GROMOV-WITTEN THEORY OF SCHEMES IN MIXED CHARACTERISTIC

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Abstract. We define Gromov-Witten classes and invariants of smooth projective schemes of finite presentation over a Dedekind domain. We prove that they are deformation invariants and verify the fundamental axioms. For a smooth projective scheme over a Dedekind domain, we prove that the invariants of fibers in different characteristics are the same. We show that genus zero Gromov-Witten invariants define a potential which satisfies the WDVV equation and we deduce from this a reconstruction theorem for genus zero Gromov-Witten invariants in arbitrary characteristic.

Contents

1. Introduction

Gromov-Witten theory was originally introduced for compact symplectic manifolds by Chen and Ruan ([7]) and was later developed in the algebraic language for smooth projective varieties over a field of characteristic zero ([17], [6], [13]). The construction in the algebraic setting is based on Kontsevich's moduli stack, denoted by $\overline{\mathcal{M}}_{g,n}(X,\beta)$, of n-pointed stable maps of genus g and class $\beta \in A_1(X)$ into a smooth projective variety X. This stack is defined for any projective scheme X of finite presentation over a noetherian base scheme S and is a proper algebraic stack over S with finite stabilizers ([1]). When the base is a field k of characteristic zero, the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is Deligne-Mumford and admits a perfect obstruction theory ([5]). This leads to a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}} \in A_*(\overline{\mathcal{M}}_{g,n}(X,\beta))$ and the Gromov-Witten invariants of X are obtained by integrating cohomology classes on X against $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}$.

In this paper we define Gromov-Witten classes and invariants associated to smooth projective schemes of finite presentation over a Dedekind domain. The main motivation for us is to compare the invariants in different characteristics for schemes defined in mixed characteristic. We hope that this approach could give a useful insight into the Gromov-Witten theory in characteristic zero, providing a new technique for computing Gromov-Witten invariants.

The main problem in developing Gromov-Witten theory in positive or mixed characteristic is that in general the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is not Deligne-Mumford. When the base is a field of characteristic

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$p > 0$, then $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is still Deligne-Mumford for certain values of the fixed discrete parameters g, n, β which are big with respect to p. However, this is not satisfactory from the point of view of Gromov-Witten theory, because most of the properties of Gromov-Witten invariants (e.g. WDVV equation, Getzler relations) involve all the invariants at the same time.

The strategy we follow to construct Gromov-Witten invariants for varieties in mixed characteristic is to extend the construction in [5] to the case of a morphism of Deligne-Mumford type of Artin stacks over a scheme S. In particular, we describe a way of constructing virtual fundamental classes of Artin stacks which admits a Deligne-Mumford type morphism into a smooth Artin stack and a relative perfect obstruction theory. We apply this to the natural forgetful functor $\theta\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$, which is representable and quasi-projective, after we exhibited a perfect relative obstruction theory for θ, and we construct a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}} \in A_*(\overline{\mathcal{M}}_{g,n}(X,\beta))$.

1.1. **Outline of the paper.** In section 2 we recall the definition of Kontsevich's moduli stack of stable maps for schemes X of finite presentation over a Dedekind domain. We define the stack $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S},\beta_\eta)$ which parametrizes stable maps to X, but we take β_η to be a cycle class over the generic fiber X_η of X rather than over X itself. This stack turns out to be more convenient when we want to compare the Gromov-Witten invariants in mixed characteristic. We prove that $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S},\beta_\eta)$ is a proper Artin stack with finite stabilizers and that it has the resolution property. Section 3 is devoted to illustrating the definition of the relative intrisic normal cone and extending to Deligne-Mumford type morphisms of Artin stacks the techniques used in [5] for constructing virtual fundamental classes of Deligne-Mumford stacks. In this way, we are able to construct virtual fundamental classes of Artin stacks which admits a Deligne-Mumford type morphism into a smooth Artin stack and a relative perfect obstruction theory. Moreover, using the deformation theory of stacks, we prove a criterion for verifying whether a complex is an obstruction theory. In section 4 we construct explicitly a perfect obstruction theory relative to the natural forgetful functor $\theta\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}$. This leads to a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}} \in A_*(\overline{\mathcal{M}}_{g,n}(X,\beta))$. In section 5 we define Gromov-Witten classes and invariants associated to smooth projective schemes of finite presentation over a Dedekind domain. We prove that they are deformation invariants and verify the fundamental axioms. For a smooth projective scheme over a Dedekind domain, we prove that the invariants of fibers in different characteristics are the same. Section 6 contains some results for genus zero Gromov-Witten invariants. We show that the Gromov-Witten potential satisfies the WDVV equation and we deduce from this a reconstruction theorem for genus zero Gromov-Witten invariants in arbitrary characteristic.

In appendix A we recall the formal criterion for smoothness of schemes and prove it for stacks; moreover, we study the deformation theory of Artin stacks and of Deligne-Mumford type morphisms of Artin stacks (over a base scheme), which is a key point in the construction of a perfect relative obstruction theory. These results are well-known to the experts and we include a proof for completeness. Appendix B is devoted to intersection theory on Artin stacks over a Dedekind domain. In particular we observe that Kresch's intersection theory for stacks over a field ([18]) extends naturally to stacks over a Dedekind domain. As a consequence we are able to generalize Manolache's construction of the virtual pullback ([21]) for Deligne-Mumford type morphisms of Artin stacks over a Dedekind domain. An essential ingredient for defining Gromov-Witten invariants in positive and mixed characteristic is the non-representable proper pushforward for morphisms of Artin stacks. We describe the construction of this pushforward as suggested in [10]. In addition, we prove Costello's pushforward formula for proper morphisms of Artin stacks with quasi-finite diagonal.

1.2. **Future work.** A natural generalization would be to develop a Gromov-Witten theory for tame Deligne-Mumford stacks in positive or mixed characteristic, using the moduli stack of twisted stable maps constructed in [2].

In another direction, it would be interesting to prove a degeneration formula in the mixed characteristic setting. This would give a useful tool to compute Gromov-Witten invariants of varieties in characteristic zero out of *simpler* invariants of varieties in positive characteristic. We imagine this is far from easy, but we hope to return to these points in a future paper.

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1.4. **Notations.** We write $(^{\mathrm{Sch}}/_S)$ for the category of schemes over a base scheme S. For a scheme $X \in (^{\mathrm{Sch}}/_S)$, we denote by $A_*(^X/_S)$ the group of numerical equivalence classes of cycles. All stacks are Artin stacks in the sense of [3], [19] and are of finite type over a base scheme. Unless otherwise specified, the words "stack of stable maps" refer to $\overline{\mathcal{M}}_{g,n}(^X/_S, \beta_\eta)$ in Definition 2.6.

2. Stable pointed maps

2.1. **Stacks of stable maps** ([1] 2). Let D be a Dedekind domain and set $S = \operatorname{Spec} D$. Let X be a projective S-scheme of finite presentation and let $\mathscr{O}(1)$ be a very ample sheaf on X. We fix $\beta \in A_1(^X/_S)$ and integers $g \geq 0$, $n \geq 0$, $d \geq 1$.

2.1. **Definition.** Let T be a scheme over S. Let $\xi = (C \xrightarrow{\pi} T, t_i, f)$, where

(1) the morphism π is a projective flat family of curves,
(2) the geometric fibers of π are reduced with at most nodes as singularities,
(3) the sheaf $\pi_*\omega_{C/T}$ is locally free of rank g (where $\omega_{C/T}$ is the relative dualizing sheaf),
(4) the morphisms $t_1, \dots, t_n$ are sections of π which are disjoint and land in the smooth locus of π,
(5) $f\colon C \to X$ is a morphism of S-schemes,
(6) the group scheme $\operatorname{Aut}(C, f, \pi, t_i)$ of automorphisms of C, which commute with f, π and t_i, is finite over T.

We say that ξ is a *stable n-pointed map of genus g and*

(a) *degree d* if the degree of $f^*\mathscr{O}(1)$ on geometric fibers of π is d;
(b) *class β* if, for every geometric point $\overline{t} \to T$, we consider the following induced morphisms

$$C_{\overline{t}} = C \times_T \overline{t} \xrightarrow{f_{\overline{t}}} X_{\overline{t}} = X \times_S \overline{t} \xrightarrow{\tau} X_{\overline{s}} = X \times_S \overline{s} \to X_s = X \times_S s \xrightarrow{i} X,$$

where $s = \operatorname{Spec} k \in S$ is the image of $\overline{t}$ and $\overline{s} = \operatorname{Spec} \overline{k}$, with $\overline{k}$ a separable closure of k, then we have $f_{\overline{t}*}[C_{\overline{t}}] = \tau^*(\overline{i^*\beta})$, where $i^*\beta \in A_1(^{X_s}/_s)$ induces $\overline{i^*\beta} \in A_1(^{X_{\overline{s}}}/_{\overline{s}})$.

2.2. **Definition.** Let T and T' be schemes over S. Given two stable maps $\xi = (C \xrightarrow{\pi} T, t_i, f)$ and $\xi' = (C' \xrightarrow{\pi'} T', t'_i, f')$, a *morphism* of stable maps $\alpha\colon \xi \to \xi'$ is a pair of morphisms of S-schemes $(C \xrightarrow{\alpha_C} C', T \xrightarrow{\alpha_T} T')$, inducing an isomorphism $C \cong C' \times_{T'} T$ and such that $\pi' \circ \alpha_C = \alpha_T \circ \pi$, $t'_i \circ \alpha_T = \alpha_C \circ t_i$, $f' \circ \alpha_C = f$.

2.3. **Definition.** We denote by $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, d)$ the category fibered in groupoids over $({}^{\mathrm{Sch}}/{}_{S})$ of stable n-pointed maps of genus g and degree d into X. We denote by $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta)$ the category of stable maps of class β.

2.4. **Theorem** ([1] 2.5)**.** *The category $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, d)$ is a proper Artin stack over S with finite stabilizers, admitting a projective coarse moduli scheme $\overline{M}_{g,n}({}^{X}/{}_{S}, d) \to S$. The category $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta)$ is an open and closed substack of $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, d)$.*

2.2. **An other stack of stable maps.** Let η be the generic point of S and set $X_\eta = X \times_S \eta$. Fix $\beta_\eta \in A_1({}^{X_\eta}/{}_{\eta})$. For any closed point $s \in S$, we denote by X_s the fiber over s. Let $\mathfrak{m}_s \subset D$ be the maximal ideal corresponding to s and consider the localization $R = D_{\mathfrak{m}_s}$ of D at $\mathfrak{m}_s$. Let $\widetilde{X}_s = X \times_S \operatorname{Spec} R$ and let $X_s \xrightarrow{i} X$ and $X_\eta \xrightarrow{j} X$ be the natural inclusions. Notice that R is a dicrete valuation ring and, by [12] 20.3, there exists a specialization homomorphism

$$\sigma_s \colon A_*({}^{X_\eta}/{}_{\eta}) \to A_*({}^{X_s}/{}_{s}),$$

sending a cycle α to $i^!\widetilde{\alpha}$, for some $\widetilde{\alpha} \in A_*({}^{\widetilde{X}_s}/{}_{R})$ such that $j^*\widetilde{\alpha} = \alpha$. By [12] 20.3.5, there exists an induced specializzation homomorphism

$$\overline{\sigma_s} \colon A_*({}^{X_{\overline{\eta}}}/{}_{\overline{\eta}}) \to A_*({}^{X_{\overline{s}}}/{}_{\overline{s}}),$$

where $\overline{\eta}$ and $\overline{s}$ are geometric points over η and s. We denote by $\overline{\beta_\eta} \in A_1({}^{X_{\overline{\eta}}}/{}_{\overline{\eta}})$ the cycle class induced by β_η and we notice that $\overline{\sigma_s}\left(\overline{\beta_\eta}\right) = \overline{\sigma_s(\beta_\eta)}$.

2.5. **Definition.** A *stable n-pointed map of genus g and class β_η into X* is $\xi = (C \xrightarrow{\pi} T, t_i, f)$ as in Definition 2.1, which satisfies conditions (1)–(5) and the following

- (c) with notations as in condition (b) in Definition 2.1, for every geometric fiber $C_{\overline{t}}$ of π, we have $f_{\overline{t}*}[C_{\overline{t}}] = \tau^*\overline{\sigma_s}\left(\overline{\beta_\eta}\right)$.

2.6. **Definition.** We denote by $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta)$ the category fibered in groupoids over $({}^{\mathrm{Sch}}/{}_{S})$ of stable n-pointed maps of genus g and class β_η into X.

2.7. **Corollary.** *The category $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta)$ is a proper Artin stack over S with finite stabilizers, admitting a projective coarse moduli scheme $\overline{M}_{g,n}({}^{X}/{}_{S}, \beta_\eta) \to S$.*

Proof. Let $d = \deg \beta_\eta$. It is enough to show that $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta)$ is an open and closed substack of $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, d)$. Notice that $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta) = \bigsqcup \overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta)$, where the union is over $\beta \in A_1({}^{X}/{}_{S})$ such that $j^*\beta = \beta_\eta$. By Theorem 2.4, $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta)$ is an open substack of $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, d)$, because it is a union of open substacks. On the other hand $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, d) \setminus \overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta) = \bigsqcup \overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta)$ is open, where the union is over $\beta \in A_1({}^{X}/{}_{S})$ such that $\deg \beta = d$, $j^*\beta \neq \beta_\eta$. It follows that $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta)$ is a closed substack of $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, d)$. □

2.8. As noted in [13] and [6], there is a representable and quasi-projective morphism

$$\theta \colon \overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta) \to \mathfrak{M}_{g,\, n/S},$$

which forgets the morphism into X. Recall, moreover, that the stack $\mathfrak{M}_{g,\, n/S}$ is smooth of dimension $3g - 3 + n$ over S.

2.9. For every S-scheme T, a morphism $T \to \overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta)$ corresponds to a stable map $(C_T \xrightarrow{\pi_T} T, t_i, f_T)$ over T, then, by descent theory, the identity of $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta)$ corresponds to a universal stable map $(\mathscr{C} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta), \sigma_i, \psi)$.

2.10. Remark. Let $\mathscr{L} = \omega_{C/T}(\sum_{i=1}^n T_i) \otimes f^*\mathscr{O}(3)$, T_i is the image of t_i. By [1] 2.2, the sheaf $\mathscr{L}^{\otimes \nu}$ is relative very ample and has no higher cohomology along geometric fibers, for $\nu \geq 3$ fixed. Moreover $\dim H^0(C, \mathscr{L}^{\otimes \nu}) = M + 1$ and $\deg \mathscr{L}^{\otimes \nu}$ are constant along geometric fibers, and $\overline{\mathcal{M}}_{g,n}({}^{X}/{}_{S}, \beta_\eta) =$

$[{}^{V}/_{\mathrm{PGL}(M+1)}]$, where V is a quasi-projective scheme parametrizing stable maps $(C \to T, t_i, f)$ together with a choice of a basis $\underline{s} = (s_0, \ldots, s_M)$ of $H^0(C, \mathscr{L}^{\otimes \nu})$ up to scalar multiplication (the action of $\mathrm{PGL}(M+1)$ on V is the natural action on the bases of $H^0(C, \mathscr{L}^{\otimes \nu})$ up to scalar multiplication).

2.11. **Theorem.** *The resolution property holds for* $\overline{\mathcal{M}}_{g,n}({}^{X}/_{S}, \beta_\eta)$.

Proof. Set $G = \mathrm{PGL}(M+1)$. By Remark 2.10 and [25], it is enough to show that every G-equivariant coherent sheaf on V is a quotient of a G-equivariant vector bundle on V.

We need to construct a G-equivariant ample line bundle L on V. Recall that V is a locally closed subscheme of $\mathrm{Hom}_H(\mathcal{U}, X)$, where $H = \mathrm{Hilb}^P(\mathbb{P}^M_S)$ is the Hilbert scheme of closed subschemes of $\mathbb{P}^M_S$ with Hilbert polynomial P and $\mathcal{U}$ is the universal family of H ([1] 2.2). Notice that $\mathrm{Hom}_H(\mathcal{U}, X) \subset \mathrm{Hilb}(\mathcal{U} \times_S X)$ and recall that $\mathrm{Hilb}(\mathcal{U} \times_S X)$ is locally closed in the Grassmannian $\mathrm{Gr}(r, \Gamma(\mathcal{U} \times_S X, E))$, where $E = (\mathscr{O}_{\mathcal{U}}(1) \otimes \mathscr{O}_X(1))^{\otimes m}$, for sufficiently large m. Thus we have a sequence of inclusions

$$V \hookrightarrow \mathrm{Hom}_H(\mathcal{U}, X) \hookrightarrow \mathrm{Hilb}(\mathcal{U} \times_S X) \hookrightarrow \mathrm{Gr}(r, \Gamma(\mathcal{U} \times_S X, E)) \xrightarrow{i} \mathbb{P}\left(\bigwedge \Gamma(\mathcal{U} \times_S X, E)\right) = \mathbb{P}^a_S,$$

where i is the Plücker embedding. Moreover G acts naturally on each of these schemes (the action is induced by the action on $\mathcal{U}$ and the trivial action on X) and observe that all the above inclusions are G-equivariant. It follows that $i^*\mathscr{O}_{\mathbb{P}^a_S}(1)$ is a G-equivariant ample line bundle on the Grassmannian and thus the pullback of $i^*\mathscr{O}_{\mathbb{P}^a_S}(1)$ to V is a G-equivariant ample line bundle L on V.

Let F be a G-equivariant coherent sheaf on V. Then there exists $m \in \mathbb{Z}$ such that $F \otimes L^{\otimes m}$ is generated by G-equivariant global sections. This gives a G-equivariant surjection

$$H^0(V, F \otimes L^{\otimes m}) \otimes \mathscr{O}_V \to F \otimes L^{\otimes m},$$

which induces a G-equivariant surjection $H^0(V, F \otimes L^{\otimes m}) \otimes L^{\otimes -m} \to F$, and the statement follows. □

2.12. **Corollary.** *There exists a quasi-affine scheme W with an action of the group* $\mathrm{GL}(m)$, *for some m, such that* $\overline{\mathcal{M}}_{g,n}({}^{X}/_{S}, \beta_\eta) \cong [{}^{W}/_{\mathrm{GL}(m)}]$. *In particular* $\overline{\mathcal{M}}_{g,n}({}^{X}/_{S}, \beta_\eta)$ *is a global quotient stack in the sense of* [18] *3.5.4.*

Proof. Follows from Theorem 2.11 and [26] 1.1. □

3. Relative intrinsic normal cone

3.1. In [5], the authors construct a cone stack, associated to a given Deligne-Mumford stack, called the intrinsic normal cone and give a relative version of it for morphisms of Deligne-Mumford type of Artin stacks over a field. They also introduce the notion of perfect obstruction theory and use this to construct virtual fundamental classes of Deligne-Mumford stacks. In this section we extend the construction in [5] to the case of a morphism of Deligne-Mumford type of Artin stacks over a scheme S. In particular, we describe a way of constructing virtual fundamental classes for Artin stacks which admits a Deligne-Mumford type morphism into a smooth Artin stack, with the additional condition that $\mathcal{M}$ satisfies the resolution property. The latter assumption is technical and not indeed required; we assume it — since we are interested in constructing a virtual fundamental class for the stack of stable maps, which has the resolution property (Theorem 2.11) — because it simplifies the proofs. Moreover, we give a criterion to verify whether a complex is an obstruction theory.

3.1. **Cones and cone stacks** ([5], [27]). Let S be a scheme and let $\mathcal{M}$ be an Artin S-stack. We consider the lisse-étale topos $\mathcal{M}_{\text{lis-ét}}$ of $\mathcal{M}$. Let $\mathscr{S}^\bullet$ be a quasi-coherent sheaf of graded $\mathscr{O}_{\mathcal{M}}$-algebras in the topos $\mathcal{M}_{\text{lis-ét}}$ such that

(1) the canonical morphism $\mathscr{O}_{\mathcal{M}} \to \mathscr{S}^0$ is an isomorphism,

(2) $\mathscr{S}^1$ is coherent,
(3) $\mathscr{S}^\bullet$ is locally generated by $\mathscr{S}^1$.

3.2. **Definition.** The *cone* associated to $\mathscr{S}^\bullet$ is the S-stack $C(\mathscr{S}^\bullet)$ associated to the groupoid $\operatorname{Spec}\mathscr{S}^\bullet_R \rightrightarrows \operatorname{Spec}\mathscr{S}^\bullet_U$, where $R \rightrightarrows U$ is a presentation of $\mathcal{M}$ and $\mathscr{S}^\bullet_U$ (respectively $\mathscr{S}^\bullet_R$) is the restriction of $\mathscr{S}^\bullet$ to U (respectively R). A morphism of cones over $\mathcal{M}$ is induced by a graded morphism of sheaves of graded $\mathscr{O}_\mathcal{M}$-algebras.

3.3. Remark. The natural morphism $\mathscr{S}^\bullet \to \mathscr{S}^0$ induces a morphism of S-stacks $0\colon \mathcal{M} \to C(\mathscr{S}^\bullet)$ called the *vertex* of $C(\mathscr{S}^\bullet)$. Moreover the morphism $\mathscr{S}^\bullet \to \mathscr{S}^\bullet[x]$ induces an action $\gamma\colon \mathbb{A}^1_S \times_S C(\mathscr{S}^\bullet) \to C(\mathscr{S}^\bullet)$.

3.4. **Definition.** If $\mathscr{F}$ is a coherent sheaf of $\mathscr{O}_\mathcal{M}$-modules over $\mathcal{M}$, the cone $C(\mathscr{F})$ associated to $\operatorname{Sym}(\mathscr{F})$ is called an *abelian cone*. An abelian cone $C(\mathscr{F})$ is a *vector bundle* over $\mathcal{M}$ if $\mathscr{F}$ is a locally free coherent sheaf over $\mathcal{M}$.

3.5. Remark. The natural morphism $\operatorname{Sym}(\mathscr{S}^1) \to \mathscr{S}^\bullet$ is surjective, because $\mathscr{S}^\bullet$ is locally generated by $\mathscr{S}^1$, hence the induced morphism of cones $C(\mathscr{S}^\bullet) \to C(\mathscr{S}^1)$ is a closed immersion. The abelian cone $C(\mathscr{S}^1)$ is called the *abelianization* of $C = C(\mathscr{S}^\bullet)$ and it is denoted by $A(C)$. Moreover a morphism of cones $C \to C'$ induces a morphism $A(C) \to A(C')$. In particular the abelianization defines a functor A from the category of cones over $\mathcal{M}$ to the category of abelian cones over $\mathcal{M}$.

3.6. **Definition.** A sequence of morphisms of cones

$$0 \to E \xrightarrow{i} C \to C' \to 0$$

is *exact* if E is a vector bundle and locally over $\mathcal{M}$ there is a morphism of cones $C \to E$ splitting i and inducing an isomorphism $C \cong E \times C'$.

3.7. Remark. A sequence of coherent sheaves on $\mathcal{M}$

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{E} \to 0,$$

with $\mathscr{E}$ locally free, is exact if and only if

$$0 \to C(\mathscr{E}) \to C(\mathscr{F}) \to C(\mathscr{F}') \to 0$$

is exact ([12] Example 4.1.7).

3.8. For the definitions of $\mathbb{A}^1$-action and $\mathbb{A}^1$-equivariant morphism and 2-isomorphism we refer to [5].

3.9. **Definition.** A *cone stack* over $\mathcal{M}$ is an algebraic $\mathcal{M}$-stack $\mathfrak{C}$ together with a section and an $\mathbb{A}^1_S$-action such that, smooth locally on $\mathcal{M}$, there exist a cone C, a vector bundle E over $\mathcal{M}$ and a morphism of abelian cones $E \to A(C)$ such that C is invariant under the induced action of E on $A(C)$, and there exists an $\mathbb{A}^1_S$-equivariant morphism $[C/E] \to \mathfrak{C}$ which is an isomorphism. A morphism of cone stacks is an $\mathbb{A}^1_S$-equivariant morphism of $\mathcal{M}$-stacks. A 2-isomorphism of cone stacks is an $\mathbb{A}^1_S$-equivariant 2-isomorphism. An *abelian cone stack* over $\mathcal{M}$ is a cone stack $\mathfrak{C}$ such that smooth locally $\mathfrak{C} \cong [C/E]$, where C is an abelian cone. A *vector bundle stack* over $\mathcal{M}$ is a cone stack $\mathfrak{C}$ such that smooth locally $\mathfrak{C} \cong [C/E]$, where C is a vector bundle.

3.10. Abelian cone stacks over $\mathcal{M}$ form a 2-category denoted by $(ACS/\mathcal{M})$. We consider the associated homotopy category $\operatorname{Ho}(ACS/\mathcal{M})$.

3.2. **Abelian cone stacks and complexes of sheaves.** Let $C^{[-1,0]}(\operatorname{Coh}(\mathcal{M}_{\text{lis-ét}}))$ be the category of complexes $(E^\bullet, d_E)$ of coherent sheaves in the topos $\mathcal{M}_{\text{lis-ét}}$ such that $h^i(E^\bullet, d_E) = 0$, for $i \neq 0, -1$; consider the subcategory $\hat{C}^{[-1,0]}(\operatorname{Coh}(\mathcal{M}_{\text{lis-ét}}))$ of complexes $(E^\bullet, d_E)$ with $\ker d^0_E$ locally free.

3.11. **Definition.** Let $\psi, \varphi\colon (E^\bullet, d_E) \to (F^\bullet, d_F)$ be morphisms in $\hat{C}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$. A *homotopy* $\varkappa\colon \psi \to \varphi$ is a morphism $\varkappa\colon E^\bullet \to F^\bullet[1]$ in $\hat{C}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ such that

$$\begin{cases} \varkappa^{i+1} d_E^i = \varphi^i - \psi^i \\ d_F^i \varkappa^{i+1} = \varphi^{i+1} - \psi^{i+1}. \end{cases}$$

3.12. We can view $\hat{C}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ as a 2-category, where the 2-morphisms are homotopies. We define a morphism of 2-categories

$$\hat{h}\colon \hat{C}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))^{\text{opp}} \to ({}^{ACS}/_{\mathcal{M}})$$

such that $\hat{h}(E^\bullet) = [{}^{C(E^{-1})}/_{C(E^0)}]$ if $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$, and $\hat{h}(E^\bullet) = \hat{h}(\tau_{[-1,0]}E^\bullet)$ in general. In the following we can assume, for every complex $E^\bullet$, that $E^i = 0$ for $i \neq -1, 0$. If $\psi\colon E^\bullet \to F^\bullet$ is a morphism of complexes, then it induces a commutative diagram of abelian cones

$$\begin{array}{ccc} C(F^0) & \longrightarrow & C(F^{-1}) \\ \downarrow & & \downarrow \\ C(E^0) & \longrightarrow & C(E^{-1}) \end{array}$$

which gives a morphism of cones $\hat{h}(\psi)\colon \hat{h}(F^\bullet) \to \hat{h}(E^\bullet)$. Finally, $\varkappa\colon E^0 \to F^{-1}$ is a homotopy of morphisms ψ, φ of complexes from $E^\bullet$ to $F^\bullet$, then $\varkappa \circ d_E = \varphi^{-1} - \psi^{-1}$ and $d_F \circ \varkappa = \varphi^0 - \psi^0$. The 2-morphism $\hat{h}(\varkappa)\colon \hat{h}(\psi) \to \hat{h}(\varphi)$ is defined in the following way. For every $\mathcal{M}$-scheme U and every $(P, f) \in \hat{h}(F^\bullet)(U)$, let $\{U_i\}$ be an open cover of U such that $U_i \times_U P \cong U_i \times_{\mathcal{M}} C(F^0)$, then

$$\hat{h}(\varkappa)(U)(P,f)\colon \hat{h}(\psi)(U)(P,f) \to \hat{h}(\varphi)(U)(P,f)$$

is obtained by gluing the isomorphisms

$$U_i \times_{\mathcal{M}} C(E^0) \xrightarrow{(\mathrm{id}_{U_i}, C(\varkappa)\circ f_i|_{U_i \times_{\mathcal{M}} \{0_F\}} \circ p_1 + p_2)} U_i \times_{\mathcal{M}} C(E^0),$$

where $C(\varkappa)$ is the morphism of cones induced by $\varkappa$. In particular $\hat{h}(\varkappa)$ is a 2-isomorphism.

3.13. **Proposition** ([5] 1.6, 2.1)**.** *Let $\hat{D}^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}ét}))$ be the derived category of complexes $(E^\bullet, d_E)$ of coherent sheaves in the topos $\mathcal{M}_{lis\text{-}ét}$ such that* $\ker d_E^0$ *is locally free and $h^i(E^\bullet, d_E) = 0$ for $i \neq -1, 0$. Let $Ho({}^{ACS}/_{\mathcal{M}})$ be the homotopy category associated to $({}^{ACS}/_{\mathcal{M}})$. The functor $\hat{h}$ induces a functor of categories*

$$\hat{D}^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}ét}))^{opp} \to Ho({}^{ACS}/_{\mathcal{M}}).$$

3.14. **Lemma.** *Let $D^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}ét}))$ be the derived category of complexes of coherent sheaves in the topos $\mathcal{M}_{lis\text{-}ét}$ with cohomology sheaves concentrated in degree -1 and 0. If $\mathcal{M}$ has the resolution property then the natural functor*

$$\hat{D}^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}ét})) \to D^{[-1,0]}(Coh(\mathcal{M}_{lis\text{-}ét}))$$

is an equivalence of categories.

Proof. Notice that the functor is fully faithfull. We want to show that every complex $E^\bullet$ in $D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ is in the essential image. We can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$ because $\tau_{\leq 0}E^\bullet$ is quasi-isomorphic to $E^\bullet$ and $\tau_{[-1,0]}E^\bullet$. Since $\mathcal{M}$ has the resolution property, there exists a locally free sheaf F^0 and a surjective morphism $\varphi^0\colon F^0 \to E^0$. We form the cartesian diagram

$$\begin{array}{ccc} F^{-1} & \xrightarrow{d_F} & F^0 \\ {\scriptstyle\varphi^{-1}}\downarrow & & \downarrow{\scriptstyle\varphi^0} \\ E^{-1} & \xrightarrow{d_E} & E^0 \end{array}$$

then $F^\bullet = [F^{-1} \xrightarrow{d_F} F^0] \in \hat{D}^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$. We claim that $\varphi\colon F^\bullet \to E^\bullet$ is a quasi-isomorphism. Since φ^0 is surjective, we have immediately that $h^0(\varphi)$ is surjective and the following sequence

$$0 \to F^{-1} \xrightarrow{(d_F,\varphi^{-1})} F^0 \oplus E^{-1} \xrightarrow{\varphi^0 - d_E} E^0 \to 0$$

is exact. Using this we get that $F^\bullet$ is quasi-isomorphic to $F^0 \oplus E^\bullet$, which is quasi isomorphic to $E^\bullet$. □

3.15. **Lemma.** *Let $D^{[-1,0]}_{coh}(\mathcal{M}_{lis\text{-}ét})$ be the derived category of complexes of sheaves of $\mathscr{O}_{\mathcal{M}}$-modules in the topos $\mathcal{M}_{lis\text{-}ét}$ with coherent cohomology sheaves concentrated in degree -1 and 0. If $\mathcal{M}$ has the resolution property then the natural functor*

$$D^{[-1,0]}(\mathit{Coh}(\mathcal{M}_{lis\text{-}ét})) \to D^{[-1,0]}_{coh}(\mathcal{M}_{lis\text{-}ét})$$

is an equivalence of categories.

Proof. First we show that the functor is fully faithfull. Let $E^\bullet, F^\bullet \in D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$, we want to show that the canonical map

$$\mathrm{Hom}_{D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))}(E^\bullet, F^\bullet) \to \mathrm{Hom}_{D^{[-1,0]}_{\text{coh}}(\mathcal{M}_{\text{lis-ét}})}(E^\bullet, F^\bullet)$$

is a bijection. We can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$ and $F^\bullet = [F^{-1} \xrightarrow{d_F} F^0]$. Recall that $\mathrm{Hom}(\bullet, F^\bullet)$ is a cohomological functor. Using the following distinguished triangle

$$E^{-1} \xrightarrow{d_E} E^0 \to E^\bullet \xrightarrow{+1} E^{-1}[1],$$

we can reduce to the case where $E^\bullet$ is a coherent sheaf E, similarly $F^\bullet = F$. By resolution property, there exists a locally free sheaf P^0 and a surjective morphism $\psi\colon P^0 \to E$. Set $P^{-1} = \ker\psi$, then $P^\bullet = [P^{-1} \to P^0]$ is a complex of locally free sheaves quasi-isomorphic to E, hence $E = P^\bullet$ in $D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$. Using the distinguished triangle

$$P^{-1} \to P^0 \to P^\bullet \xrightarrow{+1} P^{-1}[1],$$

we can reduce to the case where $E^\bullet$ is a locally free sheaf E. Let $E' = {}^{E}/_{\mathscr{O}_{\mathcal{M}}}$, then $\mathrm{rk}\, E' < \mathrm{rk}\, E$, hence we can reeduce to $E = \mathscr{O}_{\mathcal{M}}$. That is, we have reduce to showing that

$$\mathrm{Hom}_{D^{[-1,0]}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))}(\mathscr{O}_{\mathcal{M}}, F[n]) \to \mathrm{Hom}_{D^{[-1,0]}_{\text{coh}}(\mathcal{M}_{\text{lis-ét}})}(\mathscr{O}_{\mathcal{M}}, F[n])$$

is a bijection for every coherent sheaf F and $n = -1, 0$. If $n = -1$, both groups are zero. If $n = 0$ then both sides are $\Gamma(\mathcal{M}, F)$.

It remains to show that every complex $E^\bullet \in D^{[-1,0]}_{\text{coh}}(\mathcal{M}_{\text{lis-ét}})$ is in the essential image. We can assume $E^\bullet = [E^{-1} \xrightarrow{d_E} E^0]$. We have the following exact sequence of complexes of sheaves

$$0 \to h^{-1}(E^\bullet)[1] \to E^\bullet \to [\mathrm{im}\, d_E \to E^0] \to 0,$$

which induces a distinguished triangle

$$h^{-1}(E^\bullet)[1] \to E^\bullet \to [\mathrm{im}\, d_E \to E^0] \xrightarrow{+1} h^{-1}(E^\bullet)[2].$$

Notice that $[\operatorname{im} d_E \to E^0] = h^0(E^\bullet)$ in $D^{[-1,0]}_{\text{coh}}(\mathcal{M}_{\text{lis-ét}})$. Then we have a distinguished triangle

$$h^{-1}(E^\bullet)[1] \to E^\bullet \to h^0(E^\bullet) \xrightarrow{+1} h^{-1}(E^\bullet)[2].$$

Since $h^0(E^\bullet)$ and $h^{-1}(E^\bullet)$ are coherent, the morphism $h^0(E^\bullet)[-1] \xrightarrow{+1} h^{-1}(E)[1]$ corresponds to a morphism $\psi\colon h^0(E^\bullet)[-1] \to h^{-1}(E)[1]$ in $D^{\leq 0}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$. Completing ψ to a distinguished triangle in $D^{\leq 0}(\mathrm{Coh}(\mathcal{M}_{\text{lis-ét}}))$ and mapping it to $D^{\leq 0}_{\text{coh}}(\mathcal{M}_{\text{lis-ét}})$, we deduce that $E^\bullet$ is quasi-isomorphic to the mapping cone of ψ, hence it is in the essential image. □

3.16. If $\mathcal{M}$ has the resolution property then the functor $\hat{h}$ induces a functor

$$h^1/h^0\colon D^{[-1,0]}_{\text{coh}}(\mathcal{M}_{\text{lis-ét}})^{\text{opp}} \to \mathrm{Ho}(ACS/\mathcal{M}).$$

3.17. **Proposition** ([5] 2.6, 2.7). *Let $\psi\colon E^\bullet \to F^\bullet$ be a morphism in $D^{[-1,0]}_{coh}(\mathcal{M}_{lis\text{-}ét})$. If $\mathcal{M}$ has the resolution property then*

(1) $h^1/h^0(\psi)$ is representable if and only if $h^0(\psi)$ is surjective,
(2) $h^1/h^0(\psi)$ is a closed immersion if and only if $h^{-1}(\psi)$ is surjective and $h^0(\psi)$ is an isomorphism,
(3) $h^1/h^0(\psi)$ is an isomorphism if and only if $h^0(\psi)$ and $h^{-1}(\psi)$ are isomorphisms.

3.18. **Theorem.** *If $\mathcal{M}$ has the resolution property then the functor*

$$h^1/h^0\colon D^{[-1,0]}_{coh}(\mathcal{M}_{lis\text{-}ét})^{opp} \to Ho(ACS/\mathcal{M})$$

is an equivalence of categories.

Proof. By [5] 1.6, and Proposition 3.17, it follows that h^1/h^0 is fully faithfull. It remains to show that every abelian cone stack $\mathfrak{C}$ over $\mathcal{M}$ is in the essential image of h^1/h^0. By definition, for every smooth $\mathcal{M}$-scheme U, there exist a coherent sheaf E_U^{-1} and a locally free sheaf E_U^0 over U such that $\mathfrak{C} \times_{\mathcal{M}} U \cong [C(E_U^{-1})/C(E_U^0)]$. The collection $\{E_U^{-1} \to E_U^0\}_U$ defines a complex $[E^{-1} \to E^0] \in D^{[-1,0]}_{\text{coh}}(\mathcal{M}_{\text{lis-ét}})$. □

3.3. Relative intrinsic normal cone.

3.19. **Theorem** ([19] 17.3, [23], [20] 2.2.5). *Let S be a scheme and let $\mathcal{M}$, $\mathfrak{M}$ be Artin S-stacks. Let $f\colon \mathcal{M} \to \mathfrak{M}$ be a quasi-compact and quasi-separated morphism of algebraic stacks. Then there exists $L_f^\bullet \in D^{\leq 1}_{qcoh}(\mathcal{M}_{lis\text{-}ét})$ such that*

(1) f is of Deligne-Mumford type if and only if $L_f^\bullet \in D^{\leq 0}_{qcoh}(\mathcal{M}_{lis\text{-}ét})$;
(2) for every cartesian diagram

$$\begin{array}{ccc} \mathcal{M}' & \xrightarrow{f'} & \mathfrak{M}' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{M} & \xrightarrow{f} & \mathfrak{M} \end{array}$$

there exists a morphism $Lg^* L_f^\bullet \to L_{f'}^\bullet$; if h is flat, this is an isomorphism;

(3) given two morphisms of S-stacks $\mathcal{M} \xrightarrow{f} \mathfrak{M} \xrightarrow{g} Z$ with $h = g \circ f$, there exists a natural distinguished triangle

$$Lf^* L_g^\bullet \to L_h^\bullet \to L_f^\bullet \to Lf^* L_g^\bullet[1].$$

If moreover f is of Deligne-Mumford type, then

(1) f is smooth if and only if $L_f^\bullet$ is locally free in degree 0;
(2) f is étale if and only if $L_f^\bullet = 0$;

(3) if f factors as $\mathcal{M} \xrightarrow{i} M \xrightarrow{p} \mathfrak{M}$ with i representable and a closed embedding and p of Deligne-Mumford type and smooth, then

$$\tau_{\geq -1} L_f^\bullet \cong [\mathscr{I}/\mathscr{I}^2 \to \Omega_p|_{\mathcal{M}}],$$

where $\mathscr{I}$ is the ideal sheaf corresponding to i.

3.20. Remark. Notice that if f is of Deligne-Mumford type then $\tau_{\geq -1} L_f^\bullet \in D_{\mathrm{coh}}^{[-1,0]}(\mathcal{M}_{\text{lis-ét}})$.

3.21. **Definition.** Let $f\colon \mathcal{M} \to \mathfrak{M}$ be a morphism of Artin S-stacks. If f is of Deligne-Mumford type, we define the *relative intrinsic normal sheaf* of f as the abelian cone stack $\mathfrak{N}_f = {}^{h^1}/_{h^0}(\tau_{\geq -1} L_f^\bullet)$.

3.22. Remark. Smooth locally on $\mathcal{M}$ and $\mathfrak{M}$, the morphism f factors as $\mathcal{M} \xrightarrow{i} M \xrightarrow{p} \mathfrak{M}$, with i a closed embedding and p representable and smooth. More explicitly, let V be a smooth atlas for $\mathfrak{M}$ and let U be an affine scheme which is an ètale atlas for $\mathcal{M} \times_{\mathfrak{M}} V$. In particular there exists a closed embedding $j\colon U \hookrightarrow \mathbb{A}_S^n$. Let $M = \mathbb{A}_S^n \times_S V$ and let $f_U\colon U \to V$ be the morphism induced by f, then f_U factors as $U \xrightarrow{i} M \xrightarrow{p} V$, where i is a closed embedding and p is smooth. Moreover, by Theorem 3.19, we have $\mathfrak{N}_{f_U} \cong [{}^{A(C_i)}/_{T_p|_U}]$, where $C_i = C(\mathscr{I}/\mathscr{I}^2)$ and $\mathscr{I}$ is the ideal sheaf corresponding to i.

3.23. **Proposition** ([5] 7)**.** *There exists a unique closed subcone stack $\mathfrak{C}_f \subseteq \mathfrak{N}_f$ such that*

(1) if f factors as $p \circ i$, with i representable closed embedding and p representable smooth, then $\mathfrak{C}_f = [{}^{C_i}/_{T_p|_{\mathcal{M}}}]$;
(2) for every smooth morphism $V \to \mathfrak{M}$, let $g\colon U = V \times_{\mathfrak{M}} \mathcal{M} \to V$ be the induced morphism, then $\mathfrak{C}_g \cong \mathfrak{C}_f \times_{\mathcal{M}} U$.

3.24. **Definition.** The unique closed subcone stack $\mathfrak{C}_f$ of $\mathfrak{N}_f$ is called the *relative intrinsic normal cone* of f.

3.25. ([5] 7) If $\mathfrak{M}$ is purely dimensional of pure dimension n, then $\mathfrak{C}_f$ is purely dimensional of pure dimension n.

3.26. **Proposition** ([5] 7.1)**.** *Consider the following commutative diagram of algebraic Artin stacks over a scheme S,*

$$\begin{array}{ccc} \mathcal{M}' & \xrightarrow{f'} & \mathfrak{M}' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{M} & \xrightarrow{f} & \mathfrak{M} \end{array} \tag{1}$$

where the morphisms f and f' are of Deligne-Mumford type. Then there exists a natural morphism $\alpha\colon \mathfrak{C}_{f'} \to g^*\mathfrak{C}_f$ such that

(1) if (1) is cartesian then α is a closed immersion;
(2) if moreover the morphism h is flat then α is an isomorphism.

3.4. **Perfect obstruction theories.** Let $f\colon \mathcal{M} \to \mathfrak{M}$ be a morphism of Artin stacks over S. Assume that f is of Deligne-Mumford type.

3.27. **Definition.** Let $E^\bullet \in D_{\mathrm{coh}}^{[-1,0]}(\mathcal{M})$. A morphism $\varphi\colon E^\bullet \to \tau_{\geq -1} L_f^\bullet$ in $D_{\mathrm{coh}}^{[-1,0]}(\mathcal{M})$ is called a *relative obstruction theory* for f if $h^0(\varphi)$ is an isomorphism and $h^{-1}(\varphi)$ is surjective.

3.28. Remark. If $(E^\bullet, \varphi)$ is a relative obstruction theory for f, then, by Proposition 3.17, the morphism ${}^{h^1}/_{h^0}(\varphi)\colon \mathfrak{N}_f \to {}^{h^1}/_{h^0}(E^\bullet)$ is a closed embedding.

3.29. **Theorem.** *A pair* $(E^\bullet,\varphi)$ *is a relative obstruction theory for* f *if and only if, for any geometric point* $\overline{s}$ *of* S, *for any small extension* $A' \to A = {}^{A'}/_{I}$ *in* $({}^{Art}/_{\hat{\mathcal{O}}_{S,\overline{s}}})$ *and any commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} A & \xrightarrow{g} & \mathcal{M} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A' & \xrightarrow{h'} & \mathfrak{M} \end{array}$$

the obstruction $h^1(\varphi^\vee)(ob_f(g,h')) \in h^1((g^*E^\bullet)^\vee)\otimes I$ *vanishes if and only if there exists a morphism* $g'\colon \operatorname{Spec} A' \to \mathcal{M}$ *such that* $g'\circ i = g$, $f\circ g' = h'$, *and moreover if* $h^1(\varphi^\vee)(ob_f(g,h')) = 0$ *then the set of isomorphism classes of such morphisms* g' *is a torsor under* $h^0((g^*E^\bullet)^\vee)\otimes I$.

Proof. Follows by Proposition A.23 and by [5] 4. □

3.30. **Definition.** Let $(E^\bullet,\varphi)$ be a relative obstruction theory for f. We say that $(E^\bullet,\varphi)$ is *perfect* (of perfect amplitude contained in $[-1,0]$) if, smooth locally over $\mathcal{M}$, it is isomorphic to $[E^{-1}\to E^0]$ with E^{-1}, E^0 locally free sheaves over $\mathcal{M}$.

3.31. Remark. A relative obstruction theory $(E^\bullet,\varphi)$ is perfect if and only if ${}^{h^1}/_{h^0}(E^\bullet)$ is a vector bundle stack over $\mathcal{M}$.

3.5. **Virtual fundamental class.** Let D be a Dedekind domain and set $S = \operatorname{Spec} D$. Let $f\colon \mathcal{M}\to \mathfrak{M}$ be a morphism of Deligne-Mumford type of Artin stacks over S. Assume that $\mathfrak{M}$ is purely dimensional of pure dimension m and that $\mathcal{M}$ has the resolution property. Let $(E^\bullet,\varphi)$ be a perfect relative obstruction theory for f, we denote by

$$\mu\colon \mathfrak{E}_f = {}^{h^1}/_{h^0}(E^\bullet) \to \mathcal{M}$$

the associated vector bundle stack of rank r. By Remark 3.28, the relative intrinsic normal cone $\mathfrak{C}_f$ is a closed substack of $\mathfrak{E}_f$. Moreover, by Theorem B.2 and [18] Proposition 3.5.10, the flat pullback

$$\mu^*\colon A_*({}^{\mathcal{M}}/_{k}) \to A_{*+r}({}^{\mathfrak{E}_f}/_{k})$$

is an isomorphism and we denote the inverse by $0^!$.

3.32. **Definition.** The *virtual fundamental class* of $\mathcal{M}$ relative to $(E^\bullet,\varphi)$ is the cycle class

$$[\mathcal{M},E^\bullet]^{\mathrm{virt}} = 0^![\mathfrak{C}_f] \in A_*({}^{\mathcal{M}}/_{S}).$$

3.33. The intrinsic cone $\mathfrak{C}_f$ is purely dimensional of pure dimension m, hence $[\mathcal{M},E^\bullet]^{\mathrm{virt}} \in A_{m-r}({}^{\mathcal{M}}/_{S})$ and $m-r$ is called the *virtual dimension* of $\mathcal{M}$.

3.34. Remark. By resolution property and Theorem 3.18, we can assume that $E^\bullet = [E^{-1}\to E^0]$ with E^{-1}, E^0 locally free sheaves over $\mathcal{M}$, up to replacing $E^\bullet$ by a quasi-isomorphic complex in $D^{[-1,0]}_{\mathrm{coh}}(\mathcal{M})$. In particular $\mathfrak{E}_f$ is isomorphic to a globally presented vector bundle stack $[{}^{C(E^{-1})}/_{C(E^0)}]$ with $C(E^0)\to C(E^{-1})$ morphism of vector bundles over $\mathcal{M}$.

3.35. **Proposition.** *Let* $(E^\bullet,\varphi)$ *be a perfect relative obstruction theory for* f, *such that* $h^0(E^\bullet)$ *is locally free. Then*

(1) *if* $h^{-1}(E^\bullet) = 0$, *then* $\mathcal{M}$ *is smooth over* $\mathfrak{M}$ *and* $[\mathcal{M},E^\bullet]^{virt} = [\mathcal{M}]$;
(2) *if* $\mathcal{M}$ *is smooth over* $\mathfrak{M}$, *then* $h^{-1}(E^\bullet)$ *is locally free and*

$$[\mathcal{M},E^\bullet]^{virt} = c_r(C(h^{-1}(E^\bullet)))\cdot[\mathcal{M}],$$

where r *is the rank of* $C(h^{-1}(E^\bullet))$.

Proof. By Remark 3.34, we can assume $E^\bullet = [E_0 \to E_1]$ with E^{-1}, E^0 vector bundles over $\mathcal{M}$. If $h^{-1}(E^\bullet) = 0$, then $h^{-1}(\tau_{\geq -1} L_f^\bullet) = h^{-1}(L_f^\bullet) = 0$, since $h^{-1}(\varphi)$ is surjective. Hence $\tau_{\geq -1} L_f^\bullet$ is quasi-isomorphic to $h^0(L_f^\bullet)$, which is locally free since $h^0(\varphi)$ is an isomorphism. Then, from the following distinguished triangles

$$\tau_{\leq -2} L_f^\bullet \to L_f^\bullet \to \tau_{\geq -1} L_f^\bullet \xrightarrow{+1}$$
$$\tau_{\leq -1} L_f^\bullet \to L_f^\bullet \to h^0(L_f^\bullet) \xrightarrow{+1},$$

we get that $\tau_{\leq -1} L_f^\bullet$ and $\tau_{\leq -2} L_f^\bullet$ are quasi-isomorphic and hence $\tau_{\geq -1} L_f^\bullet = L_f^0$. It follows that $L_f^0 = h^0(L_f^\bullet)$ is locally free and, by Theorem 3.19, f is smooth. In particular $\mathcal{M}$ is smooth over S. Moreover, we have

$$\operatorname{rk} \mathfrak{E}_f = -\operatorname{rk} h^0(E^\bullet) = -\operatorname{rk} h^0(L_f^\bullet) = -\operatorname{rk} T_f = -\dim_{\mathfrak{M}} \mathcal{M},$$

hence the virtual dimension of $\mathcal{M}$ is equal to $\dim_S \mathcal{M}$. Notice that in this case $\mathfrak{E}_f \cong \mathfrak{N}_f = \mathfrak{C}_f = [{}^{\mathcal{M}}/_{T_f}]$, hence $\mu^*[\mathcal{M}] = [\mathfrak{C}_f]$ and

$$[\mathcal{M}, E^\bullet]^{\mathrm{virt}} = 0^![\mathfrak{C}_f] = [\mathcal{M}].$$

Assume now that f is smooth then, by Theorem 3.19, $h^{-1}(L_f^\bullet) = 0$ and $\mathfrak{N}_f = \mathfrak{C}_f = [{}^{\mathcal{M}}/_{T_f}]$. Since $h^0(E^\bullet)$ is locally free and $E^\bullet$ is perfect, we get that $h^{-1}(E^\bullet)$ is locally free. Let us consider the natural morphism $\rho\colon C(h^{-1}(E^\bullet)) \to \mathfrak{E}_f$ and set $\widetilde{\mu} = \mu \circ \rho$. Notice that $\mathfrak{C}_f \times_{\mathfrak{E}_f} C(h^{-1}(E^\bullet)) \cong \mathcal{M}$, hence we have a cartesian diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{s} & C(h^{-1}(E^\bullet)) \\ \downarrow & & \downarrow \rho \\ \mathfrak{C}_f & \hookrightarrow & \mathfrak{E}_f \end{array}$$

where s is the zero-section of $C(h^{-1}(E^\bullet))$. Let $\widetilde{0}^!$ be the inverse of $\widetilde{\mu}^*$, then $0^! = \widetilde{0}^! \circ \rho^*$. By [12] Corollary 6.3,

$$c_r(C(h^{-1}(E^\bullet))) \cdot [\mathcal{M}] = \widetilde{0}^! s_*[\mathcal{M}] = \widetilde{0}^! \rho^*[\mathfrak{C}_f] = 0^![\mathfrak{C}_f] = [\mathcal{M}, E^\bullet]^{\mathrm{virt}}. \qquad \square$$

3.36. **Proposition.** *Consider the following cartesian diagram of Artin stacks over S,*

$$\begin{array}{ccc} \mathcal{M}' & \xrightarrow{f'} & \mathfrak{M}' \\ g \downarrow & & \downarrow h \\ \mathcal{M} & \xrightarrow{f} & \mathfrak{M} \end{array}$$

where f and f' are of Deligne-Mumford type, $\mathfrak{M}$ and $\mathfrak{M}'$ are smooth and purely dimensional of pure dimension m, $\mathcal{M}$ and $\mathcal{M}'$ have the resolution property. Let $(E^\bullet, \varphi)$ be a perfect relative obstruction theory for f. If h is flat or a regular local immersion (of constant dimension) then

$$g^*[\mathcal{M}, E^\bullet]^{virt} = [\mathcal{M}', Lg^* E^\bullet]^{virt}.$$

Proof. Let us notice that $Lg^*E^\bullet$ is a perfect relative obstruction theory for f'. The statement follows by [5] 7.2 and Theorem B.2. $\square$

4. A virtual fundamental class

4.1. Let D be a Dedekind domain and set $S = \operatorname{Spec} D$. Let X be a smooth projective connected scheme of finite presentation over S. We want to define a virtual fundamental class for $\overline{\mathcal{M}}_{g,n}(X/S, \beta_\eta)$ relative to the forgetful morphism

$$\theta \colon \overline{\mathcal{M}}_{g,n}(X/S, \beta_\eta) \to \mathfrak{M}_{g,\,n/S},$$

following the construction of 3. For this, we need a perfect relative obstruction theory for θ.

4.1. **The stack of morphisms.**

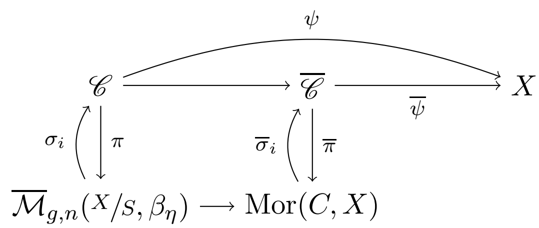
4.2. **Definition.** Let C be the universal curve of $\mathfrak{M}_{g,\,n/S}$, we define the algebraic stack $\operatorname{Mor}(C, X)$ over $\mathfrak{M}_{g,\,n/S}$ as follows

(1) for every S-scheme T, an object of $\operatorname{Mor}(C,X)(T)$ is a pre-stable curve $(C_T \xrightarrow{\pi_T} T, t_i)$ over T together with a morphism of S-schemes $f_T \colon C_T \to X$;

(2) for every S-scheme T, a morphism from $((C_T \xrightarrow{\pi_T} T, t_i), f_T)$ to $((C'_T \xrightarrow{\pi'_T} T, t'_i), f'_T)$ is an isomorphism of pre-stable curves $C_T \xrightarrow{\alpha} C'_T$ such that $f'_T \circ \alpha = f_T$.

4.3. There is a canonical functor $\overline{\theta} \colon \operatorname{Mor}(C,X) \to \mathfrak{M}_{g,\,n/S}$ which forgets the map into X. Let us notice that $\overline{\theta}$ is representable and quasi-projective. Moreover, since stability is an open condition, the stack $\overline{\mathcal{M}}_{g,n}(X/S, \beta_\eta)$ is an open substack of $\operatorname{Mor}(C,X)$.

Notice that $\overline{\mathscr{C}} = C \times_{\mathfrak{M}_{g,\,n/S}} \operatorname{Mor}(C,X)$ is a universal family for $\operatorname{Mor}(C,X)$ and we have the following commutative diagram



where $\mathscr{C} = \overline{\mathscr{C}} \times_{\operatorname{Mor}(C,X)} \overline{\mathcal{M}}_{g,n}(X/S, \beta_\eta)$ is the universal stable map.

4.4. **Lemma.** *We have $F^\bullet = R\overline{\pi}_*(\overline{\psi}^*\Omega_{X/S} \otimes \omega_{\overline{\pi}})[-1] \in D^{(-1,0)}_{coh}(\operatorname{Mor}(C,X))$ and $h^1/h^0(F^\bullet)$ is a vector bundle stack.*

Proof. Since X is smooth over S, the sheaf $\Omega_{X/S}$ is a vector bundle over X. The dualizing sheaf $\omega_{\overline{\pi}}$ is a line bundle over $\overline{\mathscr{C}}$, because $\overline{\mathscr{C}}$ is a family of curves with at most nodal singularities (which are Gorenstein). Hence $\overline{\psi}^*\Omega_{X/S} \otimes \omega_{\overline{\pi}}$ is a vector bundle on $\overline{\mathscr{C}}$. Recall that the cohomology of the total pushforward is the higher pushforward sheaf. Moreover, $\overline{\pi}$ is a flat projective morphism of relative dimension 1, so the i-pushforward vanishes for $i > 1$ by the cohomology and base-change theorem ([11] Corollary 8.3.4), therefore

$$R\overline{\pi}_*(\overline{\psi}^*\Omega_{X/S} \otimes \omega_{\overline{\pi}}) \in D^{(0,1)}_{\mathrm{coh}}(\operatorname{Mor}(C,X)).$$

Set $\mathscr{F} = \overline{\psi}^*\Omega_{X/S} \otimes \omega_{\overline{\pi}}$. Let $\mathscr{L}$ be a $\overline{\pi}$-ample line bundle (for istance, we can take $\mathscr{L} = \omega_{\overline{\pi}}(\sum_{i=1}^n \overline{\sigma}_i) \otimes \overline{\psi}^*\mathscr{O}(3)$) then, for n big enough, $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections and $R^0\overline{\pi}_*(\mathscr{F} \otimes \mathscr{L}^{-n}) = 0$. Thus we have a surjection

$$\mathscr{G} = (\overline{\pi}^*\overline{\pi}_*(\mathscr{F} \otimes \mathscr{L}^n) \otimes \mathscr{L}^{-n} \to \mathscr{F},$$

and we denote by $\mathscr{K}$ the kernel. Notice that $\mathscr{K}$ is a vector bundle because it is the kernel of a surjection of vector bundles. Hence we get the following exact sequence

$$0 \to R^0\overline{\pi}_*\mathscr{K} \to R^0\overline{\pi}_*\mathscr{G} \to R^0\overline{\pi}_*\mathscr{F} \to R^1\overline{\pi}_*\mathscr{K} \to R^1\overline{\pi}_*\mathscr{G} \to R^1\overline{\pi}_*\mathscr{F} \to 0.$$

Since $R^0\overline{\pi}_*(\mathscr{F}\otimes\mathscr{L}^{-n})=0$, we have that $R^0\overline{\pi}_*\mathscr{G}=0$ and thus $R^0\overline{\pi}_*\mathscr{K}=0$. As a consequence, $R^1\overline{\pi}_*\mathscr{K}$ and $R^1\overline{\pi}_*\mathscr{G}$ are vector bundles and $F^\bullet$ is quasi-isomorphic to $[R^1\overline{\pi}_*\mathscr{K}\to R^1\overline{\pi}_*\mathscr{G}]$. □

4.5. We define a morphism $\overline{\varphi}\colon F^\bullet\to\tau_{\geq -1}L^\bullet_{\overline{\theta}}$ in $D^{(-1,0)}_{\mathrm{coh}}(\mathrm{Mor}(C,X))$ as follows. By adjuction, this is equivalent to define a morphism

$$(\overline{\psi}^*\Omega_{X/S}\otimes\omega_{\overline{\pi}})[-1]\to L\overline{\pi}^!(L^\bullet_{\overline{\theta}}).$$

Recall that if $\overline{\pi}$ is a flat proper Gorenstein morphism of relative dimension N, then $L\overline{\pi}^!=\overline{\pi}^*\otimes\omega_{\overline{\pi}}[-N]$. This applies in our case with $N=1$ and we get $L\overline{\pi}^!=\overline{\pi}^*\otimes\omega_{\overline{\pi}}[-1]$. Hence to give the morphism $\overline{\varphi}$ is equivalent to giving a morphism $\overline{\psi}^*\Omega_{X/S}\to\overline{\pi}^*L^\bullet_{\overline{\theta}}$. Notice that $\Omega_{X/S}=L^\bullet_{X/S}$, since X is smooth over S (Theorem 3.19). We define the morphism $\overline{\psi}^*L^\bullet_{X/S}\to\overline{\pi}^*L^\bullet_{\overline{\theta}}$ as the composition

$$\overline{\psi}^*L^\bullet_{X/S}\to L^\bullet_{\overline{\mathscr{C}}/S}\to L^\bullet_{\overline{\mathscr{C}}/C}\cong\overline{\pi}^*L^\bullet_{\overline{\theta}},$$

where C is the universal curve of $\mathfrak{M}_{g,\,n/S}$, the isomorphism $L^\bullet_{\overline{\mathscr{C}}/C}\cong\overline{\pi}^*L^\bullet_{\overline{\theta}}$ follows from the fact that $\overline{\mathscr{C}}=C\times_{\mathfrak{M}_{g,\,n/S}}\mathrm{Mor}(C,X)$ and the morphism $C\to\mathfrak{M}_{g,\,n/S}$ is flat (Theorem 3.19).

4.6. **Proposition.** *The pair $(F^\bullet,\overline{\varphi})$ defined above is a perfect relative obstruction theory for $\overline{\theta}$.*

Proof. Let $\operatorname{Spec}\overline{k}\xrightarrow{\overline{x}}\mathrm{Mor}(C,X)$ be a geometric point. Then $\overline{x}$ corresponds to a pre-stable curve $C_{\overline{x}}$ over $\overline{k}$ together with a morphism $\overline{\psi}_{\overline{x}}\colon C_{\overline{x}}\to X$, obtained by pulling back $(\overline{\mathscr{C}},\overline{\psi})$ along $\overline{x}$. Using Serre duality and cohomology and base chage theorem ([11] Corollary 8.3.4), we have

$$H^i(C_{\overline{x}},\overline{\psi}^*_{\overline{x}}T_{X/S})=H^{1-i}(C_{\overline{x}},\overline{x}^*(\overline{\psi}^*\Omega_{X/S}\otimes\omega_{\overline{\pi}}))^\vee=h^{i-1}((F^\bullet[-1])^\vee)=h^i((L\overline{x}^*F^\bullet)^\vee).$$

Now, let $A'\to A={}^{A'}/_{I}$ be a small extension in $({}^{\mathrm{Art}}/_{\hat{\mathscr{O}}_{S,\overline{s}}})$ and consider a commutative diagram

$$\begin{array}{ccc}\operatorname{Spec}A & \xrightarrow{g} & \mathrm{Mor}(C,X)\\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\theta}}\\ \operatorname{Spec}A' & \xrightarrow{h'} & \mathfrak{M}_{g,\,n/S}\end{array}$$

In particular h' corresponds to a family of pre-stable curves $C_{A'}$ over A' obtained by pulling back $C\to\mathfrak{M}_{g,\,n/S}$ along h', and g corresponds to a family of pre-stable curves C_A over A together with a morphism $\overline{\psi}_A\colon C_A\to X$ obtained by pulling back $(\overline{\mathscr{C}},\overline{\psi})$ along g. Thus g extends to $g'\colon\operatorname{Spec}A'\to\mathrm{Mor}(C,X)$ if and only if $\overline{\psi}_A$ extends to $\overline{\psi}_{A'}\colon C_{A'}\to X$ if and only if, by Proposition A.23 and Proposition A.25, $h^1(\overline{\varphi}^\vee)(\mathrm{ob}_{\overline{\theta}}(g,h'))$ is zero in $H^1(C_{\overline{x}},\overline{\psi}^*_{\overline{x}}T_{X/S})\otimes I$. Moreover the extensions, if they exist, form a torsor under $H^0(C_{\overline{x}},\overline{\psi}^*_{\overline{x}}T_{X/S})\otimes I$. By Theorem 3.29, this implies that $(F^\bullet,\overline{\varphi})$ is a relative obstruction theory for $\overline{\theta}$ and, by Lemma 4.4, $F^\bullet$ is perfect. □

4.2. A perfect obstruction theory for $\overline{\mathcal{M}}_{g,n}({}^{X}/_{S},\beta_\eta)$.

4.7. **Corollary.** *Let $E^\bullet=R\pi_*(\psi^*\Omega_{X/S}\otimes\omega_\pi)[-1]$ and let $\varphi\colon E^\bullet\to\tau_{\geq-1}L^\bullet_\theta$ be the morphism induced by $\overline{\varphi}$. Then $(E^\bullet,\varphi)$ is a perfect relative obstruction theory for θ.*

Proof. Since the natural inclusion $j\colon\overline{\mathcal{M}}_{g,n}({}^{X}/_{S},\beta_\eta)\hookrightarrow\mathrm{Mor}(C,X)$ is an open immersion, it follows that $Lj^*L^\bullet_{\overline{\theta}}=L^\bullet_\theta$, $Lj^*F^\bullet=E^\bullet$, and $\varphi=j^*\overline{\varphi}$. Hence, by Lemma 4.4, we have $E^\bullet\in D^{(-1,0)}_{\mathrm{coh}}(\overline{\mathcal{M}}_{g,n}({}^{X}/_{S},\beta_\eta))$. By Proposition 4.6, we know that $(F^\bullet,\overline{\varphi})$ is a perfect obstruction theory for $\overline{\theta}$, hence $h^0(\overline{\varphi})$ is an isomorphism and $h^{-1}(\overline{\varphi})$ is surjective. Since the pullback j^* is an exact functor, we have that $h^0(\varphi)$ is an isomorphism and $h^{-1}(\varphi)$ is surjective, which implies the statement. □

4.8. **Definition.** We define the *virtual fundamental class* of $\overline{\mathcal{M}}_{g,n}({}^X\!/\!_S, \beta_\eta)$ to be

$$[\overline{\mathcal{M}}_{g,n}({}^X\!/\!_S, \beta_\eta)]^{\text{virt}} = [\overline{\mathcal{M}}_{g,n}({}^X\!/\!_S, \beta_\eta), E^\bullet]^{\text{virt}} \in A_{\text{vdim}}({}^{\overline{\mathcal{M}}_{g,n}({}^X\!/\!_S,\beta_\eta)}\!/\!_S).$$

4.9. Consider the vector bundle stack $\mu\colon \mathfrak{E}_\theta = {}^{h^1}\!/\!_{h^0}(E^\bullet) \to \overline{\mathcal{M}}_{g,n}({}^X\!/\!_S, \beta_\eta)$. Then, by cohomology and base-change theorem ([11] Corollary 8.3.4) and Riemann-Roch theorem,

$$\begin{aligned}
\operatorname{rk} \mathfrak{E}_\theta &= \dim h^{-1}(L\overline{x}^* E^\bullet) - \dim h^0(L\overline{x}^* E^\bullet) \\
&= \dim H^1(\mathscr{C}_{\overline{x}}, \psi_{\overline{x}}^* T_{X/S}) - \dim H^0(\mathscr{C}_{\overline{x}}, \psi_{\overline{x}}^* T_{X/S}) \\
&= -\deg(\operatorname{ch}(\psi_{\overline{x}}^* T_{X/S}) \cdot \operatorname{td}(T_{\mathscr{C}_{\overline{x}}})) \\
&= -\deg\left(\operatorname{rk}(\psi_{\overline{x}}^* T_{X/S}) + c_1(\psi_{\overline{x}}^* T_{X/S}) \cdot \left(1 - \frac{[K_{\mathscr{C}_{\overline{x}}}]}{2}\right)\right) \\
&= \dim_S X(g-1) + c_1(T_{X/S}) \cdot \psi_{\overline{x}*}[\mathscr{C}_{\overline{x}}] \\
&= \dim_S X(g-1) + c_1(T_{X_\eta/\eta}) \cdot \beta_\eta,
\end{aligned}$$

for a geometric point $\overline{x}$ of $\overline{\mathcal{M}}_{g,n}({}^X\!/\!_S, \beta_\eta)$. Thus

$$\text{vdim} = \dim_S \mathfrak{M}_{g,\,n/S} - \operatorname{rk} \mathfrak{E}_\theta = (\dim_S X - 3)(1-g) - c_1(T_{X_\eta/\eta}) \cdot \beta_\eta + n.$$

5. Gromov-Witten classes and invariants

5.1. **Definitions.** Let D be a Dedekind domain, set $S = \operatorname{Spec} D$ and denote by η the generic point of S. Let X be a smooth projective connected scheme of finite presentation over S and set $X_\eta = X \times_S \eta$. Fix $\beta_\eta \in A_1({}^{X_\eta}\!/\!_\eta)$ and $g, n \geq 0$.

5.1. Remark. Notice the following facts:

(1) the natural functor

$$\nu = (\hat{\theta}, \text{ev})\colon \overline{\mathcal{M}}_{g,n}({}^X\!/\!_S, \beta_\eta) \to \overline{\mathcal{M}}_{g,\,n/S} \times_S X^n$$

is proper because $\hat{\theta}$ is proper (since $\overline{\mathcal{M}}_{g,n}({}^X\!/\!_S, \beta_\eta)$ is proper and $\overline{\mathcal{M}}_{g,\,n/S}$ is separated) and X is a projective scheme;

(2) the projection

$$p\colon \overline{\mathcal{M}}_{g,\,n/S} \times_S X^n \to X^n$$

is flat of relative dimension $3g-3+n$ because $\overline{\mathcal{M}}_{g,\,n/S}$ is smooth of dimension $3g-3+n$ (since stability condition is open);

(3) the projection

$$q\colon \overline{\mathcal{M}}_{g,\,n/S} \times_S X^n \to \overline{\mathcal{M}}_{g,\,n/S}$$

is projective because X is projective.

5.2. If $S = \operatorname{Spec} k$ with k an algebraically closed field and if l is a prime different from the characteristic of k, we can define the l-adic étale cohomology as

$$H^r(X, \mathbb{Z}_l) = \varprojlim_m H^r_{\text{ét}}(X, {}^{\mathbb{Z}}\!/\!_{l^m\mathbb{Z}}).$$

Moreover $H^r(X, \mathbb{Q}_l) = H^r(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ and we have the cycle map

$$\text{cl}\colon A^r({}^X\!/\!_k)_{\mathbb{Q}} \to H^{2r}(X, \mathbb{Q}_l(r))$$

as described in [22] VI.9. We set $H^*(X) = \sum_r H^r(X, \mathbb{Q}_l(\overline{r}))$, where $\overline{r}$ is the integral part of ${}^r\!/\!_2$.

5.3. **Definition** (Gromov-Witten classes)**.** Let $\mathrm{C}^X_{g,n,\beta_\eta} \in A^*({}^{\overline{\mathcal{M}}_{g,\,n/S} \times_S X^n}\!/\!_S)_{\mathbb{Q}}$ be the class defined by $\nu_*[\overline{\mathcal{M}}_{g,n}({}^X\!/\!_S, \beta_\eta)]^{\text{virt}} \in A_*({}^{\overline{\mathcal{M}}_{g,\,n/S} \times_S X^n}\!/\!_S)_{\mathbb{Q}}$, via Poincaré duality.

(1) If $2g-3+n \geq 0$, we define the linear operator

$$\mathrm{I}^X_{g,n,\beta_\eta}\colon A^*({}^X\!/\!_S)^{\otimes n}_{\mathbb{Q}} \to A^*(\overline{\mathcal{M}}_{g,\,n/S}/S)_{\mathbb{Q}}$$

to be $\mathrm{I}^X_{g,n,\beta_\eta}(\bullet) = q_*\left(p^*(\bullet) \cap \mathrm{C}^X_{g,n,\beta_\eta}\right)$.

(2) If moreover $S = \operatorname{Spec} k$ with k an algebraically closed field, we can define

$$\mathrm{I}^X_{g,n,\beta_\eta}\colon H^*(X)^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,\,n/S}))$$

as above (abusing the notation, we write $\mathrm{C}^X_{g,n,\beta_\eta}$ instead of $\mathrm{cl}(\mathrm{C}^X_{g,n,\beta_\eta})$).

5.4. Remark. By Poicaré duality, we can consider $\mathrm{I}^X_{g,n,\beta_\eta}$ as a homomorphism $A_*({}^X\!/\!_S)^{\otimes n}_{\mathbb{Q}} \to A_*(\overline{\mathcal{M}}_{g,\,n/S}/S)_{\mathbb{Q}}$, defined as

$$\mathrm{I}^X_{g,n,\beta_\eta}(\bullet) = q_*\left(p^*(\bullet) \cap \nu_*[\overline{\mathcal{M}}_{g,n}({}^X\!/\!_S,\beta_\eta)]^{\mathrm{virt}}\right).$$

5.5. **Definition** (Gromov-Witten invariants)**.** If $2g-3+n \geq 0$, we define

$$\langle \mathrm{I}^X_{g,n,\beta_\eta}\rangle(\underline{\gamma}) = \int_{\overline{\mathcal{M}}_{g,\,n/S}} \mathrm{I}^X_{g,n,\beta_\eta}(\underline{\gamma}),$$

for $\underline{\gamma} = \gamma_1 \otimes \cdots \otimes \gamma_n \in A^*({}^X\!/\!_S)^{\otimes n}_{\mathbb{Q}}$. If $S = \operatorname{Spec} k$ with k an algebraically closed field then $\langle \mathrm{I}^X_{g,n,\beta_\eta}\rangle(\underline{\gamma})$ is defined for every $\underline{\gamma} \in H^*(X)^{\otimes n}$.

5.6. Notation. Let $\mathcal{M}$ be a proper Artin stack over a field k. Let L be a finite algebraic extension of k, then $\mathcal{M}_L = \mathcal{M} \times_k L \xrightarrow{\rho_L} X$ is smooth and finite of degree $[L:k]$. By [12] 1.7.4, $\rho_{L*}\rho_L^* = [L:k]$, therefore ρ_L^* gives an isomorphism $A_*({}^{\mathcal{M}}\!/\!_k)_{\mathbb{Q}} \cong A_*({}^{\mathcal{M}_L}\!/\!_L)_{\mathbb{Q}}$. Let $\overline{k}$ be an algebraic closure of k and set $\overline{\mathcal{M}} = \mathcal{M} \times_k \overline{k}$, then $A_*({}^{\overline{\mathcal{M}}}\!/\!_{\overline{k}}) = \varinjlim_L A_*({}^{\mathcal{M}_L}\!/\!_L)$, where the limit is over all finite algebraic extensions L of k such that $L \subset \overline{k}$. There is an induced homomorphism $\rho\colon A_*({}^{\mathcal{M}}\!/\!_k) \to A_*({}^{\overline{\mathcal{M}}}\!/\!_{\overline{k}})$ which gives an isomorphism $A_*({}^{\mathcal{M}}\!/\!_k)_{\mathbb{Q}} \cong A_*({}^{\overline{\mathcal{M}}}\!/\!_{\overline{k}})_{\mathbb{Q}}$; for all $\beta \in A_*({}^X\!/\!_k)$ we set $\overline{\beta} = \rho(\beta)$. The same holds for bivariant Chow groups $A^*(\bullet)_{\mathbb{Q}}$.

5.7. **Proposition.** *Let X be a smooth projective scheme of finite presentation over a field k. Let $\overline{k}$ be an algebraic closure of k and set $\overline{X} = X \times_k \overline{k}$. Then, for all $\gamma_1 \otimes \cdots \otimes \gamma_n \in A^*({}^X\!/\!_k)^{\otimes n}_{\mathbb{Q}}$,*

$$I^X_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_n) = I^{\overline{X}}_{g,n,\overline{\beta}}(\gamma_1 \otimes \cdots \otimes \gamma_n).$$

Proof. Let L be a finite algebraic extension of k and set $X_L = X \times_k L$. Let $\beta_L = \rho_L^*\beta$. Notice that $\overline{\mathcal{M}}_{g,n}({}^{X_L}\!/\!_L,\beta_L) \cong \overline{\mathcal{M}}_{g,n}({}^X\!/\!_k,\beta) \times_k L$ and thus, by Proposition 3.36,

$$[\overline{\mathcal{M}}_{g,n}({}^X\!/\!_k,\beta)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{g,n}({}^{X_L}\!/\!_L,\beta_L)]^{\mathrm{virt}} \in A_*(\overline{\mathcal{M}}_{g,n}({}^X\!/\!_k,\beta)/k)_{\mathbb{Q}} \cong A_*(\overline{\mathcal{M}}_{g,n}({}^{X_L}\!/\!_L,\beta_L)/L)_{\mathbb{Q}}.$$

Then for all $\underline{\gamma} \in A^*({}^X\!/\!_k)_{\mathbb{Q}}$, we have $\mathrm{I}^{X_L}_{g,n,\beta_L}(\underline{\gamma}) = \mathrm{I}^X_{g,n,\beta}(\underline{\gamma})$ and hence $\mathrm{I}^{\overline{X}}_{g,n,\overline{\beta}}(\underline{\gamma}) = \mathrm{I}^X_{g,n,\beta}(\underline{\gamma})$. □

5.2. **Deformation invariance and comparison of invariants in mixed characteristic.** Let D be a Dedekind domain, set $B = \operatorname{Spec} D$. We denote by $\eta = \operatorname{Spec} K$ the generic point of B and let $b_0, b_1 \in B$ be closed points of B. Let $\pi\colon Y \to B$ be a smooth projective family of smooth projective schemes and set $Y_\eta = Y \times_B \eta$, $Y_h = Y \times_B b_h$ for $h = 0,1$. By [12] 20.3, there are specialization morphisms $\sigma_h\colon A_*({}^{Y_\eta}\!/\!_\eta) \to A_*({}^{Y_h}\!/\!_{b_h})$ for $h = 0,1$. Let $b_h = \operatorname{Spec} k_h$ and let $\overline{k}_h$ be an algebraic closure of k_h for $h = 0,1$. We set $\overline{b}_h = \operatorname{Spec} \overline{k}_h$. Recall that the cospecialization map gives an isomorphism $H^*(\overline{Y}_0) \cong H^*(\overline{Y}_1)$, where $\overline{Y}_h = Y_h \times_{k_h} \overline{k}_h$ for $h = 0,1$ ([22] VI.4.1).

5.8. **Theorem.** *Let $\beta \in A_1({}^{Y_\eta}\!/\!{}_{\eta})$ and set $\beta_h = \sigma_h(\beta)$ for $h = 0, 1$. Then*

$$I^{\overline{Y}_0}_{g,n,\overline{\beta}_0}(\underline{\gamma}) = I^{\overline{Y}_1}_{g,n,\overline{\beta}_1}(\underline{\gamma}),$$

for every $\underline{\gamma} \in H^*(\overline{Y}_0)^{\otimes n} \cong H^*(\overline{Y}_1)^{\otimes n}$.

Proof. Let R_h be the localization of D at b_h for $h = 0, 1$, then R_h is a discrete valuation ring with generic point η and closed point b_h. Let $\hat{R}_h$ be the completion of R_h, then $\hat{R}_h$ is a complete discrete valuation ring with closed point b_h and generic point $\eta \times_{R_h} \hat{R}_h$. Moreover $R_0 \otimes_D R_1 = K$ and hence $\eta \times_{R_0} \hat{R}_0 = \eta \times_{R_1} \hat{R}_1$. We denote by $\hat{\eta} = \operatorname{Spec} \hat{K}$ the generic point of $\hat{R}_h$. Set $\hat{Y}_h = Y \times_D \hat{R}_h$ and $\hat{Y}_\eta = Y \times_D \hat{\eta}$. Let $i_h \colon Y_h \to \hat{Y}_h$ and $j_h \colon \hat{Y}_\eta \to \hat{Y}_h$ be the natural inclusions. Let $\hat{\beta} \in A_1({}^{\hat{Y}_\eta}\!/\!{}_{\hat{\eta}})$ be the pullback of β. We have the following cartesian diagram

$$\begin{array}{ccccc}
\overline{\mathcal{M}}_{g,n}({}^{\hat{Y}_\eta}\!/\!{}_{\hat{\eta}}, \hat{\beta}) & \xrightarrow{\hat{\jmath}} & \overline{\mathcal{M}}_{g,n}({}^{\hat{Y}_h}\!/\!{}_{\hat{R}_h}, \hat{\beta}) & \xleftarrow{\hat{\imath}} & \overline{\mathcal{M}}_{g,n}({}^{Y_h}\!/\!{}_{b_h}, \beta_h) \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{M}_{g,\,n/\hat{\eta}} & \xrightarrow{\tilde{\jmath}} & \mathfrak{M}_{g,\,n/\hat{R}_h} & \xleftarrow{\tilde{\imath}} & \mathfrak{M}_{g,\,n/b_h}
\end{array}$$

Let $\overline{K}$ be an algebraic closure of $\hat{K}$. We set $\overline{\beta} = \overline{\hat{\beta}} \in A_1({}^{\overline{Y}_\eta}\!/\!{}_{\overline{\eta}})$, where $\overline{\eta} = \operatorname{Spec} \overline{K}$ and $\overline{Y}_\eta = Y \times_D \overline{\eta}$. By [12] 20.3.5 and Theorem B.2, there exists a specialization homomorphism

$$\hat{\sigma}_h \colon A_*(\overline{\mathcal{M}}_{g,n}({}^{\overline{Y}_\eta}\!/\!{}_{\overline{\eta}}, \overline{\beta})/\overline{\eta})_{\mathbb{Q}} \to A_*(\overline{\mathcal{M}}_{g,n}({}^{\overline{Y}_h}\!/\!{}_{\overline{b}_h}, \overline{\beta}_h)/\overline{b}_h)_{\mathbb{Q}},$$

and, by the functoriality of the virtual fundamental class (Proposition 3.36),

$$\hat{\sigma}_h([\overline{\mathcal{M}}_{g,n}({}^{\overline{Y}_\eta}\!/\!{}_{\overline{\eta}}, \overline{\beta})]^{\text{virt}}) = [\overline{\mathcal{M}}_{g,n}({}^{\overline{Y}_h}\!/\!{}_{\overline{b}_h}, \overline{\beta}_h)]^{\text{virt}}.$$

By [22] VI.4.1, there are isomorphisms $H^*(\overline{Y}_\eta) \cong H^*(\overline{Y}_h)$ and $H^*(\overline{\mathcal{M}}_{g,\,n/\overline{\eta}}) \cong H^*(\overline{\mathcal{M}}_{g,\,n/\overline{b}_h})$ compatible with $p_h^*, p_\eta^*, q_{h*}, q_{\eta*}, \nu_{h*}, \nu_{\eta*}$. It follows that

$$\mathrm{C}^{\overline{Y}_\eta}_{g,n,\overline{\beta}} = \mathrm{C}^{\overline{Y}_h}_{g,n,\overline{\beta}_h} \in H^*(\overline{\mathcal{M}}_{g,\,n/\overline{\eta}}) \otimes H^*(\overline{Y}_\eta)^{\otimes n} \cong H^*(\overline{\mathcal{M}}_{g,\,n/\overline{b}_h}) \otimes H^*(\overline{Y}_h)^{\otimes n}.$$

Then, for $\underline{\gamma} \in H^*(\overline{Y}_h)^{\otimes n}$, we have $\mathrm{I}^{\overline{Y}_h}_{g,n,\overline{\beta}_h}(\underline{\gamma}) = \mathrm{I}^{\overline{Y}_\eta}_{g,n,\overline{\beta}}(\underline{\gamma})$ for $h = 0, 1$. □

5.9. **Corollary.** *Let X be a smooth projective scheme of finite presentation over a field k. Then the Gromov-Witten invariants $\langle I^X_{g,n,\beta_\eta} \rangle$ are invariant under deformations of X.*

5.3. **Axioms.** Let X be a smooth projective scheme of finite presentation over an algebraically closed field k.

(GW0) Effectivity. Let $A_1({}^X\!/\!{}_k)_+$ be the set of $\beta \in A_1({}^X\!/\!{}_k)$ such that $\beta \cdot c_1(\mathscr{L}) \geq 0$ for every ample line bundle $\mathscr{L}$. Then $\mathrm{I}^X_{g,n,\beta} = 0$, for all $\beta \notin A_1({}^X\!/\!{}_k)_+$.

Proof. If $\overline{\mathcal{M}}_{g,n}({}^X\!/\!{}_k, \beta) \neq \emptyset$ then $\beta = f_*[C]$ for some stable map (C, x_i, f), hence $\beta \in A_1({}^X\!/\!{}_k)_+$. It follows that $\overline{\mathcal{M}}_{g,n}({}^X\!/\!{}_k, \beta) = \emptyset$ for every $\beta \notin A_1({}^X\!/\!{}_k)_+$, and thus $[\overline{\mathcal{M}}_{g,n}({}^X\!/\!{}_k, \beta)]^{\text{virt}} = 0$. □

(GW1) S_n-covariance. For all $\gamma_j \in H^{m_j}(X)$, we have

$$\mathrm{I}^X_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_i \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_n) = (-1)^{m_i m_{i+1}} \mathrm{I}^X_{g,n,\beta}(\gamma_1 \otimes \cdots \otimes \gamma_{i+1} \otimes \gamma_i \otimes \cdots \otimes \gamma_n).$$

Proof. The statement follows from the following ([22] VI.8)

$$\gamma_1 \otimes \cdots \otimes \gamma_i \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_n = (-1)^{m_i m_{i+1}} \gamma_1 \otimes \cdots \otimes \gamma_{i+1} \otimes \gamma_i \otimes \cdots \otimes \gamma_n \in H^*(X^n).$$ □

(GW2) Grading. We have

$$\mathrm{I}^X_{g,n,\beta}\colon \bigotimes_{i=1}^{n} H^{m_i}(X) \to H^{\sum m_i + 2((g-1)\dim_k X - \beta\cdot c_1(T_{X/k}))}(\overline{\mathcal{M}}_{g,\,n/k}).$$

Proof. The virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X/k,\beta)]^{\mathrm{virt}}$ is a cycle class of dimension

$$\mathrm{vdim} = (\dim_k X - 3)(1-g) + n + \beta\cdot c_1(T_{X/k}).$$

Recall that $\overline{\mathcal{M}}_{g,\,n/k}$ is smooth of dimension $3g-3+n$, then, by Poincaré duality,

$$\nu_*[\overline{\mathcal{M}}_{g,n}(X/k,\beta)]^{\mathrm{virt}} \in H^{2(3g-3+n+n\dim_k X - \mathrm{vdim})}(\overline{\mathcal{M}}_{g,\,n/k}\times_k X^n).$$

Finally, if $\underline{\gamma}\in\bigotimes_{i=1}^n H^{m_i}(X)$ then $p^*(\underline{\gamma})\in H^{\sum m_i}(\overline{\mathcal{M}}_{g,\,n/k})$ and therefore

$$\mathrm{I}^X_{g,n,\beta}(\underline{\gamma}) \in H^{\sum m_i + 2((g-1)\dim_k X - \beta\cdot c_1(T_{X/k}))}(\overline{\mathcal{M}}_{g,\,n/k}). \qquad \square$$

5.10. Remark. Notice that $\overline{\mathcal{M}}_{g,\,n/S}$ is smooth since $\mathfrak{M}_{g,\,n/S}$ is and stability condition is open. Thus, using the formal criterion of smoothness (Proposition A.14, Proposition A.22), one can show that the following morphisms are smooth

(1) the natural functor that forgets the last marked point and stabilizes

$$\varphi_n\colon \overline{\mathcal{M}}_{g,\,n+1/S} \to \overline{\mathcal{M}}_{g,\,n/S};$$

(2) the natural functor that identifies the last marked points

$$\varphi\colon \overline{\mathcal{M}}_{g_1,\,n_1+1/S}\times_S \overline{\mathcal{M}}_{g_2,\,n_2+1/S} \to \overline{\mathcal{M}}_{g,\,n/S},$$

with $2g_j+n_j+1\geq 3$ for $j=1,2$, $g=g_1+g_2$, $n=n_1+n_2$;

(3) the natural functor that identifies the last marked points

$$\psi\colon \overline{\mathcal{M}}_{g-1,\,n+2/S} \to \overline{\mathcal{M}}_{g,\,n/S}.$$

Moreover, using the valuative criterion for properness, we can prove that φ_n is proper, since $\overline{\mathcal{M}}_{g,\,n/S}$ is proper.

(GW3) Fundamental class. With notations as in Remark 5.10, we have

$$\mathrm{I}^X_{g,n+1,\beta}(\bullet\otimes \mathrm{id}_X) = \varphi_n^*\circ \mathrm{I}^X_{g,n,\beta}(\bullet),$$

$$\mathrm{I}^X_{0,3,\beta}(\gamma_1\otimes\gamma_2\otimes\mathrm{id}_X) = \begin{cases}\int_X \gamma_1\cup\gamma_2 & \text{if } \beta=0\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Let us form the cartesian diagram

$$\begin{array}{ccccc}
\mathcal{M} & \xrightarrow{j} & \mathcal{N} & \xrightarrow{\widetilde{\varphi}_n} & \overline{\mathcal{M}}_{g,n}(X/k,\beta)\\
\downarrow{\scriptstyle\hat\theta} & & \downarrow{\scriptstyle\widetilde\theta} & & \downarrow{\scriptstyle\theta_n}\\
\mathfrak{M}_{g,\,n+1/k} & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{M}_{g,\,n/k}\\
 & & \downarrow & & \downarrow\\
 & & \overline{\mathcal{M}}_{g,\,n+1/k} & \xrightarrow{\varphi_n} & \overline{\mathcal{M}}_{g,\,n/k}
\end{array}$$

and notice that $\mathcal{M}$ is the algebraic stack of stable maps of genus g and class β with $n+1$ marked points which remain stable if we forget the last marked point. In particular there is a regular embedding $i\colon \mathcal{M} \to \overline{\mathcal{M}}_{g,n+1}({}^{X}\!/_{k}, \beta)$ which commute with θ_{n+1} and $\hat{\theta}$. If we define a virtual fundamental class $[\mathcal{M}]^{\mathrm{virt}}$ relative to $\hat{\theta}$ as described in section 4 then

$$i^{!}[\overline{\mathcal{M}}_{g,n+1}({}^{X}\!/_{k}, \beta)]^{\mathrm{virt}} = [\mathcal{M}]^{\mathrm{virt}}.$$

If we define a virtual fundamental class $[\mathcal{N}]^{\mathrm{virt}}$ relative to $\widetilde{\theta}$ then, by Proposition 3.36,

$$\widetilde{\varphi}_n^{*}[\overline{\mathcal{M}}_{g,n}({}^{X}\!/_{k}, \beta)]^{\mathrm{virt}} = [\mathcal{N}]^{\mathrm{virt}},$$

and, by Proposition B.14, $j_*[\mathcal{M}]^{\mathrm{virt}} = [\mathcal{N}]^{\mathrm{virt}}$. Let us consider the following natural morphisms

$$\begin{array}{ccccc} \mathcal{N} & \xrightarrow{\widetilde{\nu}} & \overline{\mathcal{M}}_{g,\,n+1/k} \times_k X^n & \xrightarrow{\widetilde{p}} & X^n \\ & & \big\downarrow \widetilde{q} & & \\ & & \overline{\mathcal{M}}_{g,\,n+1/k} & & \end{array}$$

and set $\hat{\nu} = \widetilde{\nu} \circ j$. Then the morphisms $\widetilde{p}$ and $\widetilde{q}$ commute with p_n, p_{n+1}, q_n and q_{n+1} via the following maps

$$\overline{\mathcal{M}}_{g,\,n+1/k} \times_k X^{n+1} \xrightarrow{\mathrm{id}\times\pi} \overline{\mathcal{M}}_{g,\,n+1/k} \times_k X^{n} \xrightarrow{\varphi_n\times\mathrm{id}} \overline{\mathcal{M}}_{g,\,n/k} \times_k X^{n}$$

where $\pi\colon X^{n+1} \to X^n$ is the projection on the first n copies of X. We have

$$\begin{aligned} (\varphi_n \times \mathrm{id})^* \nu_{n*}[\overline{\mathcal{M}}_{g,n}({}^{X}\!/_{k}, \beta)]^{\mathrm{virt}} &= \widetilde{\nu}_* \widetilde{\varphi}_n^{*}[\overline{\mathcal{M}}_{g,n}({}^{X}\!/_{k}, \beta)]^{\mathrm{virt}} \\ &= \widetilde{\nu}_* j_* i^{!}[\overline{\mathcal{M}}_{g,n+1}({}^{X}\!/_{k}, \beta)]^{\mathrm{virt}} \\ &= \hat{\nu}_* i^{!}[\overline{\mathcal{M}}_{g,n+1}({}^{X}\!/_{k}, \beta)]^{\mathrm{virt}} \\ &= (\mathrm{id}\times\pi)_* \nu_{n+1*}[\overline{\mathcal{M}}_{g,n+1}({}^{X}\!/_{k}, \beta)]^{\mathrm{virt}}, \end{aligned}$$

hence $(\mathrm{id}\times\pi)_* \mathrm{C}^{X}_{g,n+1,\beta} = (\varphi_n \times \mathrm{id})^* \mathrm{C}^{X}_{g,n,\beta}$. Let $\underline{\gamma} = \gamma_1 \otimes \cdots \otimes \gamma_n$, then

$$\begin{aligned} \varphi_n^* \mathrm{I}^{X}_{g,n,\beta}(\underline{\gamma}) &= \varphi_n^* q_{n*}\left(p_n^*(\underline{\gamma}) \cap \mathrm{C}^{X}_{g,n,\beta}\right) \\ &= \widetilde{q}_*(\varphi_n \times \mathrm{id})^* \left(p_n^*(\underline{\gamma}) \cap \mathrm{C}^{X}_{g,n,\beta}\right) \\ &= \widetilde{q}_* \left(\widetilde{p}^*(\underline{\gamma}) \cap (\mathrm{id}\times\pi)_* \mathrm{C}^{X}_{g,n+1,\beta}\right) \\ &= \widetilde{q}_*(\mathrm{id}\times\pi)_* \left((\mathrm{id}\times\pi)^* \widetilde{p}^*(\underline{\gamma}) \cap \mathrm{C}^{X}_{g,n+1,\beta}\right) \\ &= q_{n+1*}\left(p_{n+1}^*(\underline{\gamma} \otimes \mathrm{id}_X) \cap \mathrm{C}^{X}_{g,n+1,\beta}\right) \\ &= \mathrm{I}^{X}_{g,n+1,\beta}(\underline{\gamma} \otimes \mathrm{id}_X). \end{aligned}$$

Notice that $\overline{\mathcal{M}}_{0,\,3/k} = \operatorname{Spec} k$ and $\overline{\mathcal{M}}_{0,3}({}^{X}\!/_{k}, 0) = X$, hence by Proposition 3.35, we have $[\overline{\mathcal{M}}_{0,3}({}^{X}\!/_{k}, 0)]^{\mathrm{virt}} = [X]$. Then

$$\begin{aligned} \mathrm{I}^{X}_{0,3,0}(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_X) &= q_{3*}(p_3^*(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_X) \cap \nu_{3*}[X]) \\ &= q_{3*}((\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_X) \cap \nu_{3*}[X]) \\ &= \int_X \gamma_1 \cup \gamma_2. \end{aligned}$$

Let assume $\beta \neq 0$. Recall that $H^i(\operatorname{Spec} k) = 0$ for all $i \neq 0$. If $X = \mathbb{P}^r_k$ then $\beta = d \in \mathbb{Z}$, $d > 0$ and hence $\beta \cdot c_1(T_{\mathbb{P}^r_k}) = d(r+1) > r$. Let $\gamma_1 \otimes \gamma_2 \in H^{m_1}(\mathbb{P}^r_k) \otimes H^{m_2}(\mathbb{P}^r_k)$, then

$$\mathrm{I}^{\mathbb{P}^r_k}_{0,3,d}(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_{\mathbb{P}^r_k}) \in H^{m_1+m_2-2r-2d(r+1)}(\operatorname{Spec} k).$$

If $m_1 < 0$ or $m_1 > 2r$, then $H^{m_1}(\mathbb{P}^r_k) = 0$, which implies that $\mathrm{I}^{\mathbb{P}^r_k}_{0,3,d}(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_{\mathbb{P}^r_k}) = 0$. Otherwise, if $0 \le m_1, m_2 \le 2r$ then $m_1 + m_2 - 2r - 2d(r+1) < 0$ and $H^{m_1+m_2-2r-2d(r+1)}(k) = 0$. In general let $i\colon X \to \mathbb{P}^r_k$ be a closed embedding and set $d = i_*\beta$; since X is smooth over k, i is a regular embedding ([12] B.7.2). Form the fiber diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\hat{\imath}} & \overline{\mathcal{M}}_{0,3}(\mathbb{P}^r_k/k, d) \\ {\scriptstyle\hat{\nu}}\downarrow & & \downarrow{\scriptstyle\nu_{\mathbb{P}^r_k}} \\ X^3 & \xrightarrow{i} & {\mathbb{P}^r_k}^3 \end{array}$$

where $\mathcal{M} = \cup_{i_*\beta=d}\overline{\mathcal{M}}_{0,3}(X/k, \beta)$. Let $j\colon \overline{\mathcal{M}}_{0,3}(X/k, \beta) \to \mathcal{M}$ be the natural inclusion, then j is finite and flat. Set $\widetilde{\imath} = \hat{\imath} \circ j$, then

$$j^*\hat{\imath}^![\overline{\mathcal{M}}_{0,3}(\mathbb{P}^r_k/k, d)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{0,3}(X/k, \beta)]^{\mathrm{virt}}.$$

It follows that

$$\nu_*[\overline{\mathcal{M}}_{0,3}(X/k, \beta)]^{\mathrm{virt}} = \nu_* j^*\hat{\imath}^![\overline{\mathcal{M}}_{0,3}(\mathbb{P}^r_k/k, d)]^{\mathrm{virt}} = \hat{\nu}_*\hat{\imath}^![\overline{\mathcal{M}}_{0,3}(\mathbb{P}^r_k/k, d)]^{\mathrm{virt}} = i^!\nu_{\mathbb{P}^r_k *}[\overline{\mathcal{M}}_{0,3}(\mathbb{P}^r_k/k, d)]^{\mathrm{virt}}.$$

Finally, we have

$$\begin{aligned} \mathrm{I}^X_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_X) &= q_*(p^*(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_X) \cap \nu_*[\overline{\mathcal{M}}_{0,3}(X/k, \beta)]^{\mathrm{virt}}) \\ &= q_{\mathbb{P}^r_k *} i_*(p^*(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_X) \cap i^!\nu_{\mathbb{P}^r_k *}[\overline{\mathcal{M}}_{0,3}(\mathbb{P}^r_k/k, d)]^{\mathrm{virt}}) \\ &= q_{\mathbb{P}^r_k *}(i_*p^*(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_X) \cap \nu_{\mathbb{P}^r_k *}[\overline{\mathcal{M}}_{0,3}(\mathbb{P}^r_k/k, d)]^{\mathrm{virt}}) \\ &= q_{\mathbb{P}^r_k *}(p^*_{\mathbb{P}^r_k} i_*(\gamma_1 \otimes \gamma_2 \otimes \mathrm{id}_X) \cap \nu_{\mathbb{P}^r_k *}[\overline{\mathcal{M}}_{0,3}(\mathbb{P}^r_k/k, d)]^{\mathrm{virt}}) \\ &= \mathrm{I}^{\mathbb{P}^r_k}_{0,3,d}(i_*\gamma_1 \otimes i_*\gamma_2 \otimes \mathrm{id}_{\mathbb{P}^r_k}) = 0. \end{aligned}$$

□

(GW4) Divisor. With notations as in Remark 5.10, we have, for all $\gamma \in H^2(X)$,

$$\varphi_{n*}\mathrm{I}^X_{g,n+1,\beta}(\bullet \otimes \gamma) = (\beta \cdot \gamma)\mathrm{I}^X_{g,n,\beta}(\bullet).$$

Proof. Consider the functor

$$\overline{\varphi}_n\colon \overline{\mathcal{M}}_{g,n+1}(X/k, \beta) \to \overline{\mathcal{M}}_{g,n}(X/k, \beta)$$

which forgets the last marked point and stabilizes. We have the following commutative diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}}_{g,n+1}(X/k, \beta) & \xrightarrow{\nu_{n+1}} & \overline{\mathcal{M}}_{g,\,n+1/k} \times_k X^{n+1} & \xrightarrow{q_{n+1}} & \overline{\mathcal{M}}_{g,\,n+1/k} \\ \downarrow{\scriptstyle\widetilde{\varphi}_n} & & \downarrow{\scriptstyle\varphi_n \times \mathrm{id}} & & \downarrow{\scriptstyle\varphi_n} \\ \overline{\mathcal{M}}_{g,n}(X/k, \beta) \times_k X & \xrightarrow{\nu_n \times \mathrm{id}} & \overline{\mathcal{M}}_{g,\,n/k} \times_k X^{n+1} & \xrightarrow{\widetilde{q}_n} & \overline{\mathcal{M}}_{g,\,n/k} \end{array}$$

where $\widetilde{\varphi}_n = \overline{\varphi}_n \times \mathrm{ev}_{n+1}$. By the Künneth formula ([22] VI.8), we can write

$$\widetilde{\varphi_n}_*[\overline{\mathcal{M}}_{g,n+1}(X/k, \beta)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{g,n}(X/k, \beta)]^{\mathrm{virt}} \otimes \beta' + \alpha,$$

where $\beta' \in H^{2n-2}(X)$ and $\alpha \in H^m(\overline{\mathcal{M}}_{g,n}(X/k, \beta)) \otimes H^l(X)$, with m less than 2vdim. The class β' can be calculated by restricting to what happens over a generic point of $\overline{\mathcal{M}}_{g,n}(X/k, \beta)$. Representing such a point by $\xi = (C, x_1, \dots, x_n, f)$, we have the cartesian diagram

$$\begin{array}{ccccc} C & \xrightarrow{\quad f \quad} & \xi \times_k X & \longrightarrow & \xi \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle i} \\ \overline{\mathcal{M}}_{g,n+1}(X/k, \beta) & \xrightarrow{\widetilde{\varphi}_n} & \overline{\mathcal{M}}_{g,n}(X/k, \beta) \times_k X & \longrightarrow & \overline{\mathcal{M}}_{g,n}(X/k, \beta) \end{array}$$

where, for ξ generic, the map i is a regular embedding, hence

$$i^! \widetilde{\varphi}_{n*} [\overline{\mathcal{M}}_{g,n+1}(X/k, \beta)]^{\mathrm{virt}} = f_* i^! [\overline{\mathcal{M}}_{g,n+1}(X/k, \beta)]^{\mathrm{virt}} = f_*[C] = \beta,$$

on the other hand

$$i^! \widetilde{\varphi}_{n*} [\overline{\mathcal{M}}_{g,n+1}(X/k, \beta)]^{\mathrm{virt}} = i^! \left([\overline{\mathcal{M}}_{g,n}(X/k, \beta)]^{\mathrm{virt}} \otimes \beta' + \alpha \right) = \beta'.$$

It follows that $\beta' = \beta$ and

$$\begin{aligned} (\varphi_n \times \mathrm{id})_* \nu_{n+1*} [\overline{\mathcal{M}}_{g,n+1}(X/k, \beta)]^{\mathrm{virt}} &= (\nu_n \times \mathrm{id})_* \widetilde{\varphi}_{n*} [\overline{\mathcal{M}}_{g,n+1}(X/k, \beta)]^{\mathrm{virt}} \\ &= \nu_{n*} [\overline{\mathcal{M}}_{g,n}(X/k, \beta)]^{\mathrm{virt}} \otimes \beta + \alpha', \end{aligned}$$

and hence $(\varphi_n \times \mathrm{id})_* \mathrm{C}^X_{g,n+1,\beta} = \mathrm{C}^X_{g,n,\beta} \otimes \beta + \alpha'$. Let $\underline{\gamma} = \gamma_1 \otimes \cdots \otimes \gamma_n$, then

$$\begin{aligned} \varphi_{n*} \mathrm{I}^X_{g,n+1,\beta}(\underline{\gamma} \otimes \gamma) &= \varphi_{n*} q_{n+1*} \left(p^*_{n+1}(\underline{\gamma} \otimes \gamma) \cap \mathrm{C}^X_{g,n+1,\beta} \right) \\ &= \widetilde{q}_{n*} \left((p_n \times \mathrm{id})^* (\underline{\gamma} \otimes \gamma) \cap (\varphi_n \times \mathrm{id})_* \mathrm{C}^X_{g,n+1,\beta} \right) \\ &= \widetilde{q}_{n*} \left((p^*_n(\underline{\gamma}) \otimes \gamma) \cap \left(\mathrm{C}^X_{g,n,\beta} \otimes \beta + \alpha' \right) \right) \\ &= (\beta \cdot \gamma) q_{n*} \left(p^*_n(\underline{\gamma}) \cap \mathrm{C}^X_{g,n,\beta} \right) \\ &= (\beta \cdot \gamma) \mathrm{I}^X_{g,n,\beta}(\underline{\gamma}). \end{aligned}$$

□

(GW5) Mapping to point. Let $\beta = 0$, then $\overline{\mathcal{M}}_{g,n}(X/k, 0) = \overline{\mathcal{M}}_{g,\,n/k} \times_k X$. Let us consider the universal stable map

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ f\ } & X \\ \uparrow\downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{g,n}(X/k, 0) & & \end{array}$$

and notice that $E = R^1\pi_*(\psi^* T_{X/k})$ is a vector bundle of rank $g \dim X$. Then

$$\mathrm{I}^X_{g,n,0}(\gamma_1 \otimes \cdots \otimes \gamma_n) = \hat{q}_* \left(\hat{p}^*(\gamma_1 \cup \cdots \cup \gamma_n) \cap \left([\overline{\mathcal{M}}_{g,n}(X/k, 0)] \cdot c_{\mathrm{top}}(E) \right) \right),$$

where $\hat{p}\colon \overline{\mathcal{M}}_{g,\,n/k} \times_k X \to X$ and $\hat{q}\colon \overline{\mathcal{M}}_{g,\,n/k} \times_k X \to \overline{\mathcal{M}}_{g,\,n/k}$ are the projections.

Proof. Recall that $E^\bullet = R\pi_*(\psi^* \Omega_{X/k} \otimes \omega_\pi)[-1]$. By Poincaré duality,

$$h^{-1}(E^\bullet) = h^0(R\pi_*(\psi^* \Omega_{X/k} \otimes \omega_\pi)) = h^1(R\pi_*(\psi^* T_{X/k})) = R^1\pi_*(\psi^* T_{X/k}).$$

In this case θ is smooth, since it is the composition $\overline{\mathcal{M}}_{g,\,n/k} \times_k X \to \overline{\mathcal{M}}_{g,\,n/k} \to \mathfrak{M}_{g,\,n/k}$, where the first arrow is the projection, which is smooth because X is smooth, and the second arrow is the natural inclusion, which is an open immersion because stability condition is open. Hence, by Proposition 3.35,

$$[\overline{\mathcal{M}}_{g,n}(X/k, 0)]^{\mathrm{virt}} = c_{\mathrm{top}}(R^1\pi_*(f^* T_{X/k})) \cdot [\overline{\mathcal{M}}_{g,n}(X/k, 0)].$$

Notice that $q \circ \nu = \hat{q}$, therefore $q_* \nu_* = \hat{q}_*$; we have also the following cartesian diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,\, n/k} \times_k X & \xrightarrow{\nu} & \overline{\mathcal{M}}_{g,\, n/k} \times_k X^n \\ \downarrow{\scriptstyle \hat{p}} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\hat{\nu}} & X^n \end{array}$$

where $\hat{\nu} = (\mathrm{id}_X)^n$. Then

$$\begin{aligned} \mathrm{I}^X_{g,n,0}(\gamma_1 \otimes \cdots \otimes \gamma_n) &= q_* \left(p^*(\gamma_1 \otimes \cdots \otimes \gamma_n) \cap \nu_* [\overline{\mathcal{M}}_{g,n}(X/k, 0)]^{\mathrm{virt}} \right) \\ &= q_* \nu_* \left(\nu^* p^*(\gamma_1 \otimes \cdots \otimes \gamma_n) \cap [\overline{\mathcal{M}}_{g,n}(X/k, 0)]^{\mathrm{virt}} \right) \\ &= \hat{q}_* \left(\hat{p}^*(\gamma_1 \cup \cdots \cup \gamma_n) \cap \left([\overline{\mathcal{M}}_{g,n}(X/k, 0)] \cdot c_{\mathrm{top}}(E) \right) \right). \end{aligned}$$

□

(GW6) Splitting. Let $g_1, g_2, n_1, n_2 \geq 0$ be integers such that $2g_i + n_i + 1 \geq 3$ and set $g = g_1 + g_2$, $n = n_1 + n_2$. With notations as in Remark 5.10, let $\underline{\gamma} = \gamma_1 \otimes \cdots \otimes \gamma_n$, then

$$\varphi^* \circ \mathrm{I}^X_{g,n,\beta}(\underline{\gamma}) = \sum_{\beta_1 + \beta_2 = \beta} \mathrm{I}^X_{g_1, n_1+1, \beta_1} \otimes \mathrm{I}^X_{g_2, n_2+1, \beta_2} (\underline{\gamma}^{(1,n_1)} \otimes [\Delta] \otimes \underline{\gamma}^{(n_1+1,n)}),$$

where Δ is the diagonal in X^2 and $\underline{\gamma}^{(i,j)} = \gamma_i \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_j$.

5.11. **Costello's decorated pre-stable curves.** Before proving Axiom (GW6), we need to recall Costello's construction of $\mathfrak{M}_{g,\, n,\, \beta/k}$. Roughly speaking, $\mathfrak{M}_{g,\, n,\, \beta/k}$ is a smooth Artin stack over an algebraically closed field of characteristic zero, which parametrizes pre-stable curves C of genus g with n marked points, together with a labelling of each irreducible component of C by an element of a semigroup $\mathcal{A}$, such that the sum over irreducible components of the associated elements of $\mathcal{A}$ must be $\beta \in \mathcal{A}$. In the following we extend Costello's construction over an arbitrary base scheme S.

Let $\mathcal{A}$ be a commutative semigroup, with unit $0 \in \mathcal{A}$, such that

(1) $\mathcal{A}$ has indecomposable zero: $\beta + \beta' = 0$ implies $\beta = \beta' = 0$, for all $\beta, \beta' \in \mathcal{A}$;
(2) $\mathcal{A}$ has finite decomposition: for every $\beta \in \mathcal{A}$, the set of $\beta_1, \beta_2 \in \mathcal{A}$ such that $\beta_1 + \beta_2 = \beta$ is finite.

Let us fix $\beta \in \mathcal{A}$.

5.12. **Definition.** Given integers g, $n \geq 0$, we say that the triple (g, n, β) is *stable* if either $\beta \neq 0$ or $\beta = 0$ and $2g - 3 + n \geq 0$.

5.13. **Definition.** The category fibered in groupoids $\mathfrak{M}_{g,\, n,\, \beta_{\eta}/S}$ is defined inductively as follows.

(1) If (g, n, β) is unstable, then $\mathfrak{M}_{g,\, n,\, \beta_{\eta}/S}$ is empty.
(2) Suppose (g, n, β) is stable. Let T be an S-scheme, then an object of $\mathfrak{M}_{g,\, n,\, \beta_{\eta}/S}(T)$ is a pre-stable curve $(C \to T, t_i)$ of genus g with n marked points together with a constructible function $\lambda\colon C_{gen} \to \mathcal{A}$, where $C_{gen} \to T$ is the complement of the nodes and the marked points of C, such that λ is locally constant on the geometric fibers of $C_{gen} \to T$.
(3) If $T_0 \subset T$ is the open subscheme parametrizing non-singular curves and $C_0 = C \times_T T_0$, then $\lambda\colon C_{0gen} \to \mathcal{A}$ must be constant with value β.
(4) Let $g_1, g_2, n_1, n_2 \geq 0$ be such that $g = g_1 + g_2$ and $n = n_1 + n_2$. Let T and T' be S-schemes and let $T' \xrightarrow{h} T$ be a morphism of S-schemes. Given an object $(C \to T, t_i, \lambda) \in \mathfrak{M}_{g,\, n,\, \beta_{\eta}/S}(T)$, we denote by $(C' \to T', t'_i)$ the pre-stable curve obtained by pulling-back the curve $(C \to T, t_i)$ via h. Assume that there exist two pre-stable curves $(C_j \to T', t'_{j,i}) \in \mathfrak{M}_{g_j,\, n_j+1/S}(T')$, for $j = 1, 2$, such that C' is obtained by C_1 and C_2 identifying the last marked points. We

require that the induced constructible functions $\lambda_j\colon C_{j\,gen} \to \mathcal{A}$, for $j = 1, 2$, define a morphism

$$T' \to \coprod_{\beta_1+\beta_2=\beta} \mathfrak{M}_{g_1,\, n_1\,+\,1,\, \beta_1/S} \times_S \mathfrak{M}_{g_2,\, n_2\,+\,1,\, \beta_2/S}.$$

(5) Let T and T' be S-schemes and let $T' \xrightarrow{h} T$ be a morphism of S-schemes. Given an object $(C \to T, t_i, \lambda) \in \mathfrak{M}_{g,\, n,\, \beta_{\eta}/S}(T)$, we denote by $(C' \to T', t'_i)$ the pre-stable curve obtained by pulling-back $(C \to T, t_i)$ via h. Assume that there exists a pre-stable curve $(C'' \to T', t''_i) \in \mathfrak{M}_{g\,-\,1,\, n\,+\,2/S}(T')$ such that C' is obtained by C'' identifying the last two marked points. We require that the induced constructible function $\lambda'\colon C''_{gen} \to \mathcal{A}$ defines a morphism

$$T' \to \mathfrak{M}_{g\,-\,1,\, n\,+\,2,\, \beta/S}.$$

One can show directly that the category $\mathfrak{M}_{g,\, n,\, \beta_{\eta}/S}$ is a stack over the base scheme S.

5.14. **Proposition.** *The natural functor $\mathfrak{M}_{g,\, n,\, \beta_{\eta}/S} \to \mathfrak{M}_{g,\, n/S}$ which forgets the labelling with values in $\mathcal{A}$ is étale. In particular $\mathfrak{M}_{g,\, n,\, \beta_{\eta}/S}$ is a smooth algebraic stack over S.*

Proof. We use the formal criterion for étaleness ([22] I.3.22 and Proposition A.14). Let $\overline{x}\colon \operatorname{Spec} \overline{k} \to \mathfrak{M}_{g,\, n,\, \beta_{\eta}/S}$ be a geometric point and let

$$0 \to I \to A' \to A \to 0$$

be a square-zero extension in $({}^{\mathrm{Art}}/_{\hat{\mathcal{O}}_{S,\overline{x}}})$. Given a decorated pre-stable curve $(C \to \operatorname{Spec} A, a_i, \lambda)$ and a pre-stable curve $(C' \to \operatorname{Spec} A', a'_i)$ such that the diagram

$$\begin{array}{ccc} C & \longrightarrow & C' \\ {\scriptstyle a_i}\uparrow\downarrow & & {\scriptstyle a'_i}\uparrow\downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A' \end{array}$$

is cartesian, we want to show that there exists a unique constructible function $\lambda'\colon C'_{gen} \to \mathcal{A}$ such that λ factors through λ' and $(C' \to \operatorname{Spec} A', a'_i, \lambda')$ is a decorated pre-stable curve. Let us notice that C and C' has the same underlying topological space and that the labelling with values in $\mathcal{A}$ depends only on the topology of the family of pre-stable curves, hence there exists a unique function λ' with the required properties. □

Proof of Axiom (GW6). Let us notice that $A_1({}^X/_k)_+$ is a commutative semigroup then, by effectivity, the sum is finite. Denote for simplicity

$$\overline{\mathcal{M}}^{(\beta_1,\beta_2)} = \overline{\mathcal{M}}_{g_1,n_1+1}({}^X/_k, \beta_1) \times_k \overline{\mathcal{M}}_{g_2,n_2+1}({}^X/_k, \beta_2).$$

Let us form the fiber diagram

$$\begin{array}{ccc} \mathcal{N}^{(\beta_1,\beta_2)} & \xrightarrow{\hat{\Delta}} & \overline{\mathcal{M}}^{(\beta_1,\beta_2)} \\ \downarrow & & \downarrow {\scriptstyle \mathrm{ev}_{n_1+1}\times \mathrm{ev}_{n_2+1}} \\ X & \xrightarrow{\Delta} & X \times_k X \end{array}$$

where Δ is the diagonal. The stack $\mathcal{N}^{(\beta_1,\beta_2)}$ is the moduli stack of pairs

$$((C_1, x_i, f_1), (C_2, y_i, f_2)) \in \overline{\mathcal{M}}^{(\beta_1,\beta_2)}$$

such that $f_1(x_{n_1+1}) = f_2(y_{n_2+1})$.

Let us notice that the morphism $\theta\colon \overline{\mathcal{M}}_{g,n}(X/k,\beta) \to \mathfrak{M}_{g,\,n/k}$ factors through $\mathfrak{M}_{g,\,n,\,\beta/k}$. Moreover, by Proposition 5.14, the natural forgetful map $\mathfrak{M}_{g,\,n,\,\beta/k} \to \mathfrak{M}_{g,\,n/k}$ is étale. Therefore we can construct a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X/k,\beta)]_\beta^{\mathrm{virt}}$ relative to the morphism

$$\theta^\beta\colon \overline{\mathcal{M}}_{g,n}(X/k,\beta) \to \mathfrak{M}_{g,\,n,\,\beta/k}$$

(as described in section 3) and, by Theorem 3.19, we get $[\overline{\mathcal{M}}_{g,n}(X/k,\beta)]_\beta^{\mathrm{virt}} = [\overline{\mathcal{M}}_{g,n}(X/k,\beta)]^{\mathrm{virt}}$. Notice that the composition

$$\mathcal{N}^{(\beta_1,\beta_2)} \xrightarrow{\hat{\Delta}} \overline{\mathcal{M}}^{(\beta_1,\beta_2)} \to \mathfrak{M}_{g_1,\,n_1\,+\,1,\,\beta_1/k} \times_k \mathfrak{M}_{g_2,\,n_2\,+\,1,\,\beta_2/k}$$

is the natural forgetful functor that we denote by $\theta^{(\beta_1,\beta_2)}$. Moreover we can define a virtual fundamental class $[\mathcal{N}^{(\beta_1,\beta_2)}]^{\mathrm{virt}}$ relative to the morphism $\theta^{(\beta_1,\beta_2)}$, as described in section 4 and, since Δ is a regular embedding, we get

$$\Delta^![\overline{\mathcal{M}}^{(\beta_1,\beta_2)}]^{\mathrm{virt}} = [\mathcal{N}^{(\beta_1,\beta_2)}]^{\mathrm{virt}}.$$

Let us consider the commutative diagram

$$\begin{array}{ccc}
\displaystyle\coprod_{\beta_1+\beta_2=\beta} \mathcal{N}^{(\beta_1,\beta_2)} & \xrightarrow{\quad\hat{\varphi}\quad} & \overline{\mathcal{M}}_{g,n}(X/k,\beta) \\
\big\downarrow \theta^{(\beta_1,\beta_2)} & & \big\downarrow \theta^\beta \\
\displaystyle\coprod_{\beta_1+\beta_2=\beta} \mathfrak{M}_{g_1,\,n_1\,+\,1,\,\beta_1/k} \times_k \mathfrak{M}_{g_2,\,n_2\,+\,1,\,\beta_2/k} & \xrightarrow{\varphi} & \mathfrak{M}_{g,\,n,\,\beta/k}
\end{array}$$

and note that it is actually cartesian. By Proposition 3.36,

$$\hat{\varphi}^*[\overline{\mathcal{M}}_{g,n}(X/k,\beta)]^{\mathrm{virt}} = \sum_{\beta_1+\beta_2=\beta} [\mathcal{N}^{(\beta_1,\beta_2)}]^{\mathrm{virt}}.$$

Let $\overline{\mathcal{M}}^{(1,2)} = \overline{\mathcal{M}}_{g_1,\,n_1\,+\,1/k} \times_k \overline{\mathcal{M}}_{g_2,\,n_2\,+\,1/k}$ and consider the following natural morphisms

$$\begin{array}{ccccccc}
\mathcal{N}^{(\beta_1,\beta_2)} & \xrightarrow{\hat{\nu}} & \overline{\mathcal{M}}^{(1,2)} \times_k X^{n+1} & \xrightarrow{\mathrm{id}\times\pi} & \overline{\mathcal{M}}^{(1,2)} \times_k X^n & \xrightarrow{\widetilde{q}} & \overline{\mathcal{M}}^{(1,2)} \\
 & & \big\downarrow \hat{p} & & \big\downarrow \widetilde{p} & & \\
 & & X^{n+1} & \xrightarrow{\quad\pi\quad} & X^n & &
\end{array}$$

where $\hat{p} \circ \hat{\nu}$ is the evaluation at the first $n+1$ marked points and π is the projection on the first n components, and set $\widetilde{\nu} = (\mathrm{id} \times \pi) \circ \hat{\nu}$, $\hat{q} = \widetilde{q} \circ (\mathrm{id} \circ \pi)$. It is easily seen that the morphisms $\hat{p}$, $\widetilde{p}$, $\hat{q}$ and $\widetilde{q}$ commute with p_n, q_n, $p_{n_1+1} \times p_{n_2+1}$ and $q_{n_1+1} \times q_{n_2+1}$ via the following maps

$$\overline{\mathcal{M}}^{(1,2)} \times_k X^n \xrightarrow{\varphi\times\mathrm{id}} \overline{\mathcal{M}}_{g,\,n/k} \times_k X^n$$
$$\overline{\mathcal{M}}^{(1,2)} \times_k X^{n+1} \xrightarrow{\mathrm{id}\times\Delta} \overline{\mathcal{M}}^{(1,2)} \times_k X^{n+2}.$$

Let us denote for simplicity

$$\begin{cases}
\nu_{1,2} = \nu_{n_1+1} \times \nu_{n_2+1} \\
p_{1,2} = p_{n_1+1} \times p_{n_2+1} \\
q_{1,2} = q_{n_1+1} \times q_{n_2+1} \\
\mathrm{C}^X_{\beta_1,\beta_2} = \mathrm{C}^X_{g_1,n_1+1,\beta_1} \times \mathrm{C}^X_{g_2,n_2+1,\beta_2}.
\end{cases}$$

We have

$$
\begin{aligned}
(\varphi\times\mathrm{id})^*\nu_{n*}[\overline{\mathcal{M}}_{g,n}(X/k,\beta)]^{\mathrm{virt}} &= \hat{\nu}_*\hat{\varphi}^*[\overline{\mathcal{M}}_{g,n}(X/k,\beta)]^{\mathrm{virt}}\\
&= \sum_{\beta_1+\beta_2=\beta}(\mathrm{id}\times\pi)_*\widetilde{\nu}_*\Delta^![\overline{\mathcal{M}}_{(\beta_1,\beta_2)}]^{\mathrm{virt}}\\
&= \sum_{\beta_1+\beta_2=\beta}(\mathrm{id}\times\pi)_*(\mathrm{id}\times\Delta)^!\nu_{1,2_*}[\overline{\mathcal{M}}_{(\beta_1,\beta_2)}]^{\mathrm{virt}},
\end{aligned}
$$

hence $(\varphi\times\mathrm{id})^*\mathrm{C}^X_{g,n,\beta}=\sum_{\beta_1+\beta_2=\beta}(\mathrm{id}\times\pi)_*(\mathrm{id}\times\Delta)^!\mathrm{C}^X_{\beta_1,\beta_2}$. Let $\underline{\gamma}=\gamma_1\otimes\cdots\otimes\gamma_n$, then

$$
\begin{aligned}
\varphi^*\mathrm{I}^X_{g,n,\beta}(\underline{\gamma}) &= \varphi^*q_{n*}\left({p_n}^*(\underline{\gamma})\cap\mathrm{C}^X_{g,n,\beta}\right)\\
&= \hat{q}_*(\varphi\times\mathrm{id})^*\left({p_n}^*(\underline{\gamma})\cap\mathrm{C}^X_{g,n,\beta}\right)\\
&= \sum_{\beta_1+\beta_2=\beta}\hat{q}_*\left(\hat{p}^*(\underline{\gamma})\cap(\mathrm{id}\times\pi)_*(\mathrm{id}\times\Delta)^!\mathrm{C}^X_{\beta_1,\beta_2}\right)\\
&= \sum_{\beta_1+\beta_2=\beta}\widetilde{q}_*\left(\widetilde{p}^*\pi^*(\underline{\gamma})\cap(\mathrm{id}\times\Delta)^!\mathrm{C}^X_{\beta_1,\beta_2}\right)\\
&= \sum_{\beta_1+\beta_2=\beta}q_{1,2_*}(\mathrm{id}\times\Delta)_*\left(\widetilde{p}^*(\underline{\gamma}\otimes\mathrm{id})\cap(\mathrm{id}\times\Delta)^!\mathrm{C}^X_{\beta_1,\beta_2}\right)\\
&= \sum_{\beta_1+\beta_2=\beta}q_{1,2_*}\left((\mathrm{id}\times\Delta)_*\widetilde{p}^*(\underline{\gamma}\otimes\mathrm{id})\cap\mathrm{C}^X_{\beta_1,\beta_2}\right)\\
&= \sum_{\beta_1+\beta_2=\beta}q_{1,2_*}\left({p_{1,2}}^*(\underline{\gamma}^{(n_1)}\otimes\Delta\otimes\underline{\gamma}^{(n_2)})\cap\mathrm{C}^X_{\beta_1,\beta_2}\right)\\
&= \sum_{\beta_1+\beta_2=\beta}\mathrm{I}^X_{g_1,n_1+1,\beta_1}\otimes\mathrm{I}^X_{g_2,n_2+1,\beta_2}(\underline{\gamma}^{(n_1)}\otimes[\Delta]\otimes\underline{\gamma}^{(n_2)}).
\end{aligned}
$$

□

(GW7) Genus reduction. With notations as in Remark 5.10, we have

$$
\psi^*\circ\mathrm{I}^X_{g,n,\beta}(\bullet)=\mathrm{I}^X_{g-1,n+2,\beta}(\bullet\otimes[\Delta]),
$$

where Δ is the diagonal in X^2.

Proof. Let us form the fiber diagram

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\hat{\Delta}} & \overline{\mathcal{M}}_{g-1,n+2}(X/k,\beta)\\
\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_{n+1}\times\mathrm{ev}_{n+2}}\\
X & \xrightarrow{\Delta} & X\times_k X
\end{array}
$$

where Δ is the diagonal. Notice that $\mathcal{N}$ is the moduli stack of stable maps of genus $g-1$ with $n+2$ marked points such that the images in X of the last two marked points coincide. Thus $\mathcal{N}$ fits in the following cartesian diagram

$$\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\hat{\psi}} & \overline{\mathcal{M}}_{g,n}({}^{X}/_{k},\beta) \\
{\scriptstyle\hat{\theta}}\big\downarrow & & \big\downarrow{\scriptstyle\theta} \\
\mathfrak{M}_{g\,-\,1,\,n\,+\,2/k} & \longrightarrow & \mathfrak{M}_{g,\,n/k} \\
{\scriptstyle \mathrm{st}}\big\downarrow & & \big\downarrow \\
\overline{\mathcal{M}}_{g\,-\,1,\,n\,+\,2/k} & \xrightarrow{\psi} & \overline{\mathcal{M}}_{g,\,n/k}
\end{array}$$

and the composition

$$\mathcal{N} \xrightarrow{\hat{\Delta}} \overline{\mathcal{M}}_{g-1,n+2}({}^{X}/_{k},\beta) \to \mathfrak{M}_{g\,-\,1,\,n\,+\,2/k}$$

is the functor $\hat{\theta}$. We can define a virtual fundamental class $[\mathcal{N}]^{\mathrm{virt}}$ relative to the morphism $\hat{\theta}$, as described in section 4. Since Δ is a regular embedding and by Proposition 3.36, we get

$$\Delta^{!}[\overline{\mathcal{M}}_{g-1,n+2}({}^{X}/_{k},\beta)]^{\mathrm{virt}} = [\mathcal{N}]^{\mathrm{virt}} = \hat{\psi}^{*}[\overline{\mathcal{M}}_{g,n}({}^{X}/_{k},\beta)]^{\mathrm{virt}}.$$

Consider the following natural morphisms

$$\begin{array}{ccccccc}
\mathcal{N} & \xrightarrow{\hat{\nu}} & \overline{\mathcal{M}}_{g\,-\,1,\,n\,+\,2/k} \times_k X^{n+1} & \xrightarrow{\mathrm{id}\times\pi} & \overline{\mathcal{M}}_{g\,-\,1,\,n\,+\,2/k} \times_k X^{n} & \xrightarrow{\widetilde{q}} & \overline{\mathcal{M}}_{g\,-\,1,\,n\,+\,2/k} \\
& & {\scriptstyle\hat{p}}\big\downarrow & & \big\downarrow{\scriptstyle\widetilde{p}} & & \\
& & X^{n+1} & \xrightarrow{\pi} & X^{n} & &
\end{array}$$

where $\hat{p}\circ\hat{\nu}$ is the evaluation at the first $n+1$ marked points and π is the projection on the first n components, and set $\widetilde{\nu} = (\mathrm{id}\times\pi)\circ\hat{\nu}$, $\hat{q} = \widetilde{q}\circ(\mathrm{id}\circ\pi)$. It is easily seen that the morphisms $\hat{p}$, $\widetilde{p}$, $\hat{q}$ and $\widetilde{q}$ commute with p_n, q_n, p_{n+2} and q_{n+2} via the following maps

$$\overline{\mathcal{M}}_{g\,-\,1,\,n\,+\,2/k} \times_k X^{n} \xrightarrow{\psi\times\mathrm{id}} \overline{\mathcal{M}}_{g,\,n/k} \times_k X^{n}$$
$$\overline{\mathcal{M}}_{g\,-\,1,\,n\,+\,2/k} \times_k X^{n+1} \xrightarrow{\mathrm{id}\times\Delta} \overline{\mathcal{M}}_{g\,-\,1,\,n\,+\,2/k} \times_k X^{n+2}.$$

Moreover, we have

$$\begin{aligned}
(\psi\times\mathrm{id})^{*}\nu_{n*}[\overline{\mathcal{M}}_{g,n}({}^{X}/_{k},\beta)]^{\mathrm{virt}} &= \widetilde{\nu}_{*}\hat{\psi}^{*}[\overline{\mathcal{M}}_{g,n}({}^{X}/_{k},\beta)]^{\mathrm{virt}} \\
&= (\mathrm{id}\times\pi)_{*}\hat{\nu}_{*}\hat{\Delta}^{!}[\overline{\mathcal{M}}_{g-1,n+2}({}^{X}/_{k},\beta)]^{\mathrm{virt}} \\
&= (\mathrm{id}\times\pi)_{*}(\mathrm{id}\times\Delta)^{!}\nu_{n+2*}[\overline{\mathcal{M}}_{g-1,n+2}({}^{X}/_{k},\beta)]^{\mathrm{virt}},
\end{aligned}$$

hence $(\psi\times\mathrm{id})^{*}\mathrm{C}^{X}_{g,n,\beta} = (\mathrm{id}\times\pi)_{*}(\mathrm{id}\times\Delta)^{!}\mathrm{C}^{X}_{g-1,n+2,\beta}$. Let $\underline{\gamma} = \gamma_1\otimes\cdots\otimes\gamma_n$, then

$$\begin{aligned}
\psi^{*}\mathrm{I}^{X}_{g,n,\beta}(\underline{\gamma}) &= \psi^{*}q_{n*}\left(p_n{}^{*}(\underline{\gamma})\cap\mathrm{C}^{X}_{g,n,\beta}\right) \\
&= \widetilde{q}_{*}(\psi\times\mathrm{id})^{*}\left(p_n{}^{*}(\underline{\gamma})\cap\mathrm{C}^{X}_{g,n,\beta}\right) \\
&= \widetilde{q}_{*}\left(\widetilde{p}^{*}(\underline{\gamma})\cap(\mathrm{id}\times\pi)_{*}(\mathrm{id}\times\Delta)^{!}\mathrm{C}^{X}_{g-1,n+2,\beta}\right) \\
&= \hat{q}_{*}\left(\hat{p}^{*}(\mathrm{id}\times\pi)^{*}(\underline{\gamma})\cap(\mathrm{id}\times\Delta)^{!}\mathrm{C}^{X}_{g-1,n+2,\beta}\right) \\
&= q_{n+2*}\left((\mathrm{id}\times\Delta)_{*}\hat{p}^{*}(\underline{\gamma}\otimes\mathrm{id})\cap\mathrm{C}^{X}_{g-1,n+2,\beta}\right) \\
&= q_{n+2*}\left(p_{n+2}{}^{*}(\underline{\gamma}\otimes[\Delta])\cap\mathrm{C}^{X}_{g-1,n+2,\beta}\right) \\
&= \mathrm{I}^{X}_{g-1,n+2,\beta}(\underline{\gamma}\otimes[\Delta]).
\end{aligned}$$

□

(GW8) Motivic axiom. There exists a class $\mathrm{C}^X_{g,n,\beta} \in A^*(\overline{\mathcal{M}}_{g,\,n/k} \times_k X^n/k)_{\mathbb{Q}}$ such that

$$\mathrm{I}^X_{g,n,\beta}(\bullet) = q_*(p^*(\bullet) \cap \mathrm{C}^X_{g,n,\beta}).$$

6. Genus zero invariants in positive characteristic

6.1. **Gromov-Witten potential.** Let k be an algebraically closed field (of arbitrary characteristic) and let X be a smooth projective connected scheme of finite type over k. Fix $\beta \in A_1(X/k)$ and $n \geq 0$. Let l be a prime different from the characteristic of k.

6.1. By [22] V.1.11, $H^*(X) = \sum_r H^r(X, \mathbb{Q}_l(\overline{r}))$ is finitely generated over $\mathbb{Q}_l$. Let $T_0 = 1, T_1, \ldots, T_m$ be generators for $H^*(X)$. For each $i = 1, \ldots, m$, we introduce a variable t_i of the same degree of T_i, such that the t_i supercommute, which means

$$t_i t_j = (-1)^{\deg t_i \deg t_j} t_j t_i,$$

and $t_i^2 = 0$ if t_i has odd degree.

6.2. Remark. If $\gamma_i \in H^{m_i}(X)$ then $\langle \mathrm{I}^X_{g,n,\beta}\rangle(\gamma_1 \otimes \cdots \otimes \gamma_n) \in \mathbb{Q}_l$ is zero unless

$$\sum_{i+1}^{n} m_i = 2(\mathrm{vdim} - 3g + 3 - n).$$

6.3. Notation. We denote the vector $(a_0, \ldots, a_m)$ as $\underline{a}$; we set $|\underline{a}| = a_0 + \cdots + a_m$ and $\underline{a}! = a_0! \cdots a_m!$. Moreover we set $\langle \mathrm{I}^X_{0,n,\beta}\rangle = 0$ for $n < 3$.

6.4. **Definition.** Let $\gamma = \sum_{i=0}^m t_i T_i$ (regarding T_i and t_i as supercommuting variables). We define the *genus* 0 *Gromov-Witten potential* as the formal series

$$\Phi(\gamma) = \sum_{n\geq 0} \sum_{\beta \in A_1(X/k)} \frac{1}{n!} \langle \mathrm{I}^X_{0,n,\beta}\rangle(\gamma^n) q^\beta,$$

where q^β is a free variable and

$$\frac{1}{n!}\langle \mathrm{I}^X_{0,n,\beta}\rangle(\gamma^n) = \sum_{|\underline{a}|=n} \epsilon(\underline{a}) \langle \mathrm{I}^X_{0,n,\beta}\rangle(T^{\underline{a}}) \frac{t^{\underline{a}}}{\underline{a}!},$$

with $\epsilon(\underline{a}) = \pm 1$ determined by

$$(t_0 T_0)^{a_0} \cdots (t_m T_m)^{a_m} = \epsilon(\underline{a}) T_0^{a_0} \cdots T_m^{a_m} t_0^{a_0} \cdots t_m^{a_m}.$$

6.5. Remark. By effectivity axiom, the Gromov-Witten potential is a formal series in $\mathcal{R} = R[\![t_0, \ldots, t_m]\!]$, with $R = \mathbb{Q}_l[\![q^\beta;\, \beta \in A_1(X/k)_+]\!]$.

6.2. **Quantum product.** By [22] VI.8, $H^*(X \times_k X) = H^*(X) \otimes H^*(X)$. Let $\Delta \subset X \times_k X$ be the diagonal, then

$$[\Delta] = \sum_{e,f} g^{ef} T_e \otimes T_f.$$

6.6. **Definition.** We define

$$T_i * T_j = \sum_{e,f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^{ef} T_f.$$

Extending this linearly gives the *(big) quantum product* on $H^*(X, \mathcal{R})$.

6.7. Remark. Notice that the Gromov-Witten invariants with $n < 3$ do not affect the quantum product.

6.8. **Lemma.** *For all i, j, h, we have*

$$\frac{\partial^3 \Phi(\gamma)}{\partial t_i \partial t_j \partial t_h} = \sum_{n \geq 0} \sum_{\beta \in A_1(X/k)} \frac{1}{n!} \langle I^X_{0,n+3,\beta} \rangle (T_i \otimes T_j \otimes T_h \otimes \gamma^n) q^\beta .$$

Proof. For simplicity, we will assume that $H^*(X, \mathcal{R})$ has only even cohomology so that we don't have to worry about signs. We have

$$\frac{\partial^3 \Phi(\gamma)}{\partial t_i \partial t_j \partial t_h} = \frac{\partial^3}{\partial t_i \partial t_j \partial t_h} \sum_{n,\ \beta} \sum_{|\underline{a}|=n} \langle \mathrm{I}^X_{0,n,\beta} \rangle (T^{\underline{a}}) \frac{t^{\underline{a}}}{\underline{a}!} q^\beta = \sum_{n,\ \beta} \sum_{|\underline{a}|=n} \langle \mathrm{I}^X_{0,n,\beta} \rangle (T^{\underline{a}'}) \frac{t^{\underline{a}'}}{\underline{a}'!} q^\beta ,$$

where $\underline{a}' = \underline{a} - e_i - e_j - e_h$ and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the i-th position. Moreover

$$\begin{aligned} \sum_{n,\ \beta} \frac{1}{n!} \langle \mathrm{I}^X_{0,n+3,\beta} \rangle (T_i \otimes T_j \otimes T_h \otimes \gamma^n) q^\beta &= \sum_{n,\ \beta} \sum_{|\underline{a}|=n} \langle \mathrm{I}^X_{0,n+3,\beta} \rangle (T_i \otimes T_j \otimes T_h \otimes T^{\underline{a}}) \frac{t^{\underline{a}}}{\underline{a}!} q^\beta \\ &= \sum_{n,\ \beta} \sum_{|\underline{a}|=n+3} \langle \mathrm{I}^X_{0,n+3,\beta} \rangle (T^{\underline{a}+e_i+e_j+e_h}) \frac{t^{\underline{a}}}{\underline{a}!} q^\beta . \end{aligned}$$

□

6.9. **Theorem** (WDVV equation)**.** *The Gromov-Witten potential satisfies the equation*

$$\sum_{e,f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^{ef} \frac{\partial^3 \Phi}{\partial t_f \partial t_h \partial t_l} = (-1)^{\deg t_i (\deg t_j + \deg t_h)} \sum_{e,f} \frac{\partial^3 \Phi}{\partial t_j \partial t_h \partial t_e} g^{ef} \frac{\partial^3 \Phi}{\partial t_f \partial t_i \partial t_l},$$

for all i, j, h, l.

Proof. For simplicity, we will assume that $H^*(X, \mathcal{R})$ has only even cohomology so that we don't have to worry about signs. If we set

$$F(ij|hl) = \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^{ef} \frac{\partial^3 \Phi}{\partial t_f \partial t_h \partial t_l},$$

then we want to show that $F(ij|hl) = F(jh|il)$. Consider the following cartesian diagram

$$\begin{array}{ccc} D(ij|hl) & \longrightarrow & \overline{\mathcal{M}}_{0,\, n\,+\,4/k} \\ \big\downarrow & & \big\downarrow \rho \\ \operatorname{Spec} k = \overline{\mathcal{M}}_{0,\, \{i,\, j\} \,\cup\, \bullet/k} \times_k \overline{\mathcal{M}}_{0,\, \{h,\, l\} \,\cup\, \bullet/k} & \xrightarrow{\varphi_0} & \overline{\mathcal{M}}_{0,\, 4/k} \end{array}$$

where the image of φ_0 is a boundary point of $\overline{\mathcal{M}}_{0,\,4/k} \cong \mathbb{P}^1_k$. Since the boundary points are linearly equivalent, the same is true for the fibers of ρ over these points, hence $D(ij|hl)$ and $D(jh|il)$ are linearly equivalent divisors in $\overline{\mathcal{M}}_{0,\,n\,+\,4/k}$. Let $A \cup B$ be a partition of $\{1, \ldots, n+4\}$ such that $i, j \in A$ and $h, l \in B$. Form the following fiber square

$$\begin{array}{ccc} D(A|B) & \longrightarrow & D(ij|hl) \\ \big\downarrow & & \big\downarrow \\ \overline{\mathcal{M}}_{0,\, A \,\cup\, \bullet/k} \times_k \overline{\mathcal{M}}_{0,\, B \,\cup\, \bullet/k} & \xrightarrow{\varphi} & \overline{\mathcal{M}}_{0,\, n\,+\,4/k} \end{array}$$

then $D(ij|hl) = \sum_{\substack{A \cup B = \{1, \ldots, n+4\} \\ i,j \in A,\ h,l \in B}} D(A|B)$. We set

$$\overline{\mathcal{M}}_{(\beta_1, \beta_2)} = \overline{\mathcal{M}}_{0, A \cup \bullet}(X/k, \beta_1) \times_k \overline{\mathcal{M}}_{0, B \cup \bullet}(X/k, \beta_2).$$

With notations as in the proof of splitting axiom, we have the following commutative diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}}_{0,n+4}(X/k,\beta) & \xleftarrow{\hat{\varphi}} & \mathcal{N}_{A,B}^{(\beta_1,\beta_2)} & \xrightarrow{\hat{\Delta}} & \overline{\mathcal{M}}_{(\beta_1,\beta_2)} \\ \downarrow \mathrm{ev} & & \downarrow \widehat{\mathrm{ev}} & & \downarrow \mathrm{ev}_{1,2} \\ X^n & \xleftarrow{\pi} & X^{n+1} & \xrightarrow{\mathrm{id}\times\Delta} & X^{n+2} \end{array}$$

where the left square is cartesian. Therefore

$$\begin{aligned} \sum_{\substack{\beta_1+\beta_2=\beta \\ A\cup B=\{1,\dots,n+4\} \\ i,j\in A,\ h,l\in B}} \int_{[\overline{\mathcal{M}}_{(\beta_1,\beta_2)}]^{\mathrm{virt}}} \hat{\Delta}_*\hat{\varphi}^*\mathrm{ev}^*(\bullet) &= \sum_{\substack{A\cup B=\{1,\dots,n+4\} \\ i,j\in A,\ h,l\in B}} \int_{\hat{\varphi}^*[\overline{\mathcal{M}}_{0,n+4}(X/k,\beta)]^{\mathrm{virt}}} \hat{\varphi}^*\mathrm{ev}^*(\bullet) \\ &= \sum_{\substack{A\cup B=\{1,\dots,n+4\} \\ i,j\in A,\ h,l\in B}} \int_{\hat{\varphi}_*\hat{\varphi}^*[\overline{\mathcal{M}}_{0,n+4}(X/k,\beta)]^{\mathrm{virt}}} \mathrm{ev}^*(\bullet) \\ &= \sum_{\substack{A\cup B=\{1,\dots,n+4\} \\ i,j\in A,\ h,l\in B}} \int_{D(A|B)} \mathrm{I}^X_{0,n+4,\beta}(\bullet) \\ &= \int_{D(ij|hl)} \mathrm{I}^X_{0,n+4,\beta}(\bullet). \end{aligned}$$

Let us set for simplicity $\gamma_{n_1} = T_i \otimes T_j \otimes \gamma^{n_1}$ and $\gamma_{n_2} = T_h \otimes T_l \otimes \gamma^{n_2}$. Then, by Lemma 6.8 and splitting axiom,

$$\begin{aligned} F(ij|hl) &= \sum_{\beta_1,\beta_2,n_1,n_2,e,f} \frac{1}{n_1!n_2!}\langle \mathrm{I}^X_{0,n_1+3,\beta_1}\rangle(T_e\otimes\gamma_{n_1})g^{ef}\langle \mathrm{I}^X_{0,n_2+3,\beta_2}\rangle(T_f\otimes\gamma_{n_2})q^{\beta_1+\beta_2} \\ &= \sum_{\beta,n}\sum_{\substack{\beta_1+\beta_2=\beta \\ n_1+n_2=n}}\sum_{e,f} \frac{1}{n_1!n_2!}\langle \mathrm{I}^X_{0,n_1+3,\beta_1}\rangle(T_e\otimes\gamma_{n_1})g^{ef}\langle \mathrm{I}^X_{0,n_2+3,\beta_2}\rangle(T_f\otimes\gamma_{n_2})q^{\beta} \\ &= \sum_{\beta,n}\sum_{\substack{\beta_1+\beta_2=\beta \\ n_1+n_2=n}}\sum_{e,f} \frac{1}{n_1!n_2!}\int_{[\overline{\mathcal{M}}_{(\beta_1,\beta_2)}]^{\mathrm{virt}}} g^{e,f}\,\mathrm{ev}^*_{(1,2)}(T_e\otimes\gamma_{n_1}\otimes T_f\otimes\gamma_{n_2})q^{\beta} \\ &= \sum_{\beta,n}\sum_{\substack{\beta_1+\beta_2=\beta \\ n_1+n_2=n}} \frac{1}{n_1!n_2!}\int_{[\overline{\mathcal{M}}_{(\beta_1,\beta_2)}]^{\mathrm{virt}}} \mathrm{ev}^*_{(1,2)}(\gamma_{n_1}\otimes[\Delta]\otimes\gamma_{n_2})q^{\beta} \\ &= \sum_{\beta,n}\sum_{\substack{\beta_1+\beta_2=\beta \\ n_1+n_2=n}} \frac{1}{n_1!n_2!}\int_{[\overline{\mathcal{M}}_{(\beta_1,\beta_2)}]^{\mathrm{virt}}} \hat{\Delta}_*\hat{\varphi}^*\mathrm{ev}^*(T_i\otimes T_j\otimes T_h\otimes T_l\otimes\gamma^n)q^{\beta} \\ &= \sum_{\beta,n}\sum_{\substack{\beta_1+\beta_2=\beta \\ A\cup B=\{1,\dots,n+4\} \\ i,j\in A,\ h,l\in B}} \frac{1}{n!}\int_{[\overline{\mathcal{M}}_{(\beta_1,\beta_2)}]^{\mathrm{virt}}} \hat{\Delta}_*\hat{\varphi}^*\mathrm{ev}^*(T_i\otimes T_j\otimes T_h\otimes T_l\otimes\gamma^n)q^{\beta} \\ &= \sum_{\beta,n}\frac{1}{n!}\int_{D(ij|hl)} \mathrm{I}^X_{0,n+4,\beta}(T_i\otimes T_j\otimes T_h\otimes T_l\otimes\gamma^n)q^{\beta}. \end{aligned}$$

Since $D(ij|hl)$ and $D(jh|il)$ are linerly equivalent, it follows that $F(ij|hl) = F(jh|il)$. □

6.10. **Proposition.** *The quantum product is supercommutative with identity T_0 and associative.*

Proof. By Lemma 6.8 and S_n-covariance axiom,

$$\begin{aligned} T_i * T_j &= \sum_{\beta,n,e,f} \frac{1}{n!} \langle \mathrm{I}^X_{0,n+3,\beta} \rangle (T_i \otimes T_j \otimes T_e \otimes \gamma^n) g^{ef} T_f q^\beta \\ &= \sum_{\beta,n,e,f} \frac{1}{n!} (-1)^{\deg T_i \deg T_j} \langle \mathrm{I}^X_{0,n+3,\beta} \rangle (T_j \otimes T_i \otimes T_e \otimes \gamma^n) g^{ef} T_f q^\beta \\ &= (-1)^{\deg T_i \deg T_j} T_j * T_i. \end{aligned}$$

By the mapping to point axiom,

$$\begin{aligned} T_i \cup T_j &= p_{2*}(p_1^*(T_i \cup T_j) \cup [\Delta]) \\ &= \sum_{e,f} g^{ef} p_{2*}(T_i \cup T_j) \cup (T_e \otimes T_f) \\ &= \sum_{e,f} \left(\int_X T_i \cup T_j \cup T_e \right) g^{ef} T_f \\ &= \sum_{e,f} \langle \mathrm{I}^X_{0,3,0} \rangle (T_i \otimes T_j \otimes T_e) g^{ef} T_f. \end{aligned}$$

Moreover, we have $\langle \mathrm{I}^X_{0,n+3,\beta} \rangle (\bullet \otimes T_0) = 0$ unless $\beta = 0$ and $n = 3$. Therefore

$$\begin{aligned} T_0 * T_i &= \sum_{\beta,n,e,f} \frac{1}{n!} \langle \mathrm{I}^X_{0,n+3,\beta} \rangle (T_0 \otimes T_i \otimes T_e \otimes \gamma^n) g^{ef} T_f q^\beta \\ &= \sum_{e,f} \langle \mathrm{I}^X_{0,3,0} \rangle (T_0 \otimes T_i \otimes T_e) g^{ef} T_f \\ &= T_0 \cup T_i = T_i. \end{aligned}$$

Finally, we prove that the quantum product is associative. For simplicity, we will assume that $H^*(X, \mathcal{R})$ has only even cohomology so that we don't have to worry about signs. We have

$$(T_i * T_j) * T_h = \sum_{e,f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^{ef} T_e * T_h = \sum_{c,d,e,f} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_e} g^{ef} \frac{\partial^3 \Phi}{\partial t_f \partial t_h \partial t_c} g^{cd} T_d$$

and

$$T_i * (T_j * T_h) = (-1)^{\deg T_i (\deg T_j + \deg T_h)} (T_j * T_h) * T_i,$$

since the quantum product is supercommutative. Therefore, associativity follows from Theorem 6.9. □

6.3. Reconstruction for genus zero Gromov-Witten invariants.

6.11. **Theorem.** *If $H^*(X)$ is generated by $H^2(X)$ then every genus zero Gromov-Witten invariant can be uniquely reconstructed starting with the following system of Gromov-Witten invariants*

$$\left\{ I^X_{0,3,\beta}(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \,\middle|\, \beta \cdot c_1(T_{X/k}) \leq \dim_k X + 1,\ \deg \gamma_3 = 2 \right\}.$$

Proof. Apply the WDVV equation (Theorem 6.9) to $\gamma_1 \otimes \cdots \otimes \gamma_{n+1}$ with indeces $\{i, j, h, l\} = \{1, 2, n, n+1\}$. Let us define a partial order on pairs (β, n), with $\beta \in A_1({}^X\!/\!_k)_+$ and $n \geq 3$, by setting $(\beta, n) > (\beta', n')$ if and only if either $\beta = \beta' + \beta''$ or $\beta = \beta'$ and $n > n'$. Then there are four terms of higher order in the WDVV equation each of the form

$$\mathrm{I}_{a,b} = \sum_{e,f} \langle \mathrm{I}^X_{0,3,0} \rangle (\gamma_a \otimes \gamma_b \otimes T_e) g^{ef} \langle \mathrm{I}^X_{0,n-1,\beta} \rangle (T_f \otimes (\otimes_{s \neq a,b} \gamma_s)),$$

with $(a,b) \in \{(1,2),(n,n+1),(2,n),(1,n+1)\}$. As shown in the proof of Proposition 6.10, we have

$$\gamma_a \cup \gamma_b = \sum_{e,f} \langle \mathrm{I}^X_{0,3,0} \rangle (\gamma_a \otimes \gamma_b \otimes T_e) g^{ef} T_f,$$

hence $\mathrm{I}_{a,b} = \langle \mathrm{I}^X_{0,n-1,\beta} \rangle (\gamma_a \cup \gamma_b \otimes (\otimes_{s \neq a,b} \gamma_s))$. Let consider $\langle \mathrm{I}^X_{0,n,\beta} \rangle (\gamma_1 \otimes \cdots \otimes \gamma_n)$. If $\deg \gamma_n = 2$, then we can apply divisor axiom to reduce n. Otherwise, since $H^*(X)$ is generated by $H^2(X)$, we can write $\gamma_n = \sum_i \delta'_i \cup \delta_i$, with $\deg \delta_i = 2$. By linearity, we can assume $\gamma_n = \delta' \cup \delta$, with $\deg \delta = 2$. Apply the construction above with $\gamma_n = \delta'$ and $\gamma_{n+1} = \delta$. Then, by WDVV equation, we get

$$\begin{gathered}\pm \langle \mathrm{I}^X_{0,n-1,\beta} \rangle (\gamma_1 \cup \gamma_2 \otimes \gamma_3 \otimes \cdots \gamma_{n-1} \otimes \delta' \otimes \delta) \pm \langle \mathrm{I}^X_{0,n-1,\beta} \rangle (\gamma_1 \otimes \cdots \otimes \gamma_{n-1} \otimes \delta' \cup \delta) \\ \pm \langle \mathrm{I}^X_{0,n-1,\beta} \rangle (\gamma_1 \cup \delta \otimes \gamma_2 \otimes \cdots \gamma_{n-1} \otimes \delta') \pm \langle \mathrm{I}^X_{0,n-1,\beta} \rangle (\gamma_1 \otimes \gamma_2 \cup \delta' \otimes \gamma_\otimes \cdots \gamma_{n-1} \otimes \delta) = \\ = \text{a combination of higher order terms.}\end{gathered}$$

By divisor axiom, the first and the fourth summands are lifted from $\overline{\mathcal{M}}_{0,\, n\,-\,1/k}$. Moreover in the third summand we have $\deg \delta' < \deg \gamma_n$. If $\deg \delta' = 2$ then, by divisor axiom, we can reduce n, otherwise we repeat this trick and in a finite number of iterations we will reduce n. Finally, we can apply the procedure described above to $\langle \mathrm{I}^X_{0,3,\beta} \rangle (\gamma_1 \otimes \gamma_2 \otimes \gamma_3)$ and diminish $\deg \gamma_3 \geq 2$. ☐

Appendix A. Deformation theory

A.1. **Formal criteria of smoothness.** We recall a few results on smoothness and formal smoothness of morphisms of schemes and algebraic stacks.

A.1. **Definition.** A morphism of schemes $f\colon X \to Y$ is *smooth* if it is flat, locally of finite presentation and the fibers of f are geometrically regular (i.e. for every point $x \in X$, all the localizations of $\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \overline{k(f(x))}$ are regular, where $\overline{k(f(x))}$ is an algebraic closure of the residue field of $f(x)$).

A.2. Remark. If Y is locally Noetherian, then f is locally of finite presentation if and only if it is locally of finite type ([14] 1.6).

A.3. **Definition.** A morphism of schemes $f\colon X \to Y$ is *formally smooth* if for every ring A, for every nilpotent ideal $I \subset A$ and for every morphism of schemes $\operatorname{Spec} A \to Y$, the canonical map

$$\operatorname{Hom}_Y(\operatorname{Spec} A, X) \to \operatorname{Hom}_Y(\operatorname{Spec} {}^{A}\!/_{I}, X)$$

is surjective.

A.4. Remark. Both smoothness and formal smoothness can be checked locally.

A.5. Remark. It is enough to verify the condition of formal smoothness only on ideals $I \subset A$ with $I^2 = 0$. Indeed, let A be a ring and $I \subset A$ a nilpotent ideal. In particular $I^n = 0$ for some n. Consider $A_i = {}^{A}\!/_{I^i}$ for $i = 1, \ldots, n$. Then $A_{i-1} = {}^{A_i}\!/_{J_i}$, where $J_i = {}^{I^i}\!/_{I^{i+1}}$. Notice that $J_i^2 = 0$ and we have the following sequence of surjective maps

$$\operatorname{Hom}_Y(\operatorname{Spec} A, X) \to \operatorname{Hom}_Y(\operatorname{Spec} {}^{A}\!/_{I^{n-1}}, X) \to \cdots \to \operatorname{Hom}_Y(\operatorname{Spec} {}^{A}\!/_{I}, X).$$

A.6. **Proposition** ([15] Corollary 17.5.2)**.** *Let $f\colon X \to Y$ be a morphism of finite type. Then f is smooth if and only if it is formally smooth.*

A.7. **Proposition** ([22] Proposition I.3.24)**.** *Let $f\colon X \to Y$ be a morphism of finite type. Then f is smooth of dimension d if and only if locally it is of the form $\operatorname{Spec} S \to \operatorname{Spec} R$ with $S = {}^{R[x_1,\ldots,x_{d+r}]}\!/_{(p_1,\ldots,p_r)}$ and the matrix of partial derivatives $({}^{\delta p_i}\!/_{\delta x_j})$ has rank r at every point of $\operatorname{Spec} S$.*

A.8. **Lemma.** *Let $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ be a morphism of finite type. The following are equivalent:*

(1) f is smooth;
(2) for every prime ideal $\mathfrak{p} \subset R$, the induced morphism $f_{\mathfrak{p}} \colon \operatorname{Spec} S \otimes_R R_{\mathfrak{p}} \to \operatorname{Spec} R_{\mathfrak{p}}$ is smooth;
(3) for every prime ideal $\mathfrak{p} \subset R$, the induced morphism $\hat{f}_{\mathfrak{p}} \colon \operatorname{Spec} S \otimes_R \hat{R}_{\mathfrak{p}} \to \operatorname{Spec} \hat{R}_{\mathfrak{p}}$ is smooth.

Proof. Notice that $S \otimes_R \hat{R}_{\mathfrak{p}} = (S \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} \hat{R}_{\mathfrak{p}}$. Since smoothness is stable under base change ([15] Proposition 17.3.3) we have (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3).

We want to prove (2) $\Rightarrow$ (1). Assume by contradiction that f is not smooth then, by Proposition A.7, we have $S = {}^{R[x_1,\dots,x_n]}/_{(p_1,\dots,p_r)}$ and there exists a prime ideal $\mathfrak{q} \subset R[x_1,\dots,x_n]$ such that $p_1,\dots,p_r \in \mathfrak{q}$ (hence $\mathfrak{q}$ corresponds to a prime ideal $\mathfrak{q}' \subset S$) and ${}^{\delta p_r}/_{\delta x_j} \in \mathfrak{q}$ for all j. In particular $p_r \in \mathfrak{q}^2$. Let $f^{\sharp} \colon R \to S$ be the homomorphism induced by f and let $\mathfrak{p} = {f^{\sharp}}^{-1}(\mathfrak{q}')$. Notice that $\operatorname{Spec} S \otimes_R R_{\mathfrak{p}} = {}^{R_{\mathfrak{p}}[x_1,\dots,x_n]}/_{(p_1,\dots,p_r)}$ with $p_r \in \mathfrak{q}^2 R_{\mathfrak{p}}[x_1,\dots,x_n]$. Since the relative dimension of f and $f_{\mathfrak{p}}$ at $\mathfrak{q}'$ are the same, we get that $f_{\mathfrak{p}}$ is not smooth, which contradicts (2). Hence f is smooth.

Finally we prove (3) $\Rightarrow$ (2). We assume that R is a local ring and that $\hat{f} \colon \operatorname{Spec} S \otimes_R \hat{R} \to \operatorname{Spec} \hat{R}$ is smooth, then we want to prove that $f \colon \operatorname{Spec} S \to R$ is smooth. Assume by contradiction that f is not smooth then, by Proposition A.7, we have $S = {}^{R[x_1,\dots,x_n]}/_{(p_1,\dots,p_r)}$ and there exists a prime ideal $\mathfrak{q} \subset R[x_1,\dots,x_n]$ as above such that $p_r \in \mathfrak{q}^2$. Notice that $\operatorname{Spec} S \otimes_R \hat{R} = {}^{\hat{R}[x_1,\dots,x_n]}/_{(p_1,\dots,p_r)}$ with $p_r \in \mathfrak{q}^2 \hat{R}[x_1,\dots,x_n]$. Since the relative dimension of f and $\hat{f}$ at $\mathfrak{q}'$ are the same ([4] Corollary 11.19), we get that $\hat{f}$ is not smooth, which contradicts (3). Hence f is smooth. $\square$

A.9. **Proposition.** *Let $f \colon X \to Y$ be a morphism of finite type. Then f is smooth if and only if for every point $x \in X$, for every Artinian local $\mathscr{O}_{Y,f(x)}$-algebra A and for every ideal $I \subset A$ such that $I^2 = 0$, the canonical map*

$$Hom_Y(\operatorname{Spec} A, X) \to Hom_Y(\operatorname{Spec} {}^{A}/_{I}, X)$$

is surjective.

Proof. Since both conditions are local, we can assume $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$. Moreover, by Lemma A.8, we can assume that R is a local ring. If f is not smooth then, by Proposition A.7, we have $S = {}^{R[x_1,\dots,x_n]}/_{(p_1,\dots,p_r)}$ and there exists a prime ideal $\mathfrak{q} \subset R[x_1,\dots,x_n]$ such that $p_1,\dots,p_r \in \mathfrak{q}$ (hence $\mathfrak{q}$ corresponds to a prime ideal $\mathfrak{q}' \subset S$) and $p_i \in \mathfrak{q}^2$ for some i. In other words, we can write $S = {}^{T}/_{J}$ with $\operatorname{Spec} T \to \operatorname{Spec} R$ smooth and $0 \neq J \subset \mathfrak{q}^2$. Notice that we can assume that $\mathfrak{q} \subset T$ is maximal. Then there exists i such that $J \subset \mathfrak{q}^i$ but $J \not\subset \mathfrak{q}^{i+1}$. Consider the following commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & {}^{T}/_{\mathfrak{q}^{i+1}} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\psi} & {}^{T}/_{J + \mathfrak{q}^i} = {}^{T}/_{\mathfrak{q}^i} \\ \downarrow & \nearrow_{\varphi} & \\ S = {}^{T}/_{J} & & \end{array}$$

Notice that $A = {}^{T}/_{\mathfrak{q}^{i+1}}$ is an Artinian local R-algebra and ${}^{T}/_{\mathfrak{q}^i} = {}^{A}/_{I}$, where $I = {}^{\mathfrak{q}^i}/_{\mathfrak{q}^{i+1}} \subset A$ and $I^2 = 0$. Then by assumption there exists a lifting $\hat{\varphi} \colon S \to A$ of φ. Moreover, since T is smooth over R, there exists a lifting $\hat{\psi} \colon T \to A$ of ψ. in particular we get the following commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & A = {}^{T}/_{\mathfrak{q}^{i+1}} \\ \downarrow & \nearrow & \\ S = {}^{T}/_{J} & & \end{array}$$

which is absurd since $J \not\subset \mathfrak{q}^{i+1}$. Hence f is smooth. $\square$

A.10. **Proposition.** *Let $f\colon X \to Y$ be a morphism of finite type. Then f is smooth if and only if for every geometric point $\overline{x}$ of X, for every Artinian local $\hat{\mathscr{O}}_{Y,f(\overline{x})}$-algebra A and for every ideal $I \subset A$ such that $I^2 = 0$, the canonical map*

$$\mathit{Hom}_Y(\operatorname{Spec} A, X) \to \mathit{Hom}_Y(\operatorname{Spec} {}^{A}\!/_{I}, X)$$

is surjective.

Proof. Follows from Lemma A.8 and Proposition A.9. □

A.11. **Definition.** A morphism of Artin stacks $f\colon X \to Y$ is *smooth* if for every commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{f}} & V \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array}$$

where U, V are schemes, u, v are representable and smooth, and $U \to X \times_Y V$ is smooth, then $\widetilde{f}$ is smooth.

A.12. Remark. If f is representable then the condition above is equivalent to require that for every morphism $V \to Y$ from a scheme V, the induced morphism $V \times_Y X \to V$ is smooth.

A.13. **Proposition.** *Let $f\colon X \to Y$ be a representable morphism of finite type of Artin stacks. The following are equivalent:*

(1) f is smooth;
(2) for every geometric point $\overline{x}$ of X, for every Artinian local $\hat{\mathscr{O}}_{Y,f(\overline{x})}$-algebra A and for every ideal $I \subset A$ such that $I^2 = 0$, the canonical map

$$\mathit{Hom}_Y(\operatorname{Spec} A, X) \to \mathit{Hom}_Y(\operatorname{Spec} {}^{A}\!/_{I}, X)$$

is surjective;
(3) for every smooth representable morphism $w\colon W \to X$ from a scheme W, the morphism $f \circ w$ is smooth.

Proof. We prove (1) ⇒ (2) ⇒ (3) ⇒ (1). Let $\overline{x}$ be a geometric point of X and let A and I be as in the statement. Consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} {}^{A}\!/_{I} & \xrightarrow{g_X} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A & \xrightarrow{g'_Y} & Y \end{array}$$

We claim that if f smooth then there exists a morphism $g'_X\colon \operatorname{Spec} A \to X$ such that $g'_X \circ i = g_x$, $f \circ g'_X = g'_Y$. Set $V = \operatorname{Spec} {}^{A}\!/_{I}$ and $U = V \times_Y X$. If f is smooth then, by definition, the induced morphism $\widetilde{f}\colon U \to V$ is smooth. Notice that i and g_X define a unique morphism $g_U\colon \operatorname{Spec} {}^{A}\!/_{I} \to U$ such that $\widetilde{f} \circ g_U = i$, $u \circ g_U = g_X$. By Proposition A.10, there exists a morphism $g'_U\colon \operatorname{Spec} A \to U$ such that $g'_U \circ i = g_U$, $\widetilde{f} \circ g'_U = \mathrm{id}$. Then we can take $g'_X = u \circ g'_U$.

Let $w\colon W \to X$ be a smooth representable morphism from a scheme W. The morphism $f \circ w$ is smooth if and only if, for every morphism $v\colon V \to Y$ from a scheme V, the induced morphism $V \times_Y W \to V$ is smooth. Set $U = V \times_Y X$, $Z = W \times_Y V$, and let $\widetilde{f}\colon U \to V$, $\widetilde{w}\colon Z \to U$ be the induced morphisms. The morphism $\widetilde{f} \circ \widetilde{w}$ is smooth if and only if, for every geometric point $\overline{x}$ of

$Z = V \times_Y W$, for every Artinian local $\hat{\mathscr{O}}_{V,\overline{x}}$-algebra A, for every ideal $I \subset A$ such that $I^2 = 0$, and for every commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A/I & \xrightarrow{g_Z} & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \widetilde{f}\circ\widetilde{w}} \\ \operatorname{Spec} A & \xrightarrow{g'_V} & V \end{array}$$

there exists a morphism $g'_Z\colon \operatorname{Spec} A \to Z$ such that $g'_Z \circ i = g_Z$, $\widetilde{f} \circ \widetilde{w} \circ g'_Z = g'_V$. Let us fix such data. By assumptions, there exists a morphism $g'_U\colon \operatorname{Spec} A \to U$ such that $g'_U \circ i = \widetilde{w} \circ g_Z$, $\widetilde{f} \circ g'_U = g'_V$. Moreover $\widetilde{w}$ is smooth thus there exists a morphism g'_Z with the required properties.

Finally, by Proposition A.10, f is smooth if and only if, for every morphism $v\colon V \to Y$ from a scheme V, for every geometric point $\overline{x}$ of $U = V \times_Y X$, for every Artinian local $\hat{\mathscr{O}}_{V,\overline{x}}$-algebra A, for every ideal $I \subset A$ such that $I^2 = 0$, and for every commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A/I & \xrightarrow{g_U} & U \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ \operatorname{Spec} A & \xrightarrow{g'_V} & V \end{array}$$

there exists a morphism $g'_U\colon \operatorname{Spec} A \to U$ such that $g'_U \circ i = g_U$, $\widetilde{f} \circ g'_U = g'_V$. Let us fix such data. Let $w\colon W \to X$ be a representable smooth surjective morphism from a scheme W. Set $Z = W \times_X U$ and let $\widetilde{w}\colon Z \to U$ be the induced morphism, which is smooth and surjective. Then $\overline{x} \to U$ factors through $\widetilde{w}$ and there exists a morphism $g_Z\colon \operatorname{Spec} A/I \to Z$ such that $\widetilde{w} \circ g_Z = g_U$. By assumptions, $\widetilde{f} \circ \widetilde{w}$ is smooth and thus there exists a morphism $g'_Z\colon \operatorname{Spec} A \to Z$ such that $g'_Z \circ i = g_Z$, $\widetilde{f} \circ \widetilde{w} \circ g'_Z = g'_V$. Therefore we take $g'_U = \widetilde{w} \circ g'_Z$. □

A.14. **Proposition.** *Let $f\colon X \to Y$ be a morphism of finite type of Artin stacks. Then f is smooth if and only if, for every geometric point $\overline{x}$ of X, for every Artinian local $\hat{\mathscr{O}}_{Y,f(\overline{x})}$-algebra A and for every ideal $I \subset A$ such that $I^2 = 0$, the canonical map*

$$\mathit{Hom}_Y(\operatorname{Spec} A, X) \to \mathit{Hom}_Y(\operatorname{Spec} A/I, X)$$

is surjective.

Proof. Let us assume that f is smooth and let $\overline{x}$, A and I as in the statement. Consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A/I & \xrightarrow{g_X} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} A & \xrightarrow{g'_Y} & Y \end{array}$$

We want to show that there exists a morphism $g'_X\colon \operatorname{Spec} A \to X$ such that $g'_X \circ i = g_X$, $f \circ g'_X = g'_Y$. Let $v\colon V \to Y$ be a representable smooth and surjective morphism from a scheme V, then $\overline{x} \to Y$ factors through v. By Proposition A.13, there exists a morphism $g'_V\colon \operatorname{Spec} A \to V$ such that $v \circ g'_V = g'_Y$. Let us form the fibre diagram

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{f}} & V \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array}$$

Then u is smooth surjective and, by assumptions, $\widetilde{f}$ is smooth. Moreover there exists a unique morphism $g_U\colon \operatorname{Spec} A/I \to U$ such that $u \circ g_U = g_X$, $\widetilde{f} \circ g_U = g'_V \circ i$. Thus, by Proposition A.10, there exists a morphism $g'_U\colon \operatorname{Spec} A \to U$ such that $g'_U \circ i = g_U$, $\widetilde{f} \circ g'_U = g'_V$. It follows that $g'_X = u \circ g'_U$ has the required properties.

Now we want to prove the other implication. Let us consider a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\widetilde{f}} & V \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array}$$

where U, V are schemes, u, v are representable and smooth, and $\widetilde{u}\colon U \to X \times_Y V$ is smooth. By Proposition A.10, $\widetilde{f}$ is smooth if and only if, for every geometric point $\overline{x}$ of U, for every Artinian local $\hat{\mathscr{O}}_{V,\widetilde{f}(\overline{x})}$-algebra A, for every ideal $I \subset A$ such that $I^2 = 0$, and for every commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} {}^{A}\!/\!{}_{I} & \xrightarrow{g_U} & U \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ \operatorname{Spec} A & \xrightarrow{g'_V} & V \end{array}$$

there exists a morphism $g'_U\colon \operatorname{Spec} A \to U$ such that $\widetilde{f} \circ g'_U = g'_V$, $g'_U \circ i = g_U$. Let us fix such data. By assumptions, there exists $g'_X\colon \operatorname{Spec} A \to X$ such that $g'_X \circ i = u \circ g_U$, $f \circ g'_X = v \circ g'_V$. Let us form the fibre diagram

$$\begin{array}{ccc} W & \xrightarrow{\hat{f}} & V \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{f} & Y \end{array}$$

Then there exists a unique morphism $g'_W\colon \operatorname{Spec} A \to W$ such that $w \circ g'_W = g'_X$, $\hat{f} \circ g'_W = g'_V$. Since $\widetilde{u}$ is smooth, Proposition A.10 ensures the existence of a morphism $g'_U\colon \operatorname{Spec} A \to U$ with the desired properties. □

A.2. **Deformation theory.** We review some results of deformation theory (for further details see [24], [16]). Let S be a scheme. We consider Artin stacks of finite type over S. Fix a geometric point $\operatorname{Spec} \overline{k} \xrightarrow{\overline{s}} S$ of S. Let $\Lambda = \hat{\mathscr{O}}_{S,\overline{s}}$ and consider the category $({}^{\mathrm{Art}}\!/\!{}_{\Lambda})$ of local artinian Λ-algebras with residue field $\overline{k}$.

Let $\mathcal{F} \to ({}^{\mathrm{Art}}\!/\!{}_{\Lambda})^{\mathrm{opp}}$ be a category fibered in groupoids. Let $\pi'\colon A' \to A$ and $\pi''\colon A'' \to A$ be morphisms in $({}^{\mathrm{Art}}\!/\!{}_{\Lambda})$, with π'' surjective. We form the cartesian diagram

$$\begin{array}{ccc} A' \times_A A'' & \xrightarrow{q''} & A'' \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle \pi''} \\ A' & \xrightarrow{\pi'} & A \end{array}$$

then the functors

$$\mathcal{F}(\pi') \circ \mathcal{F}(q'), \mathcal{F}(\pi'') \circ \mathcal{F}(q'')\colon \mathcal{F}(A' \times_A A'') \to \mathcal{F}(A)$$

are isomorphic and we get an induced functor

$$\Psi\colon \mathcal{F}(A' \times_A A'') \to \mathcal{F}(A') \times_{\mathcal{F}(A)} \mathcal{F}(A'').$$

A.15. **Definition.** A *deformation category* over Λ is a category fibered in groupoids $\mathcal{F} \to ({}^{\mathrm{Art}}\!/\!{}_{\Lambda})^{\mathrm{opp}}$ such that, given morphisms $\pi'\colon A' \to A$ and $\pi''\colon A'' \to A$ in $({}^{\mathrm{Art}}\!/\!{}_{\Lambda})$, with π'' surjective, the functor Ψ is an equivalence of categories.

A.16. **Definition.** A *small extension* is an exact sequence

$$0 \to I \to A' \to A \to 0$$

in $({}^{\mathrm{Art}}/\Lambda)$ such that $I\mathfrak{m}_{A'} = 0$, where $\mathfrak{m}_{A'}$ is the maximal ideal of A'.

A.17. **Definition.** Let $\mathcal{F}$, $\mathcal{G}$ be deformation categories and $\nu\colon \mathcal{F} \to \mathcal{G}$ a functor of categories fibered in groupoids. Let $T^1\nu$ and $T^2\nu$ be $\overline{k}$-vector spaces. We say that ν has *tangent space* $T^1\nu$ and *obstruction space* $T^2\nu$ if, for every small extension

$$0 \to I \to A' \to A \to 0,$$

there is a functorial exact sequence

$$T^1\nu \otimes_{\overline{k}} I \to \mathcal{F}(A') \to \mathcal{F}(A) \times_{\mathcal{G}(A)} \mathcal{G}(A') \xrightarrow{\mathrm{ob}_\nu} T^2\nu \otimes_{\overline{k}} I.$$

A.18. **Definition.** Let $\mathcal{F}, \mathcal{G}$ be deformation categories and $\nu\colon \mathcal{F} \to \mathcal{G}$ a functor of categories fibered in groupoids. Let $A' \to A = {}^{A'}/I$ be a small extension and fix $\sigma' \in \mathcal{F}(A')$, $\sigma \in \mathcal{F}(A)$, $\tau \in \mathcal{G}(A)$, $\tau' \in \mathcal{G}(A')$ such that $\mathcal{F}(i)(\sigma') = \sigma$, $\nu(A')(\sigma') = \tau'$ and $\mathcal{G}(i)(\tau') = \tau = \nu(A)(\sigma)$. Let $\mathrm{Aut}_{A'}(\sigma')$ be the group of authomorphisms of σ' in $\mathcal{F}(A')$. There is a natural homomorphism

$$\mathrm{Aut}_{A'}(\sigma') \to \mathrm{Aut}_A(\sigma) \times_{\mathrm{Aut}_A(\tau)} \mathrm{Aut}_{A'}(\tau').$$

An *infinitesimal automorphism* of σ' is an element of the kernel of this homomorphism.

A.19. Notation. Let $\sigma \in \mathcal{F}(A)$, $\tau \in \mathcal{G}(A)$, $\tau' \in \mathcal{G}(A')$ such that $\mathcal{G}(i)(\tau') = \tau = \nu(A)(\sigma)$. If $\mathrm{ob}_\nu(\sigma, \tau') = 0$, we denote by $\mathcal{S}_\nu$ the set of isomorphism classes of $\sigma' \in \mathcal{F}(A')$ such that$\mathcal{F}(i)(\sigma') = \sigma$, $\nu(A')(\sigma') = \tau'$.

A.20. Let $F\colon X \to Y$ be a Deligne-Mumford type morphism of algebraic Artin stacks over S. Let $\mathrm{Spec}\,\overline{k} \xrightarrow{\overline{x}} X$ be a geometric point of X. Let $\Lambda = \hat{\mathscr{O}}_{S,\overline{x}}$. Consider the deformation category $h_{X,\overline{x}}$ such that, for all $A \in ({}^{\mathrm{Art}}/\Lambda)$, the objects of $h_{X,\overline{x}}(A)$ are morphisms $f_X\colon \mathrm{Spec}\,A \to X$ such that $f_X|_{\mathrm{Spec}\,\overline{k}} = \overline{x}$. There is a natural functor $\nu_F\colon h_{X,\overline{x}} \to h_{Y,\overline{x}}$ given by the composition with F.

A.21. **Proposition.** *Let $L^\bullet_F$ be the relative cotangent complex of F. If F is representable then, for every geometric point $\overline{x}$ of X and for every small extension $A' \to A = {}^{A'}/I$ in $({}^{Art}/\hat{\mathscr{O}}_{S,\overline{x}})$,*

(1) there is a functorial surjective set-theoretical map

$$ob_F\colon h_{X,\overline{x}}(A) \times_{h_{Y,\overline{x}}(A)} h_{Y,\overline{x}}(A') \to h^1((L\overline{x}^*L^\bullet_F)^\vee) \otimes I$$

such that $ob_F(f_X, f'_Y) = 0$ if and only if there exists $f'_X \in h_{X,\overline{x}}(A')$ such that $f'_X \circ i = f_X$ and $F \circ f'_X = f'_Y$;

(2) if $ob_F(f_X, f'_Y) = 0$ then the set of isomorphism classes of $f'_X \in h_{X,\overline{x}}(A')$, such that $f'_X \circ i = f_X$ and $F \circ f'_X = f'_Y$, is a torsor under $h^0((L\overline{x}^*L^\bullet_F)^\vee) \otimes I$;

(3) if $ob_F(f_X, f'_Y) = 0$ and $f'_X \in h_{X,\overline{x}}(A')$ is such that $F \circ f'_X = f'_Y$, $f'_X \circ i = f_X$, then the group of infinitesimal authomorphisms of f'_X with respect to (f_X, f'_Y) contains only the identity.

Proof. Let $v\colon V \to Y$ be a smooth surjective morphism from a scheme V and form the fibre diagram

$$\begin{array}{ccc} U & \xrightarrow{G} & V \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{F} & Y \end{array}$$

By Theorem 3.19, $Lu^*L^\bullet_F \cong L^\bullet_G$. By Proposition A.14, $\overline{x} \to X$ factors through u and we know, by deformation theory of schemes, that there exists a functorial exact sequence

$$0 \to h^0((L\overline{x}^*L^\bullet_G)^\vee) \otimes I \to h_{U,\overline{x}}(A') \to h_{U,\overline{x}}(A) \times_{h_{V,\overline{x}}(A)} h_{V,\overline{x}}(A') \xrightarrow{\mathrm{ob}_G} h^1((L\overline{x}^*L^\bullet_G)^\vee) \otimes I \to 0.$$

Let us consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} A & \xrightarrow{f_X} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle F} \\ \operatorname{Spec} A' & \xrightarrow{f'_Y} & Y \end{array}$$

By Proposition A.13, there exists $f'_{V,1} \colon \operatorname{Spec} A' \to V$ such that $v \circ f'_{V,1} = f'_Y$. Then there exists a unique morphism $f_{U,1} \colon \operatorname{Spec} A \to U$ such that $u \circ f_{U,1} = f_X$, $G \circ f_{U,1} = f'_{V,1} \circ i$. If $f'_{V,2} \colon \operatorname{Spec} A' \to V$ is another morphism such that $v \circ f'_{V,2} = f'_Y$ then there exists a unique morphism $f'_{V \times_Y V} \colon \operatorname{Spec} A' \to V \times_Y V$ such that $v_j \circ f'_{V \times_Y V} = f'_{V,j}$ for $j = 1, 2$, where $v_j \colon V \times_Y V \to V$ are the projections. Let us form the fibre diagram

$$\begin{array}{ccc} U \times_Y V & \xrightarrow{H} & V \times_Y V \\ {\scriptstyle u_j}\downarrow & & \downarrow{\scriptstyle v_j} \\ U & \xrightarrow{G} & V \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ X & \xrightarrow{F} & Y \end{array}$$

Then there exists a unique morphism $f_{U \times_Y V} \colon \operatorname{Spec} A \to U \times_Y V$ such that $u_j \circ f_{U \times_Y V} = f_{U,j}$ for $j = 1, 2$, and $H \circ f_{U \times_Y V} = f'_{V \times_Y V} \circ i$. Therefore

$$\mathrm{ob}_G(f_{U,j}, f'_{V,j}) = \mathrm{ob}_G(u_j \circ f_{U \times_Y V}, v_j \circ f'_{V \times_Y V}) = \mathrm{ob}_H(f_{U \times_Y V}, f'_{V \times_Y V}).$$

Hence, if we set $\mathrm{ob}_F(f_X, f'_Y) = \mathrm{ob}_G(f_{U,1}, f'_{V,1})$, this gives a well-defined surjective map

$$\mathrm{ob}_F \colon h_{X,\overline{x}}(A) \times_{h_{Y,\overline{x}}(A)} h_{Y,\overline{x}}(A') \to h^1((L\overline{x}^* L^\bullet_G)^\vee) \otimes I \cong h^1((L\overline{x}^* L^\bullet_F)^\vee) \otimes I.$$

Moreover, if $\mathrm{ob}_F(f_X, f'_Y) = 0$ then there exist $f_U \in h_{U,\overline{x}}(A)$, $f'_V \in h_{V,\overline{x}}(A')$ such that $u \circ f_U = f_X$, $v \circ f'_V = f'_Y$ and $\mathrm{ob}_G(f_U, f'_V) = 0$. Thus there exists a morphism $f'_U \in h_{U,\overline{x}}(A')$ such that $f'_U \circ i = f_U$, $G \circ f'_U = f'_V$, and $f'_X = u \circ f'_U \in h_{X,\overline{x}}(A')$ is such that $f'_X \circ i = f_X$, $F \circ f'_X = f'_Y$. Therefore $T^2_{\nu_F} = h^1((L\overline{x}^* L^\bullet_F)^\vee)$ is an obstruction space for ν_F.

Let us now fix $\overline{f}_X \in h_{X,\overline{x}}(A)$, $\overline{f}'_Y \in h_{Y,\overline{x}}(A')$, $\overline{f}'_V \in h_{V,\overline{x}}(A')$ such that $F \circ \overline{f}_X = \overline{f}'_Y \circ i$, $v \circ \overline{f}'_V = \overline{f}'_Y$, and let $\overline{f}_U \in h_{U,\overline{x}}(A)$ be the unique morphism such that $u \circ \overline{f}_U = \overline{f}_X$, $G \circ \overline{f}_U = \overline{f}'_V \circ i$. Assume that $\mathrm{ob}_F(\overline{f}_X, \overline{f}'_Y) = 0$. There is a natural map $\rho \colon \mathcal{S}_G \to \mathcal{S}_F$ such that $\rho(f'_U) = u \circ f'_U$. We know that $\mathcal{S}_G$ is a torsor under $h^0((L\overline{x}^* L^\bullet_G)^\vee) \otimes I$. We claim that ρ is an isomorphism and thus $\mathcal{S}_F$ is a torsor under $h^0((L\overline{x}^* L^\bullet_G)^\vee) \otimes I \cong h^0((L\overline{x}^* L^\bullet_F)^\vee) \otimes I$. Let $f'_X \in \mathcal{S}_F$, then $F \circ f'_X = v \circ \overline{f}'_V$ and hence there exists a unique morphism $f'_U \in h_{U,\overline{x}}(A')$ such that $G \circ f'_U = \overline{f}'_V$, $u \circ f'_U = f'_X$. It follows that $f'_U \in \mathcal{S}_G$ and $f'_X = \rho(f'_U)$.

Finally, let $f_X \in h_{X,\overline{x}}(A)$, $f_Y \in h_{Y,\overline{x}}(A)$, $f'_Y \in h_{Y,\overline{x}}(A')$ such that $F \circ f_X = f_Y = f'_Y \circ i$. Assume that $\mathrm{ob}_F(f_X, f'_Y) = 0$ and fix $f'_X \in h_{X,\overline{x}}(A')$ such that $F \circ f'_X = f'_Y$, $f'_X \circ i = f_X$. We claim that the natural homomorphism

$$\mathrm{Aut}_X(f'_X) \to \mathrm{Aut}_X(f_X) \times_{\mathrm{Aut}_Y(f_Y)} \mathrm{Aut}_Y(f'_Y),$$

where $\mathrm{Aut}_X(f'_X)$ is the authomorphism group of f'_X in X, is injective. Since F is representable, if an authomorphism α of f'_X induces the identity of $F \circ f'_X$ in Y then $\alpha = \mathrm{id}$. □

A.22. **Proposition.** *Let $L^\bullet_X$ be the cotangent complex of X. Then, for every geometric point $\overline{x}$ of X and for every small extension $A' \to A = {}^{A'}\!/_{I}$ in $({}^{Art}\!/_{\hat{\mathcal{O}}_{S,\overline{x}}})$,* *(1) there is a functorial set-theoretical map*

$$ob_X\colon h_{X,\overline{x}}(A) \to h^1((L\overline{x}^*L^\bullet_X)^\vee)\otimes I$$

such that $ob_X(f_X)=0$ *if and only if there exists* $f'_X\in h_{X,\overline{x}}(A')$ *such that* $f'_X\circ i=f_X$*;*

(2) if $ob_X(f_X)=0$ *then the set of isomorphism classes of* $f'_X\in h_{X,\overline{x}}(A')$ *such that* $f'_X\circ i=f_X$ *is a torsor under* $h^0((L\overline{x}^*L^\bullet_X)^\vee)\otimes I$*;*

(3) if $ob_X(f_X)=0$ *and* $f'_X\in h_{X,\overline{x}}(A')$ *is such that* $f'_X\circ i=f_X$*, then the group of infinitesimal authomorphisms of* f'_X *with respect to* f_X *is isomorphic to* $h^{-1}((L\overline{x}^*L^\bullet_X)^\vee)\otimes I$*.*

Proof. Let $u\colon U\to X$ be a representable smooth and surjective morphism from a scheme U, then $\overline{x}\to X$ factors through u. By Theorem 3.19, $h^1((L\overline{x}^*L^\bullet_X)^\vee)\cong h^1((L\overline{x}^*L^\bullet_U)^\vee)$ and we have the following exact sequence

$$0\to h^{-1}((L\overline{x}^*L^\bullet_X)^\vee)\to h^0((L\overline{x}^*L^\bullet_u)^\vee)\to h^0((L\overline{x}^*L^\bullet_U)^\vee)\to h^0((L\overline{x}^*L^\bullet_X)^\vee)\to 0.$$

Moreover, by deformation theory of schemes, we know that there is a functorial exact sequence

$$0\to h^0((L\overline{x}^*L^\bullet_U)^\vee)\otimes I\to h_{U,\overline{x}}(A')\to h_{U,\overline{x}}(A)\xrightarrow{\mathrm{ob}_U} h^1((L\overline{x}^*L^\bullet_U)^\vee)\otimes I\to 0.$$

Let $f_X\in h_{X,\overline{x}}(A)$. By Proposition A.13, there exists $f_{U,1}\in h_{U,\overline{x}}(A)$ such that $u\circ f_{U,1}=f_X$. If $f_{U,2}\in h_{U,\overline{x}}(A)$ is another morphism such that $u\circ f_{U,2}=f_X$ then there exists a unique morphism $f_{U\times_X U}\in h_{U\times_X U,\overline{x}}(A)$ such that $u_j\circ f_{U\times_X U}=f_{U,j}$ for $j=1,2$, where $u_j\colon U\times_X U\to U$ are the projections. By Theorem 3.19, u_1 and u_2 induces isomorphisms $h^1((L\overline{x}^*L^\bullet_{U\times_X U})^\vee)\cong h^1((L\overline{x}^*L^\bullet_U)^\vee)$. Therefore

$$\mathrm{ob}_U(f_{U,j})=\mathrm{ob}_U(u_j\circ f_{U\times_X U})=\mathrm{ob}_{U\times_X U}(f_{U\times_X U}).$$

Hence, if we set $\mathrm{ob}_X(f_X)=\mathrm{ob}_U(f_{U,1})$, this gives a well-defined surjective map

$$\mathrm{ob}_X\colon h_{X,\overline{x}}(A)\to h^1((L\overline{x}^*L^\bullet_U)^\vee)\otimes I\cong h^1((L\overline{x}^*L^\bullet_X)^\vee)\otimes I.$$

Moreover, if $\mathrm{ob}_F(f_X)=0$ then there exist $f_U\in h_{U,\overline{x}}(A)$ such that $u\circ f_U=f_X$ and $\mathrm{ob}_U(f_U)=0$. Thus there exists a morphism $f'_U\in h_{U,\overline{x}}(A')$ such that $f'_U\circ i=f_U$ and $f'_X=u\circ f'_U\in h_{X,\overline{x}}(A')$ is such that $f'_X\circ i=f_X$. Therefore $h^1((L\overline{x}^*L^\bullet_X)^\vee)$ is an obstruction space for $h_{X,\overline{x}}$.

By Theorem 3.19, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0\longrightarrow h^{-1}((L\overline{x}^*L^\bullet_X)^\vee) & \longrightarrow & h^0((L\overline{x}^*L^\bullet_{\widetilde{u}})^\vee) & \xrightarrow{\rho_{\widetilde{u}}} & h^0((L\overline{x}^*L^\bullet_{U\times_X U})^\vee) & \xrightarrow{\rho_{U\times_X U}} & h^0((L\overline{x}^*L^\bullet_X)^\vee)\longrightarrow 0\\
\Vert & & \downarrow{\scriptstyle\rho_{u_j}} & & \downarrow{\scriptstyle\rho_{u_j}} & & \Vert\\
0\longrightarrow h^{-1}((L\overline{x}^*L^\bullet_X)^\vee) & \longrightarrow & h^0((L\overline{x}^*L^\bullet_u)^\vee) & \xrightarrow{\rho_u} & h^0((L\overline{x}^*L^\bullet_U)^\vee) & \xrightarrow{\rho_U} & h^0((L\overline{x}^*L^\bullet_X)^\vee)\longrightarrow 0
\end{array}$$

where we set $\widetilde{u}=u\circ u_j$. Let us fix $\overline{f}_X\in h_{X,\overline{x}}(A)$, $\overline{f}_U\in h_{U,\overline{x}}(A)$, $\overline{f}_{U\times_X U}\in h_{U\times_X U,\overline{x}}(A)$ such that $u\circ\overline{f}_U=\overline{f}_X$, $u_j\circ\overline{f}_{U\times_X U}=\overline{f}_U$. Assume that $\mathrm{ob}_X(\overline{f}_X)=0$ and fix $\overline{f}'_X\in\mathcal{S}_X$, $\overline{f}'_U\in\mathcal{S}_U$, $\overline{f}'_{U\times_X U}\in\mathcal{S}_{U\times_X U}$ such that $u\circ\overline{f}'_U=\overline{f}'_X$, $u_j\circ\overline{f}'_{U\times_X U}=\overline{f}'_U$. There is a natural surjective map $\mathcal{S}_U\to\mathcal{S}_X$ given by composition with u. We know that $\mathcal{S}_U$ and $\mathcal{S}_U$ are torsors under $h^0((L\overline{x}^*L^\bullet_U)^\vee)\otimes I$ and $h^0((L\overline{x}^*L^\bullet_u)^\vee)\otimes I$, respectively. Let $\alpha_U\in h^0((L\overline{x}^*L^\bullet_U)^\vee)\otimes I$ be such that $\rho_U(\alpha_U)$ is the identity $e_X\in h^0((L\overline{x}^*L^\bullet_X)^\vee)\otimes I$, then $\alpha_U=\rho_u(\alpha_u)$ for some $\alpha_u\in h^0((L\overline{x}^*L^\bullet_u)^\vee)\otimes I$. It follows that $\alpha_U\cdot\overline{f}'_U=\overline{f}_U$ and therefore $\alpha_U=e_U$. As a consequence, for every $\alpha_X\in h^0((L\overline{x}^*L^\bullet_X)^\vee)\otimes I$, we can define $\alpha_X\cdot\overline{f}'_X=u\circ(\alpha_U\cdot\overline{f}'_U)\in\mathcal{S}_X$ for some $\alpha_U\in h^0((L\overline{x}^*L^\bullet_U)^\vee)\otimes I$ such that $\rho_U(\alpha_U)=\alpha_X$. We claim that this defines an action of $h^0((L\overline{x}^*L^\bullet_X)^\vee)\otimes I$ over $\mathcal{S}_X$ which is transitive and free. Let $f'_X\in\mathcal{S}_X$, then there exists $f'_U\in\mathcal{S}_U$ such that $u\circ f'_U=f'_X$. Moreover $f'_U=\alpha_U\cdot\overline{f}'_U$ for some $\alpha_U\in h^0((L\overline{x}^*L^\bullet_U)^\vee)\otimes I$, hence $f'_X=\rho_U(\alpha_U)\cdot\overline{f}'_X$. Now if $\alpha_X\cdot\overline{f}'_X=\overline{f}'_X$ then $\alpha_X=\rho_U(\alpha_U)$ and $u\circ(\alpha_U\cdot\overline{f}'_U)=u\circ\overline{f}'_U$. Hence there exists $\alpha'_{U\times_X U}\in h^0((L\overline{x}^*L^\bullet_{U\times_X U})^\vee)\otimes I$ such that $\rho_{u_1}(\alpha'_{U\times_X U})\cdot\overline{f}'_U=\alpha_U\cdot\overline{f}'_U$ and

$\rho_{u_2}(\alpha'_{U\times_X U})\cdot \overline{f}'_U = \overline{f}'_U$. Therefore $\rho_{u_2}(\alpha_{U\times_X U}) = e_U$, $\rho_{u_1}(\alpha_{U\times_X U}) = \alpha_U$, and $\alpha_{U\times_X U} = \rho_{\widetilde{u}}(\alpha_{\widetilde{u}})$. It follows that

$$\alpha_X = \rho_U(\alpha_U) = \rho_U(\rho_{u_1}(\alpha_{U\times_X U})) = \rho_{U\times_X U}(\alpha_{U\times_X U}) = e_X,$$

and this proves that $\mathcal{S}_X$ is a torsor under $h^0((L\overline{x}^* L^\bullet_X)^\vee)\otimes I$.

Finally, if $f_X \in h_{X,\overline{x}}(A)$ is such that $\mathrm{ob}_X(f_X) = 0$, let us fix $f'_X \in h_{X,\overline{x}}(A')$ such that $f'_X \circ i = f_X$. We claim that the kernel of the natural homomorphism $\mathrm{Aut}_X(f'_X) \xrightarrow{i^*_X} \mathrm{Aut}_X(f_X)$ is isomorphic to $h^{-1}((L\overline{x}^* L^\bullet_X)^\vee)\otimes I$. Let $f_U \in h_{U,\overline{x}}(A)$, $f'_U \in h_{U,\overline{x}}(A')$ such that $f'_U \circ i = f_U$, $u \circ f'_U = f'_X$. We know that we have the following commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathrm{Aut}_U(f'_U) & \longrightarrow & \mathrm{Aut}_U(f_U)\times_{\mathrm{Aut}_X(f_X)} \mathrm{Aut}_X(f'_X) & \xrightarrow{\omega} & h^0((L\overline{x}^* L^\bullet_u)^\vee)\otimes I \\
\Vert & & \downarrow & & \downarrow \rho_u \\
\mathrm{Aut}_U(f'_U) & \longrightarrow & \mathrm{Aut}_U(f_U) & \longrightarrow & h^0((L\overline{x}^* L^\bullet_U)^\vee)\otimes I
\end{array}$$

where $\mathrm{Aut}_U(f'_U) = \mathrm{Aut}_U(f_U) = \{\mathrm{id}\}$ since U is a scheme. Therefore

$$\ker(i^*_X) = \mathrm{Aut}_U(f_U)\times_{\mathrm{Aut}_X(f_X)} \mathrm{Aut}_X(f'_X) \subset h^0((L\overline{x}^* L^\bullet_u)^\vee)\otimes I.$$

Moreover, an element of $h^0((L\overline{x}^* L^\bullet_u)^\vee)\otimes I$ acts trivially on $\mathcal{S}_u$ if and only if it is in the image of ω. Therefore $\mathrm{Aut}_U(f_U)\times_{\mathrm{Aut}_X(f_X)} \mathrm{Aut}_X(f'_X) = h^{-1}((L\overline{x}^* L^\bullet_X)^\vee)\otimes I$. □

A.23. **Proposition.** *Let $L^\bullet_F$ be the relative cotangent complex of F. Then, for every geometric point $\overline{x}$ of X and for every small extension $A' \to A = {}^{A'}\!/_{I}$ in $({}^{Art}\!/_{\hat{\mathcal{O}}_{S,\overline{x}}})$,*

(1) there is a functorial surjective set-theoretical map

$$ob_F\colon h_{X,\overline{x}}(A)\times_{h_{Y,\overline{x}}(A)} h_{Y,\overline{x}}(A') \to h^1((L\overline{x}^* L^\bullet_F)^\vee)\otimes I$$

such that $ob_F(f_X, f'_Y) = 0$ if and only if there exists $f'_X \in h_{X,\overline{x}}(A')$ such that $f'_X \circ i = f_X$ and $F\circ f'_X = f'_Y$;

(2) if $ob_F(f_X, f'_Y) = 0$ then the set of isomorphism classes of $f'_X \in h_{X,\overline{x}}(A')$, such that $f'_X\circ i = f_X$ and $F\circ f'_X = f'_Y$, is a torsor under $h^0((L\overline{x}^* L^\bullet_F)^\vee)\otimes I$;

(3) if $ob_F(f_X, f'_Y) = 0$ and $f'_X \in h_{X,\overline{x}}(A')$ is such that $F\circ f'_X = f'_Y$, $f'_X\circ i = f_X$, then the group of infinitesimal authomorphisms of f'_X with respect to (f_X, f'_Y) contains only the identity.

Proof. Let $v\colon V \to Y$ be a smooth surjective morphism from a scheme V, then U is a Deligne-Mumford stack. Let $w\colon W \to U$ be an étale surjective morphism from a scheme W then, by Theorem 3.19, $Lu^* L^\bullet_F \cong L^\bullet_G$ and $Lw^* L^\bullet_G \cong L^\bullet_{G\circ w}$. By Proposition A.14, $\overline{x} \to X$ factors through u and $u\circ w$. Moreover $h_{W,\overline{x}} \cong h_{U,\overline{x}}$, because w is étale ([22] I.3.22), and, by deformation theory of schemes, we get a functorial exact sequence

$$0 \to h^0((L\overline{x}^* L^\bullet_G)^\vee)\otimes I \to h_{U,\overline{x}}(A') \to h_{U,\overline{x}}(A)\times_{h_{V,\overline{x}}(A)} h_{V,\overline{x}}(A') \xrightarrow{\mathrm{ob}_G} h^1((L\overline{x}^* L^\bullet_G)^\vee)\otimes I \to 0.$$

Therefore the first and the second part of the statement follows by the proof of Proposition A.21.

By Theorem 3.19, we have the following exact sequence

$$0 \to h^{-1}((L\overline{x}^* L^\bullet_X)^\vee) \to h^{-1}((L\overline{x}^* L^\bullet_Y)^\vee) \to h^0((L\overline{x}^* L^\bullet_F)^\vee) \to h^0((L\overline{x}^* L^\bullet_X)^\vee) \to h^0((L\overline{x}^* L^\bullet_Y)^\vee).$$

Let $f_X \in h_{X,\overline{x}}(A)$, $f_Y \in h_{Y,\overline{x}}(A)$, $f'_Y \in h_{Y,\overline{x}}(A')$ such that $F\circ f_X = f_Y = f'_Y \circ i$. Assume that $\mathrm{ob}_F(f_X, f'_Y) = 0$ and fix $f'_X \in h_{X,\overline{x}}(A')$ such that $F\circ f'_X = f'_Y$, $f'_X \circ i = f_X$. By Proposition A.22, we have the following commutative diagram with exatc rows

$$\begin{array}{ccccccc}
 & & & & \mathrm{Aut}_X(f'_X) & \longrightarrow & \mathrm{Aut}_X(f_X)\times_{\mathrm{Aut}_Y(f_Y)}\mathrm{Aut}_Y(f'_Y) \\
 & & & & \| & & \downarrow \\
0 & \longrightarrow & h^{-1}((L\overline{x}^*L^\bullet_X)^\vee)\otimes I & \longrightarrow & \mathrm{Aut}_X(f'_X) & \longrightarrow & \mathrm{Aut}_X(f_X) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & h^{-1}((L\overline{x}^*L^\bullet_Y)^\vee)\otimes I & \longrightarrow & \mathrm{Aut}_Y(f'_Y) & \longrightarrow & \mathrm{Aut}_Y(f_Y)
\end{array}$$

from which we deduce that the only infinitesimal automorphism is the identity. □

A.24. Let X be a smooth S-scheme and let $\operatorname{Spec}\overline{k}\xrightarrow{\overline{x}}X$ be a geometric point of X. Let $C\to\operatorname{Spec}\overline{x}$ be a flat morphism of schemes, with C separated. Let $\Lambda=\hat{\mathscr{O}}_{S,\overline{x}}$. We define the deformation category Def_C such that, for all $A\in({}^{\mathrm{Art}}/\Lambda)$, the objects of $\mathrm{Def}_C(A)$ are flat morphisms $C_A\xrightarrow{\pi_A}\operatorname{Spec}A$ such that the following diagram is cartesian

$$\begin{array}{ccc}
C & \xrightarrow{g} & C_A \\
\downarrow & & \downarrow{\scriptstyle \pi_A} \\
\operatorname{Spec}\overline{x} & \longrightarrow & \operatorname{Spec}A
\end{array}$$

If $\pi\colon A'\to A$ is a morphism in $({}^{\mathrm{Art}}/\Lambda)$, then

$$\mathrm{Def}_C(\pi)\colon \mathrm{Def}_C(A')\to\mathrm{Def}_C(A)$$

sends $\pi_{A'}\colon C_{A'}\to\operatorname{Spec}A'$ to $C_{A'}\times_{\operatorname{Spec}A'}\operatorname{Spec}A\to\operatorname{Spec}A$. Given a morphism of schemes $f\colon C\to X$, we define the deformation category $\mathrm{Def}_{C,f}$ such that, for all $A\in({}^{\mathrm{Art}}/\Lambda)$, the objects of $\mathrm{Def}_{C,f}(A)$ are pairs of morphisms $(C_A\xrightarrow{\pi_A}\operatorname{Spec}A, f_A)$ where π_A is an object of Def_C and $f_A\colon C_A\to X$ is such that $f_A\circ g=f$. There is a natural functor $\nu_f\colon\mathrm{Def}_{C,f}\to\mathrm{Def}_C$ which forgets the morphism to X.

A.25. **Proposition.** *The functor ν_f has tangent and obstruction spaces $T^i\nu_f=H^{i-1}(C,f^*T_{X/S})$, for $i=1,2$.*

Proof. Given a small extension

$$0\to I\to A'\to A\to 0, \tag{2}$$

we want to study the functor

$$\mathrm{Def}_{C,f}(A')\to\mathrm{Def}_{C,f}(A)\times_{\mathrm{Def}_C(A)}\mathrm{Def}_C(A').$$

An object of $\mathrm{Def}_{C,f}(A)\times_{\mathrm{Def}_C(A)}\mathrm{Def}_C(A')$ is a cartesian diagram

$$\begin{array}{ccc}
C_A & \longrightarrow & C_{A'} \\
{\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \pi_{A'}} \\
\operatorname{Spec}A & \longrightarrow & \operatorname{Spec}A'
\end{array}$$

togheter with a morphism $f_A\colon C_A\to X$. We want to know whether there exists a morphism $f_{A'}\colon C_{A'}\to X$ such that $f_{A'}|_{C_A}=f_A$ and how unique is such a morphism.

Assume $X = \operatorname{Spec} T$ and $C = \operatorname{Spec} R$, then $C_{A'} = \operatorname{Spec} R_{A'}$ and $C_A = \operatorname{Spec} R_A$ with $R_{A'} = R \otimes_\Lambda A'$ and $R_A = R \otimes_\Lambda A$. By assumption $\pi_{A'}$ is flat, then the tensor product $R_{A'} \otimes_{A'} \bullet$ is exact and from the sequence (2) we get an exact sequence

$$0 \to R \otimes_\Lambda I \to R_{A'} \to R_A \to 0.$$

Notice that $(R_{A'} \otimes_{A'} I)(R_{A'} \otimes_{A'} \mathfrak{m}_{A'}) = 0$, because $I\mathfrak{m}_{A'} = 0$. The morphism f_A induces a homomorphism $f_A^\sharp \colon T \to R_A$. We have that $T = {}^{T'}\!/_{J}$, with $T' = B[x_1, \dots, x_n]$. There exists always a lifting $g_{A'}^\sharp \colon T' \to R_{A'}$ of $f_A^\sharp$ and the set of such liftings is a principal homogeneous space under $\operatorname{Hom}_{T'}(\Omega_{T'/B}, R \otimes_\Lambda I)$. Moreover there exists a homomorphism

$$\alpha \colon \operatorname{Hom}_{T'}(\Omega_{T'/B}, R \otimes_\Lambda I) \to \operatorname{Hom}_{T'}({}^{J}\!/_{J^2}, R \otimes_\Lambda I),$$

induced by restriction, such that $\ker \alpha = T^1\nu_f \otimes I$ and $\operatorname{coker} \alpha = T^2\nu_f \otimes I$. Since

$$R \otimes_\Lambda I \cong (R \otimes_\Lambda \overline{k}) \otimes_{\overline{k}} I \cong R \otimes_{\overline{k}} I,$$

we have

$$\operatorname{Hom}_{T'}(\Omega_{T'/B}, R \otimes_\Lambda I) \cong \operatorname{Hom}_{T'}(\Omega_{T'/B}, R) \otimes_{\overline{k}} I \cong H^0(C, f^*T_{\mathbb{A}^n_S/S}|_X)$$
$$\operatorname{Hom}_{T'}({}^{J}\!/_{J^2}, R \otimes_\Lambda I) \cong \operatorname{Hom}_{T'}({}^{J}\!/_{J^2}, R) \otimes_{\overline{k}} I \cong H^0(C, f^*({}^{J}\!/_{J^2})^\vee).$$

We have an exact sequence

$$0 \to {}^{J}\!/_{J^2} \to \Omega_{T'/B} \otimes_{T'} T \to \Omega_{T/B} \to 0,$$

from which we deduce the following exact sequence

$$0 \to H^0(C, f^*T_{X/S}) \to H^0(C, f^*T_{\mathbb{A}^n_S/S}|_X) \to H^0(C, f^*({}^{J}\!/_{J^2})^\vee) \to H^1(C, f^*T_{X/S}),$$

where $H^1(C, f^*T_{X/S}) = 0$, because C is affine and $T_{X/S}$ is locally free. Hence $T^2\nu_f = 0$ and $T^1\nu_f = H^0(C, f^*T_{X/S})$.

In general, we cover X and C by open affine schemes $\{X_i = \operatorname{Spec} T_i\}$ and $\{C_i = \operatorname{Spec} R_i\}$, and we consider open affine covers $\{C_{A,i} = \operatorname{Spec} R_{A,i}\}$ and $\{C_{A',i} = \operatorname{Spec} R_{A',i}\}$ of C_A and $C_{A'}$ respectively, where $R_{A,i} = R_i \otimes_\Lambda A$ and $R_{A',i} = R_i \otimes_\Lambda A'$. We can chooce these covers in such a way that $f_A(C_{A,i}) \subset X_i$. To define $f_{A'}$ is the same as to define $f_{A',i} \colon C_{A',i} \to X_i$ such that $f_{A',i}|_{C_{A',ij}} = f_{A',j}|_{C_{A',ij}}$, where $C_{A',ij} = C_{A',i} \times_{C_{A'}} C_{A',j}$. By the affine case, there exist always a collection of liftings $\{f_{A',i}\}$. We can assume that $C_{A',ij}$ is affine (otherwise we consider an affine cover), then there exists a unique element $\eta_{ij} \in H^0(C_{ij}, f^*T_{X/S}) \otimes I$ such that $\eta_{ij}(f_{A',i}) = f_{A',j}$. Over C_{hij} we have

$$\eta_{hj}(f_{A',h}) = \eta_{ij}(\eta_{hi}(f_{A',h})),$$

hence $\eta_{hj} = \eta_{ij} + \eta_{hi}$ and so $\{\eta_{ij}\} \in C^1(\{C_i\}, f^*T_{X/S}) \otimes I$ is a cocycle. For each i, the set of liftings $\{f_{A',i}\}$ is a principal homogeneous space under $H^0(C_i, f^*T_{X/S}) \otimes I$. Therefore, given another collection of liftings $\{\tilde{f}_{A',i}\}$, there exists a unique collection $\{\eta_i \in H^0(C_i, f^*T_{X/S}) \otimes I\}$ such that $\eta_i(\{\tilde{f}_{A',i}\}) = \{f_{A',i}\}$. Let $\tilde{\eta}_{ij} \in H^0(C_{ij}, f^*T_{X/S}) \otimes I$ be such that $\tilde{\eta}_{ij}(\tilde{f}_{A',i}) = \tilde{f}_{A',j}$. Over C_{ij} we have

$$\eta_j(\tilde{\eta}_{ij}(\tilde{f}_{A',i})) = f_{A',j} = \eta_{ij}(f_{A',i}) = \eta_{ij}(\eta_i(\tilde{f}_{A',i})),$$

hence $\tilde{\eta}_{ij} = \eta_{ij} + \eta_i - \eta_j$ and the cocycle $\{\eta_{ij}\}$ is unique up to a coboundary. Now there exists a collection of morphisms $f_{A',i}$ which coincide on $C_{A',ij}$ if and only if the class of $\{\eta_{ij}\}$ in $H^1(\{C_i\}, f^*T_{X/S}) \otimes I$ is zero. In this case the set of such collections is equal to the set of $\{\eta_i \in H^0(C_i, f^*T_{X/S}) \otimes I\}$ and the gluing condition is equivalent to $\eta_i = \eta_j$ on C_{ij}. It follows that $\{\eta_i\} \in H^0(\{C_i\}, f^*T_{X/S}) \otimes I$. Finally

$$H^r(\{C_i\}, f^*T_{X/S}) \cong H^r(C, f^*T_{X/S}),$$

because C is separated. □

Appendix B. Intersection theory on Artin stacks over Dedekind domains

Intersection theory for schemes of finite type over a field was developed by Fulton and MacPherson ([12]) and was extended by Vistoli to a $\mathbb{Q}$-valued intersection theory on Deligne-Mumford stacks ([27]). In [9], Edidin and Graham define equivariant Chow groups, which provide integer-valued Chow groups for global quotient stacks. In [18], Kresch takes the idea of Edidin, Graham and Totaro further and develops an intersection theory on Artin stacks over a field together with an integer-valued intersection product on smooth Artin stacks which admit stratifications by global quotient stacks. Using an appropriate definition of relative dimension, one can define Chow groups for schemes over a Dedekind domain and show that they satisfy the properties expected from intersection theory ([12] 20). It follows that the theories in [27] and [9] are valid for stacks over a Dedekind domain ([27], [9] 6.2).

Although not mentioned in [18], one can verify that the theory can be extended to Artin stacks over a Dedekind domain. As a consequence we get that Manolache's construction of the virtual pullback in [21] is valid for Deligne-Mumford type morphisms of Artin stacks over a Dedekind domain. As a consequence we are able to extend Manolache's proof of Costello's pushforward formula to proper morphisms of Artin stacks with quasi-finite diagonal.

B.1. **Chow groups of Artin stacks with quasi-finite diagonal.** Let D be a Dedekind domain and let $\mathcal{M}$ be an Artin stack over $S = \operatorname{Spec} D$. For an integral closed substack $Z \subset \mathcal{M}$, we define the relative dimension ([12] 20.1)

$$\dim_S Z = \operatorname{trdeg}_{k(T)} k(Z) - \operatorname{codim}_S T,$$

where T is the closure of the image of $\mathcal{Z}$ in S, and $k(Z)$, $k(T)$ are function fields ([27] 1.14).

B.1. **Definition.** We denote by $Z_*(\mathcal{M}/S)$ the free abelian group on the set of integral closed substacks of $\mathcal{M}$, graded by relative dimension. Let $W_j(\mathcal{M}/S) = \bigoplus_{\mathcal{Z}} k(Z)^*$, where the sum is taken over all integral substacks Z of $\mathcal{M}$ of relative dimension $j+1$. There is a homomorphism $\partial\colon W_j(\mathcal{M}/S) \to Z_j(\mathcal{M}/S)$ which locally for the smooth topology sends a rational function to the corresponding Weil divisor. The *Chow groups* of $\mathcal{M}$ are defined to be $A_j(\mathcal{M}/S) = Z_j(\mathcal{M}/S)/\partial W_j(\mathcal{M}/S)$.

B.2. **Theorem** ([18] 3.5.7, 5.3.1)**.** *Let $\mathcal{M}$ be an Artin stack with quasi-finite diagonal over a Dedekind domain D. Then $A_*(\mathcal{M}/S) \cong A_*^{Kresch}(\mathcal{M}/S)$, where $A_*^{Kresch}(\mathcal{M}/S)$ are Kresch's Chow groups (*[18] *2.1.11).*

B.3. **Theorem** ([10] 2.7)**.** *Let $\mathcal{M}$ be an Artin stack with quasi-finite diagonal over a Dedekind domain D. Then there exists a finite surjective morphism $U \to \mathcal{M}$ from a scheme U.*

B.4. Remark. The morphism $U \to \mathcal{M}$ is strongly representable.

B.2. **Proper pushforward** ([27] 3.6–3.8, [10] 2.8). Let $\pi\colon \mathcal{N} \to \mathcal{M}$ be a proper morphism of Artin S-stacks. If $\mathcal{M}$ and $\mathcal{N}$ have quasi-finite diagonal then it is possible to define a nonrepresentable proper pushforward π_* as follows.

B.5. **Definition.** Let $u\colon U \to \mathcal{M}$ be a finite and surjective morphism from a scheme U. We define the *proper pushforward*

$$u_*\colon A_*(U/S) \to A_*(\mathcal{M}/S)$$

by $u_*[Z] = \deg(Z/u(Z))[u(Z)]$, where $\deg(Z/u(Z)) = \deg(V \times_{\mathcal{M}} U/V)$ for a smooth atlas $V \to u(Z)$.

B.6. Remark. Notice that $V \times_{\mathcal{M}} U \cong V \times_{u(Z)} Z$ is a scheme and the degree $\deg(Z/u(Z))$ is independent of the chosen atlas. Let $V' \to u(Z)$ be another smooth atlas and set $W = V \times_{u(Z)} V'$. Then

$$\deg(V \times_{\mathcal{M}} U/V) = \deg(W \times_{\mathcal{M}} U/W) = \deg(V' \times_{\mathcal{M}} U/V').$$

B.7. Remark. The proper pushforward commutes with projective pushforward and flat pullback (this follows easily from the properties of the relative degree).

B.8. Notation. We set $A_*(\mathcal{M}/S)_\mathbb{Q} = A_*(\mathcal{M}/S) \otimes_\mathbb{Z} \mathbb{Q}$.

B.9. **Lemma.** *Let $u\colon U \to \mathcal{M}$ be a finite and surjective morphism from a scheme U and let u_1, $u_2\colon U \times_\mathcal{M} U \to U$ be the projections. Then we have the following exact sequence*

$$A_j(U \times_\mathcal{M} U/S)_\mathbb{Q} \xrightarrow{u_{1*}-u_{2*}} A_j(U/S)_\mathbb{Q} \xrightarrow{u_*} A_j(\mathcal{M}/S)_\mathbb{Q} \to 0.$$

Proof. For surjectivity of u_*, let $[\mathcal{Z}] \in A_j(\mathcal{M}/S)_\mathbb{Q}$. Let Z be a j-dimensional component of $\mathcal{Z} \times_\mathcal{M} U$, then $[Z] \in A_j(U/S)_\mathbb{Q}$ and

$$u_*\left(\frac{1}{\deg(Z/\mathcal{Z})}[Z]\right) = [\mathcal{Z}] \in A_j(\mathcal{M}/S)_\mathbb{Q}.$$

Notice that also $u_*\colon W_*(U/S) \to W_*(\mathcal{M}/S)$ is surjective. Moreover, for $[Z] \in A_j(U \times_\mathcal{M} U/S)_\mathbb{Q}$,

$$\begin{aligned} u_*(u_{1*} - u_{2*})[Z] &= u_*(\deg(Z/u_1(Z))[u_1(Z)] - \deg(Z/u_2(Z))[u_2(Z)]) \\ &= \deg(Z/u(u_1(Z)))[u(u_1(Z))] - \deg(Z/u(u_2(Z))))[u(u_2(Z))] = 0. \end{aligned}$$

So it is enough to show that every $\alpha = \sum_{i=1}^s n_i[Z_i] \in Z_j(U/S)_\mathbb{Q}$ such that $u_*(\alpha) = 0$ in $Z_j(\mathcal{M}/S)_\mathbb{Q}$ lies in the image of $u_{1*} - u_{2*}$. Since

$$u_*(\alpha) = \sum_{i=1}^s n_i \deg(Z_i/u(Z_i))[u(Z_i)] = 0,$$

we may assume that $u(Z_i) = \mathcal{Z}$ for $i = 1, \ldots, s$. Therefore we get $\sum_{i=1}^s n_i d_i = 0$, where we set $d_i = \deg(Z_i/\mathcal{Z})$. For $i = 2, \ldots, s$, let V_i be a j-dimensional component of $Z_1 \times_\mathcal{Z} Z_i$, then

$$u_{2*}[V_i] = \deg(V_i/Z_i)[Z_i] = e_i d_1 [Z_i],$$

where we set $e_i = \deg(V_i/Z_i \times_\mathcal{Z} Z_1)$. By properties of relative degree,

$$u_{1*}[V_i] = \deg(V_i/Z_1)[Z_1] = e_i d_i [Z_1],$$

and it follows that

$$(u_{1*} - u_{2*}) \sum_{i=2}^s \frac{n_i}{e_i d_1}[V_i] = \alpha. \qquad \square$$

B.10. Remark. If $p\colon T \to U$ is a finite surjective morphism from a scheme T and we set $t = u \circ p$, then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} A_*(T \times_\mathcal{M} T/S)_\mathbb{Q} & \xrightarrow{t_{1*}-t_{2*}} & A_*(T/S)_\mathbb{Q} & \xrightarrow{t_*} & A_*(\mathcal{M}/S)_\mathbb{Q} & \longrightarrow & 0 \\ \downarrow & & \downarrow p_* & & \| & & \\ A_*(U \times_\mathcal{M} U/S)_\mathbb{Q} & \xrightarrow{u_{1*}-u_{2*}} & A_*(U/S)_\mathbb{Q} & \xrightarrow{u_*} & A_*(\mathcal{M}/S)_\mathbb{Q} & \longrightarrow & 0 \end{array}$$

B.11. Let $u\colon U \to \mathcal{M}$ be a finite surjective morphism from a scheme U and form the fibre diagram

$$\begin{array}{ccc} V & \xrightarrow{\pi'} & U \\ v \downarrow & & \downarrow u \\ \mathcal{N} & \xrightarrow{\pi} & \mathcal{M} \end{array}$$

Then V is a scheme and v is finite surjective. Moreover, by Lemma B.9, we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc} A_*({}^{V\times_{\mathcal{N}}V}/_S)_{\mathbb{Q}} & \xrightarrow{v_{1*}-v_{2*}} & A_*({}^{V}/_S)_{\mathbb{Q}} & \xrightarrow{v_*} & A_*({}^{\mathcal{N}}/_S)_{\mathbb{Q}} & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \pi'_*} & & & & \\ A_*({}^{U\times_{\mathcal{M}}U}/_S)_{\mathbb{Q}} & \xrightarrow{u_{1*}-u_{2*}} & A_*({}^{U}/_S)_{\mathbb{Q}} & \xrightarrow{u_*} & A_*({}^{\mathcal{M}}/_S)_{\mathbb{Q}} & \longrightarrow & 0 \end{array}$$

which induces a map $\pi_*\colon A_*({}^{\mathcal{N}}/_S)_{\mathbb{Q}} \to A_*({}^{\mathcal{M}}/_S)_{\mathbb{Q}}$.

B.12. **Lemma.** *The map π_* does not depend on the choice of the finite surjective morphism $u\colon U \to \mathcal{M}$.*

Proof. Let $u'\colon U' \to \mathcal{M}$ be a finite surjective morphism from a scheme U' and consider $T = U\times_{\mathcal{M}}U'$ with the natural morphism $p\colon T \to U$, which is finite surjective. Let us form the fibre diagram

$$\begin{array}{ccc} W & \xrightarrow{\pi''} & T \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{\pi'} & U \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{N} & \xrightarrow{\pi} & \mathcal{M} \end{array}$$

and set $t = u\circ p$, $w = v\circ q$. Let us denote by $\widetilde{\pi}_*$ the pullback defined via $t\colon T \to \mathcal{M}$. Let $\alpha \in A_*({}^{\mathcal{N}}/_S)_{\mathbb{Q}}$ and let $\alpha'' \in A_*({}^{W}/_S)_{\mathbb{Q}}$ such that $w_*\alpha'' = \alpha$, then

$$\widetilde{\pi}_*\alpha = t_*\pi''_*\alpha'' = u_*p_*\pi''_*\alpha'' = u_*\pi'_*(q_*\alpha'') = \pi_*\alpha,$$

where the last equality follows from the fact that $v_*(q_*\alpha'') = t_*\alpha'' = \alpha$. ☐

B.13. **Definition.** We call $\pi_*\colon A_*({}^{\mathcal{N}}/_S)_{\mathbb{Q}} \to A_*({}^{\mathcal{M}}/_S)_{\mathbb{Q}}$ the *proper pushforward* for π.

B.3. **Costello's pushforward formula.** In [21] Manolache uses the virtual pullback to give a short proof of Costello's pushforward formula ([8] 5.0.1). Here we apply Manolache's construction to prove the pushforward formula in a more general setting.

B.14. **Proposition.** *Let D be a Dedekind domain. Let us consider a cartesian diagram*

$$\begin{array}{ccc} \mathcal{M}_1 & \xrightarrow{f} & \mathcal{M}_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathfrak{M}_1 & \xrightarrow{g} & \mathfrak{M}_2 \end{array}$$

where

(1) *$\mathfrak{M}_1, \mathfrak{M}_2$ are Artin stacks over D of the same pure dimension,*
(2) *$\mathcal{M}_1, \mathcal{M}_2$ are Artin stacks over D with quasi-finite diagonal,*
(3) *g is a Deligne-Mumford type morphism of degree d,*
(4) *f is proper,*
(5) *for $i = 1, 2$, p_i admits perfect obstruction theory $E_i^\bullet$ such that $f^*E_2^\bullet \cong E_1^\bullet$.*

Then

$$f_*[\mathcal{M}_1, E_1^\bullet]^{virt} = d[\mathcal{M}_2, E_2^\bullet]^{virt}$$

in each of the following cases

(a) *g is projective,*

(*b*) $\mathfrak{M}_1, \mathfrak{M}_2$ *are Deligne-Mumford stacks and* g *is proper,*
(*c*) $\mathfrak{M}_1, \mathfrak{M}_2$ *have quasi-finite diagonal and* g *is proper.*

Proof. Since in each of the cases listed above we are able to pushforward along g, the statement follows by the same argument of [21] 5.29, after noticing that non-representable proper pushforward commutes with virtual pullback. □

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