Constrained BV functions on double coverings for Plateau's type problems*

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Abstract

We link Brakke's "soap films" covering construction with the theory of finite perimeter sets, in order to study Plateau's problem without fixing a priori the topology of the solution. The minimization is set up in the class of BV functions defined on a double covering space of the complement of an $(n-2)$-dimensional smooth compact manifold S without boundary. The main novelty of our approach stands in the presence of a suitable constraint on the fibers, which couples together the covering sheets. The model allows to avoid all issues concerning the presence of the boundary S. The constraint is lifted in a natural way to Sobolev spaces, allowing also an approach based on Γ-convergence theory.

Introduction

This work is inspired by the "soap films" covering space model, set up in [6] by Brakke as a new original approach to Plateau's problem in codimension one. We recall that, in its earliest and simplest formulation, Plateau's problem consists in finding a surface Σ in the ambient space $\mathbb{R}^3$, spanning a fixed reference smooth loop S, and minimizing the area. As it is well known, several models have been proposed to solve the mathematical questions related to this problem (and to its generalizations in $\mathbb{R}^n$, for $n \geq 2$), depending on the definition of surface, boundary, and area: parametric and non parametric solutions, integer rectifiable currents, varifolds, just to name a few. We refer the reader for instance to [1], [22] and [8] for a brief, but comprehensive overview on the study of Plateau's problem. In this paper we simplify and link the ideas of Brakke with the theory of finite perimeter sets and Γ-convergence, in order to solve the problem of minimal connections of an even number of points in the plane, and more generally to find an embedded solution to Plateau's problem, without fixing a priori the topology of the solution.

The starting idea of Brakke is to mathematically reproduce the physical structure of soap films, which are made of two layers with a thin liquid region between them.

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Since the model works in arbitrary dimension $n \geq 2$, it can be useful to consider the layers as $(n-1)$ dimensional integer rectifiable currents; at the same time, it is necessary not to make the corresponding orientations (one opposite to the other) cancel out. This difficulty is overcome in [6] by merging Plateau's problem in an n dimensional (abstract) manifold Y, which is a *covering space* of the open set

$$M := \Omega \setminus S,$$

$\Omega \subset \mathbb{R}^n$ being usually a bounded connected open set with Lipschitz boundary. How to choose Y is part of the model construction, and it can lead to different solutions of Plateau's problem. A typical situation is to take Y as a two−sheets, "cut and paste" covering space: let Σ be an $(n-1)-$dimensional orientable manifold with boundary equal to S, and let Y be the manifold defined by "pasting" two copies of M along the "cut" Σ.[1] The layers are then placed in different sheets of Y, thus replacing the interior and the exterior of the soap film.

At this point, the construction of [6] is not finished yet, since the sheets need to be paired up. This is done by Brakke by selecting some connected components of a *pair covering space* of M (denoted respectively by W and W_0 in [6]), where the minimization problem is set up in terms of a suitable notion of currents mass. Again, the choice of W is part of the model construction. Solvability for the minimization problem follows by standard results in the theory of integer rectifiable currents. The author gives also some results on uniqueness and regularity of minimizers, *e.g.*, showing that in some setting (namely, when $W = W_0$) the projection of minimizers onto the base set M is $(\mathbf{M}, 0, \delta)-$minimal in the sense of Almgren.[2]

We mention also the recent paper [9], where the authors, extending in a different setting some result of [12], look for a solution of Plateau's problem, minimizing the $(n-1)-$dimensional Hausdorff measure in the class of relatively closed subsets of $\mathbb{R}^n \setminus S$, with nonempty intersection with every loop having unoriented linking number with S equal to 1.[3] Here, no covering construction is performed, and the minimization problem is solved using direct variationals arguments; neverthless, the approach of [9] shares some common features with [6], as we show in the Appendix.

The aim of this work is to reformulate and simplify Brakke's construction using BV functions defined on the double covering space. In particular, we approach the problem without making use of pair covering spaces, since the sheets here are coupled by a *constraint on the fibers*. Our construction of the covering, which we denote by $(Y_{\mathbf{\Sigma}}, \pi_{\mathbf{\Sigma},M})$ in what follows, requires a suitable pair of cuts $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2)$, in order to work with local parametrizations suggesting the natural way to endow $Y_{\mathbf{\Sigma}}$ with the Euclidean metric. The metric aspects here play a role; as it will be clear from the discussion, we cannot confine ourselves to a purely topological construction of the covering (see Remark 1.6).

Our idea is to minimize the total variation of BV functions defined on $Y_{\mathbf{\Sigma}}$, taking values in $\{\pm 1\}$, and such that

$$\sum_{\pi_{\mathbf{\Sigma},M}(y)=x} u(y) = 0, \quad \text{for a.e. } x \in M. \tag{0.1}$$

[1] See Figure 3, for the case $n = 2$.
[2] See [6, Theorem 10.2].
[3] See Remark 4.5.

To have an idea of the geometric meaning of the total variation we are considering, it is useful to look at the elementary Example 1.10.

What we call a *constrained double – covering solution* to Plateau's problem *with boundary S* will be (Definition 1.23) the projection through $\pi_{\mathbf{\Sigma},M}$ of the jump set of a minimizer. In Theorem 2.1 we prove that the constrained double – covering solutions are independent of $\mathbf{\Sigma}$[4]. In some sense, this is due to the fact that, working on the covering space, all information about the exact location of the cuts becomes irrelevant, since changing the cuts corresponds just to an isometry on the covering space. In view of the conceptual importance of this independence result, in Proposition A.3 we give another proof of it. In Theorem 4.1 we show that, at least when $n < 8$, our model is equivalent to solving Plateau's problem using the theory of integral currents modulo 2 [11].

We stress that the constraint (0.1) is really crucial in our construction; for instance, it forces the minimum value to be strictly positive (Lemma 1.20), as well as it forces the boundary datum S to be covered (Corollary 1.21). Perhaps the most remarkable consequence due to the constraint is that all issues about the definition of "boundary" on S are avoided; the same qualitative property is shared by Brakke's formulation. It seemed to us not immediate to derive the constraint (0.1) on the fibers from the approach of [6].

This paper is divided as follows: in Section 1 we define the family of admissible cuts, and the space $BV(Y_{\mathbf{\Sigma}})$. For $u \in BV(Y_{\mathbf{\Sigma}})$, the functions $v_1(u), v_2(u), v_3(u), v_4(u)$ are defined in (1.13), and coincide with u read in the various domains of the atlas used to parametrize $Y_{\mathbf{\Sigma}}$. In Example 1.9 we give a representation formula for the total variation of some elementary functions in $BV(Y_{\mathbf{\Sigma}})$, in order to make the reader more confident with the covering construction. Then, for any admissible pair of cuts $\mathbf{\Sigma}$, the minimization problem is set up in

$$BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\}) := \Big\{ u \in BV(Y_{\mathbf{\Sigma}}; \{\pm 1\}) \ : \ u \text{ satisfies } (0.1) \Big\}.$$

Existence of minimizers is proved in Theorem 1.19. Regularity of constrained double – covering solutions is based on the well – established regularity theory for isoperimetric sets. Then, we lift the constraint on the fibers to the class of Sobolev functions on $Y_{\mathbf{\Sigma}}$, showing in Proposition 1.25 that our formulation naturally leads to a Γ-convergence result. In Section 2 we show that the minimization problem (1.20) is independent of the admissible pairs of cuts. In Section 3 we exploit the simplest possible case, *i.e.*, when $n = 2$ and S consists of $2k$ distinct points, for some $k \geq 1$. When $k = 1$, the (unique) constrained double – covering solution coincides, as obviously expected, with the segment joining the two points. Despite its simplicity (already pointed out in [6, Example 8.5]), the study of this example is instructive, and probably it can be considered as a starting point for more interesting situations. In Section 4 we test the model in the case of the standard Plateau's problem in $\mathbb{R}^3$. Finally, in the Appendix we present a standard abstract construction of a covering space of M, which we show to be isometric with $Y_{\mathbf{\Sigma}}$. The construction is performed avoiding the definition of admissible cuts.

We expect that our model could be generalized in a nontrivial way in various directions; in particular, as in [6], to l sheeted covering spaces, with $l > 2$, as in the case of the tripod or of space partitions; these subjects require further investigation.

[4]See [6, Proposition 12.1] for a similar result.

1 Double coverings and the minimization problem

Notation. Let $n \geq 2$. We denote by $\mathcal{H}^{n-1}$ the Euclidean $(n-1)-$dimensional Hausdorff measure in $\mathbb{R}^n$. We let $|\cdot|$ be the Euclidean norm. For any x, $x' \in \mathbb{R}^n$, we denote by $x \cdot x'$ the scalar product between x and x'. We also let $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$; for $r > 0$ and $x \in \mathbb{R}^n$, we set $B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$ and, unless otherwise specified, we let $B_r := B_r(0)$. For any $F \subseteq \mathbb{R}^n$, we denote by $\overline{F}$ the closure of F in $\mathbb{R}^n$.

We recall the following definition of unoriented linking number, see for instance [5, Section 3.17], or [15, Section 5.2].

Definition 1.1. *Let $S \subset \mathbb{R}^n$ of a boundaryless compact smooth submanifold of codimension 2. Assume that $S = \partial\Sigma$, for some compact, smooth manifold $\Sigma \subset \mathbb{R}^n$. Let $\rho \in C^1(\mathbb{S}^1; \mathbb{R}^n)$ be such that $\rho^{-1}(S) = \emptyset$, and transverse to Σ. The unoriented linking number between ρ and S is defined as*

$$\text{link}_2(\rho; S) := \begin{cases} 0 & \text{if } \mathcal{H}^0(\rho^{-1}(\Sigma) \text{ is even,} \\ 1 & \text{if } \mathcal{H}^0(\rho^{-1}(\Sigma)) \text{ is odd.} \end{cases} \tag{1.1}$$

The right hand side of (1.1) turns out to be independent of Σ. When ρ is just continuous, the unoriented linking number is defined using a C^1 loop homotopic to ρ and not intersecting S [15].

Throughout this paper, $\Omega \subseteq \mathbb{R}^n$ denotes a nonempty connected open set. Let $S \subset \Omega$ be a boundaryless, compact, embedded, smooth submanifold of dimension $n - 2$, not necessarily connected.

We define the base set as

$$M := \Omega \setminus S, \tag{1.2}$$

which is path connected.

Example 1.2. Typical choices will be:

- $n = 2$, and S an even number of points;
- $n = 3$, and S a tame link (that is, a finite number of disjoint closed smooth space curves).

Definition 1.3. *We denote by*

$$\text{Cuts}(\Omega, S) \tag{1.3}$$

the set of all $(n-1)$-dimensional compact embedded smooth[5] *submanifold $\Sigma \subset \Omega$ with*

$$\partial\Sigma = S.$$

We also let

$$\mathbf{Cuts}(\Omega, S) := \{\boldsymbol{\Sigma} = (\Sigma_1, \Sigma_2) \ : \ \Sigma_1, \Sigma_2 \in \text{Cuts}(\Omega, S), \ \Sigma_1 \cap \Sigma_2 = S\}, \tag{1.4}$$

and we call the elements of Cuts(Ω, S) *(resp. of* $\mathbf{Cuts}(\Omega, S)$*)* admissible cuts *(resp.* admissible pairs of cuts*).*

[5] The smoothness assumption on the cuts and on S is required in order to use Definition 1.1. All subsequent discussion (in particular, Theorem 2.1) can be adapted requiring just Lipschitz regularity, by properly extending unoriented intersection theory to Lipschitz manifolds. It seems nontrivial to extend the definition of admissible cuts (hence, the covering construction in Section 1.1) to the case where S is just a compact subset of $\mathbb{R}^n$, having Hausdorff dimension $n - 2$.

Throughout the paper, we shall always suppose that

$$\mathrm{Cuts}(\Omega, S) \text{ and } \mathbf{Cuts}(\Omega, S) \text{ are nonempty.}$$

A typical situation is when $S = \partial\Sigma$, for some $(n-1)-$dimensional, compact, embedded, orientable, smooth submanifold $\Sigma \subset \Omega$.[6] See Figure 1 for a very simple example.

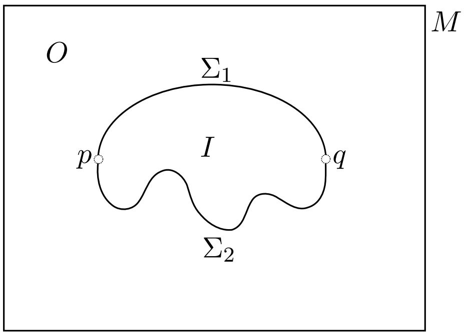

Figure 1: The base set $M = \Omega \setminus S$, when $n = 2$, Ω is a rectangle, and $S = \{p, q\}$. In the figure, an example of admissible pair of cuts is shown. $\Sigma_1 \cup \Sigma_2$ encloses an interior I (an open set), while the exterior (still an open set in Ω) is denoted by O.

1.1 "Cut and paste" construction of the double covering

In this section we explicitly construct the double covering $(Y_{\boldsymbol{\Sigma}}, \pi_{\boldsymbol{\Sigma},M})$ quoted in the introduction. As a consequence, we shall end up with local parametrizations which naturally bring the Euclidean metric on $Y_{\boldsymbol{\Sigma}}$.

Let $\boldsymbol{\Sigma} = (\Sigma_1, \Sigma_2) \in \mathbf{Cuts}(\Omega, S)$. We consider two disjoint copies of the open sets

$$D_1 := \Omega \setminus \Sigma_1, \quad D_2 := \Omega \setminus \Sigma_2, \tag{1.5}$$

which we denote respectively by

$$(D_1, 1), (D_1, 2), \qquad (D_2, 3), (D_2, 4). \tag{1.6}$$

Points in the space

$$\mathcal{X} := (D_1, 1) \cup (D_1, 2) \cup (D_2, 3) \cup (D_2, 4)$$

are identified as follows. Let $O \subset \Omega$ (resp. $I \subset \Omega$) be the open region exterior (resp. interior) to $\Sigma_1 \cup \Sigma_2$. Let $x, x' \in M$, $j \in \{1, 2\}$ and $j' \in \{3, 4\}$; then $(x, j) \sim (x', j')$

[6] Indeed, the orientability of Σ gives a unit normal vector field on $\Sigma \setminus S$ — hence, in particular, a direction to follow in order to "enlarge" the cut, separating its two faces. The construction is standard (in the case $n = 3$, it is given for instance in [16, p.147]). Necessary and sufficient conditions for the existence of this $(n-1)$-dimensional orientable submanifold can be found in [24]. When $n = 3$, and S is a tame link, there exists [21, Theorem 4, p.120] an embedded orientable surface, called *Seifert surface*, whose boundary is S.

if and only if $x = x'$, and one of the following conditions holds:

$$\begin{cases} j = j', & \\ \{j, j'\} \in \{\{1,3\},\{2,4\}\}, & x = x' \in O, \\ \{j, j'\} \in \{\{1,4\},\{2,3\}\}, & x = x' \in I. \end{cases} \tag{1.7}$$

Intuitively, if $O \subset \Omega$ (resp. $I \subset \Omega$) denotes the open region exterior (resp. interior) to $\Sigma_1 \cup \Sigma_2$, we identify: points of $(O, 1)$ with points of $(O, 3)$; points of $(O, 2)$ with points of $(O, 4)$; points of $(I, 1)$ with points of $(I, 4)$; points of $(I, 2)$ with points of $(I, 3)$; notice that each point on $(\Sigma_1 \cup \Sigma_2) \setminus S$ is identified just with itself.

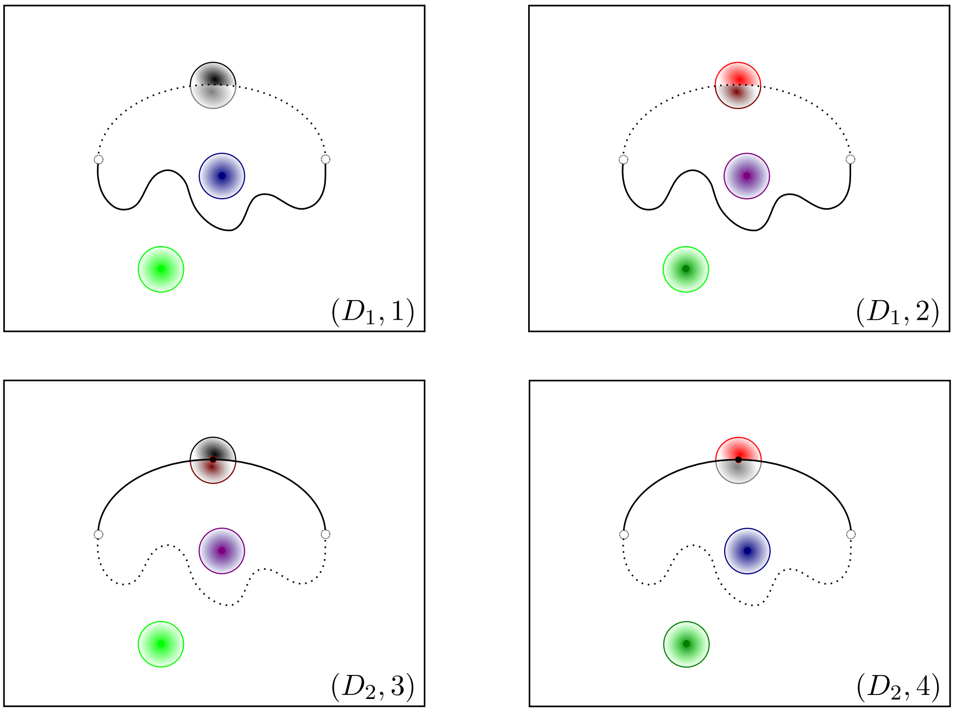

Figure 2: The double covering space $Y_{\mathbf{\Sigma}}$, for M as in Figure 1. A dashed curve denotes that an admissible cut has been removed. In the picture, some examples of admissible neighbourhoods are shown. Identifications are meant by using the same colour. Note that a complete (small) turn around p (or q) in M corresponds to a double turn on $Y_{\mathbf{\Sigma}}$.

Then $\sim$ is an equivalence relation, and the quotient space[7]

$$Y_{\mathbf{\Sigma}} := \mathcal{X} / \sim$$

is endowed with the quotient topology given by the projection $\widetilde{\pi} : \mathcal{X} \to Y_{\mathbf{\Sigma}}$ induced by $\sim$. We set $\pi \colon (x, j) \in \mathcal{X} \mapsto x \in M$, and we denote by

$$\pi_{\mathbf{\Sigma},M} \colon Y_{\mathbf{\Sigma}} \to M \tag{1.8}$$

the projection $\pi_{\mathbf{\Sigma},M}(\widetilde{\pi}(x, j)) := x$, for any $(x, j) \in \mathcal{X}$. This latter map is well defined, since if $(x, j) \sim (x', j')$, then $\pi_{\mathbf{\Sigma},M}(\widetilde{\pi}(x, j)) = x = x' = \pi_{\mathbf{\Sigma},M}(\widetilde{\pi}(x', j'))$. Therefore,

[7] $Y_{\mathbf{\Sigma}}$ depends on the choice of Ω; for notational simplicity we shall not indicate such a dependence.

we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\widetilde{\pi}} & Y_{\boldsymbol{\Sigma}} \\ & \underset{\pi}{\searrow} & \downarrow{\scriptstyle \pi_{\boldsymbol{\Sigma},M}} \\ & & M \end{array} \tag{1.9}$$

Definition 1.4 (Local parametrizations). *We set*

$$\begin{aligned} &\Psi_j \colon D_1 \to \widetilde{\pi}\big((D_1, j)\big), \qquad \Psi_j := \widetilde{\pi} \circ \pi^{-1}_{|_{(D_1,j)}}, \quad j = 1, 2, \\ &\Psi_{j'} \colon D_2 \to \widetilde{\pi}\big((D_2, j')\big), \qquad \Psi_{j'} := \widetilde{\pi} \circ \pi^{-1}_{|_{(D_2,j')}}, \quad j' = 3, 4. \end{aligned} \tag{1.10}$$

The covering space[8] $Y_{\boldsymbol{\Sigma}}$ admits a natural structure of differentiable manifold, with four local parametrizations given by

$$\Psi_1, \Psi_2, \Psi_3, \Psi_4. \tag{1.11}$$

Remark 1.5. For $j \in \{1, 2\}$ and $j' \in \{3, 4\}$, we have

$$\Psi_{j'}^{-1} \circ \Psi_j = \mathrm{id} = \Psi_j^{-1} \circ \Psi_{j'} \quad \text{on } D_1 \cap D_2,$$

where id is the identity map on $D_1 \cap D_2$. The pair $(Y_{\boldsymbol{\Sigma}}, \pi_{\boldsymbol{\Sigma},M})$ is a double covering of M, see Figure 2. Notice that $\Psi_1(D_1) \cup \Psi_1(D_2) = Y_{\boldsymbol{\Sigma}} \setminus \pi^{-1}_{\boldsymbol{\Sigma},M}(\Sigma_1 \setminus S)$.

Remark 1.6. A double covering of M can be constructed in a standard way [16, p.147] also using a single orientable cut $\Sigma \in \mathrm{Cuts}(\Omega, S)$, by suitably identifying two copies of $\Omega \setminus \Sigma$. This construction is perhaps more intuitive than the one based on (1.7) and corresponds, essentially, to the case in which Σ_1 and Σ_2 coincide (with Σ). However, in order to rigorously define the covering, one needs to slightly separate the two "faces" of Σ (as in Figure 3). Since our minimization problem (see (1.20) below) depends on the metric on the covering space, we find more convenient to use the construction via admissible pairs of cuts. However, it is worth noticing that, concretely, it will be enough to deal with only one of the two cuts of the pair $\boldsymbol{\Sigma}$.

The next observation is of importance for the computation of the total variation of BV functions on the covering space $Y_{\boldsymbol{\Sigma}}$.

Remark 1.7 (Splitting). If $E \subseteq Y_{\boldsymbol{\Sigma}}$ is a Borel set, we can write E as the union of the following four disjoint Borel sets

$$E \cap \widetilde{\pi}((D_1, 1)),\ E \cap \widetilde{\pi}((D_1, 2)),\ E \cap \widetilde{\pi}((\Sigma_1 \setminus S, 3)),\ E \cap \widetilde{\pi}((\Sigma_1 \setminus S, 4)). \tag{1.12}$$

Notice that Σ_2 does not appear in (1.12). Choosing D_2 in place of D_1 amounts in considering Σ_2 in place of Σ_1 and does not change the subsequent discussion.

[8] Since S has been removed, $Y_{\boldsymbol{\Sigma}}$ is not branched.

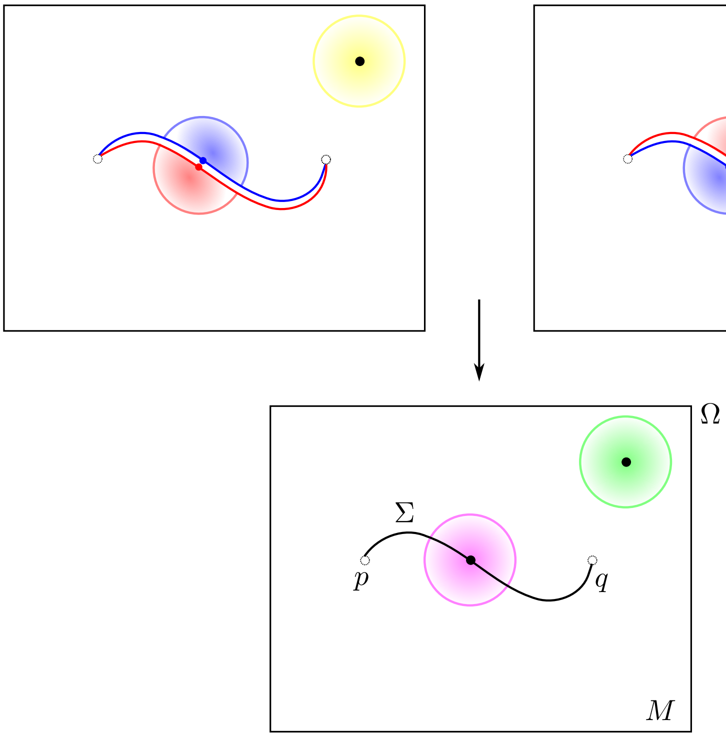

Figure 3: By slightly separating the "faces" of an admissible cut Σ, we get a double covering for the base set M defined in Figure 1. Some examples of admissible neighbourhoods are shown. Notice that each copy of M contains the whole fiber of Σ.

1.2 Total variation on the double covering

The covering space $Y_{\mathbf{\Sigma}}$ is an n-dimensional connected orientable smooth non complete manifold; it is endowed with a natural volume measure μ, which is the push-forward (indicated by $\#$) of the n-dimensional Lebesgue measure $\mathcal{L}^n$ in M via the maps (1.11). More specifically, if $E \subseteq Y_{\mathbf{\Sigma}}$ is a Borel set, using the splitting (1.12), we have

$$\begin{aligned}\mu(E) :&= \Psi_{1\#}\mathcal{L}^n(E \cap \widetilde{\pi}((D_1,1))) + \Psi_{2\#}\mathcal{L}^n(E \cap \widetilde{\pi}((D_1,2))) \\ &= \mathcal{L}^n(\pi_{\mathbf{\Sigma},M}(E \cap \widetilde{\pi}((D_1,1)))) + \mathcal{L}^n(\pi_{\mathbf{\Sigma},M}(E \cap \widetilde{\pi}((D_1,2)))).\end{aligned}$$

We set $L^1(Y_{\mathbf{\Sigma}}) := L^1_\mu(Y_{\mathbf{\Sigma}})$ and $L^1_{\mathrm{loc}}(Y_{\mathbf{\Sigma}}) := L^1_{\mu_{\mathrm{loc}}}(Y_{\mathbf{\Sigma}})$.

Definition 1.8 (The functions $v_h(u)$). *Let $u : Y_{\mathbf{\Sigma}} \to \mathbb{R}$. For $j = 1,2$ and $j' = 3,4$ we let $v_j(u) : D_1 \to \mathbb{R}$, $v_{j'}(u) : D_2 \to \mathbb{R}$ be the functions defined by*

$$v_j(u) := u \circ \Psi_j, \qquad v_{j'}(u) := u \circ \Psi_{j'}. \tag{1.13}$$

Clearly, if $u \in L^1(Y_{\mathbf{\Sigma}})$ then $v_j(u) \in L^1(D_1)$, $v_{j'}(u) \in L^1(D_2)$.
By construction (recall (1.7)), we have

$$\begin{aligned} &v_1(u) = v_3(u), \quad v_2(u) = v_4(u) \qquad \text{a.e. in } O, \\ &v_1(u) = v_4(u), \quad v_2(u) = v_3(u) \qquad \text{a.e. in } I. \end{aligned} \tag{1.14}$$

Let Ω be bounded. Our aim is to define the total variation of a function $u \in L^1(Y_{\mathbf{\Sigma}})$. We say that u is in $BV_\mu(Y_{\mathbf{\Sigma}}) =: BV(Y_{\mathbf{\Sigma}})$ if its distributional gradient[9] $Du\colon \phi \in C^1_c(Y_{\mathbf{\Sigma}}) \mapsto -\int_{Y_{\mathbf{\Sigma}}} uD\phi\, d\mu \in \mathbb{R}^n$ is a bounded vector–valued Radon measure on $Y_{\mathbf{\Sigma}}$. Let us denote by $|Du|$ the *total variation* measure corresponding to Du [3, p.3]; we recall [3, Proposition 1.47, p.21] that, for any open subset $E \subseteq Y_{\mathbf{\Sigma}}$, we have

$$|Du|(E) = \sup\left\{\int_E u \ \mathrm{div}\Phi\, d\mu \ : \ \Phi \in C^1_c(E;\mathbb{R}^n),\ |\Phi| \le 1\right\}, \tag{1.15}$$

which is $L^1(Y_{\mathbf{\Sigma}})$–lower semicontinuous.[10]

Remark 1.9 (Representation of the total variation, I). Let $u \in BV(Y_{\mathbf{\Sigma}})$ and $E \subseteq Y_{\mathbf{\Sigma}}$ be a Borel set. Then

$$\begin{aligned}|Du|(E) = &\sum_{j=1,2} |Dv_j(u)|\Big(\pi_{\mathbf{\Sigma},M}\big(E \cap \widetilde{\pi}((D_1, j))\big)\Big)\\ &+ \sum_{j'=3,4} |Dv_{j'}(u)|\Big(\pi_{\mathbf{\Sigma},M}\big(E \cap \widetilde{\pi}((\Sigma_1 \setminus S, j'))\big)\Big).\end{aligned} \tag{1.16}$$

In order to prove (1.16), let us first assume $E \subseteq \widetilde{\pi}((D_1, 1))$ is open. Then, recalling (1.15), we have

$$\begin{aligned}|Du|(E) &= \sup\left\{\int_{\Psi_1^{-1}(E)} v_1(u)\ \mathrm{div}\varphi\, d\mathcal{L}^n \ : \ \varphi \in C^1_c(\Psi_1^{-1}(E);\mathbb{R}^n),\ |\varphi| \le 1\right\}\\ &= |Dv_1(u)|(\Psi_1^{-1}(E)) = |Dv_1(u)|(\pi_{\mathbf{\Sigma},M}(E)),\end{aligned} \tag{1.17}$$

which gives (1.16).

From (1.17) and [3, Proposition 1.43], we get (1.16) for every Borel set $E \subseteq Y_{\mathbf{\Sigma}}$ contained in a single chart. The general case follows by the splitting[11] in (1.12).

[9] Let $\phi \in C^1_c(Y_{\mathbf{\Sigma}})$. For $i = 1, \ldots, n$, let e_i be the i–th element of the canonical basis of $\mathbb{R}^n$. Then $D_i\phi(y) := \lim_{h\to 0} h^{-1}\big(\phi(\Psi_j(\pi_{\mathbf{\Sigma},M}(y) + he_i)) - \phi(y)\big)$ is well–defined for every $y \in \widetilde{\pi}((D_1, j))$. Similarly for other points in $Y_{\mathbf{\Sigma}}$. We set $D\phi := (D_1\phi, \ldots, D_n\phi)$. For $\Phi := (\phi_1, \ldots, \phi_n) \in C^1_c(Y_{\mathbf{\Sigma}};\mathbb{R}^n)$, we set $\mathrm{div}\Phi := \sum_{i=1}^n D_i\phi_i$.

[10] We notice that $BV(Y_{\mathbf{\Sigma}})$ is equal to the space $BV(Y_{\mathbf{\Sigma}}, g, \omega)$ (see [4, Definition 2.1, Theorem 3.1]), where g (resp. ω) is the pull–back of the Euclidean metric (resp. Euclidean volume form) via the map $\pi_{\mathbf{\Sigma},M}$. Moreover, $BV(Y_{\mathbf{\Sigma}})$ coincides also with $BV(Y_{\mathbf{\Sigma}}, d_{Y_{\mathbf{\Sigma}}}, \mu)$ (see [4, Definition 3.2]), where $d_{Y_{\mathbf{\Sigma}}}$ is the Carnot–Carathéodory distance associated with g.See formula (A.8) in the Appendix. We notice incidentally that $(Y_{\mathbf{\Sigma}}, d_{Y_{\mathbf{\Sigma}}})$ is not a complete metric space. For the definition of Carnot–Carathéodory distance in a more general framework, see [4, p.13] By [4, Theorem 3.1], the three definitions of total variation coincide.

[11] We notice once more that, the right hand side of (1.16) is not altered replacing Σ_1 by Σ_2, D_1 by D_2, and $v_1(u)$ (resp. $v_2(u)$) by $v_3(u)$ (resp. $v_4(u)$).

Example 1.10. Referring specifically to Figure 1, let $u \in BV(Y_{\mathbf{\Sigma}})$ be such that $v_1(u)$ is equal to $a \in \mathbb{R}$ inside a disk B of radius $r > 0$ contained in I (or in O) and $b \in \mathbb{R}$ outside, and $v_2(u)$ is equal to $c \in \mathbb{R}$ in B and $d \in \mathbb{R}$ outside. Then, from (1.17),

$$\begin{aligned} |Du|(Y_{\mathbf{\Sigma}}) =& |Dv_1(u)|(B \cap D_1) + |Dv_2(u)|(B \cap D_1) \\ &+ |Dv_3(u)|(\Sigma_1 \setminus \{p,q\}) + |Dv_4(u)|(\Sigma_1 \setminus \{p,q\}) \\ =& (|b-a| + |d-c|)\ 2\pi r + 2\mathcal{H}^1(\Sigma_1)|d-b|. \end{aligned} \tag{1.18}$$

On the other hand, if B is centered at a point of Σ_1, and $B \cap \Sigma_2 = \emptyset$, then

$$\begin{aligned} |Du|(Y_{\mathbf{\Sigma}}) =& |Dv_1(u)|(B \cap D_1) + |Dv_2(u)|(B \cap D_1) \\ &+ |Dv_3(u)|(\Sigma_1 \setminus \{p,q\}) + |Dv_4(u)|(\Sigma_1 \setminus \{p,q\}) \\ =& (|b-a| + |d-c|)\ 2\pi r + 2|c-a|\mathcal{H}^1(\Sigma_1 \cap B) \\ &+ 2|d-b| \left(\mathcal{H}^1(\Sigma_1) - \mathcal{H}^1(\Sigma_1 \cap B)\right). \end{aligned} \tag{1.19}$$

If in particular $a = 1 = -b$, $c = -1 = -d$, (1.18) and (1.19) become

$$|Du|(Y_{\mathbf{\Sigma}}) = 4\left(2\pi r + \mathcal{H}^1(\Sigma_1)\right).$$

1.3 The constrained minimum problem

We define

$$BV(Y_{\mathbf{\Sigma}}; \{\pm 1\}) := \Big\{ u \in BV(Y_{\mathbf{\Sigma}}) \ : \ |u| = 1 \ \mu\text{-a.e. in } Y_{\mathbf{\Sigma}} \Big\}.$$

Our constrained minimization problem, which in principle could depend on the choice of $\mathbf{\Sigma}$, can be stated as follows:

$$\mathscr{A}^{\Omega}_{\rm constr}(S, \mathbf{\Sigma}) := \inf \Big\{ |Du|\big(Y_{\mathbf{\Sigma}}\big) \ : \ u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\}) \Big\}, \tag{1.20}$$

where

$$BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\}) := \Big\{ u \in BV(Y_{\mathbf{\Sigma}}; \{\pm 1\}) \ : \ \sum_{\pi_{\mathbf{\Sigma},M}(y) = x} u(y) = 0 \ \text{ for a.e. } x \text{ in } M \Big\}.$$

Notice that there are only two terms in the sum inside parentheses, and that the functional in (1.20) attains the same value when evaluated at u and at $-u$. By virtue of the constraint, it turns out that $\mathscr{A}^{\Omega}_{\rm constr}(S, \mathbf{\Sigma})$ is strictly positive (Theorem 1.19).

From the constraint (0.1), for any $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ we have

$$v_1(u) = -v_2(u) \text{ a.e. in } D_1, \qquad v_3(u) = -v_4(u) \text{ a.e. in } D_2. \tag{1.21}$$

For this reason, in formulas (1.25) and (1.27) the functions $v_2(u)$ and $v_4(u)$ are not present.

For any $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$, let $J_u \subset Y_{\mathbf{\Sigma}}$ be the *set of approximate jump points*[12] of u in $Y_{\mathbf{\Sigma}}$.

[12] Here we follow [3, Definition 3.67, p.163]. Let $d_{Y_{\mathbf{\Sigma}}}$ be the distance defined in (A.8). Given $y \in Y_{\mathbf{\Sigma}}$, $r > 0$, and a unit vector $\nu \in \mathbb{R}^n$, set $B(y,r) := \{y' \in Y_{\mathbf{\Sigma}} \ : \ d_{Y_{\mathbf{\Sigma}}}(y,y') < r\}$, $B^+_\nu(y,r) := \{y' \in B(y,r) \ : \ (\pi_{\mathbf{\Sigma},M}(y') - \pi_{\mathbf{\Sigma},M}(y)) \cdot \nu > 0\}$, $B^-_\nu(y,r) := \{y' \in B(y,r) \ : \ (\pi_{\mathbf{\Sigma},M}(y') - \pi_{\mathbf{\Sigma},M}(y)) \cdot \nu < 0\}$. Notice that, for $r > 0$ sufficiently small (depending on y), $B(y,r)$ is contained in a single chart of $Y_{\mathbf{\Sigma}}$. Now, we define J_u as the set of points $y \in Y_{\mathbf{\Sigma}}$ such that there exists a unit vector $\nu \in \mathbb{R}^n$, satisfying $\lim_{r \to 0^+} r^{-n} \int_{B^+_\nu(y,r)} |u-1|\, d\mu = 0 = \lim_{r \to 0^+} r^{-n} \int_{B^-_\nu(y,r)} |u+1|\, d\mu$.

Remark 1.11 (**Unbounded open sets**)**.** Let Ω be unbounded. Then, instead of (1.20), we shall consider the minimization problem

$$\mathscr{A}^{\Omega}_{\rm constr}(S,\mathbf{\Sigma}) := \inf\left\{|Du|(Y_{\mathbf{\Sigma}}) \ : \ u \in BV^{\rm loc}_{\rm constr}(Y_{\mathbf{\Sigma}},\{\pm 1\})\right\}, \tag{1.22}$$

where

$$BV^{\rm loc}_{\rm constr}(Y_{\mathbf{\Sigma}},\{\pm1\}) := \Big\{u \in L^1_{\rm loc}(Y_{\mathbf{\Sigma}}) \ : \ |Du|(E) < \infty, \ E \subset Y_{\mathbf{\Sigma}} \text{ open rel. compact}, \\ |u| = 1 \ \mu\text{−a.e. in } Y_{\mathbf{\Sigma}}, \sum_{\pi_{\mathbf{\Sigma},M}(y)=x} u(y) = 0 \ \text{ for a.e. } x \text{ in } M\Big\}.$$

We notice that the previous discussion (in particular, formula (1.16)) still holds true assuming Ω unbounded.

Remark 1.12. Let Ω, $\Omega' \subseteq \mathbb{R}^n$ be connected open sets, such that $\Omega \subseteq \Omega'$. Then

$$\mathscr{A}^{\Omega}_{\rm constr}(S,\mathbf{\Sigma}) \le \mathscr{A}^{\Omega'}_{\rm constr}(S,\mathbf{\Sigma}). \tag{1.23}$$

Indeed, let us assume that Ω' is bounded. For $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega,S) \subseteq \mathbf{Cuts}(\Omega',S)$, let us denote by $Y'_{\mathbf{\Sigma}}$ the double covering space of $M' := \Omega' \setminus S$. It is natural to see $Y_{\mathbf{\Sigma}}$ as a subset of $Y'_{\mathbf{\Sigma}}$, so that, for any $u \in BV_{\rm constr}(Y'_{\mathbf{\Sigma}};\{\pm1\})$, we have $u_{|Y_{\mathbf{\Sigma}}} \in BV_{\rm constr}(Y_{\mathbf{\Sigma}};\{\pm1\})$. In particular, $|Du|(Y_{\mathbf{\Sigma}}) \le |Du|(Y'_{\mathbf{\Sigma}})$, which gives (1.23). Similarly for the case in which Ω or Ω' are unbounded.

1.4 Existence of minimizers

Concerning functions defined on the base set, clearly $BV(M;\{\pm1\}) = BV(\Omega;\{\pm1\})$. Moreover

$$BV(\Omega;\{\pm1\}) = BV(D_1;\{\pm1\}),$$

so that any $v \in BV(D_1;\{\pm1\})$ (or more generally any $v \in BV(\Omega \setminus C;\{\pm1\})$, with C a finite union of cuts) can be considered also as a BV function in Ω, whose total variation in general may increase by a contribution due to the two traces of v on Σ_1 (more generally on C). In the following, we denote by

$$J_v \subset \Omega$$

the set of approximate jump points of v considered as a function in $BV(\Omega;\{\pm1\})$. The next definition will be of frequent use in the sequel.

Definition 1.13 (**Constrained lifting**)**.** *Let $v \in BV(D_1;\{\pm1\})$. Then the function*

$$u := \begin{cases} v & \text{in } \Psi_1(D_1), \\ -v & \text{in } \Psi_2(D_1), \end{cases} \tag{1.24}$$

is in $BV_{\rm constr}(Y_{\mathbf{\Sigma}};\{\pm1\})$, and $v_1(u) = v$. We call u the constrained lifting of v.

In particular, when v is identically equal to 1 (or −1), we have

$$\pi_{\mathbf{\Sigma},M}(J_u) = \Sigma_1 \setminus S.$$

Lemma 1.14 (**Splitting of the projection of the jump**). *Let* $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2) \in \mathbf{Cuts}(\Omega, S)$*, and let* $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$*. Then*

$$\pi_{\mathbf{\Sigma},M}(J_u) = \Big(J_{v_1(u)} \setminus (\Sigma_1 \setminus S)\Big) \cup \Big(J_{v_3(u)} \cap (\Sigma_1 \setminus S)\Big). \tag{1.25}$$

Proof. As in (1.12), let us split J_u as the union of the following four disjoint sets:

$$J_u \cap \widetilde{\pi}((D_1, 1)),\ J_u \cap \widetilde{\pi}((D_1, 2)),\ J_u \cap \widetilde{\pi}((\Sigma_1 \setminus S, 3)),\ J_u \cap \widetilde{\pi}((\Sigma_1 \setminus S, 4)). \tag{1.26}$$

By the constraint (0.1), to each point in the first set of (1.26) there corresponds a unique point in the second set, belonging to the same fiber, and viceversa. A similar correspondence holds between the third and the fourth set. Hence

$$\pi_{\mathbf{\Sigma},M}(J_u) = \pi_{\mathbf{\Sigma},M}\Big(J_u \cap \widetilde{\pi}((D_1, 1))\Big) \cup \pi_{\mathbf{\Sigma},M}\Big(J_u \cap \widetilde{\pi}((\Sigma_1 \setminus S, 3))\Big).$$

By definition of J_u, $J_{v_1(u)}$, $J_{v_3(u)}$, using also the local parametrizations Ψ_1, Ψ_3, it follows that $\pi_{\mathbf{\Sigma},M}\big(J_u \cap \widetilde{\pi}((D_1, 1))\big) = J_{v_1(u)} \setminus (\Sigma_1 \setminus S)$, and $\pi_{\mathbf{\Sigma},M}\big(J_u \cap \widetilde{\pi}((\Sigma_1 \setminus S, 3))\big) = J_{v_3(u)} \cap (\Sigma_1 \setminus S)$, and (1.25) follows. □

The next lemma seems to be consistent with [6, Lemma 10.1].

Lemma 1.15 (**Representation of the total variation on the covering, II**). *Let* $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2) \in \mathbf{Cuts}(\Omega, S)$*, and let* $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$*. Then*

$$\begin{aligned} |Du|(Y_{\mathbf{\Sigma}}) =& 4\Big(\mathcal{H}^{n-1}(J_{v_1(u)} \setminus \Sigma_1) + \mathcal{H}^{n-1}(J_{v_3(u)} \cap \Sigma_1)\Big) \\ =& 4\,\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_u)). \end{aligned} \tag{1.27}$$

Proof. Recall the splitting in (1.16), with the choice $E := Y_{\mathbf{\Sigma}}$. By (1.21), we have

$$|Dv_1(u)|(D_1) = |Dv_2(u)|(D_1), \qquad |Dv_3(u)|(\Sigma_1) = |Dv_4(u)|(\Sigma_1). \tag{1.28}$$

By [3, Theorem 3.84], we have

$$|Dv_1(u)|(D_1) = 2\mathcal{H}^{n-1}(J_{v_1(u)} \setminus \Sigma_1), \quad |Dv_3(u)|(\Sigma_1) = 2\mathcal{H}^{n-1}(J_{v_3(u)} \cap \Sigma_1). \tag{1.29}$$

Substituting (1.29) into (1.16), and recalling (1.28), we get the first equality in (1.27). The second equality is now a consequence of (1.25). □

Remark 1.16. The factor 4 in (1.27) is obtained by multiplying the absolute value between the difference of the values of u (which gives a factor 2), with the number of the covering sheets. Up to this multiplicative constant, the total variation of a function in $BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ is equal to the push−forward of the $\mathcal{H}^{n-1}$ measure of its jump set, via the local parametrizations (1.11).

From formula (1.27), we see that $|Du|(Y_{\mathbf{\Sigma}})$ is indeed independent of the orientation of Σ_1 — hence, in particular, of the orientation of S.

Remark 1.17. If we choose u as the constrained lifting of v, with v identically equal to 1, we obtain

$$\mathscr{A}^{\Omega}_{\rm constr}(S, \mathbf{\Sigma}) \le 4\mathcal{H}^{n-1}(\Sigma_1). \tag{1.30}$$

The natural bound from above for $\mathscr{A}^{\Omega}_{\rm constr}(S, \mathbf{\Sigma})$ will be given in (2.10).

Corollary 1.18 (**Compactness**). *Let Ω be bounded with Lipschitz boundary. Let $(u_k)_{k\in\mathbb{N}} \subset BV_{\rm constr}(Y_{\mathbf{\Sigma}};\{\pm1\})$ be such that $\sup_{k\in\mathbb{N}} |Du_k|(Y_{\mathbf{\Sigma}}) < +\infty$. Then there exist $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}};\{\pm1\})$ and a subsequence of $(u_k)_{k\in\mathbb{N}}$ converging to u in $L^1(Y_{\mathbf{\Sigma}})$.*

Proof. Let $v_1(u_k) \in BV(D_1;\{\pm1\}) = BV(\Omega;\{\pm1\})$ be as in (1.13). From (1.16) and (1.28) we have

$$\sup_{k\in\mathbb{N}} |Dv_1(u_k)|(\Omega) = \sup_{k\in\mathbb{N}} \Big[|Dv_1(u_k)|(D_1) + |Dv_1(u_k)|(\Sigma_1)\Big] \leq \sup_{k\in\mathbb{N}} \frac{|Du_k|(Y_{\mathbf{\Sigma}})}{4} + \mathcal{H}^{n-1}(\Sigma_1) < +\infty.$$

Since Ω is a bounded Lipschitz domain, there exists $v \in BV(\Omega;\{\pm1\})$ such that, up to a not relabelled subsequence, $v_1(u_k) \to v$ in $L^1(\Omega)$. The proof is completed, letting u be defined as in (1.24). □

We are now in the position to show that problem (1.20) has a solution; a key result is represented by Lemma 1.20 below.

Theorem 1.19 (**Existence of minimizers**). *Let Ω be a bounded connected open set with Lipschitz boundary. Let $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$. Then $\mathscr{A}^{\Omega}_{\rm constr}(S,\mathbf{\Sigma})$ is a minimum, and $\mathscr{A}^{\Omega}_{\rm constr}(S,\mathbf{\Sigma}) > 0$.*

Proof. Recalling the lower semicontinuity and Corollary 1.18, existence of minimizers for problem (1.20) follows by direct methods. Positivity of $\mathscr{A}^{\Omega}_{\rm constr}(S,\mathbf{\Sigma})$ follows from (1.32) below, with the choice $A := \Omega$. □

The next lemma shows, in particular, that in the fibers over any open subset of Ω containing a loop around a point of S, the jump set of any function in $BV_{\rm constr}(Y_{\mathbf{\Sigma}};\{\pm1\})$ has strictly positive $\mathcal{H}^{n-1}$−measure. We stress that this is due just to the constraint (0.1).

Lemma 1.20. *Let $A \subseteq \Omega$ be a nonempty open set such that $\pi^{-1}_{\mathbf{\Sigma},M}(A\backslash S)$ is connected. Then for any $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}};\{\pm1\})$,*

$$\mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_u)\big) > 0. \tag{1.31}$$

Moreover, if A is bounded with Lipschitz boundary, then

$$\inf\big\{\mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_u)\big) \ : \ u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}};\{\pm1\})\big\} > 0. \tag{1.32}$$

Proof. By contradiction, suppose that

$$\mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_u)\big) = 0. \tag{1.33}$$

Applying (1.25) to (1.33), we get

$$0 = \mathcal{H}^{n-1}(A \cap (J_{v_1(u)} \setminus \Sigma_1)) + \mathcal{H}^{n-1}(A \cap J_{v_3(u)} \cap \Sigma_1). \tag{1.34}$$

Now, set $A^S := A \setminus S$. Applying (1.16) with the choice $E := \pi^{-1}_{\mathbf{\Sigma},M}(A^S)$, we get

$$\begin{aligned} |Du|(\pi^{-1}_{\mathbf{\Sigma},M}(A^S)) =& 2|Dv_1(u)|\Big(\pi_{\mathbf{\Sigma},M}(\pi^{-1}_{\mathbf{\Sigma},M}(A^S) \cap \widetilde{\pi}((D_1,1)))\Big) \\ & + 2|Dv_3(u)|\Big(\pi_{\mathbf{\Sigma},M}(\pi^{-1}_{\mathbf{\Sigma},M}(A^S) \cap \widetilde{\pi}(\Sigma_1 \setminus S,3)))\Big) \\ =& 2\left(|Dv_1(u)|\left(A^S \setminus \Sigma_1\right) + |Dv_3(u)|\left(A^S \cap \Sigma_1\right)\right) \\ =& 4\left(\mathcal{H}^{n-1}(A \cap (J_{v_1(u)} \setminus \Sigma_1)) + \mathcal{H}^{n-1}(A \cap J_{v_3(u)} \cap \Sigma_1)\right), \end{aligned} \tag{1.35}$$

which, coupled with (1.34), implies $|Du|(\pi_{\mathbf{\Sigma},M}^{-1}(A^S)) = 0$. Then[13] u is constant on $\pi_{\mathbf{\Sigma},M}^{-1}(A^S)$, which contradicts the validity of the constraint (0.1). This proves (1.31).

Now, let us suppose, still by contradiction, that there exists a sequence $(u_k)_k \subset BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ such that $\lim_{k\to+\infty} \mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_{u_k})\big) = 0$. Thanks to the assumption on A, $\pi_{\mathbf{\Sigma},M}^{-1}(A^S)$ is a double covering of A^S. In particular, for each $k \in \mathbb{N}$, the restriction $\hat{u}_k := u_k|_{\pi_{\mathbf{\Sigma},M}^{-1}(A^S)}$ belongs to $BV_{\rm constr}(\pi_{\mathbf{\Sigma},M}^{-1}(A^S); \{\pm 1\})$, and reasoning as above, $|D\hat{u}_k|(\pi_{\mathbf{\Sigma},M}^{-1}(A^S)) = 4\mathcal{H}^{n-1}(A \cap \pi_{\mathbf{\Sigma},M}(J_{u_k}))$. Let us apply Corollary 1.18, replacing Ω with A. Then, up to a not relabelled subsequence, there exists $\hat{u} \in BV_{\rm constr}(\pi_{\mathbf{\Sigma},M}^{-1}(A^S); \{\pm 1\})$ such that $\hat{u}_k \to u$ in $L^1(\pi_{\mathbf{\Sigma},M}^{-1}(A^S))$, and by lower semicontinuity,

$$|D\hat{u}|(\pi_{\mathbf{\Sigma},M}^{-1}(A^S)) \le \liminf_{k\to+\infty} |D\hat{u}_k|(\pi_{\mathbf{\Sigma},M}^{-1}(A^S)) = 4 \lim_{k\to+\infty} \mathcal{H}^{n-1}\big(A \cap \pi_{\mathbf{\Sigma},M}(J_{u_k})\big) = 0.$$

Hence $\hat{u}$ is constant on $\pi_{\mathbf{\Sigma},M}^{-1}(A^S)$, a contradiction with (0.1). □

As a further consequence of Lemma 1.20, the boundary datum S is covered by any constrained function in the covering space. In Theorem 4.1, using also (1.37) below, we shall prove that equality holds in (1.36) when $2 \le n \le 7$ and u is a minimizer.

Corollary 1.21. *Let Ω be bounded (resp. unbounded), and let $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ (resp. $u \in BV^{\rm loc}_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$). Then*

$$\overline{\pi_{\mathbf{\Sigma},M}(J_u)} \setminus \pi_{\mathbf{\Sigma},M}(J_u) \supseteq S. \tag{1.36}$$

Proof. The relation $S \cap \pi_{\mathbf{\Sigma},M}(J_u) = \emptyset$ is trivial, recall also (1.25). Now, suppose by contradiction that there exists a point $p \in S \setminus \overline{\pi_{\mathbf{\Sigma},M}(J_u)}$. Take an open ball B centered at p, with $B \subset \Omega \setminus \overline{\pi_{\mathbf{\Sigma},M}(J_u)}$, and apply Lemma 1.20 with the choice $A := B$. Then, since $A \cap \pi_{\mathbf{\Sigma},M}(J_u) = \emptyset$, we end up with a contradiction with (1.31). □

Recalling Remark 1.5, we observe that the proof of analytic regularity for the reduced boundary of isoperimetric sets[14] applies in our setting. Here we have to observe that, even if $\int_{Y_{\mathbf{\Sigma}}} u\, d\mu = 0$ for any $u \in BV_{\rm constr}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$, we are not minimizing the total variation under a volume constraint, and therefore we are not exactly in the isoperimetric situation. Nevertheless, since the classical arguments (such as monotonicity formula, excess decay, tilt lemma) are *local*, they can be symmetrically reproduced on the two sheets of the covering space, thus respecting the constraint (0.1) on the fibers. In particular, the following result holds[15].

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[13] See [3, Proposition 3.2]; this constancy result can be generalized to our setting, considering first the case in which a connected open set $E \subseteq Y_{\mathbf{\Sigma}}$ is contained in a single chart, and then reasoning for each connected component of $E \cap \widetilde{\pi}((D_1, 1))$, $E \cap \widetilde{\pi}((D_2, 3))$.

[14] See for instance [17], and references therein. See also [20] for the case of isoperimetric hypersurfaces in smooth Riemannian manifolds.

[15] Theorem 1.22 is consistent with [6, Theorem 10.2], which implies that $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is $(\mathbf{M}, 0, \delta)$–minimal in the sense of Almgren [2]. Our result is stronger than Brakke's one, which considers more general coverings constructions, just requiring connectedness on the pair covering space W_0 cited in the introduction. In our setting, we have $W_0 := \{(y, y') \in Y_{\mathbf{\Sigma}} \times Y_{\mathbf{\Sigma}} \ : \ y \neq y',\ \pi_{\mathbf{\Sigma},M}(y) = \pi_{\mathbf{\Sigma},M}(y')\}$, which is always connected.

Theorem 1.22 (**Analytic regularity**). *Let Ω be a bounded connected open set with Lipschitz boundary (resp. an unbounded connected open set), and let $u_{\min}$ be a minimizer of problem* (1.20) *(resp. assume there exists $u_{\min}$ minimizing* (1.22)*). Then $J_{u_{\min}}$, and hence, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$, is an analytic submanifold, possibly excepting for a set of Hausdorff dimension at most $n-8$. Moreover,*

$$\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}}) \subseteq S \cup \partial\Omega. \tag{1.37}$$

Proof. We can limit ourselves to the proof of (1.37). Recalling Lemma 1.14, we have

$$\Big(\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})\Big) \cap D_1 = \Big(\overline{J_{v_1(u_{\min})}} \setminus J_{v_1(u_{\min})}\Big) \cap D_1. \tag{1.38}$$

By the regularity of local minimizers of perimeter[16], $J_{v_1(u_{\min})} \cap D_1$ coincides with the relative boundary in D_1 of the set $\{v_1(u_{\min}) = 1\}$. In particular, $J_{v_1(u_{\min})} \cap D_1$ is relatively closed in D_1, which by (1.38) implies

$$\Big(\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})\Big) \cap D_1 = \emptyset.$$

Similarly we argue on D_2, and (1.37) follows. □

In view of Lemma 1.15, also comparing (1.20) with Brakke's formulation [6, Section 6.1], we give the following definition.

Definition 1.23 (**Constrained double−covering solutions**). *Let Ω be bounded with Lipschitz boundary and let $u_{\min}$ be a minimizer of problem* (1.20) *We call*

$$\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$$

a constrained double−covering solution *(in Ω) to Plateau's problem* with boundary S.

A similar definition is given when Ω is unbounded, assuming existence of $u_{\min}$ minimizing (1.22).

Remark 1.24. No topological restrictions on $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ are required.

1.5 Regularization

The interest in using finite perimeter sets in the context of covering spaces is substantiated by a $\Gamma-$convergence [7] result. Let Ω be bounded. The main idea is to lift the constraint (0.1) onto the Sobolev space $H^1(Y_{\mathbf{\Sigma}}) := \{u \in L^2_\mu(Y_{\mathbf{\Sigma}}) \ : Du \in L^2_\mu(Y_{\mathbf{\Sigma}};\mathbb{R}^n)\ \}$, setting

$$H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}) := \Big\{u \in H^1(Y_{\mathbf{\Sigma}}) \ : \sum_{\pi_{\mathbf{\Sigma},M}(y)=x} u(y) = 0 \text{ for a.e. } x \in M\Big\}. \tag{1.39}$$

For $\epsilon \in (0,1)$, let us consider the functionals $F_\epsilon \colon L^1(Y_{\mathbf{\Sigma}}) \to [0,+\infty]$, defined as

$$F_\epsilon(u) := \int_{Y_{\mathbf{\Sigma}}} \left[\epsilon|\nabla u|^2 + \frac{1}{\epsilon}(1-u^2)^2\right] d\mu \qquad \text{if } u \in H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}),$$

and extended to $+\infty$ in $L^1(Y_{\mathbf{\Sigma}}) \setminus H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}})$.

[16] See for instance [17, Theorem 28.1].

Proposition 1.25 (**$\Gamma-$convergence**). *If $(u_{\epsilon_h})_h \subset L^1(Y_{\mathbf{\Sigma}})$ is such that $\sup_\epsilon F_h(u_{\epsilon_h}) < +\infty$, then there exist $u \in L^1(Y_{\mathbf{\Sigma}})$ and a subsequence of $(u_{\epsilon_{h'}})_{h'}$ converging to u in $L^1(Y_{\mathbf{\Sigma}})$. Moreover,*

$$\big(\Gamma(L^1(Y_{\mathbf{\Sigma}})) - \lim_{\epsilon_h \to 0^+} F_\epsilon\big)(u) = \begin{cases} \frac{c_0}{2}|Du|(Y_{\mathbf{\Sigma}}), & u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\}), \\ +\infty, & \textit{otherwise in } L^1(Y_{\mathbf{\Sigma}}), \end{cases}$$

where $c_0 := \chi(1) - \chi(-1)$, and $\chi(t) := 2\int_0^t |1 - s^2|\, ds$.

Proof. The proof of the equicoerciveness statement is standard (see, *e.g.*, [18]), as well as the $\Gamma-$limit assertion. The $\Gamma - \liminf$ inequality follows using the lower semicontinuity of the total variation, and the fact that the constraint (0.1) is closed under almost everywhere convergence in $Y_{\mathbf{\Sigma}}$. The $\Gamma - \limsup$ construction follows by recalling that the local parametrizations of $Y_{\mathbf{\Sigma}}$ are the identity (Remark 1.5); in order to get the validity of the constraint in (1.39), it is sufficient to use the standard construction, since the optimal one-dimensional profile is odd (hence, the corresponding recovering sequence is in $H^1_{\mathrm{constr}}(Y_{\mathbf{\Sigma}})$). □

2 Independence of the pair of cuts

In this section we show that constrained double−covering solutions are independent of admissible cuts. A different proof of such an independence is given in Proposition A.3.

Theorem 2.1. *Let Ω be bounded. Let $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2)$, $\mathbf{\Sigma}' = (\Sigma_1', \Sigma_2') \in \mathbf{Cuts}(\Omega, S)$. Let $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$. Then there exists $u' \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}'}; \{\pm 1\})$ such that, up to a $\mathcal{H}^{n-1}-$negligible set,*

$$\pi_{\mathbf{\Sigma},M}(J_u) = \pi_{\mathbf{\Sigma}',M}(J_{u'}). \tag{2.1}$$

Proof. Without loss of generality, we can suppose that $\Sigma_1 \neq \Sigma_1'$. Fix $x_0 \in M \setminus (\Sigma_1 \cup \Sigma_1')$. Let $x \in M \setminus (\Sigma_1 \cup \Sigma_1')$, and let $\gamma_x \in C^1([0,1]; M)$ be such that $\gamma_x(0) = x_0$, $\gamma_x(1) = x$, and γ_x is transverse both to Σ_1 and Σ_1'; such a γ_x will be called an admissible path from x_0 to x. We set

$$h(\gamma_x; \Sigma_1, \Sigma_1') := \mathcal{H}^0(\gamma_x^{-1}(\Sigma_1)) + \mathcal{H}^0(\gamma_x^{-1}(\Sigma_1')).$$

If we consider another admissible path λ_x from x_0 to x, we have that $h(\gamma_x; \Sigma_1, \Sigma_1')$ and $h(\lambda_x; \Sigma_1, \Sigma_1')$ have the same parity. Indeed, let ρ be the closed curve going from x_0 to x following γ_x, and then backward from x to x_0 along λ_x. Applying (1.1) with the choice $\Sigma = \Sigma_1$ and $\Sigma = \Sigma_1'$, and recalling that $\mathrm{link}_2(\rho; \Sigma_1) = \mathrm{link}_2(\rho; \Sigma_1')$, it follows

$$h(\gamma_x; \Sigma_1, \Sigma_1') + h(\lambda_x; \Sigma_1, \Sigma_1') = \mathcal{H}^0(\rho^{-1}(\Sigma_1)) + \mathcal{H}^0(\rho^{-1}(\Sigma_1')) \ \text{ is even.} \tag{2.2}$$

By (2.2), we are allowed to set

$$h(x; \Sigma_1, \Sigma_1') := \begin{cases} 0 & \text{if } h(\gamma_x; \Sigma_1, \Sigma_1') \text{ is even}, \\ 1 & \text{if } h(\gamma_x; \Sigma_1, \Sigma_1') \text{ is odd}, \end{cases} \tag{2.3}$$

for any admissible γ_x from x_0 to x^{17}.

Set $\mathcal{O} := \{x \in M \setminus (\Sigma_1 \cup \Sigma_1') \ : \ h(x; \Sigma_1, \Sigma_1') = 0\}$, which is an open set, and $\partial\mathcal{O} \subseteq \Sigma_1 \cup \Sigma_1'$; moreover $\mathcal{O}$ has finite perimeter in Ω by [3, Proposition 3.62]. Let

$$v_1' := \begin{cases} v_1(u) & \text{in } \mathcal{O}, \\ -v_1(u) & \text{in } \Omega \setminus \mathcal{O}. \end{cases}$$

From [3, Theorem 3.84] it follows that $v_1' \in BV(\Omega; \{\pm 1\})$. It also immediately follows[18] that

$$J_{v_1'} \setminus (\Sigma_1 \cup \Sigma_1') = J_{v_1(u)} \setminus (\Sigma_1 \cup \Sigma_1'). \tag{2.4}$$

We define $u' \in BV_{\rm constr}(Y_{\mathbf{\Sigma}'}; \{\pm 1\})$ as the constrained lifting of v_1' when D_1 is replaced by $D_1' := \Omega \setminus \Sigma_1'$, and we claim that u' satisfies (2.1).

Recalling also (1.14), set

$$v_3' := \begin{cases} v_1' & \text{in the exterior region } O' \text{ to } \Sigma_1' \cup \Sigma_2', \\ -v_1' & \text{in the interior region } I' \text{ to } \Sigma_1' \cup \Sigma_2'. \end{cases}$$

Notice that $v_3' \in BV(\Omega; \{\pm 1\})$. By construction, we have

$$v_1' = v_1(u'), \qquad v_3' = v_3(u').$$

Applying Lemma 1.14 we have

$$\pi_{\mathbf{\Sigma}',M}(J_{u'}) = \big(J_{v_1'} \setminus (\Sigma_1' \setminus S)\big) \cup \big(J_{v_3'} \cap (\Sigma_1' \setminus S)\big),$$

and our proof is concluded provided we show that, up to a $\mathcal{H}^{n-1}-$negligible set,

$$\big(J_{v_1'} \setminus \Sigma_1'\big) \cup \big(J_{v_3'} \cap \Sigma_1'\big) = \big(J_{v_1(u)} \setminus \Sigma_1\big) \cup \big(J_{v_3(u)} \cap \Sigma_1\big). \tag{2.5}$$

Let us split the sets on the left hand side of (2.5) as follows:

$$\begin{aligned} J_{v_1'} \setminus \Sigma_1' &= \Big((J_{v_1'} \cap \Sigma_1) \setminus \Sigma_1'\Big) \cup \Big(J_{v_1'} \setminus (\Sigma_1 \cup \Sigma_1')\Big), \\ J_{v_3'} \cap \Sigma_1' &= \Big(J_{v_3'} \cap \Sigma_1 \cap \Sigma_1'\Big) \cup \Big((J_{v_3'} \cap \Sigma_1') \setminus \Sigma_1\Big). \end{aligned} \tag{2.6}$$

Let us show that, up to a $\mathcal{H}^{n-1}-$negligible set,

$$(J_{v_1'} \cap \Sigma_1) \setminus \Sigma_1' = (J_{v_3(u)} \cap \Sigma_1) \setminus \Sigma_1'. \tag{2.7}$$

Let $x \in (J_{v_1'} \cap \Sigma_1) \setminus \Sigma_1'$. Up to a $\mathcal{H}^{n-1}-$negligible set[19], we can assume that the approximate tangent spaces to $J_{v_1'}$ and Σ_1 at x coincide. Let $B(x)$ be an open ball centered at x, not intersecting Σ_1', and such that $B(x) \setminus \Sigma_1$ consists of two connected

[17]Once x_0 is fixed, the function h allows to define an "exterior" and an "interior" of $\Sigma_1 \cup \Sigma_1'$, even when Σ_1 and Σ_1' intersect on a set of positive $\mathcal{H}^{n-1}-$measure.

[18] Indeed, let $x \in J_{v_1'} \setminus (\Sigma_1 \cup \Sigma_1')$ and let γ_x be an admissible path from x_0 to x. Let $B(x)$ be an open ball centered at x and disjoint from $\Sigma_1 \cup \Sigma_1'$; in particular, every $z \in B(x)$ can be reached by a path obtained attaching to γ_x the segment between x and z; notice that such a path γ_z is admissible from x_0 to z, and $h(\gamma_z; \Sigma_1, \Sigma_1') = h(\gamma_x; \Sigma_1, \Sigma_1')$. Therefore, either $v_1' = v_1(u)$ in $B(x)$ or $v_1' = -v_1(u)$ in $B(x)$, which implies $x \in J_{v_1(u)}$. Hence $J_{v_1'} \setminus (\Sigma_1 \cup \Sigma_1') \subseteq J_{v_1(u)} \setminus (\Sigma_1 \cup \Sigma_1')$. Similarly, also the converse inclusion holds, and (2.4) is proven.

[19] Here we use again [3, Theorem 3.84].

components. The same argument used in the proof of (2.4) shows that on one component $v'_1 = v_1(u)$, while on the other $v'_1 = -v_1(u)$. Since $x \in J_{v'_1}$, we have

$$x \notin J_{v_1(u)}.$$

On the other hand, by (1.14), in one component we have $v_1(u) = v_3(u)$, while in the other component $v_3(u) = v_2(u) = -v_1(u)$ (where in the last equality we used (1.21)). Thus, $x \in J_{v_3(u)}$. So, up to a $\mathcal{H}^{n-1}-$negligible set, $(J_{v'_1} \cap \Sigma_1) \setminus \Sigma'_1 \subseteq (J_{v_3(u)} \cap \Sigma_1) \setminus \Sigma'_1$. Arguing similarly for the other inclusion, we get (2.7).

The same argument applies also to prove that, up to a $\mathcal{H}^{n-1}-$negligible set,

$$J_{v'_3} \cap \Sigma_1 \cap \Sigma'_1 = J_{v_3(u)} \cap \Sigma_1 \cap \Sigma'_1, \tag{2.8}$$

and

$$(J_{v'_3} \cap \Sigma'_1) \setminus \Sigma_1 = (J_{v_1(u)} \cap \Sigma'_1) \setminus \Sigma_1. \tag{2.9}$$

From (2.4) – (2.9), we finally get (2.5). □

A statement similar to Theorem 2.1 holds (with a similar proof) when Ω is unbounded, the space $BV_{\mathrm{constr}}(Y_{\boldsymbol{\Sigma}}; \{\pm 1\})$ begin replaced by $BV^{\mathrm{loc}}_{\mathrm{constr}}(Y_{\boldsymbol{\Sigma}}; \{\pm 1\})$ and $BV_{\mathrm{constr}}(Y_{\boldsymbol{\Sigma}'}; \{\pm 1\})$ by $BV^{\mathrm{loc}}_{\mathrm{constr}}(Y_{\boldsymbol{\Sigma}'}; \{\pm 1\})$.

Corollary 2.2 (Independence). *$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \boldsymbol{\Sigma})$ in (1.20) is independent of $\boldsymbol{\Sigma} \in$ $\mathbf{Cuts}(\Omega, S)$.*

Proof. We consider the case in which Ω is bounded, the unbounded case being similar. Let $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}' \in \mathbf{Cuts}(\Omega, S)$. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\boldsymbol{\Sigma}}; \{\pm 1\})$ be such that $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \boldsymbol{\Sigma}) = 4\mathcal{H}^{n-1}(\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}}))$. Let $u' \in BV_{\mathrm{constr}}(Y_{\boldsymbol{\Sigma}'}; \{\pm 1\})$ be the function given by Theorem 2.1 (applied with $u = u_{\min}$). Then, by (1.27) and (2.1), we have

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \boldsymbol{\Sigma}') \le 4\mathcal{H}^{n-1}(\pi_{\boldsymbol{\Sigma}',M}(J_{u'})) = 4\mathcal{H}^{n-1}(\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}})) = \mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \boldsymbol{\Sigma}).$$

Arguing similarly for the converse inequality, we get $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \boldsymbol{\Sigma}') = \mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \boldsymbol{\Sigma})$. □

In accordance with Corollary 2.2, we set

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) := \mathscr{A}^{\Omega}_{\mathrm{constr}}(S, \boldsymbol{\Sigma}).$$

Recalling Remark 1.17, we get the following inequality.

Corollary 2.3 (Upper bound). *We have*

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \le 4 \inf \Big\{ \mathcal{H}^{n-1}(\Sigma) \ : \ \Sigma \in \mathrm{Cuts}(\Omega, S) \Big\}. \tag{2.10}$$

In Sections 3 and 4 we shall prove that, when $n = 2, 3$ and $\Omega = \mathbb{R}^n$, (2.10) holds as an equality (see Proposition 3.3 and Theorem 4.4). Notice that, by the regularity of area minimizing currents modulo 2 [23, Theorem 6.2.1], the infimum on the right hand side of (2.10) is a minimum.

3 The case $n = 2$

In this section we exploit the simplest setting, corresponding to the case in which we remove from the plane k copies of $\mathbb{S}^0 = \{\pm 1\}$, for some $k \in \mathbb{N}$, $k \geq 1$.[20]

3.1 The plane without two points

Let $n = 2$ and $p, q \in \Omega$, with $p \neq q$. We set $S := \{p, q\}$. When Ω contains the segment connecting p and q, the simplest example of admissible cut corresponds, clearly, to the segment itself. Hereafter, we shall assume that:

$$|q - p| < 2\,\mathrm{dist}(S, \partial\Omega). \tag{3.1}$$

Assumption (3.1) is needed, in order to get the segment as the unique constrained covering solution with boundary S. A counterexample when (3.1) is not satisfied is shown in Figure 4.

Proposition 3.1 (**Geodesics**). *Let $\Omega \subset \mathbb{R}^2$ be bounded, connected, with Lipschitz boundary, $S = \{p, q\}$, and let $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$. Assume* (3.1)*. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of* (1.20)*. Then $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is the open segment connecting p and q. In particular $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) = 4|q - p|$.*

Proof. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (1.20). By Theorem 1.22, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is a finite collection of disjoint analytic curves, whose endpoints lie on $\{p, q\} \cup \partial\Omega$. Moreover, by Corollary 1.21, p and q are touched by at least one of these curves.

Set $r := |q - p|/2$, and let $\epsilon \in (0, r)$. Suppose first that $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ consists only of loops with base point p (resp. q), and contained in $B_{r-\epsilon}(p)$ (resp. in $B_{r-\epsilon}(q)$). Let A be the circular annulus centered at p with radii $(r - \epsilon)$ and r. By construction, $A \cap \pi_{\mathbf{\Sigma},M}(J_{u_{\min}}) = \emptyset$, which contradicts (1.31).

Hence, either there is a loop contained in a ball of radius larger than or equal to r and not contained in the concentric call of radius $r - \epsilon$ (thus, giving a contribution larger than $|q - p|$), or there is at least one curve connecting p either to q or to a point on $\partial\Omega$. Using assumption (3.1), we get the statement. □

Corollary 3.2. $\mathscr{A}^{\mathbb{R}^2}_{\mathrm{constr}}(S) = 4|q - p|$.

Proof. Let Ω be as in Proposition 3.1. Then, recalling (1.23) and (2.10), we get

$$4\,|q - p| = \mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \leq \mathscr{A}^{\mathbb{R}^2}_{\mathrm{constr}}(S) \leq 4\,|q - p|.$$

□

[20] The case in which S consists of an *odd* number of points is excluded. Actually, it requires a different and more complicated covering construction, together with a generalization of the constraint on the fibers, which seems nontrivial already when S consists of three points. See for instance [6] or the final discussion in the introduction.

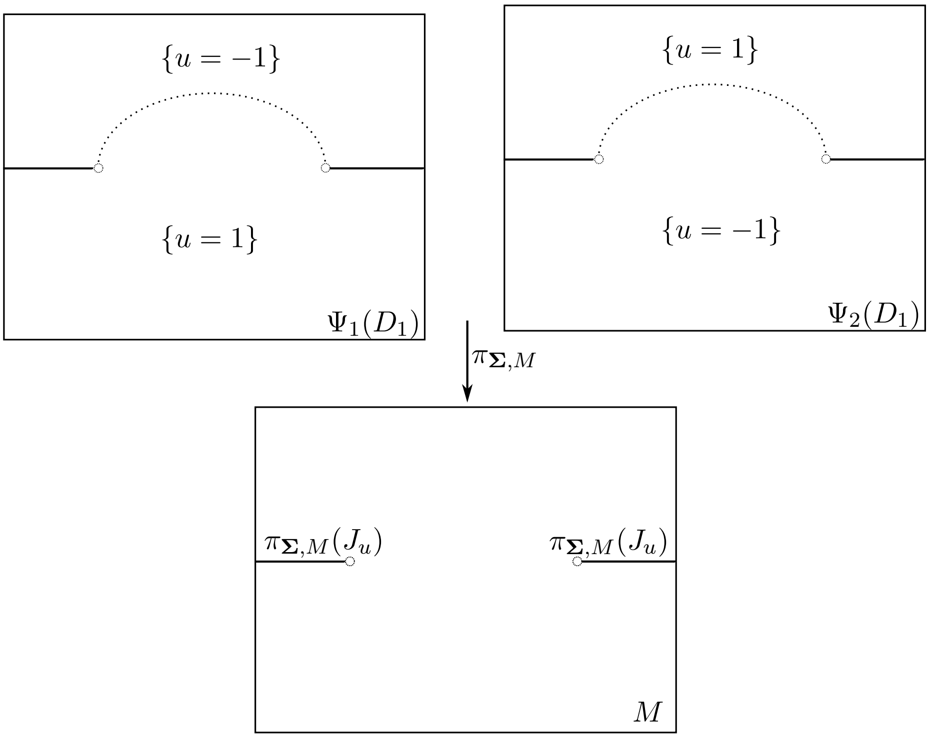

Figure 4: Referring to the cut Σ_1 in Figure 2, a minimizer of (1.20) is shown, whose jump set does not project onto the segment between p and q (the function u is defined just on $\Psi_1(D_1)$ and $\Psi_2(D_1)$, by the constraint (0.1)).

3.2 The plane without an even number of points

We generalize Proposition 3.1 to the case in which, for some integer $k \geq 1$, we fix $2k$ distinct points $p_1, \ldots, p_{2k} \in \Omega$, and

$$S := \{p_j \ : \ j = 1, \ldots, 2k\}. \tag{3.2}$$

Our aim is to explicitely characterize the constrained double−covering solutions with boundary S.

Let us denote by Σ^{opt} any union of k segments connecting the elements of S, and having minimal total length. Notice that, when Ω contains the convex envelope of S, then

$$\Sigma^{\mathrm{opt}} \in \mathrm{Cuts}(\Omega, S), \tag{3.3}$$

since any segment in Σ^{opt} cannot intersect any other different segment. Indeed, denote by $[\![p, q]\!]$ the segment connecting p with q; by the triangular inequality, for any distinct $q_1, \ldots, q_4 \in S$ such that $[\![q_1, q_2]\!]$ has nonempty intersection with $[\![q_3, q_4]\!]$, either $[\![q_1, q_3]\!] \cup [\![q_2, q_4]\!]$ or $[\![q_1, q_4]\!] \cup [\![q_2, q_3]\!]$ has total length strictly less than the length of $[\![q_1, q_2]\!] \cup [\![q_3, q_4]\!]$.

Proposition 3.3. *Let Ω be bounded, connected and with Lipschitz boundary. Assume that*

$$\mathcal{H}^1(\Sigma^{\mathrm{opt}}) < 2\,\mathrm{dist}(S, \partial\Omega). \tag{3.4}$$

Let $\mathbf{\Sigma} = (\Sigma_1, \Sigma_2) \in \mathbf{Cuts}(\Omega, S)$, and let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (1.20). Then $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is a union of k segments connecting the elements of S and having minimal total length. In particular,

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) = 4\mathcal{H}^1(\Sigma^{\mathrm{opt}}).$$

Proof. By Theorem 1.22, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is union of pairwise disjoint, simple, analytic curves, whose endpoints lie on $S \cup \partial\Omega$.

First, we claim that any closed simple curve in $\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})}$ passing through a point of S must intersect or enclose another point of S.

Suppose by contradiction that there exist $q \in S$ and a closed simple curve $\rho_q \colon [0,1] \to \Omega$, such that $\rho_q((0,1)) \subset \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$, $\rho_q(0) = \rho_q(1) = q$, and such that the open region $\mathcal{I}$ enclosed by $\rho_q([0,1])$ does not contain any other point of S. We distinguish two cases.

First case: $\overline{\mathcal{I}} \cap \Sigma_1 = \emptyset$. Define

$$\tilde{v} := \begin{cases} v_1(u_{\min}) & \text{in } \Omega \setminus \mathcal{I}, \\ \text{constantly extended by continuity} & \text{in } \mathcal{I}. \end{cases}$$

The definition is meaningful, because $v_1(u_{\min})$ does not jump in a (one-sided) neighbourhood of $\rho_q([0,1])$ in $\Omega \setminus \mathcal{I}$; moreover $\tilde{v} \in BV(\Omega; \{\pm 1\})$. Let $\tilde{u} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be the constrained lifting of $\tilde{v}$. Clearly $\pi_{\mathbf{\Sigma},M}(J_{\tilde{u}})$ is properly contained in $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$, which contradicts minimality of $u_{\min}$.

Second case: $\overline{\mathcal{I}} \cap \Sigma_1 \neq \emptyset$. By Theorem 2.1, it is not restrictive to suppose that $\mathcal{I}$ and Σ_1 meet at a finite number of points, and each intersection is transverse. We argue as in the previous case in each connected component of $\mathcal{I} \setminus \Sigma_1$, constantly extending by continuity the function $v_1(u_{\min})_{|\Omega \setminus \mathcal{I}}$ starting from its value on each connected component of $\rho_q([0,1]) \setminus \Sigma_1$.

Hence our claim is proven and, as a consequence,

$$\text{the number of loops in } \overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \text{ is finite.} \tag{3.5}$$

Given $q \in S$, we define

$$\mathscr{R}(q) = \{\Gamma_1, \ldots, \Gamma_{N_q}\} \tag{3.6}$$

as the set of all curves in $\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})}$ having q as an endpoint and which are not loops, and we let

$$\mathscr{R} := \bigcup_{q \in S} \mathscr{R}(q),$$

which is a finite set of curves.

We divide the proof into four steps.

Step 1. Let $q \in S$, and let $\rho \colon \mathbb{S}^1 \to M = \Omega \setminus S$ be a circle around q, such that the closure of the open region enclosed by $C := \rho(\mathbb{S}^1)$ does not contain any other point of S or any portion of curve belonging to $\mathscr{R} \setminus \mathscr{R}(q)$. Then, for any Γ_i as in (3.6), we have

$$N_q \equiv \sum_{i=1}^{N_q} \mathcal{H}^0(\rho^{-1}(\Gamma_i)) \equiv \mathcal{H}^0\big(\rho^{-1}(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}}))\big) \pmod 2. \tag{3.7}$$

Without loss of generality, we can assume that C intersects Σ_1 and $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ at a finite number of points, and each intersection is transverse. Therefore, since $\Gamma_i \in \mathscr{R}(q)$ is not a loop, we have that $\mathcal{H}^0(\rho^{-1}(\Gamma_i))$ is odd; hence N_q has the same parity of $\sum_{i=1}^{N_q} \mathcal{H}^0(\rho^{-1}(\Gamma_i)) < +\infty$. This latter sum, in turn, has the same parity of $\mathcal{H}^0\big(\rho^{-1}(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}}))\big)$, where we use (3.5), and we notice that each loop in $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ intersects C an even number of times. Then (3.7) follows.

Step 2. N_q is odd for every $q \in S$.

Let ρ and C be as in step 1. By regularity, the two traces of $v_1(u_{\min})$ (resp. $v_3(u_{\min})$) on C are equal, and locally constant on $C \setminus J_{v_1(u_{\min})}$ (resp. on $C \setminus J_{v_3(u_{\min})}$). Moreover, by transversality, they are discontinuous exactly at $J_{v_1(u_{\min})} \cap C$ (resp. at $J_{v_3(u_{\min})} \cap C$); whence, by (1.21),

$$\rho^{-1}(J_{v_3(u_{\min})} \cap \Sigma_1) = \rho^{-1}(\Sigma_1 \setminus J_{v_1(u_{\min})}). \tag{3.8}$$

Then, recalling (1.25), and using also (3.8), we have

$$\begin{aligned}
\mathcal{H}^0\big(\rho^{-1}(\pi_{\mathbf{\Sigma},M}(J_{u_{\min}}))\big) =&\mathcal{H}^0\big(\rho^{-1}(J_{v_1(u_{\min})} \setminus \Sigma_1)\big) + \mathcal{H}^0\big(\rho^{-1}(J_{v_3(u_{\min})} \cap \Sigma_1)\big)\\
=&\mathcal{H}^0\big(\rho^{-1}(J_{v_1(u_{\min})} \setminus \Sigma_1)\big) + \mathcal{H}^0\big(\rho^{-1}(\Sigma_1 \setminus J_{v_1(u_{\min})})\big)\\
=&\mathcal{H}^0\big(\rho^{-1}(J_{v_1(u_{\min})})\big) + \mathcal{H}^0\big(\rho^{-1}(\Sigma_1)\big)\\
&- 2\,\mathcal{H}^0\big(\rho^{-1}(J_{v_1(u_{\min})} \cap \Sigma_1)\big)\\
\equiv&\mathcal{H}^0\big(\rho^{-1}(J_{v_1(u_{\min})})\big) + \mathcal{H}^0\big(\rho^{-1}(\Sigma_1)\big) \pmod 2.
\end{aligned} \tag{3.9}$$

Hence, coupling (3.7) with (3.9),

$$N_q \equiv \mathcal{H}^0\big(\rho^{-1}(J_{v_1(u_{\min})})\big) + \mathcal{H}^0\big(\rho^{-1}(\Sigma_1)\big) \pmod 2.$$

Step 2 is proven, noticing that C intersects Σ_1 an odd number of times, while $J_{v_1(u_{\min})} \cap C$ consists of an even number of points.

Step 3. We have

$$\Gamma \cap \partial\Omega = \emptyset, \quad \Gamma \in \mathscr{R}(q),\ q \in S. \tag{3.10}$$

By assumption (3.4), we just need to exclude the case of a single curve belonging to $\mathscr{R}$, and reaching the boundary of Ω. Suppose by contradiction that there exist $q_0 \in S$ and $\Gamma_0 \in \mathscr{R}(q_0)$, such Γ_0 has an endpoint on $\partial\Omega$, and that Γ has no endpoints on $\partial\Omega$, for every $\Gamma \in \mathscr{R} \setminus \{\Gamma_0\}$. Then, using (3.2) and step 2,

$$0 \equiv 2k = \sum_{q\in S} 1 \equiv \sum_{q \in S} N_q \pmod 2. \tag{3.11}$$

On the right hand side of (3.11), each curve in $\mathscr{R}$ different from Γ_0 is counted twice (corresponding to its endpoints), while Γ_0 is counted only once. Hence, the right hand side of (3.11) is odd, a contradiction.

Step 4. $N_q = 1$ for any $q \in S$.

Suppose by contradiction that there exists $q_0 \in S$ such that $\mathscr{R}(q_0)$ contains three distinct curves Γ_1, Γ_2, Γ_3 (recall step 2). For $j = 1, 2, 3$, let q_j be the endpoint of Γ_j different from q_0; from (3.10), we know that $q_j \in S$. We distinguish three cases:

(i) if $q_1 = q_2 = q_3$, set $\mathscr{R}'(q_h) := \mathscr{R}(q_h) \setminus \{\Gamma_2, \Gamma_3\}$, for each $h = 0, 1, 2, 3$;

(ii) if q_1, q_2, q_3 are distinct, then: if q_0 is aligned to q_2 and q_3, $\Gamma_2 \cup \Gamma_3 = [\![q_2, q_3]\!]$, and $|q_2 - q_1| \leq |q_3 - q_1|$ (resp., $|q_2 - q_1| > |q_3 - q_1|$), we set $\mathscr{R}'(q_0) := \mathscr{R}(q_0) \setminus \{\Gamma_1, \Gamma_2\}$, $\mathscr{R}'(q_j) := \mathscr{R}(q_j) \cup \{[\![q_2, q_1]\!]\} \setminus \{\Gamma_j\}$, for $j = 1, 2$ (resp., $\mathscr{R}'(q_j) := \mathscr{R}(q_j) \cup \{[\![q_3, q_1]\!]\} \setminus \{\Gamma_j\}$, for $j = 1, 3$), and $\mathscr{R}'(q_3) := \mathscr{R}(q_3)$ (resp., $\mathscr{R}'(q_2) := \mathscr{R}(q_2)$); otherwise, we set $\mathscr{R}'(q_0) := \mathscr{R}(q_0) \setminus \{\Gamma_2, \Gamma_3\}$, $\mathscr{R}'(q_1) := \mathscr{R}(q_1)$, $\mathscr{R}'(q_2) := (\mathscr{R}(q_2) \cup \{[\![q_2, q_3]\!]\}) \setminus \{\Gamma_2\}$, and $\mathscr{R}'(q_3) := (\mathscr{R}(q_3) \cup \{[\![q_2, q_3]\!]\}) \setminus \{\Gamma_3\}$;

(iii) only two points among q_1, q_2, q_3 coincide. To fix ideas, suppose $q_2 = q_3$. Then, we set $\mathscr{R}'(q_h) := \mathscr{R}(q_h) \setminus \{\Gamma_2, \Gamma_3\}$ for each $h = 0, 2, 3$, and $\mathscr{R}'(q_1) := \mathscr{R}(q_1)$.

For each point (if any) $q \in S \setminus \{q_0, q_1, q_2, q_3\}$, we set $\mathscr{R}'(q) := \mathscr{R}(q)$. In all cases,

- the number of elements of $\mathscr{R}'(q_0)$ is less than the number of elements of $\mathscr{R}(q_0)$,
- the number of elements of $\mathscr{R}'(q)$ is less than or equal to the number of elements of $\mathscr{R}(q)$ for $q \neq q_0$,
- $\mathscr{R}'(q)$ has an odd number of elements, for any $q \in S$.

Moreover, by the triangular inequality, the sum of the lengths of the curves of $\mathscr{R}' := \cup_{q \in S} \mathscr{R}'(q)$ is not increased.

Now, suppose that there exists $q_0' \in S$ such that $\mathscr{R}'(q_0')$ contains at least three distinct curves connecting q_0' to some point q_1', q_2', q_3' of S. Then, we can repeat the arguments (i)-(ii)-(iii) replacing q_j by q_j', and $\mathscr{R}(q_j)$ by $\mathscr{R}'(q_j')$, for $j = 0, \ldots, 4$, obtaining a new collection of families $\mathscr{R}''(\cdot)$.

For $i \in \mathbb{N} \cup \{0\}$, set $\mathscr{R}^{(0)}(\cdot) := \mathscr{R}(\cdot)$, $\mathscr{R}^{(i)}(\cdot) := \big(\mathscr{R}^{(i-1)}(\cdot)\big)'$, $\mathscr{R}^{(i)} := \cup_{q \in S} \mathscr{R}^{(i)}(q)$ for $i \in \{1, \ldots, m\}$. By construction, we can perform the above reasoning a finite number $m \in \mathbb{N} \cup \{0\}$ of times, until $\mathscr{R}^{(m)}(q)$ consists of one curve only, for all $q \in S$. In particular, $\mathscr{R}^{(m)} \in \mathrm{Cuts}(\Omega, S)$, which implies

$$\mathcal{H}^1\Big(\bigcup_{\Gamma \in \mathscr{R}^{(m)}} \Gamma\Big) \leq \mathcal{H}^1\Big(\bigcup_{\Gamma \in \mathscr{R}} \Gamma\Big) \leq \mathcal{H}^1(\pi_{\mathbf{\Sigma}, M}(J_{u_{\min}})) \leq \mathcal{H}^1\Big(\bigcup_{\Gamma \in \mathscr{R}^{(m)}} \Gamma\Big),$$

where the second inequality follows by the inclusion $\bigcup_{\Gamma \in \mathscr{R}} \Gamma \subseteq \pi_{\mathbf{\Sigma}, M}(J_{u_{\min}})$, while the last inequality follows from (2.10). Therefore, the total length is not strictly decreased during the algorithm. This happens if and only if only $m = 0$, and this proves step 4.

Steps 3 and 4 imply that $\mathscr{R}$ consists of k curves connecting the points of S. Therefore, by definition of Σ^{opt}, we have

$$\mathcal{H}^1(\Sigma^{\mathrm{opt}}) \leq \mathcal{H}^1\Big(\bigcup_{\Gamma \in \mathscr{R}} \Gamma\Big). \tag{3.12}$$

Conversely, observe that, by (3.4), $\Sigma^{\mathrm{opt}} \subset \Omega$, so that

$$\mathcal{H}^1\Big(\bigcup_{\Gamma \in \mathscr{R}} \Gamma\Big) \leq \mathcal{H}^1(\pi_{\mathbf{\Sigma}, M}(J_{u_{\min}})) \leq \mathcal{H}^1(\Sigma^{\mathrm{opt}}), \tag{3.13}$$

where the last inequality follows recalling (3.3) and (2.10).

Coupling (3.13) with (3.12), we get the statement.

□

As in the case $k = 1$, we can drop assumption (3.4), setting the problem in the whole plane. We skip the proof, since it is the same as that in Corollary 3.2.

Corollary 3.4. $\mathscr{A}^{\mathbb{R}^2}_{\mathrm{constr}}(S) = 4\,\mathcal{H}^1(\Sigma^{\mathrm{opt}})$.

4 Constrained double covering solutions for the Plateau's problem

Let $S \subset \mathbb{R}^3$ be a tame link. Let $\Omega \subset \mathbb{R}^3$ be bounded with Lipschitz boundary, and $\mathbf{\Sigma} \in \mathbf{Cuts}(\Omega, S)$. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (1.20). By Theorem 1.22, $\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ is an embedded analytic surface in M. We ask now whether $\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}})$ coincides with S (compare with (1.37)). To this aim, we need an assumption, analogous to (3.1), in order to avoid components of $\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})}$ touching $\partial\Omega$; roughly speaking, we have to show that "long thin" parts reaching the boundary of Ω cannot occur in a constrained double−covering solution.

Theorem 4.1 (**Attaining the boundary condition**). *Let $2 \le n \le 7$. Let $\bar r > 0$ be such that $S \subset B_{\bar r}$. There exists $R > \bar r$ such that, if $\Omega \supset B_R$, then any minimizer $u_{\min} \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ of problem* (1.20) *satisfies*

$$\overline{\pi_{\mathbf{\Sigma},M}(J_{u_{\min}})} \setminus \pi_{\mathbf{\Sigma},M}(J_{u_{\min}}) = S. \tag{4.1}$$

Proof. Fix $\mathbf{\Sigma} \in \mathrm{Cuts}(B_{\bar r}, S) \subset \mathrm{Cuts}(\Omega, S)$. Set

$$\mathscr{A}_{\mathrm{constr}}(r) := \mathscr{A}^{B_r}_{\mathrm{constr}}(S), \qquad r \ge \bar r,$$

and let $u_r \in BV_{\mathrm{constr}}(Y^r_{\mathbf{\Sigma}}; \{\pm 1\})$ be a minimizer of problem (1.20) for $\Omega = B_r$; here, $Y^r_{\mathbf{\Sigma}}$ denotes the double covering space of the base set $B_r \setminus S$.

By (1.23), $\mathscr{A}_{\mathrm{constr}}(\cdot)$ is nondecreasing; in addition, it is bounded (see (2.10)). Set $\epsilon := 4\,\mathcal{H}^{n-1}(\Sigma_1) - \mathscr{A}_{\mathrm{constr}}(\bar r) \ge 0$, so that by (1.30),

$$\mathscr{A}_{\mathrm{constr}}(r) - \mathscr{A}_{\mathrm{constr}}(\bar r) \le \epsilon, \qquad r \ge \bar r. \tag{4.2}$$

Write $\mathscr{A}_{\mathrm{constr}}(r) = 4\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar r}) + 4t(r, \bar r)$, where $t(r, \bar r) := \mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \setminus B_{\bar r})$. Since $\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar r}$ is a competitor for the computation of $\mathscr{A}_{\mathrm{constr}}(\bar r)$, we have

$$\begin{aligned} \mathscr{A}_{\mathrm{constr}}(\bar r) &\le 4\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar r}) \\ &\le 4\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_{\bar r}) + 4t(r, \bar r) = \mathscr{A}_{\mathrm{constr}}(r). \end{aligned} \tag{4.3}$$

Coupling (4.2) and (4.3), we get

$$4t(r, \bar r) \le \mathscr{A}_{\mathrm{constr}}(r) - \mathscr{A}_{\mathrm{constr}}(\bar r) \le \epsilon, \qquad r \ge \bar r. \tag{4.4}$$

Suppose $\epsilon = 0$. Then, by (4.4), we have $\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \setminus B_{\bar r}) = 0$, which, by the assumption $2 \le n \le 7$ and Theorem 1.22, implies that the constrained double−covering solution does not reach ∂B_r, for any $r > \bar r$. We have the statement, taking an arbitrary $R > \bar r$.

Suppose $\epsilon > 0$, and let $r > \bar r$ be such that $\left(\overline{\pi_{\mathbf{\Sigma},M}(J_{u_r})} \setminus B_{\bar r}\right) \cap \partial B_r \neq \emptyset$. By the assumption $2 \le n \le 7$ and Theorem 1.22, there exists $x \in (\pi_{\mathbf{\Sigma},M}(J_{u_r}) \setminus B_{\bar r}) \cap$

$\partial B_{(r+\bar{r})/2}$. Take $\delta \in (0, (r-\bar{r})/2)$. By the lower density estimate for local minimizers of the perimeter functional (see for instance [17, Theorem 21.11]), we have

$$c_n \delta^{n-1} \leq 4\mathcal{H}^{n-1}(\pi_{\mathbf{\Sigma},M}(J_{u_r}) \cap B_\delta(x)) \leq 4t(r,\bar{r}) \leq \epsilon, \tag{4.5}$$

for some positive constant c_n depending only on n. Notice that (4.5) has to hold for each $\delta \in (0, (r-\bar{r})/2)$. This is possible only if $r \leq r_\epsilon := \bar{r} + 2(\epsilon/c_n)^{\frac{1}{n-1}}$. Hence, taking $R > r_\epsilon$, the proof is completed. □

Now, we compare the constrained double−covering solutions with other classical notions of solutions to Plateau's problem.

Remark 4.2 (**Area-minimizing currents**). Let $n = 3$, and assume that Ω contains the convex envelope of S. Let $T_{\min}$ be a rectifiable two-current [11] solving Plateau's problem with boundary S in the sense of currents. By [19, Theorem 5.6], the support of $T_{\min}$ is contained in Ω. Moreover, by [14], it is an embedded, orientable smooth surface $\Sigma_{\min} \subset \Omega$ up to the boundary S. In particular, $\Sigma_{\min} \in \mathrm{Cuts}(\Omega, S)$. Hence, by (2.10)

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \leq 4\,\mathbf{M}(T_{\min}), \tag{4.6}$$

where $\mathbf{M}(T_{\min})$ is the mass of $T_{\min}$.

It is worth noticing that there is not an absolute positive constant $c \in (0, 4]$, satisfying

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \geq c\,\mathbf{M}(T_{\min}) \tag{4.7}$$

for any S. As a counterexample, let $\widehat{B_1} := \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 < 1, x_3 = 0\}$, let $S := \partial\widehat{B_1}$, and, for $\epsilon > 0$, let $\Omega := (1+\epsilon)\widehat{B_1} \times (-2, 2)$. As admissible pair of cuts, we take as Σ_1 the closure of $\widehat{B_1}$, and $\Sigma_2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1^2 + x_2^2 \leq 1, \ x_3 = -\sqrt{1 - x_1^2 - x_2^2}\}$. Now, let $v \in BV(\Omega; \{\pm 1\})$ be defined as $v(x_1, x_2, x_3) := 1$ if $x_3 > 0$, and -1 elsewhere. Finally, let $u \in BV_{\mathrm{constr}}(Y_{\mathbf{\Sigma}}; \{\pm 1\})$ be the constrained lifting of v. Then, recalling (1.27), it is immediate to verify that

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \leq \frac{|Du|(Y_{\mathbf{\Sigma}})}{4} = \pi\big((1+\epsilon)^2 - 1\big) \to 0 \quad \text{as } \epsilon \to 0^+.$$

At the same time, the minimal mass in the sense of currents is π (the area of $\widehat{B}_1$), independently of ϵ.

Another (not rigorous but more intuitive) example of the lackness of inequality (4.7) can be obtained taking as S the boundary of a very thin Möbius band: in this case a surface similar to the Möbius band is expected to be the double covering solution with boundary S, while the support of the minimal current is expected to be approximately a double disk.

Remark 4.3 (**Disk-type area-minimizers**). Let $n = 3$ and suppose that S is connected. Since the right hand side of (4.6) is not larger than the area of a disk−type area-minimizing surface with boundary S (see [10]), we have

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \leq 4 \min\{\mathrm{area}(X(\hat{B}_1)) \ : \ X \in H^1(\hat{B}_1; \mathbb{R}^3), \ X(\partial \hat{B}_1) = S\}, \tag{4.8}$$

where $\hat{B}_1 \subset \mathbb{R}^2$ is the unit disk, and $\mathrm{area}(X(\hat{B}_1)) := \int_{\hat{B}_1} |\partial_{x_1} X \wedge \partial_{x_2} X|\, dx_1\, dx_2$. We observe that (4.8) can be obtained independently of (4.6), reproducing the proof of Theorems 2.1 and 4.4.

Now, we show that, when $n < 8$, constrained double−covering solutions give an equivalent way to solve Plateau's problem in the sense of integral currents modulo 2 [11].

Theorem 4.4 (**Area-minimizing integral currents mod** 2)**.** *Let $2 \le n \le 7$, and let Ω be as in Theorem 4.1. Let $u_{\min} \in BV_{\mathrm{constr}}(Y_{\boldsymbol{\Sigma}}; \{\pm 1\})$ be a minimizer of problem* (1.20)*. Then $\overline{\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}})}$ can be seen as an integral current modulo 2 with boundary S, and $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S)$ coincides with $4\mathbf{M}_2(T_{2,\min})$, where $\mathbf{M}_2$ is the mass and $T_{2,\min}$ is a mass-minimizing integral current modulo 2 having boundary S.*

Proof. By Theorems 1.22 and 4.1, $\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}})$ is an embedded analytic hypersurface satisfying (4.1). In particular, $\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}})$ can be considered as the support of an integral current modulo 2 having S as boundary support. This gives

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) = 4\mathcal{H}^{n-1}(\pi_{\boldsymbol{\Sigma},M}(J_{u_{\min}})) \ge 4\,\mathbf{M}_2(T_{2,\min}).$$

The converse inequality follows by the interior regularity of minimal integral currents modulo 2 (see [23, Theorem 6.2.1]). □

Remark 4.5. Let $n \ge 2$. Recalling Theorem 1.22 and Lemma 1.20, we have

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \ge 4 \inf \Big\{ \mathcal{H}^{n-1}(K) \;:\; K \subset M \text{ rel. closed, } K \cap \rho(\mathbb{S}^1) \ne \emptyset \\ \text{for every } \rho \in C(\mathbb{S}^1; M) \text{ such that } \mathrm{link}_2(\rho; S) = 1 \Big\}. \tag{4.9}$$

In particular,

$$\mathscr{A}^{\Omega}_{\mathrm{constr}}(S) \ge 4 \inf \Big\{ \mathcal{H}^{n-1}(K) \;:\; K \subset M \text{ rel. closed, } K \cap \rho(\mathbb{S}^1) \ne \emptyset \\ \text{for every simple link } \rho \Big\} =: m(S), \tag{4.10}$$

where, according to [12, p.4], a loop $\rho \in C(\mathbb{S}^1; M)$ is said to be a *simple link* if there is a connected component C of S such that $\mathrm{link}_2(\rho; C) = 1$, and $\mathrm{link}_2(\rho; C') = 0$ for all connected components C' of $S \setminus C$. The right hand side of (4.10) has been recently investigated, in [12] and [9], for more general choices of S.

We notice that, in general, we cannot expect the inequality in (4.10) to be an equality. A counterexample, in dimension $n = 2$, is obtained taking S as the set of (six) vertices of two triangles, as in Figure 5. Then, $m(S)$ is attained by the union of G_1, G_2, the two Steiner graphs corresponding to the triangles. On the other hand, by Proposition 3.3, $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S)$ is strictly larger than $\mathcal{H}^1(G_1) + \mathcal{H}^1(G_2)$.

A Appendix: isometry between $Y_{\boldsymbol{\Sigma}}$ and an abstract covering space

In this appendix we set the minimization problem using an alternative construction of a covering of M. The construction is standard (see, e.g., [13], [16]), and has the advantage to avoid all issues about the definition of admissible cuts. Setting up the minimization problem on the covering space M_H below could have an independent interest; we have preferred to use the "cut and paste" construction (and next proving

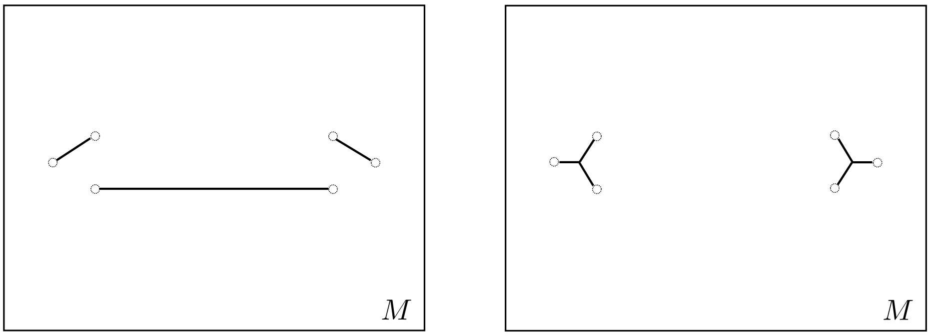

Figure 5: Let S be the set of vertices of two triangles, which are sufficiently far one from the other. In the left picture, the constrained double covering solution is shown, in the case $M = \mathbb{R}^2$. Notice that $\mathscr{A}^{\Omega}_{\mathrm{constr}}(S)$ is strictly larger than $m(S)$ (which is realized by the two Steiner graphs drawn in the right picture).

independence of the cuts) in order to deal with a more "handly" formula (like (1.27)) for the total variation of BV functions defined on the covering space.

Let $M := \Omega \setminus S$ and S be as in Section 1. Fix $x_0 \in M$, and set $C_{x_0}([0,1];M) := \{\gamma \in C\big([0,1];M\big) \ : \ \gamma(0) = x_0\}$. For $\gamma \in C_{x_0}([0,1];M)$, let $[\gamma]$ be the class of paths in $C_{x_0}([0,1];M)$ which are homotopic to γ with fixed endpoints. We recall that the universal covering of M is the pair $(\widetilde{M}, \mathfrak{p})$, where $\widetilde{M} := \big\{[\gamma] \ : \ \gamma \in C_{x_0}([0,1];M)\big\}$ and $\mathfrak{p}\colon [\gamma] \in \widetilde{M} \mapsto \mathfrak{p}([\gamma]) := \gamma(1) \in M$. A basis for the topology of $\widetilde{M}$ is given by the family $\{[\gamma\lambda] : [\gamma] \in \widetilde{M}, B \text{ open ball} \ni \gamma(1), \lambda \in C([0,1];B),\ \lambda(0) = \gamma(1)\}$.

Let $\pi_1(M, x_0)$ be the first fundamental group of M with base point $x_0 \in M$, and let

$$H := \{[\rho] \in \pi_1(M, x_0) \ : \ \mathrm{link}_2(\rho; S) = 0\}.$$

Remark A.1. H is a (normal) subgroup of $\pi_1(M, x_0)$ of index two. Moreover, H is independent of the orientation of S.

For $\gamma \in C_{x_0}([0,1];M)$, set $\bar{\gamma}(t) := \gamma(1-t)$ for all $t \in [0,1]$. Associated with H, we can consider the following equivalence relation $\sim_H$ on $\widetilde{M}$: for $[\gamma], [\lambda] \in \widetilde{M}$,

$$[\gamma] \sim_H [\lambda] \iff \gamma(1) = \lambda(1), \quad \mathrm{link}_2(\gamma\bar{\lambda}; S) = 0.$$

We denote by $[\gamma]_H$ the equivalence class of $[\gamma] \in \widetilde{M}$ induced by $\sim_H$, and we set

$$M_H := \widetilde{M} / \sim_H .$$

Letting $\widetilde{\mathfrak{p}}_H \colon \widetilde{M} \to M_H$ be the projection induced by $\sim_H$, we endow M_H with the corresponding quotient topology. We set $\mathfrak{p}_{H,M} \colon [\gamma]_H \in M_H \mapsto \gamma(1) \in M$, so that we have the following commutative diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\ \widetilde{\mathfrak{p}}_H\ } & M_H \\ & \searrow^{\mathfrak{p}} & \downarrow{\scriptstyle \mathfrak{p}_{H,M}} \\ & & M \end{array} \tag{A.1}$$

and the pair $(M_H, \mathfrak{p}_{H,M})$ is a covering of M, see [13, Proposition 1.36].

Let (Y, π_Y) be a covering of M, and let $y_0 \in \pi_Y^{-1}(x_0)$. By $(\pi_Y)_* \colon \pi_1(Y, y_0) \to \pi_1(M, x_0)$ we denote the homomorphism defined as $(\pi_Y)_*([\varrho]) := [\pi_Y \circ \varrho]$. By [13, Proposition 1.36], we have

$$(\mathfrak{p}_{H,M})_*(\pi_1(M_H, [x_0]_H)) = H. \tag{A.2}$$

Proposition A.2. *Let* $\boldsymbol{\Sigma} \in \mathrm{Cuts}(\Omega, S)$. *Then* $Y_{\boldsymbol{\Sigma}}$ *and* M_H *are homeomorphic.*

Proof. By [13, p. 28], we can assume that $x_0 \notin \Sigma_1 \cup \Sigma_2$. Now, let $[\varrho] \in \pi_1(Y_{\boldsymbol{\Sigma}}, y_0)$. Then, $[\varrho]$ changes sheet in $Y_{\boldsymbol{\Sigma}}$ an even (or zero) number of times; therefore, assuming without loss of generality ϱ of class C^1 and transverse to Σ_1, recalling also (1.1), we have

$$0 \equiv \mathcal{H}^0\big((\pi_M \circ \varrho)^{-1}(\Sigma_1)\big) \equiv \mathrm{link}_2(\pi_M \circ \varrho; S) \ (\mathrm{mod}\ 2),$$

which implies $\pi_M \circ \varrho \in H$. Hence, $(\pi_M)_*\big(\pi_1(Y_{\boldsymbol{\Sigma}}, y_0)\big) \le H$, and since H and $(\pi_M)_*\big(\pi_1(Y_{\boldsymbol{\Sigma}}, y_0)\big)$ have the same index, they must coincide. Recalling (A.2), we have

$$(\mathfrak{p}_{H,M})_*(\pi_1(M_H, [x_0]_H)) = (\pi_M)_*\big(\pi_1(Y_{\boldsymbol{\Sigma}}, y_0)\big).$$

By [13, Proposition 1.36], the proof is complete. □

The homeomorphism between the two covering spaces, which we denote by

$$f_{\boldsymbol{\Sigma}} \colon M_H \to Y_{\boldsymbol{\Sigma}}, \tag{A.3}$$

is given for instance in [13, Proposition 1.33]: for $[\gamma]_H \in M_H$, let $\beta \in C([0,1]; M_H)$ be a path from $[x_0]_H$ to $[\gamma]_H$; we uniquely lift $\mathfrak{p}_{H,M} \circ \beta$ to a path on $Y_{\boldsymbol{\Sigma}}$ with base point y_0.[21] Then, $f_{\boldsymbol{\Sigma}}([\gamma]_H)$ is defined as the endpoint of the lifted path, which turns out to be independent of β.

Now, we define the distance d_{M_H} on M_H as follows: for $[\gamma]_H$, $[\lambda]_H \in M_H$,

$$d_{M_H}([\gamma]_H, [\lambda]_H) := \inf_{\beta} \sup \Big\{ \sum_l |\mathfrak{p}_{H,M}(\beta(t_l)) - \mathfrak{p}_{H,M}(\beta(t_{l-1}))| \ : \ (t_l)_l \in \mathrm{Part}(\beta) \Big\}, \tag{A.4}$$

where the infimum runs among all $\beta \in C([0,1]; M_H)$ connecting $[\gamma]_H$ and $[\lambda]_H$; for any such β, $\mathrm{Part}(\beta)$ denotes the collection of all finite partitions $(t_l)_l$ of $[0,1]$ such that, for every l, there exists $([\gamma_l], B_l) \in \widetilde{\mathcal{U}}$ such that $\beta([t_{l-1}, t_l]) \subset \widetilde{\mathfrak{p}}_H(([\gamma_l], B_l))$.

For the sake of completeness, let us check that d_{M_H} is a distance. Symmetry, positivity, and triangular inequality are direct consequences of the definition. Let us show that $d_{M_H}([\gamma]_H, [\lambda]_H) = 0$ implies $[\gamma]_H = [\lambda]_H$. Clearly, we have $\gamma(1) = \lambda(1)$. Fix $\epsilon > 0$, and let $\beta \in C([0,1], M_H)$, $N \in \mathbb{N}$, $(t_l)_l \in \mathrm{Part}(\beta)$, $l \le N$, be such that $\sum_{l=1}^N |\mathfrak{p}_{H,M}(\beta(t_l)) - \mathfrak{p}_{H,M}(\beta(t_{l-1}))| \le \epsilon$. In particular, for $\epsilon > 0$ sufficiently small, the closed curve ρ defined as[22]

$$\rho := [\![\gamma(1), \mathfrak{p}_{H,M}(\beta(t_1))]\!] \cdots [\![\mathfrak{p}_{H,M}(\beta(t_{N-1})), \lambda(1)]\!]$$

is contractible in M, which implies that

$$\mathrm{link}_2(\rho; S) = 0. \tag{A.5}$$

[21] For instance, $\beta(t) := [\gamma_{|[0,t]} \gamma(t)_{|[t,1]}]_H$, for $t \in [0,1]$; with this choice, $\mathfrak{p}_{H,M} \circ \beta = \gamma$.

[22] With slight abuse of notation (compare Section 3.1), here by $[\![x, x']\!]$ we mean the path corresponding to the segment from x to x', for every x, $x' \in M$.

By definition of $\mathrm{Part}(\beta)$, for every $l \leq N$ there exist $\lambda_{l,1},\ \lambda_{l,2} \in C([0,1];B_l)$, with $\lambda_{l,1}(0) = \lambda_{l,2}(0) = \gamma_l(1)$, and such that $\beta(t_{l-1}) = [\gamma_l\lambda_{l,1}]_H$, $\beta(t_l) = [\gamma_l\lambda_{l,2}]_H$; notice that, since $[\gamma_{l-1}\lambda_{l-1,2}]_H = \beta(t_{l-1}) = [\gamma_l\lambda_{l,1}]_H$, we have

$$\mathrm{link}_2(\gamma_{l-1}\lambda_{l-1,2}\bar{\lambda}_{l,1}\bar{\gamma}_l; S) = 0, \quad l \leq N. \tag{A.6}$$

For every $l \leq N$, let us set $\rho_l := \gamma_l\lambda_{l,1}[\![\lambda_{l,1}(1), \lambda_{l,2}(1)]\!]\bar{\lambda}_{l,2}\bar{\gamma}_l$, which is a closed curve in M. In particular,

$$\mathrm{link}_2(\rho_l; S) = \mathrm{link}_2(\lambda_{l,1}[\![\lambda_{l,1}(1), \lambda_{l,2}(1)]\!]\bar{\lambda}_{l,2}; S) = 0, \tag{A.7}$$

where last equality follows recalling that B_l is contractible in M.

Coupling (A.5), (A.6), and (A.7), we finally get

$$\begin{aligned}\mathrm{link}_2(\gamma\bar{\lambda}; S) =& \mathrm{link}_2(\gamma_0\lambda_{0,1}\bar{\lambda}_{N,2}\bar{\lambda}; S)\\ =& \sum_{l=1}^{N}\Big(\mathrm{link}_2(\rho_l; S) + \mathrm{link}_2(\gamma_{l-1}\lambda_{l-1,2}\bar{\lambda}_{l,1}\bar{\gamma}_l; S)\Big) + \mathrm{link}_2(\rho; S) = 0.\end{aligned}$$

Hence, $[\gamma] \sim_H [\lambda]$, and the conclusion follows.

Now, we are in the position to establish the isometry bewteen the two covering spaces. We endow $Y_{\mathbf{\Sigma}}$ with the distance $d_{Y_{\mathbf{\Sigma}}}$ defined as follows: for any $y,\ y' \in Y_{\mathbf{\Sigma}}$, we set

$$d_{Y_{\mathbf{\Sigma}}}(y, y') = \inf_{\eta}\ \sup\Big\{\sum_l |\pi_M(\eta(t_l)) - \pi_M(\eta(t_{l-1}))|\ :\ (t_l)_l \in \mathrm{Part}(\eta)\Big\}, \tag{A.8}$$

where the infimum runs among all $\eta \in C([0,1]; Y_{\mathbf{\Sigma}})$ connecting y and y', and $\mathrm{Part}(\eta)$ is the family of all finite partitions $(t_l)_l$ of $[0,1]$ such that, for every l, $\eta([t_{l-1}, t_l])$ is contained in a single chart of $Y_{\mathbf{\Sigma}}$.

Proposition A.3 (Isometry). *The map $f_{\mathbf{\Sigma}}$ in* (A.3) *is an isometry between (M_H, d_{M_H}) and $(Y_{\mathbf{\Sigma}}, d_{Y_{\mathbf{\Sigma}}})$.*

Proof. Let $[\gamma]_H,\ [\lambda]_H \in M_H$. For $\epsilon > 0$, let $\beta \in C([0,1]; M_H)$ be a path from $[\gamma]_H$, $[\lambda]_H$, realizing the infimum in (A.4) up to a contribution of order ϵ. Now, set $\eta := f_{\mathbf{\Sigma}} \circ \beta$; accordingly to (A.8), let $(t_l)_l \in \mathrm{Part}(\eta)$ such that

$$d_{Y_{\mathbf{\Sigma}}}(f_{\mathbf{\Sigma}}([\gamma]_H), f_{\mathbf{\Sigma}}([\lambda]_H)) \leq \sum_l |\pi_M(\eta(t_l)) - \pi_M(\eta(t_{l-1}))| + \epsilon.$$

Clearly, it is not restrictive to assume that, for every l, $\pi_M(\eta([t_{l-1}, t_l])) \subset B_l$, for some open ball $B_l \subset M$. Therefore, accordingly to (A.4), we have $(t_l)_l \in \mathrm{Part}(\beta)$; hence, for every l,

$$|\pi_M(\eta(t_l)) - \pi_M(\eta(t_{l-1}))| = |\mathfrak{p}_{H,M}(\beta(t_l)) - \mathfrak{p}_{H,M}(\beta(t_{l-1}))|,$$

which implies

$$d_{Y_{\mathbf{\Sigma}}}(f_{\mathbf{\Sigma}}([\gamma]_H), f_{\mathbf{\Sigma}}([\lambda]_H)) \leq d_{M_H}([\gamma]_H, [\lambda]_H) + 2\epsilon.$$

By the arbitrariness of ϵ, we get $d_{Y_{\mathbf{\Sigma}}}(f_{\mathbf{\Sigma}}([\gamma]_H), f_{\mathbf{\Sigma}}([\lambda]_H)) \leq d_{M_H}([\gamma]_H, [\lambda]_H)$. Similarly, we get the converse inequality. ☐

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