# EULERIAN, LAGRANGIAN AND BROAD CONTINUOUS SOLUTIONS TO A BALANCE LAW WITH NON CONVEX FLUX II

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ABSTRACT. We consider a *continuous* solution to the balance law

$$\partial_t u + \partial_x(f(u)) = g \qquad g \text{ bounded, } f \in C^2.$$

We obtain correspondences among the source terms in the Eulerian and Lagrangian settings, extending previous works relative to convex fluxes when possible. The most surprising feature consists in new behaviors of solutions when $f$ is non-convex, and when the set of inflection points of $f$ is not negligible: there is a clear difference among the Lagrangian and Eulerian correspondence of source terms even in the apparently trivial setting of continuous solutions to a single balance law in one space dimension.
KEYWORDS: Balance law, Lagrangian description, Eulerian formulation, counterexample
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## 1. INTRODUCTION

In the present paper we consider a *continuous* function $u : \mathbb{R}^2 \to \mathbb{R}$ and a flux function $f \in C^2(\mathbb{R})$ and we clarify the equivalence of the Lagrangian and Eulerian formulation of the conservation law

$$\partial_t u(t,x) + \partial_x(f(u(t,x))) = \mathfrak{g}(t,x) \ .$$

The object of investigation is the meaning of the source term $\mathfrak{g}$, as it is unclear.

For the Eulerian formulation, $\mathfrak{g}$ is naturally an $\mathcal{L}^1$-function on the plane, thus defined $\mathcal{L}^2$-almost everywhere. For the Lagrangian formulation what matters is the restriction of $\mathfrak{g}$ on characteristic curves of the conservation law—namely the integral curves of the vector field $(1, f'(u))$. Such a restriction is expected to be the derivative of $u$ on such characteristic curves, but... what is the restriction of $\mathcal{L}^1$-functions on curves? On the contrary, even when $u$ is Lipschitz continuous on characteristic curves and thus the derivative of $u$ on characteristic curves is well defined, it is not clear if the points where $u$ is differentiable along a characteristic curve have full $\mathcal{L}^2$-measure, so that such a derivative along characteristic curves could provide an $\mathcal{L}^1$-function on the plane. We indeed provide a negative answer

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in § 4: unexpectedly, we show that a positive $\mathcal{L}^2$-measure set of the plane can stay out of this definition because there is no candidate value for the derivative of $u$ along characteristic curves. When $\mathfrak{g}$ is just bounded, there are therefore important differences among the two formulations. Even with continuous sources, one shall moreover prove that sources in the Lagrangian and Eulerian formulations are the same: this is not trivial.

We already know that if the distributional derivative of the continuous function $u$ restricted on any characteristic curve is bounded by some constant $G$, then the distribution $\partial_t u(t,x) + \partial_x(f(u(t,x)))$ is bounded by the same constant $G$ ([2, Lemma 17-Corollary 28]). Under a non-degeneracy condition on the inflection points of the flux $f$, also the converse holds ([2, Theorem 37]). In § 5 of this paper, a counterexample shows that the last equivalence fails if such non-degeneracy condition is not satisfied and the source is not continous: an Eulerian solution might be not Lipschitz continuous on characteristic curves, even though it must be Lipschitz continuous on a suitable selection of them ([2, Theorem 30]). Such distinction disappears with continuous source terms. Nevertheless, identifying that the Eulerian and the Lagrangian source terms are the same is still not trivial, being $u$ only continous: for the case of the quadratic flux, it was first proved by a series of works by Francesco Bigolin and Francesco Serra Cassano [5, 6], extending $C^1$-source terms [3]. For more general fluxes, there is a full correspondence among Eulerian and Lagrangian formulations.

The main positive result in this paper states that when the inflection points of $f$ are negligible there exists indeed a Borel, bounded function $\mathfrak{g}$ that is both the source term for the conservation law

$$\partial_t u(t,x) + \partial_x(f(u(t,x))) = \mathfrak{g}(t,x) \qquad \text{in } \mathcal{D}'(\mathbb{R}^2)\text{, which means that}$$

$$\forall v \text{ Lipschitz continuous in } \mathbb{R}^2 \qquad \iint (u\,\partial_t v + f(u)\,\partial_x v + \mathfrak{g}v) = 0$$

and which is also the derivative of $u$ on characteristic curves:

$$\text{if } \dot\gamma(t) = f'(u(t,\gamma(t)),\ \gamma \in C^1(\mathbb{R})\text{, then} \qquad \frac{\mathrm{d}}{\mathrm{d}t} u(t,\gamma(t)) = \mathfrak{g}(t,\gamma(t)) \qquad \text{in } \mathcal{D}'(\mathbb{R}).$$

See Theorem 2.1. Of course, the statement includes analytic fluxes. For continuous sources, we recover the continuous representative for both the Lagrangian and the Eulerian formulations.

### 1.1. Revision of conventions and established equivalences.

Be given a *continuous* function $u : \mathbb{R}^2 \to \mathbb{R}$ and a flux function $f \in C^2(\mathbb{R})$. Some statement requires the following assumption of negligibility of inflection points of $f$:

$$\text{(H)} \qquad \mathcal{L}^1(\mathrm{clos}(\mathrm{Infl}(f))) = 0 \qquad \mathrm{clos}(\mathrm{Infl}(f)) \equiv \mathrm{Acc}(\{f''>0\}) \bigcap \mathrm{Acc}(\{f''<0\})$$

We denoted by $\mathrm{Infl}(f)$ the set of inflection points of $f$, $\mathrm{clos}(\mathrm{Infl}(f))$ is its closure. In particular, $z^* \in \mathbb{R}$ is an inflection point of a function $f \in C^2(\mathbb{R})$ if $f''(z^*) = 0$ but it is neither a local maximum nor a local minimum for $f(z) - f'(z^*)(z-z^*)$. The assumption that $u$ is defined on all $\mathbb{R}^2$ is mostly a notational convenience: as we discuss local properties, it would be equivalent having $u$ in an open set.

Define *characteristic curves* as $C^1$ integral curves of the vector field $(1, f'(u))$. We collect a monotone family of such characteristic curves and we define a change of variables in $\mathbb{R}^2$.

**Definition 1.1** (Lagrangian Parameterization)**.** We call *Lagrangian parameterization* associated with $u$ a surjective continuous function $\chi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, or equivalently $\chi : \mathbb{R} \to C(\mathbb{R}^+)$, such that[1]

- for each $y \in \mathbb{R}$, the curve defined by $t \mapsto \chi(t,y) = \chi_y(t)$ is a characteristic curve:
$$\dot\chi_y(t) = \partial_t \chi(t,y) = \lambda(t,\chi(t,y)) = \lambda(i_{\chi(y)}(t));$$
- for each $t \in \mathbb{R}^+$, $y \mapsto \chi(t,y) = \chi_t(y)$ is nondecreasing.

[1]There is no reason for asking the following condition only for $\mathcal{L}^1$-a.e. $y$: if it holds for $\mathcal{L}^1$-a.e. $y$ then it holds naturally for every $y$. As well, it would be odd requiring the second condition only $\mathcal{L}^1$-a.e. $t$.

We set the notation $i_\chi(t,y) \equiv i_{\chi(y)}(t) \equiv (t,\chi(t,y))$. A Lagrangian parameterization $\chi$ is *absolutely continuous* if $(i_\chi^{-1})_\sharp \mathcal{L}^2 \ll \mathcal{L}^2$. Equivalently, $\chi^{-1}(S)$ must have positive $\mathcal{L}^2$-measure if $\mathcal{L}^2(S) > 0$: $\chi$ maps negligible sets into negligible sets.

Assume that for some $G > 0$ one of the following condition holds, which are equivalent:

(i) **Eulerian solution**: The distribution $\partial_t u(t,x) + \partial_x(f(u(t,x)))$ is bounded by $G$.
(ii) Characteristic curves along which u is $G$-Lipschitz continuous are dense in $\mathbb{R}^2$.
(iii) **Lagrangian solution**: $u$ is $G$-Lipschitz continuous along 'a Lagrangian parameterization', see Definition 1.1 below.

The equivalence was proved in [2, Lemma 45–Corollary 46–Corollary 28], where it is also shown that there is no entropy dissipation. Assuming $\mathcal{L}^2(\mathrm{clos}(\mathrm{Infl}(f))) = 0$, they are also equivalent to [2, § 3]:

(iv) **Broad solution**: $u$ is $G$-Lipschitz continuous along all characteristic curves.

Summarizing, without discussing the identification of sources, we established in [2] the equivalences

$$\text{Broad} \quad \frac{\Longrightarrow \text{ always}}{\Longleftarrow \text{ if (H) holds}} \quad \text{Lagrangian} \quad \Longleftrightarrow \quad \text{distributional}$$

Notice that in the three different situations of the theorem one can associate to $u$ three different subsets of Borel bounded functions on the plane. More precisely we define:

(i) The family of **Eulerian sources**: If $\partial_t u + \partial_x f(u)$ is a bounded distribution, one can associate to $u$ those functions $\mathfrak{g}$ such that $u$ solves the PDE

$$\partial_t u(t,x) + \partial_x(f(u(t,x))) = \mathfrak{g}(t,x) \qquad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{1.1}$$

Such a family is just an element $g_E$ in the Lebesgue space $L^\infty(\mathbb{R}^2)$, and of course two of such sources $\mathfrak{g}_1$ and $\mathfrak{g}_2$ differ on $\mathcal{L}^2$-negligible sets.

(ii) The family of **Broad sources**: If $u$ is Lipschitz continuous along all $C^1$ integral curves of the vector field $(1, f(u))$, one can associate to $u$ those Borel functions $\mathfrak{g}$ such that

$$\text{for all characteristic curves } \gamma \text{ of (1.1)} \qquad \frac{\mathrm{d}}{\mathrm{d}t} u(t,\gamma(t)) = \mathfrak{g}(t,\gamma(t)) \qquad \text{in } \mathcal{D}'(\mathbb{R}).$$

We denote such a family by $\mathfrak{L}_B^\infty(\mathbb{R}^2)$, and of course two of such sources $\mathfrak{g}_1$ and $\mathfrak{g}_2$ differ on $\mathcal{L}^1$-negligible sets if restricted to any characteristic curve of (1.1).

(iii) The family of **sources associated to a Lagrangian parameterization** $\chi$: If $u$ is Lipschitz continuous along the 'Lagrangian parameterization' $\chi$, then one can associate to $u$ those Borel functions $\mathfrak{g}$ such that

$$\forall y \in \mathbb{R} \qquad \frac{\mathrm{d}}{\mathrm{d}t} u(t,\chi(t,y)) = \mathfrak{g}(t,\chi(t,y)) \qquad \text{in } \mathcal{D}'(\mathbb{R}).$$

We denote such a family by $\mathfrak{L}_{L\chi}^\infty(\mathbb{R}^2)$, and of course two of such sources $\mathfrak{g}_1$ and $\mathfrak{g}_2$ differ on $\mathcal{L}^2$-negligible sets if composed with the Lagrangian parameterization $\chi$.

We call family of **Lagrangian sources** those functions $\mathfrak{g}$ which are sources associated to some Lagrangian parameterization $\chi$ along which $u$ is Lipschitz continuous.

The present paper answers the following question: if $u$ is both a Lagrangian solution and an Eulerian solution, is the Lagrangian source an Eulerian source and viceversa?

### 1.2. **A positive statement.**

We already established the following relations among sources in the different formulations, that we picture in Figure 1 when inflections of the flux $f$ are negligible.

**Theorem 1.2** ([2, Lemma 16-Theorem 37-Corollary 46]). *The family of Broad sources is contained in the family of sources associated to a Lagrangian parameterization $\chi$, for any Lagrangian parameterization $\chi$. If there exists an Eulerian source, then:*

- *The family of Lagrangian sources is non-empty.*
- *If* $\mathcal{L}^2(\mathrm{clos}(\mathrm{Infl}(f))) = 0$ *holds, then the family of Broad sources is also nonempty.*

We emphasize that any Broad source $\mathfrak{g}$ is a good Lagrangian source independently of the choice of the Lagrangian parameterization.

Section 2. We prove the following statement, see Theorem 2.8 below. We stress that analytic functions do satisfy Assumption (H). We emphasis that in the case of continuous source the continuous representative of the source is, as expected, both an Eulerian and a Broad source.

**Theorem 1.3.** *If Assumption* (H) *of negligibility of 'inflection points of* $f$*' holds, then the family of Eulerian sources has nonempty intersection with the family of Broad sources.*

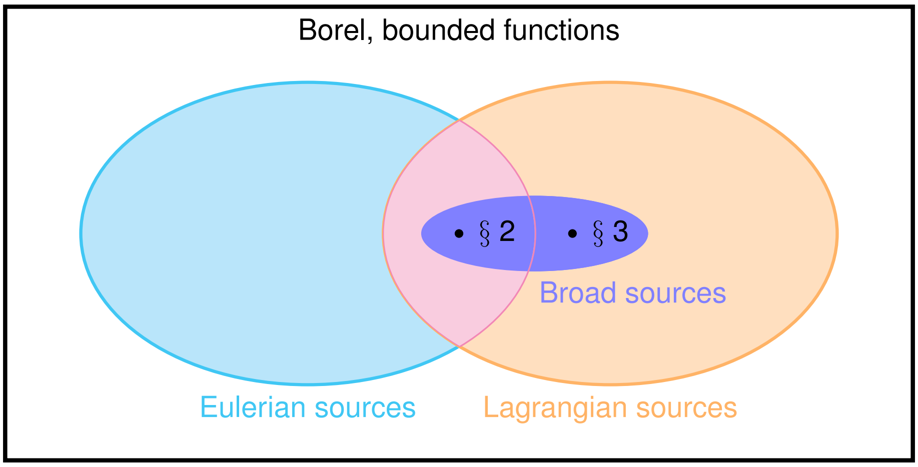


FIGURE 1. We picture relations among the sources that we determine for a fixed continuous solution of the balance law (1.1) under the non-degeneracy Assumption (H). When the source is continuous, it is 'the' source term in all the formulations.

1.3. **Counterexamples.** Counterexamples are the most unexpected part of this work. Of course the family of Eulerian sources is not generally contained in the family of Lagrangian sources, since changing values on a curve affects both the Lagrangian and Broad formulation but not the Eulerian one. Other relations in Figure 1 are less trivial.

Section 3. The first counterexample concerns Lagrangian parameterizations, but not only. We already mentioned that one can build up a monotone family of such characteristic curves in order to define a 'monotone' change of variables $i_\chi$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. We know from [4, Example A.2] that in general one cannot choose a Lagrangian parameterization $\chi$ which is Locally Lipschitz continuous. In § 3 we construct an example—for the quadratic flux!!!—where the measure $\partial_y\chi$ has a Cantor part. In particular, the Lagrangian parameterization is not absolutely continuous, according to Definition 1.1. Even for the quadratic flux thus, surprisingly, there exists a subset $K$ of the plane of positive $\mathcal{L}^2$-measure which intersects each characteristic curve in a single point. For this reason, even for the quadratic flux, the family of Broad sources is not contained in the family of Eulerian sources: the Lagrangiangian source can be defined arbitrarily on the $\mathcal{L}^2$-positive measure set $K$, but of course not all such definitions provide an Eulerian source.

Section 4. For uniformly convex fluxes, there is a natural element lying both in the family of Eulerian sources and in the family of Lagrangian sources ([4, Corollary 6.7], [7, Theorem 1.2], [8]): the pointwise derivative of $u$ along any characteristic curve through the point, when possible, and vanishing in the remaining $\mathcal{L}^2$-negligible set. We construct in this paper a second example which shows that no such natural element exists in general even for convex fluxes, not uniformly, see § 4.

`Section 5`. In the third place, we show that when Assumption (H) fails a continuous function $u$ which is both an Eulerian and a Lagrangian solution is not necessarily a Broad solution: the family of Broad sources might be empty, if inflections of $f$ are not negligible. Namely, in § 5 of this paper, Example 5.2 provides a continuous solution $u$ to a balance law which is *not* Lipschitz continuity on many of its characteristics. It also proves that, whenever one chooses a 'good' Lagrangian parameterization, the Lagrangian source should be fixed accordingly: no universal choice is possible even within all admissible Lagrangian parameterizations. This is even more astonishing considering that it holds also when source terms are autonomous. With continuous sources, such phenomenon disappears.

Summarizing, we point out unexpected differences among the sources in the different formulations:

§ 3 Even with quadratic flux $f(z) = z^2/2$, one may have that any Lagrangian parameterization $\chi$ is not an absolutely continuous functions, but it has a Cantor part. Even for the quadratic flux, thus, if the pointwise representative of the source is not carefully chosen it might work for either the Lagrangian or Eulerian formulation, but not for both of them. Lagrangian parameterizations can instead be taken absolutely continuous if $u$ has also bounded variation, see [2, § 2.1], but for uniformly convex fluxes $u$ is only Hölder continuous in general.

§ 4 Even if $f$ is strictly convex, if the flux $f$ is not $\alpha$-convex, $\alpha > 1$, there can be a positive measure subset $K \subset \mathbb{R}^2$ of points where $u$ is not differentiable along any characteristic curve. This does not contradict the Lipschitz continuity of $u$ along any characteristic curves [2, Theorem 30], because Lagrangian parameterizations are not absolutely continuous. Nevertheless, the pointwise derivative of $u$ along characteristic curves does not define a function $\mathcal{L}^2$-a.e. Moreover, continuous solutions might fail to be Hölder continuous.

§ 5: In the worst case of fluxes whose inflection points have positive measure, changing the Lagrangian parameterization, the Lagrangian source $\mathfrak{g}$ changes. We provide a flux function $f$ with non-negligible inflection points and a continuous (Lagrangian and Eulerian) solution $u$ which is not Lipschitz continuous when restricted to *some* characteristic curves. The notion of Broad solution does not make sense for such fluxes. This does not happen when inflection points of $f$ are negligible [2, Th. 30] . We also stress that this behavior holds even for autonomous solutions $u(t, x) \equiv u(x)$.

## 2. Compatibility of Broad and Eulerian sources when inflections are negligible

**Theorem 2.1.** *Assume that $\mathcal{L}^1(\mathrm{clos}(\mathrm{Infl}(f))) = 0$. Let $u$ be a continuous function for which the first order operator $u \mapsto \partial_t u(t, x) + \partial_x(f(u(t, x)))$ is a bounded distribution. Then the family of Broad sources and the family of Eulerian sources have nonempty intersection.*

In [2, Definition 36, Theorem 37] we constructed the Broad source

$$\mathfrak{g}_B(t,x) := \begin{cases} \mathfrak{g}(t,x) & (t,x) \in E \setminus u^{-1}(\mathrm{clos}(\mathrm{Infl}(f))) \\ 0 & \text{otherwishe} \end{cases}$$

where $E$ is the Borel set [2, equation (3.4) and Lemma 38] of points $(t, x)$ through which there exists a $C^1$ (time-translated) characteristic curve $\gamma$, where $\gamma(0) = (t, x)$, for which $s \mapsto u(t + s, \gamma(s))$ is differentiable at $s = 0$ and $(t, x) \mapsto \mathfrak{g}(t, x) = \frac{\mathrm{d}}{\mathrm{d}s} u(t + s, \gamma(s))\Big|_{s=0}$ was defined by a selection theorem on such set $E$.

In order to have that the source is also an Eulerian source, of course we have to modify $\mathfrak{g}_B$ outside $E$ setting it equal to any Eulerian source $\mathfrak{g}_E$ instead of fixing a 'random' value. Changing the value outside $E$ does not affect the fact of being a Broad source term, because $u$ is Lipschitz continuous along any characteristic curve [2, Theorem 30] and therefore the complementary of $E$ is $L^1$-negligible along any characteristic curve. Nevertheless, we could also need to turn $\mathfrak{g}_B$ into $\mathfrak{g}_E$ within $E$: we indeed construct a subset $\widehat{D}_{\mathfrak{g}_E}$ of $\{\mathfrak{g}_B \neq \mathfrak{g}_E\}$ of its same (maybe positive!) $\mathcal{L}^2$-measure and which is negligible along any integral curve of $(1, f'(u))$. Once we turn $\mathfrak{g}_B$ into $\mathfrak{g}_E$ also on $\widehat{D}_{\mathfrak{g}_E}$ we are done and we get a Borel function which is both a Broad and an Eulerian source term.

We first notice that $0$ is a good value both for Eulerian and Broad sources on $u^{-1}(\mathrm{clos}(\mathrm{Infl}(f)))$.

**Lemma 2.2.** *Consider any closed set $N \subset \mathbb{R}$ which is $\mathcal{L}^1$-negligible. Then*

(i) *any Eulerian source $\mathfrak{g}_E$ vanishes at $\mathcal{L}^2$-Lebesgue points of $u^{-1}(N) \subseteq \mathbb{R}^2$.*

(ii) *the $t$-derivative of $u \circ i_\gamma(t) = u(t, \gamma(t))$ vanishes at $\mathcal{L}^1$-Lebesgue points of $(u \circ i_\gamma)^{-1}(N) \subseteq \mathbb{R}$, for any any $C^1$ integral curve $\gamma$ of $(1, f'(u))$*

*Proof.* By $\sigma$-additivity, we directly assume that $N$ is compact, not only closed.
(i) We apply the entropy equality [2, Lemma 42]. Choose in particular the entropies

$$\eta'_\varepsilon(z) = (\mathbb{1}_{O_\varepsilon} * \rho_\varepsilon)(z), \qquad \eta_\varepsilon(-\infty) = 0,$$

where $\rho_\varepsilon$ is a smooth convolution kernel concentrated on $[-\varepsilon, \varepsilon]$ and $O_\varepsilon \supset N$ is a sequence of open sets such that $\mathcal{L}^1(O_\varepsilon) < \varepsilon$. Since $\mathcal{L}^1(N) = 0$ and $N$ is compact, then $\eta'_\varepsilon(u)$ converges pointwise to $\mathbb{1}_{u^{-1}(N)}$ so that $\eta_\varepsilon(z)$ converges locally uniformly to $0$. In particular, the distributions $\partial_t\, \eta_\varepsilon(u)$ and $\partial_x(q_\varepsilon(u))$ must vanish in the limit as $\varepsilon \downarrow 0$. From the entropy equality

$$\partial_t\, \eta_\varepsilon(u) + \partial_x(q_\varepsilon(u)) = \eta'_\varepsilon(u)\mathfrak{g}_E \qquad \text{in } \mathcal{D}'(\mathbb{R}^2)$$

we deduce the claim in the limit $\varepsilon \downarrow 0$: $\mathfrak{g}_E$ must vanish $\mathcal{L}^2$-a.e. on $u^{-1}(N)$ because

$$0 = \mathbb{1}_{u^{-1}(N)}(t,x)\mathfrak{g}_E(t,x) \qquad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

(ii) Recall that $u$ is $G$-Lipschitz continuous along characteristic curves [2, Theorem 30]: since $N$ is $\mathcal{L}^1$-negligible by hypothesis then the derivative of $u \circ i_{\chi(\bar{y})}(t)$ must vanish at Lebesgue points of $(u \circ i_{\chi(\bar{y})})^{-1}(N)$ by an easy computation, see [2, Lemma 41]. □

We now construct in Lemma 2.6 a subset $\widehat{D}_{\mathfrak{g}_E}$ of $E$ where we need to turn $\mathfrak{g}_B$ into $\mathfrak{g}_E$ in two steps:

- in Lemma 2.3 we remove from $E$
  - the set where $\mathfrak{g}_E = \mathfrak{g}_B$,
  - the set $u^{-1}(\mathrm{clos}(\mathrm{Infl}(f)))$ and
  - the points which are not Lebesgue points of the time-restrictions of $\mathfrak{g}_E$.

  We prove that the remaining set $D_{\mathfrak{g}_E}$ is negligible along any Lagrangian parameterization.
- Lemma 2.5 is the main ingredient in order to prove that $D_{\mathfrak{g}_E}$ is negligible not only along any Lagrangian parameterization, but also along any integral curve of $(1, f'(u))$ provided that we cut a further $\mathcal{L}^2$-negligible set: the set $D_{\mathfrak{g}_E} \setminus \widehat{D}_{\mathfrak{g}_E}$ containing the points through which there is a characteristic curve along which $D_{\mathfrak{g}_E}$ is not $\mathcal{L}^1$-negligible.

Once constructed $\widehat{D}_{\mathfrak{g}_E}$, having a compatible source in Theorem 2.8 will be straightforward.

**Lemma 2.3.** *Let $S^{\complement}$ denote the complementary of a set $S$. Consider any Eulerian source term $\mathfrak{g}_E$ and define the subset of $E$*

$$D_{\mathfrak{g}_E} := E \bigcap \{\mathfrak{g}_E = \mathfrak{g}_B\}^{\complement} \bigcap \left(u^{-1}\left(\mathrm{clos}(\mathrm{Infl}(f))\right)\right)^{\complement} \bigcap \left\{ \exists \lim_{h\to 0} \frac{1}{h} \int_{\bar{x}}^{\bar{x}+h} \mathfrak{g}_E(\bar{t}, x) dx = \mathfrak{g}_E(\bar{t}, \bar{x}) \right\} .$$

*Then $\mathcal{L}^2(\Psi^{-1}(D_{\mathfrak{g}_E})) = 0$ for any Lagrangian parameterization $\chi$, where $\Psi(t, y) = (t, \chi(t, y))$.*

*Proof.* Notice that we are not considering the intersection of the accumulation points of $\{f'' > 0\}$ and $\{f'' < 0\}$ since such intersection is precisely $\mathrm{clos}(\mathrm{Infl}(f))$, which lies in the complementary of $D_{\mathfrak{g}_E}$. We can thus focus in an open set where $\{f'' \geq 0\}$: open sets where $\{f'' \leq 0\}$ are analogous.

Let $(\bar t,\bar x)\in \mathrm{r.i.}\{f''\geq 0\}\cap D_{\mathfrak{g}_E}$. Suppose $(\bar t,\bar x)=(\bar t,\chi(\bar t,\bar y))=\Psi(\bar t,\bar y)$ with

$$\exists \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\bar t}^{\bar t+\varepsilon}\mathfrak{G}_E(t,\bar y)dt=\mathfrak{g}_E(\bar t,\bar x)\qquad \text{where } \mathfrak{G}_E(t,y)=\mathfrak{g}_E(t,\chi(t,y)) \tag{2.1}$$

$$\exists \lim_{\varepsilon\downarrow 0}\frac{\mathcal{L}^1\left(\Psi^{-1}(D_{\mathfrak{g}_E})\cap[\bar t-\varepsilon,\bar t+\varepsilon]\times\{\bar y\}\right)}{2\varepsilon}=1 \tag{2.2}$$

$$\exists \lim_{\varepsilon\downarrow 0}\frac{u(\Psi(\bar t+\varepsilon,\bar y))-u(\Psi(\bar t,\bar y))}{\varepsilon}=\mathfrak{g}_B(\bar t,\bar x) \tag{2.3}$$

Since this is satisfied at $\mathcal{L}^2$-a.e. $(t,y)$ in $\Psi^{-1}(D_{\mathfrak{g}_E})$, the thesis will not be affected.

One can go back to Dafermos' computation [2, (3.1a)], which means in the integral formulation of the PDE, and exploit first the sign information on $f''$: what we get is

$$\frac{1}{\varepsilon(\tau-\sigma)}\left\{\int_{\gamma(\tau)}^{\gamma(\tau)+\varepsilon}u(\tau,x)dx-\int_{\gamma(\sigma)}^{\gamma(\sigma)+\varepsilon}u(\sigma,x)dx-\int_\sigma^\tau\int_{\gamma(t)}^{\gamma(t)+\varepsilon}\mathfrak{g}_E(t,x)dxdt\right\}\leq 0\ . \tag{2.4}$$

We apply the integral relation fixing the characteristic curve $\gamma(t)=\chi(t,\bar y)$. We show below that by the choice of $(\bar t,\bar x)$ one can pass to the limit, first as $\varepsilon\downarrow 0$, then as $\tau\downarrow\sigma$.

*1: Last addend.* When $(t,\gamma(t))\in D_{\mathfrak{g}_E}$, the space average converges by definition of $D_{\mathfrak{g}_E}$:

$$\exists \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\gamma(t)}^{\gamma(t)+\varepsilon}\mathfrak{g}_E(t,x)dx=\mathfrak{g}_E(t,\gamma(t))$$

In particular, when $(t,\gamma(t))\in D_{\mathfrak{g}_E}$

$$\mathfrak{g}_E(t,\gamma(t))=\liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\gamma(t)}^{\gamma(t)+\varepsilon}\mathfrak{g}_E(t,x)dx=\limsup_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\gamma(t)}^{\gamma(t)+\varepsilon}\mathfrak{g}_E(t,x)dx$$

When $(t,\gamma(t))\notin D_{\mathfrak{g}_E}$ perhaps the limit does not exists, but the liminf and the limsup are bounded by $\pm G$. Thanks to this bound and by (2.2), the time-average is the same when the average is done considering only the points $t$ with $(t,\gamma(t))\in D_{\mathfrak{g}_E}$: by (2.1) we thus deduce

$$\begin{aligned}\mathfrak{g}_E(\bar t,\bar x)&=\lim_{\tau\downarrow\sigma}\frac{1}{|\tau-\sigma|}\int_{[\sigma,\tau]\cap(\Psi^{-1}(D_{\mathfrak{g}_E}))_{\bar y}}\mathfrak{g}_E(t,\gamma(t))dt\\&=\lim_{\tau\downarrow\sigma}\frac{1}{|\tau-\sigma|}\int_{[\sigma,\tau]}\liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\gamma(t)}^{\gamma(t)+\varepsilon}\mathfrak{g}_E(t,x)dxdt\\&=\lim_{\tau\downarrow\sigma}\frac{1}{|\tau-\sigma|}\int_{[\sigma,\tau]}\limsup_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\gamma(t)}^{\gamma(t)+\varepsilon}\mathfrak{g}_E(t,x)dxdt\ .\end{aligned}$$

We conclude by Fatou's lemma, which yields the double limit for the last addend

$$\begin{aligned}\mathfrak{g}_E(\bar t,\bar x)&=\lim_{\tau\downarrow\sigma}\liminf_{\varepsilon\downarrow 0}\frac{1}{|\tau-\sigma|\varepsilon}\int_\sigma^\tau\int_{\gamma(t)}^{\gamma(t)+\varepsilon}g(t,x)dxdt\\&=\lim_{\tau\downarrow\sigma}\limsup_{\varepsilon\downarrow 0}\frac{1}{|\tau-\sigma|\varepsilon}\int_\sigma^\tau\int_{\gamma(t)}^{\gamma(t)+\varepsilon}g(t,x)dxdt.\end{aligned}$$

*2: First addends.* Recall that $u$ is differentiable at $(\bar t,\bar x)$ along $\chi(t,\bar y)$ by (2.3) with derivative $\mathfrak{g}_B(\bar t,\bar x)$. By the continuity of $u$ then one has that the first two addends in the LHS converge to $\mathfrak{g}_B(\bar t,\bar x)$:

$$\begin{aligned}&\lim_{\tau\downarrow\sigma}\lim_{\varepsilon\downarrow 0}\frac{1}{|\tau-\sigma|\varepsilon}\left[\int_{\gamma(\tau)}^{\gamma(\tau)+\varepsilon}u(\tau,x)dx-\int_{\gamma(\sigma)}^{\gamma(\sigma)+\varepsilon}u(\sigma,x)dx\right]\\&=\lim_{\tau\downarrow\sigma}\frac{u(\tau,\gamma(\tau))-u(\sigma,\gamma(\sigma))}{|\tau-\sigma|}=\mathfrak{g}_B(\bar t,\bar x).\end{aligned}$$

*3: Conclusion.* By the double limits proved in the previous sub steps, the inequality (2.4) yields

$$\mathfrak{g}_B(\bar t,\bar x)-\mathfrak{g}_E(\bar t,\bar x)\le 0.$$

In particular, $\mathfrak{g}_B(\bar t,\bar x)\le\mathfrak{g}_E(\bar t,\bar x)$. The reverse inequality comes considering the similar region bounded by $\gamma(t)$ and $\gamma(t)-\varepsilon$ instead of $\gamma(t)$ and $\gamma(t)+\varepsilon$. As the set $\mathrm{r.i.}\{f''\le 0\}\cap D_{\mathfrak{g}_E}$ is entirely analogous, we conclude thus $\mathfrak{g}_B(\bar t,\bar x)=\mathfrak{g}_E(\bar t,\bar x)$ at those points $\Psi(\bar t,\bar y)=(\bar t,\bar x)\in D_{\mathfrak{g}_E}$ satisfying (2.1)-(2.2)-(2.3): in particular, this holds at $\mathcal{L}^2$-a.e. point in $\Psi^{-1}(D_{\mathfrak{g}_E})$. □

**Lemma 2.4.** *Let $\{\widetilde\gamma_{x_q}\}_{q\in\mathbb{N}}$ be characteristic curves such that $x_q=\widetilde\gamma_q(\bar t)$ and $\mathcal{H}^1(\gamma_q(\mathbb{R})\cap K)\ge\varepsilon$, $q\in\mathbb{N}$, where $K\subset\mathbb{R}^2$ is compact. There exists a sequence of characteristic curves $\{\gamma_{x_q}\}_{q\in\mathbb{N}}$ such that*

$$x_q=\gamma_q(\bar t)\ ,\quad \gamma_{q_1}\le\gamma_{q_2}\quad \textit{if } x_{q_1}\le x_{q_2}\qquad\textit{and}\qquad \mathcal{H}^1(\gamma_q(\mathbb{R})\cap K)\ge\varepsilon\quad\forall q,q_1,q_2\in\mathbb{N}\ .$$

*Proof.* We modify the family of curves recursively for having monotonicity, without destroying the measure property, and we then pass to the limit by a diagonal procedure and exploiting upper semi-continuity of Hausdorff measure on compact sets.

Step $1$: Set $\gamma^1_{x_1}:=\widetilde\gamma_{x_1}$, as there is no ordering to impose with a single curve.

Step $2$: Consider the least time $t^+>\bar t$ of intersection among $\widetilde\gamma_{x_2}$ and $\gamma^1_{x_1}$, if any, and the last time $t^-<\bar t$ as well, if any, otherwise set them $\pm\infty$. If $t^+$ or $t^-$ are finite, associate to them

$$\gamma^+(t):=\begin{cases}\gamma^1_{x_1}(t) & \text{if } \mathcal{H}^1\left(\gamma^1_{x_1}((t^+,+\infty))\cap K\right)\ge\mathcal{H}^1\left(\widetilde\gamma_{x_2}((t^+,+\infty))\cap K\right)\\ \widetilde\gamma_{x_2}(t) & \text{otherwise}\end{cases}\qquad\text{for } t\ge t^+$$

$$\gamma^-(t):=\begin{cases}\gamma^1_{x_1}(t) & \text{if } \mathcal{H}^1\left(\gamma^1_{x_1}((-\infty,t^-))\cap K\right)\ge\mathcal{H}^1\left(\widetilde\gamma_{x_2}((-\infty,t^-))\cap K\right)\\ \widetilde\gamma_{x_2}(t) & \text{otherwise}\end{cases}\qquad\text{for } t\le t^-$$

Set

$$\gamma^2_{x_1}(t):=\begin{cases}\gamma^1_{x_1}(t) & \text{for } t^-\le t\le t^+\\ \gamma^+(t) & \text{for } t>t^+\\ \gamma^-(t) & \text{for } t<t^-\end{cases}\qquad \gamma^2_{x_2}(t):=\begin{cases}\widetilde\gamma_{x_2}(t) & \text{for } t^-\le t\le t^+\\ \gamma^+(t) & \text{for } t>t^+\\ \gamma^-(t) & \text{for } t<t^-\end{cases}$$

Of course these two curve might touch each other but they do not cross each other, and they satisfy

$$\mathcal{H}^1(\gamma^2_{x_1}(\mathbb{R})\cap K)\ge\mathcal{H}^1(\gamma^1_{x_1}(\mathbb{R})\cap K)\ge\varepsilon\qquad \mathcal{H}^1(\gamma^2_{x_2}(\mathbb{R})\cap K)\ge\mathcal{H}^1(\widetilde\gamma_{x_2}(\mathbb{R})\cap K)\ge\varepsilon$$

Step $n$: We already constructed $\gamma^{n-1}_{x_{j(1)}}\le\dots\le\gamma^{n-1}_{x_{j(n-1)}}$ satisfying $\mathcal{H}^1(\gamma^{n-1}_{x_j}(\mathbb{R})\cap K)$, $j=1,\dots,n-1$. Let's modify first $\gamma^{n-1}_{x_{j(1)}}$. Consider the least time $t^+>\bar t$ of intersection among $\widetilde\gamma_{x_n}$ and $\gamma^{n-1}_{x_{j(1)}}$, if any, and the last time $t^-<\bar t$ as well, if any, otherwise set them $\pm\infty$. If $t^+$ or $t^-$ are finite, associate to them

$$\gamma^+(t):=\begin{cases}\gamma^{n-1}_{x_{j(1)}}(t) & \text{if } \mathcal{H}^1\left(\gamma^{n-1}_{x_{j(1)}}((t^+,+\infty))\cap K\right)\ge\mathcal{H}^1\left(\widetilde\gamma_{x_n}((t^+,+\infty))\cap K\right)\\ \widetilde\gamma_1 x_n(t) & \text{otherwise}\end{cases}\qquad\text{for } t\ge t^+$$

$$\gamma^-(t):=\begin{cases}\gamma^{n-1}_{x_{j(1)}}(t) & \text{if } \mathcal{H}^1\left(\gamma^{n-1}_{x_{j(1)}}((-\infty,t^-))\cap K\right)\ge\mathcal{H}^1\left(\widetilde\gamma_{x_n}((-\infty,t^-))\cap K\right)\\ \widetilde\gamma_{x_n}(t) & \text{otherwise}\end{cases}\qquad\text{for } t\le t^-$$

Set

$$\gamma^n_{x_{j(1)}}(t):=\begin{cases}\gamma^{n-1}_{x_{j(1)}}(t) & \text{for } t^-\le t\le t^+\\ \gamma^+(t) & \text{for } t>t^+\\ \gamma^-(t) & \text{for } t<t^-\end{cases}\qquad \widetilde\gamma^1_{x_n}(t):=\begin{cases}\widetilde\gamma_{x_n}(t) & \text{for } t^-\le t\le t^+\\ \gamma^+(t) & \text{for } t>t^+\\ \gamma^-(t) & \text{for } t<t^-\end{cases}$$

For $i=2,\dots,n-1$ repeat the procedure: compare $\gamma^{n-1}_{x_{j(i)}}$ and $\widetilde\gamma^{i-1}_{x_n}$, define two curves $\gamma^n_{x_{j(i)}}\ge\gamma^n_{x_{j(i-1)}}$ and $\widetilde\gamma^i_{x_n}$ which might touch each other but which do not cross each other and such that

$$\mathcal{H}^1(\gamma^n_{j(i)}(\mathbb{R})\cap K)\ge\mathcal{H}^1(\gamma^{n-1}_{j(i)}(\mathbb{R})\cap K)\ge\varepsilon\qquad \mathcal{H}^1(\widetilde\gamma^i_n(\mathbb{R})\cap K)\ge\mathcal{H}^1(\widetilde\gamma^{i-1}_n(\mathbb{R})\cap K)\ge\varepsilon\ .$$

Notice that the value of curves at time $\bar t$ never changes. Set $\gamma^n_n:=\widetilde\gamma^{n-1}_n$.

By the above recursive procedure, we manage to construct sequence of equi-Lipschitz curves $\{\gamma^n_{x_i}\}^{n\in\{i,i+1,\dots\}}_{i\in\mathbb{N}}$ which preserve the ordering when $n$ is fixed and such that

$$\gamma^n_{x_i}(\bar t) = x_i\ , \qquad \mathcal{H}^1(\gamma^n_{x_i}(\mathbb{R})\cap K) \geq \varepsilon\ .$$

By Ascoli-Arzelà theorem, up to subsequence and applying also Cantor diagonal argument, we extract a subsequence $\{n_i\}_{i\in\mathbb{N}}$ such that for all $j\in\mathbb{N}$

$$\exists\gamma_{x_j}\ \text{characteristic curve}\ :\ \gamma^{n_i}_{x_j} \xrightarrow[i\to+\infty]{} \gamma_{x_j} \qquad \gamma_{x_j}(\bar t) = x_i\ , \qquad \mathcal{H}^1(\gamma_{x_j}(\mathbb{R})\cap K) \geq \varepsilon\ .$$

We obtained the last point by the upper semicontinuity of $\mathcal{H}^1$ for Hausdorff convergence on compact sets, thanks to the fact that $K$ is compact and that each $\{\gamma^{n_i}_{x_j}\}_{i=j,j+1,\dots}$ converges uniformly. □

**Lemma 2.5.** *Let $K$ be any closed set such that through each point $(t,x)\in K$ there exists a $C^1$ integral curve $\gamma$ of $(1,f'(u))$ with $\mathcal{H}^1(\gamma(\mathbb{R})\cap K) > 0$. Then either $\mathcal{L}^2(K) = 0$ or there exists a Lagrangian parameterization $\chi$ such that $\mathcal{L}^2(\Psi^{-1}(K)) > 0$, where $\Psi(t,y) = (t,\chi(t,y))$.*

*Proof.* By $\sigma$-additivity of measures, we directly assume that $K$ is compact. Suppose $\mathcal{L}^2(K) > 0$: we construct a Lagrangian parameterization $\chi$ for which $\mathcal{L}^2(\Psi^{-1}(K)) > 0$, where $\Psi(t,y) = (t,\chi(t,y))$.

Let $\varepsilon > 0$, small enough, and fix any $\bar t$ such that $\mathcal{L}^1(K^\varepsilon_{\bar t}) > 0$, where

$$K^\varepsilon_{\bar t} := \{x\ :\ (\bar t,x)\in K \text{ and } \exists\gamma_x \text{ characteristic curve through } (\bar t,x) \text{ such that } \mathcal{H}^1(\gamma_q(\mathbb{R})\cap K)\geq\varepsilon\}\ .$$

Of course by definition of $K^\varepsilon_{\bar t}$ one can pick up a sequence $\{x_q\}_{q\in\mathbb{Q}}$ dense in $K^\varepsilon_{\bar t}$ such that there exists a characteristic curves $\widetilde\gamma_q$, $q\in\mathbb{N}$, with $x_q = \widetilde\gamma_q(\bar t)$ and $\mathcal{H}^1(\widetilde\gamma_q(\mathbb{R})\cap K)\geq\varepsilon$. In general, this is not a monotone family: we need to modify it in order to have monotonicity, but without destroying the positive intersection with $K$, as done in Lemma 2.4. We thus get a sequence of characteristic curves $\{\gamma_{x_q}\}_{q\in\mathbb{N}}$, such that $x_q = \gamma_q(\bar t)$ is dense in $K^\varepsilon_{\bar t}$ and such that moreover

$$\gamma_{q_1}\leq\gamma_{q_2}\quad \text{if } x_{q_1}\leq x_{q_2} \qquad \text{and} \qquad \mathcal{H}^1(\gamma_q(\mathbb{R})\cap K)\geq\varepsilon \quad \forall q\in\mathbb{N}\ .$$

One can construct [4, Lemma A.1] a Lagrangian parameterization $\chi$ such that $\chi(t,j(q)) = \gamma_q(t)$ for $q\in Q$ and for some monotone function $j : Q\to\mathbb{R}$. Let $\Psi(t,y) = (t,\chi(t,y))$ and consider $\Psi^{-1}(K)$. Since

$$\mathcal{L}^1\left(\Psi^{-1}(K)\bigcap\mathbb{R}\times\{j(q)\}\right) = \mathcal{H}^1(\gamma_q(\mathbb{R})\cap K)\geq\varepsilon$$

and since whenever $j(q_n)\to\bar y$ by compactness and upper semicontinuity of $\mathcal{L}^1$ on compact sets

$$\mathcal{L}^1\left(\Psi^{-1}(K)\bigcap\mathbb{R}\times\{\bar y\}\right) \geq \limsup_{n\in\mathbb{N}}\mathcal{L}^1\left(\Psi^{-1}(K)\bigcap\mathbb{R}\times\{j(q_n)\}\right)\geq\varepsilon$$

then by Tonelli's theorem we get the thesis provided that

$$\mathcal{L}^1(\mathrm{clos}(j(Q))) > 0\ .$$

This is incidentally the case since following the construction [4, Lemma A.1], which extends the partial Lagrangian parameterization $\widetilde\chi(t,q) = \gamma_q(t)$, it turns out that

$$\mathcal{L}^1(\mathrm{clos}(j(Q))) \geq \mathcal{L}^1(K\cap\{\bar t\}\times\mathbb{R}) > 0\ .$$

□

**Lemma 2.6.** *There exists a Borel subset $\widehat D_{\mathfrak{g}_E}$ of $E\cap\{\mathfrak{g}_E\neq\mathfrak{g}_B\}$ with the same $\mathcal{L}^2$-measure of $E\cap\{\mathfrak{g}_E\neq\mathfrak{g}_B\}$ and which is $\mathcal{L}^1$-negligible along any $C^1$ integral curve of $(1,f'(u))$.*

*Proof.* For brevity, set $Z = u^{-1}(\mathrm{clos}(\mathrm{Infl}(f)))$. By definition of $D_{\mathfrak{g}_E}$ in Lemma 2.3

$$E\setminus D_{\mathfrak{g}_E} = E\bigcap\left(\{\mathfrak{g}_E=\mathfrak{g}_B\}\bigcup(Z\setminus\{\mathfrak{g}_E=\mathfrak{g}_B\})\bigcup\left\{\exists\lim_{h\to0}\frac{1}{h}\int_{\bar x}^{\bar x+h}\mathfrak{g}_E(\bar t,x)dx = \mathfrak{g}_E(\bar t,\bar x)\right\}^{\complement}\right)\ .$$

Notice that $\mathcal{L}^2\left(Z \setminus \{\mathfrak{g}_E = \mathfrak{g}_B\}\right)$ by definition of $\mathfrak{g}_B$ and by Lemma 2.2. By Lebesgue differentiation theorem and by Fubini theorem, also the last set where $\mathfrak{g}_E$ differs from the Lebesgue value on its $t$-sections is $\mathcal{L}^2$-negligible. We conclude thus that

$$\mathcal{L}^2(E \setminus D_{\mathfrak{g}_E}) = \mathcal{L}^2(E \cap \{\mathfrak{g}_E \neq \mathfrak{g}_B\}).$$

The set $D_{\mathfrak{g}_E}$ is negligible along any Lagrangian parameterization $\chi$ by Lemma 2.3. Nevertheless, the set $D_{\mathfrak{g}_E}$ could have intersection of positive measure with some characteristic curve: this is why fixing $\widehat{D}_{\mathfrak{g}_E} = D_{\mathfrak{g}_E}$ in the statement generally does not work. We now show that $D_{\mathfrak{g}_E}$ becomes negligible along all characteristic curves by removing a further $\mathcal{L}^2$-negligible set.

By inner regularity of the Lebesgue measure, one can write

$$D_{\mathfrak{g}_E} = R \bigcup \left( \bigcup_n K_n \right) \qquad \text{where } \mathcal{L}^2(R) = 0$$

and $\{K_n\}_{n\in\mathbb{N}}$ is an increasing sequence of compact sets. Consider their compact subsets

$$\widetilde{K}_n^\varepsilon = \{(t,x) \in K_n \quad : \quad \exists \gamma \text{ characteristic curve through } (t,x) \text{ such that } \mathcal{H}^1(i_\gamma(\mathbb{R}) \cap K_n) \geq \varepsilon\} \; .$$

Notice moreover that for all $(t,x) \in \widetilde{K}_n^\varepsilon$ there exists a characteristic curve through $(t,x)$ such that $\mathcal{H}^1(i_\gamma(\mathbb{R}) \cap K_n^\varepsilon) \geq \varepsilon$, since whenever $(t,x) \in \widetilde{K}_n^\varepsilon$ then $i_\gamma(\mathbb{R}) \cap K_n$ is contained in $\widetilde{K}_n^\varepsilon$.

*Claim* 2.7. Each set $K_n^\varepsilon$ is indeed compact.

*Proof of the claim.* If $(t_j, x_j) \in \widetilde{K}_n^\varepsilon$ converges to $(t,x)$, $(t,x)$ necessarily belongs to $K_n$ by compactness. There exist then characteristic curves $\gamma_j$ through $(t_j, x_j)$ which converge locally uniformly to a characteristic curve $\gamma$ through $(t,x)$ such that $\mathcal{H}^1(i_{\gamma_j}(\mathbb{R}) \cap K_n) \geq \varepsilon$. By compactness, the sets $i_{\gamma_j}(\mathbb{R}) \cap K_n$ converge in the Hausdorff distance to a subset of $i_\gamma(\mathbb{R}) \cap K_n$. By upper semicontinuity of the measure of compact sets for Hausdorff convergence, we get that $\mathcal{H}^1(i_\gamma(\mathbb{R}) \cap K_n) \geq \varepsilon$, proving that $(t,x) \in \widetilde{K}_n^\varepsilon$. □

Being subsets of $D_{\mathfrak{g}_E}$, of course $K_n^\varepsilon$ are negligible along any Lagrangian parameterization $\chi$ by Lemma 2.3. By Lemma 2.5 we thus get $\mathcal{L}^2(\widetilde{K}_n^\varepsilon) = 0$, so that the set

$$\widehat{D}_{\mathfrak{g}_E} := \bigcup_{n\in\mathbb{N}} \left( K_n \setminus \bigcup_{m\in\mathbb{N}} \widetilde{K}_n^{2^{-m}} \right)$$

satisfies $\mathcal{L}^2(\widehat{D}_{\mathfrak{g}_E}) = \mathcal{L}^2(D_{\mathfrak{g}_E})$ and $\widehat{D}_{\mathfrak{g}_E}$ has negligible intersection with any characteristic curve. □

**Theorem 2.8.** *Assume that $\mathcal{L}^1(\mathrm{clos}(\mathrm{Infl}(f))) = 0$. Let $\mathfrak{g}_E$ be any Eulerian source term. Then there exists a subset $\widehat{D}_{\mathfrak{g}_E}$ of the set $D_{\mathfrak{g}_E}$ in Lemma 2.3, with the same $\mathcal{L}^2$-measure, such that the function*

$$\mathfrak{g}_U(t,x) := \begin{cases} \mathfrak{g}_B(t,x) & \text{on } E \setminus \widehat{D}_{\mathfrak{g}_E} \text{ and on } u^{-1}\left(\mathrm{clos}(\mathrm{Infl}(f))\right) \\ \mathfrak{g}_E(t,x) & \text{on } \widehat{D}_{\mathfrak{g}_E} \text{ and on } \mathbb{R}^2 \setminus E \end{cases}$$

*is both an Eulerian source term and a Broad source term.*

*Proof.* Notice that $\mathfrak{g}_E = \mathfrak{g}_B$ $\mathcal{L}^2$-a.e. on $u^{-1}\left(\mathrm{clos}(\mathrm{Infl}(f))\right)$ by definition of $\mathfrak{g}_B$ and by Lemma 2.2. Moreover $\mathfrak{g}_E = \mathfrak{g}_B$ $\mathcal{L}^2$-a.e. on $E \setminus \widehat{D}_{\mathfrak{g}_E}$ by Lemma 2.6. The function $\mathfrak{g}_U$ is therefore an Eulerian source because $\mathfrak{g}_U = \mathfrak{g}_E$ in $\mathcal{D}'(\mathbb{R}^2)$.

As $\widehat{D}_{\mathfrak{g}_E}$ is negligible along any characteristic curve by Lemma 2.6 and $\mathbb{R}^2 \setminus E$ is negligible as well along any characteristic curve by [2, Theorem 30] and by definition of $E$ [2, equation (3.4) and Lemma 38], then $\mathfrak{g}_U$ is still a Broad source, as $\mathfrak{g}_B$ was. □

## 3. Lagrangian parameterizations may be Cantor functions

This section aims at constructing

- a continuous solution $u$ of a balance law

$$\partial_t u(t,x) + \partial_x(u^2(t,x)) = g_E(t,x), \qquad |g_E(t,x)| \leq 1 \ . \tag{3.1}$$

- a compact set $K \subset \mathbb{R}^2$ of positive Lebesgue measure whose intersection with *any* characteristic curve of $u$ is $\mathcal{H}^1$-negligible.

This shows that in some cases the change of variable providing any Lagrangian parameterization fails to be a special function of bounded variation, but it has instead a Cantor part. This is not due to strange behaviors of the flux function: we provide it with the quadratic flux. A more complex example is provided in § 4 for the non-uniformly convex case. This lack of regularity is a serious obstacle in the correspondence of the Lagrangian and Eulerian source terms, because values attained at points negligible for the Lagrangian formulation may be not negligible for the Eulerian one and viceversa. Even if Lagrangian parameterization fails to be a special function of bounded variation, this behavior is instead absent for $\alpha$-convex fluxes, $\alpha > 1$, [7, Theorem 1.2].

We briefly outline the construction before presenting it precisely:

(i) We partition $\mathbb{R}^2$ in a rectangle $Q_0$ and its complementary. We define $u = 0$ on the complementary of $Q_0$.

(ii) At the first step we subdivide $Q_0$ into finitely many sub-strips, say $d_1$ sub-strips equal to $S_1$. The strip $S_1$ is made of two sub-rectangles which are a translation of a given rectangle $Q_1 \subset Q_0$ and a remaining 'corridor'. We assign a value to $u$ in each 'corridor' as the derivative of a suitable family of curves $x = \gamma(t)$ covering the region of the 'corridor'. We do it in such a way that—in this closed region with $2d_1$ equal rectangular holes—$u$ will be a $C^1$ function. By Cauchy uniqueness theorem for ODEs, all characteristic curves of $u$ in this region of the 'corridor' must then coincide with the family that we assign.

(iii) At the $i$-th step, $i \in \mathbb{N}$, $u$ is defined as a $C^1$ function on the complementary of finitely many disjoint equal rectangles which are translations of a given rectangle $Q_i \subset Q_{i-1}$. We subdivide $Q_i$ recursively into finitely many sub-strips, say $d_{i+1}$ sub-strips. Each strip is made by two rectangles translation of $Q_{i+1} \subset Q_i$ and a remaining 'corridor'. We assign a value to $u$ on each 'corridor' so that this extension of $u$ becomes a $C^1$-function on the closed region with $2^i d_1 \cdot \dots \cdot d_i$ equal rectangular holes which are a translation of $Q_{i+1}$.

(iv) By the previous steps we will have assigned a value to $u$ on the whole complementary of a compact Cantor-like set $K \subset \mathbb{R}^2$ of positive Lebesgue measure but with empty interior: we will be able to assign a unique value of $u$ on $K$ extending $u$ to $K$ by continuity.

(v) By the details of our construction, *every* characteristic curve of $u$ will intersect $K$ in at most one point. We obtain this property by requiring that *every* characteristic curve must intersect at most *one* of the disjoint translation of $Q_i$ considered at the $i$-th step. This is possible because we impose $u \in C^1$ on compact subsets of the open set $\mathbb{R}^2 \setminus K$, therefore characteristic curves of $u$ are uniquely defined in the open set $\mathbb{R}^2 \setminus K$: we force that every characteristic curve reaching the boundary of a translation of $Q_i$ does not intersect any other translation of $Q_i$.

After perfuming this program the counterexample will be ready: the counter-image of the $\mathcal{L}^2$-non-negligible set $K$ by any Lagrangian parameterization must have null measure, because each vertical section of this counterimage is made by the single point of intersection with $K$. This show that there is no Lagrangian parameterization satisfying the absolute continuity of Definition 1.1

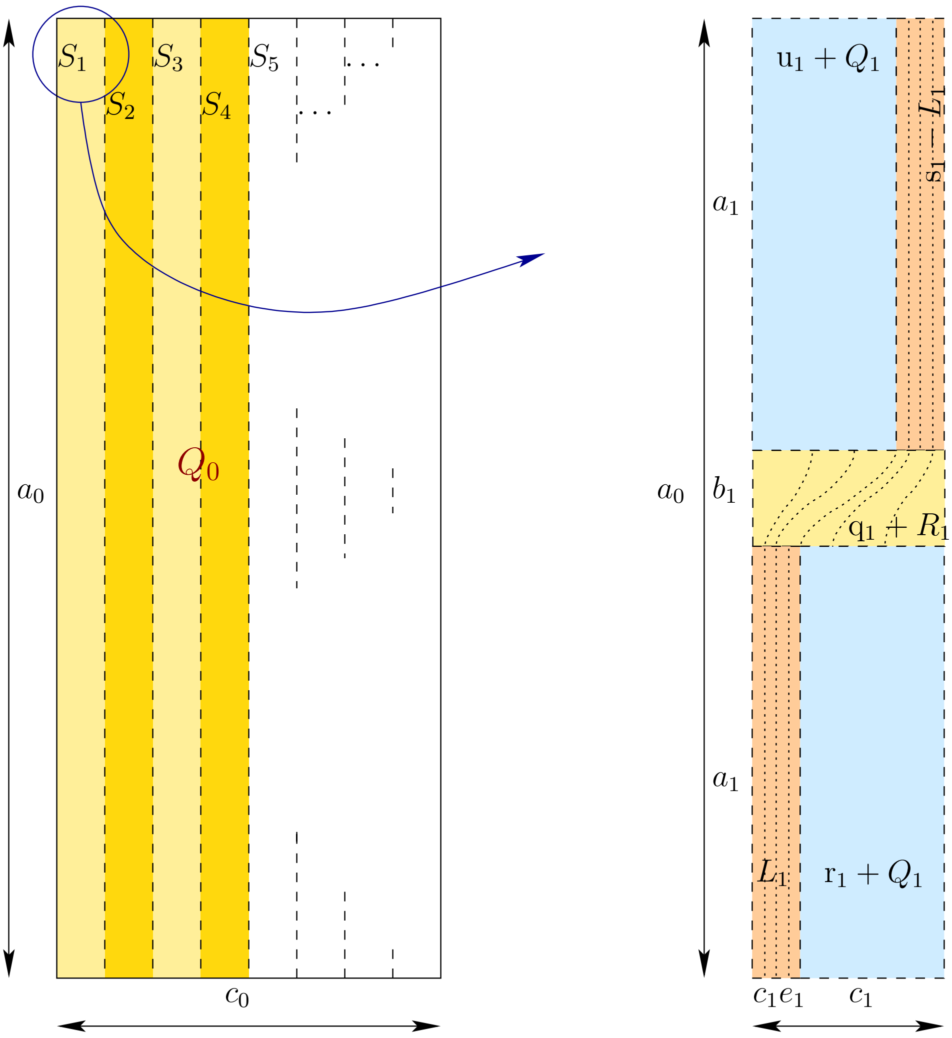


FIGURE 2. The initial region $Q_0$ and one of its strips $S_1$. Proportions are distorted. Dashed lines suggest the qualitative behavior of characteristic curves.

*1: Definition of the iterative regions (Figure 2).* We define here in an iterative way a finer and finer partition of a rectangle $Q_0$. This construction is based on parameters $a_i, b_i, c_i, d_i, e_i$ that we write explicitly in the next step. We only mention here the relation

$$d_i c_i(1+e_i) = c_{i-1} \qquad \Rightarrow \qquad c_i(1+e_i) = \frac{c_{i-1}}{d_i}. \tag{3.2}$$

The procedure is visualized in Figure 2. We consider the basic domain

$$Q_{i-1} = [0, a_{i-1}] \times [0, c_{i-1}] \qquad i \in \mathbb{N}$$

and we partition it at the $i$-th step into $d_i$ vertical sub-strips

$$S_i = [0, a_{i-1}] \times [0, c_i\,(1+e_i)]$$

of size $a_i \times c_i(1+e_i)$: this is possible by the relation (3.2). We obtain the flowing partition:

$$Q_{i-1} = [0, a_{i-1}] \times [0, c_{i-1}] = \bigcup_{j=0}^{d_i-1} (j\mathrm{v}_i + S_i) \qquad \mathrm{v}_i = \left(0, \frac{c_{i-1}}{d_i}\right) = (0, c_i\,(1+e_i))\,.$$

The vertical strip $S_i$ is then partitioned into three horizontal strips as follows:

- The intermediate strip is the translation of a rectangle

$$R_i = [0, b_i] \times [0, c_i\,(1+e_i)]\,.$$

  This strip provides space for smooth junctions among vertical curves.

- The two extremal horizontal strips have size $a_i \times c_i(1+e_i)$. Each extremal horizontal strip is the union of two vertical rectangles which are translations of

$$L_i = [0, a_i] \times [0, c_i e_i] \qquad Q_i = [0, a_i] \times [0, c_i]$$

This is possible if $2a_i + b_i = a_{i-1}$. One can define vectors for the translation and one can write

$$\mathrm{q}_i = (a_i, 0) \qquad \mathrm{r}_i = (0, c_i e_i) \qquad \mathrm{u}_i = (a_i + b_i, 0) \qquad \mathrm{s}_i = (a_{i-1}, c_i(1+e_i))$$
$$S_i = L_i \bigcup [\mathrm{r}_i + Q_i] \bigcup [\mathrm{q}_i + R_i] \bigcup [\mathrm{u}_i + Q_i] \bigcup [\mathrm{s}_i - L_i] \,.$$

Let us term $Q^{h_1\dots h_i}_{j_1\dots j_i}$ the rectangles generated at the $i$-th step, for $h_\ell \in \{0,1\}$, $j_\ell \in \{0, \dots, d_\ell - 1\}$:

$$Q^{h_1\dots h_i}_{j_1\dots j_i} = \mathrm{p}^{h_1\dots h_i}_{j_1\dots j_i} + Q_i, \quad \mathrm{p}^{h_1\dots h_i}_{j_1\dots j_i} = \sum_{\ell=1}^{i} j_\ell \mathrm{v}_\ell + \sum_{\ell=1}^{i} h_{j_\ell} \mathrm{u}_\ell + \sum_{\ell=1}^{i} (1 - h_{j_\ell}) \mathrm{r}_\ell.$$

Since at each step such rectangles are nested into previous ones, the translation vector $\mathrm{p}^{h_1\dots h_i}_{j_1\dots j_i}$ takes into account which is the list of nesting rectangles: $j_\ell$ tells us in which strip of the $\ell$-th step we are it, $h_{j_\ell}$ tells us if at the $\ell$-th step we are in the low right triangle of the corresponding strip—in case $h_{j_\ell} = 1$—or if we are instead in the upper left one—in case $h_{j_\ell} = 0$.

The remaining regions of shape $L_i$ and $R_i$ are not partitioned anymore at future steps.

*2: Setting up the parameters.* Set for $i \in \mathbb{N} \cup \{0\}$

$$a_i := 2^{-i-1}(1 + 2^{-i}).$$

We impose that for $i \in \mathbb{N}$ the interval of length $a_{i-1}$ is divided into three subintervals, two of which of length $a_i$ and one of length $b_i$:

$$b_i := a_{i-1} - 2a_i = 2^{-i}(1 + 2^{-i+1}) - 2 \cdot 2^{-i-1}(1 + 2^{-i}) = 2^{-i}[\cancel{1} + 2^{-i+1} \cancel{-1} - 2^{-i}] = 2^{-2i}.$$

We determine now values of $c_i, d_i, e_i$ satisfying (3.2) plus an additional requirement that we need later. Example A.3 constructs a curve $x = \gamma(t) \in C^2([0, b_i])$ which satisfies

$$0 \leq \frac{\mathrm{d}}{\mathrm{d}t} u \circ (t, \gamma(t)) \leq 1 \qquad u \circ (t, \gamma(t)) := \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \qquad \forall t \in [a, b_i] \tag{3.3a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t, \gamma(t))\Big|_{t=0} = u(0, \gamma(0)) = 0 = u(b_i, \gamma(b_i)) = \frac{\mathrm{d}}{\mathrm{d}t} u(t, \gamma(t))\Big|_{t=b_i} . \tag{3.3b}$$

Easy computations, reported for completeness in Example A.3, show the estimate

$$\max_{s,t \in [0, b_i]} |\gamma(t) - \gamma(s)| < 2f(b_i/2) = 2^{-4i-1}.$$

For every $0 \leq c < 2f(b_i/2)$, Example A.3 also constructs a $C^2$ curve satisfying (3.3) and

$$\gamma\,(b_i) - \gamma(0) = c.$$

Define therefore the following sequence of positive numbers, lower than $2f(b_i/2)$:

$$c_i := \frac{2f(b_i/2)}{\Pi_{j=1}^{i}(1 + e_j)} = \frac{2^{-4i-1}}{\Pi_{j=1}^{i}(1 + 2^{-j})}, \qquad c_0 = 1/2, \qquad e_i := 2^{-i}, \qquad i \in \mathbb{N}.$$

Define finally the integer ratio

$$d_i = \frac{c_{i-1}}{c_i(1 + e_i)} = \frac{16 \cdot \cancel{2^{-4i-1}}}{\cancel{\Pi_{j=1}^{i-1}(1 + 2^{-j})}} \frac{\cancel{(1 + 2^{-i})\Pi_{j=1}^{i-1}(1 + 2^{-j})}}{\cancel{2^{-4i-1}}} \frac{1}{\cancel{1 + 2^{-i}}} = 16.$$

A table of the first values is the following

| | $a_i$ | $b_i$ | $c_i$ | $d_i$ | $e_i$ |
|---|---|---|---|---|---|
| 0 | 1 | – | 1/2 | – | – |
| 1 | 3/8 | 1/8 | 1/48 | 16 | 1/2 |
| 2 | 5/32 | 1/32 | 1/960 | 16 | 1/4 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

*3: Measure of the Cantor set.* We compute the measure of the set

$$K = \bigcap_{i\in\mathbb{N}} \bigcup_{j_1=0}^{d_1-1} \cdots \bigcup_{j_i=0}^{d_i-1} \bigcup_{h_1,\dots,h_i=0}^{1} Q^{h_1\dots h_i}_{j_1\dots j_i}$$

Let us describe the above intersection step by step. At the first step there are $d_1$ stripes translations of $S_1$, each of which generates two rectangles which are translation of $Q_1 = [0, a_1] \times [0, c_1]$. Therefore

$$\mathcal{L}^2\left(\bigcup_{j=0}^{d_1-1}\bigcup_{h=0}^{1} Q^h_j\right) = 2d_1c_1a_1 \overset{(3.2)}{=} 2\frac{c_0}{(1+e_1)}a_1 = 2\frac{c_0}{1+e_1}a_1.$$

At the second step, each rectangle $j_\ell \mathrm{r}_1 + h_\ell \mathrm{r}_1 + Q_0$, for $h_\ell \in \{0, 1\}$ and $j_\ell \in \{0, \dots, d_1 - 1\}$, produces $2d_2$ smaller rectangles of size $a_2 \times c_2$: there are thus $2d_1 \cdot 2d_2$ rectangles of size $a_2 \times c_2$. More generally, each rectangle $Q^{h_1\dots h_{i-1}}_{j_1\dots j_{i-1}}$ generates $2d_i$ rectangles each of size $c_i a_i$, and there are $2d_1 \cdot \dots \cdot 2d_{i-1}$ such rectangles. We can hence conclude that at the $i$-th step

$$\begin{aligned}\mathcal{L}^2\left(\bigcup_{j_1=0}^{d_1-1}\cdots\bigcup_{j_i=0}^{d_i-1}\bigcup_{h_1,\dots,h_i=0}^{1} Q^{h_1\dots h_i}_{j_1\dots j_i}\right) &= 2^i d_1\cdots d_{i-1}d_ic_ia_i \\ &\overset{(3.2)}{=} 2^i d_1 \dots d_{i-1}\frac{c_{i-1}}{1+e_i}a_i \\ &\overset{(3.2)}{=} 2^i a_i \frac{c_0}{\prod_{j=1}^i(1+e_j)}\end{aligned}$$

As the series $\sum_{j=0}^\infty e_j$ converges, by the elementary estimate $\prod_{j=1}^i(1+e_j) \leq \exp(\sum_{j=1}^i e_j)$ the infinite product converges and in the limit we get

$$\mathcal{L}^2(K) = \lim_{i\to\infty} \cancel{2^i} \cdot \cancel{2^{-i}}{}^{-1}(1+2^{-i}) \cdot \frac{2^{-1}}{\prod_{j=1}^i(1+2^{-j})} = \frac{3\cdot 2^{-3}}{\prod_{j=1}^\infty(1+2^{-j})} > \frac{3}{8e}\,. \tag{3.4}$$

In particular $K$ is non-negligible. We also observe that it is compact, since the $i$-th element of the intersection is the union of finitely many closed rectangles contained in $Q_0$.

*4: Assigning $u$ and characteristic curves.* We divided $Q_0$ into different regions in order to facilitate the definition of the characteristic curves. Set:

- $u \equiv 0$ in each region which is created at the $i$-th step as a translation of $L_i$, $i \in \mathbb{N}$.
- define in $R_i$ characteristic curves providing smooth junctions, as in Example A.3, from

  $$\begin{matrix} u = 0 \text{ on } \{0\}\times[0, c_ie_i] \\ u = 0 \text{ on } \{0\}\times[c_ie_i, c_i(1+e_i)] \end{matrix} \quad\text{to}\quad \begin{matrix} u = 0 \text{ on } \{b_i\}\times[0, c_i] \\ u = 0 \text{ on } \{b_i\}\times[c_i, c_i(1+e_i)]. \end{matrix}$$

  We associated in this way characteristic curves, and therefore $u$, to each fundamental domain $R_i$. Characteristic curves are defined in the region $\mathrm{p}^{h_1\dots h_i}_{j_1\dots j_i} + \mathrm{q}_i + R_i$, translation of $R_i$, as the above characteristic curves translated by the same vector $\mathrm{p}^{h_1\dots h_i}_{j_1\dots j_i} + \mathrm{q}_i$.

The dashed lines in the RHS of Figure 2 give an idea of the qualitative behaviour. We have

$$u \in C^1(Q_0 \setminus K) \cap C(Q_0).$$

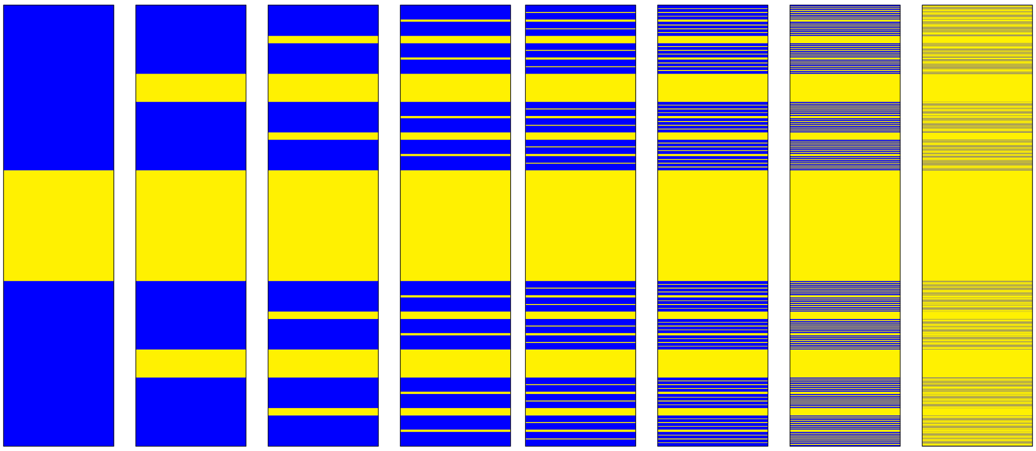

FIGURE 3. From left to right, figures illustrate the iterative horizontal subdivision of the height $a_0$—left figure—first in two extremal horizontal strips of height $a_1$ (blue ones) and a central strip of height $b_1$ (central one), then—second figure—the subdivision of each horizontal strip of height $a_1$ into two horizontal strips of height $a_2$ (blue ones) and a central strip of height $b_2$, and so on al later iterations. $K$ lies within blue regions. The regions $L_i$ are so thin, even after two iterations, that are not visible in such a picture.

The unique continuous extension of $u$ to $Q_0$ vanishes on $K$.

*5: Conclusion.* By (3.4) the set $K$ has positive measure. Notice that every characteristic curve intersects $K$ in a single point, and countably many of them in two points: the iterative construction is made in such a way that each characteristic curve intersecting a region $Q^{h_1\dots h_i}_{j_1\dots j_i}$ is uniquely defined out of it. In particular, if *any* characteristic curve of the continuous function $u$ intersects a rectangle $Q^{h_1\dots h_i}_{j_1\dots j_i}$ it does not intersect in the complementary of $Q^{h_1\dots h_i}_{j_1\dots j_i}$ other regions constructed as translation of any $Q_i$, $i \in \mathbb{N}$—with the exception of the curves on the boundary of $Q^{h_1\dots h_i}_{j_1\dots j_i}$, which run on the boundary of another equal rectangle. This implies that the counter-image of $K$ by *any* Lagrangian parameterization must have null measure, as each vertical section of this counterimage is made by the single point which is the intersection of $K$ with the relative characteristic composing the parameterization—or by two such points, for countably many curves.

We show in Figure 3 at a better scale the iterative horizontal subdivision of the height $a_0$ first in two extremal horizontal strips of height $a_1$ (blue ones) and a central strip of height $b_1$ (central one), then the subdivision of each horizontal strip of height $a_1$ into two horizontal strips of height $a_2$ (blue ones) and a central strip of height $b_2$, and so on al later iterations. The compact $K$ lies within blue regions.

## 4. NON-NEGLIGIBLE POINTS OF NON-DIFFERENTIABILITY ALONG CHARACTERISTICS

The example in this section shows the following: even when $u$ is Lipschitz continuous, or even $C^\infty$, along characteristics, there could be a compact, $\mathcal{L}^2$-positive measure set $K \subset \mathbb{R}^2$ of points where $u$ fails to be differentiable along characteristics, whichever characteristic curves one choses through the point. One can also have $u \in C^\infty(\mathbb{R}^2 \setminus K)$, but clearly it will be just continuous on the whole region.

This behavior is prevented by the $\alpha$-convexity of the flux [7, Theorem 1.2]: we give an example where the convex flux function vanishes at $0$ together with all its derivatives, while it is uniformly convex out of the origin. This provides as well a second example of non-absolute continuity of Lagrangian parametrizations.

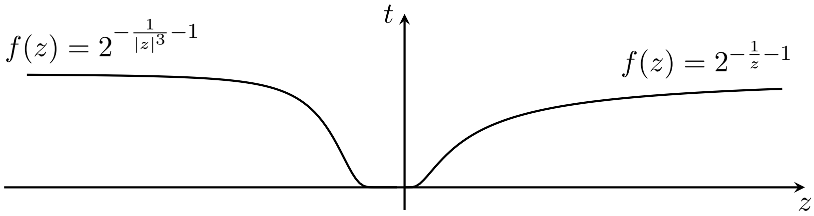


FIGURE 4. Flux function $f$ considered in § 4. Close to the origin, $f$ is strictly convex, but not uniformly convex. This flux function is $C^\infty(\mathbb{R})$, with all derivatives vanishing at the origin, but of course it is not analytic

Define the flux function, for $|z| \le 1$, given by (Figure 4)

$$f(z) = \begin{cases} 2^{-\frac{1}{z}-1} & z \ge 0, \\ 2^{-\frac{1}{|z|^3}-1} & z < 0. \end{cases}$$

We mimic the construction of § 3, modifying the regions $L_i$ and the parameters.

*1: Setting up the parameters.* Similarly to § 3 we set up parameters $a_i, b_i, c_i$ satisfying

$$a_{i-1} = 2a_i + b_i, \qquad c_{i-1} = c_i(1+d_i), \qquad c_i = 2f(b_i/2) \qquad i \in \mathbb{N} \cup \{0\},$$

so that the following properties for the recursive construction are satisfied:

- An interval of length $a_i$ is the disjoint union of two intervals of length $a_i$ plus an interval of length $b_i$.
- There exists a $C^2$ curve $x = \gamma(t)$, for $t \in [0, b_i]$ which satisfies relations

$$0 \le \frac{\mathrm{d}}{\mathrm{d}t} u \circ (t, \gamma(t)) \le 1 \qquad f'(u) \circ (t, \gamma(t)) := \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \qquad \forall t \in [a, b_i] \tag{4.1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t,\gamma(t))\Big|_{t=0} = u(0,\gamma(0)) = 0 = u(b_i, \gamma(b_i)) = \frac{\mathrm{d}}{\mathrm{d}t} u(t,\gamma(t))\Big|_{t=b_i} . \tag{4.1b}$$

and

$$\gamma(b_i) - \gamma(0) = \frac{c_i}{\lambda_i} = c_{i+1}, \qquad \lambda_i := 1 + d_i.$$

The last point is given again by Example A.3. In particular, one can fix

$$a_i = 2^{-i-1}(1+2^{-i}) \qquad b_i = 2^{-2i} \qquad a_0 = 1 \tag{4.2a}$$

$$c_i = 2f(b_i/2) = \cancel{2} \cdot 2^{-\frac{1}{b_i/2}\cancel{-1}} = 2^{-2^{2i+1}}, \qquad c_0 = \frac{1}{4}, \qquad i \in \mathbb{N}. \tag{4.2b}$$

$$\lambda_i = \frac{c_{i-1}}{c_i} = \frac{2^{-2^{2i-1}}}{2^{-2^{2i+1}}} = 2^{2^{2i-1}(4-1)} = 2^{3\cdot 2^{2i-1}}, \tag{4.2c}$$

$$d_i = \frac{c_{i-1} - c_i}{c_i} = \lambda_{i+1} - 1 = 2^{3\cdot 2^{2i-1}} - 1 . \tag{4.2d}$$

Notice finally that the difference $c_{i-1} - c_i$ is asyntotic to $c_{i-1}$:

$$\frac{c_{i-1} - c_i}{c_{i-1}} = 1 - \frac{c_i}{c_{i-1}} = 1 - \frac{1}{1+d_1} = 1 - \lambda_i^{-1}$$

and thus

$$d_i c_i = c_{i-1} - c_i = c_{i-1}(1 - \lambda_i^{-1}) . \tag{4.3}$$

A table of the first values is the following

| | $a_i$ | $b_i$ | $c_i$ | $d_i$ |
|---|---|---|---|---|
| 0 | 1 | – | 1/4 | – |
| 1 | 3/8 | 1/4 | 1/256 | 63 |
| 2 | 5/32 | 1/16 | $2^{-32}$ | $2^{24}-1$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

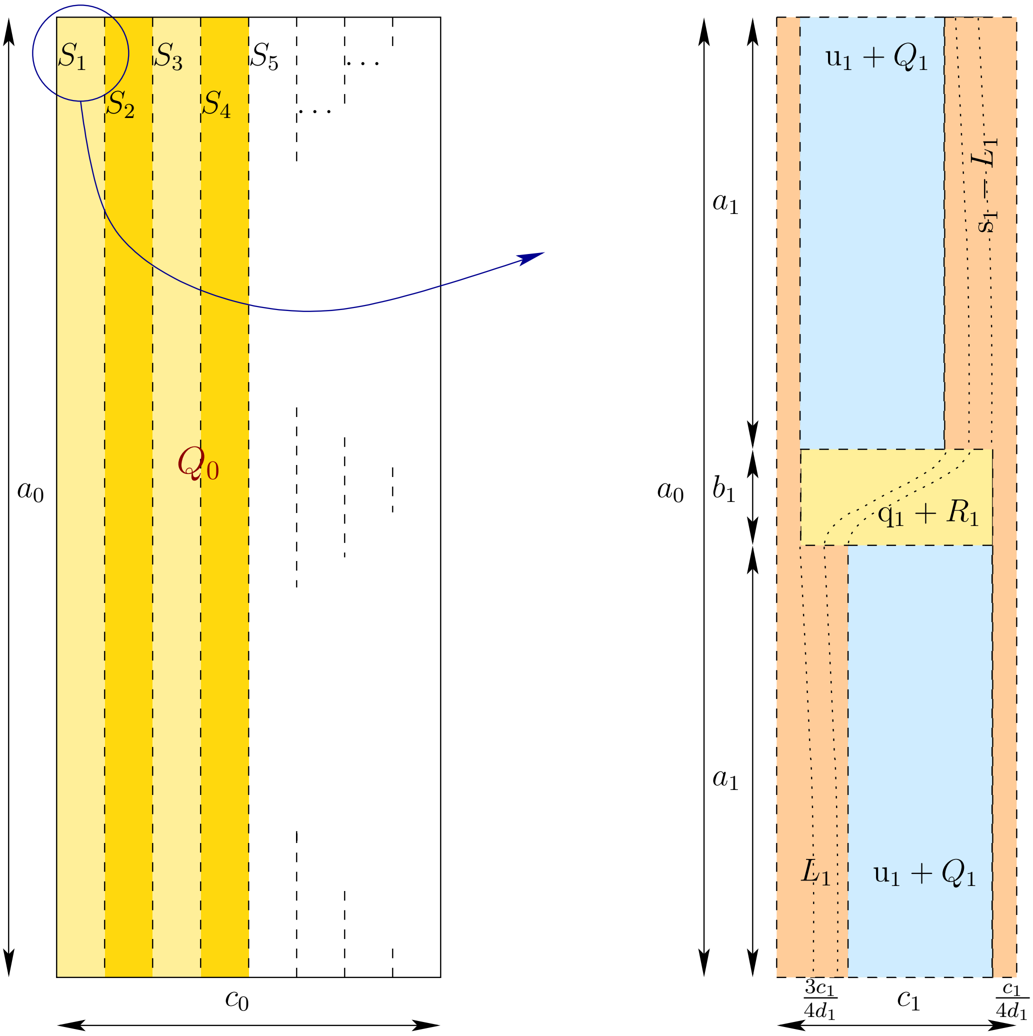


FIGURE 5. The initial region $Q_0$ and one of its strips $S_1$. Proportions are distorted. Dashed lines suggest the qualitative behavior of characteristic curves.

*2: Definition of the iterative regions (Figure 5).* We consider the basic domain

$$Q_{i-1} = [0, a_{i-1}] \times [0, c_{i-1}] \qquad i \in \mathbb{N}$$

and we partition it at the $i$-th step into $d_1$ vertical sub-strips which are translations of

$$S_i = [0, a_{i-1}] \times \left[0, c_i\left(1 + \frac{1}{d_i}\right)\right]$$

of size $a_0 c_1(1 + 1/d_1)$: indeed $a_0 = 1$ and

$$d_1 c_1(1 + 1/d_1) = d_1 c_1 + c_1 = c_0 \cancel{-c_1+c_1}.$$

We obtain the flowing partition of $Q_0$, that we write with a notation suitable for later iterations, when we will then have $i > 1$:

$$Q_{i-1} = [0, a_{i-1}] \times [0, c_{i-1}] = \bigcup_{j=0}^{d_i - 1} (j\mathrm{v}_i + S_i) \qquad \mathrm{v}_i = \left(0, \frac{c_{i-1}}{d_i}\right).$$

The vertical strip $S_i$ is then partitioned into three horizontal strips as follows, see Figure 5:

- The intermediate horizontal strip has size $b_i \times c_i(1 + 1/d_i)$. It is in turn made of three vertical sub-strips:
  - two extremal strips of size $b_i \times c_i/4d_i$, included in the regions $L_i$ of the next point, and
  - a central one strip which is the translation of a rectangle

$$R_i = [a_i, a_i + b_i] \times \left[0, c_i\left(1 + \frac{1}{2d_i}\right)\right].$$

    This strip provides space for smooth junctions among vertical curves.
- The two extremal strips have size $a_i \times c_i(1 + 1/d_i)$. Each horizontal strip is in turn made of three vertical sub-strips:
  - a central iterative strip which is a translation of the rectangle

$$Q_i = [0, a_i] \times [0, c_i];$$

  - two extremal strips which are translations and reflections of

$$L_i = [0, a_i] \times \left[0, \frac{3c_i}{4d_i}\right] \bigcup [0, a_{i-1}] \times \left[0, \frac{c_i}{4d_i}\right].$$

We can hence write the decomposition

$$S_i = L_i \bigcup [\mathrm{s}_i - L_i] \bigcup [\mathrm{q}_i + R_i] \bigcup [\mathrm{r}_i + Q_i] \bigcup [\mathrm{u}_i + Q_i]$$

$$\mathrm{s}_i = \left(a_0, c_1 + \frac{c_1}{d_i}\right) \qquad \mathrm{q}_i = \left(0, \frac{c_i}{4d_i}\right) \qquad \mathrm{r}_i = \left(0, \frac{3c_i}{4d_i}\right) \qquad \mathrm{u}_i = \left(a_i + b_i, \frac{c_i}{4d_i}\right).$$

We term $Q^{h_1 \dots h_i}_{j_1 \dots j_i}$, for $h_\ell \in \{0, 1\}$ and $j_\ell \in \{0, \dots, d_\ell - 1\}$ when $\ell = 1, \dots, i$, the rectangles that are generated at the $i$-th step: with the above notation one can write

$$Q^{h_1 \dots h_i}_{j_1 \dots j_i} = \sum_{\ell=1}^{i} j_\ell \mathrm{v}_\ell + \sum_{\ell=1}^{i} h_{j_\ell} \mathrm{u}_\ell + \sum_{\ell=1}^{i} (1 - h_{j_\ell}) \mathrm{r}_\ell + Q_i.$$

Each of this rectangles $Q^{h_1 \dots h_i}_{j_1 \dots j_i}$ will be further partitioned at the next step. The remaining regions of shape $L_i, -L_i$ and $R_i$, suitably translated, are not partitioned any more.

*3: Measure of the Cantor set.* We compute the measure of the set

$$K = \bigcap_{i \in \mathbb{N}} \bigcup_{j_1=0}^{d_1-1} \cdots \bigcup_{j_i=0}^{d_i-1} \bigcup_{h_1, \dots, h_i=0}^{1} Q^{h_1 \dots h_i}_{j_1 \dots j_i}$$

As each $Q^{h_1 \dots h_{i-1}}_{j_1 \dots j_{i-1}}$ generates $2d_i$ rectangles each of size $c_i a_i$, then at the $i$-th step

$$\begin{aligned}
\mathcal{L}^2 \left( \bigcup_{j_1=0}^{d_1-1} \cdots \bigcup_{j_i=0}^{d_i-1} \bigcup_{h_1, \dots, h_i=0}^{1} Q^{h_1 \dots h_i}_{j_1 \dots j_i} \right) &= 2^i d_1 \cdots d_{i-1} d_i c_i a_i \\
&\overset{(4.3)}{=} 2^i d_1 \dots d_{i-1} [c_{i-1}(1 - \lambda_i^{-1})] a_i \\
&\overset{(4.3)}{=} 2^i a_i c_0 \prod_{j=0}^{i} (1 - \lambda_j^{-1})
\end{aligned}$$

As the series $\sum_{j=0}^{\infty}\lambda_j^{-1}$ converges, by the elementary estimate $\prod_{j=1}^{i}(1-\lambda_j^{-1}) \le \exp(-\sum_{j=1}^{i}\lambda_j^{-1})$ the infinite product also converges and in the limit we get

$$\mathcal{L}^2(K) = \lim_{i\to\infty} \cancel{2^i}\cdot 2^{\cancel{-i}-1}(1+2^{-i})\cdot \prod_{j=1}^{i}\left(1-2^{-3\cdot 2^{2i-1}}\right)\cdot 2^{-2} \tag{4.4}$$

$$= \frac{1+2^{-i}}{8}\prod_{j=1}^{\infty}\left(1-2^{-3\cdot 2^{2i-1}}\right) > \frac{1}{8e}\,.$$

In particular $K$ is non-negligible.

*4: Assigning $u$ and characteristic curves.* We subdivided $Q_0$ into different regions in order to facilitate the definition of the characteristic curves. We care now of defining

$$u \in C^1(Q_0\setminus K)\cap C(Q_0).$$

It will vanish on $K$ by continuity. We define simultaneously characteristic curves in $Q_0\setminus K$. They will be defined separately in the different regions, and they will have smooth junctions.

Up to translations, focus on the fundamental regions $R_i$, $L_i$, $-L_i$. We first specify the following common properties that characteristic curves should satisfy in a region $R\in\{R_i, L_i, -L_i\}$:

- Characteristic curves do not intersect.
- Characteristic curves through points $(t,x)$ with $t\in\{0; a_i; a_i+b_i; a_{i-1}\}$ have there vertical tangent. This means that $u$ vanishes on those horizontal lines. We impose moreover that at those points also the derivative of $u$ along the characteristic vanishes.
- The image of the curves cover the whole region $R$, defining a $C^1(R)$ function

  $u(t,x) := \left(f'\right)^{-1}(\dot\gamma(t))$ where $\gamma$ is the characteristic defined through $(t,x)\in R$.

  There is no ambiguity in this structure due to the strict, even if not uniform, convexity of $f$.

We describe now the shape of the curves, depending on the region $R\in\{R_i, L_i, -L_i\}$. The dashed lines in the RHS of Figure 5 give an idea of this qualitative behavior. The precise expression of the characteristic curves that we describe can be computed by elementary auxiliary computations that we report for completeness in § A.

*4.1: Region $R_i$ (Figure 5).* The unique characteristic curve through a point in

| | | |
|---|---|---|
| $\{a_i\}\times[0,\frac{c_i}{4d_i}]$ | | $\{a_i+b_i\}\times[0,c_i]$ |
| $\{a_i\}\times[\frac{c_i}{4d_i},\frac{c_i}{2d_i}]$ | reaches increasingly a point in | $\{a_i+b_i\}\times[c_i,c_i+\frac{c_i}{4d_i}]$ |
| $\{a_i\}\times[\frac{c_i}{2d_i},c_i+\frac{c_i}{2d_i}]$ | | $\{a_i+b_i\}\times[c_i+\frac{c_i}{4d_i},c_i+\frac{c_i}{2d_i}]$. |

This is compatible with the previous common requirements by the choice (4.2b): an explicit construction of a curve joining $(a_i,\lambda\frac{c_i}{4d_i})$ and $(a_i+b_i,\lambda c_i)$, for $\lambda\in[0,1]$, is provided by Example A.3.

*4.2: Region $L_i$ (Figure 5).* We describe characteristics separately in the strips

$$[0,a_i]\times\left[0,\frac{c_i}{4d_i}\right],\quad [0,a_i]\times\left[\frac{c_i}{4d_i},\frac{c_i}{2d_i}\right],\quad [0,a_i]\times\left[\frac{3c_i}{4d_i},c_{i-1}\right],\quad [a_i,a_{i-1}]\times\left[0,\frac{c_i}{4d_i}\right].$$

The unique characteristic curve through a point in

| | | |
|---|---|---|
| $\{0\}\times[0,e_i]$ | reaches decreasingly a point in | $\{a_i\}\times[0,\frac{c_i}{4d_i}]$ |
| $\{0\}\times[e_i,e_i+\frac{c_i}{4d_i}]$ | reaches decreasingly a point in | $\{a_i\}\times[\frac{c_i}{4d_i},\frac{c_i}{2d_i}]$ |
| $\{0\}\times[e_i+\frac{c_i}{4d_i},\frac{3c_i}{4d_i}]$ | reaches decreasingly a point in | $\{a_i\}\times[\frac{c_i}{2d_i},\frac{3c_i}{4d_i}]$ |
| $\{a_i\}\times[0,\frac{c_i}{4d_i}]$ | remains constant up to | $\{a_{i-1}\}\times[0,\frac{c_i}{4d_i}]$ |

for a value of $e_i$ that we specify now. We are in a situation completely analogous to Example A.3. We show it in the most interesting region, which is the second one, and we define together $e_i$. Consider the curve $\gamma(t)=\gamma(t;a_i/8)$ given by (A.3), but substituting $f(-z)$ to $f(z)$: this corresponds to the fact that $u$ is decreasing along the curve instead of increasing, and thus $u$ is negative—which in turn corresponds to $\gamma$ decreasing. Set then

$$e_i = \frac{c_i}{4d_i} - [\gamma(a_i) - \gamma(0)] > \frac{c_i}{4d_i}.$$

Notice that the intervals above are well defined, because $e_i < \frac{c_i}{2d_i}$:

$$|\gamma(a_i) - \gamma(0)| < f(-a_i/2) = 2^{-\frac{1}{(a_i/2)^3}-1} = 2^{-\frac{2^{3i+6}}{(1+2^{-i})^3}-1} < \frac{2^{-2^{2i+1}}}{4(2^{3\cdot 2^{2i-1}}-1)} = \frac{c_i}{4d_i}.$$

We observe finally that $u$ decreases along $\gamma(t)$ up to $t = a_i/2$ and its derivative is $-1$ for $t$ in $[a_i/8, 3a_i/8]$: then the minimum value of $u$, reached at the centre of the interval, is

$$u(\gamma(a_i/2)) = -\frac{3a_i}{8} < -\frac{a_i}{4}. \tag{4.5}$$

*4.3: Region $-L_i$ (Figure 5).* The region $-L_i$ is entirely similar to the region $L_i$ already described, therefore we will be quick. We require that the unique characteristic curve through a point in

| | | |
|---|---|---|
| $\{-a_i\} \times [-\frac{3c_i}{4d_i}, -\frac{c_i}{2d_i}]$ | reaches decreasingly a point in | $\{0\} \times [-\frac{3c_i}{4d_i}, -e_i - \frac{c_i}{4d_i}]$ |
| $\{-a_i\} \times [-\frac{c_i}{2d_i}, -\frac{c_i}{4d_i}]$ | reaches decreasingly a point in | $\{0\} \times [-e_i - \frac{c_i}{4d_i}, -e_i]$ |
| $\{-a_i\} \times [-\frac{c_i}{4d_i}, 0]$ | reaches decreasingly a point in | $\{0\} \times [-e_i, 0]$ |
| $\{-a_{i-1}\} \times [-\frac{c_i}{4d_i}, 0]$ | remains constant up to | $\{-a_i\} \times [-\frac{c_i}{4d_i}, 0]$ |

for the value of $e_i$ already defined. Along characteristic curves passing through $\{-a_i\} \times [-\frac{c_i}{2d_i}, -\frac{c_i}{4d_i}]$ the function $u$ reaches a minimum value which is less than $-a_i/4$.

*5: $u$ on $K$ is not differentiable along characteristics.* We remind that $K$ has positive measure by (4.4). We check now that $K$ is made of non-differentiability points of $u$ along characteristics. Consider a point $(t,x) \in K$: it is the countable intersection of countable unions of rectangles, and the $2d_1 \cdots d_i$ rectangles in the $i$-th union are translations of $Q_i$. With the notation above, one can write

$$\forall \ell \in \mathbb{N} \quad \exists h_\ell \in \{0,1\},\ j_\ell \in \{0,\dots,d_\ell - 1\} \quad : \quad (t,x) \in \bigcap_{i=1}^{\infty} Q^{h_1\dots h_i}_{j_1\dots j_i}.$$

For simplicity of notation focus on $j_\ell = 0$, $h_\ell = 1$ for every $\ell \in \mathbb{N}$, which means

$$(t,x) = \left(a_0,\ \sum_{i\in\mathbb{N}} \frac{c_i}{4d_i}\right).$$

The general case is entirely analogous. Denote by $\gamma(t)$ a characteristic curve through $(t,x)$. We show the non-differentiability by proving the following:

(i) There is a sequence of points $\{t_i^0\}_{i\in\mathbb{N}}$ converging to $t$ such that $u(t_i^0, \gamma(t_i^0)) = 0$.
(ii) There is a sequence of points $\{t_i^-\}_{i\in\mathbb{N}}$ such that

$$|t_i^- - t| = \frac{a_i}{2} \le a_i, \qquad u(t_i^-, \gamma(t_i^-)) = -\frac{3a_i}{8} < -\frac{a_i}{4}. \tag{4.6}$$

This implies that $u \circ \gamma$ cannot have zero derivative at $t$: since $u(t,x) = 0$ we get

$$\liminf_{i\to\infty} \left| \frac{u(t,x) - u(t_i^-, \gamma(t_i^-))}{t - t_i^-} \right| > \liminf_{i\to\infty} \frac{a_i/4}{a_i} = \frac{1}{4}.$$

The two points together imply that $u \circ \gamma$ cannot be differentiable at $t$, because along the two different sequences $\{t_i^-\}_{i\in\mathbb{N}}$ and $\{t_i^0\}_{i\in\mathbb{N}}$ the different quotients have two different limits: respectively $0$ and something less than or equal to $-\frac{1}{4}$. The two sequences are defined as follows.

(i) By construction (Figure 5) $\gamma$ intersects each lower side of the rectangles $Q_0^1$, $Q_{00}^{11}$, $Q_{000}^{111}$, ... at times $a_0 - a_1$, $a_0 - a_2$, .... On that side we set $u$ vanishing: then $u$ vanishes on the sequence of times $t_i^0 = a_0 - a_i$, $i \in \mathbb{N}$, which converges to $t = a_0$.

(ii) Characteristic curves were conveyed to the lower side of $Q^{1\cdots 1}_{0\cdots 0}$, which is say a translation of $Q_i$, from a specific part of the region

$$\left(\sum_{j=1}^{i-1}(a_j+b_j),\ \sum_{j=1}^{i-1}\frac{c_j}{4d_j}\right)+L_i.$$

There are times inside this region translated of $L_i$ where our requirement is satisfied, like

$$t_i^- := \sum_{j=1}^{i-1}(a_j+b_j)+\frac{a_i}{2}=a_0-b_i-\frac{3a_i}{2}$$

works by (4.5). Notice that $|t-t_i^-|=b_i+3a_i/2<a_{i-1}\to 0$.

This concludes proof that $u$ at any point of $K$ is not differentiable along characteristics, and hence this concludes the example.

*Remark* 4.1. We notice that the function $u$ constructed in the present section is not Hölder continuous: in the same setting where (4.6) was derived one has

$$u(t_i^-,\gamma(t_i^-))=-\frac{3a_i}{8}\overset{\text{(4.2a)}}{=}2^{-i-1}(1+2^{-i})\qquad\sim\qquad 2^{-i-1}\,.$$

Moreover, $u$ vanishes on the left side of each $L_i$ which contains part of $\gamma$, so that if we denote by $x_i^*$ the intersection of that left side with the fixed time $t=t_i^-$, namely $x^*=\sum_{j=1}^{i-1}\frac{c_j}{4d_j}$, then

$$0<\gamma(t_i^-)-x_i^*<\frac{3}{4}\frac{c_i}{d_i}=\frac{3}{4}\frac{c_i^2}{c_{i-1}-c_i}\overset{\text{(4.2b)–(4.2d)}}{=}\frac{3}{4}\frac{2^{-2^{2(i+1)}}}{2^{3\cdot 2^{2i-1}}-1}\qquad\sim\qquad\frac{3}{4}\cdot 2^{-11\cdot 2^{2i-1}}\,.$$

We thus conclude that for every constant $\alpha>0$

$$\lim_i\frac{|u(t_i^-,\gamma(t_i^-))-u(t_i^-,x^*)|}{|\gamma(t_i^-)-x_i^*|^\alpha}=\lim_i\frac{2^{-i+1}}{3\cdot 2^{-11\alpha\cdot 2^{2i-1}}}=+\infty\,.$$

## 5. Failure of Lipschitz continuity along characteristics

A continuous distributional solution $u$ to

$$\partial_t u(t,x)+\partial_x(f(u(t,x)))=g_E(t,x)\qquad f\in C^2(\mathbb{R}),\qquad |g_E(t,x)|\le G \tag{1.1}$$

is always Lipschitz continuous along characteristics if

$$\mathcal{L}^1(\mathrm{clos}(\mathrm{Infl}(f)))=0 \tag{H}$$

is satisfied [2, Theorem 30]. Our first aim is to show now that this assumption is needed.

Example 5.2 shows that for some Lagrangian parameterization it does not make sense defining a Lagrangian source, since Lipschitz continuity of $u$ on its characteristics might fail. It also proves that, whenever one chooses a 'good' Lagrangian parameterization, the Lagrangian source should be fixed accordingly: no universal choice is possible even within all admissible Lagrangian parameterizations. This is even more astonishing considering that it holds also when source terms are autonomous.

*Remark* 5.1. We first remind that there might be continuous solutions to (1.1), even in the autonomous case $u(t,x)\equiv u(x)$, which are Cantor-like functions. Consider a flux function $f\in C^2(\mathbb{R})$ which is strictly increasing and which satisfies

$$S:=\{z\ :\ f'(z)=0\}\subset[0,1]\,,\qquad \mathcal{L}^1(S)>0\,,\quad S\text{ does not contain intervals.}$$

Notice that $f''(z)=0$ at all points $z\in S$ and all points of $S$ are inflection points of $f$, so that condition (H) is violated. Consider the function

$$w(z)=z-\mathcal{L}^1(\{q<z\ :\ f'(q)=0\})\,,$$

which of course is strictly increasing and 1-Lipschitz continuous, since

$$z_1 < z_2 \qquad \Rightarrow \qquad 0 < w(z_2) - w(z_1) = z_2 - z_1 - \mathcal{L}^1([z_1, z_2] \cap S) \leq z_2 - z_1 \ .$$

Being $w'(z) = 0$ and $f'(z) = 0$ when $z$ is a Lebesgue point of $S$, then by the area formula

$$\mathcal{L}^1(w(S)) = 0 \ , \qquad \mathcal{L}^1(f(S)) = 0 \ . \tag{5.1}$$

This vanishing condition implies that the (continuous, strictly increasing) inverse $w^{-1}$ of $w$ is a Cantor-Vitali like function, since $w^{-1}$ maps the $\mathcal{L}^1$-negligible set $w(S)$ to the non $\mathcal{L}^1$-negligible set $S$. Set

$$u(t, x) = w^{-1}(x) \ .$$

Notice that $u$ defines a continuous function which is constant in $t$ and strictly increasing in $x$. The composition $f \circ u$ is Lipschitz continuous and $u$ is a distributional solution of

$$\partial_t \, u(t, x) + \partial_x(f(u(t, x))) = f(w^{-1})'(x).$$

*Example* 5.2 (A distributional solution is not necessarily broad). Consider the same flux as in Remark 5.1, where $f$ could be the inverse of a Cantor-Vitali-like function.

Define the continuous function

$$u(t, x) = f^{-1}(x), \tag{5.2}$$

which is a distributional solution of the equation

$$\partial_t \, u + \partial_x \, f(u) = \partial_x \, x = 1.$$

We now show that $u$ is not a broad solution, because it is not Lipschitz continuous on every characteristic curve. We then compute that it is indeed a Lagrangian solution, as it must be [2, Cor. 46].

*1: $u$ is not a broad solution.* Consider the Lipschitz continuous and monotone function

$$w \quad : \quad z \quad \mapsto \quad z - \mathcal{L}^1(N \cap [0, z]).$$

The derivative of $w$ at the density points of $N$ is 0, while it is 1 at the Lebesgue points of the complementary. As $N$ is mapped into a $\mathcal{L}^1$-negligible set on which the singular part of $\partial_z w$ is concentrated: in particular, $w^{-1}$ is not absolutely continuous. The following curve is well defined, because $w$ is a bijection:

$$\gamma(t) = f(w^{-1}(t)). \tag{5.3}$$

Form the definition of $N$, $f'$ vanishes on it. As a consequence of (5.1) the curve $\gamma$ is Lipschitz continuous because the composition of $f$ and $w^{-1}$ is absolutely continuous, and

$$\dot{\gamma}(t) = f'(w^{-1}(t)) \overset{(5.3)}{=} f'(f^{-1}(f(\gamma(t)))) \overset{(5.2)}{=} f'(u(t, \gamma(t))).$$

Nevertheless, $u$ is not absolutely continuous on this characteristic curve $\gamma$:

$$u(t, \gamma(t)) \overset{(5.2)}{=} f^{-1}(\gamma(t)) \overset{(5.3)}{=} f^{-1}(f(w^{-1}(t))) = w^{-1}(t).$$

*2: $u$ is a Lagrangian solution.* For each $\tau \in \mathbb{R}$ the curve

$$\chi(t, \tau) = f(t + \tau) \tag{5.4}$$

is a characteristic curve: indeed by (5.2) and by definition of this Lagrangian parameterizatin $\chi$

$$u(t, \chi(t, \tau)) = f^{-1}(f(t + \tau)) = t + \tau$$

and thus

$$\partial_t \, \chi(t, \tau) = f'(t + \tau) = f'(u(\chi(t, \tau))) \ .$$

Since $\chi$ is also monotone in $\tau$, $\chi$ is a Lagrangian parameterization associated with $u$. Notice that the Lagrangian source coincides with the natural representative of the distributional one:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(\chi(t,\tau))=\frac{\mathrm{d}}{\mathrm{d}t}(t+\tau)=1\qquad\forall t>0,\tau\in\mathbb{R}.$$

This yields to the natural definition $\mathfrak{g}(t,x)=1$ for $(t,x)\in\mathbb{R}^+\times\mathbb{R}$.

*3: $u$ admits different Lagrangian sources in $\mathfrak{L}^\infty_B$.* On the compact, $\mathcal{L}^1$-negligible set of $x$ defined by $f(N)$ one has also the vertical characteristics. One can clearly define a Lagrangian parameterization $\tilde\chi$ which includes these characteristics. Being $u$ identically zero on these, then it would be necessary to define pointwise the source term

$$\tilde{\mathfrak{g}}(t,x)=0\qquad\forall(t,x)\in\mathbb{R}\times f(N).$$

This is different from what we obtain with the previous Lagrangian parameterization. However, being $\mathcal{L}^1(f(N))=0$, the functions $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ differ on an $\mathcal{L}^2$-negligible set. The Lagrangian sources of the two Lagrangian parameterizations identify the same distributional source:

$$\tilde{\mathfrak{g}}=\mathfrak{g}\qquad\mathcal{L}^2\text{-a.e.}.$$

Attaining different values on the characteristics $\{(t,\tau)\}$ for $\tau\in f(N)$, $\mathfrak{g}$ fails to be a source associated to the Lagrangian parameterization $\tilde\chi$ as well as $\tilde{\mathfrak{g}}$ fails to be a source associated to the Lagrangian parameterization $\chi$. Indeed, there are positive measure times $t\in(N-\tau)$ where the sources differ on the characteristic curves (5.4): $1=\mathfrak{g}(t,\chi(t,\tau))\neq\mathfrak{g}(t,\chi(t,\tau))=0$.

We conclude observing that Lagrangian sources relative to $\chi$ and to $\tilde\chi$ are not compatible among themselves: there exists no Borel, bounded function $\mathfrak{g}$ such that

- $\mathfrak{g}\circ i_\chi=1$ $\mathcal{L}^2$-a.e. and
- which satisfies $\mathfrak{g}(t,x)=0$ for $\mathcal{L}^1$-a.e. $t$ if $x\in f(N)$.

## Appendix A. Auxiliary computations

We collect here elementary computations which exhibit junctions, with a generic flux $f$, among characteristic curves defined in separate regions. Each characteristic curve $\gamma$ must satisfy

$$\dot\gamma(t)=f'(u((t,\gamma(t))))$$

by definition. In particular, if one prescribes $u((t,\gamma(t)))$ smooth enough then the curve $\gamma$ is determined, up to translations, simply by integrating in time the prescriber function $f'(u((t,\gamma(t))))$. We see below some examples: $u$ growing linearly, quadratically, and a combination of the two.

*Example* A.1. Let us start with a trivial example. Consider a characteristic curve $\gamma$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,\gamma(t))=1\text{ for }t\in(a,b).$$

In this case, the characteristic curve $\gamma$ can be easily computed. Indeed, one has

$$u(t,\gamma(t))=u_a+t-a,\text{ where }u_a:=u(a,\gamma(a)),$$

and therefore for $t\in[a,b]$

$$\begin{aligned}\gamma(t)-\gamma(a)&=\int_a^t\dot\gamma(s)ds=\int_a^t f'(u(s,\gamma(s)))ds=\int_a^t f'(u_a+s-a)ds\\&=f(u_a+t-a)-f(u_a).\end{aligned}$$

If more generally

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,\gamma(t))=v(t)>0\text{ for }t\in(a,b),$$

then

$$t \mapsto u(t,\gamma(t)) = u_a + \int_a^t v(s)ds, \qquad u_a = u(a,\gamma(a))$$

is invertible with inverse that we denote $U^{-1}$. Even if the expression of the characteristic curve $\gamma$ is not as explicit as before, one can compute the variation of $\gamma(t)$ from time $a$ to time $t$ by

$$\text{(A.1)} \qquad \gamma(t)-\gamma(a) = \int_a^t \dot\gamma(s)ds = \int_a^t f'(u(s,\gamma(s)))ds = \int_a^t f'\left(u_a + \int_a^s v(r)dr\right)ds \overset{z=u(s,\gamma(s))}{=} \int_{u_a}^{u_t} \frac{f'(z)}{v(U^{-1}(z))}dz,$$

where we termed

$$u_a := u(a,\gamma(a)), \qquad u_t := u(t,\gamma(t)) = u_a + \int_a^t v(s)ds.$$

*Example* A.2. Let us consider another easy example where the trace of $u$ on a characteristic curve $\gamma$ determines the characteristic curve $\gamma$ itself. If

$$u((t,\gamma(t))) = \frac{t^2}{2\tau} \text{ for } t\in(0,\tau) \quad \Rightarrow \quad v(t) := \frac{\mathrm{d}}{\mathrm{d}t}u((t,\gamma(t)) = \frac{t}{\tau},\ v(U^{-1}(z)) = \sqrt{2z/\tau}$$

and equation (A.1) gives us

$$\gamma(\tau)-\gamma(0) = \int_0^{\tau/2} \frac{f'(z)}{\sqrt{2z/\tau}}dz, \qquad \frac{\mathrm{d}}{\mathrm{d}t}u(0,\gamma(0)) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}u(\tau,\gamma(\tau)) = 1.$$

Similarly, if one fixes

$$u((t,\gamma(t))) = C - \frac{(t-b)^2}{2\tau} \qquad \text{for } t\in(b-\tau,b),$$

then

$$v(t) = (b-t)/\tau \qquad \text{and} \qquad v(U^{-1}(z)) = -\sqrt{2(C-z)/\tau}\ .$$

*Example* A.3. Suppose one requires that $0 \le \frac{\mathrm{d}}{\mathrm{d}t}u((t,\gamma(t))) \le 1$ for $t\in[0,b]$ and

$$\text{(A.2)} \qquad \gamma(b)-\gamma(0) = c, \qquad \frac{\mathrm{d}}{\mathrm{d}t}u(t,\gamma(t))\Big|_{t=0} = u(0,\gamma(0)) = 0 = u(b,\gamma(b)) = \frac{\mathrm{d}}{\mathrm{d}t}u(t,\gamma(t))\Big|_{t=b}.$$

We describe the curve in the interval $[0,b/2]$, and then we take the symmetric

$$\text{(A.3a)} \qquad \gamma(t) = \gamma(b/2) + \int_{b-t}^{b/2} \dot\gamma(s)ds \qquad t\in(b/2,b].$$

The (positive) values of $c$ one can hope to achieve are less than $2f(b/2)$ since

$$\gamma(b/2)-\gamma(0) = \int_0^{b/2}\dot\gamma(s)ds = \int_0^{b/2} f'(u(s,\gamma(s)))ds \overset{0\le u(s,\gamma(s))<s}{<} \int_0^{b/2} f'(s)ds = f(b/2).$$

We give below $C^2$-curves achieving each precise value $0<c<2f(b/2)$, distinguishing $c$ small or big.

*Case* $c \sim 2f(b/2)$. Combine Examples A.1 and A.2: the continuous function

$$\text{(A.3b)} \qquad \gamma(t;\tau) = \begin{cases} \gamma(0) + \int_0^{t^2/(2\tau)} \frac{f'(z)}{\sqrt{2z/\tau}}dz & 0\le t\le\tau \\ \gamma(\tau) + f(\tau/2+t-\tau) - f(\tau/2) & \tau<t\le b/2-\tau \\ \gamma(b/2-\tau) + \int_0^{b/2-\tau-(t-b/2)^2/(2\tau)} f'(z)\sqrt{\frac{\tau}{b-2\tau-2z}}dz & b/2-\tau<t\le b/2 \end{cases}$$

will satisfy the requirements in (A.2) for one fixed $\tau\in(0,b/4]$, provided that

$$f\left(\frac{b}{2}\right) > \frac{c}{2} \ge \gamma\left(\frac{b}{2};\frac{b}{4}\right) - \gamma(0) = 2\int_0^{b/2}\frac{f'(z)}{2\sqrt{2z/b}}dz.$$

This choice is equivalent to assigning the $C^1([0,b/2])$ function

$$u(i_{\gamma_\tau}(t)) = \begin{cases} \frac{t^2}{2\tau} & 0 \le t \le \tau\ , \\ \frac{\tau}{2} + t - \tau \equiv t - \frac{\tau}{2} & \tau < t \le \frac{b}{2} - \tau\ , \\ \frac{b}{2} - \tau - \frac{(t-\frac{b}{2})^2}{2\tau} & \frac{b}{2} - \tau < t \le \frac{b}{2}\ , \end{cases} \qquad \frac{\mathrm{d}}{\mathrm{d}t} u(i_{\gamma_\tau}(t)) = \begin{cases} \frac{t}{\tau} & 0 \le t \le \tau\ , \\ 1 & \tau < t \le \frac{b}{2} - \tau\ , \\ \frac{b}{2\tau} - \frac{t}{\tau} & \frac{b}{2} - \tau < t \le \frac{b}{2}\ . \end{cases}$$

*Case $c$ small.* If instead $c$ is small one may have

$$f\left(\frac{b}{2}\right) > 2\int_0^{b/2} \frac{f'(z)}{2\sqrt{2z/b}} dz > \frac{c}{2}\ .$$

In this case one can just consider, for the suitable $\tau \in [0, 2/b]$, the choice of the $C^1([0,b/2])$ function

$$u(i_{\gamma_\tau}(t)) = \begin{cases} \tau t^2 & 0 \le t \le \frac{b}{4}\ , \\ \frac{\tau b^2}{8} - \tau\left(t - \frac{b}{2}\right)^2 & \frac{b}{4} < t \le \frac{b}{2}\ , \end{cases} \qquad \frac{\mathrm{d}}{\mathrm{d}t} u(i_{\gamma_\tau}(t)) = \begin{cases} 2\tau t & 0 \le t \le \frac{b}{4}\ , \\ (b-2t)\tau & \frac{b}{4} < t \le \frac{b}{2}\ . \end{cases}$$

This defines the curve, for example in the first half interval $[0, b/4]$,

$$\gamma(t;\tau) = \gamma(0) + \int_0^{\tau t^2} \frac{f'(z)}{2\sqrt{\tau z}} dz \qquad 0 \le t \le b/4.$$

Since $\gamma(b/2;\tau) \downarrow \gamma(0)$ as $\tau \downarrow 0$ and $\gamma(b/2; 1/b) > c/2$, by the continuity there is indeed a suitable $\tau$.

If $g_E$ is continuous, then $\mathfrak{g}$ is automatically defined as a Borel function with a unique definition. If the source term is only bounded, then

- $g_E$ by itself is not enough in order to identify the Lagrangian source, for which one needs looking at the values of $u$ along characteristic curves (§ 3).
- In the case of $\alpha$-convex fluxes the broad point of view—intended in a strong sense where the pointwise value of the broad source is the classical derivative of $u$ along characteristics whenever it is possible—identifies the correct distributional source $g_E$ [7, Theorem 1.2].
- If inflections are negligible, there exists a Borel function $\mathfrak{g}$ which provides the correct source in all the formulations [2, § 3.2]. It is not determined however by just one of the points of view, as it was instead happening in the uniformly convex case.
- In the general case, there is no universal source $\mathfrak{g}$, and in particular there is no Borel function providing the source for all the Lagrangian parameterizations (Example 5.2).

## NOMENCLATURE

| | |
|---|---|
| $(t, \gamma(t))$ | Characteristic curve of (1.1): $\gamma \in C^1(\mathbb{R};\mathbb{R}^2)$ solves $\dot\gamma(t) = f'(u(t,\gamma(t)))$ |
| $\chi$ | Lagrangian parameterization for a continuous solution $u$ to (1.1), see Definition 1.1 |
| $\mathrm{clos}(X)$ | Closure of $X$ |
| $\mathcal{D}(\Omega)$ | Distributions on $\Omega$ |
| $\frac{\mathrm{d}}{\mathrm{d}t}$ | Classical derivative in the real variable $t$ |
| $L^\infty(X)$ | Bounded functions on $X$ identified $\mathcal{L}^2$-a.e., with the norm of the essential supremum |
| $\mathrm{Infl}(f)$ | Inflection points of $f$, see Assumption (H) |
| $\lambda(t,x)$ | The composite function $f'(u(t,x))$ |
| $\mathcal{L}^1$, $\mathcal{L}^2$ | 1- or 2-dimensional Lebesgue measure |
| $\Omega$ | Open subset of either $\mathbb{R}^2$ or $\mathbb{R}$, if needed connected |
| $\partial_t, \partial_x$ | Partial derivatives in the sense of distributions |
| $\mathfrak{L}^\infty_B(X)$ | Borel bounded functions on $X$ identified if, when restricted on any characteristic curve of $u$, they coincide $\mathcal{L}^1$-a.e., see Broad sources at Page ii. |

| | |
|---|---|
| $\mathfrak{L}^{\infty}_{L\chi}(X)$ | Borel bounded functions on $X$ identified if their composition with the Lagrangian parameterization $\chi$ coincides $\mathcal{L}^2$-a.e., see Lagrangian sources at Page iii. |
| $i_\chi$ | $i_\chi(t,y) \equiv i_{\chi(y)}(t) \equiv (t, \chi(t,y))$, see Definition 1.1 |
| $u$, $f$ | $u$ is a fixed continuous solution for the balance law (1.1) and $f$ is the $C^2$-flux |
| $X$ | Subset either of $\mathbb{R}^2$ or of $\mathbb{R}$, usually Borel. |
| (H) | Assumption of negligibility of (the closure of) inflection points of $f$ at Page 2 |

We also collect here interesting properties of the solution, depending on the assumptions:

| | $\alpha$-convexity | Negligible inflections | General case |
|---|---|---|---|
| absolutely continuous Lagrangian parameterization | ✗ (§ 3) | ✗ | ✗ |
| $u$ Hölder continuous | ✓ ([7, Th. 1.2]) | ✗ (§ 4) | ✗ |
| $u$ $\mathcal{L}^2$-a.e. differentiable along characteristic curves | ✓([7, Th. 1.2]) | ✗ (§ 4) | ✗ |
| $u$ Lipschitz continuous along characteristic curves | ✓ | ✓ [2, Th. 30] | ✗ (§ 5) |
| entropy equality | ✓ | ✓ | ✓([2, Lemma 42]) |
| compatibility of sources | ✓ ✓ ([7, Th.3.1]) | ✓ (§ 2) | |

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