COMPACTNESS BY MAXIMALITY

SANDRO ZAGATTI
SISSA, VIA BONOMEA 265 – 34136 TRIESTE (ITALY)

ABSTRACT. We derive a compactness property in the Sobolev space $W^{1,1}(\Omega)$ in order to study the Dirichlet problem for the eikonal equation

$$\begin{cases} \frac{1}{2}|\nabla u(x)|^2 - a(x) = 0 & \text{in } \Omega \\ u(x) = \varphi(x) & \text{on } \partial\Omega, \end{cases}$$

without continuity assumptions on the map a.

AMS 2010 Classification: 35F21, 49L25, 46B50, 35F30

1. Introduction

The possibility of extracting strong convergence out from weak convergence in Sobolev spaces arises as a crucial device in many applications of functional analysis to Partial Differential Equations and to Calculus of Variations. A typical example comes form the study of the minimum problem for classical functional of the form

$$\mathcal{F}(u) = \int_\Omega f(x, u, Du)\, dx,$$

where Ω is an open subset of $\mathbb{R}^d$ and $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times d}$ is a Caratheodory function satisfying a growth condition of the type $f(x, u, \xi) \geq c|\xi|^r$ ($c > 0$, $r > 1$), with competing functions ranging in the Sobolev space $W^{1,r}(\Omega, \mathbb{R}^m)$ and satisfying prescribed boundary conditions. Any minimizing sequence is relatively compact in the weak topology of $W^{1,r}(\Omega, \mathbb{R}^m)$ and, extracting a subsequence, one obtains a weak limit which turns out to be a minimizer provided $\mathcal{F}$ is sequentially weakly lower semicontinuous, a condition which is equivalent to the quasiconvexity (or convexity, in the scalar case) of the integrand f with respect to the last variable. On the other hand, if we would be able to find a strongly converging minimizing sequence, the existence of a minimizer would be guaranteed by a trivial application of Fatou's lemma without any further assumption on f.

Unfortunately it is very hard to show the existence of strongly converging minimizing sequences and, usually, in order to prove the existence of minimizers of $\mathcal{F}$, the procedure is the following. Let $\overline{f}(x, u, \xi)$ be the quasiconvex envelope (or convex envelope, in the scalar case) of f with respect to the last variable and consider the relaxed functional

$$\overline{\mathcal{F}}(u) = \int_\Omega \overline{f}(x, u, Du)\, dx.$$

Since the set $S_{\overline{\mathcal{F}}}$ of minimizers of $\overline{\mathcal{F}}$ is nonempty, a minimizer of $\mathcal{F}$ is an element $u \in S_{\overline{\mathcal{F}}}$ which satisfies the following equation:

$$\overline{f}(x, u(x), Du(x)) - f(x, u(x), Du(x)) = 0 \quad \text{a.e. } x \in \Omega. \tag{1.1}$$

By this approach we have reduced the problem to the solution of a first order equation of Hamilton-Jacobi type in the restricted set $S_{\overline{\mathcal{F}}}$ of minimizers of $\overline{\mathcal{F}}$.

In papers [9], [10], [11], [12], [13] and [14] we have developed the method of integro-maximality (or, more generally, integro-extremality) in order to treat this kind of problems, creating a link between the study of non semicontinuous variational problems and the theory of viscosity solutions for Hamilton-Jacobi equation. The idea is the following: since the set $S_{\overline{\mathcal{F}}}$ is compact in the strong $L^1(\Omega)$ topology and the map

$$S_{\overline{\mathcal{F}}} \ni u \mapsto \int_\Omega u(x)\, dx$$

is obviously continuous in the same topology, there exist at least one element $\overline{u}$ of $S_{\overline{\mathcal{F}}}$ which maximizes the integral on Ω in the set $S_{\overline{\mathcal{F}}}$, i.e.

$$\int_\Omega \overline{u}(x)\, dx \geq \int_\Omega u(x)\, dx \quad \forall u \in S_{\overline{\mathcal{F}}}.$$

Then, by a contradictory argument which requires the differentiability almost everywhere of $\overline{u}$, the continuity of f and $\overline{f}$ and suitable assumptions on the structure of $\overline{f}$, it is possible to show that $\overline{u}$ solves equation (1.1) almost everywhere (and, in the scalar case, in viscosity sense); hence it minimizes the non semicontinuous functional $\mathcal{F}$.

This fact suggests the use of the same integro-extremality technique for the solution of Hamilton-Jacobi first order equations and actually this has been done in papers [15] and [16], where we devoted ourselves to Dirichlet problems of the kind

$$\begin{cases} H(x,\nabla u)=0 & \text{in} \quad \Omega \\ u=\varphi & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where φ is a given boundary datum and $H:\Omega\times\mathbb{R}^d\to\mathbb{R}$ is a continuous function, convex and satisfying suitable growth conditions with respect to the second variable. The idea is to consider the set of almost everywhere subsolutions

$$S_{H,\varphi}\doteq\left\{v\in\varphi+W^{1,\infty}(\Omega):H(x,\nabla v(x)\leq 0 \text{ a.e } x\in\Omega\right\},$$

showing that the pointwise maximal element $\overline{u}$ of $S_{H,\varphi}$, whose existence is guaranteed also in this case by the compactness of $S_{H,\varphi}$ in the $L^1(\Omega)$ strong topology, is a viscosity solution of the Dirichlet problem (1.2).

The rising of the notion of viscosity in these contexts, in our humble opinion, is more than a curiosity. Indeed viscosity solutions enjoy the property of stability, which may be interpreted, in perspective, as a first step toward the construction of strongly converging sequences, a tool that we claim to be powerful in the minimization of non semicontinuous functionals and in other nonlinear problems.

A further remark concerns the hypotheses on the Hamiltonian H in (1.2). Indeed the viscosity theory requires the continuity of $H=H(x,p)$ also with respect to the variable x, and this, when dealing with equation (1.1), corresponds to the continuity with respect to x of the integrands f and $\overline{f}$, in contrast with the natural assumption that they are Caratheodory functions. On the other hand it should be stressed that the maximal element $\overline{u}$ of $S_{H,\varphi}$ mentioned above exists also when H is assumed to be only measurable in the variable x, and then it is worth to ask if $\overline{u}$ is still a generalized solution of (1.2) even in this case, in which the notion of viscosity solution makes no sense. A possible way to face the problem could consist in the construction of approximate solutions and in a limit procedure and this would still require strong convergence.

This brief discussion shows how it may be useful to find conditions which ensure that a weakly converging sequence in a Sobolev space is actually strongly converging, or, with different words, properties which ensure the strong relative compactness of suitable bounded subsets of Sobolev spaces.

Now we list three simple examples in which the (integral) maximality appears as a possible tool in this direction.

Example 1: Integral maximality and uniform convexity.

Let $\Omega\subset\mathbb{R}^d$ be open and bounded, $1<r<+\infty$, and consider the ball $B\doteq\left\{u\in W^{1,r}(\Omega):\|u\|_{W^{1,r}(\Omega)}\leq 1\right\}$. The set B is clearly compact in the $L^1(\Omega)$ strong topology and then there exists an element $\overline{u}\in B$ such that

$$\int_\Omega \overline{u}(x)\,dx\geq\int_\Omega u(x)\,dx \quad \forall u\in B. \tag{1.3}$$

It is easy to see that $\|\overline{u}\|_{W^{1,r}(\Omega)}=1$. Indeed, suppose by contradiction that $\|\overline{u}\|_{W^{1,r}(\Omega)}<1$ and take any nonnegative nonzero test function $\phi\in C^1_c(\Omega)$. Clearly there exists a sufficiently small positive t such that $\|\overline{u}+t\phi\|_{W^{1,r}(\Omega)}\leq\|\overline{u}\|_{W^{1,r}(\Omega)}+$

$t\|\phi\|_{W^{1,r}(\Omega)} \leq 1$; hence $\overline{u}+t\phi \in B$ and this contradicts (1.3). Now let any sequence (u_k) in B such that $u_k \rightharpoonup \overline{u}$ in $W^{1,r}(\Omega)$. The uniform convexity of $W^{1,r}(\Omega)$ and the inequality $\|\overline{u}\|_{W^{1,r}(\Omega)} \geq \|u_k\|_{W^{1,r}(\Omega)}$ for every $k \in \mathbb{N}$ imply that actually (u_k) converges strongly to u in $W^{1,r}(\Omega)$.

Example 2: Maximality and extremality.

Let Ω be an open, bounded subset of $\mathbb{R}^d$ and let $K : \Omega \to \mathcal{K}(\mathbb{R}^d)$ be a continuous multifunction with values in the set $\mathcal{K}(\mathbb{R}^d)$ of compact, convex subsets of $\mathbb{R}^d$. Assume in addition that K is uniformly bounded and that, for every $x \in \Omega$, the boundary $\partial K(x)$ of $K(x)$ coincides with the set $\text{extr}(K(x))$ of its extreme points. Consider the differential inclusion:

$$\nabla u(x) \in \partial K(x) \text{ a.e. } x \in \Omega,$$

assuming that the "relaxed" set

$$S_K \doteq \left\{u \in W^{1,\infty}(\Omega) : \nabla u(x) \in K(x) \text{ a.e. } x \in \Omega\right\}$$

is nonempty. It is easy to see that also in this case S_K is compact in the $L^1(\Omega)$ strong topology and then there exists an element $\overline{u} \in S_K$ such that

$$\int_\Omega \overline{u}(x)\,dx \geq \int_\Omega u(x)\,dx \quad \forall u \in S_K.$$

In addition $\overline{u}$ turns out to be also pointwise maximal, in the sense that $\overline{u}(x) \geq u(x)$ for every $x \in \Omega$ and for every $u \in S_K$ (in this paper we assume to identify a Sobolev function with its precise representative which in this case is Lipschitz continuous). Actually we have $\nabla\overline{u}(x) \in \partial K(x)$ for every $x \in \Omega$ at which $\overline{u}$ is differentiable, i.e. for almost every $x \in \Omega$. Indeed, suppose by contradition that there exists a point $x_0 \in \Omega$ of differentiability of $\overline{u}$ for which $\nabla\overline{u}(x_0) \in \text{int}(K(x_0))$. By an argument analogous to the one used, for example, in the proof of Lemma 1 in [12] or [15] we may construct a map $\hat{u} \in S_K$ such that $\hat{u} \geq \overline{u}$ on Ω and $\hat{u} > \overline{u}$ on a small ball centered at x_0. This violates the maximality of $\overline{u}$ and the contradiction proves the claim. Now consider any sequence (u_k) in S_K such that $u_k \overset{*}{\rightharpoonup} \overline{u}$ in $W^{1,\infty}(\Omega)$. It follows from the above argument that

$$\nabla\overline{u}(x) \in \text{extr}\left(\overline{\text{co}}\left\{\nabla u_k(x), k \in \mathbb{N}\right\}\right).$$

Applying the classical Tartar-Visintin theorem (see [8]), we obtain that, actually, $\nabla u_k(x) \overset{n\to\infty}{\longrightarrow} \nabla u(x)$ for almost every $x \in \Omega$; then the convergence of the sequence is strong in $W^{1,r}(\Omega)$ for every $r \in [1, \infty[$.

Example 1: Maximality and semiconcavity.

Let Ω be a bounded, open, simply connected subset of $\mathbb{R}^d$ and let (α_n) be a bounded sequence in $C(\overline{\Omega})$ of nonnegative Lipschitz continuous functions with Lipschitz constant independent on n. For every $n \in \mathbb{N}$ consider the Dirichlet problems for the eikonal equation:

$$\begin{cases} |\nabla u(x)| - \alpha_n(x) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where the boundary datum $\varphi \in W^{1,\infty}(\Omega)$ is an almost everywhere subsolution for every $n \in \mathbb{N}$. By classical and well known results (see for example [7], [4], [3] and [15]) for every $n \in \mathbb{N}$ there exists a unique viscosity solution u_n of the Dirichlet problem which is maximal in the set of subsolutions, i.e. $u_n(x) \geq v(x)$ for every $x \in \Omega$ and for every $v \in \varphi + W_0^{1,\infty}(\Omega)$ such that $|\nabla v| \leq \alpha_n$ almost everywhere in Ω.

Passing if necessary to a subsequence, we may assume that (u_n) converges weakly$*$ in $W^{1,\infty}(\Omega)$, and then uniformly on $\overline{\Omega}$, to some limit $u \in \varphi + W_0^{1,\infty}(\Omega)$. By theorem 5.3.7 (p. 116) in [4] we have that the functions u_n and u are semiconcave with modulus $\omega(\cdot)$ independent on n and then, applying theorem 3.3.3 (p. 57) in [4], and passing if necessary to a further subsequence, we deduce that $\nabla u_n(x) \overset{n\to\infty}{\longrightarrow} \nabla u(x)$ for almost every $x \in \Omega$. Then, also in this case, we conclude that the convergence of the sequence is strong in $W^{1,r}(\Omega)$ for every $r \in [1, \infty[$.

This last example give us a hint to face a special case of problem (1.2): consider an open bounded subset Ω of $\mathbb{R}^d$, a nonnegative function $a \in L^\infty(\Omega)$ and the following Dirichlet problem for the eikonal equation:

$$\mathcal{P}(a, \varphi, \Omega) : \qquad \begin{cases} \frac{1}{2} |\nabla u(x)|^2 - a(x) = 0 & \text{in} \quad \Omega \\ u = \varphi & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

where φ is an almost everywhere subsolution, i.e. satisfy the compatibility condition

$$\frac{1}{2} |\nabla \varphi(x)|^2 \leq a(x) \quad \text{a.e. } x \in \Omega. \tag{1.5}$$

Let $S(a, \varphi, \Omega) \doteq \left\{ v \in \varphi + W_0^{1,\infty}(\Omega) : |\nabla v|^2 \leq 2a \text{ a.e. in } \Omega \right\}$ be the set of a.e. subsolutions and let $\overline{u}$ be the maximal element of $S(a, \varphi, \Omega)$ obtained as in previous discussion. Is it true that $\overline{u}$ solves the equation in $\mathcal{P}(a, \varphi, \Omega)$ almost everywhere? In section 5 we give an answer to this question under suitable assumption on the map a (see hypothesis 1 in section 4): making use of the representation formula for viscosity solutions provided by P.L. Lions in [7] and of several properties of semiconcave functions, collected in the excellent monograph [4], we develop a theory based on the construction of strongly converging sequences by the use of maximality. With respect to the examples listed above, we use maximality as a property of the elements of the sequence, and not of the limit, without any additional condition. By this way maximality appears, as claimed in the discussion developed above, as the searched tool in order to construct strongly converging sequences in Sobolev spaces.

2. Notations and preliminaries

In this paper $\mathbb{R}^d$ is the euclidean d-dimensional space; we denote respectively by $\langle \cdot, \cdot \rangle$ and by $| \cdot |$ the inner product and the euclidean norm in $\mathbb{R}^d$, while $\mathcal{E} \doteq \{e_1, \ldots, e_d\}$ is the canonical basis in $\mathbb{R}^d$ and a point $x \in \mathbb{R}^d$ is written as $x = (x_1, \ldots, x_d)$. Given $E \subseteq \mathbb{R}^d$, $\mathrm{m}_k(E)$ is the k-dimensional Lebesgue measure, ∂E is the boundary, E^c is the complement, and $\mathrm{co}(E)$ is the convex hull of E; by $\mathrm{dist}\,(x, E)$ we mean the distance of the point x from the set E. Given an open bounded subset U of $\mathbb{R}^d$; we use the spaces $C^k(U)$, $L^r(U)$, $W^{1,r}(U)$, $W_0^{1,r}(U)$, for $k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $1 \leq r \leq \infty$, with their usual (strong and weak) topologies. Dealing with a Sobolev function we assume to use the precise representative. We recall from [1] the well known precompactness criterion in spaces L^r that we formulate in a convenient form for our purposes.

Theorem 1. *Let $A \subseteq \mathbb{R}^d$ be open, $r \in [1, \infty[$ and let $\mathcal{K}$ be a bounded subset of $L^\infty(A)$ (whose element are assumed to be extended by zero on A^c) satisfying the following property: for every $\epsilon > 0$ there exist $\delta > 0$ and $G \subset\subset A$ such that*

$$m_d\,(A \setminus G) \leq \epsilon$$

and for every $h \in \mathbb{R}^d$ with $|h| \le \delta$ we have

$$\int_G |v(x+h) - v(x)|\, dx \le \epsilon \quad \forall v \in \mathcal{K}.$$

Then $\mathcal{K}$ is relatively compact in $L^r(A)$.

We shall use many arguments taken from the theories of viscosity solutions for Hamilton-Jacobi equations and of semiconcave functions, for which we refer to the wide existing literature, mentioning for example the monographs [2], [3], [4], [5], [7]. In addition we shall need the results contained in [15] and the representation formula for viscosity solutions provided by P.L. Lions in [7]. To this aim we introduce suitable notations and recall the following results.

Definition 1. *Let $\Lambda \subseteq \mathbb{R}^d$ be open and bounded. Given $T > 0$ and $x, y \in \overline{\Lambda}$ we set*

$$\Xi(x, y, T, \Lambda) \doteq \left\{ \xi \in W^{1,\infty}([0,T], \mathbb{R}^d) : \xi(t) \in \Lambda \text{ a.e. } t \in [0,T],\ \xi(0) = x, \xi(T) = y \right\}.$$

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain and let $H \in C(\overline{\Omega} \times \mathbb{R}^d)$ be a function satisfying the following conditions:*

(i) $\mathbb{R}^d \ni p \mapsto H(x,p)$ is convex for every $x \in \Lambda$;

(ii) there exist constants $\alpha > 0$, $\beta \ge 0$ such that

$$H(x,p) \ge \alpha |p| - \beta \quad \forall x \in \Omega,\ \forall p \in \mathbb{R}^d;$$

(iii) $\inf_{p \in \mathbb{R}^d} H(x,p) \le 0$ for every $x \in \Omega$.

Let $\varphi \in W^{1,\infty}(\Omega)$ satisfy the compatibility condition

$$H(x, \nabla\varphi(x)) \le 0 \text{ a.e. } x \in \Omega$$

and introduce the (nonempty) set

$$S_{H,\varphi} \doteq \left\{ v \in \varphi + W_0^{1,\infty}(\Omega) : H(x, \nabla v(x)) \le 0 \text{ a.e. } x \in \Omega \right\}.$$

Then there exists a unique element $\overline{u} \in S_{H,\varphi}$ such that $\overline{u}(x) \ge v(x)$ for every $x \in \Omega$ and for every $v \in S_{H,\varphi}$ which turns out to be a viscosity solution of the Dirichlet problem

$$\begin{cases} H(x, \nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

In addition, calling H^* the Fenchel transform of H, i.e.

$$H^*(x,q) \doteq \sup_{p \in \mathbb{R}^d} \left\{ -\langle p, q \rangle - H(x,p) \right\},$$

we have the following formula:

(2.1)

$$\overline{u}(x) = \inf \left\{ \int_0^T e^{-t} H^*(\xi(t), \xi'(t)\, dt + \overline{u}(w) : T > 0, w \in \partial\Lambda, \xi \in \Xi(x, w, T, \Lambda) \right\},$$

for every $x \in \Lambda$ and for every $\Lambda \subset\subset \Omega$ open, bounded, simply connected with smooth boundary $\partial\Lambda$.

The proof of the first part is performed in [15] (theorems 1 and 2), while the second is treated in section 5 (p. 115 and ff.) of [7].

Notations. Let $E \subseteq \mathbb{R}^d$ be an open convex set and $v : E \to \mathbb{R}$. Given $j \in \{1, \dots, d\}$ and $x \in \mathbb{R}^d$ we write $x'_j = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) \in \mathbb{R}^{d-1}$; given $x'_j \in \mathbb{R}^{d-1}$

consider the line $r_{x'_j}$ touching the point x'_j and parallel to the direction e_j and call $J_{x'_j}$ the line segment $r_{x'_j} \cap E$; in addition set

$$v_{x'_j}(t) = v(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d),$$

with $t \in J_{x'_j}$ and x'_j ranging in the set $E_j^{d-1} \subset \mathbb{R}^{d-1}$ for which $J_{x'_j} \neq \emptyset$.

We shall need the following consequences of Fubini-Tonelli theorem.

Lemma 1. *Let $E \subseteq \mathbb{R}^d$ be open, bounded and convex and let (v_n) be a sequence in $W^{1,\infty}(E)$ such that*

$$v_n \overset{*}{\rightharpoonup} v \text{ in } W^{1,\infty}(E).$$

Suppose that for every $j \in \{1, \dots, d\}$ and for almost every $x'_j \in E_j^{d-1}$, the sequence (v'_{n,x'_j}) is relatively compact in $L^1(J_{x'_j})$. Then

$$v_n \longrightarrow v \text{ strongly in } W^{1,1}(E).$$

Proof. Assume $E = Q^q \doteq]0,1[^d$ and $v = 0$.

First of all remark that the sequence (v_n) converge uniformly to zero on E and, consequently, for every $j \in \{1, \dots, d\}$ and for every $x'_j \in E_j^{d-1}$, the sequence v_{n,x'_j} converges uniformly to zero on $J_{x'_j}$. Hence, by relative compactness of the sequence of derivatives (v'_{n,x'_j}) in $L^1(J_{x'_j})$, the sequence v_{n,x'_j} converges strongly to zero in $W^{1,1}(J_{x'_j})$ for every $j \in \{1, \dots, d\}$ and for almost every $x'_j \in E_j^{d-1}$. Suppose, by contradiction, that there exist a positive ρ such that

$$\int_{Q^d} |\nabla v_n(x)| \, dx \geq \rho.$$

Necessarily there exist a positive η, an index $j \in \{1, \dots, d\}$ and a subsequence, still denoted by (v_n), such that

$$\int_{Q^d} |D_j v_n(x)| \, dx \geq \eta,$$

that is to say

$$\int_{Q^{d-1}} \int_0^1 |D_j v_n(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)| \, dt \, dx'_j \geq \eta > 0. \tag{2.2}$$

It follows that there exists a subset G of Q^{d-1} of positive measure $m_{d-1}(G) > 0$ and, for almost every $x'_j \in G$, positive numbers $\alpha(x'_j) > 0$ such that

$$\int_0^1 |D_j v_n(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)| \, dt = \int_0^1 |v'_{n,x'_j}(t)| \, dt \geq \alpha(x'_j) > 0. \tag{2.3}$$

Indeed, if not, we would have, for almost every $x'_j \in Q^{d-1}$,

$$\lim_{n \to 0} \int_0^1 |D_j v_n(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)| \, dt = 0,$$

and then, remembering the boundedness of the integrands and by dominated convergence,

$$\lim_{n \to 0} \int_{Q^{d-1}} \int_0^1 |D_j v_n(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_d)| \, dt \, dx'_j = 0,$$

contradicting (2.2).

By hypothesis the sequences (v_{n,x'_j}) converges strongly to zero in $W^{1,1}(0,1)$ for almost every $x'_j \in E_j^{d-1}$ and then inequalities (2.3) give the required contradiction. The general case of an arbitrary open, bounded, convex subset can be easily obtained by extension or invoking Vitali covering lemma.

□

Remark 1. It is obvious that actually the convergence of the sequence is strong in $W^{1,r}(\Omega)$ for every $r \in [1,\infty[$ and, in addition, it is easy to see that the result, by the aid of covering theorems, can be generalized to arbitrary open subsets of $\mathbb{R}^d$.

Lemma 2. *Let $E \subset \mathbb{R}^d$ be open bounded and convex and let (v_n) be a sequence of nonnegative functions $v_n : E \times [0,+\infty[\to \mathbb{R}$, $v_n = v_n(x,h)$ such that*

$$\sup_n \int_E v_n(x,h)\, dx \stackrel{h\to 0+}{\longrightarrow} 0.$$

Then, for every $j \in \{1,\ldots,d\}$ and for almost every $x'_j \in \mathbb{R}^{d-1}$ we have

$$\sup_n \int_{J_{x'_j}} v_{n,x'_j}(t,h)\, dx \stackrel{h\to 0+}{\longrightarrow} 0,$$

where x'_j is such that the integral in (2) makes sense and $J_{x'_j}$ is the line segment contained in E parallel to the direction e_j passing through the point x'_j.

Proof. Assume $E = Q^d =]0,1[^d$. By Fubini Tonelli theorem we write

$$\int_E v_n(x,h)\, dx = \int_{Q^{d-1}} \int_0^1 v_{n,x'_j}(t,h)\, dt\, dx'_j$$

and assume, by contradiction, that there exists $D \subseteq Q^{d-1}$ with $M_{d-1}(D) > 0$ such that the sequence of supremums

$$\sup_n \int_0^1 v_{n,x'_j}(t,h)\, dt \tag{2.4}$$

does not converges to zero for almost every $x'_j \in D$. This means that there exists $n_0 \in \mathbb{N}$ and a nonnegative sequence (h_k) converging to zero such that

$$\int_0^1 v_{n_0,x'_j}(t,h_k)\, dt \geq \alpha(x'_j) > 0 \quad \forall k \in \mathbb{N}, \tag{2.5}$$

where $\alpha(x'_j) > 0$ is a positive constant defined for almost every point $x'_j \in D$. It follows, by Fubini Tonelli theorem, that there exists a positive $\alpha > 0$ such that

$$\int_E v_{n_0}(x,h_k)\, dx \geq \int_D \int_0^1 v_{n_0,x'_j}(t,h_k)\, dt\, dx'_j \geq \alpha > 0 \quad \forall k \in \mathbb{N};$$

a contradiction.

The general case of an arbitrary open, bounded, convex subset can be easily obtained by extension or invoking Vitali covering lemma.

□

3. C^1 A PRIORI ESTIMATES

In this section we consider problem $\mathcal{P}(a, \varphi, \Omega)$, assuming that the given nonnegative function a is sufficiently regular, and derive estimates that will be used in the proof of our main result. We take $a \in C^1(\Omega) \cap L^\infty(\Omega)$ and $\varphi \in W^{1,\infty}(\Omega)$ satisfying the compatibility condition (1.5) and introduce the nonempty set $S(a, \varphi, \Omega)$ of a.e. subsolutions:

Definition 2.

$$S(a, \varphi, \Omega) \doteq \left\{ v \in \varphi + W_0^{1,\infty}(\Omega) : \frac{1}{2} |\nabla v(x)|^2 \leq a(x) \text{ a.e. } x \in \Omega \right\}. \tag{3.1}$$

Applying theorem 2 we infer the existence of a unique map $u \in S(a, \varphi, \Omega)$ such that $u(x) \geq v(x)$ for every $x \in \Omega$ and for every $v \in S(a, \varphi, \Omega)$ which turns out to be a viscosity and a generalized solution of problem $\mathcal{P}(a, \varphi, \Omega)$; in addition formula (2.1) holds on every set $\Lambda \subseteq \Omega$ open, bounded, simply connected with smooth boundary. It is immediate to see that given $A \geq \|a\|_{L^\infty(\Omega)}$ there exist positive constants $K_0 = K_0(A, \varphi, \Omega)$ and $K_1 \doteq (2A)^{\frac{1}{2}}$ such that

$$\|u\|_{L^\infty(\Omega)} \leq K_0 = K_0(A, \varphi, \Omega), \qquad \|\nabla u\|_{L^\infty(\Omega)} \leq K_1 = (2A)^{\frac{1}{2}}. \tag{3.2}$$

We shall need also the following notion.

Remark 2. Let $R > 0$ and $\tau > 0$. Clearly there exists a nonnegative function $g : \Omega \times]0, \tau] \to \mathbb{R}$ such that the following condition holds:
for every $x \in \Omega$, for every $\delta \in]0, \tau]$ and for every $\zeta \in W^{1,\infty}([0, \delta])$ with $|\zeta(t)| \leq R$ and $\zeta(t) \in \Omega$ for every $t \in [0, \delta]$ we have

$$\int_0^\delta |\nabla a(x + \zeta(t)| \, dt \leq g(x, \delta). \tag{3.3}$$

It is evident that the function g could be defined as a supremum on the set of considered paths ζ's and would depend on R and τ. However we will recover this notion in section 4 (see hypothesis 1) with additional assumptions on g and it will be clear at that point the use of this device.

Lemma 3. *Let Ω, a, φ, u, g as above and let $\Lambda \subset\subset \Omega$ be an open, convex subset of Ω with smooth boundary $\partial\Lambda$. Let $A \geq \|a\|_{L^\infty(\Omega)}$, $\gamma > 0$ and*

$$\Lambda_\gamma \doteq \{x \in \Lambda : \mathit{dist}(x, \Lambda^c) > \gamma\}.$$

Then there exist $T_0 = T_0(\gamma, A, \varphi, \Omega) > 0$, $\tilde{h} > 0$ and $\tilde{\delta} > 0$ such that for every $\lambda \in]0, 1[$, for almost every $x \in \Lambda_\gamma$, for every $h \in \mathbb{R}^d$ such that $|h| \leq \tilde{h}$ and $y \doteq x + h \in \Lambda_\gamma$, for every $\delta \in]0, \min\{1, T_0, \tilde{\delta}\}]$, we have

$$\lambda u(x) + (1 - \lambda) u(y) - u(\lambda x + (1 - \lambda) y) \leq \lambda(1 - \lambda) \left[\frac{1}{2} \frac{|h|^2}{\delta} + 2|h| g(x, \delta) \right]. \tag{3.4}$$

Proof.

Step 1. We recall form section 2 that the solution u may be represented according to formula (2.1). In our case we have

$$H(x, p) = \frac{1}{2}|p|^2 - a(x),$$

and by easy computations (see for example chapter 5 in [6]) we find that the Fenchel transform H^* of H is

$$H^*(x,q) = \frac{1}{2}|q|^2 + a(x).$$

Hence the following formula holds true:

$$u(x) = \inf\{J(\xi,T,w); T>0, w\in\partial\Lambda, \xi\in\Xi(x,w,\Lambda,T)\} \quad \forall x\in\Lambda, \tag{3.5}$$

where

$$J(\xi,T,w) \doteq \int_0^T e^{-t}\left[a(\xi(t) + \frac{1}{2}|\xi'(t)|^2\right] dt + u(w). \tag{3.6}$$

Step 2. We prove now the existence of M such that the infimum in formula (3.5)-(3.6) can be taken on the elements $\xi\in\Xi(x,w,T,\Lambda)$ such that

$$\|\xi'\|_{L^2([0,T^*])} \le M, \tag{3.7}$$

where

$$T^* \doteq \min\{1,T\}. \tag{3.8}$$

Clearly, in (3.5), we may limit ourselves to paths $\xi \in \Xi(x,w,T,\Lambda)$ such that $J(\xi,T,w) \le u(x)+1 \le K_0+1$. We have

$$\int_0^T \frac{1}{2}e^{-t}|\xi'(t)|^2\,dt \le J(\xi,T,w) - u(w) \le u(x)+1-u(w),$$

and it follows that

$$\int_0^T \frac{1}{2}e^{-t}|\xi'(t)|^2\,dt \le 2K_0+1.$$

Since $e^{-T^*} \le e^{-t}$ for every $t\in[0,T^*]$, we have

$$e^{-T^*}\int_0^{T^*}\frac{1}{2}|\xi'(t)|^2\,dt \le \int_0^{T^*}\frac{1}{2}e^{-t}|\xi'(t)|^2\,dt \le \int_0^T \frac{1}{2}e^{-t}|\xi'(t)|^2 \le 1+2K_0,$$

and finally

$$\int_0^{T^*}|\xi'(t)|^2\,dt \le 2e^{T^*}(1+2K_0) \le 2e(1+2K_0).$$

Hence we set $M \doteq (2e(1+2K_0))^{\frac{1}{2}}$ and, recalling (3.2), the claim is proved.

Step 3. We prove now the existence of $T_0 = T_0(\gamma, A, \varphi, \Omega)$ as in the statement such that the infimum in formula (3.5)-(3.6) can be taken on the values T such that $T\ge T_0$ (and, correspondingly, on paths $\xi\in\Xi(x,w,T,\Lambda)$) for every $x\in\Lambda_\gamma$.

Let $x\in\Lambda_\gamma$, $T>0$, $w\in\partial\Lambda$ and $\xi\in\Xi(x,w,T,\Lambda)$ be arbitrary; write $I=[0,T]$ and, given $R\ge 0$, set

$$I_\rho \doteq \{t\in I : |\xi'(t)|\ge\rho\}.$$

We have

$$\begin{aligned}\gamma \le |x-w| \le |\xi(0)-\xi(T)| = \left|\int_0^T \xi'(t)\,dt\right| \le \int_0^T |\xi'(t)|\,dt \quad &\le \\ \int_{I_\rho}|\xi'(t)|\,dt + \int_{I\setminus I_\rho}|\xi'(t)|\,dt \quad &\le \\ \int_{I_\rho}|\xi'(t)|\,dt + \rho T.&\end{aligned}$$

Hence

$$\int_{I_{2T^{-\frac{1}{2}}}} |\xi'(t)|\,dt \geq \gamma - \rho T. \tag{3.9}$$

Let us now minorize $J(\xi,T,w)$.

$$\begin{aligned} J(\xi,T,w) = \int_0^T e^{-t}\left[a(\xi(t)) + \frac{1}{2}|\xi'(t)|^2\right] dt + u(w) \quad &\geq \\ -K_0 + \int_0^T \frac{1}{2}e^{-t}|\xi'(t)|^2\,dt \quad &\geq \\ -K_0 + \frac{e^{-T}}{2}\int_0^T |\xi'(t)|^2\,dt \quad &\geq \\ -K_0 + \frac{e^{-T}}{2}\int_{I_\rho} |\xi'(t)|^2\,dt \quad &\geq \\ -K_0 + \frac{e^{-T}}{2}\int_{I_\rho} \rho|\xi'(t)|\,dt \quad &= \\ -K_0 + e^{-T}\frac{\rho}{2}\int_{I_\rho} |\xi'(t)|\,dt \quad &\geq \\ -K_0 + e^{-T}\frac{\rho}{2}\left(\gamma - 2T^{\frac{1}{2}}\right) \quad &= \\ -K_0 - 2e^{-T} + \gamma e^{-T}T^{-\frac{1}{2}}, & \end{aligned} \tag{3.10}$$

where we have used (3.9). Recall again that we may limit ourselves to elements ξ, T and w such that

$$J(\xi,T,w) \leq K_0 + 1; \tag{3.11}$$

then collect (3.10) and (3.11) obtaining the following chain

$$-K_0 - \frac{\rho^2}{2}Te^{-T} + \gamma e^{-T}\frac{\rho}{2} \leq J(\xi,T,w) \leq K_0 + 1.$$

Setting

$$\rho \doteq T^{-\frac{1}{2}}$$

it follows that

$$e^{-T}T^{-\frac{1}{2}} \leq \frac{4K_0 + 6}{\gamma}. \tag{3.12}$$

Since $e^{-T}T^{-\frac{1}{2}} \to +\infty$ as $T \to 0+$, (3.12) implies the existence of the required positive $T_0 = T_0(\gamma, A, \varphi, \Omega)$.

Step 4. Let now M and T_0 be as given by steps 2 and 3; fix $\tilde{\delta} \in\,]0,\tau]$ and $\tilde{h} > 0$ such that

$$M\tilde{\delta}^{\frac{1}{2}} + 2\tilde{h} \leq R. \tag{3.13}$$

Then take $\delta \in\,]0, \min\{1, T_0, \tilde{\delta}\}]$; let $\lambda \in\,]0,1[$, $x \in \Lambda_\gamma$ and $h \in \mathbb{R}^d$ such that $|h| \leq \tilde{h}$, $y \doteq x + h \in \Lambda_\gamma$. Choose arbitrary $w \in \partial\Lambda$, $T \geq T_0$ and $\xi \in \Xi(z, w, T, \Lambda)$, where

$$z = \lambda x + (1-\lambda)y = x + (1-\lambda)h. \tag{3.14}$$

Define the following paths:

$$\xi_1(t) = \begin{cases} \xi(t) - (1-\lambda)\left(\frac{\delta-t}{\delta}\right) h & t \in [0,\delta] \\ \xi(t) & t \in]\delta, T], \end{cases} \tag{3.15}$$

$$\xi_2(t) = \begin{cases} \xi(t) + \lambda\left(\frac{\delta-t}{\delta}\right) h & t \in [0,\delta] \\ \xi(t) & t \in]\delta, T]; \end{cases} \tag{3.16}$$

remarking that

$$\xi_1 \in \Xi(x, w, T, \Lambda), \quad \xi_2 \in \Xi(y, w, T, \Lambda) \tag{3.17}$$

and that

$$\xi(t) = \lambda\xi_1(t) + (1-\lambda)\xi_2(t) \quad t \in I \doteq [0, T]. \tag{3.18}$$

Now we estimate the quantity

$$\mathcal{A} \doteq \lambda J(\xi_1, T, w) + (1-\lambda) J(\xi_2, T, w) - J(\xi, T, w). \tag{3.19}$$

By definitions (3.15) and (3.16), the three considered paths ξ_1, ξ_2 and ξ coincide for values of the parameter t larger than δ. Then we have

$$\begin{aligned} \mathcal{A} = \lambda \int_0^\delta e^{-t}\left[a(\xi_1(t)) + \frac{1}{2}|\xi_1'(t)|^2\right] dt \quad + \\ (1-\lambda)\int_0^\delta e^{-t}\left[a(\xi_2(t)) + \frac{1}{2}|\xi_2'(t)|^2\right] dt \quad - \\ \int_0^\delta e^{-t}\left[a(\xi(t)) + \frac{1}{2}|\xi'(t)|^2\right] dt \quad = \\ \int_0^\delta e^{-t}\left[\lambda a(\xi_1(t)) + (1-\lambda)a(\xi_2(t)) - a(\xi(t))\right] dt \quad + \\ \int_0^\delta \frac{1}{2}e^{-t}\left[\lambda|\xi_1'(t)|^2 + (1-\lambda)|\xi_2'(t)|^2 - |\xi'(t)|^2\right] dt \quad = \\ \int_0^\delta e^{-t}\lambda\left[a(\xi_1(t)) - a(\xi(t))\right] dt \quad + \\ \int_0^\delta e^{-t}(1-\lambda)\left[a(\xi_2(t)) - a(\xi(t))\right] dt \quad + \\ \int_0^\delta \frac{1}{2}e^{-t}\left[\lambda|\xi_1'(t)|^2 + (1-\lambda)|\xi_2'(t)|^2 - |\lambda\xi_1'(t) + (1-\lambda)\xi_2'(t)|^2\right] dt \quad = \\ \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3. \end{aligned} \tag{3.20}$$

Step 5: estimate of $\mathcal{A}_3$.

Observe that for every $q_1, q_2 \in \mathbb{R}^d$ and for every $\lambda \in [0,1]$ we have

$$|\lambda|q_1|^2 + (1-\lambda)|q_2|^2 - |\lambda q_1 + (1-\lambda)q_2|^2 = \lambda(1-\lambda)|q_1 - q_2|^2$$

and that, by direct computations,

$$\xi_1'(t) - \xi_2'(t) = \frac{h}{\delta}, \quad \text{a.e.} \quad t \in I.$$

Recalling (3.18), it follows that

$$\mathcal{A}_3 \le \frac{\lambda(1-\lambda)}{2}\int_0^\delta e^{-t}\frac{|h|^2}{\delta^2}\, dt \le \frac{\lambda(1-\lambda)}{2}\frac{|h|^2}{\delta}. \tag{3.21}$$

Step 6: estimate of $\mathcal{A}_1$ and $\mathcal{A}_2$. Let us consider the term

$$\mathcal{A}_1 = \lambda \int_0^\delta e^{-t}\,[a(\xi_1(t)) - a(\xi(t))]\; dt.$$

Recalling (3.15), by mean value formula, we may write

$$\begin{aligned} a(\xi_1(t)) - a(\xi(t)) = a\left(\xi(t) - (1-\lambda)\left(\frac{\delta - t}{\delta}\right) h\right) - a(\xi(t)) \quad = \\ \left\langle \nabla a \left(\xi(t) - (1-\lambda)\theta(t)\left(\frac{\delta - t}{\delta}\right) h\right), (1-\lambda)\left(\frac{t-\delta}{\delta}\right) h\right\rangle; \end{aligned}$$

where $\theta(t) \in [0,1]$ for every $t \in [0,\delta]$. We set

$$\zeta_1(t) = \xi(t) - x - (1-\lambda)\theta(t)\left(\frac{\delta - t}{\delta}\right) h, \tag{3.22}$$

and then we have

$$\mathcal{A}_1 = \lambda(1-\lambda)\int_0^\delta e^{-t}\left\langle \nabla a(x + \zeta_1(t)), \left(\frac{t-\delta}{\delta}\right) h\right\rangle\; dt. \tag{3.23}$$

Now we estimate the modulus of $\zeta_1(t)$:

$$\begin{aligned} |\zeta_1(t)| \quad &= \quad \left|\xi(t) - z + z - x - (1-\lambda)\theta(t)\left(\frac{\delta - t}{\delta}\right) h\right| \le \\ &\int_0^t |\xi'(s)|\, ds + |z - x| + |h| \le \\ &M\delta^{\frac{1}{2}} + 2|h|, \end{aligned}$$

where we have used the fact that $\xi(0) = z$ and that, by (3.7) and by Hölder inequality,

$$\begin{aligned} |\xi(t) - \xi(0)| = \left|\int_0^t \xi'(s)\, ds\right| \le \int_0^t |\xi'(s)|\, ds \quad \le \\ \left(\int_0^t |\xi'(s)|^2\, ds\right)^{\frac{1}{2}} t^{\frac{1}{2}} \le M\delta^{\frac{1}{2}} \quad \forall t \in [0,\delta]. \end{aligned}$$

Recalling (3.13) and that we have chosen $\delta \le \tilde{\delta}$ and $|h| \le \tilde{h}$, we have that $|\zeta_1(t)| \le R$ for every $t \in [0,\delta]$; then we may invoke (3.3) and obtain that

$$\begin{aligned} \mathcal{A}_1 \quad &= \quad \lambda(1-\lambda)\int_0^\delta e^{-t}\left\langle \nabla a(x+\zeta_1(t)), \left(\frac{t-\delta}{\delta}\right) h\right\rangle\, dt \le \\ &\lambda(1-\lambda)\int_0^\delta |\nabla a(x + \zeta_1(t))|\; dt\, |h| \le \\ &\lambda(1-\lambda)|h| g(x,\delta). \end{aligned} \tag{3.24}$$

The estimate of $\mathcal{A}_2$ gives the same result and then, putting together (3.21) and (3.24), we conclude that

$$\mathcal{A} \le \lambda(1-\lambda)\left[\frac{1}{2}\frac{|h|^2}{\delta} + 2\,|h| g(x,\delta)\right]. \tag{3.25}$$

Step 7. By the arbitrariness of $w \in \partial\Lambda$, $T \ge T_0$ and $\xi \in \Xi(z,w,T,\Lambda)$, recalling (3.17) and representation (3.5)-(3.6), formulas (3.19), (3.20) and (3.25) imply that for almost every $x \in \Lambda_\gamma$, for every $\delta \in]0, \min\{1, T_0, \tilde{\delta}\}]$, $\lambda \in]0,1[$, $h \in \mathbb{R}^d$ such that $|h| \le \tilde{h}$ and $y = x + h \in \Lambda_\gamma$, the claimed inequality (3.4) holds.

$\square$

Lemma 4. *Assume hypotheses and notations of lemma 3. Then there exists $\overline{h} > 0$ such that for every $h \in \mathbb{R}^d$ such that $|h| \leq \overline{h}$, for almost every $x \in \Lambda_\gamma$ such that $y = x + h \in \Lambda_\gamma$ and such that u is differentiable at the points x and $x + h$, the following inequality holds true:*

$$\left\langle \nabla u(x+h) - \nabla u(x), \frac{h}{|h|} \right\rangle \leq |h|^{\frac{1}{2}} + 4g(x, |h|^{\frac{1}{2}}). \tag{3.26}$$

Proof. We maintain notations and hypotheses of lemma 3 and set $\delta = |h|^{\frac{1}{2}}$; hence we fix $\overline{h}$ imposing that

$$M\overline{h}^{\frac{1}{4}} + 2\overline{h} \leq R, \tag{3.27}$$

so that the conclusions of lemma 3 hold true for any $h \in \mathbb{R}^d$ with $|h| \leq \overline{h}$ and $\delta = |h|^{\frac{1}{2}}$. We define the function

$$G(x,k) \doteq \frac{1}{2}k^{\frac{1}{2}} + 2g(x, k^{\frac{1}{2}}), \quad x \in \Omega,\ k > 0, \tag{3.28}$$

so that formula (3.4) takes the form

$$\lambda u(x) + (1-\lambda)u(x+h) - u(\lambda x + (1-\lambda)(x+h)) \leq \lambda(1-\lambda)|h|G(x,|h|)$$

and can be rewritten as

$$(1-\lambda)\left[u(x+h) - u(x)\right] \leq u(x + (1-\lambda)h) - u(x) + \lambda(1-\lambda)|h|G(x,|h|). \tag{3.29}$$

From (3.29) we have

$$\frac{u(x+h) - u(x)}{|h|} \leq \frac{u(x+(1-\lambda)h) - u(x)}{(1-\lambda)|h|} + \lambda G(x,|h|). \tag{3.30}$$

We recall that whenever a map v is differentiable at the point x we have, for every nonzero vector $\eta \in \mathbb{R}^d$,

$$\limsup_{\alpha \to 0+} \frac{v(x+\alpha\eta) - v(x) - \langle \nabla v(x), \alpha\eta \rangle}{\alpha|\eta|} \leq 0; \tag{3.31}$$

hence it follows from (3.30) and (3.31) that

$$\frac{u(x+h) - u(x)}{|h|} \leq \limsup_{\lambda \to 1-} \left[\frac{u(x+(1-\lambda)h) - u(x)}{(1-\lambda)|h|} + \lambda G(x,|h|) \right] \leq \left\langle \nabla u(x), \frac{h}{|h|} \right\rangle + G(x,|h|). \tag{3.32}$$

Analogously we may write inequality (3.29) in the form

$$\frac{u(x) - u(x+h)}{|h|} \leq \frac{u(x+h-\lambda h) - u(x+h)}{\lambda|h|} + (1-\lambda)G(x,|h|), \tag{3.33}$$

obtaining, as above,

$$\frac{u(x) - u(x+h)}{|h|} \leq \limsup_{\lambda \to 0+} \left[\frac{u(x+h-\lambda h) - u(x+h)}{\lambda|h|} + (1-\lambda)G(x,|h|) \right] \leq \left\langle \nabla u(x+h), \frac{-h}{|h|} \right\rangle + G(x,|h|). \tag{3.34}$$

Formulas (3.32) and (3.34) imply the inequalities

$$\begin{aligned}\frac{u(x+h)-u(x)}{|h|} &\leq \left\langle \nabla u(x), \frac{h}{|h|} \right\rangle + G(x,|h|),\\ \frac{u(x)-u(x+h)}{|h|} &\leq \left\langle \nabla u(x+h), \frac{-h}{|h|} \right\rangle + G(x,|h|),\end{aligned}$$

from which we obtain the following chain

$$-|h|G(x,|h|)+\langle \nabla u(x+h), h\rangle \leq u(x+h)-u(x) \leq \langle \nabla u(x), h\rangle + |h|G(x,|h|). \tag{3.35}$$

Finally, (3.35) and (3.28) imply the claimed inequality (3.26):

$$\langle \nabla u(x+h) - \nabla u(x), h\rangle \leq 2|h|G(x,|h|).$$

□

4. Approximate solutions

Hypothesis 1. Let $\varphi \in W^{1,\infty}(\Omega)$ and let (a_n) be a sequence of nonnegative functions in $L^\infty(\Omega)\bigcap C^1(\Omega)$. We impose the following conditions.

(i) There exists a positive constant $A > 0$ such that $\|a_n\|_{\in L^\infty(\Omega)} \leq A$ for every $n \in \mathbb{N}$;

(ii) the following compatibility conditions hold:

$$\frac{1}{2}\left|\nabla\varphi(x)\right|^2 - a_n(x) \leq 0 \quad \text{a.e.} \quad x \in \Omega \quad \forall n \in \mathbb{N}; \tag{4.1}$$

(iii) there exist $R > 0$, $\tau > 0$ and a nonnegative measurable (in the first variable) function $g : \Omega \times]0,\tau] \to \mathbb{R}$ such that following conditions hold:

- for every $n \in \mathbb{N}$, for almost every $x \in \Omega$, for every $\delta \in]0,\tau]$ and for every $\zeta \in W^{1,\infty}([0,\delta])$ with $|\zeta(t)| \leq R$ and $\zeta(t) \in \Omega$ for every $t \in [0,\delta]$, we have

$$\int_0^\delta |\nabla a_n(x+\zeta(t)|\, dt \leq g(x,\delta); \tag{4.2}$$

- the map $\Omega \ni x \mapsto g(x,\delta)$ belongs to $L^1_{\text{loc}}(\Omega)$ for every $\delta \in]0,\tau]$ and

$$\int_\Lambda g(x,\delta)\, dx \overset{\delta\to 0+}{\longrightarrow} 0 \tag{4.3}$$

for every $\Lambda \subset\subset \Omega$.

Given $n \in \mathbb{N}$ consider the problem

$$\mathcal{P}(a_n,\varphi,\Omega) : \qquad \begin{cases} \frac{1}{2}\left|\nabla u(x)\right|^2 - a_n(x) = 0 & \text{in} \quad \Omega \\ u(x) = \varphi(x) & \text{on} \quad \partial\Omega, \end{cases} \tag{4.4}$$

and let $u_n \in \varphi + W^{1,\infty}_0(\Omega)$ be the (maximal) viscosity solution of $\mathcal{P}(a_n,\varphi,\Omega)$ ensured by theorem 2 and for which formula 2.1 holds true. Recall from (3.2) that there exist $K_0 = K_0(A,\varphi,\Omega)$ and $K_1 = (2A)^{\frac{1}{2}}$ such that

$$\|u_n\|_{L^\infty(\Omega)} \leq K_0, \qquad \|\nabla u_n\|_{L^\infty(\Omega)} \leq K_1 \quad \forall n \in \mathbb{N}. \tag{4.5}$$

Theorem 3. *Assume hypothesis 1 and let (u_n) be the sequence of maximal viscosity solutions of $\mathcal{P}(a_n, \varphi, \Omega)$ introduced above. Then there exists a subsequence, still denoted by (u_n), and a map $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that $u_n \longrightarrow u$ strongly in $W^{1,r}(\Omega)$ for every $r \in [1, \infty[$ and*

$$\nabla u_n(x) \overset{n\to\infty}{\longrightarrow} \nabla u(x) \text{ for a.e. } x \in \Omega. \tag{4.6}$$

Proof.

Step 1: weak limit. Consider the sequence (u_n) and observe that it is bounded in $W^{1,\infty}(\Omega)$; hence there exists $u \in \varphi + W_0^{1,\infty}(\Omega)$ and a subsequence, still denoted by (u_n), such that $u_n \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(\Omega)$ and uniformly on $\overline{\Omega}$.

Step 2: notations. Consider $\Lambda \subset\subset \Omega$ open, convex with smooth boundary $\partial\Lambda$, $\gamma > 0$ and Λ_γ as in lemma 3. For $j \in \{1, \ldots, d\}$, $\delta \in]0, \tau]$ and $x'_j \in \Lambda_j^{d-1}$ we set

$$f_{x'_j}(t, \delta) = g(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d, \delta), \tag{4.7}$$

where we recall from section 2 that Λ_j^{d-1} is the set of points $x'_j \in \mathbb{R}^{d-1}$ such that definition (4.7) makes sense.

Remark that, by point *(iii)* in hypothesis 1 and by lemma 2, we have, for a.e. $x'_j \in \Lambda_j^{d-1}$.

$$\int f_{x'_j}(t, \delta)\, dt \overset{\delta\to 0^+}{\longrightarrow} 0, \tag{4.8}$$

where the integral is performed on intervals of $\mathbb{R}$ determined by the direction e_j, the point x'_j and the set Ω.

Now fix $j \in \{1, \ldots, d\}$ and $e_j \in \mathcal{E}$, vector of the canonical basis of $\mathbb{R}^d$; recall formula (4.5) and, by abuse of notation, write

$$h \doteq he_j, \quad h > 0.$$

Recall from (3.28) the definition of the function $G(x, h) = \frac{1}{2}h^{\frac{1}{2}} + 2g(x, h^{\frac{1}{2}})$, for $x \in \Omega$ and $h > 0$, and apply lemma 4, assuming to deal with points x and $x + h$ for which formula (4.2) is valid; hence formula (3.26) takes the form

$$\langle \nabla u_n(x + he_j) - \nabla u_n(x), e_j \rangle \leq 2G(x, h),$$

or

$$D_j u_n(x + he_j) - D_j u_n(x) \leq 2G(x, h). \tag{4.9}$$

Now fix $x'_j \in \Lambda_j^{d-1}$ and, recalling the notations introduced in section 2, introduce the functions

$$v_{n,x'_j}(t) = u_n(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d).$$

By this way, for almost every $x'_j \in \mathbb{R}^{d-1}$, we have

$$v'_{n,x'_j}(t) = D_j u_n(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d),$$

for almost every $t \in J_{x'_j}(\Lambda, \gamma)$, where $J_{x'_j}(\Lambda, \gamma)$ is the interval in $\mathbb{R}$ for which the points $(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d)$ lie in Λ_γ for every $t \in J_{x'_j}(\Lambda, \gamma)$. Recalling the definition (4.7), formula (4.9) takes the form

$$v'_{n,x'_j}(t + h) - v'_{n,x'_j}(t) \leq h^{\frac{1}{2}} + 4h f_{x'_j}(t, h^{\frac{1}{2}}), \tag{4.10}$$

for $h > 0$ and for a.e. t for which $t, t + h \in J_{x'_j}(\Lambda, \gamma)$.

For the sake of simplicity we omit the dependence on x'_j, setting

$$J = J_{x'_j}(\Lambda, \gamma), \quad v_n = v_{n,x'_j}, \quad f = f_{x'_j},$$

and define

$$F(t,h) = h^{\frac{1}{2}} + 4f(t, h^{\frac{1}{2}}). \tag{4.11}$$

By this way formula (4.10) takes the form:

$$v'_n(t+h) - v'_n(t) \leq F(t,h), \quad h > 0, \text{ a.e. } t, t+h \in J. \tag{4.12}$$

Step 3: L^1 estimate of derivatives.

By (4.12), for $h > 0$ and for a.e. $t, t+h \in J$, we have

$$\begin{aligned} \left| v'_n(t+h) - v'_n(t) \right| \leq \left| v'_n(t+h) - v'_n(t) - F(t,h) \right| + F(t,h) \quad \leq \\ \Big(v'_n(t) - v'_n(t+h) \Big) + 2F(t,h). \end{aligned} \tag{4.13}$$

Now fix $\hat{h} > 0$ small and consider the largest subinterval $\hat{J} \subset J$ such that $\hat{J} \pm \hat{h} \subseteq J$. Integrating (4.13) on $\hat{J}$, we have, for $0 \leq h \leq \hat{h}$,

$$\begin{aligned} \int_{\hat{J}} |v'_n(t+h) - v_n(t)| \, dt \quad \leq \\ \left| \int_{\hat{J}} v'_n(t) \, dt - \int_{\hat{J}} v_n(t+h) \, dt \right| + 2 \int_{\hat{J}} F(t,h) \, dt. \end{aligned} \tag{4.14}$$

By a change of variable we write

$$\int_{\hat{J}} v'_n(t+h) \, dt = \int_{\hat{J}-h} v'_n(t) \, dt,$$

so that (4.14) gives the following inequality:

$$\begin{aligned} \int_{\hat{J}} |v'_n(t+h) - v'_n(t)| \, dt \quad \leq \\ 2 \int_{\hat{J}} F(t,h) \, dt + 2\|v'_n\|_{L^\infty(\hat{J})} \left[m_1 \Big((\hat{J} - h) \setminus \hat{J} \Big) + m_1 \Big((\hat{J}) \setminus (\hat{J} - h) \Big) \right] \quad \leq \\ 2 \int_{\hat{J}} F(t,h) \, dt + 4K_1 h; \quad h \leq \hat{h}. \end{aligned} \tag{4.15}$$

In formula (4.15, by a change of variable, we may interchange t and $t+h$, so that the inequality holds also for $h < 0, |h| \leq \hat{h}$; then, recalling (4.8) and (4.11), we have immediately that

$$\sup_n \int_{\hat{J}} |v'_n(t+h) - v'_n(t)| \, dt \overset{h \to 0}{\longrightarrow} 0. \tag{4.16}$$

Step 4: conclusion. The arbitrariness of $\hat{h}$, theorem 1 and (4.16) imply that the sequence (v'_n) is relatively compact in $L^1(J)$ and by lemma 1 we obtain, recalling step 1, that the sequence (u_n) converges to u strongly in $W^{1,1}(\Lambda_\gamma)$. Since this holds for every $\gamma > 0$ we deduce that (u_n) converges to u strongly in $W^{1,1}(\Lambda)$. Hence, by the arbitrariness of Λ (open, bounded, convex subset of Ω with smooth boundary), we conclude that it converges to u strongly in $W^{1,1}(\Omega)$ and also in $W^{1,r}(\Omega)$ for

$1 \le r < \infty$. In addition, passing if necessary to a further subsequence, we have also that

$$\nabla u_n(x) \overset{n\to\infty}{\longrightarrow} \nabla u(x) \text{ for a.e. } x \in \Omega.$$

□

5. Eikonal equation

Now we apply the results obtained in previous sections to the solution of the Dirichlet problem $\mathcal{P}(a, \varphi, \Omega)$.

Theorem 4. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $a \in L^\infty(\Omega)$ be a nonnegative function. Assume that there exists a sequence (a_n) satisfying hypothesis 1 such that*

$$\lim_{n\to\infty} a_n(x) = a(x) \quad a.e.\ x \in \Omega \tag{5.1}$$

and let $\varphi \in W^{1,\infty}(\Omega)$ be a map satisfying the compatibility conditions (4.1). Then there exists a generalized solution of $\mathcal{P}(a, \varphi, \Omega)$, i.e. a map $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that

$$\frac{1}{2} |\nabla u(x)|^2 = a(x) \quad a.e.\ x \in \Omega. \tag{5.2}$$

If in addition

$$a(x) \le a_n(x) \quad a.e.\ x \in \Omega,\ \forall n \in \mathbb{N}, \tag{5.3}$$

then the solution u is the unique maximal element of $S(a, \varphi, \Omega)$. In particular, if in addition the map a is assumed to be continuous, then u is the unique viscosity solution of $\mathcal{P}(a, \varphi, \Omega)$.

Proof.
Consider the sequence (a_n), the corresponding Dirichlet problems $\mathcal{P}(a_n, \varphi, \Omega)$ and the sequence (u_n) in $\varphi + W_0^{1,\infty}(\Omega)$ of (maximal) viscosity solutions considered in section 4. By theorem 3 there exists $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that ∇u_n converges to ∇u in $L^1(\Omega)$ and almost everywhere. Hence, by (5.1) and passing to the limit $n \to \infty$ in the equation

$$\frac{1}{2} |\nabla u_n(x)|^2 = a_n(x),$$

we obtain (5.2).

In order to prove the second part, recall definition 2 and observe that condition (5.3) implies that

$$S(a, \varphi, \Omega) \subseteq S(a_n, \varphi, \Omega) \quad \forall n \in \mathbb{N}.$$

Hence, calling $\overline{u}$ the maximal element of $S(a, \varphi, \Omega)$, we have necessarily $\overline{u} \le u_n$ on Ω for every $n \in \mathbb{N}$. On the other hand (5.2) implies that $u \in S(a, \varphi, \Omega)$, so that

$$u(x) \le \overline{u}(x) \le u_n(x) \quad \text{a.e. } x \in \Omega,\ \forall n \in \mathbb{N}.$$

Since u_n converges uniformly to u on $\overline{\Omega}$, the thesis is achieved. The last assertion is a trivial consequence of theorems 1 and 2 in [15].

□

Acknowledgments. I am grateful to Professor Paolo Marcellini and to Professor Gianni Dal Maso for interesting conversations on some aspects of the paper.

References

[1] R. Adams, *Sobolev spaces.* Academic Press, New York. 1975.

[2] G. Barles, *Solutions de viscosité des équations de Hamilton-Jacobi*, Mathematiques et Applications, 17, Springer-Verlag, Berlin, 1994.

[3] M. Bardi, I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*, Birkäuser, Boston, 1997.

[4] P. Cannarsa, C. Sinestrari, *Semiconcave functions, Hamilton-Jacobi Equations and Optimal Control*, Birkhäuser, Basel, 2004.

[5] B. Dacorogna, P. Marcellini, *Implicit partial differential equations*, Birkhäuser, Basel, 1999.

[6] R. Lucchetti, *Convexity and well posed problems*, Springer-Verlag, New York, 2006.

[7] P.L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research notes in Math. 69, Pitman, London, 1982.

[8] A. Visintin, *Strong convergence results related to strict convexity*, Comm. Partial Diff. Equations **9(5)** (1984), 439-466.

[9] S. Zagatti, *On the minimum problem for non convex scalar functionals*, SIAM J. on Math. Anal. **37** (2005), 982-995.

[10] S. Zagatti, *Uniqueness and continuous dependence on boundary data for integro-extremal minimizer of the functional of the gradient.* J. Convex Anal. **14** (2007), 705-727.

[11] S. Zagatti, *Solutions of Hamilton-Jacobi equations and minimizers of non quasiconvex functionals.* J. Math. Anal. Appl. **335** (2007), 1143-1160.

[12] S. Zagatti, *Minimizers of non convex scalar functionals and viscosity solutions of Hamilton-Jacobi equations.* Calc. Var. and PDE's. **31** n. 4 (2008), 511-519.

[13] S. Zagatti, *Qualitative properties of integro-esxtremal minimizers of non-homogeneous scalar functionals*, Int. J. Pure Appl. Math. **51** n. 1 (2009), 103-116.

[14] S. Zagatti, *Minimization of non quasiconvex functionals by integro-extremization method.* DCDS-A, **21** n. 2 (2008), 625-641.

[15] S. Zagatti, *On viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **361** n. 1, (2009), 41-59.

[16] S. Zagatti, *An integro-extremization approach for non coercive and evolution Hamilton-Jacobi equations*, J. Convex Analysis **18** n. 4, (2011).