Canonical κ-Minkowski Spacetime

Ludwik Dąbrowski[*], Gherardo Piacitelli[*]

April 29, 2010

Abstract

A complete classification of the regular representations of the relations

$$[T, X_j] = \frac{i}{\kappa} X_j, \quad j = 1, \ldots, d$$

is given. The quantisation of $\mathbb{R} \times \mathbb{R}^d$ canonically (in the sense of Weyl) associated with the universal representation of the above relations is intrinsically "radial", this meaning that it only involves the time variable and the distance from the origin; angle variables remain classical.

The time axis through the origin is a spectral singularity of the model: in the large scale limit it is topologically disjoint from the rest.

The symbolic calculus is developed; in particular there is a trace functional on symbols.

For suitable choices of states localised very close to the origin, the uncertainties of all spacetime coordinates can be made simultaneously small at wish. On the contrary, uncertainty relations become important at "large" distances: Planck scale effects should be visible at LHC energies, if processes are spread in a region of size 1mm (order of peak nominal beam size) around the origin of spacetime.

Contents

[*]SISSA, Via Bonomea 265, 34136 Trieste, Italy

1 Introduction

The κ-Minkowski spacetime, where the selfadjoint coordinates fulfil

$$[T, X_j] = \frac{i}{\kappa} X^j, \quad [X_j, X_k] = 0, \qquad j, k = 1, \ldots, d, \tag{1.1}$$

has been analysed for almost 20 years [1, 2], mainly from the algebraic point of view. Here we will take natural units where $\kappa = 1$, in addition to $c = \hbar = 1$.

Not much attention has been paid to representations, with the notable exception of the work of Agostini [3], where the representations in $d = 1$ space dimensions were constructed by means of the Jordan–Schwinger map, and classified by means of the theory of induced representations. In that paper (as well as in many others; see especially [4]), Weyl operators were then defined by making an arbitrary choice in the order of operator products; the corresponding quantisation then lacks the fundamental property of sending real functions into selfadjoint operators.

Our approach is closer in spirit to that of Weyl [5] and von Neumann [6]. After focusing on the appropriate regular commutation relations à la Weyl

$$e^{i\alpha T} e^{i\boldsymbol{\beta X}} = e^{ie^{-\alpha}\boldsymbol{\beta X}} e^{i\alpha T}, \quad \alpha \in \mathbb{R}, \boldsymbol{\beta} \in \mathbb{R}^d, \tag{1.2}$$

we will show that the most general representation of (1.2) is of the form

$$(T, X_1, \ldots, X_d), \quad X_j = C_j R, \tag{1.3}$$

where R is strictly positive, T has spectrum $\mathbb{R}$,

$$[T, R] = iR, \tag{1.4}$$

and each $C_j = X_j R^{-1}$ is bounded, non negative, and strongly commute swith both T, R. Moreover, up to a unitary equivalence, the operators C_j may be chosen such that

$$E = \sum_i C_i^2 \tag{1.5}$$

is an orthogonal projection. With this choice, $\sum_j X_j^2 = R^2 E$. In particular, the universal representation is the direct sum of a trivial and a non trivial component, corresponding to the two eigenspaces of E.

The analogy with polar coordinates is evident; for this reason we say that the quantisation is radial, since it only involves the commutation relations between time and the space radius; the angular variables remain commutative, i.e. classical. Note that, if $E \neq I$, then $\mathbf{0}$ is an isolated point of the joint spectrum $j\sigma(C_1, \ldots, C_d)$; we may regard this as the noncommutative shadow of the singularity of classical radial coordinates in the origin of space. However, this singularity is not a consequence of some arbitrary choice of representations: it is built in the commutation relations, which define an intrinsically radial model.

The classification problem is so reduced to the case of $1+1$ dimensions. The only regular irreducible representation of (1.4) with strictly positive R is (up to equivalence)

$$T = P, \quad R = e^{-Q}, \tag{1.6}$$

where P, Q is a pair of Schrödinger operators, fulfilling $[P, Q] = -iI$. Although this was proved in [3] (under a different, though equivalent definition of regularity), we present a more direct, elementary proof which relies of von Neumann uniqueness. This proved, by Schur lemma the irreducible non trivial representations $(T^{(\boldsymbol{c})}, \boldsymbol{X}^{(\boldsymbol{c})}) = (P, \boldsymbol{c}e^{-Q})$ are labeled by vectors $\boldsymbol{c} = (c_1, \ldots, c_d) \in S^{d-1} = \{\boldsymbol{c} \in \mathbb{R}^d : \|\boldsymbol{c}\| = 1\}$, while trivial irreducible representations $(\tau, \mathbf{0})$, as operators on $\mathbb{C}$, are labeled by a real parameter τ.

Our subsequent discussion is based on the explicit computation of the radial Weyl operators

$$e^{i(\alpha T + \beta R)} = e^{i\alpha T} e^{i\frac{e^{\alpha}-1}{\alpha}\beta R}. \tag{1.7}$$

Indeed, together with the Weyl relations (1.2), they *realise* precisely the composition rule described e.g. in [7] on the basis of the integration of the BCH series done in [4]. Note that to formally apply the BCH formula to unbounded operators is roughly equivalent (through the theory of analytic vectors) to assuming that the representation is regular.

Contrary to the case of the CCR, where products of Weyl operators are Weyl operators up to a phase only (the twist), the family of radial Weyl operators $e^{i(\alpha T+\beta R)}$ is a subgroup of the unitary group. Moreover, the correspondence with $\mathbb{R}^2$ is bijective, so that $\mathbb{R}^2$ inherits a group law; the resulting group[1] $\mathfrak{R}$ is isomorphic with the connected, simply connected Lie group whose real Lie algebra is generated by the relations $[u,v]=-v$. Then the full Weyl operators $e^{i(\alpha T+\boldsymbol{\beta}\boldsymbol{X})}$ form a group which is isomorphic with a central extension of the radial group $\mathfrak{R}$.

The natural (i.e. à la Weyl) prescription for the quantisation of a function $f=f(t,\boldsymbol{x})$ is

$$f(T,\boldsymbol{X})=\frac{1}{\sqrt{(2\pi)^{d+1}}}\int\limits_{\mathbb{R}^{d+1}} d\alpha\, d\boldsymbol{\beta}\, \hat{f}(\alpha,\beta)e^{i\alpha T+\boldsymbol{\beta}\boldsymbol{X}}, \tag{1.8}$$

where $\hat{f}$ is the Fourier transform of $f\in L^1(\mathbb{R}^d)\cap\widehat{L^1(\mathbb{R}^d)}$; we call such functions symbols, for short. It has all the necessary good properties: in particular, it sends real functions into selfadjoint operators. No prescription deserves the name of quantisation if not enjoying this property. Moreover, the quantisation prescription (extended to the multiplier algebra) sends plane waves $e^{i(\alpha t+\boldsymbol{\beta}\boldsymbol{x})}$ precisely into the corresponding Weyl operators.

The usual translation invariant Lebesgue measure $d\alpha d\boldsymbol{\beta}$ which shows up in the quantisation prescription (1.8) is *not* the Haar measure of the group of Weyl operators, which is not even unimodular. Notwithstanding this fact, the *-algebra of the symbols with product defined by

$$f(T,\boldsymbol{X})g(T,\boldsymbol{X})=(f\star g)(T,\boldsymbol{X}), \tag{1.9}$$

and pointwise conjugation as involution, reproduces precisely the group algebra of the group of Weyl operators, up to (completion and) isomorphism.

From the interplay between the radial nature of the quantisation and the connection with CCR quatisation, we obtain an unbounded linear funtional

$$\tau_c(f)=\frac{2\pi^{d/2}}{\Gamma(d/2)}\int dt\, d\boldsymbol{x}|\boldsymbol{x}|^{-d}f(t,\boldsymbol{x}) \tag{1.10}$$

[1]This group plays a rôle analogous to that of the Heisenberg group for the CCR. We refrain however from calling it the κ-Heisenberg group because it is not a deformation of the Heisenberg group; moreover our definition is slightly different from the κ-Heisenberg group defined in [3].

fulfilling

$$\tau_c(f \star g) = \tau_c(g \star f), \quad \tau_c(\bar{f} \star f) \geqslant 0,$$

which extends to an unbounded trace on the universal C*-algebra.

The universal C*-algebra of the algebra of symbols is

$$\mathcal{A}_d = \mathcal{C}(S^{d-1}, \mathcal{K}) \oplus \mathcal{C}_\infty(\mathbb{R}),$$

where $\mathcal{C}(S^{d-1}, \mathcal{K})$ is the algebra of the continuous functions of the sphere, with values in the algebra $\mathcal{K}$ of compact operators on the separable, infinite dimensional Hilbert space; in other words, a trivial continuous field of C*-algebras on S^{d-1}, with standard fibre $\mathcal{K}$.

The picture is that each fibre $\mathcal{K}$ over the base point $\boldsymbol{c} \in S^{d-1}$ describes the quantised open half plane which contains $\boldsymbol{c}$ and is bounded by the time axis. The time axis does not belong to any fibre, and remains classical: it is associated with the abelian C*-algebra $\mathcal{C}_\infty(\mathbb{R})$ of continuous functions vanishing at infinity; it arises from the trivial representations.

To say it differently, let $\overset{\circ}{M}{}^{(1,d)} = \mathbb{R} \times (\mathbb{R}^d - \{\mathbf{0}\})$ be the classical Minkowski spacetime with the time axis trhough the origin removed. Then $\mathcal{C}(S^{d-1}, \mathcal{K})$ is to be thought of as the quantisation of $\overset{\circ}{M}{}^{(1,d)}$, while $\mathbb{R}$ remains classical. This remains true for every value of κ and is thus bound to survive the large scale limit $\kappa \to \infty$. The resulting large scale limit is indeed $\mathbb{R}^{d+1}$ as a set, but equipped with a topology which makes the time axis topologically disconnected from the rest. More precisely, it is the disjoint topological union $\overset{\circ}{M}{}^{(1,d)} \sqcup \mathbb{R}$.

As a special case, the $1+1$ dimensional κ-Minkowski spacetime corresponds to $S^0 = \{\pm 1\}$, and has C*-algebra

$$\mathcal{A}_1 = \mathcal{K} \oplus \mathcal{C}_\infty(\mathbb{R}) \oplus \mathcal{K}.$$

In the large scale limit, the spacetime hase three disjoint connected components, corresponding to being on the left of the origin, in the origin, or on the right of the origin; see also [8].

The uncertainty relations are shortly discussed in section 7. We show that there are localisations states which make both the uncertainties $\Delta T, \Delta X$ arbitrarily small; these states are localised around the origin of space, and at any time. This means that the noncommutative intrinsic limitations to localisation arising in this model allow for localisation processes which in principle could transfer arbitrary high energies to sharp localised regions of the geometric background by effect of localisation. This is in plain contrast with the standard motivations underlying the quest of spacetime quantisation (see [9] for a particularly careful discussion, where the probe is not implicitly assumed to posses spherical symmetry). On the other side, we provide estimates on the effects of noncommutativity at large scale.

Notations

We choose natural units so that $\kappa = 1$, with the exception of section 6, where we discuss the large scale limit, and of section 7, where we discuss uncertainty relations.

An important auxiliary rôle will be played by the Schrödinger operators

$$P = -id/ds, \quad Q = s\cdot$$

on their usual domain in $L^2(\mathbb{R})$.

We take the following conventions for Fourier transformations of functions of $\mathbb{R} \times \mathbb{R}^d$:

$$\hat{f}(\lambda_1, \ldots, \lambda_n) = (2\pi)^{-\frac{n}{2}} \int dx_1 \cdots dx_n \, f(x_1 \ldots, x_n) e^{-i(\lambda_1 x_1 + \cdots + \lambda_n x_n)}, \quad (1.11a)$$

$$\check{\varphi}(x_1, \ldots, x_n) = (2\pi)^{-\frac{n}{2}} \int d\lambda_1 \cdots d\lambda_n \, \varphi(\lambda_1 \ldots, \lambda_n) e^{i(\lambda_1 x_1 + \cdots + \lambda_n x_n)}. \quad (1.11b)$$

Furthermore, we will write $\mathscr{F}_j$ for the Fourier transform in the j^{th} variable of a generic function:

$$(\mathscr{F}_j f)(x_1, \ldots, x_{j-1}, \lambda, x_{j+1}, \ldots, x_n) = \frac{1}{\sqrt{2\pi}} \int dx_j f(x_1, \ldots, x_n) e^{-i\lambda x_j},$$

and analogously we define $\mathscr{F}_{j_1, j_2, \ldots, j_r}$, for $1 \leqslant j_1 < j_2 < \ldots < j_r \leqslant n$.

2 Representations

We will begin by classifying the regular representations of the (formal) relations

$$[T, R] = iR. \quad (2.1)$$

Definition 2.1. *A pair* (T, R) *of selfadjoint operators fulfilling*

$$e^{i\alpha T} e^{i\beta R} = e^{i\beta e^{-\alpha} R} e^{i\alpha T}, \quad \alpha, \beta \in \mathbb{R}, \tag{2.2}$$

is said to fulfil (2.1) *in regular form.*

This definition is in agreement with a formal application of the BCH formula.

Let P, Q be a pair of Schrödinger operators. Then the pair (P, e^{-Q}) provides a regular representation of (2.1), with R positive. This can be checked directly, using that

$$\left(e^{i\alpha P}\xi\right)(s) = \xi(s+\alpha), \quad \left(e^{i\beta e^{-Q}}\xi\right)(s) = e^{i\beta e^{-s}}\xi(s).$$

A distinguished rôle will be played by trivial regular representations, namely those where the radius operator R is zero. By Schur's lemma, irreducible trivial regular representations are one dimensional, in which case T is a real number.

Proposition 2.2. *(i) Any irreducible non trivial regular representation of* (2.1) *is equivalent to one of the following:*

$$(T, R) = (P, \pm e^{-Q}); \tag{2.3}$$

in particular there is one only irreducible representation with R positive, up to equivalence. Moreover, (ii) any trivial irreducible representation is one dimensional, with T a real number.

Proof. We already proved (ii). Let (T, R) be a non trivial irreducible representation: we rewrite the relations (2.2) in the form

$$e^{i\alpha T} e^{i\beta R} e^{-i\alpha T} = e^{i\beta e^{-\alpha} R}.$$

Holding α fixed, the generator for the resulting group with parametre β fulfils

$$e^{i\alpha T} R e^{-i\alpha T} = e^{-\alpha} R. \tag{2.4}$$

Consequently, for f a (Borel) function of the spectrum of R,

$$f(e^{i\alpha T} R e^{-i\alpha T}) = e^{i\alpha T} f(R) e^{-i\alpha T} = f(e^{-\alpha} R).$$

Since (T, R) is irreducible and not trivial, 0 is *not* in the spectrum of R and we may apply the above remark to the function $f(x) = e^{-i\beta \log|x|}$, obtaining

$$e^{i\alpha T} e^{i\beta(-\log|R|)} = e^{i\alpha\beta} e^{i\beta(-\log|R|)} e^{i\alpha T},$$

namely the Weyl relations for the CCR: by von Neumann uniqueness [6], we may assume (up to equivalence) that $T = P$ and $Q = -\log|R|$.

Let $C = \operatorname{sign}(R)$, which commutes strongly with Q. We rewrite again (2.4) in terms of $T = P$, $R = Ce^{-Q}$:

$$e^{i\alpha P} C e^{-Q} e^{-i\alpha P} = e^{-\alpha} C e^{-Q}$$

and, using $e^{-Q} e^{-i\alpha P} = e^{-\alpha} e^{-i\alpha P} e^{-Q}$ and strict positivity of e^{-Q},

$$e^{i\alpha P} C = C e^{i\alpha P},$$

namely C strongly commutes with P, too. By the generalised Schur's lemma, C is a multiple of the identity, $C = \pm I$, and $R = \pm e^{-Q}$. ■

Since any real linear combination of pairwise strongly commuting selfadjoint operators is essentially selfadjoint, let us introduce the following notations:

$$\boldsymbol{\beta} = (\beta_1, \dots, \beta_d), \quad \boldsymbol{X} = (X_1, \dots, X_d), \quad \boldsymbol{\beta X} = \left(\sum_j \beta_j X_j \right)^{**}.$$

Moreover, $\mathbf{0} = (0, 0, \dots, 0)$, while $\boldsymbol{e}_j$ denotes the usual canonical basis for $\mathbb{R}^d$, e.g. $\boldsymbol{e}_d = (0, 0, \dots, 0, 1)$.

The regular form of the relations

$$[T, X_j] = iX_j, \quad [X_j, X_k] = 0 \tag{2.5}$$

among the selfadjoint operators $T, X_1, \dots, X_d$ can be easily generalised:

Definition 2.3. *A set $(T, \boldsymbol{X})$ of $d+1$ selfadjoint operators fulfilling*

$$e^{i\alpha T} e^{i\boldsymbol{\beta X}} = e^{ie^{-\alpha}\boldsymbol{\beta X}} e^{i\alpha T}, \qquad \alpha \in \mathbb{R}, \boldsymbol{\beta} \in \mathbb{R}^d, \tag{2.6a}$$

$$[e^{i\boldsymbol{\beta X}}, e^{i\boldsymbol{\beta' X}}] = 0, \qquad \boldsymbol{\beta}, \boldsymbol{\beta'} \in \mathbb{R}^d, \tag{2.6b}$$

is said a regular representation of the relations (2.5).

We will call a representation $(T, \boldsymbol{X})$ trivial if $\boldsymbol{X} = 0$. Irreducible trivial representations are then of the form $T = t, X_1 = \ldots = X_d = 0$ as operators on the one dimensional Hilbert space $\mathbb{C}$. It is convenient to introduce the notation

$$T^{(0)} = P, \boldsymbol{X}^{(0)} = \mathbf{0}. \tag{2.7}$$

Since P may be replaced with Q by means of a canonical transformation, the trivial representation $(T^{(0)}, \boldsymbol{X}^{(0)})$ contains every irreducible trivial representation precisely once (up to equivalence).

To every $\boldsymbol{c} = (c_1, \ldots, c_d) \in \mathbb{R}^d \backslash \{0\}$ there is an irreducible regular representation

$$T^{(\boldsymbol{c})} = P, \quad X_j^{(\boldsymbol{c})} = c_j e^{-Q}$$

where P, Q are the Schrödinger operators on $L^2(\mathbb{R})$. It is clear that, since the length of $\boldsymbol{c}$ may be rescaled by means of a canonical transformation (see the proof below), we may restrict ourselves to $|\boldsymbol{c}| = 1$; then, different choices of $\boldsymbol{c}$ give inequivalent representations. In other words, there is a family of pairwise inequivalent representations labeled by the unit sphere $S^{d-1} = \{\boldsymbol{c} \in \mathbb{R}^d : |\boldsymbol{c}| = 1\}$.

We now prove that there are no other irreducible representations, up to unitary equivalence.

Proposition 2.4. *Let $(T, \boldsymbol{X})$ be a non trivial irreducible regular representation of the relations* (2.6)*; then there is a unique $\boldsymbol{c} \in S^{d-1}$ such that*

$$(T, \boldsymbol{X}) \simeq (T^{(\boldsymbol{c})}, \boldsymbol{X}^{(\boldsymbol{c})}). \tag{2.8}$$

Moreover, the representation $(T^{(\mathbf{0})}, \boldsymbol{X}^{(\mathbf{0})})$ contains a representative for each class of trivial irreducible representations, without multiplicity.

Proof. With $R^2 = (X_1^2 + \ldots + X_d^2)^{**}$, let $R = \int_0^\infty r \, dF(r)$ be the spectral resolution of R, and $E = F(\infty) - F(0) = \chi_{(0,\infty)}(R)$, where $\chi_{(0,\infty)}$ is the characteristic function of the set $(0, \infty)$. By (2.4)

$$e^{i\alpha T} E e^{-i\alpha T} = \chi_{(0,\infty)}(e^{-\alpha} R) = E,$$

so that E commutes strongly both with R and T. Hence, by the generalised Schur's lemma, either $E = I$ or $E = 0$. If $E = 0$, the representation is trivial: T is multiplication by a real number on a one dimensional Hilbert space and $\boldsymbol{X} = \boldsymbol{C} = 0$. Otherwise R is invertible, and the bounded operators

$$C_k = X_k R^{-1}, \quad k = 1, \ldots, d,$$

strongly commute pairwise and with R. By (2.4) and the properties of functional calculus,

$$e^{i\alpha T}C_k e^{-i\alpha T} = e^{i\alpha T}X_k e^{-i\alpha T}e^{i\alpha T}R^{-1}e^{-i\alpha T} = (e^{-\alpha}X_k)(e^{-\alpha}R)^{-1} = C_k,$$

so that C_k strongly commutes with T, too. Hence by Schur's lemma $C_k = c_k I$ for some c_k, and $X_k = c_k R$.

Since the representation is not trivial, there is at least some $c_j \neq 0$: thus (2.6a) written for $\boldsymbol{\beta} = \beta \boldsymbol{e}_j$ gives precisely (2.2). It follows that (up to equivalence) $T = P$, $R = e^{-Q}$, by positivity of R and proposition 2.2.

With $|\boldsymbol{c}|^2 = \sum_j |c_j|^2$, the unitary operator $U = e^{iP\log|\boldsymbol{c}|}$ fulfils $Ue^{-Q}U^* = (1/|\boldsymbol{c}|)e^{-Q}$, so that we may assume $\boldsymbol{c} \in S^{d-1}$, the unit sphere. ■

Note that, for $d = 1$, $S^0 = \{\pm 1\}$, so that there are two only equivalence classes of non trivial, irreducible representations: $(P, \pm e^{-Q})$. We thus recovered the special case discussed in [3, 8].

Definition 2.5. *Let $d\boldsymbol{c}$ be the rotation invariant Lebesgue measure on $S^{(d-1)}$. Let $d\mu(\boldsymbol{c})$ be the measure on*

$$S^{d-1} \sqcup \{\mathbf{0}\},$$

defined by $\int d\mu(\boldsymbol{c})f(\boldsymbol{c}) = f(\mathbf{0}) + \int d\boldsymbol{c} f(\boldsymbol{c})$. The universal representation of the relations (2.6) *is*

$$T^u = I \otimes P, \tag{2.9a}$$

$$R^u = I \otimes e^{-Q}, \tag{2.9b}$$

$$C_j^u = (c_j \cdot) \otimes I, \tag{2.9c}$$

$$X_j^u = C_j^u R^u = (c_j \cdot) \otimes e^{-Q}, \tag{2.9d}$$

on the Hilbert space

$$\mathfrak{H}^u = L^2(S^{d-1} \sqcup \{\mathbf{0}\}, d\mu(\boldsymbol{c})) \otimes L^2(\mathbb{R}), \tag{2.9e}$$

where $c_j\cdot$ is the operator of multiplication by c_j.

By construction, the above representation contains precisely one representative for every equivalence class of irreducible representations; for this reason we called it universal. By taking a suitable amplification, we can easily obtain a representation which is covariant under a unitary representation of the group G_d of orthogonal space transformations, time translations, and space dilations.

Proposition 2.6. *Let $G_d = O(\mathbb{R}^d) \times \mathbb{R} \times (0, \infty)$ be the Kronecker product of the orthogonal group, the additive group $\mathbb{R}$ and the multiplicative group $(0, \infty)$, so that*

$$(A_1, a_1, \lambda_1)(A_2, a_2, \lambda_2) = (A_1 A_2, a_1 + \lambda_1 a_2, \lambda_1 \lambda_2), \quad (A_j, a_j, \lambda_j) \in G_d. \tag{2.10}$$

There exists a G_d-covariant representation, namely a strongly continuous unitary representation U of G_d and a representation $(T, \boldsymbol{X})$ such that, for every $(A = (a_{jk}), a, \lambda) \in G_d$,

$$U(A, a, \lambda)^{-1} X_j U(A, a, \lambda) = \lambda \sum_k a_{jk} X_k, \quad j = 1, \ldots, k, \tag{2.11a}$$

$$U(A, a, \lambda)^{-1} T U(A, a, \lambda) = T + aI. \tag{2.11b}$$

Proof. Take

$$T = I \otimes P \otimes I,$$
$$X_j = C_j \otimes e^{-Q} \otimes e^{-Q},$$

on $\mathfrak{H}^u \otimes L^2(\mathbb{R})$, and

$$U(A, a, \lambda) \eta \otimes \xi \otimes \xi' = \eta(A^{-1} \cdot) \otimes e^{iaQ} \otimes e^{i(\log \lambda) P}.$$

■

3 Radial Weyl Operators, Quantisation and Trace

According to the discussion of section 2, the quantisation will take place (in the non trivial component) in the radial directions labeled by vectors $\boldsymbol{c} \in S^{d-1}$. Hence, we discuss preliminarly the quantisation corresponding to the operators $T = P, R = e^{-Q}$. Our first task is to compute the Weyl operators $e^{i(\alpha T + \beta R)}$.

Proposition 3.1. *Let T, R be selfadjoint operators on some Hilbert space, fulfilling (2.2) with $R > 0$. Then, for every $(\alpha, \beta) \in \mathbb{R}^2$, the operator $\alpha T + \beta R$ is essentially selfadjoint and fulfils*

$$e^{i\alpha T + \beta R} = e^{i\alpha T} e^{i \frac{e^{\alpha} - 1}{\alpha} \beta R}. \tag{3.1}$$

Once the right ansatz has been guessed, the proof is a standard application of the Stone–von Neumann theorem [10, theorem VIII.10], which we omit. Indeed, we find it more instructive to describe a method for finding the right ansatz, which does not rely on a formal application of the BCH formula.

Assume that there there is a common dense domain $\mathscr{X}$ on which $\alpha T+\beta X$ is essentially selfadjoint for every α, β. The operators $W(\alpha, \beta) := e^{i\alpha T+\beta X}$ should fulfill the following properties:

$$W(\alpha, 0) = e^{i\alpha T}, \quad W(0, \beta) = e^{i\beta X}, \tag{3.2}$$

$$W(\alpha, \beta)^{-1} = W(\alpha, \beta)^{*}, \tag{3.3}$$

$$W(\lambda\alpha, \lambda\beta)W(\lambda'\alpha, \lambda'\beta) = W((\lambda+\lambda')\alpha, (\lambda+\lambda')\beta) \tag{3.4}$$

identically for $\alpha, \beta, \lambda, \lambda' \in \mathbb{R}$. To solve the above problem, we took the following ansatz:

$$W(\alpha, \beta) = e^{ir(\alpha,\beta)T} e^{is(\alpha,\beta)X}.$$

Some little effort leads to the given solution.

We now discuss the properties of the map

$$f \mapsto f(T, R) = \frac{1}{2\pi} \int d\alpha\, d\beta\, \hat{f}(\alpha, \beta) e^{i(\alpha T+\beta R)},$$

defined on the class $L^1(\mathbb{R}^2) \cap \widehat{L^1(\mathbb{R}^2)}$ of symbols, where

$$\hat{f}(\alpha, \beta) = \frac{1}{2\pi} \int dt\, dr\, f(t, r) e^{-i(\alpha t+\beta r)}.$$

In this section we always will take $T = P, R = e^{-Q}$ on $L^2(\mathbb{R})$.

With the explicit action

$$(e^{i(\alpha P+\beta e^{-Q})}\xi)(s) = e^{i\frac{1-e^{-\alpha}}{\alpha}\beta e^{-s}}\xi(s+\alpha),$$

standard computations yield

$$(f(T, R)\xi)(s) = \int du\, K_f(s, u)\xi(u),$$

where

$$K_f(s, u) = (\mathscr{F}_1 f)\left(u - s, \frac{e^{-s} - e^{-u}}{u - s}\right); \tag{3.5}$$

here above, $\mathscr{F}_1$ denotes the Fourier transform in the first variable (for conventions, see the end of the introduction).

Inspection of (3.5) gives immediately

Lemma 3.2. *Let $f_1, f_2 \in L^1(\mathbb{R}^2) \cap \widehat{L^1(\mathbb{R}^2)}$. We have*

$$f_1(T, R) = f_2(T, R)$$

if and only if

$$f_1(t, r) = f_2(t, r), \quad t \in \mathbb{R}, r \in (0, \infty).$$

Equivalently, the restriction of the map $f \mapsto f(T, R)$ to the symbols f which are even in the second variable (namely $f(t, \cdot) = f(t, |\cdot|)$ for all t's) is injective.

Moreover,

Lemma 3.3. *Let $f \in L^1(\mathbb{R}^2) \cap \widehat{L^1(\mathbb{R}^2)}$. If*

$$\int_{r>0} dt\, dr\, \frac{1}{r} |f(t, r)|^2 < \infty,$$

then the operator $f(T, R)$ is Hilbert-Schmidt, with Schatten norm

$$\|f(T, R)\|_2 = \left(\int_{r>0} dt\, dr\, \frac{1}{r} |f(t, r)|^2 \right)^{1/2}. \tag{3.6}$$

Proof. By lemma 3.2, we may assume $f(t, r) = f(t, -r)$ identically, without loss of generality. With the substitution

$$\alpha = u - s, \quad r = \frac{e^{-s} - e^{-u}}{u - s},$$

we find

$$\begin{aligned} \|K_f\|^2_{L^2(\mathbb{R}^2)} &= \int du\, ds\, \left| (\mathscr{F}_1 f) \left(u - s, \frac{e^{-s} - e^{-u}}{u - s} \right) \right|^2 = \\ &= \int_{r>0} d\alpha\, dr\, \frac{1}{r} |(\mathscr{F}_1 f)(\alpha, r)|^2 = \int_{r>0} dt\, dr\, \frac{1}{r} |f(t, r)|^2 . \end{aligned}$$

where we used that $\mathscr{F}_1$ is unitary on $L^2(\mathbb{R}^2, |r|^{-1} dt\, dr)$. The result then follows from classical theorems (see e.g. [10, Theorem VI.23]). ■

We next ask ourselves when $f(T, R)$ has trace, and how to compute it. We will give a somewhat indirect argument, which is of some interest on its own.

In some sense, by its very definition the operator $f(T, R)$ appears as a "function" of the Schrödinger operators P, Q. Hence, it is natural to expect (at least for a suitable subclass of symbols) that there exists a function g such that $f(T, R) = g(P, Q)$, where the latter is intended as the CCR–Weyl quantisation

$$g(P, Q) = \frac{1}{2\pi} \int d\alpha \, d\beta \; \hat{g}(\alpha, \beta) e^{i(\alpha P + \beta Q)}.$$

Such a map$f \mapsto g$ would allow for computing the trace of $f(T, R)$, by known results on Weyl quantisation[2].

Indeed it is well known (see e.g. [12, 4.1, eq. (59)]) that

$$(g(P, Q)\xi)(s) = \int dt \; H_g(s, u)\xi(u),$$

where

$$H_g(s, u) = (\mathscr{F}_1 g) \left(u - s, \frac{u + s}{2} \right). \tag{3.7}$$

The operators $g(P, Q), f(T, R)$ are the same if and only if they have the same integral kernel (a.e.). Setting $\lambda = u - s, q = (u + s)/2$, from the condition

$$H_g(s, u) \equiv K_f(s, u) \tag{3.8}$$

we get

$$(\mathscr{F}_1 g)(\lambda, q) \equiv (\mathscr{F}_1 f) \left(\lambda, e^{-q} \frac{\sinh(\lambda/2)}{\lambda/2} \right). \tag{3.9}$$

Proposition 3.4. *For any $f \in L^1(\mathbb{R}^2) \cap \widehat{L^1(\mathbb{R}^2)}$ fulfilling*

$$\int_{\{r>0\}} dt \; dr \; \frac{1}{r} |f(t, r)| < \infty,$$

[2]Note that it would also allow for describing the star product of κ-Minkowski symbols as the pull-back ot the twisted product defined by the Weyl quantisation associated with CCR; the relations with the approach of [11] (see especially eq. (5.1) therein) will be discussed elsewhere.

the operator $f(T,R)$ is trace class, with trace

$$Tr(f(T,R) = \int_{\{r>0\}} dt\ dr\ \frac{1}{r} f(t,r). \qquad (3.10)$$

Proof. Use (3.9) and $\mathrm{Tr}(g(P,Q)) = \int dp\ dq\ g(p,q)$. ■

4 The Radial Algebra

While in the canonical case the product of Weyl operators is a Weyl operator only up to a phase (called the twist), in this case the product of two radial Weyl operators $e^{i(\alpha P + \beta e^{-Q})}$ is again a radial Weyl operator. The reason is that in the case of κ-Minkowski the identity is not involved in the commutation relations. Hence the set of Weyl operators is a subgroup of the unitary group. Since the correspondence between $\mathbb{R}^2$ and the radial Weyl operators is one to one, $\mathbb{R}^2$ inherits from the composition rule of radial Weyl operators a group law:

Definition 4.1. *The (locally compact) group $\mathfrak{R}$ is the group obtained by equipping $\mathbb{R}^2$ with the usual topology and the group law*

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, w(\alpha_1 + \alpha_2, \alpha_1)e^{\alpha_2}\beta_1 + w(\alpha_1 + \alpha_2, \alpha_2)\beta_2),$$

where

$$w(\alpha, \alpha') = \frac{\alpha(e^{\alpha'} - 1)}{\alpha'(e^{\alpha} - 1)}. \qquad (4.1)$$

Note that w (which is always understood to be extended to the full $\mathbb{R}^2$ by continuity) fulfills

$$w(\alpha_1, \alpha_2)w(\alpha_2, \alpha_3) = w(\alpha_1, \alpha_3), \quad w(\alpha_1, \alpha_2)e^{\alpha_1} = -w(-\alpha_1, \alpha_2) \qquad (4.2)$$

identically, and is always positive. The identity of $\mathfrak{R}$ is $(0, 0)$, and $(\alpha, \beta)^{-1} = (-\alpha, -\beta)$. Finally

$$\lim_{\kappa \to \pm\infty} w(\alpha_1/\kappa, \alpha_2/\kappa) = 1, \qquad (4.3)$$

so that we may regard $\mathfrak{R}$ as a non abelian deformation ot the additive group $(\mathbb{R}^2, +)$.

As anticipated, the above definition finds its motivations in the following

Lemma 4.2. *The map*

$$\mathfrak{R} \ni (\alpha, \beta) \mapsto e^{i(\alpha P + \beta e^{-Q})}$$

is a strongly continuous[3]*, faithful unitary representation of the radial group.*

The proof consists of a direct check. The group $\mathfrak{R}$ and its Lie algebra are well known:

Proposition 4.3. *(i) The generators u, v of the real Lie algebra $\mathfrak{r} = Lie(\mathfrak{R})$, corresponding to the one–parameter subgroups $\lambda \mapsto (\lambda, 0)$ and $\lambda \mapsto (0, \lambda)$ respectively, fulfil*

$$[u, v] = -v.$$

(ii) $\mathfrak{R}$ is isomorphic to the subgroup[4]

$$G = \left\{ \begin{pmatrix} e^a & 0 \\ b & 1 \end{pmatrix} : (a, b) \in \mathbb{R}^2 \right\} \tag{4.4}$$

of $GL(\mathbb{R}^2)$; an explicit isomorphism $j : \mathfrak{R} \to G$ is given by

$$j((\alpha, \beta)) = \begin{pmatrix} e^\alpha & 0 \\ \frac{e^\alpha - 1}{\alpha}\beta & 1 \end{pmatrix}. \tag{4.5}$$

Proof. The proof of (ii) is a direct check. Then (i) follows immediately, since connected, simply connected Lie groups are isomorphic if and only if their Lie algebras also are isomorphic. ■

Note that the choice of the generators u, v is such that for any strongly continuous unitary representation W of $\mathfrak{R}$, the selfadjoint operators T, R defined by $W(\mathrm{Exp}\{\lambda u\}) = e^{i\lambda T}$, $W(\mathrm{Exp}\{\lambda v\}) = e^{i\lambda R}$ are a regular representation of the relations $[T, R] = iR$, where Exp is the Lie exponential map.

Note also that under the isomorphism j of (i) above, the one parameter subgroups $\lambda \mapsto (\lambda\alpha, \lambda\beta)$ of $\mathfrak{R}$ are mapped into

$$\lambda \mapsto \begin{pmatrix} e^{\lambda\alpha} & 0 \\ \frac{e^{\lambda\alpha} - 1}{\alpha}\beta & 1 \end{pmatrix}. \tag{4.6}$$

[3] Note however that, from proposition 3.1 and [13], it follows that a nontrivial representation cannot be continuous in the operator norm topology.

[4] The group G is a variant of the so called "$ax + b$" group.

This should completely clarify the relations of our definition of Weyl operators with other (non canonical) definitions available in the literature, which also are related with such Lie groups, but do not fulfill the essential condition that $\lambda \mapsto W(\lambda\alpha, \lambda\beta)$ is a one parameter group. See e.g. [4, 3], where the Weyl operators are defined with the choice "time first": $e^{i\alpha T}e^{i\alpha R}$.

The group $\mathfrak{R}$ is not unimodular. Indeed

Lemma 4.4. *(i) The left Haar measure and modular function of the group* $\mathfrak{R}$ *are*

$$d\mu(\alpha,\beta) = \frac{e^\alpha - 1}{\alpha} d\alpha\, d\beta, \quad \Delta(\alpha,\beta) = e^\alpha. \tag{4.7}$$

(ii) With the above choice of the normalisation, for every bounded function φ *with compact support,*

$$\lim_{\kappa\to\infty} \kappa^2 \int d\mu(\alpha,\beta)\varphi(\alpha\kappa,\beta\kappa) = \int d\alpha\, d\beta\; \varphi(\alpha,\beta). \tag{4.8}$$

Proof. The proof of (i) is a direct check. (ii) follows from the fact that the function $(\alpha \mapsto \kappa(e^\alpha - 1)/\alpha)$ converges to $(\alpha \mapsto 1)$ uniformly on compact sets as $\kappa \to \infty$. ■

We describe explicitly in our case some basic facts of abstract harmonic analysis (see e.g. [14, §§29-30]):

1. Let μ, Δ be the left Haar measure and modular function, respectively, described in lemma 4.4. The group *-algebra $L^1(\mathfrak{R})$ of $\mathfrak{R}$ is obtained by equipping the Banach space $L^1(\mathbb{R}^2,\mu)$ with the (convolution) product

$$\begin{aligned}(\varphi\times\psi)(\alpha,\beta) &= \int d\mu(\alpha',\beta')\varphi(\alpha',\beta')\psi((\alpha',\beta')^{-1}(\alpha,\beta)) = \\ &= \int d\alpha'\, d\beta'\; \frac{e^{\alpha'}-1}{\alpha'}\varphi(\alpha',\beta') \\ &\quad \psi(\alpha-\alpha', w(\alpha-\alpha',\alpha)\beta - w(\alpha-\alpha',-\alpha')e^\alpha\beta')\end{aligned} \tag{4.9a}$$

and the involution

$$\varphi^\dagger(\alpha,\beta) := \Delta((\alpha,\beta)^{-1})\overline{\varphi((\alpha,\beta)^{-1})} = e^{-\alpha}\overline{\varphi(-\alpha,-\beta)}; \tag{4.9b}$$

the group algebra is a Banach *-algebra, namely the product is continuous and

$$\|\varphi^\dagger\|_{L^1(\mathfrak{R})} = \|\varphi\|_{L^1(\mathfrak{R})}, \quad (\varphi\times\psi)^\dagger = \psi^\dagger\times\varphi^\dagger,$$

where
$$\|\varphi\|_{L^1(\mathfrak{R})} = \int d\mu(\alpha,\beta)|\varphi(\alpha,\beta)|.$$

2. Let W be a unitary representation of the group $\mathfrak{R}$; then
$$\Pi(\varphi) := \int d\mu(\alpha,\beta)\varphi(\alpha,\beta)W(\alpha,\beta), \quad \varphi \in L^1(\mathfrak{R}),$$
defines a *-representation of the group algebra $L^1(\mathfrak{R})$:
$$\Pi(\varphi)\Pi(\psi) = \Pi(\varphi \times \psi), \quad \varphi,\psi \in L^1(\mathfrak{R}),$$
$$\Pi(\varphi^\dagger) = \Pi(\varphi)^*;$$
the preservation of involution is a consequence of unitarity of W. Since $\|W(\alpha,\beta)\| = 1$,
$$\|\Pi(\varphi)\| \leqslant \|\varphi\|_{L^1(\mathfrak{R})}$$
and the representation is continuous (actually, any *-representation of a Banach *-algebra by bounded operators on a Hilbert space is continuous for general reasons, see e.g. [15, 1.3.7]).

Definition 4.5. *The algebra $\mathscr{B}$ is the Banach *-algebra obtained by equipping $L^1(\mathbb{R}^2)$ with the product*
$$(\varphi_1 * \varphi_2)(\alpha,\beta) = \int d\alpha' d\beta' w(\alpha-\alpha',\alpha)\varphi_1(\alpha',\beta') \varphi_2(\alpha-\alpha', w(\alpha-\alpha',\alpha)\beta - w(\alpha'-\alpha,\alpha')\beta'); \quad (4.10a)$$
and the involution
$$\varphi^*(\alpha,\beta) = \overline{\varphi(-\alpha,-\beta)}. \quad (4.10b)$$

The above definition is motivated by the following lemma, which also proves that it is well posed.

Lemma 4.6. *The Banach *-algebra $\mathscr{B}$ and the group algebra $L^1(\mathfrak{R})$ are isomorphic. With the usual normalisation of the Lebesgue measure on $\mathbb{R}^2$ and the normalisation of the Haar measure described in lemma 4.4, the isomorphism can be chosen isometric.*

Proof. The linear map
$$u : L^1(\mathbb{R}^2) \to L^1(\mathfrak{R})$$
defined by
$$(u\varphi)(\alpha, \beta) = \frac{\alpha}{e^\alpha - 1}\varphi(\alpha, \beta)$$
is evidently an equivalence of Banach spaces. The proof that
$$(u\varphi) \times (u\psi) = u(\varphi * \psi), \quad u(\varphi^*) = u(\varphi)^\dagger$$
for $\phi, \psi \in L^1(\mathfrak{R})$ consists of a direct check. ■

Definition 4.7. *A representation π of $\mathscr{B}$ is said trivial if $\pi(\mathscr{B}) \subset \pi(\mathscr{B})'$.*

Proposition 4.8. *Let W be a strongly continuous unitary representation of the radial group $\mathfrak{R}$. Then*
$$\pi(\varphi) = \int d\alpha \, d\beta \, \varphi(\alpha, \beta) W(\alpha, \beta), \quad \varphi \in L^1(\mathbb{R}^2)$$
*defines a *-representation π of the *-algebra $\mathscr{B}$. This establishes a bijection between equivalence classes of strongly continuous unitary representations of the group $\mathfrak{R}$ and the equivalence classes of *-representations of the *-algebra $\mathscr{B}$. In particular, a representation π of $\mathscr{B}$ is trivial if and only if the corresponding representation W of $\mathfrak{R}$ gives a trivial regular representation of the Weyl relations.*

Any trivial irreducible representation is equivalent, for some $t \in \mathbb{R}$, to the one dimensional representation $\pi(\varphi) = \check{\varphi}(t, 0)$.

Any non trivial irreducible representation is equivalent to one of the representations $\pi_\pm$, where
$$\pi_\pm(\varphi) = \int d\alpha \, d\beta \, \varphi(\alpha, \beta) e^{i(\alpha P \pm \beta e^{-Q})}$$
on $L^2(\mathbb{R})$. Note that
$$\pi_+(\hat{f}) = f(T, R).$$

The proof is an immediate consequence of the remark that $\pi \circ u = \Pi$, where Π is the representation of the group algebra associated with W.

We are now ready to give the following, crucial

Definition 4.9. *The radial algebra is the algebra $\mathscr{R} \subset \mathscr{B}$ of fixed elements under the automorphism $\digamma : \mathscr{B} \to \mathscr{B}$, where $(\digamma\varphi)(\alpha, \beta) = \varphi(\alpha, -\beta)$.*

It is clear that the representations $\pi_\pm$ of $\mathscr{B}$, described in the above corollary, are related by $\pi_- = \pi_+ \circ \digamma$. It follows that their restrictions

$$\pi_r = \pi_\pm \restriction_{\mathscr{R}} \tag{4.11}$$

to the radial algebra coincide. Indeed, there are no other irreducible (classes of) irreducible representation.

Proposition 4.10. *Every non trivial irreducible representation of the radial algebra is equivalent to* $\pi_r = \pi_+ \restriction_{\mathscr{R}}$.

Proof. The representation π_r has the same image as any of $\pi_\pm$, hence it is clearly irreducible. Conversely, let us first show that the operators $E_\pm$ on $L^1(\mathbb{R}^2)$ defined by

$$(E_\pm\varphi)\check{}\,(t,x) = \check{\varphi}(t, \pm|x|). \tag{4.12}$$

are *-homomorphism from $\mathscr{B}$ onto its Banach *-subalgebra $\mathscr{R}$. Surjectivity is obvious. To prove multiplicativity, namely

$$E_\pm(\varphi * \psi) = (E_\pm\varphi) * (E_\pm\psi), \tag{4.13}$$

we observe that, by the same argument of lemma 3.2 (or using that $f(T,R) = \pi_+(\hat{f})$), $\pi_\pm$ fulfils $\pi_\pm \circ E_\pm = \pi_\pm$; hence,

$$\begin{aligned}\pi_r(E_\pm(\varphi * \psi)) = \pi_\pm(\varphi * \psi) = \pi_\pm(\varphi)\pi_\pm(\psi) = \pi_\pm(E_\pm\varphi)\pi_\pm(E_\pm\psi) = \\ = \pi_\pm((E_\pm\varphi) * (E_\pm\psi)) = \pi_r((E_\pm\varphi) * (E_\pm\psi)),\end{aligned}$$

where in the last step we used that $(E_\pm\varphi) * (E_\pm\psi) \in \mathscr{R}$. Again by the argument of lemma 3.2, π_r is injective, so that multiplicativity is proved. Finally, by $(\varphi^*)\check{}\, = \overline{\check{\varphi}}$, it follows that involutions are respected.

Now, let π be an irreducible, non trivial representation of $\mathscr{R}$. For every $\varphi \in \mathscr{B}$, set $\tilde{\pi}(\varphi) = \pi(E_-\varphi) \oplus \pi(E_+\varphi)$. From the preceding remark it follows that $\tilde{\pi}$ is a non trivial representation of $\mathscr{B}$. Hence we have $\pi \circ E_\pm \simeq \pi_+$. ■

Proposition 4.11. *Let* π_r *be the representation of the radial algebra* $\mathscr{R}$ *described in proposition 4.10. Then* $\pi_r(\mathscr{R})$ *is norm-dense in the* C^**-algebra* $\mathcal{K}$ *of compact operators on a separable, infinite dimensional Hilbert space.*

In particular, for every $f \in L^1(\mathbb{R}^2) \cap \widehat{L^1(\mathbb{R}^2)}$, $f(T,R)$ *is a compact operator.*

Proof. By [10, VI.12(a)] it is sufficient to show that $\pi_r(\varphi)$ is compact for every φ in some total subset of $\mathscr{R}$; for example if $\varphi = \varphi_1 \otimes \varphi_2$ with $\varphi_j \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and φ_2 even. Since in general $\|\pi_r(\varphi)\| \leqslant \|\varphi\|_{L^1}$, by [10, VI.22(e)] it is sufficient to show that there exists a sequence (φ_n) in L^1 such that $\lim_n \|\varphi_n - \varphi\|_{L^1} = 0$ and $\pi_r(\varphi_n)$ is Hilbert-Schmidt. Let $j \in \mathcal{C}_b(\mathbb{R})$ be a bounded, positive continuous function with $\lim_{r\to 0} j(|r|)/\sqrt{|r|} = 1$, $0 \leqslant j(|r|) \leqslant 1$, and $j(r) = 1, r > 1$. We define $f_n(t,r) = j(n|r|)\check{\varphi}(t,r)$. By construction $\lim_n \|f_n - \check{\varphi}\|_{L^\infty} = 0$. The functions $f_n, \check{\varphi}$ are continuous and L^1, hence the preceding remark implies $\lim_n \|f_n - \check{\varphi}\|_{L^1} = 0$. Since the Fourier transform is bounded as a linear map from L^1 to L^∞ (see e.g. [10, Theorem IX.8]), we also have $\lim_n \|\hat{f}_n - \varphi\|_{L^\infty} = 0$, which again implies $\lim_n \|\hat{f}_n - \varphi\|_{L^1} = 0$. Hence we have our candidate sequence $\varphi_n = \hat{f}_n$. Finally,

$$\int_{r>0} \frac{dt\ dr}{r} |f_n(t,r)|^2 \leqslant \|\varphi_1\|_{L^2} \left(\int_{0<r<\frac{1}{n}} \frac{dr}{r} |j(nr)\check{\varphi}_2(r)|^2 + n\|\varphi_2\|^2_{L^2} \right) < \infty.$$

Hence each $\pi_r(\varphi_n) = f_n(T,R)$ is Hilbert-Schmidt by lemma 3.3. ∎

5 Cartesian vs Radial Weyl Quantisation

In this section $T, \boldsymbol{X}, \boldsymbol{C}, R$ are the operators of the universal representation described in proposition 2.5, (we drop the apex u, for simplicity). An easy generalisation of the argument of proposition 3.1 gives the Weyl operators

$$e^{i(\alpha T + \boldsymbol{\beta} \boldsymbol{X})} = e^{i\alpha T} e^{i \frac{e^{\alpha}-1}{\alpha} \boldsymbol{\beta} \boldsymbol{X}}, \quad (\alpha, \boldsymbol{\beta}) \in \mathbb{R} \times \mathbb{R}^d \tag{5.1}$$

in $d+1$ dimensions.

The Weyl operators, by definition, are the quantum replacement of Fourier characters. Hence we mimic Weyl proposal [5]:

Definition 5.1. *The Cartesian quantisation of the function* $f \in L^1(\mathbb{R}^{d+1}) \cap \widehat{L^1(\mathbb{R}^{d+1})}$ *is the operator*

$$f(T, \boldsymbol{X}) = \frac{1}{\sqrt{(2\pi)^{d+1}}} \int d\alpha\ d\boldsymbol{\beta}\ \hat{f}(\alpha, \boldsymbol{\beta}) e^{i(\alpha T + \boldsymbol{\beta} \boldsymbol{X})}. \tag{5.2}$$

We may define the star product by setting

$$(f \star g)(T, \boldsymbol{X}) = f(T, \boldsymbol{X}) g(T, \boldsymbol{X}), \tag{5.3}$$

where the operator product is taken on the right hand side, and injectivity of the Weyl quantisation (with universal Weyl operators) is used.

The above definition does not emphasise the radial nature of the quantisation. We give an alternative definition of quantisation for an alternative class of symbols, called radial symbols, and we will compare the two descriptions.

Definition 5.2. *Let $\mathcal{F} \in \mathcal{C}(S^{d-1} \sqcup \{\mathbf{0}\}, L^1(\mathbb{R}^2))$ a continuous function of $S^{d-1} \sqcup \{\mathbf{0}\}$ with values in $L^1(\mathbb{R}^2)$. If, for any $\boldsymbol{c} \in S^{d-1} \sqcup \{\mathbf{0}\}$ fixed, we also have $\widehat{\mathcal{F}(\boldsymbol{c})} \in L^1(\mathbb{R}^2)$, then we call $\mathcal{F}$ a radial symbol.*

A radial symbol $\mathcal{F}$ is said continuous at zero if $\lim_{r\to 0} \mathcal{F}(\boldsymbol{c}(r))(t,r)$ exists and does not depend on the particular choice of the continuous S^{d-1}-valued function $\boldsymbol{c}(r)$ of $(0,\infty)$.

For any radial symbol $\mathcal{F}$, we define

$$\mathcal{F}(\boldsymbol{c})(T,R) = \frac{1}{2\pi}\int \widehat{\mathcal{F}}(\boldsymbol{c})(\alpha,\beta)e^{i(\alpha T+\beta R)}. \tag{5.4a}$$

Finally we define the (universal) radial quantisation $\mathcal{F}(\boldsymbol{C})(T,R)$ of $\mathcal{F}$ by means of the obvious generalisation of the continuous functional calculus: with $P(d\boldsymbol{c})$ the joint spectral measure of the pairwise commuting operators $(C_1,\ldots,C_n)$,

$$\mathcal{F}(\boldsymbol{C})(T,R) = \int \mathcal{F}(\boldsymbol{c})(T,R)P(d\boldsymbol{c}). \tag{5.4b}$$

Note that the definition is well posed, since each C_j strongly commutes with both T and R. In particular joint spectral projections for $\boldsymbol{C}$ commute with $\mathcal{F}(\boldsymbol{c})(T,R)$.

The following remarks should not come as a surprise.

Lemma 5.3. *For every $f \in L^1(\mathbb{R}^{d+1}) \cap \widehat{L^1(\mathbb{R}^{d+1})}$,*

$$\mathcal{F}^f(\boldsymbol{c})(t,r) = f(t,r\boldsymbol{c})$$

is a radial symbol continuous at zero, such that

$$\mathcal{F}^f(\boldsymbol{C})(T,R) := f(T,\boldsymbol{X}). \tag{5.5}$$

Note that the above establishes a bijective correspondence between cartesian symbols and radial symbols continuous at zero; in particular to every $\mathcal{F}$ there corresponds $f(t,\boldsymbol{x}) = \mathcal{F}(|\boldsymbol{x}|^{-1}\boldsymbol{x})(t,|\boldsymbol{x}|)$.

Proof. By reduction theory, it is sufficient to prove (5.5) in the irreducible case, where $\boldsymbol{C} = \boldsymbol{c}I$ and $\boldsymbol{X}^{(\boldsymbol{c})} = \boldsymbol{c}R$. Let $S \in O(3)$ be such that $S\boldsymbol{e}_d = \boldsymbol{c}$. With the change of integration variables $\boldsymbol{\beta}' = S^t\boldsymbol{\beta}$, we get

$$f(T^{(\boldsymbol{c})}, \boldsymbol{X}^{(\boldsymbol{c})}) = \frac{1}{2\pi}\int d\alpha\, d\boldsymbol{\beta}\, \hat{f}(\alpha, \boldsymbol{\beta})e^{-i(\alpha T^{(\boldsymbol{c})} + (\boldsymbol{\beta}\boldsymbol{c})R)} = \int \mathcal{F}(\boldsymbol{c})(T^{(\boldsymbol{c})}, R^{(\boldsymbol{c})}),$$

where

$$\widehat{\mathcal{F}(\boldsymbol{c})}(\alpha, \beta_d') = \int d\beta_1' \cdots d\beta_{d-1}' \hat{f}(\alpha, S\boldsymbol{\beta}').$$

Standard computations then yield $\mathcal{F} = \mathcal{F}^f$. ■

Of course, there is a radial star product defined by

$$\mathcal{F}_1(\boldsymbol{C})(T, R)\mathcal{F}_2(\boldsymbol{C})(T, R) = (\mathcal{F}_1 \star \mathcal{F}_2)(\boldsymbol{C})(T, R) \tag{5.6}$$

which is intertwined with the cartesian star product by the above correspondence:

$$\mathcal{F}^f \star \mathcal{F}^g = \mathcal{F}^{f\star g}. \tag{5.7}$$

Although $f(T, \boldsymbol{X}) = \mathcal{F}^f(\boldsymbol{C})(T, R)$ is not compact any more because of the amplification with infinite multiplicity, we still may define an unbounded trace on the algebra of radial symbols by

$$\tau_r(\mathcal{F}) = \int_{S^{d-1}} d\boldsymbol{c} \int_{r>0} dt\, dr \frac{1}{r}\mathcal{F}(\boldsymbol{c})(t, r) = \int_{S^{d-1}} d\boldsymbol{c}\, \mathrm{Tr}(\mathcal{F}(\boldsymbol{c})(T, R), \tag{5.8}$$

which is finite on the symbols $\mathcal{F}$ with $\mathcal{F}/r$ is summable.

It follows from the definition and proposition 3.4,

$$\tau_r(\mathcal{F}_1 \star \mathcal{F}_2) = \tau_r(\mathcal{F}_2 \star \mathcal{F}_1), \quad \tau_r(\overline{\mathcal{F}} \star \mathcal{F}) \geqslant 0. \tag{5.9}$$

The above trace functional can be written also in terms of cartesian symbols:

$$\tau_c(f) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int dt\, d\boldsymbol{x}|\boldsymbol{x}|^{-d} f(t, \boldsymbol{x}),$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

If we now stick to the covariant representation described in proposition 2.6, we find

$$U(g)f(T, \boldsymbol{X})U(g)^{-1} = \rho_g(f)(T, \boldsymbol{X}), \quad g \in G_d,$$

where, for $g = (A, a, \lambda)$,

$$\rho_g(f)(t, \boldsymbol{x}) = f(t - a, \lambda^{-1} A^{-1} \boldsymbol{x}). \tag{5.10}$$

The corresponding action on radial symbols can be easily computed using lemma 5.3.

6 The C*-Algebra and the Large Scale Limit

The most natural and canonical definition of C*-algebra of the κ-Minkowski spacetime is the smallest C*-algebra containing all universal quantisations of cartesian symbols.

Definition 6.1. *Let $(T, \boldsymbol{X})$ be the universal representation on $\mathfrak{H}$. The C*-algebra of the κ-Minkowski spacetime in $d+1$ dimensions is defined as*

$$\mathcal{A}_d = \overline{\{f(T, \boldsymbol{X}) : f \in L^1(\mathbb{R}^{d+1}) \cap \widehat{L^1(\mathbb{R}^{d+1})}\}}^{\|\cdot\|_{B(\mathfrak{H})}} \tag{6.1}$$

From the discussion of the preceding section, it is clear that $\mathcal{A}_d$ can be obtained equivalently from the universal quantisations of the radial symbols continuous at zero.

The next lemma will be crucial, in that it will both allow for the explicit characterisation of $\mathcal{A}_d$, and to dismiss the condition of continuity at zero of radial symbols. The two facts are related; the latter will also be responsible of the exotic topology of the large scale limit.

Lemma 6.2. *Let $\mathcal{C}(S^{d-1}, L^1(\mathscr{R}))$ be the Banach space of continuous functions $\Phi : S^{d-1} \mapsto \mathscr{R}$ with values in the radial algebra $\mathscr{R}$, and with norm*

$$\|\Phi\| = \sup_{\boldsymbol{c} \in S^{d-1}} \|\Phi(\boldsymbol{c})\|_{\mathscr{R}}$$

We denote by $\hat{\mathcal{F}}$ the fibrewise Fourier transform on radial symbols, namely

$$\hat{\mathcal{F}}(\boldsymbol{c})(\alpha, \beta) = \widehat{\mathcal{F}(\boldsymbol{c})}(\alpha, \beta) = \frac{1}{2\pi} \int dt dr \mathcal{F}(\boldsymbol{c})(t, r) e^{-i(\alpha t + \beta r)}.$$

Then the set of all fibrewise Fourier transforms of radial symbols continuous at zero is dense in $\mathcal{C}(S^{d-1}, L^1(\mathscr{R}))$.

Proof. Since removing the request of continuity at zero the set $\{\hat{\mathcal{F}}\}$ is dense, it is sufficient to show that, for any radial symbol $\mathcal{F}$ (possibly not continuous at zero), there exists a sequence $(\mathcal{F}_n)$ of radial symbols continuous at zero such that $\lim_n \|\hat{\mathcal{F}} - \hat{\mathcal{F}}_n\| = 0$. It is sufficient to further restrict ourselves to the case where $\mathcal{F}$ is of the form $\mathcal{F}(\boldsymbol{c})(t,r) = f(\boldsymbol{c})g(t,r)$, since they provide, through Fourier transformation, a total set in $\mathcal{C}(S^{d-1}, L^1(\mathscr{R}))$. Let j be defined as in the proof of proposition 4.11, and $\mathcal{F}_n(\boldsymbol{c})(t,r) = f(\boldsymbol{c})j(n|r|)(g(t,r)$. Then the proof follows by the same arguments used to prove proposition 4.11. ■

Combining the preceding lemma with propositions 4.10, 4.11, we see that the Banach *-algebra $\mathcal{C}(S^{d-1}, \mathscr{R})$ (equipped with pointwise product in $\mathscr{R}$) has a unique C*-completion $\mathcal{C}(S^{d-1}, \mathcal{K})$. As a consequence,

Proposition 6.3. *The C*-algebra $\mathcal{A}_d$ is isomorphic with $\mathcal{C}(S^{d-1}, \mathcal{K}) \oplus \mathcal{C}_\infty(\mathbb{R})$. The action* (5.10) *of G_d extends to an action by automorphism of $\mathcal{A}_d$, still denoted by ρ.*

Note that the commutative component $\mathcal{C}_\infty(\mathbb{R})$ shows up in the universal C*-completion because of the presence of continuously many equivalence classes of irreducible trivial representations.

We may then extend the quantisation of a function $f \in \mathcal{C}(\overset{\circ}{M}{}^{(1,d)} \sqcup \mathbb{R}) \simeq \mathcal{C}(\overset{\circ}{M}{}^{(1,d)}) \oplus \mathcal{C}_\infty(\mathbb{R})$ according to the above proposition. It is then clear that the large κ limit (which is the same as the large scale limit), can be performed separately in the trivial and non trivial component and gives $\mathcal{C}(\overset{\circ}{M}{}^{(1,d)} \sqcup \mathbb{R})$).

7 Remarks on the Uncertainty Relations

We now consider the Heisenberg uncertainty relations arising from the commutation relations between time T and radial distance $R > 0$, which read

$$c\Delta_\omega(T)\Delta_\omega(R) \geqslant \frac{c}{2}\omega(|[T,R]|) = \frac{1}{2\kappa}\omega(R). \tag{7.1}$$

for any state ωin the domain of T, R, where c is light speed. In this section, we restore the explicit dependence on c, κ.

We may observe that, for a state whose expectation on X is very close to the origin (namely $\omega(R)$ very "small"), the above uncertainty relations tend to have little content. Indeed,

Lemma 7.1. *For every $\varepsilon, \eta > 0$ there exists a non trivial pure vector state in the domain of T, R, such that*

$$\Delta_\omega(T) < \varepsilon, \quad \Delta_\omega(X) < \eta.$$

The state can be chosen to belong to the non trivial component.

This means that there is no limit on the precision with which we may simultaneously localise all the spacetime coordinates, at least in the region close to the space origin. This is in plain contrast with the standard motivations for spacetime quantisation, namely to prevent the formation of closed horizons as an effect of localisation *alone* (see [9]).The proof will be found at the end of this section.

The above is yet another way of saying that the model is approximately commutative close to the origin, while noncommutative effects grow the more important, the more the states we consider are localised far away from the origin. This suggests that, on the other side, it could be of some interest to have estimates about how fast noncommutativity grows with distance.

Indeed, for a state localised at distance $\omega(R) = L$ from the origin, we may rewrite the uncertainty relations as

$$L \leqslant 2\kappa c \Delta T \Delta R \tag{7.2}$$

(note that $\kappa > 0$).

We may ask ourselves at which distance from the centre of κ-Minkowski is is meaningful to speak of strong interactions. Asking for $\Delta R \sim c\Delta T \ll 10^{-19}m$ we find (if $\kappa \sim 10^{35}m$)

$$L \ll 10^{-3}m. \tag{7.3}$$

The peak nominal beam size at LHC is $1.2 \cdot 10^{-3}$ m [16, 2.2.2].

The diametre of the classical orbit of the electron in the fundamental state of the Hydrogen atom is around $\ell_0 = .5 \cdot 10^{-10}m$; the classical period of the orbit is $\tau_0 \sim 1.5 \cdot 10^{-16}m$. We take τ_0, ℓ_0 as characteristic scales of atomic physics, and we ask for the avalability of localisation states which are not so undetermined to destroy the meaning of atomic physics, namely $\Delta T \ll \tau_0, \Delta R \ll \ell_0$; we find

$$L \ll \kappa c \tau_0 \ell_0 \sim \kappa \cdot (10^{-18}m). \tag{7.4}$$

With $\kappa \sim 10^{35}m$, namely of order of the inverse Planck length, we find

$$L \ll 10^{17}m \sim 10 \text{ light–years.} \tag{7.5}$$

The Galaxie is 10^3 light–years thick; α-Centauri is five light–years from Earth.

We could also turn things the other way round. We may ask LHC physics to exist and be the same no matter where the LHC is built on Earth. Since the diametre of the Earth is $L \sim 10^7m$, the condition $c\Delta T \Delta R \ll 10^{-38}m^2$ gives

$$\frac{1}{\kappa} \ll 10^{-45}m \tag{7.6}$$

which is a less than a billionth of the Planck length.

Of course, it could be objected to such estimates that the model only should be intended "locally", and that large distance effects could be "cut off". It is however not clear how to do this. Commutation relations are global, and the unavailability of a suitable generalisation of the concept of locality is precisely the obstruction preventing us from going beyond semiclassical models of flat spacetime, which are globally defined. For the model as it stands, the above estimates are meaningful.

We now come to the discussion which is summarised by lemma 7.1 above.

For any state which is pure and belongs to the trivial component (via GNS), both T, R have null uncertainty, since the trivial component is commutative. Hence the lemma is trivially true in this case.

It may seem however that the above is a pathology due to the special status of the origin. What about states localised "very close" to the origin, but not precisely there? Let $\varepsilon, \eta > 0$ be any arbitrary choice of ("small") positive numbers. There always is a choice of $\xi^\varepsilon \in \mathscr{D}(P)$ derivable and with compact support such that $\Delta_{\xi^\varepsilon}(P) < \varepsilon$. For such a choice, let $\xi^\varepsilon_\lambda(s) = (e^{-i\lambda P}\xi^\varepsilon)(s) = \xi^\varepsilon(s-\lambda)$; since unitary transformations preserve uncertainties, $\Delta_{\xi^\varepsilon_\lambda}(P) < \varepsilon$ for every λ. But we also have $\xi^\varepsilon_\lambda \in \mathscr{D}(e^{-Q})$; moreover, as a consequence of the compactness of the support,

$$\lim_{\lambda\to\infty} \Delta_{\xi^\varepsilon_\lambda}(e^{-Q}) = 0.$$

In particular, there is a λ_η such that $\Delta_{\xi^\varepsilon_{\lambda_\eta}}(e^{-Q}) < \eta$. We found a state not beloging to the trivial component, and such that

$$\Delta(T) < \varepsilon, \quad \Delta(R) < \eta.$$

8 Conclusions and Outlook

On the mathematical side, we found that the C*-algebra of the κ-Minkowski model and its representations can be discussed thoroughly, leading to a sound quantisation prescription, which is canonically associated with the abstract Lie algebra underlying the relations.

On the side of interpretation, on the contrary, we observed some features which (together with other evident remarks which we also collect here for completeness) are not fully satisfactory from the point of view of spacetime quantisation.

- The main motivation for spacetime quantisation, namely to prevent arbitrarily precise localisation (which could lead to horizon formation) is lost for this model.
- Covariance under Lorentz boosts is severely broken.
- Moreover, translation covariance is so severely broken that the origin of space, which already got a special status from the relations, remains classical at $\kappa \neq 0$, and remains topologically disjoint from the rest in the large scale limit.
- The model is classical in the origin and grows noncommutative very fastly as the distance from the origin increases.

We also have seen that the C*-algebra and the quantisation prescription can be derived from the canonical CCR quantisation. In particular, this means that deformed Lorentz covariance could be easily established by exploiting the twisted covariance of [17, 18]. We discuss this feature in [19], where we also show that the deformed–covariant κ-Minkowski spacetime can be obtained as a non invariant restriction of a fully covariant model (thus reproducing the situation described in [20, 21]). The fully covariant model will be obtained as a minimal central covariantisation of the usual κ-Minkowski model. This will hopefully shed some light on noncommutative covariance, but unfortunately will not cure the lack of stability of spacetime under localisation alone, which will survive covariantisation. By the same techniques we may obtain a fully Poincaré covariant model. Once again, the initial κ-Minkowski is contained in the covariantised model as a subrepresentation; hence the states with sharp localisation still will be available, and will be localisable everywhere.

References

[1] J. Lukierski, H. Ruegg, A. Nowicki, and V. N. Tolstoi, "q-deformation of Poincare algebra," *Phys. Lett.* **B264** (1991) 331–338.

[2] S. Majid and H. Ruegg, "Bicrossproduct structure of κ-Poincare group and noncommutative geometry," *Phys. Lett.* **B334** (1994) 348–354, `arXiv:hep-th/9405107`.

[3] A. Agostini, "kappa-Minkowski operators on Hilbert spaces," *J. Math. Phys.* **48** (2007) 052305, `arXiv:hep-th/0512114`.

[4] P. Kosinski, J. Lukierski, and P. Maslanka, "Local field theory on kappa-Minkowski space, star products and noncommutative translations," *Czech. J. Phys.* **50** (2000) 1283–1290, `arXiv:hep-th/0009120`.

[5] H. Weyl, *Gruppentheorie und Quantenmechanik.* Hirzel, Leipzig, 1928.

[6] J. Neumann, von, "Uber die eindeutigkeit der Schrödingerschen operatoren," *Math. Annalen* **104** (1931) 570–578, `GDZ:purl?GDZPPN002274701`.

[7] A. Agostini, F. Lizzi, and A. Zampini, "Generalized Weyl systems and kappa-Minkowski space," *Mod. Phys. Lett.* **A17** (2002) 2105–2126, `arXiv:hep-th/0209174`.

[8] L. Dabrowski and G. Piacitelli, "The k-Minkowski Spacetime: Trace, Classical Limit and uncertainty Relations," in Kowalski-Glikman *et. al.* [22]. `arXiv:0909.3215 [hep-th]`.

[9] S. Doplicher, K. Fredenhagen, and J. E. Roberts, "The Quantum structure of space-time at the Planck scale and quantum fields," *Commun. Math. Phys.* **172** (1995) 187–220, `arXiv:hep-th/0303037`.

[10] M. Reed and B. Simon, *Modern Methods of Mathematical Physics*, vol. I: Functional Analysis. Academic Press, New York, 1972.

[11] J. M. Gracia-Bondia, F. Lizzi, G. Marmo, and P. Vitale, "Infinitely many star products to play with," *JHEP* **04** (2002) 026, `arXiv:hep-th/0112092`.

[12] E. Stein, *Harmonic Analysis: Real–Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, Princeton (NJ), 1993.

[13] I. M. Singer, "Uniformly continuous representations of Lie groups," *Ann. Math.* **56** (1952) 242–247, `JSTOR:1969797`.

[14] L. H. Loomis, *An Introduction to Abstract Harmonic Analysis.* van Nostrand Company, Inc., Princeton (NJ), 1954.

[15] J. Dixmier, *C*-algebras.* North Holland P.C., Amsterdam, New York, Oxford, 1977.

[16] L. Evans and P. Bryant, "LHC Machine," *JINST* **3** (2008) S08001.

[17] M. Chaichian, P. P. Kulish, K. Nishijima, and A. Tureanu, "On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT," *Phys. Lett.* **B604** (2004) 98–102, `arXiv:hep-th/0408069`.

[18] J. Wess, "Deformed coordinate spaces: Derivatives," in *Mathematical, Theoretical and Phenomenological Challenges Beyond the Standard Model. Perspectives of the Balcan Collaboration, 29 August – 2 Se ptember 2003, Vrnjačka Banja*, G. Djordjevic and J. Wess, eds. Word Scientific, 2005. `arXiv:hep-th/0408080`.

[19] L. Dabrowski, M. Godlinski, and G. Piacitelli, "Lorentz Covariant k-Minkowski Spacetime," `arXiv:0912.5451 [hep-th]`.

[20] G. Piacitelli, "Twisted Covariance and Weyl Quantisation," in Kowalski-Glikman *et. al.* [22]. `arXiv:0901.3109 [hep-th]`.

[21] G. Piacitelli, "Twisted Covariance as a Non Invariant Restriction of the Fully Covariant DFR Model," *Commun. Math. Phys.* **295** (2010) 701–729, `arXiv:0902.0575 [hep-th]`.

[22] J. Kowalski-Glikman, R. Durka, and M. Szcząchor, eds., *The Planck Scale. Proceedings of the XXV Max Born Symposium, Wrocław, 29 June – 3 July 2009*, no. 1196 in Conference Proceedings. AIP, Melville, New York, 2010.