LINEAR HYPERBOLIC SYSTEMS IN DOMAINS WITH GROWING CRACKS

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Abstract. We consider the hyperbolic system $\ddot{u} - \mathrm{div}\,(\mathbb{A}\nabla u) = f$ in the time varying cracked domain $\Omega\setminus\Gamma_t$, where the set $\Omega \subset \mathbb{R}^d$ is open, bounded, and with Lipschitz boundary, the cracks Γ_t, $t \in [0,T]$, are closed subsets of $\overline{\Omega}$, increasing with respect to inclusion, and $u(t) : \Omega \setminus \Gamma_t \to \mathbb{R}^d$ for every $t \in [0,T]$. We assume the existence of suitable regular changes of variables, which reduce our problem to the transformed system $\ddot{v} - \mathrm{div}\,(\mathbb{B}\nabla v) + \mathbf{a}\nabla v - 2\nabla\dot{v}b = g$ on the fixed domain $\Omega \setminus \Gamma_0$. Under these assumptions, we obtain existence and uniqueness of weak solutions for these two problems. Moreover, we show an energy equality for the functions v, which allows us to prove a continuous dependence result for both systems. The same study has already been carried out in [3, 7] in the scalar case.

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1. Introduction

In this paper we study existence, uniqueness, and continuous dependence on the data for the solutions of a class of linear hyperbolic systems in domains with a prescribed growing crack. The systems we consider include those of elastodynamics and have the general form

$$\ddot{u}(t,x) - \mathrm{div}_x\,(\mathbb{A}(t,x)\nabla_x u(t,x)) = f(t,x), \quad t \in [0,T], x \in \Omega_t := \Omega \setminus \Gamma_t. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^d$ is an open bounded set, Γ_t, $t \in [0,T]$, is a family of (possibly irregular) closed subsets of $\overline{\Omega}$, u is an $\mathbb{R}^d$–valued vector function depending on $t \in [0,T]$ and $x \in \Omega_t$, $\mathbb{A}$ is a tensor field which satisfies the usual standard assumptions in linear elasticity, and f is a forcing term. We assume that $t \mapsto \Gamma_t$ is increasing with respect to inclusion and contained in a given C^2 manifold Γ of dimension $d-1$. The system (1.1) is supplemented by Dirichlet and Neumann boundary conditions on prescribed parts of $\partial\Omega$, and by homogeneous Neumann boundary conditions on the cracks Γ_t.

In the literature on elastodynamics with cracks we can find two different approaches to the study of this kind of problems. The first one was developed in [2] for a scalar version of (1.1) with homogeneous Neumann conditions on the boundary of Ω_t. The existence of a solution with assigned initial data is proved using a discrete time approximation and passing to the limit as the time step tends to zero. This construction leads to an existence result under very weak conditions on the cracks Γ_t, but the uniqueness of the solution is still an open problem under these assumptions.

In this work we use a different technique, considered in this context by [3, 7], based on a suitable change of variables of class C^2 which reduces the domain $\{(t,x) \in (0,T) \times \Omega : x \in \Omega_t\}$ to the cylinder $(0,T) \times \Omega_0$. This leads to the transformed system

$$\ddot{v}(t,y) - \mathrm{div}_y\,(\mathbb{B}(t,y)\nabla_y v(t,y)) + \mathbf{a}(t,y)\nabla_y v(t,y) + \nabla_y \dot{v}(t,y)b(t,y) = g(t,y) \quad t \in [0,T],\, y \in \Omega_0, \tag{1.2}$$

where the coefficients $\mathbb{B}$, $\mathbf{a}$, b, and g are constructed starting from $\mathbb{A}$, f, and the change of variables. The boundary conditions are also transformed by the change of variables and lead to Dirichlet and Neumann boundary conditions on prescribed parts of $\partial\Omega$, and homogeneous Neumann boundary conditions on the fixed crack Γ_0.

The proofs of existence, uniqueness, and continuous dependence on the data are obtained by adapting those of [3], which deals with the same problem for a scalar function u. The extension of these results to the vectorial case requires suitable strategies to overcome the difficulties which arise from the new situation and from the slightly different assumptions on Γ.

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The main changes are in the treatment of the terms involving $\mathbb{B}$. This is due to the fact that, in linear elasticity, the natural ellipticity condition on $\mathbb{A}$ is given by

$$\mathbb{A}(t,x)\eta\cdot\eta \geq c_{\mathbb{A}}|\eta^{sym}|^2 \quad \text{for every } \eta\in\mathbb{R}^{d\times d}, \tag{1.3}$$

where η^{sym} is the symmetric part of the matrix η. Unfortunately this condition is not inherited by the transformed tensor $\mathbb{B}(t,y)$. To overcome this difficulty, we assume that $\mathbb{B}$ satisfies a weaker ellipticity assumption of integral type (see (3.1)), which always holds when $\mathbb{A}$ satisfies (1.3) and the velocity of the time dependent diffeomorphisms used in the change of variables is sufficiently small (see (3.3)).

In addition, in this paper we take the opportunity to complete the study of [3] by considering also the case of non–homogeneous Neumann boundary conditions, which corresponds to traction forces on prescribed parts of the boundary.

We first prove existence and uniqueness for solutions of (1.2), with assigned initial and boundary conditions. Moreover, we prove an energy equality (see (3.62)), which is slightly different from the one in [3], and takes into account the non-homogeneous Neumann boundary terms. This energy balance allows us to prove suitable continuity conditions of the solutions v with respect to t, which are important in the proof of the existence result for (1.1).

Finally, in the last part, we prove the continuous dependence of the solutions on the cracks Γ_t and on the manifold Γ. More precisely, given a sequence Γ^n of manifolds and a sequence Γ^n_t of time dependent cracks contained in Γ^n, we use the energy equality (3.62) to prove that, under appropriate convergence conditions, the solutions u^n and v^n of problems (1.1) and (1.2) corresponding to Γ^n_t converge to the solutions u and v of the limit problems corresponding to Γ_t.

The paper is organized as follows. In Section 2 we fix the notation adopted throughout the paper and we list the standard assumptions on the set Ω, on the geometry of the crack Γ_t, and on the diffeomorphisms used for the changes of variables. Moreover, in Definitions 2.1 and 2.3 we specify the notion of weak solution for problems (1.1) and (1.2), respectively. Section 3 deals with the study of problems (1.1) and (1.2). First, In Theorem 3.3 we show that these two problems are equivalent. Then, in Theorems 3.6 and 3.7 we prove an existence and uniqueness result for (1.2), but in a weaker sense (see Definition 3.4). Furthermore, in Proposition 3.9, we prove the energy equality (3.62), which ensures that the solution given by Theorem 3.6 is indeed a weak solution. Section 4 is devoted to the proof of the continuous dependence result, which is obtained in Theorem 4.1. We conclude with Appendix 5, where we recall an auxiliary existence theorem and give the proofs of some technical lemmas.

2. Notation and preliminary results

The space of $m\times d$ matrices with real entries is denoted by $\mathbb{R}^{m\times d}$; in case $m=d$, the subspace of symmetric matrices is denoted by $\mathbb{R}^{d\times d}_{sym}$. The space of linear and continuous maps from $\mathbb{R}^{m\times d}$ into $\mathbb{R}^{n\times l}$ is denoted by $\mathscr{L}(\mathbb{R}^{m\times d};\mathbb{R}^{n\times l})$; given $\mathbb{A}\in\mathscr{L}(\mathbb{R}^{m\times d};\mathbb{R}^{n\times l})$, we write $\mathbb{A}\eta\in\mathbb{R}^{n\times l}$ to denote the image of $\eta\in\mathbb{R}^{m\times d}$ under $\mathbb{A}$. Given two vectors $a,b\in\mathbb{R}^d$, their scalar product is denoted by $a\cdot b$ and their tensor product is denoted by $a\otimes b$. We always consider the elements of $\mathbb{R}^d$ as column vectors. Given $a\in\mathbb{R}^d$ and $\eta\in\mathbb{R}^{m\times d}$, we write ηa to denote the vector of $\mathbb{R}^d$ defined as $(\eta a)_i := \sum_j \eta_{ij}a_j$. Given two square matrices η and ξ in $\mathbb{R}^{d\times d}$, we write $\eta\xi$ to denote their matrix product, namely $(\eta\xi)_{ij} := \sum_k \eta_{ik}\xi_{kj}$, and $\eta\cdot\xi$ to denote their Euclidean scalar product, namely $\eta\cdot\xi := \sum_{i,j}\eta_{ij}\xi_{ij}$. We denote by η^{-1} and η^T the inverse and the transpose matrices of η, by η^{-T} the transpose of the inverse of η, by η^{sym} its symmetric part, namely $\eta^{sym} := (\eta+\eta^T)/2$.

The partial derivatives with respect to the variable x_i are denoted by ∂_i. Given a function $F:\mathbb{R}^d\to\mathbb{R}^m$, we denote its Jacobian matrix by ∇F, whose components are $(\nabla F)_{ij} := \partial_j F_i$. For a tensor field $A\in C^1(\mathbb{R}^d;\mathbb{R}^{d\times d})$, by $\operatorname{div} A$ we mean its divergence with respect to lines, namely $(\operatorname{div} A)_i := \sum_j \partial_j A_{ij}$. We denote by id the identity function in $\mathbb{R}^m$, possibly restricted to a subset.

We adopted standard notations for Lebesgue and Sobolev spaces on a bounded open set of $\mathbb{R}^d$. The boundary values of a Sobolev function are always intended in the sense of traces. The $(d-1)$–dimensional Hausdorff measure is denoted by $\mathcal{H}^{d-1}$. Given an opens set Ω, with Lipschitz boundary, we denote by ν the outer unit normal vector to $\partial\Omega$, which is defined $\mathcal{H}^{d-1}$–a.e. on the boundary.

Given a normed vector space X, its norm is denoted by $\|\cdot\|_X$. Given an interval $I\subset\mathbb{R}$ and a Banach space X, $L^p(I;X)$ is the space of L^p functions from I to X. Given $u\in L^p(I;X)$, we denote by $\dot u\in\mathcal{D}'(I;X)$ its distributional derivative. The set of continuous and absolutely continuous functions from I to X are denote by $C^0(I;X)$ and $AC(I;X)$, respectively. When $X=\mathbb{R}^d$, we denote the uniform norm in $C^0(I;\mathbb{R}^d)$ by $\|\cdot\|_\infty$. Given two metric spaces X and Z, $\mathrm{Lip}(Y;Z)$ is the space of Lipschitz functions from Y to Z.

In this paper we assume the following hypotheses on the set Ω, on the geometry of the cracks Γ_t, and on the diffeomorphisms of Ω into itself mapping Γ_0 into Γ_t:

(H1) $\Omega\subset\mathbb{R}^d$ is a bounded open set, with Lipschitz boundary $\partial\Omega$;
(H2) $\partial_D\Omega$ is a (possibly empty) Borel subset of $\partial\Omega$ and $\partial_N\Omega$ is its complement;
(H3) $\Gamma\subset\mathbb{R}^d$ is a complete C^2 manifold, with boundary, of dimension $d-1$, such that $\partial\Gamma\cap\Omega=\emptyset$ and $\mathcal{H}^{d-1}(\Gamma\cap\partial\Omega)=0$;
(H4) for every $x\in\Gamma\cap\overline{\Omega}$ there exists an open neighborhood U of x in $\mathbb{R}^d$ such that $(U\cap\Omega)\setminus\Gamma$ is the union of two disjoint open sets U^+ and U^- with Lipschitz boundary;
(H5) $T>0$;
(H6) Γ_t, $t\in[0,T]$, is a family of (possibly irregular) closed subsets of $\Gamma\cap\overline{\Omega}$, with $\Gamma_s\subset\Gamma_t$ for every $s\le t$;
(H7) Φ, $\Psi:[0,T]\times\overline{\Omega}\to\overline{\Omega}$ are continuous and the partial derivatives $\partial_t\Phi$, $\partial_t\Psi$, $\partial_i\Phi$, $\partial_i\Psi$, $\partial_i\partial_j\Phi$, $\partial_i\partial_j\Psi$, $\partial_i\partial_t\Phi=\partial_t\partial_i\Phi$, $\partial_i\partial_t\Psi=\partial_t\partial_i\Psi$ exist and are continuous for $i,j=1,\dots,d$;
(H8) $\Phi(t,\Omega)=\Omega$, $\Phi(t,\Gamma\cap\Omega)=\Gamma\cap\Omega$, $\Phi(t,\Gamma_0)=\Gamma_t$, and $\Phi(t,y)=y$ for every $t\in[0,T]$ and every y in a neighborhood of $\partial\Omega$;
(H9) $\Psi(t,\Phi(t,y))=y$ and $\Phi(t,\Psi(t,x))=x$ for every $x,y\in\overline{\Omega}$ and every $t\in[0,T]$;
(H10) $\Phi(0,y)=y$ for every $y\in\overline{\Omega}$;
(H11) $\partial_t\Phi$, $\partial_t\Psi$, $\partial_i\Phi$, $\partial_i\Psi$, $\partial_i\partial_j\Phi$, $\partial_i\partial_j\Psi$, $\partial_i\partial_t\Phi$, $\partial_i\partial_t\Psi$ belong to $\mathrm{Lip}([0,T];C^0(\overline{\Omega};\mathbb{R}^d))$ for $i,j=1,\dots,d$;
(H12) there exists $L>0$ such that $|\partial_i\partial_t\Phi(t,x)-\partial_i\partial_t\Phi(t,y)|\le L|x-y|$ and $|\partial_i\partial_t\Psi(t,x)-\partial_i\partial_t\Psi(t,y)|\le L|x-y|$ for every $t\in[0,T]$, $x,y\in\overline{\Omega}$, and $i=1,\dots,d$.

The differential operators ∇ and div always refer to the space variable in Ω. We often use the notation $\dot u$ instead of $\partial_t u$.

Observe that $\det\nabla\Phi(t,y)\neq0$ and $\det\nabla\Psi(t,x)\neq0$ for every $t\in[0,T]$ and $x,y\in\overline{\Omega}$, thanks to (H7) and (H9). In particular, using (H10), we conclude that both determinants are positive.

Conditions (H3) and (H4) imply that the trace of $\varphi\in H^1(\Omega\setminus\Gamma)$ is well defined on $\partial\Omega$, and on $\Gamma\cap\Omega$ from both sides. Indeed, we may find a finite number of open sets $V_k\subset\Omega$, $k=1,\dots N$, with Lipschitz boundary, such that $((\Gamma\cap\Omega)\cup\partial\Omega)\setminus(\Gamma\cap\partial\Omega)\subset\cup_{k=1}^N\partial V_k$. Moreover, since $\mathcal{H}^{d-1}(\Gamma\cap\partial\Omega)=0$, we obtain that there exists a constant $C>0$, which depends only on Ω and Γ, such that

$$\|\varphi\|_{L^2(\partial\Omega)}\le C\|\varphi\|_{H^1(\Omega\setminus\Gamma)}\quad\text{for all }\varphi\in H^1(\Omega\setminus\Gamma).\tag{2.1}$$

In particular we have

$$\|\varphi\|_{L^2(\partial_N\Omega)}\le C\|\varphi\|_{H^1(\Omega\setminus\Gamma_0)}\quad\text{for all }\varphi\in H^1(\Omega\setminus\Gamma_0).\tag{2.2}$$

Similarly, we obtain the embedding $H^1(\Omega\setminus\Gamma;\mathbb{R}^d)\hookrightarrow L^p(\Omega;\mathbb{R}^d)$ for every $p\in[1,2^*]$, where 2^* is the critical Sobolev exponent. In particular there exists a constant $C_p>0$, which depends only on Ω, Γ, and p, such that

$$\|\varphi\|_{L^p(\Omega)}\le C_p\|\varphi\|_{H^1(\Omega\setminus\Gamma_0)}\quad\text{for all }\varphi\in H^1(\Omega\setminus\Gamma_0).\tag{2.3}$$

Given a point $y\in\Gamma\cap\Omega$, its trajectory in time is described by the function $t\mapsto\Phi(t,y)\in\Gamma$. We infer that its velocity is tangential to the manifold Γ at the point $\Phi(t,y)$, that is $\dot\Phi(t,y)\cdot\nu(\Phi(t,y))=0$, where $\nu(x)$ is the normal vector to Γ at x. By combining this equality with the relation

$$\nu(\Phi(t,y))=\frac{\nabla\Phi(t,y)^{-T}\nu(y)}{|\nabla\Phi(t,y)^{-T}\nu(y)|}\quad\text{for }y\in\Gamma\cap\Omega,$$

we deduce that

$$((\nabla\Phi(t,y))^{-1}\dot\Phi(t,y))\cdot\nu(y)=\dot\Psi(t,\Phi(t,y))\cdot\nu(y)=0\quad\text{for }y\in\Gamma\cap\Omega,\tag{2.4}$$

or equivalently

$$\dot\Phi(t,\Psi(t,x))\cdot\nu(x)=0\quad\text{for }x\in\Gamma\cap\Omega.\tag{2.5}$$

Let $\Omega_t:=\Omega\setminus\Gamma_t$. We introduce the space

$$H^1_D(\Omega_t;\mathbb{R}^d):=\{u\in H^1(\Omega_t;\mathbb{R}^d):u=0\ \mathcal{H}^{d-1}\text{-a.e. on }\partial_D\Omega\},$$

where the equality on $\partial_D\Omega$ refers to the trace of u on $\partial\Omega$. We have that $H^1_D(\Omega_t;\mathbb{R}^d)$ is a Banach space endowed with the norm of $H^1(\Omega_t;\mathbb{R}^d)$, and its dual is denoted by $H^{-1}_D(\Omega_t;\mathbb{R}^d)$. Observe that the canonical isomorphism between $H^1_D(\Omega_t;\mathbb{R}^d)$ and $[H^1_D(\Omega_t)]^d$ induces the isomorphism of $H^{-1}_D(\Omega_t;\mathbb{R}^d)$ into $[H^{-1}_D(\Omega_t)]^d$.

We use the notation $\langle\cdot,\cdot\rangle_t$ to denote the duality product between the spaces $H^{-1}_D(\Omega_t;\mathbb{R}^d)$ and $H^1_D(\Omega_t;\mathbb{R}^d)$, in particular $\langle\cdot,\cdot\rangle_0$ denotes the duality product between $H^{-1}_D(\Omega_0;\mathbb{R}^d)$ and $H^1_D(\Omega_0;\mathbb{R}^d)$. Moreover we use the notation $\langle\cdot,\cdot\rangle$ to denote the scalar products in $L^2(\Omega)$, $L^2(\Omega;\mathbb{R}^d)$, and $L^2(\Omega;\mathbb{R}^{d\times d})$, according to the context, and we use $\langle\cdot,\cdot\rangle_{\partial_N\Omega}$ to denote the scalar product in $L^2(\partial_N\Omega;\mathbb{R}^d)$.

The transpose of the natural embedding of $H^1_D(\Omega_t;\mathbb{R}^d)\hookrightarrow L^2(\Omega;\mathbb{R}^d)$ induces the embedding of $L^2(\Omega;\mathbb{R}^d)$ into $H^{-1}_D(\Omega_t;\mathbb{R}^d)$, which is defined by $\langle g,\varphi\rangle_t := \langle g,\varphi\rangle$ for every $g\in L^2(\Omega;\mathbb{R}^d)$ and $\varphi\in H^1_D(\Omega_t;\mathbb{R}^d)$. Given $0\le s\le t\le T$, let $P_{st}: H^{-1}_D(\Omega_t;\mathbb{R}^d)\to H^{-1}_D(\Omega_s;\mathbb{R}^d)$ be the transpose of the natural embedding $H^1_D(\Omega_s;\mathbb{R}^d)\hookrightarrow H^1_D(\Omega_t;\mathbb{R}^d)$, i.e., $\langle P_{st}(g),\varphi\rangle_s := \langle g,\varphi\rangle_t$ for every $g\in H^{-1}_D(\Omega_t;\mathbb{R}^d)$ and $\varphi\in H^1_D(\Omega_s;\mathbb{R}^d)$. We have that the operator P_{st} is continuous, with norm less than or equal to 1. In general is not injective, since $H^1_D(\Omega_s;\mathbb{R}^d)$ is not dense in $H^1_D(\Omega_t;\mathbb{R}^d)$. Note that $P_{st}(g)=g$ for every $g\in L^2(\Omega;\mathbb{R}^d)$.

Let $\mathbb{A}:[0,T]\times\overline{\Omega}\to\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})$ be a time varying tensor field such that

$$\mathbb{A}\in \mathrm{Lip}([0,T];C^0(\overline{\Omega};\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))), \tag{2.6}$$

$$\mathbb{A}(t,\cdot)\in\mathrm{Lip}(\overline{\Omega}),\quad \|\partial_i\mathbb{A}(t,\cdot)\|_{L^\infty(\Omega)}\le C \tag{2.7}$$

for every $t\in[0,T]$, for every $i=1,\dots d$, and for some $C>0$ independent of t and i. For every $t\in[0,T]$ and $x\in\overline{\Omega}$ we assume that the tensor $\mathbb{A}(t,x)$ satisfies the following properties, which are standard in linear elasticity:

$$\mathbb{A}(t,x)\eta\in\mathbb{R}^{d\times d}_{sym}\quad\text{for every } \eta\in\mathbb{R}^{d\times d}, \tag{2.8}$$

$$\mathbb{A}(t,x)\eta=\mathbb{A}(t,x)\eta^{sym}\quad\text{for every } \eta\in\mathbb{R}^{d\times d}, \tag{2.9}$$

$$(\mathbb{A}(t,x)\eta)\cdot\xi=\eta\cdot\mathbb{A}(t,x)\xi\quad\text{for every } \eta,\xi\in\mathbb{R}^{d\times d}, \tag{2.10}$$

$$(\mathbb{A}(t,x)\eta)\cdot\eta\ge c_{\mathbb{A}}|\eta|^2\quad\text{for every } \eta\in\mathbb{R}^{d\times d}_{sym}, \tag{2.11}$$

for a suitable constant $c_{\mathbb{A}}>0$. In particular condition (1.3) holds true under these assumptions.

Given

$$f\in L^2((0,T);L^2(\Omega;\mathbb{R}^d)),\quad u^0\in H^1(\Omega_0;\mathbb{R}^d),\quad u^1\in L^2(\Omega;\mathbb{R}^d), \tag{2.12}$$

$$w_D\in L^2((0,T);H^{1/2}(\partial_D\Omega;\mathbb{R}^d)),\quad F\in H^1((0,T);L^2(\partial_N\Omega;\mathbb{R}^d)), \tag{2.13}$$

we study the linear system

$$\ddot u-\mathrm{div}\,(\mathbb{A}\nabla u)=f\quad\text{in } Q_\Gamma:=\{(t,x):t\in(0,T),x\in\Omega_t\}, \tag{2.14}$$

with boundary conditions formally written as

$$u(t)=w_D(t)\quad\text{on }\partial_D\Omega\text{ for a.e. } t\in(0,T), \tag{2.15}$$

$$(\mathbb{A}(t)\nabla u(t))\nu=F(t)\quad\text{on }\partial_N\Omega\text{ for a.e. } t\in(0,T), \tag{2.16}$$

$$(\mathbb{A}(t)\nabla u(t))\nu=0\quad\text{on }\Gamma_t\text{ for a.e. } t\in(0,T), \tag{2.17}$$

and initial conditions

$$u(0)=u^0,\quad \dot u(0)=u^1\quad\text{in }\Omega_0. \tag{2.18}$$

To give a precise meaning to (2.14)–(2.18), it is convenient to introduce the following notation. Given $v\in H^1(\Omega_t;\mathbb{R}^d)$, its gradient in the sense of distributions is denoted by ∇v and it is an element of $L^2(\Omega_t;\mathbb{R}^{d\times d})$. We define the function $\widehat\nabla v\in L^2(\Omega;\mathbb{R}^{d\times d})$ by setting $\widehat\nabla v=\nabla v$ on Ω_t and $\widehat\nabla v=0$ on Γ_t. Note that $\widehat\nabla v$ is not the gradient in the sense of distributions on Ω of the function v, considered as defined almost everywhere on Ω. Indeed the equality

$$\int_\Omega\widehat\nabla v\cdot\omega dx=-\int_\Omega v\cdot\mathrm{div}\,\omega dx$$

holds for $\omega\in C^\infty_c(\Omega_t;\mathbb{R}^{d\times d})$, but in general not for $\omega\in C^\infty_c(\Omega;\mathbb{R}^{d\times d})$. Similarly, we define $\widehat{\mathrm{div}}\,v\in L^2(\Omega)$ by setting $\widehat{\mathrm{div}}\,v=\mathrm{div}\,v$ on Ω_t and $\widehat{\mathrm{div}}\,v=0$ on Γ_t.

To prove an existence and uniqueness result for (2.14)–(2.18), we assume that there exists

$$w\in L^2((0,T);H^2(\Omega_0;\mathbb{R}^d))\cap H^1((0,T);H^1(\Omega_0;\mathbb{R}^d))\cap H^2((0,T);L^2(\Omega_0;\mathbb{R}^d)) \tag{2.19}$$

such that

$$w(t)=w_D(t)\quad\text{on }\partial_D\Omega\text{ for a.e. } t\in(0,T), \tag{2.20}$$

$$(\mathbb{A}(t)\nabla w(t))\nu = 0 \quad \text{on } \Gamma_t \cup \partial_N\Omega \text{ for a.e. } t \in (0,T), \tag{2.21}$$

$$w(0) = u^0 \quad \text{on } \partial_D\Omega, \tag{2.22}$$

where these equalities have to be considered in the appropriate sense of traces. Note that the equality (2.21) for the conormal derivative must be satisfied also on $\Gamma_t \setminus \Gamma_0$.

We recall the notion of solution of (2.14)–(2.17) given in [3]. Consider a function u satisfying the following regularity assumptions:

$$u \in C^1([0,T]; L^2(\Omega;\mathbb{R}^d)), \tag{2.23}$$

$$u(t) - w(t) \in H^1_D(\Omega_t;\mathbb{R}^d) \quad \text{for every } t \in [0,T], \tag{2.24}$$

$$\widehat{\nabla} u \in C^0([0,T]; L^2(\Omega;\mathbb{R}^{d\times d})), \tag{2.25}$$

$$\dot u \in AC([s,T]; H^{-1}_D(\Omega_s;\mathbb{R}^d)) \quad \text{for every } s \in [0,T), \tag{2.26}$$

$$\frac{1}{h}[\dot u(t+h) - \dot u(t)] \rightharpoonup \ddot u(t) \text{ weakly in } H^{-1}_D(\Omega_t;\mathbb{R}^d) \text{ for a.e. } t \in (0,T) \text{ as } h \to 0, \tag{2.27}$$

$$\text{the function } t \mapsto \|\ddot u(t)\|_{H^{-1}_D(\Omega_t)} \text{ is integrable in } (0,T). \tag{2.28}$$

The relationship between $\ddot u$ and the distributional time derivative of $\dot u$ is explained in Lemma 1.2 of [3], which shows that, in particular, under the assumptions (2.23)–(2.28), the function $t \to P_{st}(\ddot u(t))$ is the distributional derivative of the function $t \to \dot u(t)$ from (s,T) to $H^{-1}_D(\Omega_s;\mathbb{R}^d)$. Moreover

$$\dot u(t) - \dot u(s) = \int_s^t P_{s\tau}(\ddot u(\tau))d\tau \tag{2.29}$$

for every $0 \le s \le t \le T$.

Definition 2.1. Let $\mathbb{A}$, f, F, and w be as in (2.6), (2.7), (2.12), (2.13), and (2.19)–(2.21). We say that u is a *weak solution* of the hyperbolic system (2.14) with boundary conditions (2.15)–(2.17) if u satisfies (2.23)–(2.28), and for a.e. $t \in (0,T)$ we have

$$\langle \ddot u(t), \varphi\rangle_t + \langle \mathbb{A}(t)\widehat{\nabla} u(t), \widehat{\nabla}\varphi\rangle = \langle f(t),\varphi\rangle + \langle F(t),\varphi\rangle_{\partial_N\Omega} \tag{2.30}$$

for every $\varphi \in H^1_D(\Omega_t;\mathbb{R}^d)$, where $\ddot u(t)$ is defined in (2.27).

Remark 2.2. Let us check that (2.30) makes sense for a.e. $t \in (0,T)$. By (2.27), $\ddot u(t) \in H^{-1}_D(\Omega_t;\mathbb{R}^d)$ for a.e. $t \in (0,T)$, and so it is in duality with $\varphi \in H^1_D(\Omega_t;\mathbb{R}^d)$. Moreover, by (2.6), (2.7), and (2.25) for every $t \in [0,T]$ we have that $\mathbb{A}(t)\widehat{\nabla} u(t)$ belongs to $L^2(\Omega;\mathbb{R}^{d\times d})$. Finally, also that the last term of (2.30) is well defined for every $t \in [0,T]$, thanks to (2.1).

Following [7], to prove the existence and uniqueness of a weak solution, we perform a change of variable. We denote by

$$v(t,y) := u(t,\Phi(t,y)), \tag{2.31}$$

where Φ is the diffeomorphism introduced in (H7)–(H12), so that

$$u(t,x) = v(t,\Psi(t,x)). \tag{2.32}$$

Notice that $v(t,\cdot) \in H^1(\Omega_0;\mathbb{R}^d)$ if and only if $u(t,\cdot) \in H^1(\Omega_t;\mathbb{R}^d)$. Moreover we reduce the domain Q_Γ to the cylinder $(0,T) \times \Omega_0$. The transformed system reads

$$\ddot v - \operatorname{div}(\mathbb{B}\nabla v) + \mathbf{a}\nabla v - 2\nabla \dot v b = g \quad \text{in } (0,T)\times\Omega_0, \tag{2.33}$$

where $\mathbb{B}(t,y) \in \mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})$, $\mathbf{a}(t,y) \in \mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)$, $b(t,y) \in \mathbb{R}^d$, and $g(t,y) \in \mathbb{R}^d$ are defined as

$$\mathbb{B}(t,y)\eta := [\mathbb{A}(t,\Phi(t,y))(\eta\nabla\Psi(t,\Phi(t,y)))]\nabla\Psi(t,\Phi(t,y))^T - \eta b(t,y)\otimes b(t,y), \tag{2.34}$$

$$\mathbf{a}(t,y)\eta := -[(\mathbb{B}(t,y)\eta)\nabla(\det\nabla\Phi(t,y))^T + \eta\partial_t(b(t,y)\det\nabla\Phi(t,y))]\det\nabla\Psi(t,\Phi(t,y)), \tag{2.35}$$

$$b(t,y) := -\dot\Psi(t,\Phi(t,y)), \tag{2.36}$$

$$g(t,y) := f(t,\Phi(t,y)), \tag{2.37}$$

for every $\eta \in \mathbb{R}^{d\times d}$. The system is supplemented by boundary conditions formally written as

$$v(t) = w_D(t) \quad \text{on } \partial_D\Omega \text{ for a.e. } t \in (0,T), \tag{2.38}$$

$$(\mathbb{B}(t)\nabla v(t))\nu = F(t) \quad \text{on } \partial_N\Omega \text{ for a.e. } t \in (0,T), \tag{2.39}$$

$$(\mathbb{B}(t)\nabla v(t))\nu = 0 \quad \text{on } \Gamma_0 \text{ for a.e. } t \in (0,T), \tag{2.40}$$

and initial conditions

$$v(0) = v^0, \quad \dot v(0) = v^1 \quad \text{in } \Omega_0, \tag{2.41}$$

with

$$v^0 := u^0, \quad v^1 := u^1 + \nabla u^0 \dot\Phi(0). \tag{2.42}$$

To give a precise meaning to the notion of solution of (2.33) with boundary conditions (2.38)–(2.40), we consider functions v which satisfy the following regularity assumptions:

$$v \in C^1([0,T]; L^2(\Omega_0;\mathbb{R}^d)), \tag{2.43}$$

$$v(t) - w(t) \in H^1_D(\Omega_0;\mathbb{R}^d) \quad \text{for every } t \in [0,T], \tag{2.44}$$

$$\nabla v \in C^0([0,T]; L^2(\Omega_0;\mathbb{R}^{d\times d})), \tag{2.45}$$

$$\dot v \in AC([0,T]; H^{-1}_D(\Omega_0;\mathbb{R}^d)). \tag{2.46}$$

Definition 2.3. Let $\mathbb{A}$, f, F, and w be as in (2.6), (2.7), (2.12), (2.13), and (2.19)–(2.21), and let $\mathbb{B}$, $\mathbf{a}$, b, and g be defined according to (2.34)–(2.37). We say that v is a *weak solution* of the transformed system (2.33) with boundary conditions (2.38)–(2.40), if v satisfies (2.43)–(2.46), and for a.e. $t \in (0,T)$ we have

$$\langle \ddot v(t), \psi\rangle_0 + \langle \mathbb{B}(t)\widehat\nabla v(t), \widehat\nabla\psi\rangle + \langle \mathbf{a}(t)\widehat\nabla v(t), \psi\rangle + 2\langle \dot v(t), \widehat{\operatorname{div}}\,[\psi\otimes b(t)]\rangle = \langle g(t),\psi\rangle + \langle F(t),\psi\rangle_{\partial_N\Omega} \tag{2.47}$$

for every $\psi \in H^1_D(\Omega_0;\mathbb{R}^d)$.

Remark 2.4. Observe that (H8) and (2.4) imply that $b(t) = 0$ on the boundary of Ω_0. This allow us to pass, in the weak formulation of (2.33), from $-2\langle \nabla\dot v(t)\, b(t), \psi\rangle$ to $2\langle \dot v(t), \widehat{\operatorname{div}}\,[\psi\otimes b(t)]\rangle$, which can be defined even for $\dot v(t) \in L^2(\Omega;\mathbb{R}^d)$.

Remark 2.5. Take a function v satisfying (2.43)–(2.46). Let us check that the scalar products in (2.47) make sense for a.e. $t \in (0,T)$. By (2.46) we have that $\ddot v(t) \in H^{-1}_D(\Omega_0;\mathbb{R}^d)$ for a.e. $t \in (0,T)$, therefore it is duality with $\psi \in H^1_D(\Omega_0;\mathbb{R}^d)$. In view of (2.43) and (2.45), for every $t \in [0,T]$ we have that $\dot v(t)$ and $\widehat\nabla v(t)$ belong to $L^2(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^{d\times d})$, respectively. Hence, to ensure that the scalar products in the left–hand side of (2.47) are well defined, we need to show that the coefficients $\mathbb{B}$, $\mathbf{a}$, b, and $\operatorname{div} b$ are essentially bounded in space for almost every time.

Observe that assumptions (H7), (H11), (H12), (2.6), and (2.7) imply that the tensor field $\mathbb{A}(t,\Phi(t,\cdot))$, the tensor field $\nabla\Psi(t,\Phi(t,\cdot))$, the vector field $\dot\Psi(t,\Phi(t,\cdot))$, and the function $\operatorname{div}(\dot\Psi(t,\Phi(t,\cdot)))$ are Lipschitz continuous from $[0,T]$ to $L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))$, $L^\infty(\Omega;\mathbb{R}^{d\times d})$, $L^\infty(\Omega;\mathbb{R}^d)$, and $L^\infty(\Omega)$ respectively. Thus we get

$$\mathbb{B} \in \operatorname{Lip}([0,T]; L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))), \tag{2.48}$$

$$b \in \operatorname{Lip}([0,T]; L^\infty(\Omega;\mathbb{R}^d)), \quad \operatorname{div} b \in \operatorname{Lip}([0,T]; L^\infty(\Omega)). \tag{2.49}$$

We split the coefficient $\mathbf{a}$, defined in (2.35), into the sum $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$, where $\mathbf{a}_1(t,y), \mathbf{a}_2(t,y) \in \mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)$ are defined as

$$\mathbf{a}_1(t,y)\eta := -[(\mathbb{B}(t,y)\eta)\nabla(\det\nabla\Phi(t,y))^T + \eta b(t,y)\partial_t(\det\nabla\Phi(t,y))]\det\nabla\Psi(t,\Phi(t,y)), \tag{2.50}$$

$$\mathbf{a}_2(t,y)\eta := -\eta\dot b(t,y). \tag{2.51}$$

In view of the discussion above, $\mathbf{a}_1$ belongs to $\operatorname{Lip}([0,T]; L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)))$, while $\mathbf{a}_2$ is an element of $L^\infty((0,T); L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)))$, being $\dot b$ the distributional derivative of a function in $\operatorname{Lip}([0,T]; L^\infty(\Omega;\mathbb{R}^d))$. Moreover there exists $C > 0$ such that $\|\mathbf{a}_2(t,\cdot)\|_{L^\infty(\Omega)} \le C$ for a.e. $t \in (0,T)$.

Finally, g defined in (2.37) is an element of $L^2((0,T); L^2(\Omega;\mathbb{R}^d))$, since f belongs to $L^2((0,T); L^2(\Omega;\mathbb{R}^d))$. Then the right–hand side of (2.47) makes sense for a.e. $t \in (0,T)$.

Remark 2.6. Using (H3) and (H4), together with a partition of unity, and integrating by parts, we get

$$\int_\Omega \widehat\nabla\eta(x)\cdot[\xi(x)h(x)]dx = -\int_\Omega \eta(x)\widehat{\operatorname{div}}\,[\xi(x)h(x)]dx \tag{2.52}$$

for every $\eta, \xi \in H^1(\Omega \setminus \Gamma)$, and for every $h \in W^{1,\infty}(\Omega; \mathbb{R}^d)$, with $h \cdot \nu = 0$ on $(\Gamma \cap \Omega) \cup \partial\Omega$. Similarly, for every $\zeta \in W^{1,1}(\Omega \setminus \Gamma)$ we have

$$\int_\Omega \widehat{\nabla}\zeta(x) \cdot h(x)dx = -\int_\Omega \zeta(x)\mathrm{div}\, h(x)dx. \tag{2.53}$$

In particular, formulas (2.52) and (2.53) hold true if h is either $\dot\Psi(t, \Phi(t, \cdot))$ or $\dot\Phi(t, \Psi(t, \cdot))$, thanks to (2.4), (2.5), and (H7).

Now we want to clarify the relation between problem (2.14) with boundary conditions (2.15)–(2.17), and problem (2.33) with boundary conditions (2.38)–(2.40). First, in the following lemma, we investigate the regularity properties of the functions u and v.

Lemma 2.7. *Suppose that u and v are related by* (2.31) *and* (2.32). *Then u satisfies* (2.23)–(2.28) *if and only if v satisfies* (2.43)–(2.46).

Proof. The proof is straightforward by applying Lemmas 1.8 and 1.11 of [3] to the components of u and v, which is possible thanks to our Lemmas 5.3 and 5.4, and formula (2.52). ☐

Arguing again as before, thanks to the identification $H^{-1}_D(\Omega_t; \mathbb{R}^d) = [H^{-1}_D(\Omega_t)]^d$, and Lemmas 1.9 and 1.12 of [3], we have the following two results.

Lemma 2.8. *Assume that u satisfies* (2.23)–(2.28). *Then for a.e. $t \in (0, T)$ we have*

$$\begin{aligned}\langle \ddot v(t,\cdot), \psi\rangle_0 = {}& \langle \ddot u(t,\cdot), \psi(\Psi(t,\cdot)) \det \nabla\Psi(t,\cdot)\rangle_t + \langle \widehat{\nabla} u(t,\cdot), \partial_t[\psi(\Psi(t,\cdot)) \otimes \dot\Phi(t,\Psi(t,\cdot)) \det \nabla\Psi(t,\cdot)]\rangle \\ &+ \langle \dot u(t,\cdot), \partial_t[\psi(\Psi(t,\cdot)) \det \nabla\Psi(t,\cdot)] - \widehat{\mathrm{div}}\, [\psi(\Psi(t,\cdot)) \otimes \dot\Phi(t,\Psi(t,\cdot)) \det \nabla\Psi(t,\cdot)]\rangle\end{aligned}$$

for all $\psi \in H^1_D(\Omega_0; \mathbb{R}^d)$.

Lemma 2.9. *Assume that v satisfies* (2.43)–(2.46). *Then for a.e. $t \in (0, T)$ we have*

$$\begin{aligned}\langle \ddot u(t,\cdot), \varphi\rangle_t = {}& \langle \ddot v(t,\cdot), \varphi(\Phi(t,\cdot)) \det \nabla\Phi(t,\cdot)\rangle_0 + \langle \widehat{\nabla} v(t,\cdot), \partial_t[\varphi(\Phi(t,\cdot)) \otimes \dot\Psi(t,\Phi(t,\cdot)) \det \nabla\Phi(t,\cdot)]\rangle \\ &+ \langle \dot v(t,\cdot), \partial_t[\varphi(\Phi(t,\cdot)) \det \nabla\Phi(t,\cdot)] - \widehat{\mathrm{div}}\, [\varphi(\Phi(t,\cdot)) \otimes \dot\Psi(t,\Phi(t,\cdot)) \det \nabla\Phi(t,\cdot)]\rangle\end{aligned}$$

for all $\varphi \in H^1_D(\Omega_t; \mathbb{R}^d)$.

Thanks to Lemmas 2.8 and 2.9, we can explain precisely the relation between problem (2.14) with boundary conditions (2.15)–(2.17), and problem (2.33) with boundary conditions (2.38)–(2.40).

Theorem 2.10. *Under the assumptions of Definition 2.3, a function u is a weak solution of problem* (2.14) *with boundary conditions* (2.15)–(2.17), *if and only if the corresponding function v introduced in* (2.31) *is a weak solution of* (2.33) *with boundary conditions* (2.38)–(2.40).

Proof. The proof is the same of [3, Theorem 1.7]. The only difference is given by the term involving F, which remains the same through the change of variables, since the diffeomorphisms Φ and Ψ are the identity in a neighborhood of $\partial\Omega$. ☐

Remark 2.11. Observe that if u is a weak solution of (2.14)–(2.17), then we can improve the integrability condition (2.28). Indeed, by (2.30), by the Lipschitz regularity of $\mathbb{A}$, by the continuity (2.25) of $\widehat{\nabla} u$, and by (2.1), we infer that for a.e. $t \in (0, T)$

$$\|\ddot u(t)\|_{H^{-1}_D(\Omega_t)} \le C(1 + \|f(t)\|_{L^2(\Omega)} + \|F(t)\|_{L^2(\partial_N\Omega)}) \tag{2.54}$$

for some constants $C > 0$ independent of t. Therefore, the function $t \mapsto \|\ddot u(t)\|_{H^{-1}_D(\Omega_t)}$ belongs to $L^2(0, T)$, since $f \in L^2((0, T); L^2(\Omega; \mathbb{R}^d))$ and $F \in H^1((0, T); L^2(\partial_N\Omega; \mathbb{R}^d)) \subset C^0([0, T]; L^2(\partial_N\Omega; \mathbb{R}^d))$. If in addition f is in $L^p((0, T); L^2(\Omega; \mathbb{R}^d))$, with $p \in (2, \infty]$, then the function $t \mapsto \|\ddot u(t)\|_{H^{-1}_D(\Omega_t)}$ belongs to $L^p(0, T)$. The same holds true also for a weak solution v of (2.33) with boundary conditions (2.38)–(2.40), exploiting the regularity properties of $\dot v$ and ∇v, and the regularity of the coefficients (2.34)–(2.37) discussed in Remark 2.5.

3. Existence and uniqueness results

To prove the existence and uniqueness results, both for problems (2.14) and (2.33), we require an additional hypothesis on the tensor field $\mathbb{B}$. We assume that there exist two constants $\gamma > 0$ and $\beta \in \mathbb{R}$ such that for every $t \in [0,T]$

$$\langle \mathbb{B}(t)\widehat{\nabla}\eta, \widehat{\nabla}\eta\rangle \geq \gamma \|\eta\|^2_{H^1_D(\Omega_0)} - \beta\|\eta\|^2_{L^2(\Omega_0)} \quad \text{for all } \eta \in H^1_D(\Omega_0;\mathbb{R}^d). \tag{3.1}$$

Observe that condition (3.1) is satisfied whenever the velocity of the diffeomorphism $\dot{\Phi}$ is sufficiently small. Indeed, by (H3) and (H4), we can find a finite number of open sets $V_k \subset \Omega$, $k = 1, \dots N$, with Lipschitz boundary, such that $\Omega \setminus \Gamma \subset \cup_{k=1}^N V_k$. Hence, using the Second Korn Inequality in each V_k (see, e.g., [8, Theorem 2.4]) and taking the sum over k, we find a constant C, depending only on Ω and Γ, such that

$$\int_\Omega |\widehat{\nabla}\eta|^2 dx \leq C\bigg(\int_\Omega |\eta|^2 dx + \int_\Omega |\widehat{E}\eta|^2 dx\bigg) \quad \text{for all } \eta \in H^1(\Omega \setminus \Gamma;\mathbb{R}^d),$$

where $\widehat{E}\eta$ is the symmetric part of $\widehat{\nabla}\eta$, i.e., $\widehat{E}\eta = (\widehat{\nabla}\eta + \widehat{\nabla}\eta^T)/2$. In particular

$$\int_\Omega |\widehat{\nabla}\eta|^2 dx \leq C\bigg(\int_\Omega |\eta|^2 dx + \int_\Omega |\widehat{E}\eta|^2 dx\bigg) \quad \text{for all } \eta \in H^1(\Omega_T;\mathbb{R}^d). \tag{3.2}$$

Define

$$m = \min_{[0,T]\times\overline{\Omega}} \det \nabla\Psi(t,x), \quad M = \max_{[0,T]\times\overline{\Omega}} \det \nabla\Psi(t,x).$$

For every $t \in [0,T]$ and $\eta \in H^1_D(\Omega_0;\mathbb{R}^d)$, by using the definition of $\mathbb{B}$ and the change of variable formula together with conditions (2.8)–(2.11) and (3.2), we have

$$\begin{aligned}
&\int_\Omega \mathbb{B}(t,y)\widehat{\nabla}\eta(y)\cdot\widehat{\nabla}\eta(y)dy \\
&\geq mc_{\mathbb{A}}\int_\Omega |\widehat{E}[\eta(\Psi(t,x))]|^2 dx - \int_\Omega |\widehat{\nabla}\eta(y)\nabla\Psi(t,\Phi(t,y))\dot{\Phi}(t,y)|^2 dy \\
&\geq \frac{mc_{\mathbb{A}}}{MC}\int_\Omega |\widehat{\nabla}\eta(y)\nabla\Psi(t,\Phi(t,y))|^2 dy - \frac{mc_{\mathbb{A}}}{M}\int_\Omega |\eta(y)|^2 dx - \int_\Omega |\widehat{\nabla}\eta(y)\nabla\Psi(t,\Phi(t,y))\dot{\Phi}(t,y)|^2 dy,
\end{aligned}$$

since the function $\eta(\Psi(t,\cdot)) \in H^1_D(\Omega_t;\mathbb{R}^d) \subset H^1(\Omega_T;\mathbb{R}^d)$ for every $t \in [0,T]$. Hence, if we assume that

$$|\dot{\Phi}(t,y)|^2 < \frac{mc_{\mathbb{A}}}{MC} \quad \text{for all } t \in [0,T] \text{ and } y \in \overline{\Omega}, \tag{3.3}$$

by (H7) we obtain that there exists $\delta > 0$ such that

$$\int_\Omega \mathbb{B}(t,y)\widehat{\nabla}\eta(y)\cdot\widehat{\nabla}\eta(y)dy \geq \delta\int_\Omega |\widehat{\nabla}\eta(y)\nabla\Psi(t,\Phi(t,y))|^2 dy - \frac{mc_{\mathbb{A}}}{M}\int_\Omega |\eta(y)|^2 dy,$$

which gives (3.1).

Remark 3.1. The assumption (3.3) imposes a condition on the velocity of the growing crack which depends on the geometry of the crack itself.

In Section 2 we have seen that problem (2.14) with boundary conditions (2.15)–(2.17), and problem (2.33) with boundary conditions (2.38)–(2.40) are equivalent. Here we will prove the following existence theorem.

Theorem 3.2. *Let be given $\mathbb{A}$, f, u^0, u^1, F, as in* (2.6), (2.7), (2.12), *and* (2.13), *and assume the existence of w satisfying* (2.19)–(2.22). *Let $\mathbb{B}$, $\mathbf{a}$, b, g, v^0, v^1 be defined according to* (2.34)–(2.37) *and* (2.42), *with $\mathbb{B}$ satisfying* (3.1). *Then problem* (2.33) *with boundary conditions* (2.38)–(2.40) *and initial conditions* (2.41) *admits a unique solution v, according to Definition 2.3.*

The proof of Theorem 3.2 will be obtained as a consequence of Theorems 3.6 and 3.7, and Proposition 3.9 below. Thanks to Theorem 2.10, as corollary we readily obtain the following result.

Theorem 3.3. *Let be given $\mathbb{A}$, f, u^0, u^1, F as in* (2.6), (2.6), (2.12), *and* (2.13), *and assume the existence of w satisfying* (2.19)–(2.22). *Suppose that $\mathbb{B}$ defined in* (2.34) *satisfies* (3.1). *Then problem* (2.14) *with boundary conditions* (2.15)–(2.17) *and initial conditions* (2.18) *admits a unique solution u, according to Definition 2.1.*

Proof. Using Theorems 2.10 and 3.2 there exists a solution u of (2.14)–(2.17). Moreover, the initial conditions (2.18) follow from the regularity conditions (2.23)–(2.28) of u and from the initial conditions of v. Finally the solution is unique, since every solution u of (2.14) with boundary conditions (2.15)–(2.17) and initial conditions (2.18), gives a solution v of (2.33) with boundary conditions (2.38)–(2.40) and initial conditions (2.41), thanks to Theorem 2.10, the regularity conditions (2.43)–(2.46) of v, and the initial conditions of u. □

In order to prove Theorem 3.2, it is convenient to define the function

$$z(t,y) := w(t,\Phi(t,y)), \tag{3.4}$$

where w is a function satisfying (2.19)–(2.21). By Lemma 2.4 of [3], z satisfies the following properties:

$$z\in L^2((0,T);H^2(\Omega_0;\mathbb{R}^d))\cap H^1((0,T);H^1(\Omega_0;\mathbb{R}^d))\cap H^2((0,T);L^2(\Omega_0;\mathbb{R}^d)), \tag{3.5}$$

$$z(t) = w_D(t) \quad \text{on } \partial_D\Omega \text{ for a.e. } t\in(0,T), \tag{3.6}$$

$$(\mathbb{B}(t)\nabla z(t))\nu = 0 \quad \text{on } \Gamma_0\cup\partial_N\Omega \text{ for a.e. } t\in(0,T), \tag{3.7}$$

where the last two equalities are satisfied in the sense of traces. Moreover, if w satisfies also (2.22), then z satisfies

$$z(0) = v^0 \quad \text{on } \partial_D\Omega, \tag{3.8}$$

again in the sense of trace.

To obtain the existence and uniqueness result of Theorem 3.2, we first introduce a notion of solution of (2.33) which is weaker than the one considered in Definition 2.3. Next, we will prove an energy equality that ensures that this type of solution is more regular, namely it satisfies the regularity conditions (2.43)–(2.46).

Definition 3.4. Let $\mathbb{A}$, f, F, w be as in (2.6), (2.7), (2.12), (2.13), and (2.19)–(2.21), and let $\mathbb{B}$, $\mathbf{a}$, b, g, z be defined according to (2.34)–(2.37) and (3.4). We say that v is a *generalized solution* of (2.33) with boundary conditions (2.38)–(2.40) if $v\in L^\infty((0,T);H^1(\Omega_0;\mathbb{R}^d))$, $v-z\in L^\infty((0,T);H^1_D(\Omega_0;\mathbb{R}^d))$, $\dot v\in L^\infty((0,T);L^2(\Omega_0;\mathbb{R}^d))$, $\ddot v\in L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$, and

$$\langle \ddot v(t),\psi\rangle_0 + \langle \mathbb{B}(t)\widehat\nabla v(t),\widehat\nabla\psi\rangle + \langle \mathbf{a}(t)\widehat\nabla v(t),\psi\rangle + 2\langle \dot v(t),\widehat{\operatorname{div}}\,[\psi\otimes b(t)]\rangle = \langle g(t),\psi\rangle + \langle F(t),\psi\rangle_{\partial_N\Omega} \tag{3.9}$$

for a.e. $t\in(0,T)$ and every $\psi\in H^1_D(\Omega_0;\mathbb{R}^d)$.

Remark 3.5. Since $\mathcal{D}(0,T)\otimes H^1_D(\Omega_0;\mathbb{R}^d)$ is dense in $L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d))$, we can recast the equality (3.9) in the framework of the duality between $L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$ and $L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d))$. Indeed, it is easy to see that (3.9) is equivalent to

$$\begin{aligned}&\int_0^T [\langle \ddot v(t),\xi(t)\rangle_0 + \langle \mathbb{B}(t)\widehat\nabla v(t),\widehat\nabla\xi(t)\rangle + \langle \mathbf{a}(t)\widehat\nabla v(t),\xi(t)\rangle + 2\langle \dot v(t),\widehat{\operatorname{div}}\,[\xi(t)\otimes b(t)]\rangle]dt\\&=\int_0^T [\langle g(t),\xi(t)\rangle + \langle F(t),\xi(t)\rangle_{\partial_N\Omega}]dt\end{aligned}$$

for every $\xi\in L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d))$.

Note that for a generalized solution of (2.33) with boundary conditions (2.38)–(2.40), the initial conditions (2.41) make sense, as explained in detail in [3, Remark 2.7]. We are now in a position to prove the first existence result.

Theorem 3.6 (Existence)**.** *Let* $\mathbb{A}$, f, u^0, u^1, F, w *be as in* (2.6), (2.7), (2.12), (2.13), *and* (2.19)–(2.22). *Let* $\mathbb{B}$, $\mathbf{a}$, b, g, v^0, v^1, z *be defined according to* (2.34)–(2.37), (2.42), *and* (3.4), *with* $\mathbb{B}$ *satisfying* (3.1). *Then there exists a generalized solution of* (2.33) *with boundary conditions* (2.38)–(2.40) *satisfying the initial conditions* (2.41).

Proof. As in the proof of [3, Theorem 2.6], it is enough to consider the case of homogeneous Dirichlet boundary conditions, i.e., $z=0$. Indeed, in view of the properties (3.5)–(3.8) of z, v is a generalized solution of (2.33) if and only if the difference $\hat v := v-z$ satisfies the statement with initial conditions $\hat v^0 = v^0 - z(0)\in H^1_D(\Omega_0;\mathbb{R}^d)$ and $\hat v^1 = v^1-\dot z(0)\in L^2(\Omega_0,\mathbb{R}^d)$, homogeneous Dirichlet-Neumann boundary conditions on $\partial_D\Omega$ and Γ_0, Neumann boundary condition (2.39) on $\partial_N\Omega$, and source term $\hat g := g-h$, where

$$h := \ddot z - \operatorname{div}(\mathbb{B}\nabla z) + \mathbf{a}\nabla z - 2\nabla\dot z b\in L^2((0,T);L^2(\Omega_0;\mathbb{R}^d)).$$

Notice that in the definition of h we have used the equality

$$\int_{\Omega_0} \mathbb{B}(t,y)\nabla z(t,y)\cdot\nabla\varphi(y)dy = -\int_{\Omega_0} \operatorname{div}\left[\mathbb{B}(t,y)\nabla z(t,y)\right]\cdot\varphi(y)dy$$

for every $\varphi\in H^1_D(\Omega_0;\mathbb{R}^d)$, which can be obtained by arguing as in Remark 2.6 and using (3.7). Therefore, from now on we assume that $z=0$.

The proof is based on a perturbation argument. Following the procedure adopted in [4, Chapitre XVIII, §5], we first study the equation (3.9) with the additional term

$$\varepsilon\langle\dot v(t),\psi\rangle+\varepsilon\langle\widehat\nabla\dot v(t),\widehat\nabla\psi\rangle,\quad \varepsilon>0,$$

and then we let the viscosity parameter ε tend to zero.

Step 1. The perturbed problem. Let $\varepsilon>0$ be fixed. We want to show that there exists a solution $v_\varepsilon\in H^1((0;T);H^1_D(\Omega_0;\mathbb{R}^d))$, with $\ddot v_\varepsilon\in L^2((0;T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$ of the equation

$$\begin{aligned}&\langle\ddot v_\varepsilon(t),\psi\rangle_0+\langle\mathbb{B}(t)\widehat\nabla v_\varepsilon(t),\widehat\nabla\psi\rangle+\langle\mathbf{a}(t)\widehat\nabla v_\varepsilon(t),\psi\rangle-2\langle\widehat\nabla\dot v_\varepsilon(t)b(t),\psi\rangle\\&+\varepsilon\langle\dot v_\varepsilon(t),\psi\rangle+\varepsilon\langle\widehat\nabla\dot v_\varepsilon(t),\widehat\nabla\psi\rangle=\langle g(t),\psi\rangle+\langle F(t),\psi\rangle_{\partial_N\Omega},\end{aligned}\tag{3.10}$$

for a.e $t\in(0,T)$ and every $\psi\in H^1_D(\Omega_0;\mathbb{R}^d)$, with initial conditions $v_\varepsilon(0)=v^0$ and $\dot v_\varepsilon(0)=v^1$. To study equation (3.10), we regularize our coefficients with respect to time using a sequence of mollifiers, as in the proof of [3, Theorem 2.6]. Let $\rho_n\in C^\infty_c(\mathbb{R})$ satisfying $\rho_n\ge0$, $\operatorname{supp}(\rho_n)\subset[-1/n,1/n]$ and $\int\rho_n=1$, and extend our coefficients $\mathbb{B}$ and $\mathbf{a}$ to all $\mathbb{R}$ as explained in [3, Theorem 2.6].

To deal with the term F, which is not present in the quoted theorem, we introduce the function $\tilde g:=g+F$, which belongs to $L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$, thanks to (2.2), (2.13), and (2.37). Therefore we may apply Theorem 5.1 (which is a slightly stronger version of [3, Theorem 4.1]) with forcing term $\tilde g$ and $k=\varepsilon$. For every $n\in\mathbb{N}$ this leads to the existence of a solution v^n_ε of (3.10), with $\mathbb{B}(t)$ replaced by $(\mathbb{B}*\rho_n)(t)$ and $\mathbf{a}(t)$ replaced by $(\mathbf{a}*\rho_n)(t)$, satisfying the initial conditions v^0 and v^1.

Taking $\dot v^n_\varepsilon$ as test function in (3.10) and integrating over $(0,t)$, we get

$$\begin{aligned}&\int_0^t[\langle\ddot v^n_\varepsilon(s),\dot v^n_\varepsilon(s)\rangle_0+\langle(\mathbb{B}*\rho_n)(s)\widehat\nabla v^n_\varepsilon(s),\widehat\nabla\dot v^n_\varepsilon(s)\rangle+\langle(\mathbf{a}*\rho_n)(s)\widehat\nabla v^n_\varepsilon(s),\dot v^n_\varepsilon(s)\rangle]ds\\&+\int_0^t[-2\langle\widehat\nabla\dot v^n_\varepsilon(s)b(s),\dot v^n_\varepsilon(s)\rangle+\varepsilon\|\dot v^n_\varepsilon(s)\|^2_{H^1_D(\Omega_0)}]ds=\int_0^t[\langle g(s),\dot v^n_\varepsilon(s)\rangle+\langle F(s),\dot v^n_\varepsilon(s)\rangle_{\partial_N\Omega}]ds.\end{aligned}\tag{3.11}$$

Integrating the first two terms by parts with respect to time, we get

$$\int_0^t\langle\ddot v^n_\varepsilon(s),\dot v^n_\varepsilon(s)\rangle_0ds=\frac12\|\dot v^n_\varepsilon(t)\|^2_{L^2(\Omega_0)}-\frac12\|v^1\|^2_{L^2(\Omega_0)},\tag{3.12}$$

$$\begin{aligned}\int_0^t\langle(\mathbb{B}*\rho_n)(s)\widehat\nabla v^n_\varepsilon(s),\widehat\nabla\dot v^n_\varepsilon(s)\rangle ds&=\frac12\langle(\mathbb{B}*\rho_n)(t)\widehat\nabla v^n_\varepsilon(t),\widehat\nabla v^n_\varepsilon(t)\rangle\\&-\frac12\langle(\mathbb{B}*\rho_n)(0)\widehat\nabla v^0,\widehat\nabla v^0\rangle-\int_0^t\langle\partial_s(\mathbb{B}*\rho_n)(s)\widehat\nabla v^n_\varepsilon(s),\widehat\nabla v^n_\varepsilon(s)\rangle ds.\end{aligned}\tag{3.13}$$

Moreover, using (3.1) we infer that

$$\frac12\langle(\mathbb{B}*\rho_n)(t)\widehat\nabla v^n_\varepsilon(t),\widehat\nabla v^n_\varepsilon(t)\rangle\ge\frac\gamma2\|v^n_\varepsilon(t)\|^2_{H^1_D(\Omega_0)}-\frac\beta2\|v^n_\varepsilon(t)\|^2_{L^2(\Omega_0)}.\tag{3.14}$$

Since $v^n_\varepsilon(t)=v^0+\int_0^t\dot v^n_\varepsilon(s)ds$, we have also that

$$\|v^n_\varepsilon(t)\|^2_{L^2(\Omega_0)}\le2\|v^0\|^2_{L^2(\Omega_0)}+2T\int_0^t\|\dot v^n_\varepsilon(s)\|^2_{L^2(\Omega_0)}ds.\tag{3.15}$$

Now, by the regularity of the coefficients $\mathbb{B}$, $\mathbf{a}$, and g there exits a constant $C>0$ independent of n, ε, and t such that

$$\left|\int_0^t\langle(\mathbf{a}*\rho_n)(s)\widehat\nabla v^n_\varepsilon(s),\dot v^n_\varepsilon(s)\rangle ds\right|\le C\int_0^t[\|v^n_\varepsilon(s)\|^2_{H^1_D(\Omega_0)}+\|\dot v^n_\varepsilon(s)\|^2_{L^2(\Omega_0)}]ds,\tag{3.16}$$

$$\left|\int_0^t\langle g(s),\dot v^n_\varepsilon(s)\rangle ds\right|\le C\left[\|f\|^2_{L^2((0,T);L^2(\Omega_0;\mathbb{R}^d))}+\int_0^t\|\dot v^n_\varepsilon(s)\|^2_{L^2(\Omega_0)}ds\right],\tag{3.17}$$

$$\left|\int_0^t \langle \partial_s(\mathbb{B}*\rho_n)(s)\widehat{\nabla} v_\varepsilon^n(s), \widehat{\nabla} v_\varepsilon^n(s)\rangle ds\right| \le C\int_0^t \|v_\varepsilon^n(s)\|^2_{H^1_D(\Omega_0)} ds. \tag{3.18}$$

Note that by (2.53), for every $\eta \in H^1_D(\Omega_0;\mathbb{R}^d)$ we have

$$2\langle \widehat{\nabla}\eta b(s), \eta\rangle = \langle b(s), \widehat{\nabla}|\eta|^2\rangle_{L^1} = -\langle \operatorname{div} b(s), |\eta|^2\rangle_{L^1}, \tag{3.19}$$

where $\langle\cdot,\cdot\rangle_{L^1}$ denotes both the duality between $L^\infty(\Omega)$ and $L^1(\Omega)$, and the duality between $L^\infty(\Omega;\mathbb{R}^d)$ and $L^1(\Omega;\mathbb{R}^d)$. Therefore, by (2.49) and (3.19), there exits a constant $C>0$ independent of n, ε, and t such that

$$\left|2\int_0^t \langle \widehat{\nabla}\dot v_\varepsilon^n(s) b(s), \dot v_\varepsilon^n(s)\rangle ds\right| \le C\int_0^t \|\dot v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)} ds. \tag{3.20}$$

Since $F \in H^1((0,T);L^2(\partial_N\Omega;\mathbb{R}^d))$, we can integrate the last term in (3.11) by parts with respect to time, and we get that

$$\int_0^t \langle F(s), \dot v_\varepsilon^n(s)\rangle_{\partial_N\Omega} ds = \langle F(t), v_\varepsilon^n(t)\rangle_{\partial_N\Omega} - \langle F(0), v^0\rangle_{\partial_N\Omega} - \int_0^t \langle \dot F(s), v_\varepsilon^n(s)\rangle_{\partial_N\Omega} ds. \tag{3.21}$$

Now, by (2.2) and the Young's inequality there exists a constant $C>0$ independent of n, ε, and t such that

$$|\langle F(t), v_\varepsilon^n(t)\rangle_{\partial_N\Omega}| \le C\|F(t)\|^2_{L^2(\partial_N\Omega)} + \frac{\gamma}{4}\|v_\varepsilon^n(t)\|^2_{H^1_D(\Omega_0)}. \tag{3.22}$$

Moreover, $F(t) = F(0) + \int_0^t \dot F(s)ds$ and so

$$\|F(t)\|^2_{L^2(\partial_N\Omega)} \le 2\|F(0)\|^2_{L^2(\partial_N\Omega)} + 2T\|\dot F\|^2_{L^2((0,T);L^2(\partial_N\Omega;\mathbb{R}^d))}. \tag{3.23}$$

Hence, by using (3.22) and (3.23) together with (3.21), we obtain the existence of two constants $C_1, C_2 > 0$ independent of n, ε, and t such that

$$\left|\int_0^t \langle F(s), \dot v_\varepsilon^n(s)\rangle_{L^2(\partial_N\Omega)} ds\right| \le \frac{\gamma}{4}\|v_\varepsilon^n(t)\|^2_{H^1_D(\Omega_0)} + C_1 + C_2\int_0^t \|v_\varepsilon^n(s)\|^2_{H^1_D(\Omega_0)} ds. \tag{3.24}$$

By combining (3.11)–(3.18) with (3.20) and (3.24), we infer that

$$\begin{aligned}\|\dot v_\varepsilon^n(t)\|^2_{L^2(\Omega_0)} &+ \frac{\gamma}{2}\|v_\varepsilon^n(t)\|^2_{H^1_D(\Omega_0)} + 2\varepsilon\int_0^t \|\dot v_\varepsilon^n(s)\|^2_{H^1_D(\Omega_0)} ds\\ &\le C_1 + C_2\int_0^t [\|\dot v_\varepsilon^n(s)\|^2_{L^2(\Omega_0)} + \|v_\varepsilon^n(s)\|^2_{H^1_D(\Omega_0)}] ds,\end{aligned}$$

for some constants C_i, $i=1,2$, independent of n, ε, and t.

Then, using Gronwall's Lemma we obtain that

$$\|\dot v_\varepsilon^n(t)\|^2_{L^2(\Omega_0)} + \frac{\gamma}{2}\|v_\varepsilon^n(t)\|^2_{H^1_D(\Omega_0)} \le C_1 e^{C_2 T} \quad \text{for every } t\in[0,T]. \tag{3.25}$$

Hence

$$v_\varepsilon^n \text{ is bounded in } L^\infty((0,T);H^1_D(\Omega_0;\mathbb{R}^d)), \tag{3.26}$$

$$\dot v_\varepsilon^n \text{ is bounded in } L^\infty((0,T);L^2(\Omega_0;\mathbb{R}^d)), \tag{3.27}$$

$$\sqrt{\varepsilon}\dot v_\varepsilon^n \text{ is bounded in } L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d)), \tag{3.28}$$

uniformly with respect to n and ε. From these properties, using (3.11) and (3.19), we obtain also that

$$\ddot v_\varepsilon^n \text{ is bounded in } L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d)), \tag{3.29}$$

uniformly with respect to n and ε. Fixed $\varepsilon>0$, by (3.26) and (3.28) there is a subsequence of v_ε^n, not relabeled, which converges to some v_ε weakly in $H^1((0;T);H^1_D(\Omega_0;\mathbb{R}^d))$ as $n\to+\infty$. Furthermore, using also (3.29), we infer that $\ddot v_\varepsilon^n$ converges weakly in $L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$ to $\ddot v_\varepsilon$.

Let us show that v_ε satisfies equation (3.10). We fix a test function $\psi\in H^1_D(\Omega_0;\mathbb{R}^d)$ for (3.10) and, since $\mathbb{B}$ is symmetric, we observe that

$$\langle(\mathbb{B}*\rho_n)(t)\widehat{\nabla} v_\varepsilon^n(t), \widehat{\nabla}\psi\rangle = \langle \widehat{\nabla} v_\varepsilon^n(t), (\mathbb{B}*\rho_n)(t)\widehat{\nabla}\psi\rangle,$$

$$\langle(\mathbf{a}*\rho_n)(t)\widehat{\nabla} v_\varepsilon^n(t), \psi\rangle = \langle \widehat{\nabla} v_\varepsilon^n(t), (\mathbf{a}^\star*\rho_n)(t)\psi\rangle,$$

where $\mathbf{a}^\star(t,y)\in\mathscr{L}(\mathbb{R}^d;\mathbb{R}^{d\times d})$ is defined as

$$\mathbf{a}^\star(t,y)c\cdot\eta=\mathbf{a}(t,y)\eta\cdot c\quad\text{for all } c\in\mathbb{R}^d,\eta\in\mathbb{R}^{d\times d}. \tag{3.30}$$

By the regularity properties of $\mathbb{B}$ and $\mathbf{a}^\star$, as $n\to+\infty$

$$(\mathbb{B}*\rho_n)(t)\nabla\psi\to\mathbb{B}(t)\nabla\psi \text{ strongly in } L^2(\Omega_0;\mathbb{R}^{d\times d}) \text{ for all } t\in(0,T), \tag{3.31}$$

$$(\mathbf{a}^\star*\rho_n)(t)\psi\to\mathbf{a}^\star(t)\psi \text{ strongly in } L^2(\Omega_0;\mathbb{R}^{d\times d}) \text{ for a.e. } t\in(0,T). \tag{3.32}$$

Passing to the limit as $n\to+\infty$ in the PDE solved by v_ε^n, using the strong convergences (3.31) and (3.32), and the weak convergences of v_ε^n, $\dot v_\varepsilon^n$ and $\ddot v_\varepsilon^n$, we obtain that the weak limit v_ε solves equation (3.10). Furthermore, the bound (3.25) and the weak convergence of v_ε^n, $\dot v_\varepsilon^n$ and $\ddot v_\varepsilon^n$, imply that for every $t\in[0,T]$

$$\begin{aligned} v_\varepsilon^n(t)&\rightharpoonup v_\varepsilon(t)\quad\text{weakly in } H^1_D(\Omega_0;\mathbb{R}^d),\\ \dot v_\varepsilon^n(t)&\rightharpoonup \dot v_\varepsilon(t)\quad\text{weakly in } L^2(\Omega_0;\mathbb{R}^d).\end{aligned}$$

Hence v_ε satisfies the initial conditions $v_\varepsilon(0)=v^0$ and $\dot v_\varepsilon(0)=v^1$.

Step 2. Vanishing viscosity. As already done in Step 1 for the sequence v_ε^n, taking as test function in (3.10) the velocity of v_ε itself and integrating in $(0,t)$ we derive the energy equality

$$\begin{aligned}&\frac12\|\dot v_\varepsilon(t)\|^2_{L^2(\Omega_0)}+\frac12\langle\mathbb{B}(t)\widehat\nabla v_\varepsilon(t),\widehat\nabla v_\varepsilon(t)\rangle+\varepsilon\int_0^t\|\dot v_\varepsilon(s)\|^2ds\\ &=\frac12\|v^1\|^2_{L^2(\Omega_0)}+\frac12\langle\mathbb{B}(0)\widehat\nabla v^0,\widehat\nabla v^0\rangle+\int_0^t\left[\frac12\langle\dot{\mathbb{B}}(s)\widehat\nabla v_\varepsilon(s),\widehat\nabla v_\varepsilon(s)\rangle-\langle\mathbf{a}(s)\widehat\nabla v_\varepsilon(s),\dot v_\varepsilon(s)\rangle\right]ds\\ &+\int_0^t[2\langle\widehat\nabla\dot v_\varepsilon(s)b(s),\dot v_\varepsilon(s)\rangle+\langle g(s),\dot v_\varepsilon(s)\rangle+\langle F(s),\dot v_\varepsilon(s)\rangle_{\partial_N\Omega}]ds.\end{aligned} \tag{3.33}$$

Arguing as before, by the uniform ellipticity (3.1) of $\mathbb{B}$ we get the estimate

$$\begin{aligned}\|\dot v_\varepsilon(t)\|^2_{L^2(\Omega_0)}&+\frac{\gamma}{2}\|v_\varepsilon(t)\|^2_{H^1_D(\Omega_0)}+2\varepsilon\int_0^t\|\dot v_\varepsilon(s)\|^2_{H^1_D(\Omega_0)}ds\\ &\le C_1+C_2\int_0^t[\|\dot v_\varepsilon(s)\|^2_{L^2(\Omega_0)}+\|v_\varepsilon(s)\|^2_{H^1_D(\Omega_0)}]ds,\end{aligned} \tag{3.34}$$

for some constants C_i, $i=1,2$, independent of ε and t. By applying Gronwall's Lemma, we conclude that for every $t\in[0,T]$

$$\|\dot v_\varepsilon(t)\|^2_{L^2(\Omega_0)}+\frac{\gamma}{2}\|v_\varepsilon(t)\|^2_{H^1_D(\Omega_0)}\le C_1e^{C_2T}. \tag{3.35}$$

Hence v_ε is bounded in $L^\infty((0,T);H^1_D(\Omega_0;\mathbb{R}^d))$, and $\dot v_\varepsilon$ is bounded in $L^\infty((0,T);L^2(\Omega_0;\mathbb{R}^d))$. Moreover, by combining (3.34) and (3.35), we infer that

$$\varepsilon\int_0^T\|\dot v_\varepsilon(s)\|^2_{H^1_D(\Omega_0)}ds\le C, \tag{3.36}$$

for some constant C independent of ε and t. Observe that $\langle\widehat\nabla\dot v_\varepsilon(t)b(t),\psi\rangle=-\langle\dot v_\varepsilon(t),\widehat{\operatorname{div}}\,[\psi\otimes b(t)]\rangle$ by (2.52). Then, using (3.10) and the previous estimates, we obtain that $\ddot v_\varepsilon$ is bounded in $L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$. Therefore, up to a subsequence (not relabeled), v_ε converges to some

$$v\in L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d))\cap H^1((0,T);L^2(\Omega_0;\mathbb{R}^d))\cap H^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$$

in the following way

$$v_\varepsilon\rightharpoonup v\quad\text{weakly in } L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d)), \tag{3.37}$$

$$\dot v_\varepsilon\rightharpoonup \dot v\quad\text{weakly in } L^2((0,T);L^2(\Omega_0;\mathbb{R}^d)), \tag{3.38}$$

$$\ddot v_\varepsilon\rightharpoonup \ddot v\quad\text{weakly in } L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d)). \tag{3.39}$$

Moreover, $v\in L^\infty((0,T);H^1_D(\Omega_0;\mathbb{R}^d))$ and $\dot v\in L^\infty((0,T);L^2(\Omega_0;\mathbb{R}^d))$. Notice that a priori the weak limit v is not unique, but might depend on the particular subsequence chosen.

Let us show that v solves equation (3.9). For every $\varepsilon > 0$ we take $\xi \in L^2((0,T); H^1_D(\Omega_0;\mathbb{R}^d))$ as test function in (3.10) (see Remark 3.5), and we obtain

$$\begin{aligned}&\int_0^T [\langle \ddot v_\varepsilon(t), \xi(t)\rangle_0 + \langle \mathbb{B}(t)\widehat\nabla v_\varepsilon(t), \widehat\nabla\xi(t)\rangle + \langle \mathbf{a}(t)\widehat\nabla v_\varepsilon(t), \xi(t)\rangle + 2\langle \dot v_\varepsilon(t), \widehat{\operatorname{div}}\,[\xi(t)\otimes b(t)]\rangle]dt\\ &+\varepsilon\int_0^T [\langle \dot v_\varepsilon(t), \xi(t)\rangle + \langle \widehat\nabla \dot v_\varepsilon(t), \widehat\nabla\xi(t)\rangle]dt = \int_0^T [\langle g(t), \xi(t)\rangle + \langle F(t), \xi(t)\rangle_{\partial_N\Omega}]dt.\end{aligned} \tag{3.40}$$

Thanks to (3.36), we get as $\varepsilon \to 0$

$$\begin{aligned}\left|\varepsilon\int_0^T [\langle \dot v_\varepsilon(t), \xi(t)\rangle + \langle \widehat\nabla \dot v_\varepsilon(t), \widehat\nabla\xi(t)\rangle]dt\right| &\le \sqrt{\varepsilon}\int_0^T \sqrt{\varepsilon}\|\dot v_\varepsilon(t)\|_{H^1_D(\Omega_0)}\|\xi(t)\|_{H^1_D(\Omega_0)}dt\\ &\le \sqrt{\varepsilon}\|\xi\|_{L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d))}\left(\int_0^T \varepsilon\|\dot v_\varepsilon(t)\|^2_{H^1_D(\Omega_0)}dt\right)^{1/2} \le \sqrt{\varepsilon}C \to 0.\end{aligned}$$

The last property, together with the convergences (3.37)–(3.39) and equality (3.40), gives that v is a solution of (3.9). Arguing as in Step 1, we obtain for every $t \in [0,T]$

$$v_\varepsilon(t) \rightharpoonup v(t) \quad \text{weakly in } H^1_D(\Omega_0;\mathbb{R}^d), \tag{3.41}$$

$$\dot v_\varepsilon(t) \rightharpoonup \dot v(t) \quad \text{weakly in } L^2(\Omega_0;\mathbb{R}^d). \tag{3.42}$$

This gives the validity of the initial conditions of v. □

The proof of uniqueness is similar to the one in [3] and relies on a standard technique due to Ladyzenskaya [5], which consists in taking as test function in (3.9) the primitive of a solution.

Theorem 3.7 (Uniqueness)**.** *Under the assumptions of Theorem 3.6, there is at most one generalized solution of* (2.33) *with boundary conditions* (2.38)–(2.40)*, satisfying the initial conditions* (2.41).

Proof. As already pointed out at the beginning of the proof of Theorem 3.6, we may restrict ourselves to the case $z = 0$. Moreover, by linearity, it is enough to show that the sole generalized solution v to the problem (2.33) with

$$z = g = F = v^0 = v^1 = 0$$

is $v = 0$. The proof is in two steps: first we show uniqueness in a small interval $[0, t_0]$; then, by a continuity argument, we deduce uniqueness in the all $[0,T]$.

Step 1. Let $s \in (0,T)$ be fixed and let $\xi \in L^2((0,T); H^1_D(\Omega_0;\mathbb{R}^d))$ be defined as

$$\xi(t) = \begin{cases} -\int_t^s v(\tau)d\tau & \text{if } t \in [0,s],\\ 0 & \text{if } t \in [s,T].\end{cases}$$

Note that $\xi(s) = \xi(T) = 0$. Moreover, $\dot\xi \in L^2((0,T); H^1_D(\Omega_0;\mathbb{R}^d))$. Indeed

$$\dot\xi(t) = \begin{cases} v(t) & \text{if } t \in [0,s),\\ 0 & \text{if } t \in (s,T].\end{cases}$$

By taking ξ as test function in (3.9), we get

$$\int_0^s [\langle \ddot v(t), \xi(t)\rangle_0 + \langle \mathbb{B}(t)\widehat\nabla v(t), \widehat\nabla\xi(t)\rangle + \langle \mathbf{a}(t)\widehat\nabla v(t), \xi(t)\rangle + 2\int_0^s \langle \dot v(t), \widehat{\operatorname{div}}\,[\xi(t)\otimes b(t)]\rangle]dt = 0. \tag{3.43}$$

Integrating by parts with respect to time in the first term, we obtain that

$$\int_0^s \langle \ddot v(t), \xi(t)\rangle_0 dt = -\int_0^s \langle \dot v(t), v(t)\rangle dt = -\frac{1}{2}\|v(s)\|^2_{L^2(\Omega_0)}, \tag{3.44}$$

since $v^0 = v^1 = \xi(s) = 0$.

Let us rewrite the second term involving $\mathbb{B}$. By Definition 3.4 of generalized solution it is easy to see that $\xi \in \operatorname{Lip}([0,T]; H^1_D(\Omega_0;\mathbb{R}^d))$. Therefore, using (2.48) we have that $\mathbb{B}\nabla\xi \in \operatorname{Lip}([0,T]; L^2(\Omega_0;\mathbb{R}^{d\times d}))$.

Integrating by parts with respect to time and using the fact that $\xi(s)=0$ in $H^1_D(\Omega_0;\mathbb{R}^d)$, we may write

$$\begin{aligned}\int_0^s\langle\mathbb{B}(t)\widehat{\nabla}v(t),\widehat{\nabla}\xi(t)\rangle dt&=-\int_0^s\langle\widehat{\nabla}\dot{\xi}(t),\mathbb{B}(t)\widehat{\nabla}\xi(t)\rangle dt\\&=-\frac{1}{2}\langle\mathbb{B}(0)\widehat{\nabla}\xi(0),\widehat{\nabla}\xi(0)\rangle-\frac{1}{2}\int_0^s\langle\dot{\mathbb{B}}(t)\widehat{\nabla}\xi(t),\widehat{\nabla}\xi(t)\rangle dt.\end{aligned}\tag{3.45}$$

Inserting (3.44) and (3.45) into (3.43), we get

$$\begin{aligned}&\frac{1}{2}\|v(s)\|^2_{L^2(\Omega_0)}+\frac{1}{2}\langle\mathbb{B}(0)\widehat{\nabla}\xi(0),\widehat{\nabla}\xi(0)\rangle\\&=\int_0^s\Big[-\frac{1}{2}\langle\dot{\mathbb{B}}(t)\widehat{\nabla}\xi(t),\widehat{\nabla}\xi(t)\rangle+\langle\mathbf{a}(t)\widehat{\nabla}v(t),\xi(t)\rangle+2\langle\dot{v}(t),\widehat{\operatorname{div}}\,[\xi(t)\otimes b(t)]\rangle\Big]dt.\end{aligned}\tag{3.46}$$

Let us now bound from above the scalar products in the right–hand side of (3.46). By the Lipschitz regularity of $\mathbb{B}$ there exists $C>0$ such that $\|\dot{\mathbb{B}}(t,\cdot)\|_{L^\infty(\Omega)}\le C$ for a.e. $t\in(0,T)$, and so

$$\left|\int_0^s\langle\dot{\mathbb{B}}(t)\widehat{\nabla}\xi(t),\widehat{\nabla}\xi(t)\rangle dt\right|\le C\int_0^s\|\xi(t)\|^2_{H^1_D(\Omega_0)}dt.\tag{3.47}$$

We split $\operatorname{div}(\xi\otimes b)$ into the sum $\xi\operatorname{div}b+\nabla\xi b$. As already pointed in (2.49), the function $\operatorname{div}b\in\mathrm{Lip}([0,T];L^\infty(\Omega))$, therefore we may repeat the same argument as before. Integrating by parts with respect to time and using the equalities $v^0=\xi(s)=0$, we obtain

$$\begin{aligned}\int_0^s\langle\dot{v}(t),\xi(t)\operatorname{div}b(t)\rangle dt&=-\int_0^s[\langle v(t),v(t)\operatorname{div}b(t)+\xi(t)\partial_t(\operatorname{div}b(t))\rangle]dt\\&\le C\int_0^s[\|v(t)\|^2_{L^2(\Omega_0)}+\|\xi(t)\|^2_{H^1_D(\Omega_0)}]dt,\end{aligned}\tag{3.48}$$

for some $C>0$ independent of s. Performing first an integration by parts with respect to time exploiting the assumptions $v^0=\xi(s)=0$, and then using formula (2.53) and the regularity properties (2.49) of b, we infer that

$$\begin{aligned}\int_0^s\langle\dot{v}(t),\widehat{\nabla}\xi(t)b(t)\rangle dt&=\frac{1}{2}\int_0^s[\langle\operatorname{div}b(t),|v(t)|^2\rangle_{L^1}-2\langle v(t),\widehat{\nabla}\xi(t)\dot{b}(t)\rangle]dt\\&\le C\int_0^s[\|v(t)\|^2_{L^2(\Omega_0)}+\|\xi(t)\|^2_{H^1_D(\Omega_0)}]dt,\end{aligned}\tag{3.49}$$

for some constant $C>0$ independent on s.

Now, we split $\mathbf{a}=\tilde{\mathbf{a}}_1+\tilde{\mathbf{a}}_2$, where $\tilde{\mathbf{a}}_1(t,y),\tilde{\mathbf{a}}_2(t,y)\in\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)$ are defined as

$$\tilde{\mathbf{a}}_1(t,y)\eta:=\mathbf{a}_1(t,y)\eta-\mathbf{a}_1(0,y)\eta=\mathbf{a}_1(t,y)\eta+\eta b(0,y)\operatorname{div}\dot{\Phi}(0,y),\tag{3.50}$$

$$\tilde{\mathbf{a}}_2(t,y)\eta:=\mathbf{a}_2(t,y)\eta+\mathbf{a}_1(0,y)\eta=-\eta[\dot{b}(t,y)+b(0,y)\operatorname{div}\dot{\Phi}(0,y)].\tag{3.51}$$

Observe that $\tilde{\mathbf{a}}_1\in\mathrm{Lip}([0,T];L^\infty(\Omega;\mathscr{L}(\mathbb{R}^d;\mathbb{R}^{d\times d})))$, therefore $\dot{\tilde{\mathbf{a}}}_1$ belongs to $L^\infty((0,T);L^2(\Omega;\mathscr{L}(\mathbb{R}^d;\mathbb{R}^{d\times d})))$ and there exists a constant $C>0$ such that $\|\dot{\tilde{\mathbf{a}}}_1(t,\cdot)\|_{L^\infty(\Omega)}\le C$ for a.e. $t\in(0,T)$. Integrating by parts with respect to time and exploiting the equalities $\xi(s)=\tilde{\mathbf{a}}_1(0)=0$, we get

$$\begin{aligned}\int_0^s\langle\tilde{\mathbf{a}}_1(t)\widehat{\nabla}v(t),\xi(t)\rangle dt&=\int_0^s\langle\widehat{\nabla}\dot{\xi}(t),\tilde{\mathbf{a}}_1^\star(t)\xi(t)\rangle dt\\&=-\int_0^s\langle\widehat{\nabla}\xi(t),\dot{\tilde{\mathbf{a}}}_1^\star(t)\xi(t)\rangle dt-\int_0^s\langle\widehat{\nabla}\xi(t),\tilde{\mathbf{a}}_1^\star(t)\dot{\xi}(t)\rangle dt\\&=-\int_0^s\langle\dot{\tilde{\mathbf{a}}}_1(t)\widehat{\nabla}\xi(t),\xi(t)\rangle dt-\int_0^s\langle\tilde{\mathbf{a}}_1(t)\widehat{\nabla}\xi(t),v(t)\rangle dt\\&\le C\int_0^s[\|\xi(t)\|^2_{H^1_D(\Omega_0)}+\|v(t)\|^2_{L^2(\Omega_0)}]dt,\end{aligned}\tag{3.52}$$

for some constant $C>0$ independent of s, where $\tilde{\mathbf{a}}_1^\star$ is defined in a similar way to (3.30). On the other hand, $\operatorname{div}\dot{b}\in L^\infty((0,T);L^2(\Omega))$ by (2.49), and there exits $C>0$ such that $\|\operatorname{div}\dot{b}(t,\cdot)\|_{L^\infty(\Omega)}\le C$ for a.e.

$t \in (0,T)$. Furthermore, $b(0)\mathrm{div}\,\dot\Phi(0) \in \mathrm{Lip}(\overline\Omega;\mathbb{R}^d)$ thanks to (H12). Hence, performing an integration by parts with respect to the space variable we obtain

$$\begin{aligned}\int_0^s \langle \tilde{\mathbf{a}}_2(t)\widehat\nabla v(t), \xi(t)\rangle dt &= -\int_0^s \langle \widehat\nabla v(t), \xi(t)\otimes[\dot b(t) + b(0)\mathrm{div}\,\dot\Phi(0)]\rangle dt\\ &= \int_0^s \langle v(t), \xi(t)\mathrm{div}\,[\dot b(t)+b(0)\mathrm{div}\,\dot\Phi(0)] + \widehat\nabla\xi(t)\otimes[\dot b(t)+b(0)\mathrm{div}\,\dot\Phi(0)]\rangle dt \\ &\le C\int_0^s [\|\xi(t)\|^2_{H^1_D(\Omega_0)} + \|v(t)\|^2_{L^2(\Omega_0)}]dt,\end{aligned} \tag{3.53}$$

for some constant $C>0$ independent of s. Notice that, to derive (3.53), we have used formula (2.52) with $h = \dot b(t) + b(0)\mathrm{div}\,\dot\Phi(0)$. Indeed for a.e. $t\in(0,T)$ the function $\dot b(t)+b(0)\mathrm{div}\,\dot\Phi(0) \in W^{1,\infty}(\Omega;\mathbb{R}^d)$ and satisfies $(\dot b(t)+b(0)\mathrm{div}\,\dot\Phi(0))\cdot\nu = 0$ on $(\Gamma\cap\Omega)\cup\partial\Omega$, since $b(t)\cdot\nu = 0$ on $(\Gamma\cap\Omega)\cup\partial\Omega$ for every $t\in[0,T]$ by (2.4) and (H7), and $\frac1h[b(t+h)-b(t)] \to \dot b(t)$ strongly in $L^\infty(\Omega;\mathbb{R}^d)$ for a.e. $t\in(0,T)$ by (H11).

By combining (3.46) with the coercivity property (3.1) of $\mathbb{B}$, the upper bounds (3.47)–(3.49), (3.52), (3.53), and the fact that

$$\|\xi(0)\|^2_{L^2(\Omega_0)} \le T\int_0^s \|v(t)\|^2_{L^2(\Omega_0)}dt,$$

we conclude that

$$\|v(s)\|^2_{L^2(\Omega_0)} + \gamma\|\xi(0)\|^2_{H^1_D(\Omega_0)} \le \overline{C}\int_0^s [\|v(t)\|^2_{L^2(\Omega_0)} + \|\xi(t)\|^2_{H^1_D(\Omega_0)}]dt, \tag{3.54}$$

where the constant $\overline{C}$ does not depends on the parameter s chosen. Now, considering

$$z(s) := \int_0^s v(\tau)d\tau,$$

we can rewrite $\xi(t) = z(t) - z(s)$ for every $t\in[0,s]$, in particular

$$\|\xi(0)\|_{H^1_D(\Omega_0)} = \|z(s)\|_{H^1_D(\Omega_0)}, \tag{3.55}$$

$$\int_0^s \|\xi(t)\|^2_{H^1_D(\Omega_0)}dt \le 2s\|z(s)\|^2_{H^1_D(\Omega_0)} + 2\int_0^s \|z(t)\|^2_{H^1_D(\Omega_0)}dt. \tag{3.56}$$

Therefore, by combining (3.54)–(3.56), we obtain

$$\|v(s)\|^2_{L^2(\Omega_0)} + (\gamma - 2\overline{C}s)\|z(s)\|^2_{H^1_D(\Omega_0)} \le 2\overline{C}\int_0^s [\|v(t)\|^2_{L^2(\Omega_0)} + \|z(t)\|^2_{H^1_D(\Omega_0)}]dt.$$

If s is small enough, e.g., $s \le t_0 := \gamma/(4\overline{C})$, we can apply Gronwall's lemma and obtain that

$$v(s) = 0 \quad \text{for every } s\in[0,t_0].$$

Step 2. By Definition 3.4, the functions $v\colon[0,T]\to L^2(\Omega_0;\mathbb{R}^d)$ and $\dot v\colon[0,T]\to H^{-1}_D(\Omega_0;\mathbb{R}^d)$ are continuous, and we can define

$$t^* := \sup\{t\in[0,T] : v(s) = 0 \text{ for every } s\in[0,t]\}.$$

Using Step 1 and the continuity of v and $\dot v$, we get that $t^*\ge t_0>0$ and $v(t^*) = \dot v(t^*) = 0$. By contradiction, assume $t^*<T$. Now, repeating the strategy adopted in Step 1 with starting point t^* and initial set Ω_{t^*}, we may find a point $t_1>t^*$ such that $v(s)=0$ for every $s\in[t^*,t_1]$, which lay to a contradiction. Therefore $t^*=T$ and so $v(s)=0$ for every $s\in[0,T]$. □

Remark 3.8. Let v be the generalized solution of (2.33) with boundary conditions (2.38)–(2.40) and initial conditions (2.41), and let v_ε be its viscous approximation obtained by solving (3.10). Using (3.41), (3.42), and the weak lower semicontinuity of the norm, we get for every $t\in[0,T]$

$$\|\dot v(t)\|^2_{L^2(\Omega_0)} + \|v(t)\|^2_{H^1_D(\Omega_0)} \le C, \tag{3.57}$$

for some constant $C>0$ independent of t. If we now consider the function u defined by (2.32), since

$$\widehat\nabla u(t,\cdot) = \widehat\nabla v(t,\Psi(t,\cdot))\nabla\Psi(t,\cdot), \tag{3.58}$$

$$\dot u(t,\cdot) = \dot v(t,\Psi(t,\cdot)) + \widehat\nabla v(t,\Psi(t,\cdot))\dot\Psi(t,\cdot), \tag{3.59}$$

it is immediate to check that for every $t \in [0,T]$

$$\|\dot u(t)\|^2_{L^2(\Omega_0)} + \|u(t)\|^2_{H^1_D(\Omega_0)} \leq C, \tag{3.60}$$

for some constant $C > 0$ independent of t.

In order to state the next result, we introduce the following energy; given $\eta \in L^\infty((0,T); H^1(\Omega_0;\mathbb{R}^d))$, with distributional derivative $\dot\eta \in L^\infty((0,T); L^2(\Omega_0;\mathbb{R}^d))$, we set for a.e. $t \in (0,T)$

$$\mathcal{E}_{\mathbb{B}}(\eta,t) := \frac{1}{2}\|\dot\eta(t)\|^2_{L^2(\Omega_0)} + \frac{1}{2}\langle \mathbb{B}(t)\widehat\nabla\eta(t), \widehat\nabla\eta(t)\rangle, \tag{3.61}$$

where $\mathbb{B}$ is the tensor field defined in (2.34). Now, following [3, Proposition 2.11], we will prove an energy equality for a generalized solution v of (2.33).

Proposition 3.9 (Energy equality). *Under the assumptions of Theorem 3.6, let v be the (unique) generalized solution of* (2.33) *with boundary conditions* (2.38)–(2.40)*, satisfying the initial conditions* (2.41)*. Then the energy $\mathcal{E}_{\mathbb{B}}(v,\cdot)$ is a continuous function from $[0,T]$ to $\mathbb{R}$. Moreover, in case $z = 0$, it reads*

$$\mathcal{E}_{\mathbb{B}}(v,t) = \mathcal{E}_{\mathbb{B}}(v,0) + \mathcal{R}(v,t), \tag{3.62}$$

where the remainder $\mathcal{R}$ is defined as

$$\begin{aligned}\mathcal{R}(v,t) := &\int_0^t \left[\frac{1}{2}\langle \dot{\mathbb{B}}(s)\widehat\nabla v(s), \widehat\nabla v(s)\rangle - \langle \mathbf{a}(s)\widehat\nabla v(s), \dot v(s)\rangle - \langle \operatorname{div} b(s), |\dot v(s)|^2\rangle_{L^1}\right] ds\\ &+ \int_0^t [\langle g(s), \dot v(s)\rangle - \langle \dot F(s), v(s)\rangle_{\partial_N\Omega}] ds + \langle F(t), v(t)\rangle_{\partial_N\Omega} - \langle F(0), v^0\rangle_{\partial_N\Omega}.\end{aligned}$$

Remark 3.10. Note that if the solution v were smooth enough, then (3.62) would be straightforward by taking $\dot v$ as test function in (3.9). In our case, we follow the proof of [3, Proposition 2.1] by approaching $\dot v$ with $H^1_D(\Omega_0;\mathbb{R}^d)$–valued functions, in the same spirit of [6, Chapter 3, Lemma 8.3].

Proof of Proposition 3.9. As observed in the proof of [3, Proposition 2.1], it is enough to prove the statement in the case of homogeneous Dirichlet boundary conditions, i.e., $z = 0$. Moreover, by [3, Remark 2.7] and (2.2), v belongs to $C_w([0,T]; L^2(\partial_N\Omega;\mathbb{R}^d)) \cap L^\infty((0,T); L^2(\partial_N\Omega;\mathbb{R}^d))$ and F belongs to $H^1((0,T); L^2(\partial_N\Omega;\mathbb{R}^d)) \subset C([0,T]; L^2(\partial_N\Omega;\mathbb{R}^d))$, so that $\langle F(t), v(t)\rangle_{\partial_N\Omega}$ is a continuous function from $[0,T]$ to $\mathbb{R}$. Hence, to prove that $\mathcal{E}_{\mathbb{B}}(v,\cdot)$ is continuous, it is enough to show that equality (3.62) holds.

Let t_0 be fixed and let θ_0 denote the characteristic function of the time interval $(0,t_0)$. For every $\delta > 0$, we call $\theta_\delta : \mathbb{R} \to \mathbb{R}$ the function equals to 1 in $[\delta, t_0-\delta]$, 0 outside $[0,t_0]$ and which is linear in $[0,\delta]$ and $[t_0-\delta, t_0]$. As $\delta \to 0$, $\theta_\delta \to \delta_0$ in $L^1(\mathbb{R})$. Let $\rho_m \in C^\infty_c(\mathbb{R})$ be a sequence of mollifiers. For brevity, in the following, we will omit the indices δ and m.

We want to approximate $\theta_0 v : (0,T) \to H^1_D(\Omega_0;\mathbb{R}^d)$ by a suitable sequence of functions belonging to $C^\infty_c(\mathbb{R}; H^1_D(\Omega_0;\mathbb{R}^d))$. To this aim, we first extend the function $\theta_0 v$ to all $\mathbb{R}$ by setting $\theta_0 v = 0$ outside of $(0,T)$. Moreover, we extend in a similar way every function multiplied by either θ_0 or θ.

In view of the definitions above, for every m and δ fixed, it holds

$$\rho * (\theta v) \in C^\infty_c(\mathbb{R}; H^1_D(\Omega_0;\mathbb{R}^d)),$$

since $\theta v : \mathbb{R} \to H^1_D(\Omega_0;\mathbb{R}^d)$ has compact support, and the regularity follows from the equality

$$\frac{d^k}{ds^k}[\rho * (\theta v)] = \left(\frac{d^k\rho}{ds^k}\right) * (\theta v).$$

Similarly, it holds

$$\rho * (\theta \dot v) \in C^\infty_c(\mathbb{R}; L^2(\Omega_0;\mathbb{R}^d)), \tag{3.63}$$

and since

$$\rho * (\theta \dot v) = \dot\rho * (\theta v) - \rho * (\dot\theta v), \tag{3.64}$$

we realize that $\rho * (\theta \dot v) \in C^\infty_c(\mathbb{R}; H^1_D(\Omega_0;\mathbb{R}^d))$.

The only difference with respect to the proof of [3, Proposition 2.11] is given by the term involving F. Observe that $\langle \rho * (\theta F), \rho * (\theta v)\rangle_{\partial_N\Omega} \in C^\infty_c(\mathbb{R})$, then

$$\int_{\mathbb{R}} \frac{d}{ds}\langle \rho * (\theta F), \rho * (\theta v)\rangle_{\partial_N\Omega} ds = 0.$$

In particular, since $\rho*(\theta\dot v)$ is well defined in $L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d))$ (see (2.2) and (3.64)), we have that

$$\begin{aligned}0=&\int_{\mathbb{R}}[\langle\rho*(\theta F),\rho*(\dot\theta v)\rangle_{\partial_N\Omega}+\langle\rho*(\dot\theta F),\rho*(\theta v)\rangle_{\partial_N\Omega}]ds\\&+\int_{\mathbb{R}}[\langle\rho*(\theta F),\rho*(\theta\dot v)\rangle_{\partial_N\Omega}+\langle\rho*(\theta\dot F),\rho*(\theta v)\rangle_{\partial_N\Omega}]ds.\end{aligned}\tag{3.65}$$

Let us study separately the behavior of each term in (3.65) as $\delta\to0$, keeping m fixed. The first term has the following asymptotics:

$$\begin{aligned}&\lim_{\delta\to0}\int_{\mathbb{R}}\langle\rho*(\theta F),\rho*(\dot\theta v)\rangle_{\partial_N\Omega}ds=\lim_{\delta\to0}\int_{\mathbb{R}}\langle\rho*(\theta_0 F),\rho*(\dot\theta v)\rangle_{\partial_N\Omega}ds\\&=\lim_{\delta\to0}\int_0^{t_0}\dot\theta\langle\rho*\rho*(\theta_0F),v\rangle_{\partial_N\Omega}ds=-\langle\rho*\rho*(\theta_0F),v\rangle_{\partial_N\Omega}(t_0)+\langle\rho*\rho*(\theta_0F),v\rangle_{\partial_N\Omega}(0).\end{aligned}\tag{3.66}$$

To obtain this result, we have split θ as $\theta=(\theta-\theta_0)+\theta_0$ and used the following facts

$$\begin{aligned}&\rho*(\dot\theta v)\quad\text{is uniformly bounded in } L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d)),\\&\rho*(\theta F)\to\rho*(\theta_0F)\quad\text{strongly in } L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d)),\\&s\mapsto\langle\rho*\rho*(\theta_0F),v\rangle_{\partial_N\Omega}(s)\quad\text{is continuous on }[0,T],\end{aligned}$$

where the last property holds true since $\rho*\rho*(\theta_0F)\in C^0(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d))$ and $\dot v\in C_w([0,T];L^2(\partial_N\Omega;\mathbb{R}^d))\cap L^\infty((0,T);L^2(\partial_N\Omega;\mathbb{R}^d))$. Similarly, the second term of (3.65) satisfies

$$\lim_{\delta\to0}\int_{\mathbb{R}}\langle\rho*(\dot\theta F),\rho*(\theta v)\rangle_{\partial_N\Omega}ds=-\langle F,\rho*\rho*(\theta_0v)\rangle_{\partial_N\Omega}(t_0)+\langle F,\rho*\rho*(\theta_0v)\rangle_{\partial_N\Omega}(0).\tag{3.67}$$

For the last two terms of (3.65), by direct computation we have

$$\begin{aligned}&\lim_{\delta\to0}\int_{\mathbb{R}}[\langle\rho*(\theta F),\rho*(\theta\dot v)\rangle_{\partial_N\Omega}+\langle\rho*(\theta\dot F),\rho*(\theta v)\rangle_{\partial_N\Omega}]ds\\&=\int_{\mathbb{R}}[\langle\rho*(\theta_0F),\rho*(\theta_0\dot v)\rangle_{\partial_N\Omega}+\langle\rho*(\theta_0\dot F),\rho*(\theta_0v)\rangle_{\partial_N\Omega}]ds.\end{aligned}\tag{3.68}$$

Indeed, $\rho*(\theta\dot v)$ is uniformly bounded in $L^2(\mathbb{R};H^1_D(\Omega_0;\mathbb{R}^d))$ by (3.64), and it converges strongly to $\rho*(\theta_0\dot v)$ in $L^2(\mathbb{R};L^2(\Omega_0;\mathbb{R}^d))$. Hence, using also (2.2), $\rho*(\theta_0\dot v)$ is an element of $L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d))$ and $\rho*(\theta\dot v)$ converges to $\rho*(\theta_0\dot v)$ weakly in $L^2(\mathbb{R};L^2(\partial_N\Omega;\mathbb{R}^d))$. By combining (3.65)–(3.68), we infer that

$$\begin{aligned}0=&-\langle\rho*\rho*(\theta_0F),v\rangle_{\partial_N\Omega}(t_0)+\langle\rho*\rho*(\theta_0F),v\rangle_{\partial_N\Omega}(0)-\langle F,\rho*\rho*(\theta_0v)\rangle_{\partial_N\Omega}(t_0)\\&+\langle F,\rho*\rho*(\theta_0v)\rangle_{\partial_N\Omega}(0)+\int_{\mathbb{R}}[\langle\rho*(\theta_0\dot F),\rho*(\theta_0v)\rangle_{\partial_N\Omega}+\langle\rho*(\theta_0F),\rho*(\theta_0\dot v)\rangle_{\partial_N\Omega}]ds.\end{aligned}\tag{3.69}$$

Now, observe that the function v solves

$$\int_{\mathbb{R}}[\langle\theta_0\ddot v,\xi\rangle_0+\langle\theta_0\mathbb{B}\widehat\nabla v,\widehat\nabla\xi\rangle+\langle\theta_0\mathbf{a}\widehat\nabla v,\xi\rangle+2\langle\theta_0\dot v,\widehat{\operatorname{div}}\,[\xi\otimes b]\rangle]ds=\int_{\mathbb{R}}[\langle\theta_0g,\xi\rangle+\langle\theta_0F,\xi\rangle_{\partial_N\Omega}]ds,$$

for every $\xi\in L^2(\mathbb{R};H^1_D(\Omega_0;\mathbb{R}^d))$ (see Remark 3.5). In particular, by taking $\xi=\rho*(\rho*(\theta_0\dot v))$, which belongs to $L^2(\mathbb{R};H^1_D(\Omega_0;\mathbb{R}^d))$ thanks to (3.64), exploiting the properties of the convolution, and using (3.69), we obtain

$$\begin{aligned}&\int_{\mathbb{R}}\big[\langle\rho*(\theta_0\ddot v),\rho*(\theta_0\dot v)\rangle+\langle\rho*(\theta_0\mathbb{B}\widehat\nabla v),\rho*(\theta_0\widehat\nabla\dot v)\rangle\big]ds\\&+\int_{\mathbb{R}}\big[\langle\rho*(\theta_0\mathbf{a}\widehat\nabla v),\rho*(\theta_0\dot v)\rangle+2\langle\rho*(\theta_0\dot v\otimes b),\rho*(\theta_0\widehat\nabla\dot v)\rangle+2\langle\rho*(\theta_0\dot v\operatorname{div}b),\rho*(\theta_0\dot v)\rangle\big]ds\\&=\int_{\mathbb{R}}\big[\langle\rho*(\theta_0g),\rho*(\theta_0\dot v)\rangle-\langle\rho*(\theta_0\dot F),\rho*(\theta_0v)\rangle_{\partial_N\Omega}\big]ds+\langle\rho*\rho*(\theta_0F),v\rangle_{\partial_N\Omega}(t_0)\\&+\langle F,\rho*\rho*(\theta_0v)\rangle_{\partial_N\Omega}(t_0)-\langle\rho*\rho*(\theta_0F),v\rangle_{\partial_N\Omega}(0)-\langle F,\rho*\rho*(\theta_0v)\rangle_{\partial_N\Omega}(0).\end{aligned}\tag{3.70}$$

Let us now perform the second passage to the limit: we let the index m associated to the convolution ρ_m tend to $+\infty$. The last four terms in the right–hand side of (3.70) converge to

$$\langle F(t_0),v(t_0)\rangle_{\partial_N\Omega}-\langle F(0),v(0)\rangle_{\partial_N\Omega}.\tag{3.71}$$

Here we have used the strong continuity of F, the weak continuity of v, the presence of θ_0, and the fact that $\rho * \rho$ is still a smooth even mollifier. Moreover, by using the strong approximation property of the convolution and the dominated convergence theorem, it is easy to check that

$$\lim_{m\to+\infty}\int_{\mathbb{R}}\langle \rho*(\theta_0\dot{F}), \rho*(\theta_0 v)\rangle_{\partial_N\Omega} ds = \int_0^{t_0}\langle \dot{F}, v\rangle_{\partial_N\Omega} ds. \tag{3.72}$$

Finally, using [3, Proposition 2.11] for the remaining terms, we conclude the proof of the representation formula (3.62), and we obtain the desired continuity of $\mathscr{E}_{\mathbb{B}}(v,\cdot)$ in $[0,T]$. □

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. In view of Theorems 3.6 and 3.7, we know that problem (2.33) admits a unique generalized solution v (cf. Definition 3.4). Hence, to conclude the proof, it is enough to show that the generalized solution v of (2.33) is indeed a weak solution (cf. Definition 2.3), more precisely it satisfies (2.43)–(2.46).

Let us first consider the case in which w, and hence z, is zero. As pointed out in [3, Remark 2.7], $v\in C^0([0,T];L^2(\Omega_0;\mathbb{R}^d))\cap C_w([0,T];H^1_D(\Omega_0;\mathbb{R}^d))$ and $\dot v\in C^0([0,T];H^{-1}_D(\Omega_0;\mathbb{R}^d))\cap C_w([0,T];L^2(\Omega_0;\mathbb{R}^d))$. In addition, thanks to Proposition 3.9, $\mathscr{E}_{\mathbb{B}}(v,\cdot)$ is a continuous function from $[0,T]$ to $\mathbb{R}$. Let us now prove that ∇v and $\dot v$ are strongly continuous from $[0,T]$ to $L^2(\Omega_0;\mathbb{R}^{d\times d})$ and $L^2(\Omega_0;\mathbb{R}^d)$, respectively.

Let $t_0\in[0,T]$ be fixed and let (t_k) be a sequence of points converging to t_0. Since $\dot v$ is weakly continuous, we have that

$$\|\dot v(t_0)\|^2_{L^2(\Omega_0)}\le \liminf_{k\to+\infty}\|\dot v(t_k)\|^2_{L^2(\Omega_0)}.$$

Moreover, condition (3.1) implies that

$$\langle\mathbb{B}(t_0)\widehat\nabla\eta,\widehat\nabla\eta\rangle+\beta\|\eta\|^2_{L^2(\Omega_0)},\quad \eta\in H^1_D(\Omega_0;\mathbb{R}^d),$$

is an equivalent norm on $H^1_D(\Omega_0;\mathbb{R}^d)$, and so

$$\begin{aligned}\langle\mathbb{B}(t_0)\widehat\nabla v(t_0),\widehat\nabla v(t_0)\rangle+\beta\|v(t_0)\|^2_{L^2(\Omega_0)}&\le\liminf_{k\to+\infty}\left\{\langle\mathbb{B}(t_0)\widehat\nabla v(t_k),\widehat\nabla v(t_k)\rangle+\beta\|v(t_k)\|^2_{L^2(\Omega_0)}\right\}\\&=\liminf_{k\to+\infty}\langle\mathbb{B}(t_0)\widehat\nabla v(t_k),\widehat\nabla v(t_k)\rangle+\beta\|v(t_0)\|^2_{L^2(\Omega_0)},\end{aligned}$$

by the strong continuity and the weak continuity of v in $L^2(\Omega_0;\mathbb{R}^d)$ and $H^1_D(\Omega_0;\mathbb{R}^d)$, respectively. Hence, using also the strong continuity of $\mathbb{B}$ in $L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))$ and (3.57), we get

$$\begin{aligned}\langle\mathbb{B}(t_0)\widehat\nabla v(t_0),\widehat\nabla v(t_0)\rangle&\le\liminf_{k\to+\infty}\left\{\langle\mathbb{B}(t_k)\widehat\nabla v(t_k),\widehat\nabla v(t_k)\rangle+\langle(\mathbb{B}(t_0)-\mathbb{B}(t_k))\widehat\nabla v(t_k),\widehat\nabla v(t_k)\rangle\right\}\\&\le\liminf_{k\to+\infty}\langle\mathbb{B}(t_k)\widehat\nabla v(t_k),\widehat\nabla v(t_k)\rangle+C\lim_{k\to+\infty}\|\mathbb{B}(t_0)-\mathbb{B}(t_k)\|_{L^\infty(\Omega)}\\&=\liminf_{k\to+\infty}\langle\mathbb{B}(t_k)\widehat\nabla v(t_k),\widehat\nabla v(t_k)\rangle.\end{aligned}$$

Then

$$\mathscr{E}_{\mathbb{B}}(v,t_0)\le\frac12\liminf_{k\to+\infty}\|\dot v(t_k)\|^2_{L^2(\Omega_0)}+\frac12\liminf_{k\to+\infty}\langle\mathbb{B}(t_k)\widehat\nabla v(t_k),\widehat\nabla v(t_k)\rangle\le\lim_{k\to+\infty}\mathscr{E}_{\mathbb{B}}(v,t_k)=\mathscr{E}_{\mathbb{B}}(v,t_0),$$

which implies the continuity of $\|\dot v(t)\|^2_{L^2(\Omega_0)}$ and $\langle\mathbb{B}(t)\widehat\nabla v(t),\widehat\nabla v(t)\rangle$ in $t_0\in[0,T]$. Thus $\dot v$ and ∇v are strongly continuous from $[0,T]$ to $L^2(\Omega_0;\mathbb{R}^d)$ and $L^2(\Omega_0;\mathbb{R}^{d\times d})$, respectively. Therefore the properties (2.43)–(2.45) are readily verified. Eventually, since $\dot v,\ddot v\in L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$, we infer that $\dot v\in W^{1,2}((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))\subset AC([0,T];H^{-1}_D(\Omega_0;\mathbb{R}^d))$. This gives (2.46) and concludes the proof. The general case, when $z\neq0$, can be deduced by difference, exploiting the regularity of $v-z$, that we have just proved, and the regularity (3.5) of z. □

4. Continuous dependence on the data

In this section, following the same procedure adopted in [3, Theorem 3.1], we use the energy equality (3.62) to obtain a continuous dependence result on the data, both for problem (2.14) with boundary conditions (2.15)–(2.17) and initial conditions (2.18), and problem (2.33) with boundary conditions (2.38)–(2.40) and initial conditions (2.41).

The initial crack Γ_0 and the Dirichlet boundary datum w_D are kept fixed. We consider a sequence Γ^n_t of closed sets such that for every $s,t\in[0,T]$ with $s\le t$

$$\Gamma^n_s\subset\Gamma^n_t\subset\Gamma^n\cap\overline{\Omega},\quad \Gamma^n_0=\Gamma_0,\quad \Gamma^n_t\setminus\Gamma_0\subset\subset K, \tag{4.1}$$

where Γ^n satisfies (H3) and (H4), and $K\subset\subset\Omega$ is a compact set such that

$$w(t)\equiv 0\quad\text{on } K,\ \text{for a.e. } t\in(0,T). \tag{4.2}$$

Moreover, we consider a sequence $\mathbb{A}^n$ of tensor fields, a sequence f_n of source terms, a sequence F^n of Neumann boundary data, and a sequence $(u^{0,n},u^{1,n})$ of initial data. The convergences of the corresponding solutions will be obtained under the assumptions detailed in the following theorem.

Theorem 4.1. *Assume that Ω, $\partial_D\Omega$, $\partial_N\Omega$, Γ, T, Γ_t, Φ, Ψ satisfy* (H1)–(H12) *and that w satisfies* (2.19)–(2.21) *and* (4.2). *Let $\mathbb{A}$ be a tensor field satisfying* (2.6)–(2.11) *and the ellipticity condition* (3.1). *Let $f\in L^2((0,T);L^2(\Omega;\mathbb{R}^d))$, $F\in H^1((0,T);L^2(\partial_N\Omega;\mathbb{R}^d))$, $u^0\in H^1_D(\Omega_0;\mathbb{R}^d)+w(0)$, and $u^1\in L^2(\Omega_0;\mathbb{R}^d)$. Let u and v be the weak solutions of problem* (2.14) *with boundary conditions* (2.15)–(2.17) *and initial conditions* (2.18), *and problem* (2.33) *with boundary conditions* (2.38)–(2.40) *and initial conditions* (2.41), *respectively. For every $n\in\mathbb{N}$, assume that Γ^n, Γ^n_t, Φ^n, Ψ^n satisfy* (H3), (H4), (H6)–(H12), *and* (4.1). *Assume also that $\mathbb{A}^n$ satisfies* (2.6)–(2.11) *and that* (3.1) *holds for the operator $\mathbb{B}^n$ constructed starting from $\mathbb{A}^n$, Φ^n, and Ψ^n, with constants γ and β independent on $n\in\mathbb{N}$. Let $f^n\in L^2((0,T);L^2(\Omega;\mathbb{R}^d))$, $F^n\in H^1((0,T);L^2(\partial_N\Omega;\mathbb{R}^d))$, $u^{0,n}\in H^1_D(\Omega_0;\mathbb{R}^d)+w(0)$, and $u^{1,n}\in L^2(\Omega_0;\mathbb{R}^d)$. For every $n\in\mathbb{N}$ let u^n be the weak solution of problem* (2.14) *with growing crack Γ^n_t, tensor field $\mathbb{A}^n$, forcing term f^n, Dirichlet–Neumann boundary conditions as in* (2.15)–(2.17) *with F^n and Γ^n_t, initial displacement $u^{0,n}$, and initial velocity $u^{1,n}$. Similarly, let v^n be the weak solution of* (2.33) *with boundary conditions* (2.38)–(2.40) *and initial conditions* (2.41), *where the coefficients* (2.34)–(2.37) *and the initial data* (2.42) *are constructed starting from Φ^n, Ψ^n, $\mathbb{A}^n$, f^n, $u^{0,n}$, and $u^{1,n}$. Assume that there exists a constant $C>0$ such that the following inequalities hold for every $n\in\mathbb{N}$*

$$\|\Phi^n(t,\cdot)-\Phi^n(s,\cdot)\|_{L^\infty(\Omega)}\le C|t-s|\quad \textit{for every } t,s\in[0,T], \tag{4.3}$$

$$\|\partial_i\Phi^n(t,\cdot)-\partial_i\Phi^n(s,\cdot)\|_{L^\infty(\Omega)}\le C|t-s|\quad \textit{for every } t,s\in[0,T], \tag{4.4}$$

$$\|\dot\Phi^n(t,\cdot)-\dot\Phi^n(s,\cdot)\|_{L^\infty(\Omega)}\le C|t-s|\quad \textit{for every } t,s\in[0,T], \tag{4.5}$$

$$\|\partial^2_{ij}\Phi^n(t,\cdot)\|_{L^\infty(\Omega)}\le C\quad \textit{for every } t\in[0,T], \tag{4.6}$$

$$\|\mathbb{A}^n(t,\cdot)-\mathbb{A}^n(s,\cdot)\|_{L^\infty(\Omega)}\le C|t-s|\quad \textit{for every } t,s\in[0,T], \tag{4.7}$$

$$\|\partial_i\mathbb{A}^n(t,\cdot)\|_{L^\infty(\Omega)}\le C\quad \textit{for every } t\in[0,T]. \tag{4.8}$$

Furthermore, assume that the following properties hold as $n\to+\infty$

$$\dot\Phi^n(t)\to\dot\Phi(t)\quad \textit{strongly in } L^2(\Omega;\mathbb{R}^d),\quad \textit{for a.e. } t\in(0,T), \tag{4.9}$$

$$\partial_i\dot\Phi^n(t)\to\partial_i\dot\Phi(t)\quad \textit{strongly in } L^2(\Omega;\mathbb{R}^d),\quad \textit{for a.e. } t\in(0,T), \tag{4.10}$$

$$\partial^2_{ij}\Phi^n(t)\to\partial^2_{ij}\Phi(t)\quad \textit{strongly in } L^2(\Omega;\mathbb{R}^d),\quad \textit{for a.e. } t\in(0,T), \tag{4.11}$$

$$\ddot\Phi^n(t)\to\ddot\Phi(t)\quad \textit{strongly in } L^2(\Omega;\mathbb{R}^d),\quad \textit{for a.e. } t\in(0,T), \tag{4.12}$$

$$\mathbb{A}^n(t)\to\mathbb{A}(t)\quad \textit{strongly in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})),\quad \textit{for a.e. } t\in(0,T), \tag{4.13}$$

$$\partial_i\mathbb{A}^n(t)\to\partial_i\mathbb{A}(t)\quad \textit{strongly in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})),\quad \textit{for a.e. } t\in(0,T), \tag{4.14}$$

$$\dot{\mathbb{A}}^n(t)\to\dot{\mathbb{A}}(t)\quad \textit{strongly in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})),\quad \textit{for a.e. } t\in(0,T), \tag{4.15}$$

$$f^n\to f\quad \textit{strongly in } L^2((0,T);L^2(\Omega;\mathbb{R}^d)), \tag{4.16}$$

$$u^{0,n}\to u^0\quad \textit{strongly in } H^1(\Omega_0;\mathbb{R}^d), \tag{4.17}$$

$$u^{1,n}\to u^1\quad \textit{strongly in } L^2(\Omega_0;\mathbb{R}^d), \tag{4.18}$$

$$F^n \to F \quad \textit{strongly in } H^1((0,T);L^2(\partial_N\Omega;\mathbb{R}^d)). \tag{4.19}$$

Finally, assume that conditions (4.3)–(4.6) *and* (4.9)–(4.12) *hold also for the sequence* Ψ^n *with limit* Ψ. *Under these assumptions, for every* $t \in [0,T]$ *we have as* $n \to \infty$

$$u^n(t) \to u(t) \textit{ and } \dot u^n(t) \to \dot u(t) \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^d), \tag{4.20}$$

$$\widehat\nabla u^n(t) \to \widehat\nabla u(t) \quad \textit{strongly in } L^2(\Omega;\mathbb{R}^{d\times d}), \tag{4.21}$$

$$v^n(t) \to v(t) \quad \textit{strongly in } H^1(\Omega_0;\mathbb{R}^d), \tag{4.22}$$

$$\dot v^n(t) \to \dot v(t) \quad \textit{strongly in } L^2(\Omega_0;\mathbb{R}^d). \tag{4.23}$$

Remark 4.2. Since $\Phi^n(0,\cdot) = id$ for very $n \in \mathbb{N}$, assumptions (4.9) and (4.10) imply that

$$\Phi^n(t) \to \Phi(t) \quad \text{and} \quad \partial_i\Phi^n(t) \to \partial_i\Phi(t) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^d), \tag{4.24}$$

for every $t \in [0,T]$ and $i = 1,\dots,d$. Moreover, in view of (4.12) and (4.15), also the convergences (4.9) and (4.13) hold true for every time. Finally, $\|\det\nabla\Phi^n(t,\cdot)\|_{L^\infty(\Omega)}, \|\det\nabla\Psi^n(t,\cdot)\|_{L^\infty(\Omega)} \le C$ for some $C > 0$ independent of n. Thus there exists a constant $\delta_0 > 0$, independent of n, such that

$$\det\nabla\Phi^n(t,\cdot) \ge \delta_0 \quad \text{and} \quad \det\nabla\Psi^n(t,\cdot) \ge \delta_0 \quad \text{for every } t \in [0,T]. \tag{4.25}$$

Proof of Theorem 4.1. We follow the lines of the proof of [3, Theorem 3.1]. As explained in the quoted paper, the statement for the sequence u^n follows from the statement for v^n. Indeed, let $t \in [0,T]$ be fixed and assume that (4.22) and (4.23) are satisfied. By (2.32), (3.58), (3.59), and the bounds (4.3)–(4.6) on the diffeomorphisms, we get that $\widehat\nabla u^n(t)$, $u^n(t)$, and $\dot u^n(t)$ are uniformly bounded in $L^2(\Omega;\mathbb{R}^{d\times d})$, $L^2(\Omega;\mathbb{R}^d)$, and $L^2(\Omega;\mathbb{R}^d)$, respectively. In particular, up to a subsequence, they converge weakly in these spaces. To determine the weak limits, we fix a smooth function $\varphi \in C_c^\infty(\Omega;\mathbb{R}^{d\times d})$. As shown in [3, Theorem 3.1] we have that

$$\langle\widehat\nabla u^n(t),\varphi\rangle \to \langle\widehat\nabla u(t),\varphi\rangle \quad \text{as } n \to \infty.$$

Hence $\widehat\nabla u^n(t)$ converges weakly in $L^2(\Omega;\mathbb{R}^{d\times d})$ to $\widehat\nabla u(t)$. Similarly, using (4.22)–(4.24) we obtain that $\|\widehat\nabla u^n(t)\|_{L^2(\Omega)} \to \|\widehat\nabla u(t)\|_{L^2(\Omega)}$ as $n \to \infty$. Then $\widehat\nabla u^n(t)$ converges strongly in $L^2(\Omega;\mathbb{R}^{d\times d})$ to $\widehat\nabla u(t)$ and the same argument applies to $u^n(t)$ and $\dot u^n(t)$, which converge strongly in $L^2(\Omega;\mathbb{R}^d)$ to $u(t)$ and $\dot u(t)$, respectively. This gives (4.20) and (4.21), since the limits do not depend on the subsequence.

Denote by $\mathbb{B}^n$, $\mathbf{a}^n$, b^n, g^n, $v^{0,n}$, and $v^{1,n}$ the coefficients of the system (2.33) constructed starting from Φ^n, Ψ^n, $\mathbb{A}^n$, f^n, $u^{0,n}$, and $u^{1,n}$. In view of (4.3)–(4.8) it is easy to check that for every $n \in \mathbb{N}$

$$\|\mathbb{B}^n(t,\cdot) - \mathbb{B}^n(s,\cdot)\|_{L^\infty(\Omega)}, \|b^n(t,\cdot) - b^n(s,\cdot)\|_{L^\infty(\Omega)} \le C|t-s|, \quad \text{for every } t,s \in [0,T], \tag{4.26}$$

$$\|\partial_i\mathbb{B}^n(t,\cdot)\|_{L^\infty(\Omega)}, \|\partial_i b^n(t,\cdot)\|_{L^\infty(\Omega)}, \|\mathbf{a}^n(t,\cdot)\|_{L^\infty(\Omega)} \le C, \quad \text{for a.e. } t \in (0,T), \tag{4.27}$$

where C is a constant independent of t, s, n, and i. Now, the convergences (4.9)–(4.15), the lower bounds (4.25), and [3, Lemma 4.7] imply that as $n \to +\infty$

$$\partial_i\mathbb{B}^n(t) \to \partial_i\mathbb{B}(t) \quad \text{strongly in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})), \quad \text{for a.e. } t \in (0,T), \tag{4.28}$$

$$\dot{\mathbb{B}}^n(t) \to \dot{\mathbb{B}}(t) \quad \text{strongly in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})), \quad \text{for a.e. } t \in (0,T), \tag{4.29}$$

$$\mathbf{a}^n(t) \to \mathbf{a}(t) \quad \text{strongly in } L^2(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^d)), \quad \text{for a.e. } t \in (0,T), \tag{4.30}$$

$$\operatorname{div} b^n(t) \to \operatorname{div} b(t) \quad \text{strongly in } L^2(\Omega), \quad \text{for a.e. } t \in (0,T). \tag{4.31}$$

Moreover, using also (4.27) and Ascoli–Arzelá's Theorem, we infer that as $n \to +\infty$

$$\mathbb{B}^n(t) \to \mathbb{B}(t) \quad \text{strongly in } L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d})), \quad \text{for a.e. } t \in (0,T), \tag{4.32}$$

$$b^n(t) \to b(t) \quad \text{strongly in } L^\infty(\Omega;\mathbb{R}^d), \quad \text{for a.e. } t \in (0,T). \tag{4.33}$$

Finally, by (4.17)–(4.19) we obtain that as $n \to \infty$

$$g^n \to g \quad \text{strongly in } L^2((0,T);L^2(\Omega;\mathbb{R}^d)), \tag{4.34}$$

$$v^{0,n} \to v^0 \quad \text{strongly in } H^1(\Omega_0;\mathbb{R}^d), \tag{4.35}$$

$$v^{1,n} \to v^1 \quad \text{strongly in } L^2(\Omega_0;\mathbb{R}^d). \tag{4.36}$$

Notice that for every $n \in \mathbb{N}$ the function $z^n(t,y) := w(t, \Phi^n(t,y))$ satisfies (3.5)–(3.7), thanks to (2.19)–(2.21), (4.2), and [3, Lemma 2.4]. Hence, as explained in [3, Theorem 3.1], we can reduce ourselves to the case of homogeneous Dirichlet boundary conditions, namely $w = 0$.

In order to prove the validity of (4.22) and (4.23), for every $\varepsilon > 0$ we consider the solution v_ε of the perturbed problem (3.10) with coefficients $\mathbb{B}$, $\mathbf{a}$, b, g, v^0, v^1, and F, and the solution v_ε^n of the perturbed problem with coefficients $\mathbb{B}^n$, $\mathbf{a}^n$, b^n, g^n, $v^{0,n}$, $v^{1,n}$, and F^n. We claim that for every $t \in [0,T]$ as $\varepsilon \to 0$

$$v_\varepsilon(t) \to v(t) \quad \text{strongly in } H^1_D(\Omega_0;\mathbb{R}^d), \tag{4.37}$$

$$\dot v_\varepsilon(t) \to \dot v(t) \quad \text{strongly in } L^2(\Omega_0;\mathbb{R}^d). \tag{4.38}$$

Moreover, we claim that there exists a sequence of parameters $\varepsilon_n > 0$, converging to 0 as $n \to +\infty$, such that for every $t \in [0,T]$ as $n \to +\infty$

$$v^n_{\varepsilon_n}(t) - v_{\varepsilon_n}(t) \to 0 \quad \text{strongly in } H^1_D(\Omega_0;\mathbb{R}^d), \tag{4.39}$$

$$\dot v^n_{\varepsilon_n}(t) - \dot v_{\varepsilon_n}(t) \to 0 \quad \text{strongly in } L^2(\Omega_0;\mathbb{R}^d), \tag{4.40}$$

$$v^n_{\varepsilon_n}(t) - v^n(t) \to 0 \quad \text{strongly in } H^1_D(\Omega_0;\mathbb{R}^d), \tag{4.41}$$

$$\dot v^n_{\varepsilon_n}(t) - \dot v^n(t) \to 0 \quad \text{strongly in } L^2(\Omega_0;\mathbb{R}^d). \tag{4.42}$$

Observe that (4.37)–(4.42) imply the strong convergences (4.22) and (4.23). Indeed, by the triangle inequality

$$\|v^n(t) - v(t)\|_{H^1_D(\Omega_0)} \le \|v^n(t) - v^n_{\varepsilon_n}(t)\|_{H^1_D(\Omega_0)} + \|v^n_{\varepsilon_n}(t) - v_{\varepsilon_n}(t)\|_{H^1_D(\Omega_0)} + \|v_{\varepsilon_n}(t) - v(t)\|_{H^1_D(\Omega_0)} \to 0$$

as $n \to \infty$, and the same holds true for $\|\dot v^n(t) - \dot v(t)\|_{L^2(\Omega_0)}$. To prove the claims (4.37)–(4.42) we divide the proof into several steps.

Step 1. Strong convergence of v_ε. Let $X^\varepsilon := v_\varepsilon - v$. By comparing the two energy equalities (3.33) and (3.62), we infer that X^ε satisfies

$$\mathcal{E}_{\mathbb{B}}(X^\varepsilon, t) + \varepsilon \int_0^t \|\dot v_\varepsilon(s)\|^2_{H^1_D(\Omega_0)} ds = \int_0^t \left[\frac12 \langle \dot{\mathbb{B}} \widehat\nabla X^\varepsilon, \widehat\nabla X^\varepsilon\rangle - \langle \mathbf{a}\widehat\nabla X^\varepsilon, \dot X^\varepsilon\rangle - \langle \operatorname{div} b, |\dot X^\varepsilon|^2\rangle_{L^1}\right] ds + R_\varepsilon(t),$$

where $\mathcal{E}_{\mathbb{B}}$ is the energy defined in (3.61) and

$$\begin{aligned} R_\varepsilon(t) &:= -\langle \dot v_\varepsilon(t), \dot v(t)\rangle - \langle \mathbb{B}(t)\widehat\nabla v_\varepsilon(t), \widehat\nabla v(t)\rangle + \|v^1\|^2_{L^2(\Omega_0)} + \langle \mathbb{B}(0)\widehat\nabla v^0, \widehat\nabla v^0\rangle \\ &+ \int_0^t [\langle \dot{\mathbb{B}}\widehat\nabla v_\varepsilon, \widehat\nabla v\rangle - \langle \mathbf{a}\widehat\nabla v_\varepsilon, \dot v\rangle - \langle \mathbf{a}\widehat\nabla v, \dot v_\varepsilon\rangle - 2\langle \operatorname{div} b, \dot v_\varepsilon \cdot \dot v\rangle_{L^1} + \langle g, \dot v_\varepsilon + \dot v\rangle] ds \\ &- \int_0^t \langle \dot F, v_\varepsilon + v\rangle_{\partial_N\Omega} ds + \langle F(t), v_\varepsilon(t) + v(t)\rangle_{\partial_N\Omega} - 2\langle F(0), v^0\rangle_{\partial_N\Omega}. \end{aligned}$$

Thanks to the weak convergences (3.41) and (3.42), the energy equality (3.62) gives that $R_\varepsilon(t) \to 0$ as $\varepsilon \to 0$. Moreover the uniform bounds on $\dot{\mathbb{B}}$, $\mathbf{a}$, and $\operatorname{div} b$, the ellipticity condition (3.1), and the estimate

$$\|X^\varepsilon(t)\|^2_{L^2(\Omega_0)} \le T \int_0^t \|\dot X^\varepsilon(s)\|^2_{L^2(\Omega_0)} ds, \tag{4.43}$$

imply that for every $t \in [0,T]$

$$\|X^\varepsilon(t)\|^2_{L^2(\Omega_0)} + \gamma \|X^\varepsilon(t)\|^2_{H^1_D(\Omega_0)} \le 2R_\varepsilon(t) + C\int_0^t [\|\dot X^\varepsilon(s)\|^2_{L^2(\Omega_0)} + \|X^\varepsilon(s)\|^2_{H^1_D(\Omega_0)}] ds,$$

with $C > 0$ independent of t and ε. Then, by applying Fatou's Lemma, for every $t \in [0,T]$ we have that

$$\begin{aligned} \limsup_{\varepsilon\to0} [\|X^\varepsilon(t)\|^2_{L^2(\Omega_0)} + \gamma\|X^\varepsilon(t)\|^2_{H^1_D(\Omega_0)}] &\le \limsup_{\varepsilon\to0}\left[2R_\varepsilon(t) + C\int_0^t [\|\dot X^\varepsilon(s)\|^2_{L^2(\Omega_0)} + \|X^\varepsilon(s)\|^2_{H^1_D(\Omega_0)}] ds\right] \\ &\le C\int_0^t \limsup_{\varepsilon\to0} [\|\dot X^\varepsilon(s)\|^2_{L^2(\Omega_0)} + \|X^\varepsilon(s)\|^2_{H^1_D(\Omega_0)}] ds. \end{aligned}$$

Consequently, using Gronwall's Lemma we conclude that

$$\lim_{\varepsilon\to0} [\|X^\varepsilon(t)\|^2_{L^2(\Omega_0)} + \gamma\|X^\varepsilon(t)\|^2_{H^1_D(\Omega_0)}] = 0.$$

Therefore we obtain the strong convergences (4.37) and (4.38).

Step 2. Strong convergence of $v^n_{\varepsilon_n} - v_{\varepsilon_n}$. Let $\varepsilon_n > 0$ be a sequence of parameters to be fixed. The functions $v^n_{\varepsilon_n}$ and v_{ε_n} satisfy the perturbed problem (3.10) with different coefficients, but with the same viscosity ε_n. Then, by linearity the difference $X^n := v^n_{\varepsilon_n} - v_{\varepsilon_n}$ solves

$$\begin{aligned}&\langle \ddot X^n(t),\psi\rangle_0 + \langle \mathbb{B}(t)\widehat\nabla X^n(t),\widehat\nabla\psi\rangle + \langle \mathbf{a}(t)\widehat\nabla X^n(t),\psi\rangle - 2\langle \widehat\nabla \dot X^n(t)b(t),\psi\rangle\\&+\varepsilon_n\langle \dot X^n(t),\psi\rangle + \varepsilon_n\langle \widehat\nabla\dot X^n(t)\widehat\nabla\psi\rangle = \langle q^n(t),\psi\rangle_0\end{aligned} \tag{4.44}$$

for a.e. $t\in(0,T)$ and every $\psi\in H^1_D(\Omega_0;\mathbb{R}^d)$, with initial data $X^{0,n} = v^{0,n} - v^0$ and $X^{1,n} = v^{1,n} - v^1$. In particular the right–hand side of (4.44), which is defined as

$$\begin{aligned}\langle q^n,\psi\rangle_0 :&= \langle (g^n - g) - (\mathbf{a}^n - \mathbf{a})\widehat\nabla v^n_{\varepsilon_n} - 2(\operatorname{div} b^n - \operatorname{div} b)\dot v^n_{\varepsilon_n},\psi\rangle\\&- \langle (\mathbb{B}^n - \mathbb{B})\widehat\nabla v^n_{\varepsilon_n} + 2\dot v^n_{\varepsilon_n}\otimes(b^n - b),\widehat\nabla\psi\rangle + \langle F^n - F,\psi\rangle_{\partial_N\Omega},\end{aligned} \tag{4.45}$$

is an element of $L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$. Observe that, to derive (4.44), we have used formula (2.53) both for b^n and b. By combining the energy equality (3.33) with (2.2), the uniform ellipticity condition (3.1) for $\mathbb{B}^n$, the uniform bounds (4.26) and (4.27), and the convergences (4.19) and (4.34)–(4.36), we conclude that $\dot v^n_{\varepsilon_n}$ and $v^n_{\varepsilon_n}$ are uniformly bounded in $L^\infty((0,T);L^2(\Omega_0;\mathbb{R}^d))$ and $L^\infty((0,T);H^1_D(\Omega_0;\mathbb{R}^d))$, respectively. Moreover, these bounds do not depend on the sequence ε_n. Then, thanks to (2.3) and (4.26)–(4.34) we get that

$$q^n \to 0 \quad \text{strongly in } L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d)) \quad \text{as } n\to\infty, \tag{4.46}$$

and the rate of this convergence is independent of the choice of ε_n. Note that, to pass to the limit in the second term in the right–hand side of (4.45), we have used the strong convergences (4.32) and (4.33).

Since $X^n \in H^1((0,T);H^1_D(\Omega_0;\mathbb{R}^d))$, we can use $\dot X^n$ as test function in (4.44) and integrating by parts we obtain

$$\begin{aligned}&\mathcal{E}_{\mathbb{B}}(X^n,t) + \varepsilon_n\int_0^t \|\dot X^n(s)\|^2_{H^1_D(\Omega_0)}ds = \mathcal{E}_{\mathbb{B}}(X^n,0)\\&+\int_0^t\left[\frac12\langle \dot{\mathbb{B}}\widehat\nabla X^n,\widehat\nabla X^n\rangle - \langle \mathbf{a}\widehat\nabla X^n,\dot X^n\rangle - \langle \operatorname{div} b,|\dot X^n|^2\rangle_{L^1} + \langle q^n,\dot X^n\rangle_0\right]ds.\end{aligned}$$

As in the previous step, the uniform bounds on $\dot{\mathbb{B}}$, $\mathbf{a}$, and $\operatorname{div} b$, the ellipticity condition (3.1), and the estimate (3.15) for X^n imply that

$$\begin{aligned}&\frac12\|\dot X^n(t)\|^2_{L^2(\Omega_0)} + \frac{\gamma}{2}\|X^n(t)\|^2_{H^1_D(\Omega_0)} + \varepsilon_n\int_0^t\|\dot X^n(s)\|^2_{H^1_D(\Omega_0)}ds\\&\le \frac12\|\dot X^n(0)\|^2_{L^2(\Omega_0)} + \left(\beta+\frac12\right)\|X^n(0)\|^2_{H^1_D(\Omega_0)} + \int_0^t|\langle q^n(s),X^n(s)\rangle_0|ds,\\&+C\int_0^t[\|\dot X^n(s)\|^2_{L^2(\Omega_0)} + \|X^n(s)\|^2_{H^1_D(\Omega_0)}]ds\end{aligned}$$

for a suitable constant $C>0$ independent of n and t. Since

$$\int_0^t|\langle q^n,X^n\rangle_0|ds \le \frac12\|q^n\|_{L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))} + \frac12\|q^n\|_{L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))}\int_0^t\|\dot X^n(s)\|^2_{H^1_D(\Omega_0)}ds,$$

by choosing ε_n such that

$$\varepsilon_n - \frac12\|q^n\|_{L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))} \ge 0 \quad \text{for all } n, \quad \varepsilon_n\to0,$$

we obtain the following estimate

$$\|\dot X^n(t)\|^2_{L^2(\Omega_0)} + \gamma\|X^n(t)\|^2_{H^1_D(\Omega_0)} \le C_n + 2C\int_0^t[\|\dot X^n(s)\|^2_{L^2(\Omega_0)} + \|X^n(s)\|^2_{H^1_D(\Omega_0)}]ds,$$

with

$$C_n := \|\dot X^n(0)\|^2_{L^2(\Omega_0)} + (2\beta+1)\|X^n(0)\|^2_{H^1_D(\Omega_0)} + \|q^n\|_{L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))}.$$

The convergences (4.35), (4.36), and (4.46) imply that the sequence $C_n\to0$ as $n\to+\infty$. Therefore, by Fatou and Gronwall's Lemma it is easy to show that $\lim_{n\to\infty}[\|\dot X^n(t)\|^2_{L^2(\Omega_0)} + \gamma\|X^n(t)\|^2_{H^1_D(\Omega_0)}] = 0$ for every $t\in[0,T]$. This proves (4.39) and (4.40).

Step 3. Weak convergence of v^n to v. For every $n\in\mathbb{N}$ the function v^n satisfies

$$\begin{aligned}&\langle\ddot v^n(t),\psi\rangle_0+\langle\mathbb{B}^n(t)\widehat\nabla v^n(t),\widehat\nabla\psi\rangle+\langle\mathbf{a}^n(t)\widehat\nabla v^n(t),\psi\rangle+2\langle\dot v^n(t),\widehat{\operatorname{div}}\,[\psi\otimes b^n(t)]\rangle\\&=\langle g^n(t),\psi\rangle+\langle F^n(t),\psi\rangle_{\partial_N\Omega}\end{aligned}\tag{4.47}$$

for a.e. $t\in(0,T)$ and for every $\psi\in H^1_D(\Omega_0;\mathbb{R}^d)$. As shown in (3.57), there exists $C>0$ such that for every $t\in[0,T]$

$$\|\dot v^n(t)\|^2_{L^2(\Omega_0)}+\|v^n(t)\|^2_{H^1_D(\Omega_0)}\le C.\tag{4.48}$$

In particular the constant C can be chosen independent of $n\in\mathbb{N}$, thanks to (2.2), the uniform ellipticity condition (3.1) for $\mathbb{B}^n$, the bounds (4.26) and (4.27), and the convergences (4.19) and (4.34)–(4.36). Moreover, using (2.47) we infer that also $\ddot v^n$ is uniformly bounded in $L^2((0,T);H^{-1}_D(\Omega;\mathbb{R}^d))$. Hence there exists

$$\xi\in L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d))\cap H^1((0,T);L^2(\Omega_0;\mathbb{R}^d))\cap H^2((0,T);H^{-1}_D(\Omega;\mathbb{R}^d))$$

such that, up to a subsequence,

$$v^n\rightharpoonup\xi\quad\text{weakly in }L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d)),\tag{4.49}$$

$$\dot v^n\rightharpoonup\dot\xi\quad\text{weakly in }L^2((0,T);L^2(\Omega_0;\mathbb{R}^d)),\tag{4.50}$$

$$\ddot v^n\rightharpoonup\ddot\xi\quad\text{weakly in }L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d)).\tag{4.51}$$

By combining the strong convergences (4.19) and (4.28)–(4.34) with the weak convergences (4.49)–(4.51), we can pass to the limit as $n\to+\infty$ in (4.47) and we obtain that ξ is a generalized solution of the limit problem (3.9), with initial conditions v^0 and v^1. Thanks to Theorem 3.7 such solution is unique, therefore $\xi=v$. Therefore, since the limit does not dependent on the subsequence, the whole sequence v^n satisfies

$$\begin{aligned}v^n\rightharpoonup v&\quad\text{weakly in }L^2((0,T);H^1_D(\Omega_0;\mathbb{R}^d)),\\\dot v^n\rightharpoonup\dot v&\quad\text{weakly in }L^2((0,T);L^2(\Omega_0;\mathbb{R}^d)).\end{aligned}$$

Furthermore, the bounds (4.48) imply that for every $t\in[0,T]$

$$\begin{aligned}v^n(t)\rightharpoonup v(t)&\quad\text{weakly in }H^1_D(\Omega_0;\mathbb{R}^d),\\\dot v^n(t)\rightharpoonup\dot v(t)&\quad\text{weakly in }L^2(\Omega_0;\mathbb{R}^d).\end{aligned}$$

Step 4. Strong convergence of $v^n_{\varepsilon_n}-v^n$. Let $X^n:=v^n_{\varepsilon_n}-v^n$, where ε_n are the parameters chosen in Step 2. Following the same procedure adopted in Step 1, we get

$$\begin{aligned}&\mathcal{E}_{\mathbb{B}^n}(X^n,t)+\varepsilon_n\int_0^t\|\dot v_{\varepsilon_n}(s)\|^2_{H^1_D(\Omega_0)}ds\\&=\int_0^t\left[\frac12\langle\dot{\mathbb{B}}^n\widehat\nabla X^n,\widehat\nabla X^n\rangle-\langle\mathbf{a}^n\widehat\nabla X^n,\dot X^n\rangle-\langle\operatorname{div}b^n,|\dot X^n|^2\rangle_{L^1}\right]ds+R_n(t),\end{aligned}$$

with

$$\begin{aligned}R_n(t):=&-\langle\dot v^n_{\varepsilon_n}(t),\dot v^n(t)\rangle-\langle\mathbb{B}^n(t)\widehat\nabla v^n_{\varepsilon_n}(t),\widehat\nabla v^n(t)\rangle+\|v^{1,n}\|^2_{L^2(\Omega_0)}+\langle\mathbb{B}^n(0)\widehat\nabla v^{0,n},\widehat\nabla v^{0,n}\rangle\\&+\int_0^t\left[\langle\dot{\mathbb{B}}^n\widehat\nabla v^n_{\varepsilon_n},\widehat\nabla v^n\rangle-\langle\mathbf{a}^n\widehat\nabla v^n_{\varepsilon_n},\dot v^n\rangle-\langle\mathbf{a}^n\widehat\nabla v^n,\dot v^n_{\varepsilon_n}\rangle-2\langle\operatorname{div}b^n,\dot v^n_{\varepsilon_n}\cdot\dot v^n\rangle_{L^1}\right]ds\\&+\int_0^t\left[\langle g^n,\dot v^n_{\varepsilon_n}+\dot v^n\rangle-\langle\dot F^n,v^n_{\varepsilon_n}+v^n\rangle_{\partial_N\Omega}\right]ds+\langle F^n(t),v^n_{\varepsilon_n}(t)+v^n(t)\rangle_{\partial_N\Omega}-2\langle F^n(0),v^{0,n}\rangle_{\partial_N\Omega}.\end{aligned}\tag{4.52}$$

By using the bounds (4.26) and (4.27), the uniform ellipticity condition (3.1) for $\mathbb{B}^n$, and the estimate (4.43) for X^n, we infer that

$$\|\dot X^n(t)\|^2_{L^2(\Omega_0)}+\gamma\|X^n(t)\|^2_{H^1_D(\Omega_0)}\le2R_n(t)+C\int_0^t[\|\dot X^n(s)\|^2_{L^2(\Omega_0)}+\|X^n(s)\|^2_{H^1_D(\Omega_0)}]ds\tag{4.53}$$

for some $C>0$ independent of n and t.

Let us show that $R_n(t)\to0$ as $n\to+\infty$. In view of Step 1 and 2, $v^n_{\varepsilon_n}(t)$ converges strongly to $v(t)$ for every $t\in[0,T]$, while, by Step 3, $v^n(t)$ converges weakly to $v(t)$. Thus, using also (4.19) we have that as $n\to+\infty$

$$\langle F^n(t),v^n_{\varepsilon_n}(t)+v^n(t)\rangle_{\partial_N\Omega}-2\langle F^n(0),v^{0,n}\rangle_{\partial_N\Omega}\to2\langle F(t),v(t)\rangle_{\partial_N\Omega}-2\langle F(0),v^0\rangle_{\partial_N\Omega}.\tag{4.54}$$

Moreover, the dominated convergence theorem in the time variable gives that

$$\int_0^t \langle \dot{F}^n, v^n_{\varepsilon_n} + v^n \rangle_{\partial_N\Omega} ds \to 2\int_0^t \langle \dot{F}, v\rangle_{\partial_N\Omega} ds. \tag{4.55}$$

Now, as explained in Remark 4.2, the convergence (4.13) holds for every $t \in [0,T]$. Then, arguing as in [3, Theorem 3.1] we get

$$\|v^{1,n}\|^2_{L^2(\Omega_0)} + \langle \mathbb{B}^n(0)\widehat{\nabla} v^{0,n}, \widehat{\nabla} v^{0,n}\rangle \to \|v^1\|^2_{L^2(\Omega_0)} + \langle \mathbb{B}(0)\widehat{\nabla} v^0, \widehat{\nabla} v^0\rangle, \tag{4.56}$$

$$\langle \dot{v}^n_{\varepsilon_n}(t), \dot{v}^n(t)\rangle + \langle \mathbb{B}^n(t)\widehat{\nabla} v^n_{\varepsilon_n}(t), \widehat{\nabla} v^n(t)\rangle \to \|\dot{v}(t)\|^2_{L^2(\Omega_0)} + \langle \mathbb{B}(t)\widehat{\nabla} v(t), \widehat{\nabla} v(t)\rangle. \tag{4.57}$$

Finally, thanks to (4.26)–(4.34) it is easy to check that

$$\begin{aligned}&\int_0^t \big[\langle \dot{\mathbb{B}}^n\widehat{\nabla} v^n_{\varepsilon_n}, \widehat{\nabla} v^n\rangle - \langle \mathbf{a}^n\widehat{\nabla} v^n_{\varepsilon_n}, \dot{v}^n\rangle - \langle \mathbf{a}^n\widehat{\nabla} v^n, \dot{v}^n_{\varepsilon_n}\rangle - 2\langle \operatorname{div} b^n, \dot{v}^n_{\varepsilon_n}\cdot \dot{v}^n\rangle_{L^1} + \langle g^n, \dot{v}^n_{\varepsilon_n} + \dot{v}^n\rangle\big] ds\\ &\to \int_0^t \big[\langle \dot{\mathbb{B}}\widehat{\nabla} v, \widehat{\nabla} v\rangle - 2\langle \mathbf{a}\widehat{\nabla} v, \dot{v}\rangle - 2\langle \operatorname{div} b, |\dot{v}|^2\rangle_{L^1} + 2\langle g, \dot{v}\rangle\big] ds.\end{aligned} \tag{4.58}$$

In view of (4.54)–(4.58), the energy equality (3.62) implies that $R_n(t) \to 0$ as $n \to +\infty$ for every $t \in [0,T]$. Hence, by Fatou and Gronwall's Lemmas we obtain that $\lim_{n\to\infty}[\|\dot{X}^n(t)\|^2_{L^2(\Omega_0)} + \gamma\|X^n(t)\|^2_{H^1_D(\Omega_0)}] = 0$. This gives the strong convergences (4.41) and (4.42), and concludes the proof of the claims and of the theorem. □

5. Appendix

For the benefit of the reader, we recall an existence result for evolution problems of second order in time, whose proof can be found in [4]. Let $\mathcal{B}(t;\cdot,\cdot)$, $\mathcal{A}_1(t;\cdot,\cdot)$, $\mathcal{A}_2(t;\cdot,\cdot)$ be three families of continuous bilinear forms over $H^1_D(\Omega_0;\mathbb{R}^d) \times H^1_D(\Omega_0;\mathbb{R}^d)$, with t varying in $[0,T]$, satisfying the following properties, where $\dot{\mathcal{B}}(\cdot;\eta,\xi)$ denotes the derivative of $\mathcal{B}(\cdot;\eta,\xi)$:

(i) for every $t \in [0,T]$ the form $\mathcal{B}(t;\cdot,\cdot)$ is symmetric;
(ii) there exist $c_0 > 0$, $c_1 \in \mathbb{R}$ such that $\mathcal{B}(t;\eta,\eta) \geq c_0\|\eta\|^2_{H^1_D(\Omega_0)} - c_1\|\eta\|^2_{L^2(\Omega_0)}$ for every $t \in [0,T]$, for every $\eta \in H^1_D(\Omega_0;\mathbb{R}^d)$;
(iii) for every $\eta,\xi \in H^1_D(\Omega_0;\mathbb{R}^d)$ the function $t \mapsto \mathcal{B}(t;\eta,\xi)$ is continuously differentiable in $[0,T]$;
(iv) there exists $c_2 > 0$ such that $|\dot{\mathcal{B}}(t;\eta,\xi)| \leq c_2\|\eta\|_{H^1_D(\Omega_0)}\|\xi\|_{H^1_D(\Omega_0)}$ for every $t \in [0,T]$, for every $\eta,\xi \in H^1_D(\Omega_0;\mathbb{R}^d)$;
(v) for every $\eta,\xi \in H^1_D(\Omega_0;\mathbb{R}^d)$ the function $t \mapsto \mathcal{A}_1(t;\eta,\xi)$ is continuous in $[0,T]$;
(vi) there exists $c_3 > 0$ such that $|\mathcal{A}_1(t;\eta,\xi)| \leq c_3\|\eta\|_{H^1_D(\Omega_0)}\|\xi\|_{L^2(\Omega_0)}$ for every $t \in [0,T]$, for every $\eta,\xi \in H^1_D(\Omega_0;\mathbb{R}^d)$;
(vii) for every $\eta,\xi \in H^1_D(\Omega_0;\mathbb{R}^d)$ the function $t \mapsto \mathcal{A}_2(t;\eta,\xi)$ is continuous in $[0,T]$;
(viii) there exists $c_4 > 0$ such that $|\mathcal{A}_2(t;\eta,\xi)| \leq c_4\|\eta\|_{H^1_D(\Omega_0)}\|\xi\|_{L^2(\Omega_0)}$ for every $t \in [0,T]$, for every $\eta,\xi \in H^1_D(\Omega_0;\mathbb{R}^d)$.

Theorem 5.1. *Let $k > 0$, $g \in L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$, $v^0 \in H^1_D(\Omega_0;\mathbb{R}^d)$, $v^1 \in L^2(\Omega_0;\mathbb{R}^d)$, and $\mathcal{B}(t;\cdot,\cdot)$, $\mathcal{A}_1(t;\cdot,\cdot)$, $\mathcal{A}_2(t;\cdot,\cdot)$, $t \in [0,T]$, be three families of continuous bilinear forms over $H^1_D(\Omega_0;\mathbb{R}^d) \times H^1_D(\Omega_0;\mathbb{R}^d)$ satisfying the assumptions (i)–(viii) above. Then there exists $v \in H^1((0,T);H^1_D(\Omega_0;\mathbb{R}^d))$ with $\ddot{v}$ which belongs to $L^2((0,T);H^{-1}_D(\Omega_0;\mathbb{R}^d))$ such that, for a.e. $t \in (0,T)$*

$$\langle \ddot{v}(t), \psi\rangle_0 + \mathcal{B}(t;v(t),\psi) + \mathcal{A}_1(t;v(t),\psi) + \mathcal{A}_2(t;\dot{v}(t),\psi) + k\langle \dot{v}(t),\psi\rangle + k\langle \widehat{\nabla}\dot{v}(t), \widehat{\nabla}\psi\rangle = \langle g(t),\psi\rangle_0, \tag{5.1}$$

for every $\psi \in H^1_D(\Omega_0;\mathbb{R}^d)$, with initial conditions $v(0) = v^0$ and $\dot{v}(0) = v^1$.

Proof. See [4, Chapitre XVIII §5, Théorème 1 and Remarque 4]. □

In the following lemmas we investigate some regularity properties of functions defined in $\Omega \setminus \Gamma$, when composed with suitable diffeomorphisms of the domain into itself. Let us specify the class of diffeomorphisms under study.

Definition 5.2. We say that $\Lambda : [0,T] \times \overline{\Omega} \to \mathbb{R}^d$ is admissible if it belongs to $C^1([0,T] \times \overline{\Omega}; \mathbb{R}^d)$ and, for every $t \in [0,T]$, the function $\Lambda(t,\cdot)$ is a C^2 diffeomorphism of $\overline{\Omega}$ in itself such that $\Lambda(t,\Omega) = \Omega$ and $\Lambda(t, \Gamma \cap \Omega) = \Gamma \cap \Omega$.

Note that, according to (H7)–(H9), both Φ and Ψ are admissible in Definition 5.2.

Lemma 5.3. *Let Λ be admissible. There exists a constant $C > 0$ such that*

$$\|f(\Lambda(t,\cdot)) - f(\Lambda(s,\cdot))\|_{L^2(\Omega)} \leq C\|\widehat{\nabla} f\|_{L^2(\Omega)}|t-s|,$$

for every $f \in H^1(\Omega \setminus \Gamma)$ and $0 \leq t \leq s \leq T$.

Proof. It is sufficient to repeat the proof of Lemmas 4.5 of [3], by approximating $f \in H^1(\Omega\setminus\Gamma)$ with functions $f_\varepsilon \in C^\infty(\Omega \setminus \Gamma) \cap H^1(\Omega \setminus \Gamma)$ given by Meyers–Serrin's Theorem (see, e.g, [1, Theorem 3.16]), and integrating over $\Omega \setminus \Gamma$. □

Lemma 5.4. *Let Λ be admissible and let $t \in [0,T]$. Then for every function $f \in H^1(\Omega \setminus \Gamma)$ as $h \to 0$*

$$\frac{1}{h}[f(\Lambda(t+h,\cdot)) - f(t,\Lambda(t,\cdot))] \to \widehat{\nabla} f(\Lambda(t,\cdot)) \cdot \dot{\Lambda}(t,\cdot) \quad \textit{strongly in } L^2(\Omega).$$

Proof. As before, we repeat the same proof of Lemmas 4.6 of [3], by approximating f with functions $f_\varepsilon \in C^\infty(\Omega \setminus \Gamma) \cap H^1(\Omega \setminus \Gamma)$ given by Meyers–Serrin's Theorem, and integrating over $\Omega \setminus \Gamma$. In particular, we claim that as $h \to 0$

$$T_h(f_\varepsilon) := \frac{1}{h}\int_0^h \widehat{\nabla} f_\varepsilon(\Lambda(t+\tau,\cdot)) \cdot \dot{\Lambda}(t+\tau,\cdot)d\tau \to L(f_\varepsilon) := \widehat{\nabla} f_\varepsilon(\Lambda(t,\cdot)) \cdot \dot{\Lambda}(t,\cdot) \quad \text{strongly in } L^2(\Omega).$$

Indeed, $\Lambda : [0,T] \times \overline{\Omega} \to \mathbb{R}^d$ is uniformly continuous, then for every $\delta > 0$ there exists $\bar{\rho} > 0$ such that

$$|\Lambda(t+\tau,y) - \Lambda(t,y)| < \delta \quad \text{for every } |\tau| < \bar{\rho} \text{ and } y \in \overline{\Omega}. \tag{5.2}$$

Similarly, fixed $t \in [0,T]$, we have that $\Lambda^{-1}(t,\cdot) : \overline{\Omega} \to \mathbb{R}^d$ is uniformly continuous. Hence for every $\eta > 0$ there is $\overline{\delta} > 0$ such that

$$|\Lambda^{-1}(t,y) - \Lambda^{-1}(t,z)| < \eta \quad \text{for every } y,z \in \overline{\Omega}, \text{ with } |y-z| < \overline{\delta}. \tag{5.3}$$

Combining (5.2) and (5.3), we get that for every $\eta > 0$ there exists $\bar{\rho} > 0$ such that

$$\Lambda(t+\tau, A) \subset \Lambda(t, I_\eta(A)) \quad \text{for every } A \subset \Omega \text{ and } |\tau| < \bar{\rho}, \tag{5.4}$$

where $I_\eta(A) := \{x \in \Omega : \mathrm{dist}\,(x,A) < \eta\}$. Define

$$K_n := \{x \in \Omega \setminus \Gamma : \mathrm{dist}\,(x, \partial(\Omega \setminus \Gamma)) \geq 1/n\}.$$

The sets K_n are compacts, $K_n \subset K_{n+1}$ and $\cup_n K_n = \Omega \setminus \Gamma$. Furthermore, fixed $n \in \mathbb{N}$ we can find $\eta > 0$ such that $I_\eta(K_n) \subset\subset \Omega \setminus \Gamma$, which implies that $\Lambda(t, I_\eta(K_n)) \subset\subset \Omega \setminus \Gamma$. Hence there exist $\bar{\rho} > 0$ such that if $|h| < \bar{\rho}$ and $y \in K_n$

$$|T_h(f_\varepsilon)(y)| \leq \frac{1}{h}\int_0^h |\nabla f_\varepsilon(\Lambda(t+\tau,y)) \cdot \dot{\Lambda}(t+\tau,y)|d\tau \leq C,$$

for some $C > 0$ independent of h. Therefore, $\|T_h(f_\varepsilon) - L(f_\varepsilon)\|_{L^2(K_n)} \to 0$ as $h \to 0$ by the dominated convergence theorem, since $T_h(f_\varepsilon)(y) \to L(f_\varepsilon)(y)$ for every $y \in \Omega \setminus \Gamma$. Similarly, there exists $\eta > 0$ such that $I_\eta(\Omega \setminus K_{n+1}) \subset \Omega \setminus K_n$, and so we can find $\bar{\rho} > 0$ such that for all $|h| < \bar{\rho}$

$$\|T_h(f_\varepsilon)\|^2_{L^2(\Omega\setminus K_{n+1})} \leq \frac{1}{h}\int_{\Omega\setminus K_{n+1}}\int_0^h |\widehat{\nabla} f_\varepsilon(\Lambda(t+\tau,y)) \cdot \dot{\Lambda}(t+\tau,y)|^2 d\tau dy \leq C\int_{\Omega\setminus K_n} |\widehat{\nabla} f_\varepsilon(\Lambda(t,y))|^2 dy.$$

Thus for every $|h| < \bar{\rho}$ we obtain

$$\begin{aligned}\|T_h(f_\varepsilon) - L(f_\varepsilon)\|_{L^2(\Omega)} &\leq \|T_h(f_\varepsilon) - L(f_\varepsilon)\|_{L^2(K_{n+1})} + \|T_h(f_\varepsilon)\|_{L^2(\Omega\setminus K_{n+1})} + \|L(f_\varepsilon)\|_{L^2(\Omega\setminus K_{n+1})}\\ &\leq \|T_h(f_\varepsilon) - L(f_\varepsilon)\|_{L^2(K_{n+1})} + 2C\|\widehat{\nabla} f_\varepsilon(\Lambda(t,\cdot))\|_{L^2(\Omega\setminus K_n)}.\end{aligned}$$

Consequently $\limsup_{h\to 0} \|T_h(f_\varepsilon) - L(f_\varepsilon)\|_{L^2(\Omega)} \leq 2C\|\widehat{\nabla} f_\varepsilon(\Lambda(t,\cdot))\|_{L^2(\Omega\setminus K_n)}$ for all $n \in \mathbb{N}$. We conclude by observing that $|\Omega \setminus K_n| \to 0$ as $n \to +\infty$, and that $\widehat{\nabla} f_\varepsilon(\Lambda(t,\cdot)) \in L^2(\Omega; \mathbb{R}^d)$. □

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