n+1}{2} \\ e^{-b|\xi|} |\xi|^a \log \xi & \text{if } a' = -\frac{n+1}{2} \\ e^{-b|\xi|} |\xi|^a & \text{if } a' < -\frac{n+1}{2} \end{cases}$$

The proof of this Lemma is technical and is postponed in the Appendix, where we also prove other estimates for integrals with three functions, see Lemma 5.1.

Proposition 4.7 *Assumption $(A1)$ holds. More precisely, for some $\delta > 0$ fixed and any $\mathbf{z} \in Z$, we have that:*

$$\|\Phi'_\epsilon(\mathbf{z})\| \leq C\, \epsilon^{s/(s-r+\delta)}, \quad \forall\, \xi \in T_\epsilon.$$

Remark 4.8 If δ is small, the preceding estimate is better that the one in Lemma 3.4. For example, if we take a degenerate $\mathcal{P}$ consisting of two points, we have $s = 2$, $r = 1$ and $s/(s - r + \delta) > 1$, if $\delta < 1$. The proof will highlight that the sharpening is due to the better choice of the manifold Z.

Proof. One has

$$\begin{aligned} \Phi'(\mathbf{z})[\varphi, \psi] &= \int \left(-\Delta z_1 + z_1 - z_1^3 - \epsilon z_2\right) \varphi \\ &\quad + \int \left(-\Delta z_2 + z_2 - z_2^3 - \epsilon z_1\right) \psi. \end{aligned} \tag{4.7}$$

Using the definitions of z_i, and the fact that V, v verify

$$\begin{cases} -\Delta V + V &= V^3 + \epsilon v, \\ -\Delta v + v &= v^3 + \epsilon V, \end{cases}$$

we find:

$$\begin{cases} -\Delta z_1 + z_1 - z_1^3 - \epsilon z_2 & = -3\sum_{i\neq j} V_i^2 V_j - 6\sum_{i<j<k} V_i V_j V_k \\ & \quad -3\sum_i (V_i v^2 - V_i^2 v) + 6\sum_{i<j} V_i V_j v, \\ -\Delta z_2 + z_2 - z_2^3 - \epsilon z_1 & = -3\sum_{i\neq j} v_i^2 v_j - 6\sum_{i<j<k} v_i v_j v_k \\ & \quad -3\sum_i (v_i^2 V - v_i V^2) + 6\sum_{i<j} v_i v_j V. \end{cases}$$

Let us emphasize that above there are no more terms like ϵz_i. Inserting the preceding equations into (4.7) we get

$$|\Phi'(\mathbf{z})[\varphi,\psi]| \le \left[\sum_{l=1}^{5} \Omega_l\right] \|\varphi\| + \left[\sum_{l=1}^{5} \omega_l\right] \|\psi\|.$$

where

$$\begin{aligned} \Omega_1 = & \ 3\sum_{i\neq j} \left(\int V_i^4 V_j^2\right)^{1/2}, & \omega_1 = 3\sum_{i\neq j} \left(\int v_i^4 v_j^2\right)^{1/2} \\ \Omega_2 = & \ 6\sum_{i<j<k} \left(\int V_i^2 V_j^2 V_k^2\right)^{1/2}, & \omega_2 = 6\sum_{i<j<k} \left(\int v_i^2 v_j^2 v_k^2\right)^{1/2} \\ \Omega_3 = & \ 3\sum_i \left(\int V_i^2 v^4\right)^{1/2}, & \omega_3 = 3\sum_{i\neq j} \left(\int v_i^2 V^4\right)^{1/2} \\ \Omega_4 = & \ 3\sum_i \left(\int V_i^4 v^2\right)^{1/2}, & \omega_4 = 3\sum_{i<j<k} \left(\int v_i^4 V^2\right)^{1/2} \\ \Omega_5 = & \ 6\sum_{i<j} \left(\int V_i^2 V_j^2 v^2\right)^{1/2}, & \omega_5 = 6\sum_{i<j} \left(\int v_i^2 v_j^2 V^2\right)^{1/2}. \end{aligned}$$

The Lemma will follow if we show that all the $\Omega_l,\ \omega_l$ satisfy $\Omega_l,\ \omega_l \le c\,\epsilon^{s/(s-r+\delta)}$. To prove this, we use the Taylor expansion of the functions $v_\epsilon,\ V_\epsilon$ of order $m = [\frac{s}{s-r+\delta}]$ ($[a]$ denotes the integer part of $a \in \mathbb{R}$):

$$V_\epsilon = U + \sum_{k=1}^{m} \frac{\epsilon^k}{k!} w_k + O(\epsilon^{m+1}),\ v_\epsilon = \sum_{k=1}^{m} \frac{\epsilon^k}{k!} \tilde{w}_k + O(\epsilon^{m+1}).$$

Recall that all functions $w_k,\ \tilde{w}_k$ exhibit an exponential decay for $|x|$ large of the form $|x|^{\frac{1-n}{2}+k} e^{-|x|}$.

Let us start with Ω_1. To estimate $\int V_i^4 V_j^2$, we use the above Taylor expansion. A first term is given by $\int U_i^4 U_j^2$. By Lemma 4.6 with $b = 2,\ b' = 4,\ a = 1-n,\ a' = 2(1-n)$, we find (observe that $U_i,\ U_j$ are centered at $\xi p_i,\ \xi p_j$ respectively)

$$\int U_i^4 U_j^2 dx \sim |\xi|^{1-n} e^{-2|p_i - p_j|\xi}. \tag{4.8}$$

Since $|p_i - p_j| \geq s$ and $\xi \in T_\epsilon$, it follows that

$$\left(\int U_i^4 U_j^2 dx \right)^{1/2} \leq c\, \epsilon^{s/(s-r+\delta)}.$$

Let us show that this is the main term in Ω_1. Actually, the other terms of $\int V_i^4 V_j^2$ are of the form:

$$\epsilon^{k_1+k_2+k_3+k_4+s_1+s_2} \int \Pi_{\ell=1}^4 w_{k_\ell}(x - \xi p_i) \times \Pi_{\ell=1}^2 w_{s_\ell}(x - \xi p_j) dx$$

where $0 \leq k_i,\ s_i \leq m$. Because of the asymptotic behavior of w_k and Lemma 4.6, we have that the above expression is of order:

$$\epsilon^{k_1+k_2+k_3+k_4+s_1+s_2} \xi^{1-n+s_1+s_2} e^{-2|p_i-p_j|\xi}.$$

Since $\xi \in T_\epsilon$, this term is smaller than (4.8).

Let us now consider the term Ω_4. As before, one can check that the main term is:

$$\left(\int V_i^4 v^2 \right)^{1/2} \sim \left(\epsilon^2 \int U_i^4 \tilde{w}_1^2 dx \right)^{1/2}.$$

Since

$$U^4 \sim |x|^{2(1-n)} e^{-4|x|}, \qquad \tilde{w}_1^2 \sim |x|^{3-n} e^{-2|x|},$$

Lemma 4.6 implies that $\int U_i^4 \tilde{w}_1^2 dx \sim \xi^{3-n}\ e^{-2\,r\,\xi}$. Then, $\xi \in T_\epsilon$ and $r > s$ yield

$$\left(\int V_i^4 v^2 \right)^{1/2} \sim \epsilon\, \xi^{(3-n)/2}\ e^{-r\xi} < c\, \epsilon^{\frac{s}{s-r+\delta}}.$$

Similarly, one finds that $\omega_3 \leq c\, \epsilon^{\frac{s}{s-r+\delta}}$. All the other terms are smaller than Ω_1, Ω_4 and ω_3. For example, dealing with $\int V_i^2 V_j^2 V_k^2$, let us consider the term $\int U_i^2 U_j^2 U_k^2$. By Lemma 5.1, we obtain:

$$\int U_i^2 U_j^2 U_k^2 < e^{-2q\xi} \tag{4.9}$$

for any $s < q < \min\{|p_i - p| + |p_j - p| - |p_k - p| :\ p \in \mathbb{R}^n\}$. In view of (1.9), $e^{-2q\xi} < \epsilon^{\frac{2s}{s-r+\delta}}$ for small $\delta > 0$ fixed. We now observe that the one considered in (4.9) is the main term of $\int V_i^2 V_j^2 V_k^2$. Actually, the other terms are of the form

$$\epsilon^{k_1+k_2+s_1+s_2+t_1+t_2} \int \Pi_{\ell=1}^2 w_{k_\ell}(x - \xi p_i) w_{s_\ell}(x - \xi p_j) w_{t_\ell}(x - \xi p_k),$$

and, as before, we conclude that the above expression is smaller than (4.9).

The rest of the Ω_i, ω_i can be estimated by using the same ideas and the lemma follows. ■

4.3 The auxiliary equation: proof of (A2)

In this section we are concerned with the invertibility of $\Phi_\epsilon''(\mathbf{z})$ on $W = (T_{\mathbf{z}} Z^\epsilon)^\perp$. With this result in hand, we will be able to solve the auxiliary equation for any $\mathbf{z} \in Z_\epsilon$.

Lemma 4.9 *There exists a positive constant C_1 such that for ϵ small enough, $P\Phi_\epsilon''(\mathbf{z})$ is uniformly invertible for all $\xi \in \mathbb{R}$ satisfying* (1.9)*, and $\|[P\Phi_\epsilon''(\mathbf{z})]^{-1}\| \leq C_1$.*

Proof.

Observe that $T_{\mathbf{z}} Z = span\left\{\dfrac{\partial \mathbf{z}}{\partial \xi}\right\}$, and

$$\frac{\partial \mathbf{z}}{\partial \xi} = -\left(\sum_{i=1}^{m} \frac{\partial V_\epsilon}{\partial p_i}(x - \xi p_i), \sum_{i=1}^{m} \frac{\partial v_\epsilon}{\partial p_i}(x - \xi p_i)\right).$$

As in the one-dimensional case, we shall use the following orthogonal decomposition of the space W: $W = E \oplus F$, $E = span\,\{(z_1, 0), (0, z_2)\}$, $F = E^\perp \cap W$.

We first apply the bilineal form given by $\Phi_\epsilon''(\mathbf{z})$ on the pairs in E: let a, b in $\mathbb{R}$, and compute:

$$\begin{aligned}
\Phi_\epsilon''(\mathbf{z})[(az_1, bz_2)]^2 &= a^2 \int \left(|\nabla z_1|^2 + z_1^2 - 3z_1^4\right) \\
&\quad + b^2 \int \left(|\nabla z_2|^2 + z_2^2 - 3z_2^4\right) - 2ab\epsilon \int z_1 z_2 \\
&= \Phi_\epsilon'(\mathbf{z})[a^2 z_1, b^2 z_2] - 2a^2 \int z_1^4 - 2b^2 \int z_2^4 \\
&\quad + \epsilon(a^2 + b^2 - 2ab) \int z_1 z_2 \leq -c(a^2 + b^2),
\end{aligned}$$

for some positive constant $c > 0$ and ϵ small enough. Recall that, by Proposition 4.7, $\Phi_\epsilon'(\mathbf{z})$ is small.

Hence, $\Phi_\epsilon''(\mathbf{z})$ is negative definite on E. We now prove that $\Phi_\epsilon''(\mathbf{z})$ is positive definite on F; from that, we will conclude the invertibility result.

Choose an arbitrary $\mathbf{w} \in F$, $\mathbf{w} = (w_1, w_2)$, where $w_1 \perp z_1$, $w_2 \perp z_2$. For simplicity, assume that $\|\mathbf{w}\| = 1$.

Since $w_1 \perp z_1$, we have that:

$$\sum_{i=1}^{m} (w_1 \mid V_i) - (w_1 \mid v) = 0$$

Recall that v_ϵ is of order ϵ, and so it is the last term in the above equality. It is well-known, that the polyhedral group $\mathcal{G}$ is transitive, namely such that for every pair p_i, p_j, $i \neq j$, there exists $g \in \mathcal{G}$ such that $gp_i = p_j$. By composing with g and using that $w_1 \circ g = w_1$, we have that:

$$(w_1 \mid V_i) = (w_1 \mid V_j).$$

Therefore, $(w_1 \mid V_i) \leq C\epsilon$ for all $i = 1 \dots m$.

Moreover, $\mathbf{w}$ is orthogonal to $\dfrac{\partial \mathbf{z}}{\partial \xi}$, which implies that:

$$(w_1 \mid \frac{\partial z_1}{\partial \xi}) + (w_2 \mid \frac{\partial z_2}{\partial \xi}) = 0.$$

Observe now that $\dfrac{\partial z_2}{\partial \xi}$ is of order ϵ, and this means that:

$$(w_1 \mid \frac{\partial z_1}{\partial \xi}) \leq C\epsilon.$$

Take again $g \in \mathcal{G}$ such that $gp_i = p_j$, $i \neq j$. Reasoning as above, we obtain:

$$\left(w_1 \mid \frac{\partial V_\epsilon}{\partial p_i}(x - \xi p_i)\right) = \left(w_1 \mid \frac{\partial V_\epsilon}{\partial p_j}(x - \xi p_j)\right).$$

Therefore, we conclude that

$$\left(w_1 \mid \frac{\partial V_\epsilon}{\partial p_i}(x - \xi p_i)\right) \leq C\epsilon, \ \forall\ i = 1 \dots n.$$

We now claim that $w_1 \perp \dfrac{\partial V_\epsilon}{\partial q}(x - \xi p_i)$ for any q orthogonal to p_i (in the $\mathbb{R}^n$ sense). To prove that, we need to make use of the following property of $\mathcal{P}$. For every vertex $p \in \mathcal{P}$ we can find a line $\ell \subset \mathbb{R}^2$, resp. two planes $\pi_1 \neq \pi_2 \subset \mathbb{R}^3$, such that $p \in \ell$, resp. $p_j \in \pi_1 \cap \pi_2$, and $\mathcal{P}$ is symmetric with respect to ℓ, resp. π_1 and π_2. Such a property is trivially verified for the regular polygons in $\mathbb{R}^2$. For the regular polyhedra in $\mathbb{R}^3$, it suffices to take two non-planar sides passing through p and consider the two planes containing each side and the origin.

Fixed i, take $q_1, \dots q_{n-1} \in \mathbb{R}^n$ unitary vectors orthogonal to p_i, linearly independent, and such that $\mathcal{P}$ is symmetric with respect to the hyperplane (plane, if $n = 3$, line, if $n = 2$):

$$\pi_j = \{x \in \mathbb{R}^n : \ \langle x, q_j \rangle = 0\}.$$

Let g_j be the reflection with respect to π_j, that is, $g_j(x) = x - 2\langle x, q_j \rangle q_j$.

Recall now that w_1 is $\mathcal{G}-$invariant. Moreover, since V_ϵ is radial, the function: $\dfrac{\partial V_\epsilon}{\partial q_j}(x - \xi p_i)$ is odd with respect to q_j, that is,

$$\frac{\partial V_\epsilon}{\partial q_j}(g_j(x) - \xi p_i) = -\frac{\partial V_\epsilon}{\partial q_j}(x - \xi p_i).$$

Therefore, $w_1 \perp \dfrac{\partial V_\epsilon}{\partial q_j}(x - \xi p_i)$ for any $j = 1 \dots n - 1$.

We now focus on w_2. Since $w_2 \perp z_2$, we have that:

$$\sum_{i=1}^{m} (w_2 \mid v_i) - (w_2 \mid V) = 0.$$

Since v_i is of order ϵ, we have that: $(w_2 \mid V)$ is also of order ϵ. Recall that both w_2 and V_ϵ are radial functions; using symmetry arguments as before, one easily obtains that $w_2 \perp \dfrac{\partial V_\epsilon}{\partial x_j}(x)$ for any $j = 1 \ldots n$.

As we mentioned before, we claim that $\Phi_\epsilon''(\mathbf{z})[\mathbf{w},\mathbf{w}] > c > 0$. Let us compute:

$$\Phi_\epsilon''(\mathbf{z})[\mathbf{w},\mathbf{w}] = \|w_1\|^2 + \|w_2\|^2 - \int \left[3z_1^2 w_1^2 + 3z_2^2 w_2^2 - 2\epsilon w_1 w_2\right] dx.$$

By neglecting the terms which tend to zero, we obtain,

$$\Phi_\epsilon''(\mathbf{z})[\mathbf{w},\mathbf{w}] \sim \|w_1\|^2 + \|w_2\|^2 - 3\int w_1^2 \left(\sum_{i=1}^m V_i^2\right) dx - 3\int V^2 w_2^2 dx. \quad (4.10)$$

First, recall that since $(w_2 \mid V) = o(1)$, $(w_2 \mid \dfrac{\partial V_\epsilon}{\partial x_j}(x)) = 0$, we can use Lemma 3.1 to prove that:

$$\int |\nabla w_2|^2 + w_2^2 - 3V^2 w_2^2 \geq c\|w_2\|^2 - C\epsilon. \quad (4.11)$$

In order to estimate the remaining part of (4.10), we argue as in the one-dimensional case.

Claim: There exists $R \in (\xi^{1/4}, \xi^{1/2})$ such that

$$\sum_{j=1}^m \int_{R<|x-\xi p_j|<R+1} [|\nabla w_i|^2 + w_i^2] dx < 2\xi^{-1/2}, \quad i = 1,\ 2$$

The proof of this claim can be done as in the ODE case.

Once we have that constant R, we can define:

$$\chi_j(x) = \begin{cases} 1 & |x - \xi p_j| < R \\ 0 & |x - \xi p_j| > R + 1 \\ |\nabla \chi_j(x)| \leq 2 & \forall\, x \in \mathbb{R}^n \end{cases}$$

Define also $\chi_0(x) = 1 - \sum_{j=1}^m \chi_j(x)$. We can now decompose $w_1 = \sum_{j=0}^m w_1 \chi_j$ and argue as in the one-dimensional case, to conclude that:

$$\int |\nabla w_1|^2 + w_1^2 - 3w_1^2 \left(\sum_{i=1}^m V_i^2\right) \geq c\|w_1\|^2 - C\epsilon \quad (4.12)$$

Estimates (4.12) and (4.11) imply that (4.10) is above a fixed positive constant. ■

4.4 Proof of Theorem 1.3

In the previous sections we have proved that $(A1)$ and $(A2)$ hold. Since $(A3)$ is trivially satisfied as in the ODE case, we can use the results of Section 2 to conclude that the auxiliary equation has a solution $\mathbf{w}_{\epsilon,\xi}$ for ϵ small and any ξ satisfying (1.9).

So, we are led with the study of critical points of the reduced functional, that is, the function:

$$\widetilde{\Phi}_\epsilon(\xi) = \Phi_\epsilon(\mathbf{z}_{\epsilon,\xi} + \mathbf{w}_{\epsilon,\xi}).$$

We first prove the analogue of Lemma 3.6.

Lemma 4.10 *There exist positive constants $C_\epsilon, \underline{C}, \overline{C} > 0$ such that*

$$\widetilde{\Phi}_\epsilon(\xi) = C_\epsilon - a_{\epsilon,\xi}\,|\xi|^{\frac{1-n}{2}} e^{-s\xi} + b_{\epsilon,\xi}\,\epsilon\,|\xi|^{\frac{3-n}{2}}\, e^{-r\xi} + o(\epsilon^{\frac{s}{s-r}}), \quad \forall \xi \in T_\epsilon,\ \epsilon \sim 0,$$

where C_ϵ is independent of ξ and $a_{\epsilon,\xi},\ b_{\epsilon,\xi} \in [\underline{C}, \overline{C}]$.

Proof. To simplify notation we will write $\mathbf{z}$ instead of $\mathbf{z}_{\epsilon,\xi}$ and $\mathbf{w}$ instead of $\mathbf{w}_{\epsilon,\xi}$. First of all, since Φ''_ϵ is uniformly bounded one has $\widetilde{\Phi}_\epsilon(\xi) = \Phi_\epsilon(\mathbf{z}) + \Phi'_\epsilon(\mathbf{z})[\mathbf{w}] + O(\|\mathbf{w}\|^2)$. Using Proposition 4.7 we deduce

$$\widetilde{\Phi}_\epsilon(\xi) = \Phi_\epsilon(\mathbf{z}) + O(\epsilon^{\frac{2s}{s-r+\delta}}). \tag{4.13}$$

So, in order to study the reduced functional $\widetilde{\Phi}_\epsilon$, we will estimate:

$$\Phi_\epsilon(\mathbf{z}) = \tfrac{1}{2}\|z_1\|^2 + \tfrac{1}{2}\|z_2\|^2 - \tfrac{1}{4}\int (z_1^4 + z_2^4)dx - \epsilon \int z_1 z_2 dx.$$

Let us evaluate separately the various terms in the right-hand side. Using the definition of z_1 and z_2, one has:

$$\begin{aligned} \tfrac{1}{2}\|z_1\|^2 + \tfrac{1}{2}\|z_2\|^2 &= C_\epsilon + \sum_{i<j}(V_i \mid V_j) - \sum_i (V_i \mid v) \\ &\quad + \sum_{i<j}(v_i \mid v_j) - \sum_i (v_i \mid V), \end{aligned} \tag{4.14}$$

where $C_\epsilon = \sum_{i=1}^m \|V_i\|^2 + \|v\|^2 + \sum_{i=1}^m \|v_i\|^2 + \|V\|^2$ which is independent of ξ.

Moreover

$$\begin{aligned}\int (z_1^4+z_2^4)dx &= C'_\epsilon + 4\int\left(\sum_{i\neq j} V_i^3 V_j - \sum_i V_i^3 v - \sum_i V_i v^3\right)dx\\ &+4\int\left(\sum_{i\neq j} v_i^3 v_j - \sum_i v_i^3 V - \sum_i v_i V^3\right)dx\\ &+6\int\left(\sum_{i<j} V_i^2 V_j^2 + \sum_i V_i^2 v^2\right)dx\\ &+6\int\left(\sum_{i<j} v_i^2 v_j^2 + \sum_i v_i^2 V^2\right)dx\\ &+12\int\left(\sum_{i\neq j\neq k\neq i} V_i^2 V_j V_k + \sum_{i\neq j} v^2 V_i V_j - \sum_{i\neq j} V_i^2 V_j v\right)dx\\ &+12\int\left(\sum_{i\neq j\neq k\neq i} v_i^2 v_j v_k + \sum_{i\neq j} V^2 v_i v_j - \sum_{i\neq j} v_i^2 v_j V\right)dx\end{aligned}$$

And finally

$$\epsilon\int z_1 z_2 dx = C''_\epsilon + \epsilon\int\left(\sum_{i\neq j} V_i v_j - \sum_i v v_i - \sum_i V V_i\right)dx.$$

As in the proof of Proposition 4.7, we will estimate the preceding integrals making use of a Taylor expansion of the functions V, v, up to a convenient order, as well as the decay of the functions w_k, $\tilde{w}_k$. We anticipate that the leading terms are

$$-\int\sum_{i>j} V_i^3 V_j + \epsilon\int\sum_i V V_i \sim -a_{\epsilon,\xi}\xi^{\frac{1-n}{2}} e^{-\xi s} + b_{\epsilon,\xi}\epsilon\xi^{\frac{3-n}{2}} e^{-\xi r}, \tag{4.15}$$

where $a_{\epsilon,\xi}, b_{\epsilon,\xi} \in [\underline{C}, \overline{C}]$. Since $\xi \in T_\epsilon$, then $e^{-\xi s} + \epsilon e^{-\xi r} \sim \epsilon^{s/(s-r)}$ and hence each term of the form $o(\epsilon^{s/(s-r)})$ are negligible with respect to (4.15). We are going to show that this is indeed the case for all the remaining integrals. Below we use Lemma 4.6 and the arguments carried out in the proof of Proposition 4.7. Hereafter, α denotes a positive constant depending only on the dimension n.

First of all, we evaluate

$$(V_i \mid V_j) - \int V_i^3 V_j = \epsilon\int v_i V_j < \epsilon^2 \xi^\alpha e^{-s\xi} < \xi^\alpha \epsilon^2 \epsilon^{\frac{s}{s-r+\delta}} \tag{4.16}$$

Let us point out that, according to (4.14), (4.16) will only be used for $i < j$.

Moreover, we find

$$-(V_i \mid v) + \int V_i^3 v = \epsilon \int v_i v < \epsilon^3 \xi^\alpha e^{-r\xi} < \xi^\alpha \epsilon^3 \epsilon^{\frac{r}{s-r+\delta}},$$

$$-(v_i \mid V) + \int v_i^3 V = \epsilon \int v v_i < \epsilon^3 \xi^a e^{-r\xi} < \xi^a \epsilon^3 \epsilon^{\frac{r}{s-r+\delta}},$$

$$(v_i \mid v_j) - \int v_i^3 v_j = \epsilon \int V_i v_j < \epsilon^2 \xi^a e^{-s\xi} < \xi^a \epsilon^2 \epsilon^{\frac{s}{s-r+\delta}}.$$

Similarly, we can evaluate the other integrals yielding

$$\begin{aligned}
\int V_i v^3 &= \int V v_i^3 < \xi^\alpha \epsilon^3 e^{-r\xi} < \xi^\alpha \epsilon^3 \epsilon^{\frac{r}{s-r+\delta}}, \\
\epsilon \int V_i v_j &< \epsilon^2 \xi^\alpha e^{-s\xi} < \xi^\alpha \epsilon^2 \epsilon^{\frac{s}{s-r+\delta}}, \\
\epsilon \int v v_i &< \epsilon^3 \xi^\alpha e^{-r\xi} < \xi^\alpha \epsilon^3 \epsilon^{\frac{r}{s-r+\delta}}, \\
\int V_i^2 V_j^2 &< \xi^\alpha e^{-2s\xi} < \xi^\alpha \epsilon^{\frac{2s}{s-r+\delta}}, \\
\int V_i^2 v^2 &= \int v_i^2 V^2 < \epsilon \xi^\alpha e^{-2r\xi} < \xi^\alpha \epsilon \epsilon^{\frac{2r}{s-r+\delta}}, \\
\int v_i^2 v_j^2 &< \epsilon^2 \xi^\alpha e^{-2s\xi} < \xi^\alpha \epsilon^2 \epsilon^{\frac{2s}{s-r+\delta}}.
\end{aligned}$$

We now consider the triple products:

$$\int V_i^2 V_j V_k < e^{-q\xi} < \epsilon^{\frac{q}{s-r+\delta}}$$

where $s < q < \min\{|p_i - p| + |p_j - p| + |p_k - p| : \ p \in \mathbb{R}^n\}$. Moreover,

$$\int V_i^2 V_j v < \epsilon e^{-q'\xi} < \epsilon \epsilon^{\frac{q'}{s-r+\delta}}$$

$$\int v^2 V_i V_j < \epsilon^2 e^{-q'\xi} < \epsilon^2 \epsilon^{\frac{q'}{s-r+\delta}}$$

where $s < q' < \min\{|p_i - p| + |p_j - p| + |p| : \ p \in \mathbb{R}^n\}$. In the same way, we can neglect the other triple products.

From the preceding estimates it follows that we can choose a fixed $\delta > 0$ small enough so that all the above expressions are negligible with respect to (4.15). This δ only depends on the geometric properties of the polytope $\mathcal{P}$.

In conclusion, taking also into account what remarked after equation (4.16), we infer that

$$\Phi_\epsilon(\mathbf{z}_{\epsilon,\xi}) \sim C_\epsilon - \int \sum_{i>j} V_i^3 V_j + \epsilon \int \sum_i V V_i.$$

Now the Lemma follows from this, (4.15) and (4.13). ∎

We are now ready to complete the proof of Theorem 1.3.

Proof of Theorem 1.3 completed. To find a critical point of Φ_ϵ bifurcating from Z, it remains to prove that for ϵ sufficiently small there exists a critical point $\tilde{\xi}_\epsilon$ of the reduced functional satisfying inequality (1.9).

For this, let us study the auxiliary function:

$$f(\xi) = -c_1 \xi^{\frac{1-n}{2}} e^{-\xi s} + c_2 \epsilon \xi^{\frac{3-n}{2}} e^{-\xi r}.$$

It is clear that this function is negative for $\xi = 1$ (for ϵ small) and positive for ξ large. Moreover, $f(\xi) \to 0$ $(\xi \to +\infty)$, and hence f has an absolute maximum at a certain $\xi_\epsilon > 0$.

Define $\zeta = \frac{-\log(\epsilon)}{s-r+\delta}$ and $\zeta' = \frac{-\log(\epsilon)}{s-r}$, so that $T_\epsilon = (\zeta, \zeta')$. One has

$$f(\zeta) = -c_1 \zeta^{\frac{1-n}{2}} \epsilon^{\frac{s}{s-r+\delta}} + c_2 \epsilon \zeta^{\frac{3-n}{2}} \epsilon^{\frac{r}{s-r+\delta}} = -c_1 \zeta^{\frac{1-n}{2}} \epsilon^{\frac{s}{s-r+\delta}} + c_2 \zeta^{\frac{3-n}{2}} \epsilon^{\frac{s+\delta}{s-r+\delta}} < 0.$$

Therefore, $\xi_\epsilon > \zeta$. Let us show that $\xi_\epsilon < \zeta'$. By elementary calculation it follows that $f'(\xi_\epsilon) = 0$ implies

$$c_1 e^{-(s-r)\xi_\epsilon} \left[\frac{1-n}{2} - s\xi_\epsilon \right] = c_2 \epsilon \left[\frac{3-n}{2} \xi_\epsilon - r\xi_\epsilon^2 \right],$$

and hence

$$e^{-(s-r)\xi_\epsilon} = \epsilon \frac{c_2 \left[\frac{3-n}{2} \xi_\epsilon - r\xi_\epsilon^2 \right]}{c_1 \left[\frac{1-n}{2} - s\xi_\epsilon \right]} \tag{4.17}$$

Moreover, since $\xi_\epsilon > \zeta$, one has that $\xi_\epsilon \to +\infty$ as $\epsilon \to 0$ and hence

$$\epsilon \frac{c_2 \left[\frac{3-n}{2} \xi_\epsilon - r\xi_\epsilon^2 \right]}{c_1 \left[\frac{1-n}{2} - s\xi_\epsilon \right]} > \epsilon, \quad (\epsilon \to 0).$$

Then this and (4.17) yield

$$\epsilon < e^{-(s-r)\xi_\epsilon} \Rightarrow \xi_\epsilon < \frac{-\log(\epsilon)}{s-r} = \zeta'.$$

In order to complete the proof of the lemma, we estimate $f(\xi_\epsilon)$ by using (4.17):

$$f(\xi_\epsilon) = e^{-\xi_\epsilon r} \left[-c_1 \xi_\epsilon^{\frac{1-n}{2}} \epsilon \frac{c_2 \left[\frac{3-n}{2} \xi_\epsilon - r\xi_\epsilon^2 \right]}{c_1 \left[\frac{1-n}{2} - s\xi_\epsilon \right]} + c_2 \epsilon \xi_\epsilon^{\frac{3-n}{2}} \right] =$$

$$c_2 e^{-\xi_\epsilon r} \xi_\epsilon^{\frac{3-n}{2}} \epsilon \left[1 - \frac{\frac{3-n}{2} - r\xi_\epsilon}{\frac{1-n}{2} - s\xi_\epsilon} \right] \sim c_2 e^{-\xi_\epsilon r} \xi_\epsilon^{\frac{3-n}{2}} \epsilon \left[1 - \frac{r}{s} \right].$$

By using again (4.17), we have:

$$f(\xi_\epsilon) \sim c_2 \left[\epsilon \frac{r}{s} \xi_\epsilon \right]^{\frac{r}{s-r}} \xi_\epsilon^{\frac{3-n}{2}} \epsilon \sim \epsilon^{\frac{s}{s-r}} \xi_\epsilon^{\frac{3-n}{2} + \frac{r}{s-r}}$$

Finally,

$$f(\zeta') = -c_1 \zeta'^{\frac{1-n}{2}} \epsilon^{\frac{s}{s-r}} + c_2 \epsilon \zeta'^{\frac{3-n}{2}} \epsilon^{\frac{r}{s-r}} \sim \zeta'^{\frac{3-n}{2}} \epsilon^{\frac{s}{s-r}}.$$

Recall now Lemma 4.10; then, the above estimates imply the existence of a maximum for $\widetilde{\Phi}_\epsilon$ at a certain $\tilde{\xi}_\epsilon \in (\zeta, \zeta')$. This completes the proof of Theorem 1.3. ■

Remark 4.11 We could consider systems with more than two equations, namely the PDE counterpart of (3.21), yielding results quite similar to the ones found in Subsection 3.4.

5 Appendix

In this appendix we will prove Lemma 4.6. We can suppose, by using a convenient rotation, that $\xi = (\xi_0, 0, \dots 0)$. The proof requires some technical work and is divided in some steps.

STEP 1. We begin by showing a bound from below in any case, namely

$$\int_{\mathbb{R}^n} u_\xi u' dx \geq c \max\{|\xi|^a e^{-b|\xi|}, |\xi|^{a'} e^{-b'|\xi|}\}. \tag{5.1}$$

To prove (5.1) it suffices to consider the ball $B \subset \mathbb{R}^n$ centered at 0 with radius 1. Since in B one has that $u' \geq c_1 > 0$ we deduce

$$\int_{\mathbb{R}^n} u_\xi u' dx > \int_B u_\xi u' dx \geq c\, |\xi|^a e^{-b|\xi|},$$

proving (5.1). Analogously,

$$\int_{\mathbb{R}^n} u_\xi u' dx \geq c\, |\xi|^{a'} e^{-b'|\xi|}.$$

STEP 2. We now give the bound from above in the case $b < b'$:

$$\int_{\mathbb{R}^n} u_\xi u' dx \leq c\, |\xi|^a e^{-b|\xi|}. \tag{5.2}$$

Proof of (5.2). We will use the following notation: $x = (r, y)$, with $r \in \mathbb{R}$ and $y \in \mathbb{R}^{n-1}$.

$$\begin{aligned}
&\int_{\mathbb{R}^n} u_\xi(x) u'(x) dx = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^{n-1}} u_\xi(r, y) u'(r, y)\, dy\, dr \\
&= \int_{-\infty}^{1} \int_{\mathbb{R}^{n-1}} u_\xi(r, y) u'(r, y)\, dy\, dr + \int_{1}^{\xi_0 - 1} \int_{\mathbb{R}^{n-1}} u_\xi(r, y) u'(r, y)\, dy\, dr \\
&+ \int_{\xi_0 - 1}^{+\infty} \int_{\mathbb{R}^{n-1}} u_\xi(r, y) u'(r, y)\, dy\, dr
\end{aligned}$$

Let us estimate the first integral term above. Observe that for $x = (r, y)$, $r < 1$, we have that $u_\xi(x) \le C|\xi|^a e^{-b|\xi|}$. Since u' is integrable, we conclude that

$$\int_{-\infty}^{1} \int_{\mathbb{R}^{n-1}} u_\xi(r,y) u'(r,y)\, dy\, dr \le C|\xi|^a e^{-b|\xi|}.$$

Analogously, one can prove that:

$$\int_{\xi_0-1}^{\infty} \int_{\mathbb{R}^{n-1}} u_\xi(r,y) u'(r,y)\, dy\, dr \le C|\xi|^{a'} e^{-b'|\xi|}.$$

We now study the remaining term, that is,

$$\int_1^{\xi_0-1}\int_{\mathbb{R}^{n-1}} u_\xi(r,y) u'(r,y)\, dy\, dr \sim \int_1^{\xi_0-1}\int_{\mathbb{R}^{n-1}} |x|^{a'} e^{-b'|x|} \cdot |\xi - x|^a e^{-b|\xi - x|}\, dy\, dr.$$

Remark that the following elementary inequalities holds true:

$$\begin{cases} |x|^2 = r^2 + |y|^2 \implies r \le |x| \le r + |y| \le r(1+|y|), \\ \\ |\xi - x|^2 = |y|^2 + |\xi_0 - r|^2 \implies |\xi_0 - r| \le |\xi - x| \le |\xi_0 - r|(1+|y|), \end{cases}$$

as well as $\xi_0 \le |r| + |\xi_0 - r| \le |x| + |\xi - x|$.

Suppose first that both a and a' are negative. Then, we have

$$\begin{aligned} |x|^{a'} |\xi - x|^a e^{-b'|x| - b|\xi - x|} &\le r^{a'} |\xi_0 - r|^a e^{-b\xi_0} e^{-(b'-b)|x|} \\ &\le r^{a'} |\xi_0 - r|^a e^{-b\xi_0} e^{\frac{-(b'-b)r}{2}} e^{\frac{-(b'-b)|y|}{2}} \end{aligned}$$

If both a and a' are nonnegative, we have instead:

$$\begin{aligned} |x|^{a'} |\xi - x|^a e^{-b'|x| - b|\xi - x|} &\le r^{a'} (1+|y|)^{a'} |\xi_0 - r|^a (1+|y|)^a e^{-b\xi_0} e^{-(b'-b)|x|} \\ &\le r^{a'} |\xi_0 - r|^a e^{-b\xi_0} e^{\frac{-(b'-b)r}{2}} (1+|y|)^{a'} (1+|y|)^{a'} e^{\frac{-(b'-b)|y|}{2}} \end{aligned}$$

If a is positive and a' is negative, or viceversa, we would have a similar expression. In any case, we have:

$$\begin{aligned} &\int_1^{\xi_0-1} \int_{\mathbb{R}^{n-1}} |x|^{a'} e^{-b'|x|} \cdot |\xi - x|^a e^{-b|\xi - x|}\, dy\, dr \\ &\le \int_1^{\xi_0-1} \int_{\mathbb{R}^{n-1}} r^{a'} |\xi_0 - r|^a e^{-b\xi_0} e^{\frac{-(b'-b)r}{2}} h(y)\, dy\, dr \end{aligned}$$

where $h(y)$ is a function integrable in $\mathbb{R}^{n-1}$. Therefore, we obtain:

$$Ce^{-b\xi_0} \int_1^{\xi_0-1} r^{a'} |\xi_0 - r|^a e^{\frac{-(b'-b)r}{2}}\, dr.$$

Let us decompose:

$$\int_1^{\xi_0-1} r^{a'}|\xi_0-r|^a e^{\frac{-(b'-b)r}{2}}\,dr =$$
$$= \int_1^{\frac{\xi_0}{2}} r^{a'}|\xi_0-r|^a e^{\frac{-(b'-b)r}{2}}\,dr + \int_{\frac{\xi_0}{2}}^{\xi_0-1} r^{a'}|\xi_0-r|^a e^{\frac{-(b'-b)r}{2}}\,dr.$$

The first integral can be easily estimated:

$$\int_1^{\frac{\xi_0}{2}} r^{a'}|\xi_0-r|^a e^{\frac{-(b'-b)r}{2}}\,dr \sim \xi_0^a \int_1^{\frac{\xi_0}{2}} r^{a'} e^{\frac{-(b'-b)r}{2}}\,dr \sim \xi_0^a.$$

We show now that the second integral is smaller: actually,

$$\int_{\frac{\xi_0}{2}}^{\xi_0-1} r^{a'}|\xi_0-r|^a e^{\frac{-(b'-b)r}{2}}\,dr \le \max\{1,\xi_0^{a'}\}\ \max\{1,\xi_0^a\}\, e^{\frac{-(b'-b)\xi_0}{4}}\ \frac{\xi_0-2}{2}.$$

The proof of Step 2 is complete.

STEP 3. We now treat the case $b=b'$. As before, we split the integral in three parts.

$$\int_{\mathbb{R}^n} u_\xi(x)u'(x)dx = \int_{-\infty}^{+\infty}\int_{\mathbb{R}^{n-1}} u_\xi(r,y)u'(r,y)\,dy\,dr$$
$$= \int_{-\infty}^{1}\int_{\mathbb{R}^{n-1}} u_\xi(r,y)u'(r,y)\,dy\,dr$$
$$+ \int_{1}^{\xi_0-1}\int_{\mathbb{R}^{n-1}} u_\xi(r,y)u'(r,y)\,dy\,dr + \int_{\xi_0-1}^{+\infty}\int_{\mathbb{R}^{n-1}} u_\xi(r,y)u'(r,y)\,dy\,dr.$$

Reasoning exactly as in Step 2, one can easily see that

$$\int_{-\infty}^{1}\int_{\mathbb{R}^{n-1}} u_\xi(r,y)u'(r,y)\,dy\,dr \le C|\xi|^a e^{-b|\xi|},$$
$$\int_{\xi_0-1}^{\infty}\int_{\mathbb{R}^{n-1}} u_\xi(r,y)u'(r,y)\,dy\,dr \le C|\xi|^{a'} e^{-b|\xi|}.$$

We now deal with the second term, that will also be divided into two terms:

$$\int_1^{\xi_0-1}\int_{\mathbb{R}^{n-1}} u_\xi u'\,dy\,dr = \int_1^{\xi_0/2}\int_{\mathbb{R}^{n-1}} u_\xi u'\,dy\,dr + \int_{\xi_0/2}^{\xi_0-1}\int_{\mathbb{R}^{n-1}} u_\xi u'\,dy\,dr. \tag{5.3}$$

By symmetry arguments, once we obtain the estimate for the first term of (5.3), the second can be estimated just by interchanging the roles of a and a'.

A last decomposition of the integrals is needed:

$$\int_1^{\xi_0/2}\int_{\mathbb{R}^{n-1}} u_\xi u'\,dy\,dr = \int_1^{\frac{\xi_0}{2}} dr \int_{|y|<r} u_\xi u'\,dy\,dr + \int_1^{\frac{\xi_0}{2}} dr \int_{|y|\geq r} u_\xi u'\,dy\,dr. \tag{5.4}$$

We first study the first term of (5.4), namely

$$\int_1^{\frac{\xi_0}{2}}\int_{|y|<r} u_\xi u'\,dy\,dr \sim \int_1^{\frac{\xi_0}{2}}\int_{|y|<r} |x|^{a'}e^{-b|x|}\cdot|\xi-x|^a e^{-b|\xi-x|}\,dy\,dr.$$

Recall that $x=(r,y)$, and observe that for $|y|<r$ we have

$$r<|x|=\sqrt{r^2+|y|^2}<\sqrt{2}r,$$

and, using also the fact that $1<r<\xi_0/2$,

$$\xi_0-r\leq|\xi-x|\leq\xi_0-r+|y| \implies \frac{\xi_0}{2}<|\xi-x|<\xi_0. \tag{5.5}$$

Moreover, we claim that there exist $0<\delta_1<\delta_2$ such that for any $x=(r,y)$ with $r\in(1,\xi_0/2)$, $|y|<r$, there holds

$$\xi_0+\delta_1\frac{|y|^2}{r}<|x|+|\xi-x|<\xi_0+\delta_2\frac{|y|^2}{r}. \tag{5.6}$$

The proof of (5.6) is postponed at the end of this section.

Using the preceding inequalities, we infer

$$\int_1^{\frac{\xi_0}{2}}\int_{|y|<r} |x|^{a'}|\xi-x|^a e^{-b(|x|+|\xi-x|)}\,dy\,dr \sim \int_1^{\frac{\xi_0}{2}}\int_{|y|<r} r^{a'}\xi_0^a e^{-b(\xi_0+\delta\frac{|y|^2}{r})}\,dy\,dr$$
$$= \xi_0^a e^{-b\xi_0}\int_1^{\frac{\xi_0}{2}}\int_{|y|<r} r^{a'}e^{-b\delta\frac{|y|^2}{r}}\,dy\,dr,$$

where $\delta=\delta_1$ if we look for a lower bound and $\delta=\delta_2$ for un upper bound. Performing the change of variable $y=\zeta\sqrt{r}$ we find

$$\xi_0^a e^{-b\xi_0}\int_1^{\frac{\xi_0}{2}} r^{a'}\int_{|y|<r} e^{-b\delta\frac{|y|^2}{r}}\,dy\,dr = \xi_0^a e^{-b\xi_0}\int_1^{\frac{\xi_0}{2}} r^{a'}r^{\frac{n-1}{2}}\int_{|\zeta|<\sqrt{r}} e^{-b\delta|\zeta|^2}d\zeta\,dr$$
$$\sim \xi_0^a e^{-b\xi_0}\int_1^{\frac{\xi_0}{2}} r^{a'+\frac{n-1}{2}}\,dr \sim \begin{cases} e^{-b\xi_0}\xi_0^{a+a'+\frac{n+1}{2}} & \text{if } a'+\frac{n+1}{2}>0,\\ e^{-b\xi_0}\xi_0^a\log\xi_0 & \text{if } a'+\frac{n+1}{2}=0,\\ e^{-b\xi_0}\xi_0^a & \text{if } a'+\frac{n+1}{2}<0.\end{cases}$$

We now study the second term of (5.4), namely:

$$\int_1^{\frac{\xi_0}{2}} \int_{|y|\geq r} u_\xi u' \, dy \, dr \sim \int_1^{\frac{\xi_0}{2}} dr \int_{|y|\geq r} |x|^{a'} e^{-b|x|} \cdot |\xi - x|^a e^{-b|\xi - x|} \, dy \, dr.$$

First of all, observe that:

$$r < |y| \Rightarrow |x| > \sqrt{2}\, r \Rightarrow \frac{1}{\sqrt{2}} |x| > r,$$

$$|x| \geq |y| \Rightarrow \left(1 - \frac{1}{\sqrt{2}}\right) |x| > \left(1 - \frac{1}{\sqrt{2}}\right) |y|.$$

We sum the above right expressions and obtain that $|x| \geq r + \alpha |y|$, with $\alpha = 1 - 1/\sqrt{2}$.

Taking also into account that $|\xi - x| > \xi_0 - r$, we deduce

$$|x| + |\xi - x| > \xi_0 + \alpha |y|. \tag{5.7}$$

Moreover, there holds:

$$r \leq |x| \leq r + |y| \leq 2r|y|,$$

$$\frac{\xi_0}{2} \leq |\xi - x| \leq \xi_0 + |y| \leq 2\xi_0 |y|.$$

Using the above inequalities and (5.7) we find

$$\begin{aligned} &\int_1^{\frac{\xi_0}{2}} \int_{|y|\geq r} |x|^{a'} |\xi - x|^a e^{-b(|x| + |\xi - x|)} \, dy \, dr \leq \\ &C \int_1^{\frac{\xi_0}{2}} \int_{|y|\geq r} r^{a'} \xi_0^a e^{-b\xi_0} e^{-\alpha|y|} |y|^{a_+ + a'_+} \, dy \, dr, \end{aligned} \tag{5.8}$$

where $a_+ = \max\{a, 0\}$, $a'_+ = \max\{a', 0\}$. We now estimate this last expression:

$$\xi_0^a e^{-b\xi_0} \int_1^{\frac{\xi_0}{2}} r^{a'} \int_{|y|\geq r} e^{-\alpha|y|} |y|^{a_+ + a'_+} \, dy \, dr$$

$$\leq \xi_0^a e^{-b\xi_0} \int_1^{+\infty} \int_r^{+\infty} \rho^{a'} e^{-\alpha\rho} \rho^{a_+ + a'_+} \rho^{n-2} \, d\rho \, dr$$

$$= \xi_0^a e^{-b\xi_0} \int_1^{+\infty} \int_1^{\rho} \rho^s e^{-\alpha\rho} \, dr \, d\rho = \xi_0^a e^{-b\xi_0} \int_1^{+\infty} \rho^s e^{-\alpha\rho} (\rho - 1) \, dr \leq C \xi_0^a e^{-b\xi_0},$$

where $s = a' + a_+ + a'_+ + n - 2$. Such a term is also a lower bound, as shown in Step 1.

Recall that the estimate of the second term of (5.3) involves analogous expressions, but interchanging the roles of a and a'. From all the previous, we can conclude the statement of Lemma 4.6.

It remains to prove (5.6). For convenience of the reader, we state it here again:

There exists $0 < \delta_1 < \delta_2$ such that for any $x = (r, y)$ with $r \in (1, \xi_0/2)$, $|y| < r$, there holds

$$\xi_0 + \delta_1 \frac{|y|^2}{r} < |x| + |\xi - x| < \xi_0 + \delta_2 \frac{|y|^2}{r}.$$

Let us set $h = |y|^2$ and

$$f(h) = r(|x| - |\xi - x|) = r\left(\sqrt{r^2 + h} + \sqrt{(\xi_0 - r)^2 + h}\right).$$

Since

$$f'(h) = \frac{r}{2\sqrt{r^2 + h}} + \frac{r}{2\sqrt{(\xi_0 - r)^2 + h}},$$

and taking into account that $0 \le h \le r^2$ and that $1 \le r \le \xi_0/2$, one has:

$$\frac{1}{2\sqrt{2}} \le f'(h) \le 1.$$

Therefore $f(0) + \frac{1}{2\sqrt{2}} h \le f(h) \le f(0) + h$, namely

$$r\,\xi_0 + \frac{1}{2\sqrt{2}}\,|y|^2 \le r\,(|x| + |\xi - x|) \le r\,\xi_0 + |y|^2.$$

■

In the paper we have also dealt with products of three functions. But in this case we only need an estimate from above, and hence the proof is easier:

Lemma 5.1 *Given u, u', u'' : $\mathbb{R}^n \to \mathbb{R}$ three positive continuous radial functions such that:*

$$u(x) + u'(x) + u''(x) \le C(1 + |x|)^a e^{-b|x|}$$

for some $a \in \mathbb{R}$, $b > 0$. Take p_1, p_2, p_3 three fixed points in $\mathbb{R}^n$.

Let $\xi \in \mathbb{R}^n$ tend to infinity, and denote $\xi_i = \xi p_i$. Then:

$$\int u_{\xi_1}(x) u'_{\xi_2}(x) u''_{\xi_3}(x)\, dx \le C e^{-bq\xi},$$

where $C = C(q)$ is a positive constant and q is any fixed quantity satisfying:

$$q < q_0 = \min\{|p - p_1| + |p - p_2| + |p - p_3|\}.$$

Proof. Fix $q < q_0$, and take $\frac{q}{q_0} < \alpha_2 < \alpha_1 < 1$. Clearly, u, u' and u'' satisfy that:

$$u(x) + u'(x) + u''(x) \le C e^{-b\alpha_1 |x|}$$

for some $C > 0$. Therefore, we have that:

$$\int u_{\xi_1}(x)u'_{\xi_2}(x)u''_{\xi_3(x)}\,dx \le C\int \exp[-b\alpha_1(|x-\xi_1|+|x-\xi_2|+|x-\xi_3|)]\,dx \le$$

$$C\int \exp[-b(\alpha_1-\alpha_2)|x-\xi_1|]\exp[-b\alpha_2(|x-\xi_1|+|x-\xi_2|+|x-\xi_3|)]\,dx \le$$

$$C\int \exp[-b(\alpha_1-\alpha_2)|x|]\,dx\,\exp[-b\alpha_2\xi q_0] \le C\exp[-b\alpha_2\xi q_0] \le C\exp[-bq\xi].$$

■

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