

ON A PHASE–FIELD MODEL OF DAMAGE FOR HYBRID LAMINATES WITH COHESIVE INTERFACE

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ABSTRACT. In this paper we investigate a rate–independent model for hybrid laminates described by a damage phase–field approach on two layers coupled with a cohesive law governing the behaviour of their interface. For the analysis we adopt the notion of energetic evolution, based on global minimisation of the involved energy. Due to the presence of the cohesive zone, as already emerged in literature, compactness mathematical issues lead to the introduction of a fictitious variable replacing the physical one which represents the maximal opening of the interface displacement discontinuity reached during the evolution. A new strategy which allows to recover the equivalence between the fictitious and the real variable under general loading–unloading regimes is illustrated. The argument is based on temporal regularity of energetic evolutions. This regularity is achieved by means of a careful balance between the convexity of the elastic energy of the layers and the natural concavity of the cohesive energy of the interface.

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INTRODUCTION

Composite fibre reinforced materials are increasingly finding applications in the manufacturing industry due to their capacity of offering high strength and stiffness with low mass density. Their only mechanical weakness is the brittleness. Indeed, rapid failure occurs without sufficient warning, due to the intrinsic nature of the adopted materials. A possible strategy to provide a ductile failure response is to consider novel composite architectures where fibres of different

stiffness and ultimate strain values are combined through cohesive interfaces (hybridisation). In this case, complex rupture processes occur with diffuse crack pattern (fragmentation) and/or delamination. A deep analytical comprehension of the failure mechanisms of these kind of materials is thus needed in order to predict and control the appearance and the evolution of the cracks.

Among the mathematical community, the variational approach to fracture, as formulated by [12, 19], is one of the most adopted viewpoint to deal with crack problems. It is based on the Griffith's idea [26] that the crack growth is governed by a reciprocal competition between the internal elastic energy of the body and the energy spent to increase the crack length. In the original theory the energy associated with the fracture is proportional to the measure of the fracture itself, while in the cohesive case (Barenblatt [6]), where the process is more gradual, the energy depends on the opening of the crack.

Due to the complexity of the phenomenon and the technical difficulties of the related mathematical analysis, especially from the numerical point of view, in the last twenty years a damage phase-field approach has been developed to overcome the aforementioned issues. Nowadays it is a well established and consolidated method to approximate both brittle (see [4, 5, 24]) and cohesive fractures (see [13, 28]). It consists in the introduction of an irreversible internal variable taking values in $[0, 1]$ and representing the damage state of the material. Usually, values 0 and 1 mean a completely sound and a completely broken state, respectively, while a value in between represents the case of a partial damage. The presence of a fracture is thus ideally replaced by those parts of the body whose damage variable has reached the value 1.

In this work a rigorous mathematical analysis of a one-dimensional model for hybrid laminates introduced and numerically investigated in [3] is proposed. Its description is given by coupling the damage phase-field approach, which models the elastic-brittle behaviour of the layers, with a cohesive law in the interface connecting the materials. The investigation is restricted to the case of incomplete damage in the sense that a reservoir of elastic material stiffness is always maintained, even if the damage variable reaches the maximum value 1. This situation can be concretely justified by considering materials formed by different components from which only a part can undergo a damage (for instance in composite materials obtained with a matrix and a reinforcement) and delamination may take place; on the other side it can be seen as a mathematical approximation of the complete damage setting in which the material goes through full rupture. We refer for instance to [11, 30] for an analysis of complete damage between two viscoelastic bodies, while we postpone the inspection of this model to future works, due to high mathematical difficulties.

Here, the model we want to analyse describes the evolution of a unidirectional hybrid laminate in hard device: a prescribed time-dependent displacement $\bar{u}(t)$ is applied on one side of the bar, whereas the other is fixed. We restrict our attention to slow prescribed displacements, so that inertial effects can be neglected and the analysis can be included in a quasi-static and rate-independent regime. For the sake of simplicity we consider a bar composed by only two layers with thickness ρ_1 and ρ_2 , respectively, bonded together along the entire length by a cohesive interface. The thickness of the interface is very thin compared with ρ_1 and ρ_2 , which in turn are way smaller than the length of the laminate $L > 0$. Thus the model can be considered as one-dimensional.

The brittle behaviour of the two elastic layers undergoes a damage phase-field approach. It suits with the rate-independent framework we are considering. For the reader interested instead in dynamic and rate-dependent damage models we refer for instance to [14, 21]. The unknowns that govern the problem are thus the displacements of the two layers, denoted by u_1 and u_2 , and their irreversible damage variables α_1 and α_2 .

In the quasi-static setting a huge variety of notions of solution can be considered, see for instance the monograph [31]. In this paper we focus our attention on the concept of energetic evolution, based on two ingredients: at every time the solution is a global minimiser of the involved total energy, and the sum of internal and dissipated energy balances the work done by the external prescribed displacement. The same kind of evolution in an analogous cohesive fracture model between two elastic bodies is studied in [16, 18]; other notions based on stationary points of the energy, always in the framework of cohesive fractures, are instead analysed in [34, 35].

The total energy we consider is composed by a first part taking into account elastic responses of the layers and dissipation due to damage, and a second part reflecting the cohesive behaviour of the interface. The cohesive interface is governed by the slip between the two layers $\delta = |u_1 - u_2|$ and its irreversible counterpart δ_h which represents the maximal slip achieved during the evolution. The presence of an irreversible historical variable can be also found in different models than cohesive fracture: we mention for instance the notion of fatigue, investigated in [1, 17].

The expression of the energy in the model under consideration is hence given by:

$$\sum_{i=1}^2 \rho_i \left(\underbrace{\frac{1}{2} \int_0^L E_i(\alpha_i(x)) (u_i'(x))^2 dx}_{\text{elastic energy of the } i\text{-th layer}} + \underbrace{\frac{1}{2} \int_0^L (\alpha_i'(x))^2 dx}_{\text{internal energy of the } i\text{-th damage variable}} + \underbrace{\int_0^L w_i(\alpha_i(x)) dx}_{\text{energy dissipated by damage in the } i\text{-th layer}} \right) + \underbrace{\int_0^L \varphi(\delta(x), \delta_h(x)) dx}_{\text{internal and dissipated energy in the interface}}$$

where the symbol prime $'$ denotes the one-dimensional spatial derivative, $E_i: [0, 1] \rightarrow (0, +\infty)$ is the elastic Young modulus of the i -th layer (which is strictly positive since we are in the incomplete damage framework), $w_i: [0, 1] \rightarrow [0, +\infty)$ is a dissipation density and $\varphi: \{(y, z) \in \mathbb{R}^2 \mid z \geq y \geq 0\} \rightarrow [0, +\infty)$ is the loading-unloading density of the cohesive interface.

As usual in the context of energetic evolutions, we follow a time-discretisation algorithm to show existence of solutions. More precisely, we consider a fine partition of the time interval $[0, T]$ and at each time step we select a global minimiser of the total energy; we then recover the time-continuous evolution by sending to zero the discretisation parameter. Due to compactness issues regarding the maximal slip δ_h , the time-discretisation process leads to the introduction of a weaker notion of solution where a fictitious historical variable γ replaces the concrete one δ_h . We point out that this fictitious device only appears when dealing with global minima of the energy, indeed it can be found in [16, 18], but not in [34, 35] where stationary points are considered. The issue has been partially overcome in [16, 18] with different approaches, but assuming the hypothesis of constant unloading response, namely when the loading-unloading density φ depends only on the second variable z .

Here, an original strategy based on temporal regularity properties of energetic evolutions in order to recover the equivalence between the fictitious variable γ and the proper one δ_h under reasonable assumptions on the density φ is illustrated. In particular, we are able to cover all the general cases considered in [34]. Moreover, the proposed approach fits well with the model under consideration, but it can be also adapted to more general situations.

An alternative strategy to deal with cohesive problems can be found in literature, where adhesion is treated with the introduction of a damage variable that macroscopically defines the bond state between two solids. Detachment corresponds to full damage state. The problem has been investigated theoretically in [7, 8, 9, 10] and numerically in [22, 23]

The paper is organised as follows. In Section 1 we introduce in a rigorous way the variational problem, presenting the global and historical variables: the displacement field u_i , the damage variables α_i , the slip δ and the historical slip δ_h . Subsequently, details of the involved energies

are given together with a precise notion of energetic evolution and of its weak counterpart, here named generalised energetic evolution, including the fictitious variable γ .

Section 2 is devoted to the proof of existence of generalised energetic evolutions under very mild assumptions on the loading–unloading cohesive density φ . We first introduce the time–discretisation algorithm based on global minimisation of the energy, and we provide uniform bounds on the sequence of discrete minimisers. Thanks to these bounds and by means of a suitable version of Helly’s selection theorem we are able to extract convergent subsequences as the time step vanishes. After the introduction of the fictitious historical variable γ and by exploiting the fact that the discrete functions selected by the algorithm are global minima of the total energy, we finally deduce that the previously obtained limit functions actually are a generalised energetic evolution.

In Section 3 attention is focused on the equations that a generalised energetic evolution must satisfy; they are a byproduct of the global minimality condition together with the energy balance. It turns out that the displacements fulfil a system of equations in divergence form, see (3.1a), while the damages satisfy a Karush–Kuhn–Tucker condition, see (3.1b), assuming a priori certain regularity in time. Of course these equations have to be meant in a weak sense. The results of this third section are a first step in order to obtain the equivalence between γ and the concrete historical variable δ_h .

Finally Section 4 illustrates the main result of the paper. We first adapt a convexity argument introduced in [32] to our setting in which a cohesive energy (concave by nature) is present, in order to gain regularity in time (absolute continuity) of generalised energetic evolutions. Once this temporal regularity is achieved, we exploit the Euler–Lagrange equations of Section 3 together with the monotonicity (in time) of γ and δ_h to deduce their equivalence under reasonable assumptions on φ . We thus obtain as a byproduct that the generalised energetic evolution found in Section 2 is actually an energetic evolution, since γ coincides with δ_h .

At the end of the work we attach an Appendix in which we gather some definitions and properties we need throughout the paper about absolutely continuous and bounded variation functions with values in Banach spaces.

1. SETTING OF THE PROBLEM

In this section we present the variational formulation of the one-dimensional continuum model described in the Introduction of two layers bonded together by a cohesive interface in a hard device setup. We list all the main assumptions we need throughout the paper. We also introduce the two notions of energetic evolution and generalised energetic evolution in our context, see Definitions 1.5 and 1.8.

For the sake of clarity, in this work every function in the Sobolev space $H^1(a, b)$ is always identified with its continuous representative. The prime symbol $'$ is used to denote spatial derivatives, while the dot symbol $\dot{\cdot}$ to denote time derivatives. In the case of a function $f: [0, T] \rightarrow H^1(a, b)$, which thus depends on both time and space, we write $f(t)'$ to denote the (weak) spatial derivative of $f(t) \in H^1(a, b)$ and with a little abuse of notation we write $f'(t, x)$ to denote its value at a.e. $x \in [a, b]$. If f is sufficiently regular in time, for instance in $C^1([0, T]; H^1(a, b))$, for the time derivative we instead adopt the scripts \dot{f} , $\dot{f}(t)$ and $\dot{f}(t, x)$, with the obvious meanings: \dot{f} is the function from $[0, T]$ to $H^1(a, b)$, $\dot{f}(t)$ is its value as a function in $H^1(a, b)$, once $t \in [0, T]$ is fixed, and $\dot{f}(t, x)$ is its value (as a real number) at $x \in [a, b]$. By $a \vee b$ and $a \wedge b$ we finally mean the maximum and the minimum between two extended real numbers a and b in $[-\infty, +\infty]$.

We fix a time $T > 0$ and the length of the laminate $L > 0$. We also normalise the thickness of the two layers ρ_1 and ρ_2 to 1, since this does not affect the results.

1.1. The variables. To describe the evolution of the system, for $i = 1, 2$ we introduce the function $u_i: [0, T] \times [0, L] \rightarrow \mathbb{R}$, where $u_i(t, x)$ denotes the displacement at time t of the point x of the i -th layer; here $\mathbf{u}(t, x)$ represents the vector in \mathbb{R}^2 with components $u_1(t, x)$ and $u_2(t, x)$. For the structure of the model itself, at every time $t \in [0, T]$ the displacement $u_i(t)$ will belong to the space $H^1(0, L)$. The function $\delta: [0, T] \times [0, L] \rightarrow [0, +\infty)$ defined as

$$\delta(t, x) = \delta[\mathbf{u}](t, x) := |u_1(t, x) - u_2(t, x)|, \quad (1.1a)$$

instead denotes the displacement slip on the interface between the two layers. Then, we introduce the non-decreasing function $\delta_h: [0, T] \times [0, L] \rightarrow [0, +\infty)$ as

$$\delta_h(t, x) := \sup_{\tau \in [0, t]} \delta(\tau, x), \quad (1.1b)$$

namely the historical variable which records the maximal slip reached at the point x in the interface till the time t . Internal constraints, such as unilateral conditions (see [8, 9]), are not necessary on the kinematics as this only permits displacement slips between the two solids and interpenetration is prevented a-priori.

Finally, for $i = 1, 2$, we consider the function $\alpha_i: [0, T] \times [0, L] \rightarrow [0, 1]$, where $\alpha_i(t, x)$ represents the damage at time t of the point x of the i -th layer. It is non-decreasing in time with values in $[0, 1]$. The value 0 means completely sound material whereas the value 1 represents fully damage state. We however point out that we confine ourselves to the incomplete damage setting, namely the fully damage state does not describe the rupture of the layer, whose stiffness indeed never vanishes; this will be clear in (1.3), in which we assume a strictly positive elastic modulus for both layers. As for the displacement, the damage variable $\alpha_i(t)$ will be in $H^1(0, L)$ for every $t \in [0, T]$. In analogy with the previous setting, $\boldsymbol{\alpha}(t, x)$ denotes the vector in \mathbb{R}^2 with components $\alpha_1(t, x)$ and $\alpha_2(t, x)$.

1.2. The energies. We now present the energies involved in our model. Given a pair $(\mathbf{u}, \boldsymbol{\alpha})$ belonging to $[H^1(0, L)]^2 \times [H^1(0, L)]^2$ and representing an admissible displacement and damage, the stored elastic energy of the two layers is given by:

$$\mathcal{E}[\mathbf{u}, \boldsymbol{\alpha}] := \sum_{i=1}^2 \frac{1}{2} \int_0^L E_i(\alpha_i(x)) (u_i'(x))^2 dx, \quad (1.2)$$

where, for $i = 1, 2$, we assume the elastic Young moduli E_i satisfy:

$$E_i \in C^0([0, 1]) \text{ such that } E_i(y) \geq \min_{\tilde{y} \in [0, 1]} E_i(\tilde{y}) =: \varepsilon_i > 0, \text{ for every } y \in [0, 1]. \quad (1.3)$$

We define

$$\varepsilon := \varepsilon_1 \wedge \varepsilon_2 > 0, \quad (1.4)$$

which is strictly positive by (1.3). This feature reflects the fact that we are considering the incomplete damage framework, and it will be used to gain coercivity of \mathcal{E} . This property of the energy is indeed missing in the complete damage setting where the functions E_i can vanish, and a completely different notion of solution and strategy must be adopted. We refer to [11, 30] for the interested reader.

We can now introduce for $i = 1, 2$ the stress $\sigma_i: [0, T] \times [0, L] \rightarrow \mathbb{R}$, defined as

$$\sigma_i(t, x) = \sigma_i[u_i, \alpha_i](t, x) := E_i(\alpha_i(t, x)) u_i'(t, x). \quad (1.5)$$

As before, by $\boldsymbol{\sigma}(t, x)$ we mean the vector with components $\sigma_1(t, x)$ and $\sigma_2(t, x)$.

An other energy term appearing in the model is the sum of the stored and the dissipated energy of the phase-field variable $\boldsymbol{\alpha} \in [H^1(0, L)]^2$ during the damaging process and expressed

by

$$\mathcal{D}[\boldsymbol{\alpha}] := \sum_{i=1}^2 \left(\frac{1}{2} \int_0^L (\alpha'_i(x))^2 dx + \int_0^L w_i(\alpha_i(x)) dx \right). \quad (1.6)$$

In literature there are very different choices of dissipation functions w_i (see for instance [2, 3, 29, 33, 36, 38]). As a simple example we can consider $w_i(y) = \frac{y^2+y}{2}$.

In this work we permit quite general assumptions on w_i as follows:

$$w_i \in C^0([0, 1]) \text{ such that } w_i(y) \geq c_i y^2 \text{ for some } c_i > 0 \text{ and for every } y \in [0, 1]. \quad (1.7)$$

Remark 1.1. The dissipated damage density, usually a process dependent function (i.e. depending on the time derivative of the damage variable $\dot{\alpha}(t)$), is here treated as a state function due to the underlying gradient damage model. See also [2], [3] and [30].

We finally introduce the cohesive energy in the interface between the two layers:

$$\mathcal{K}[\delta, \gamma] := \int_0^L \varphi(\delta(x), \gamma(x)) dx, \quad (1.8)$$

where δ and γ are two non-negative functions in $[0, L]$ such that $\gamma \geq \delta$ and representing, respectively, the slip and the historical slip of the displacement at a given instant. The non-negative function

$$\varphi : \mathcal{T} \rightarrow [0, +\infty), \quad \text{where } \mathcal{T} = \{(y, z) \in \mathbb{R}^2 \mid z \geq y \geq 0\}, \quad (1.9)$$

is the loading-unloading density of the cohesive interface; the variable y governs the unloading regime (usually convex), while z the loading regime (usually concave).

Since several assumptions on φ will be needed throughout the paper we prefer listing them here. The first set of assumptions, very mild, will be used in Section 2 to prove existence of (generalised) energetic evolutions (see Definitions 1.5 and 1.8):

- ($\varphi 1$) φ is lower semicontinuous;
- ($\varphi 2$) $\varphi(0, \cdot)$ is bounded in $[0, +\infty)$;
- ($\varphi 3$) $\varphi(y, \cdot)$ is continuous and non-decreasing in $[y, +\infty)$, for every $y \geq 0$.

We also present here the specific assumption which has been used in [16] and [18] to deal with the fictitious variable γ (see Definition 1.8), and which in this work we are able to avoid. We however include it in the list because we make use of it in Theorem 2.12, where we employ the argument of [16] in our context:

- ($\varphi 4$) there exist two functions $\varphi_1, \varphi_2 : [0, +\infty) \rightarrow [0, +\infty)$ such that φ_1 is lower semicontinuous, φ_2 is bounded, non-decreasing and concave, and $\varphi(y, z) = \varphi_1(y) + \varphi_2(z)$.

Remark 1.2. Actually, in [16, 18] the function φ_1 appearing above is chosen identically 0, so that the cohesive density φ depends only on the second variable z (constant unloading regime). However, their argument can be easily adapted to our case.

To overcome the necessity of ($\varphi 4$), in Sections 3 and 4 we develop a new argument based on time regularity of solutions. The assumptions we need to perform the whole strategy are listed just below. For the sake of brevity, given a non-negative function φ with domain \mathcal{T} , for $z \in [0, +\infty)$ we define

$$\psi(z) := \varphi(z, z),$$

namely the restriction of φ on the diagonal. Moreover we introduce the constant

$$\bar{\delta} := \inf\{z > 0 \mid \psi \text{ is constant in } [z, +\infty)\}, \quad (1.10)$$

with the convention $\inf\{\emptyset\} = +\infty$; it represents the limit slip which triggers complete delamination. According to [2, 3], complete delamination may occur for finite or infinite slip value (see Remark 1.3).

We then set

$$\mathcal{T}_{\bar{\delta}} := \{(y, z) \in \mathcal{T} \mid z < \bar{\delta}\}.$$

We thus require:

($\varphi 5$) the function $\psi \in C^1([0, +\infty))$ is λ -convex for some $\lambda > 0$, namely for every $\theta \in [0, 1]$ and $z^a, z^b \in [0, +\infty)$ it holds

$$\psi(\theta z^a + (1 - \theta)z^b) \leq \theta\psi(z^a) + (1 - \theta)\psi(z^b) + \frac{\lambda}{2}\theta(1 - \theta)|z^a - z^b|^2;$$

($\varphi 6$) for every $z \in (0, +\infty)$ the map $\varphi(\cdot, z) \in C^1([0, z])$ is non-decreasing and convex;

($\varphi 7$) for every $z \in (0, +\infty)$ there hold $\partial_y \varphi(z, z) = \psi'(z)$ and $\partial_y \varphi(0, z) = 0$;

($\varphi 8$) the partial derivative $\partial_y \varphi$ belongs to $C^0(\mathcal{T} \setminus (0, 0))$ and it is bounded in \mathcal{T} .

($\varphi 9$) for every $y \in [0, \bar{\delta})$ the map $\varphi(y, \cdot)$ is differentiable in $[y, \bar{\delta})$ and the partial derivative $\partial_z \varphi$ is continuous and strictly positive on $\mathcal{T}_{\bar{\delta}} \setminus \{(z, z) \in \mathbb{R}^2 \mid z \geq 0\}$.

Condition ($\varphi 9$) will be actually weakened in Section 4, where only a uniform strict monotonicity with respect to z will be needed, see (4.17).

We want to point out that this set of assumptions includes a huge variety of mechanically meaningful loading–unloading densities φ , as precised in the next remark. We also notice that these conditions are similar to the one considered in [34].

Remark 1.3 (Main Example). The prototypical example of a physically meaningful loading–unloading density is obtained reasoning in the opposite way of what we presented before, namely firstly a function ψ is given and then the density φ is built from ψ . As regards ψ , which governs the loading regime, natural assumptions are the following: $\psi \in C^2([0, \bar{\delta})) \cap C^1([0, +\infty))$ is a non-decreasing, concave and bounded function such that $\psi(0) = 0$, $\psi' > 0$ and ψ'' is bounded from below in $[0, \bar{\delta})$. In particular ($\varphi 5$) is satisfied with $\lambda = \sup_{z \in [0, \bar{\delta})} |\psi''(z)|$. For instance one can consider:

$$\psi(z) = c(1 - e^{-kz}), \quad \psi(z) = \begin{cases} cz(2k - z), & \text{if } z \in [0, k), \\ ck^2, & \text{if } z \in [k, +\infty), \end{cases} \quad \text{for } c, k > 0.$$

In the first example $\bar{\delta} = +\infty$, while in the second one $\bar{\delta} = k < +\infty$.

The function φ is then defined by considering a quadratic unloading regime:

$$\varphi(y, z) := \begin{cases} \frac{1}{2} \frac{\psi'(z)}{z} y^2 + \psi(z) - \frac{1}{2} z \psi'(z), & \text{if } (y, z) \in \mathcal{T} \setminus (0, 0), \\ 0, & \text{if } (y, z) = (0, 0). \end{cases} \quad (1.11)$$

By construction φ is continuous on \mathcal{T} and ($\varphi 6$), ($\varphi 7$) and ($\varphi 8$) are satisfied. To verify also ($\varphi 9$) we notice that it holds:

$$\partial_z \varphi(y, z) = \frac{\psi'(z) - z\psi''(z)}{2} \left(1 - \frac{y^2}{z^2}\right), \quad \text{for every } (y, z) \in \mathcal{T}_{\bar{\delta}}.$$

Thus we deduce $\partial_z \varphi$ is continuous in $\mathcal{T}_{\bar{\delta}} \setminus (0, 0)$; if moreover $z > y$, since $\psi'(z)$ is strictly positive in $[0, \bar{\delta})$, we get that $\partial_z \varphi(y, z) > 0$ and so ($\varphi 9$) is fulfilled.

We finally observe that by the boundedness of ψ we also obtain ($\varphi 2$).

We now present a very simple lemma regarding the behaviour of φ in the case $\bar{\delta} < +\infty$.

Lemma 1.4. *Assume φ satisfies $(\varphi 5)$, $(\varphi 6)$ and $(\varphi 7)$ and assume $\bar{\delta}$ is finite. Then φ is constant in $\mathcal{T} \setminus \mathcal{T}_{\bar{\delta}}$, and in particular:*

$$\varphi(y, z) = \psi(\bar{\delta}), \text{ for every } (y, z) \in \mathcal{T} \setminus \mathcal{T}_{\bar{\delta}}.$$

Proof. Since ψ is $C^1[0, +\infty)$, then by definition of $\bar{\delta}$ it holds $\psi'(z) = 0$ for every $z \in [\bar{\delta}, +\infty)$. We now fix $z \in [\bar{\delta}, +\infty)$; by $(\varphi 7)$ we deduce that $\partial_y \varphi(z, z) = 0$. Condition $(\varphi 6)$ thus yields $\partial_y \varphi(y, z) = 0$ for every $y \in [0, z]$, and hence we conclude. \square

We finally introduce the function $\varphi_{\bar{\delta}}$, defined as:

$$\varphi_{\bar{\delta}}(y, z) := \varphi(y \wedge \bar{\delta}, z \wedge \bar{\delta}), \quad \text{for every } (y, z) \in \mathcal{T}.$$

Thanks to previous lemma, it is easy to deduce that if conditions $(\varphi 5)$, $(\varphi 6)$ and $(\varphi 7)$ are fulfilled, then actually φ and $\varphi_{\bar{\delta}}$ coincide, namely it holds:

$$\varphi(y, z) = \varphi_{\bar{\delta}}(y, z), \quad \text{for every } (y, z) \in \mathcal{T}. \quad (1.12)$$

This last equality will be widely exploited in Section 4.

1.3. Energetic evolutions. We are now in a position to introduce the notion of quasistatic solution we want to investigate in this work. Before presenting it we need to consider the prescribed displacement acting on one extrema of the laminate, namely a function $\bar{u} \in AC([0, T])$; we also need to consider initial displacements and damages, namely functions u_i^0, α_i^0 which must satisfy, for $i = 1, 2$, the following regularity and compatibility conditions:

$$u_i^0, \alpha_i^0 \in H^1(0, L), \quad (1.13a)$$

$$u_1^0(0) = u_2^0(0) = 0, \quad u_1^0(L) = u_2^0(L) = \bar{u}(0), \quad (1.13b)$$

$$0 \leq \alpha_i^0(x) \leq 1, \quad \text{for every } x \in [0, L]. \quad (1.13c)$$

Once the initial displacements are given, we define the initial slip

$$\delta^0 := |u_1^0 - u_2^0|.$$

For $t \in [0, T]$, we denote by $H_{0, \bar{u}(t)}^1(0, L)$ the space of functions $v \in H^1(0, L)$ attaining the boundary values $v(0) = 0$ and $v(L) = \bar{u}(t)$. We instead denote by $H_{[0, 1]}^1(0, L)$ the space of functions $v \in H^1(0, L)$ such that $0 \leq v(x) \leq 1$ for every $x \in [0, L]$.

Definition 1.5. *Given a prescribed displacement $\bar{u} \in AC([0, T])$ and initial data $\mathbf{u}^0, \boldsymbol{\alpha}^0$ satisfying (1.13), we say that a pair $(\mathbf{u}, \boldsymbol{\alpha}): [0, T] \times [0, L] \rightarrow \mathbb{R}^2 \times \mathbb{R}^2$ is an **energetic evolution** if:*

(CO) $\mathbf{u}(t) \in [H_{0, \bar{u}(t)}^1(0, L)]^2, \boldsymbol{\alpha}(t) \in [H_{[0, 1]}^1(0, L)]^2$, for every $t \in [0, T]$;

(ID) $\mathbf{u}(0) = \mathbf{u}^0, \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0$;

(IR) for $i = 1, 2$ the damage function α_i is non-decreasing in time, namely,

$$\text{for every } 0 \leq s \leq t \leq T \text{ it holds: } \alpha_i(s, x) \leq \alpha_i(t, x), \text{ for every } x \in [0, L];$$

(GS) for every $t \in [0, T]$, for every $\tilde{\mathbf{u}} \in [H_{0, \bar{u}(t)}^1(0, L)]^2$ and for every $\tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2$ such that $\alpha_i(t) \leq \tilde{\alpha}_i \leq 1$ in $[0, L]$, $i = 1, 2$, one has:

$$\mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \delta_h(t)] \leq \mathcal{E}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}[\tilde{\delta}, \delta_h(t) \vee \tilde{\delta}];$$

here we mean $\tilde{\delta} = |\tilde{u}_1 - \tilde{u}_2|$;

(EB) the function $\tau \mapsto \frac{\dot{u}(\tau)}{L} \int_0^L \sum_{i=1}^2 \sigma_i(\tau, x) dx$ belongs to $L^1(0, T)$ and for every $t \in [0, T]$ it holds:

$$\mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \delta_h(t)] = \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t),$$

where

$$\mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t) := \int_0^t \frac{\dot{u}(\tau)}{L} \int_0^L \sum_{i=1}^2 \sigma_i(\tau, x) dx d\tau, \quad (1.14)$$

is the work done by the external prescribed displacement.

In the above Definition (CO) stands for compatibility, (ID) for initial data and (IR) for irreversibility (of the damage variables); the main conditions which characterise this sort of solution are of course the global stability (GS) and the energy balance (EB).

We notice that, by (GS), a necessary condition for the existence of such an evolution is the global minimality of the initial data at time $t = 0$, namely:

$$\mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] \leq \mathcal{E}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}[\tilde{\delta}, \delta^0 \vee \tilde{\delta}], \quad (1.15)$$

for every $\tilde{\mathbf{u}} \in [H_{0, \bar{u}(0)}^1(0, L)]^2$ and for every $\tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2$ such that $\alpha_i^0 \leq \tilde{\alpha}_i \leq 1$ in $[0, L]$, $i = 1, 2$.

We also observe that the definition yields some very weak time regularity on the solution, namely \mathbf{u} and $\boldsymbol{\alpha}$ are bounded in time with values in $[H^1(0, L)]^2$, as stated in the next Proposition. As a byproduct we also obtain both temporal and spatial regularity on the historical variable δ_h , which actually is bounded in time with values in $C_0^{1/2}([0, L])$, namely the space of 1/2-Hölder functions vanishing at $x = 0$ and $x = L$.

Proposition 1.6. *Assume E_i satisfies (1.3), w_i satisfies (1.7), φ satisfies ($\varphi 2$) and let $(\mathbf{u}, \boldsymbol{\alpha})$ be an energetic evolution. Then there exists a positive constant C such that:*

$$\sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{[H^1(0, L)]^2} \leq \frac{C}{\sqrt{\varepsilon}}, \quad \text{and} \quad \sup_{t \in [0, T]} \|\boldsymbol{\alpha}(t)\|_{[H^1(0, L)]^2} \leq C, \quad (1.16a)$$

where $\varepsilon > 0$ has been introduced in (1.4). In particular, $\delta_h(t)$ belongs to $C_0^{1/2}([0, L])$, for every $t \in [0, T]$, and the following estimate holds true:

$$\sup_{t \in [0, T]} |\delta_h(t, x) - \delta_h(t, y)| \leq \frac{C}{\sqrt{\varepsilon}} \sqrt{|x - y|}, \quad \text{for every } x, y \in [0, L]. \quad (1.16b)$$

Proof. Choosing as competitors in (GS) the functions

$$\tilde{u}_i(x) = \frac{\bar{u}(t)}{L} x, \quad \tilde{\alpha}_i \equiv 1, \quad \text{for } i = 1, 2,$$

and exploiting (1.3), (1.7) and ($\varphi 2$) we deduce that

$$\begin{aligned} C_1 \sum_{i=1}^2 \left(\varepsilon \|u_i(t)\|_{H^1(0, L)}^2 + \|\alpha_i(t)\|_{H^1(0, L)}^2 \right) &\leq \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \delta_h(t)] \\ &\leq \mathcal{E}[(\tilde{u}_1, \tilde{u}_2), (1, 1)] + \mathcal{D}[(1, 1)] + \mathcal{K}[0, \delta_h(t)] \\ &\leq C_2 \left(\|\bar{u}\|_{C^0([0, T])}^2 + 1 \right), \end{aligned}$$

for every $t \in [0, T]$, where C_1 and C_2 are suitable positive constants independent of t . Hence (1.16a) is proved.

By (1.16a) and Sobolev embedding Theorems, we now know that $u_i(t)$ are uniformly 1/2-equiHölder, for every $t \in [0, T]$. We thus fix $t \in [0, T]$ and $x, y \in [0, L]$; by definition of $\delta_h(t, x)$, for every $\eta > 0$ there exists $\tau_\eta \in [0, t]$ such that

$$\delta_h(t, x) - \eta \leq |u_1(\tau_\eta, x) - u_2(\tau_\eta, x)|.$$

Hence we can estimate:

$$\begin{aligned} \delta_h(t, x) - \eta &\leq |u_1(\tau_\eta, x) - u_1(\tau_\eta, y)| + |u_1(\tau_\eta, y) - u_2(\tau_\eta, y)| + |u_2(\tau_\eta, y) - u_2(\tau_\eta, x)| \\ &\leq \frac{C}{\sqrt{\varepsilon}} \sqrt{|x - y|} + \delta_h(t, y), \end{aligned}$$

for any $t \in [0, T]$ and $x, y \in [0, L]$. By the arbitrariness of η and reverting the role of x and y we deduce that $\delta_h(t)$ is 1/2-Hölder and (1.16b) holds true. Trivially $\delta_h(t, 0) = \delta_h(t, L) = 0$ and so we conclude. \square

Remark 1.7. In the previous proposition we stressed the dependence on $\varepsilon > 0$ to point out the importance of assumption (1.3), which ensures the coerciveness of the elastic energy. In the complete damage setting, where E_i can vanish, one needs to consider the sequence of functions $E_i + \varepsilon$, fulfilling (1.3), and then to perform an analysis of the limit $\varepsilon \rightarrow 0^+$, usually via Γ -convergence. We refer for instance to [11, 30] for a model of contact between two viscoelastic bodies.

As we said in the Introduction, the common procedure used to prove existence of energetic evolutions (and which we will perform in Section 2) is based on a time discretisation algorithm and then on a limit passage as the time step goes to 0. Due to lack of compactness for the historical variable δ_h , one needs to weaken the notion of energetic evolution and to introduce a fictitious variable γ replacing δ_h (see also [16, 18]). Thanks to Proposition 1.6 we however expect that $\gamma(t)$ should be at least continuous in $[0, L]$; we are thus led to the following definition:

Definition 1.8. *Given a prescribed displacement $\bar{u} \in AC([0, T])$ and initial data $\mathbf{u}^0, \boldsymbol{\alpha}^0$ satisfying (1.13), we say that a triple $(\mathbf{u}, \boldsymbol{\alpha}, \gamma): [0, T] \times [0, L] \rightarrow \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ is a **generalised energetic evolution** if:*

(CO') $\mathbf{u}(t) \in [H_{0, \bar{u}(t)}^1(0, L)]^2, \boldsymbol{\alpha}(t) \in [H_{[0, 1]}^1(0, L)]^2, \gamma(t) \in C^0([0, L])$, for every $t \in [0, T]$;

(ID') $\mathbf{u}(0) = \mathbf{u}^0, \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0, \gamma(0) = \delta^0$;

(IR') for $i = 1, 2$ the damage function α_i and the generalised historical variable γ are non-decreasing in time, namely,

$$\text{for every } 0 \leq s \leq t \leq T \text{ it holds: } \alpha_i(s, x) \leq \alpha_i(t, x), \text{ for every } x \in [0, L];$$

$$\text{for every } 0 \leq s \leq t \leq T \text{ it holds: } \gamma(s, x) \leq \gamma(t, x), \text{ for every } x \in [0, L];$$

(GS') for every $t \in [0, T]$ one has $\gamma(t) \geq \delta(t)$ in $[0, L]$ and:

$$\mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \leq \mathcal{E}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}[\tilde{\delta}, \gamma(t) \vee \tilde{\delta}],$$

for every $\tilde{\mathbf{u}} \in [H_{0, \bar{u}(t)}^1(0, L)]^2$ and for every $\tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2$ such that $\alpha_i(t) \leq \tilde{\alpha}_i \leq 1$ in $[0, L]$ for $i = 1, 2$;

(EB') the function $\tau \mapsto \frac{\dot{u}(\tau)}{L} \int_0^L \sum_{i=1}^2 \sigma_i(\tau, x) dx$ belongs to $L^1(0, T)$ and for every $t \in [0, T]$ it holds:

$$\mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] = \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t),$$

where $\mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t)$ is defined as in (1.14).

Remark 1.9. If conditions $(\varphi 5)$, $(\varphi 6)$ and $(\varphi 7)$ are satisfied, then equality (1.12) allows us to replace the function φ in the functional \mathcal{K} (see (1.8)) by $\varphi_{\bar{\delta}}$. This means that the functions which actually play a role in the cohesive energy are $\delta \wedge \bar{\delta}$, $\delta_h \wedge \bar{\delta}$ and $\gamma \wedge \bar{\delta}$. This observation will be useful in Section 4.

From the very definition it is easy to see that a pair $(\mathbf{u}, \boldsymbol{\alpha})$ is an energetic evolution if and only if the triple $(\mathbf{u}, \boldsymbol{\alpha}, \delta_h)$ is a generalised energetic evolution. It is also easy to see that given a generalised energetic evolution $(\mathbf{u}, \boldsymbol{\alpha}, \gamma)$ it necessarily holds $\gamma(t, x) \geq \delta_h(t, x)$, for every $(t, x) \in [0, T] \times [0, L]$. Unfortunately, there are no easy arguments which ensure that $\gamma = \delta_h$ in a general case. This will be the topic of Section 4 and the main outcome of the paper.

We finally notice that the same argument used to prove Proposition 1.6 leads to the bound (1.16a) also for a generalised energetic evolution. However (1.16b) only holds for δ_h due to its explicit definition (1.1b), and nothing can be said, in general, about the generalised historical variable γ .

2. EXISTENCE RESULT

In this section we show existence of generalised energetic evolutions under very weak assumptions on the data, especially on the density φ . We indeed require (1.3), (1.7) and only $(\varphi 1)$, $(\varphi 2)$, $(\varphi 3)$, see Theorem 2.11. Of course we always assume that the prescribed displacement \bar{u} belongs to $AC([0, T])$. We then prove the existence of an energetic evolution assuming the specific assumption $(\varphi 4)$, following the same approach of [16], see Theorem 2.12. We will overcome the necessity of $(\varphi 4)$ in Section 4, recovering the existence of energetic evolutions in meaningful mechanical situations (namely assuming $(\varphi 5)$ – $(\varphi 9)$) and thus obtaining our main result, Theorem 4.8.

The classical tool used to prove existence of energetic evolutions is a time–discretisation procedure. Here we combine the ideas of [30] to deal with the irreversible damage variables and of [16, 18] to handle the historical variable.

2.1. Time-discretisation. We consider a sequence of partition $0 = t_0^n < t_1^n < \dots < t_n^n = T$ such that

$$\lim_{n \rightarrow +\infty} \max_{k=1, \dots, n} (t_k^n - t_{k-1}^n) = 0, \quad (2.1)$$

and for $k = 1, \dots, n$ we perform the following implicit Euler scheme: given $(\mathbf{u}^{k-1}, \boldsymbol{\alpha}^{k-1}, \delta_h^{k-1})$, we first select $(\mathbf{u}^k, \boldsymbol{\alpha}^k)$ by minimising the total energy among suitable natural competitors:

$$(\mathbf{u}^k, \boldsymbol{\alpha}^k) \in \underset{\substack{\tilde{\mathbf{u}} \in [H_{0, \tilde{u}(t_k^n)}^1(0, L)]^2, \\ \tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2 \text{ s.t. } \alpha_i^{k-1} \leq \tilde{\alpha}_i \leq 1}}{\text{argmin}} \left\{ \mathcal{E}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}[\tilde{\delta}, \delta_h^{k-1} \vee \tilde{\delta}] \right\}. \quad (2.2a)$$

Here we want to recall that we mean $\tilde{\delta} = |\tilde{u}_1 - \tilde{u}_2|$.

We then define δ_h^k as:

$$\delta_h^k := \delta_h^{k-1} \vee |u_1^k - u_2^k| = \delta_h^{k-1} \vee \delta^k. \quad (2.2b)$$

The initial values in the minimisation algorithm are functions $(\mathbf{u}^0, \boldsymbol{\alpha}^0)$ satisfying the compatibility conditions (1.13); moreover we set $\delta_h^0 := \delta^0 = |u_1^0 - u_2^0|$.

Proposition 2.1. *Assume E_i satisfies (1.3), w_i satisfies (1.7) and φ satisfies $(\varphi 1)$. Then there exists a solution to the minimisation algorithm (2.2a).*

Proof. We fix $n \in \mathbb{N}$ and for every $k = 1, \dots, n$ we prove the existence of a minimum by means of the direct method of Calculus of Variations. For the sake of clarity we denote by \mathcal{F}^{k-1} the

functional we want to minimise, namely

$$\mathcal{F}^{k-1}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] = \mathcal{E}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}[\tilde{\delta}, \delta_h^{k-1} \vee \tilde{\delta}]. \quad (2.3)$$

Sequential coerciveness of \mathcal{F}^{k-1} in the weak topology of $[H^1(0, L)]^4$ follows by means of (1.3) and (1.7):

$$\begin{aligned} \mathcal{F}^{k-1}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] &\geq \mathcal{E}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] \\ &\geq C \sum_{i=1}^2 \left(\varepsilon \|\tilde{u}_i\|_{H^1(0, L)}^2 + \|\tilde{\alpha}_i\|_{H^1(0, L)}^2 \right). \end{aligned}$$

As regards the lower semicontinuity of \mathcal{F}^{k-1} we exploit the compact embedding $H^1(0, L) \subset\subset C^0(0, L)$. By $(\varphi 1)$ and Fatou's Lemma we deduce that \mathcal{K} is sequentially lower semicontinuous with respect to the considered topology. The same holds for \mathcal{D} by using again Fatou's Lemma together with weak lower semicontinuity of the norm. To prove finally lower semicontinuity of \mathcal{E} it is enough to show that, given weakly convergent sequences $\tilde{u}_i^j \rightharpoonup \tilde{u}_i$, $\tilde{\alpha}_i^j \rightharpoonup \tilde{\alpha}_i$ in $H^1(0, L)$, we have that $\sqrt{E_i(\tilde{\alpha}_i^j)}(\tilde{u}_i^j)'$ weakly converges to $\sqrt{E_i(\tilde{\alpha}_i)}\tilde{u}_i'$ in $L^2(0, L)$ as $j \rightarrow +\infty$, for $i = 1, 2$. To prove it we fix $\phi \in L^2(0, L)$ and we estimate by exploiting (1.3):

$$\begin{aligned} &\left| \int_0^L \sqrt{E_i(\tilde{\alpha}_i^j(x))}(\tilde{u}_i^j)'(x)\phi(x) \, dx - \int_0^L \sqrt{E_i(\tilde{\alpha}_i(x))}\tilde{u}_i'(x)\phi(x) \, dx \right| \\ &\leq \left\| \sqrt{E_i(\tilde{\alpha}_i^j)} - \sqrt{E_i(\tilde{\alpha}_i)} \right\|_{C^0([0, L])} \|\tilde{u}_i^j\|_{H^1(0, L)} \|\phi\|_{L^2(0, L)} \\ &\quad + \left| \int_0^L (\tilde{u}_i^j)'(x)\sqrt{E_i(\tilde{\alpha}_i(x))}\phi(x) \, dx - \int_0^L \tilde{u}_i'(x)\sqrt{E_i(\tilde{\alpha}_i(x))}\phi(x) \, dx \right|. \end{aligned}$$

The first term goes to zero as $j \rightarrow +\infty$ since $\tilde{\alpha}_i^j$ uniformly converges to $\tilde{\alpha}_i$ as $j \rightarrow +\infty$ and the function E_i is continuous. The second term vanishes too as $j \rightarrow +\infty$ since $\sqrt{E_i(\tilde{\alpha}_i)}\phi$ belongs to $L^2(0, L)$ by the boundedness of E_i .

We conclude by noticing that, exploiting again the compactness of the embedding $H^1(0, L) \subset\subset C^0(0, L)$, the set on which we minimise is sequentially closed with respect to the considered topology. \square

To pass from discrete to continuous evolutions we now introduce the (right-continuous) piecewise constant interpolation $(\mathbf{u}^n, \boldsymbol{\alpha}^n)$ of the discrete displacement and damage, and the piecewise constant interpolation δ_h^n of the discrete historical variable, namely:

$$\begin{cases} \mathbf{u}^n(t) := \mathbf{u}^k, & \boldsymbol{\alpha}^n(t) := \boldsymbol{\alpha}^k, & \delta_h^n(t) := \delta_h^k, & \text{if } t \in [t_k^n, t_{k+1}^n), \\ \mathbf{u}^n(T) := \mathbf{u}^n, & \boldsymbol{\alpha}^n(T) := \boldsymbol{\alpha}^n, & \delta_h^n(T) := \delta_h^n. \end{cases} \quad (2.4a)$$

Of course, in the following, by the expression δ^n we mean the piecewise constant slip, namely

$$\delta^n(t, x) = |u_1^n(t, x) - u_2^n(t, x)|. \quad (2.4b)$$

Analogously, we consider a piecewise constant version \bar{u}^n of the prescribed displacement:

$$\begin{cases} \bar{u}^n(t) := \bar{u}(t_k^n), & \text{if } t \in [t_k^n, t_{k+1}^n), \\ \bar{u}^n(T) := \bar{u}(T). \end{cases} \quad (2.4c)$$

We also adopt the following notation:

$$\tau^n(t) := \max\{t_k^n \mid t_k^n \leq t\}. \quad (2.4d)$$

Next proposition provides useful uniform bounds on the just introduced piecewise constant interpolations. It is the analogue of Proposition 1.6 in this discrete setting.

Proposition 2.2. *Assume E_i satisfies (1.3), w_i satisfies (1.7) and φ satisfies $(\varphi 1)$, $(\varphi 2)$. Then there exists a positive constant C independent of n such that:*

$$\max_{t \in [0, T]} \|\mathbf{u}^n(t)\|_{[H^1(0, L)]^2} \leq \frac{C}{\sqrt{\varepsilon}}, \quad \max_{t \in [0, T]} \|\boldsymbol{\alpha}^n(t)\|_{[H^1(0, L)]^2} \leq C, \quad (2.5a)$$

$$\max_{t \in [0, T]} \left(\sup_{x, y \in [0, L], x \neq y} \frac{|\delta_h^n(t, x) - \delta_h^n(t, y)|}{\sqrt{|x - y|}} \right) \leq \frac{C}{\sqrt{\varepsilon}}, \quad (2.5b)$$

where $\varepsilon > 0$ has been introduced in (1.4).

Proof. The result follows by using exactly the same argument of Proposition 1.6. We only notice that here we need to choose as competitors for $(\mathbf{u}^k, \boldsymbol{\alpha}^k)$ in (2.2a) the functions

$$\tilde{u}_i(x) = \frac{\bar{u}(t_k^n)}{L}x, \quad \tilde{\alpha}_i \equiv 1, \quad \text{for } i = 1, 2,$$

and then we argue in the same way. \square

Since the piecewise constant interpolations are built starting from the minimisation algorithm (2.2a), they automatically fulfil the following inequality, which is related to the energy balance (EB):

Lemma 2.3 (Discrete Energy Inequality). *Assume E_i satisfies (1.3), w_i satisfies (1.7) and φ satisfies $(\varphi 1)$, $(\varphi 2)$. Then there exists a vanishing sequence of positive real numbers R^n such that for every $t \in [0, T]$ and for every $n \in \mathbb{N}$ the following inequality holds true:*

$$\begin{aligned} & \mathcal{E}[\mathbf{u}^n(t), \boldsymbol{\alpha}^n(t)] + \mathcal{D}[\boldsymbol{\alpha}^n(t)] + \mathcal{K}[\delta^n(t), \delta_h^n(t)] \\ & \leq \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \int_0^t W^n(\tau) d\tau + R^n, \end{aligned}$$

where $W^n(\tau) := \frac{\dot{u}(\tau)}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i^n(\tau, x)) (u_i^n)'(\tau, x) dx$.

Proof. We fix $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$; for $j = 1, \dots, k$ we then choose as competitors for $(\mathbf{u}^j, \boldsymbol{\alpha}^j)$ in (2.2a) the functions $\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}$, with components:

$$\tilde{u}_i(x) = u_i^{j-1}(x) + (\bar{u}(t_j^n) - \bar{u}(t_{j-1}^n))x/L, \quad \text{and } \tilde{\alpha}_i = \alpha_i^{j-1}, \quad \text{for } i = 1, 2.$$

We thus obtain:

$$\mathcal{E}[\mathbf{u}^j, \boldsymbol{\alpha}^j] + \mathcal{D}[\boldsymbol{\alpha}^j] + \mathcal{K}[\delta^j, \delta_h^j] \leq \mathcal{E}[\mathbf{u}^{j-1} + \mathbf{v}^{j-1}, \boldsymbol{\alpha}^{j-1}] + \mathcal{D}[\boldsymbol{\alpha}^{j-1}] + \mathcal{K}[\delta^{j-1}, \delta_h^{j-1}],$$

where we denoted by $\mathbf{v}^{j-1}(x)$ the vector in \mathbb{R}^2 with both components equal to $(\bar{u}(t_j^n) - \bar{u}(t_{j-1}^n))\frac{x}{L}$. From the above inequality we now get:

$$\begin{aligned} & \mathcal{E}[\mathbf{u}^j, \boldsymbol{\alpha}^j] + \mathcal{D}[\boldsymbol{\alpha}^j] + \mathcal{K}[\delta^j, \delta_h^j] - \mathcal{E}[\mathbf{u}^{j-1}, \boldsymbol{\alpha}^{j-1}] - \mathcal{D}[\boldsymbol{\alpha}^{j-1}] - \mathcal{K}[\delta^{j-1}, \delta_h^{j-1}] \\ & \leq \mathcal{E}[\mathbf{u}^{j-1} + \mathbf{v}^{j-1}, \boldsymbol{\alpha}^{j-1}] - \mathcal{E}[\mathbf{u}^{j-1}, \boldsymbol{\alpha}^{j-1}] \\ & = \int_{t_{j-1}^n}^{t_j^n} \frac{\dot{u}(\tau)}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i^{j-1}(x)) \left((u_i^{j-1})'(x) + \frac{\bar{u}(\tau) - \bar{u}(t_{j-1}^n)}{L} \right) dx d\tau. \end{aligned}$$

Summing the obtained inequality from $j = 1$ to $j = k$ we hence deduce:

$$\begin{aligned} & \mathcal{E}[\mathbf{u}^k, \boldsymbol{\alpha}^k] + \mathcal{D}[\boldsymbol{\alpha}^k] + \mathcal{K}[\delta^k, \delta_h^k] - \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] - \mathcal{D}[\boldsymbol{\alpha}^0] - \mathcal{K}[\delta^0, \delta^0] \\ & \leq \sum_{j=1}^k \left(\int_{t_{j-1}^n}^{t_j^n} W^n(\tau) d\tau + \int_{t_{j-1}^n}^{t_j^n} \frac{\dot{u}(\tau) \bar{u}(\tau) - \bar{u}^n(\tau)}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i^n(\tau, x)) dx d\tau \right) \\ & = \int_0^{t_k^n} W^n(\tau) d\tau + \int_0^{t_k^n} \frac{\dot{u}(\tau) \bar{u}(\tau) - \bar{u}^n(\tau)}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i^n(\tau, x)) dx d\tau. \end{aligned}$$

Recalling the definition of the interpolations \mathbf{u}^n , $\boldsymbol{\alpha}^n$ and τ^n , see (2.4), by the arbitrariness of k we finally obtain for every $t \in [0, T]$:

$$\begin{aligned} & \mathcal{E}[\mathbf{u}^n(t), \boldsymbol{\alpha}^n(t)] + \mathcal{D}[\boldsymbol{\alpha}^n(t)] + \mathcal{K}[\delta^n(t), \delta_h^n(t)] \\ & \leq \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \int_0^t W^n(\tau) d\tau \\ & \quad + \int_0^{\tau^n(t)} \frac{\dot{u}(\tau) \bar{u}(\tau) - \bar{u}^n(\tau)}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i^n(\tau, x)) dx d\tau - \int_{\tau^n(t)}^t W^n(\tau) d\tau. \end{aligned}$$

We thus conclude by defining:

$$R^n := \int_0^T \frac{|\dot{u}(\tau)| |\bar{u}(\tau) - \bar{u}^n(\tau)|}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i^n(\tau, x)) dx d\tau + \sup_{t \in [0, T]} \int_{\tau^n(t)}^t |W^n(\tau)| d\tau. \quad (2.6)$$

Indeed we now show that $\lim_{n \rightarrow +\infty} R^n = 0$. First of all by the very definition of W^n and exploiting (2.5a) it is easy to see that $|W^n(\tau)| \leq C|\dot{u}(\tau)|$, with $C > 0$ independent of n ; hence by the absolute continuity of the integral the second term in (2.6) vanishes as $n \rightarrow +\infty$ (we recall that by assumption the sequence of partitions satisfies (2.1)). Then we notice that the first term is bounded by

$$C \|\dot{u}\|_{L^1(0, T)} \sup_{t \in [0, T]} |\bar{u}(t) - \bar{u}^n(t)|,$$

which vanishes since \bar{u} is absolutely continuous and the sequence of partitions satisfies (2.1). \square

2.2. Extraction of convergent subsequences. By the uniform bounds obtained in Proposition 2.2 we are able to deduce the existence of convergent subsequences of the piecewise constant interpolations \mathbf{u}^n , $\boldsymbol{\alpha}^n$ and δ_h^n . We first need the following Helly–type compactness result:

Lemma 2.4 (Helly). *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of non-decreasing functions from $[0, T]$ to $C^0([0, L])$, meaning that for every $0 \leq s \leq t \leq T$ it holds $f_n(s, x) \leq f_n(t, x)$ for all $x \in [0, L]$, such that:*

- the families $\{f_n(0)\}_{n \in \mathbb{N}}$ and $\{f_n(T)\}_{n \in \mathbb{N}}$ are equibounded;
- the family $\{f_n(t)\}_{n \in \mathbb{N}}$ is equicontinuous uniformly with respect to $t \in [0, T]$.

Then there exist a subsequence (not relabelled) and a non-decreasing function f from $[0, T]$ to $C^0([0, L])$ such that $f_n(t)$ converges uniformly to $f(t)$ as $n \rightarrow +\infty$ for every $t \in [0, T]$.

Moreover for every $t \in [0, T]$ the right and left limits $f^\pm(t)$, which are well defined pointwise by monotonicity, actually belong to $C^0([0, L])$ and it holds

$$f^\pm(t) = \lim_{h \rightarrow 0^\pm} f(t+h), \quad \text{uniformly in } [0, L]. \quad (2.7)$$

Proof. We only sketch the proof, being very similar to the one of Lemma 4.6 in [18].

We consider a countable and dense set $D \subseteq [0, T]$ containing 0 and T and by using Ascoli–Arzelá theorem and a diagonal argument we can extract a subsequence (not relabelled) and a

function f from D to $C^0([0, L])$ such that $f_n(t)$ converges uniformly to $f(t)$ for every $t \in D$. Since each f_n is non-decreasing, trivially f is non-decreasing on D .

For every $t \in [0, T]$ we now define

$$f^+(t) := \inf_{s \geq t, s \in D} f(s), \quad f^-(t) := \sup_{s \leq t, s \in D} f(s), \quad (2.8)$$

and we easily observe that

- (i) $f^-(t) = f(t) = f^+(t)$ for every $t \in D$;
- (ii) $f^-(t) \leq f^+(t)$ for every $t \in [0, T]$;
- (iii) if $0 \leq s < t \leq T$, then $f^+(s) \leq f^-(t)$.

Since the family $\{f_n(t)\}_{n \in \mathbb{N}}$ is equicontinuous uniformly with respect to time we obtain that the limit family $\{f(t)\}_{t \in D}$ is equicontinuous. This actually ensures that (2.8) can be improved in the following way:

$$f^+(t) = \lim_{s \searrow t, s \in D} f(s), \quad f^-(t) = \lim_{s \nearrow t, s \in D} f(s), \quad \text{uniformly in } [0, L]. \quad (2.9)$$

In particular for every $t \in [0, T]$ the functions $f^+(t)$ and $f^-(t)$ are continuous in $[0, L]$.

We now introduce the set $E := \{t \in [0, T] \mid f^+(t) = f^-(t)\}$ and for every $t \in E$ we define $f(t) := f^+(t) = f^-(t)$. Of course, by (i), the set D is contained in E and the definition of f agrees with the one we already had on D . We now prove that for every $t \in E$ we have $f_n(t) \rightarrow f(t)$ uniformly in $[0, L]$. We already know it is true for $t \in D$, so we assume $t \in E \setminus D$. We fix two points $s' < t < t'$ such that $s', t' \in D$ and since the original sequence was non-decreasing in time we easily get:

$$\begin{aligned} \|f_n(t) - f(t)\|_{C^0([0, L])} &\leq \max \{ \|f_n(t') - f(t)\|_{C^0([0, L])}, \|f_n(s') - f(t)\|_{C^0([0, L])} \} \\ &\leq \max \{ \|f_n(t') - f(t')\|_{C^0} + \|f(t') - f(t)\|_{C^0}, \|f_n(s') - f(s')\|_{C^0} + \|f(s') - f(t)\|_{C^0} \}. \end{aligned}$$

Since s' and t' belong to D we infer:

$$\limsup_{n \rightarrow +\infty} \|f_n(t) - f(t)\|_{C^0([0, L])} \leq \max \{ \|f(t') - f(t)\|_{C^0([0, L])}, \|f(s') - f(t)\|_{C^0([0, L])} \}.$$

Thanks to (2.9) and since t is in E , letting $t' \searrow t$ and $s' \nearrow t$ we finally conclude that $f_n(t)$ converges uniformly to $f(t)$ for every $t \in E$.

Let us now show that the set $E^c = [0, T] \setminus E$ is countable. First of all it is easy to see that E^c coincides with $\bigcup_{k \in \mathbb{N}} A_k$ where for every $k \in \mathbb{N}$ we define

$$A_k = \left\{ t \in [0, T] \mid \int_0^L (f^+(t, x) - f^-(t, x)) \, dx \geq \frac{1}{k} \right\}.$$

We conclude if we prove that each A_k is finite. So we fix $t_1 < t_2 < \dots < t_r \in A_k$ and thanks to (iii) we estimate:

$$\begin{aligned} \frac{r}{k} &\leq \sum_{j=1}^r \int_0^L (f^+(t_j, x) - f^-(t_j, x)) \, dx \leq \int_0^L (f^+(t_r, x) - f^-(t_1, x)) \, dx \\ &\leq \|f(T) - f(0)\|_{L^1(0, L)}, \end{aligned}$$

thus r is bounded from above and thus A_k is finite.

So by using again Ascoli–Arzelá theorem and a diagonal argument we can extract a further subsequence and a function f from E^c to $C^0([0, L])$ such that $f_n(t)$ converges uniformly to $f(t)$ for every $t \in E^c$. Since in E we already obtained the result, we conclude by noticing that such a f is non-decreasing in the whole $[0, T]$ recalling that the original sequence was non-decreasing in time. Indeed (2.7) easily follows by (2.9). \square

Proposition 2.5. *Assume E_i satisfies (1.3), w_i satisfies (1.7) and φ satisfies $(\varphi 1)$, $(\varphi 2)$. Consider the sequences of functions \mathbf{u}^n , $\boldsymbol{\alpha}^n$, δ_h^n introduced in (2.4a). Then there exist a subsequence n_j and for every $t \in [0, T]$ a further subsequence $n_j(t)$ (depending on time) such that:*

- (a) $\mathbf{u}^{n_j(t)}(t) \rightharpoonup \mathbf{u}(t)$ in $[H^1(0, L)]^2$ as $n_j(t) \rightarrow +\infty$;
- (b) $\boldsymbol{\alpha}^{n_j(t)}(t) \rightharpoonup \boldsymbol{\alpha}(t)$ in $[H^1(0, L)]^2$ as $n_j(t) \rightarrow +\infty$;
- (c) $\delta_h^{n_j(t)}(t) \rightarrow \gamma(t)$ uniformly in $[0, L]$ as $n_j \rightarrow +\infty$.

Moreover the limit functions satisfy:

- (1) $\mathbf{u}(t) \in [H_{0, \bar{u}(t)}^1(0, L)]^2$, $\boldsymbol{\alpha}(t) \in [H_{[0, 1]}^1(0, L)]^2$ and $\gamma(t) \in C_0^{1/2}([0, L])$ for every $t \in [0, T]$;
- (2) $\mathbf{u}(0) = \mathbf{u}^0$, $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}^0$ and $\gamma(0) = \delta^0$;
- (3) α_i and γ are non-decreasing in time;
- (4) $\gamma(t) \geq \delta_h(t) = \sup_{\tau \in [0, t]} |u_1(\tau) - u_2(\tau)|$ for every $t \in [0, T]$.

Remark 2.6. For the sake of clarity, in order to avoid too heavy notations, from now on we prefer not to stress the occurrence of the subsequence via the subscript j ; namely we still write n instead of n_j and $n(t)$ instead of $n_j(t)$.

Proof of Proposition 2.5. The validity of (c), the 1/2–Hölderianity of the limit function $\gamma(t)$ and the fact that $\gamma(t, 0) = \gamma(t, L) = 0$ are a byproduct of (2.5b) and Lemma 2.4; (a) and (b) instead follow by (2.5a) together with the weak sequential compactness of the unit ball in $H^1(0, L)$. Since $H^1(0, L) \subset\subset C^0([0, L])$ we also deduce (1), (2) and (3).

We only need to prove (4). So let us assume by contradiction that there exists a pair $(t, x) \in [0, T] \times [0, L]$ such that:

$$\delta_h(t, x) > \gamma(t, x) = \lim_{n \rightarrow +\infty} \delta_h^n(t, x). \quad (2.10)$$

By (2.10) and the definition of δ_h , there exists a time $\tau_t \in [0, t]$ for which $|u_1(\tau_t, x) - u_2(\tau_t, x)| > \gamma(t, x)$; thus we infer:

$$\begin{aligned} |u_1(\tau_t, x) - u_2(\tau_t, x)| &> \lim_{n \rightarrow +\infty} \delta_h^n(t, x) \geq \lim_{n \rightarrow +\infty} \delta_h^n(\tau_t, x) \\ &\geq \limsup_{n \rightarrow +\infty} |u_1^n(\tau_t, x) - u_2^n(\tau_t, x)| \\ &\geq \lim_{n(\tau_t) \rightarrow +\infty} |u_1^{n(\tau_t)}(\tau_t, x) - u_2^{n(\tau_t)}(\tau_t, x)| = |u_1(\tau_t, x) - u_2(\tau_t, x)|, \end{aligned}$$

which is absurd. \square

Remark 2.7. We want to point out that also the subsequence of the damage variable in (b) could be chosen independent of time, since each term of the sequence is non-decreasing in time. This follows by means of a suitable version of Helly’s selection theorem (see for instance Theorem B.5.13 in the Appendix B of [31]), and arguing as in [30], Proposition 3.2. However, for the sake of simplicity we preferred to consider a time–dependent subsequence; indeed this will be enough for our purposes.

The fact that the subsequence in (c) does not depend on time was instead crucial for the validity of (4), as the reader can check from the proof.

2.3. Existence of generalised energetic evolutions. The aim of this subsection is proving that the limit functions obtained in Proposition 2.5 are actually a generalised energetic evolution. We only need to show that global stability (GS’) and energy balance (EB’) hold true, being the other conditions automatically satisfied due to Lemma 2.5. This first proposition deals with the global stability:

Proposition 2.8. *Assume E_i satisfies (1.3), w_i satisfies (1.7), and φ satisfies $(\varphi 1)$ – $(\varphi 3)$. Assume the initial data $\mathbf{u}^0, \boldsymbol{\alpha}^0$ fulfil the stability condition (1.15). Then the limit functions $\mathbf{u}, \boldsymbol{\alpha}, \gamma$ obtained in Proposition 2.5 satisfy (GS').*

Proof. If $t = 0$ there is nothing to prove, so we consider $t \in (0, T]$ and we first notice that by (4) in Proposition 2.5 we know $\gamma(t) \geq \delta(t)$. Then we fix $\tilde{\mathbf{u}} \in [H^1_{0, \bar{u}(t)}(0, L)]^2$ and $\tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2$ such that $\alpha_i(t) \leq \tilde{\alpha}_i \leq 1$ for $i = 1, 2$.

By weak lower semicontinuity of the energy we get:

$$\begin{aligned} & \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \\ & \leq \liminf_{n(t) \rightarrow +\infty} \left(\mathcal{E}[\mathbf{u}^{n(t)}(t), \boldsymbol{\alpha}^{n(t)}(t)] + \mathcal{D}[\boldsymbol{\alpha}^{n(t)}(t)] + \mathcal{K}[\delta^{n(t)}(t), \delta_h^{n(t)}(t)] \right) =: (\star). \end{aligned}$$

Now we can use the minimality properties of the discrete functions, considering as competitors the functions $\hat{\mathbf{u}}^{n(t)}$ and $\hat{\boldsymbol{\alpha}}^{n(t)}$ whose components are

$$\hat{u}_i^{n(t)}(x) := \tilde{u}_i(x) - (\bar{u}(t) - \bar{u}(\tau^{n(t)}(t))) \frac{x}{L}, \quad \hat{\alpha}_i^{n(t)} := \min \left\{ \tilde{\alpha}_i + \max_{[0, L]} \left| \alpha_i^{n(t)}(t) - \alpha_i(t) \right|, 1 \right\}.$$

It is easy to see that they are admissible; moreover, since $\tau^{n(t)}(t) \rightarrow t$ and $\alpha_i^{n(t)}(t) \rightarrow \alpha_i(t)$ uniformly as $n(t) \rightarrow +\infty$, they strongly converge to $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\alpha}}$ in $[H^1(0, L)]^2$. See also [30], Lemma 3.5.

By minimality, going back to the previous estimate, we obtain:

$$\begin{aligned} (\star) & \leq \liminf_{n(t) \rightarrow +\infty} \left(\mathcal{E}[\hat{\mathbf{u}}^{n(t)}, \hat{\boldsymbol{\alpha}}^{n(t)}] + \mathcal{D}[\hat{\boldsymbol{\alpha}}^{n(t)}] + \mathcal{K}[\tilde{\delta}, \delta_h^{n(t)}(t) \vee \tilde{\delta}] \right) \\ & = \mathcal{E}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}[\tilde{\delta}, \gamma(t) \vee \tilde{\delta}], \end{aligned}$$

where in the last equality we exploited the strong convergence of $\hat{\mathbf{u}}^{n(t)}$ and $\hat{\boldsymbol{\alpha}}^{n(t)}$ towards $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\alpha}}$, plus assumption $(\varphi 3)$. Thus we conclude. \square

To show the validity of (EB') we prove separately the two inequalities. The first one follows from the discrete energy inequality presented in Lemma 2.3:

Proposition 2.9 (Upper Energy Estimate). *Assume E_i satisfies (1.3), w_i satisfies (1.7), and φ satisfies $(\varphi 1)$, $(\varphi 2)$. Then for every $t \in [0, T]$ the limit functions $\mathbf{u}, \boldsymbol{\alpha}, \gamma$ obtained in Proposition 2.5 satisfy the following inequality:*

$$\mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \leq \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t).$$

Proof. We fix $t \in [0, T]$ and by lower semicontinuity of the energy and Lemma 2.3 we deduce:

$$\begin{aligned} & \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \\ & \leq \liminf_{n(t) \rightarrow +\infty} \left(\mathcal{E}[\mathbf{u}^{n(t)}(t), \boldsymbol{\alpha}^{n(t)}(t)] + \mathcal{D}[\boldsymbol{\alpha}^{n(t)}(t)] + \mathcal{K}[\delta^{n(t)}(t), \delta_h^{n(t)}(t)] \right) \\ & \leq \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \liminf_{n(t) \rightarrow +\infty} \int_0^t W^{n(t)}(\tau) d\tau. \end{aligned}$$

By means of the reverse Fatou's Lemma we thus get:

$$\liminf_{n(t) \rightarrow +\infty} \int_0^t W^{n(t)}(\tau) d\tau \leq \int_0^t \limsup_{n(t) \rightarrow +\infty} W^{n(t)}(\tau) d\tau =: (*).$$

In order to deal with (*) we argue as follows (see also [16], Section 4). We consider the subsequence n obtained in Proposition 2.5 (see also Remark 2.6) and for every $\tau \in [0, T]$ we first

set

$$W(\tau) := \limsup_{n \rightarrow +\infty} W^n(\tau), \quad (2.11)$$

which belongs to $L^1(0, T)$ since we recall that $|W^n(\tau)| \leq C|\dot{u}(\tau)|$. Without loss of generality we can assume that the time-dependent subsequences obtained in Proposition 2.5 also satisfy

$$W(\tau) = \lim_{n(\tau) \rightarrow +\infty} W^{n(\tau)}(\tau), \quad \text{for every } \tau \in [0, T].$$

Thus exploiting (a) and (b) in Proposition 2.5 for a.e. $\tau \in [0, T]$ we obtain:

$$\begin{aligned} W(\tau) &= \lim_{n(\tau) \rightarrow +\infty} W^{n(\tau)}(\tau) = \lim_{n(\tau) \rightarrow +\infty} \frac{\dot{u}(\tau)}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i^{n(\tau)}(\tau, x))(u_i^{n(\tau)})'(\tau, x) dx \\ &= \frac{\dot{u}(\tau)}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i(\tau, x))(u_i)'(\tau, x) dx. \end{aligned} \quad (2.12)$$

Combining (2.11) and (2.12) we finally get

$$(*) \leq \int_0^t W(\tau) d\tau = \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t),$$

and we conclude. \square

The opposite inequality is instead a byproduct of the global stability condition we proved in Proposition 2.8:

Proposition 2.10 (Lower Energy Estimate). *Assume E_i satisfies (1.3), w_i satisfies (1.7), and φ satisfies $(\varphi 1)$ – $(\varphi 3)$. Assume the initial data $\mathbf{u}^0, \boldsymbol{\alpha}^0$ fulfil the stability condition (1.15). Then for every $t \in [0, T]$ the limit functions $\mathbf{u}, \boldsymbol{\alpha}, \gamma$ obtained in Proposition 2.5 satisfy:*

$$\mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \geq \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t).$$

Proof. If $t = 0$ the inequality is trivial, so we fix $t \in (0, T]$ and we consider a sequence of partitions of $[0, t]$ of the form $0 = t_0^n < t_1^n < \dots < t_n^n = t$ satisfying:

- (i) $\lim_{n \rightarrow +\infty} \max_{k=1, \dots, n} |t_k^n - t_{k-1}^n| = 0;$
- (ii) $\lim_{n \rightarrow +\infty} \sum_{k=1}^n \left| (t_k^n - t_{k-1}^n) \dot{u}(t_k^n) - \int_{t_{k-1}^n}^{t_k^n} \dot{u}(\tau) d\tau \right| = 0;$
- (iii) $\lim_{n \rightarrow +\infty} \sum_{k=1}^n (t_k^n - t_{k-1}^n) W(t_k^n) = \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t),$

where W is the function introduced in (2.11) and (2.12). The existence of such a sequence of partitions follows from Lemma 4.5 in [20], since both \dot{u} and W belong to $L^1(0, T)$. In particular, by (i) and the absolute continuity of the integral, we can assume without loss of generality that:

- (iv) for every $n \in \mathbb{N}$ it holds $\int_{t_{k-1}^n}^{t_k^n} |\dot{u}(\tau)| d\tau \leq \frac{1}{n}$ for every $k = 1, \dots, n$.

For a given partition we fix $k = 1, \dots, n$ and, recalling Proposition 2.8, we choose as competitors for $\mathbf{u}(t_{k-1}^n), \boldsymbol{\alpha}(t_{k-1}^n)$ and $\gamma(t_{k-1}^n)$ in (GS') the functions $\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}$, with components:

$$\tilde{u}_i(x) = u_i(t_k^n, x) + (\bar{u}(t_{k-1}^n) - \bar{u}(t_k^n)) \frac{x}{L}, \quad \tilde{\alpha}_i = \alpha_i(t_k^n), \quad \text{for } i = 1, 2.$$

Recalling that $\gamma(t_{k-1}^n) \vee \delta(t_k^n) \leq \gamma(t_k^n)$, and hence $\mathcal{K}[\delta(t_k^n), \gamma(t_{k-1}^n) \vee \delta(t_k^n)] \leq \mathcal{K}[\delta(t_k^n), \gamma(t_k^n)]$ by $(\varphi 3)$, arguing as in the proof of Lemma 2.3 we thus deduce:

$$\begin{aligned} & \mathcal{E}[\mathbf{u}(t_{k-1}^n), \boldsymbol{\alpha}(t_{k-1}^n)] + \mathcal{D}[\boldsymbol{\alpha}(t_{k-1}^n)] + \mathcal{K}[\delta(t_{k-1}^n), \gamma(t_{k-1}^n)] - \mathcal{E}[\mathbf{u}(t_k^n), \boldsymbol{\alpha}(t_k^n)] - \mathcal{D}[\boldsymbol{\alpha}(t_k^n)] - \mathcal{K}[\delta(t_k^n), \gamma(t_k^n)] \\ & \leq - \int_{t_{k-1}^n}^{t_k^n} \frac{\dot{u}(\tau)}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i(t_k^n, x)) \left(u'_i(t_k^n, x) + \frac{\bar{u}(\tau) - \bar{u}(t_k^n)}{L} \right) dx d\tau. \end{aligned}$$

Summing the above inequality from $k = 1$ to $k = n$ we obtain:

$$\begin{aligned} & \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] - \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] - \mathcal{D}[\boldsymbol{\alpha}^0] - \mathcal{K}[\delta^0, \gamma^0] \\ & \geq \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} \frac{\dot{u}(\tau)}{L} \sum_{i=1}^2 \int_0^L E_i(\alpha_i(t_k^n, x)) \left(u'_i(t_k^n, x) + \frac{\bar{u}(\tau) - \bar{u}(t_k^n)}{L} \right) dx d\tau =: J_n \end{aligned}$$

Now we easily notice that J_n can be written as:

$$\begin{aligned} J_n &= \sum_{k=1}^n (t_k^n - t_{k-1}^n) W(t_k^n) \\ &+ \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} \frac{\dot{u}(\tau) - \dot{u}(t_k^n)}{L} d\tau \int_0^L \sum_{i=1}^2 E_i(\alpha_i(t_k^n, x)) u'_i(t_k^n, x) dx \\ &+ \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} \frac{\dot{u}(\tau)}{L} \frac{\bar{u}(\tau) - \bar{u}(t_k^n)}{L} d\tau \int_0^L \sum_{i=1}^2 E_i(\alpha_i(t_k^n, x)) dx =: J_n^1 + J_n^2 + J_n^3. \end{aligned}$$

By (iii) we know that $\lim_{n \rightarrow +\infty} J_n^1 = \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t)$, so we conclude if we prove that $\lim_{n \rightarrow +\infty} J_n^2 = \lim_{n \rightarrow +\infty} J_n^3 = 0$. With this aim we estimate:

$$\begin{aligned} |J_n^2| &\leq C \sum_{k=1}^n \left| \int_{t_{k-1}^n}^{t_k^n} (\dot{u}(\tau) - \dot{u}(t_k^n)) d\tau \right| \left(\sum_{i=1}^2 \|u_i(t_k^n)\|_{H^1(0,L)} \right) \\ &\leq C \sum_{k=1}^n \left| (t_k^n - t_{k-1}^n) \dot{u}(t_k^n) - \int_{t_{k-1}^n}^{t_k^n} \dot{u}(\tau) d\tau \right|, \end{aligned}$$

which goes to 0 by (ii). As regards J_n^3 , by using (iv) we get:

$$\begin{aligned} |J_n^3| &\leq C \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} |\dot{u}(\tau)| |\bar{u}(\tau) - \bar{u}(t_k^n)| d\tau = C \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} |\dot{u}(\tau)| \left| \int_{\tau}^{t_k^n} \dot{u}(s) ds \right| d\tau \\ &\leq C \sum_{k=1}^n \left(\int_{t_{k-1}^n}^{t_k^n} |\dot{u}(\tau)| d\tau \right)^2 \leq \frac{C}{n} \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} |\dot{u}(\tau)| d\tau = \frac{C}{n} \|\dot{u}\|_{L^1(0,t)}, \end{aligned}$$

and the proof is complete. \square

Putting together what we obtained in this section we infer our first result of existence of generalised energetic evolutions:

Theorem 2.11 (Existence of Generalised Energetic Evolutions). *Let the prescribed displacement \bar{u} belong to $AC([0, T])$ and the initial data $\mathbf{u}^0, \boldsymbol{\alpha}^0$ fulfil (1.13) together with the stability condition (1.15). Assume E_i satisfies (1.3), w_i satisfies (1.7), and φ satisfies $(\varphi 1)$ – $(\varphi 3)$. Then the triplet composed by the functions \mathbf{u} , $\boldsymbol{\alpha}$ and γ obtained in Proposition 2.5 is a generalised energetic evolution.*

We conclude this section by showing that, assuming in addition the specific condition $(\varphi 4)$, which we recall also here for the sake of clarity:

$(\varphi 4)$ there exist two functions $\varphi_1, \varphi_2: [0, +\infty) \rightarrow [0, +\infty)$ such that φ_1 is lower semicontinuous, φ_2 is bounded, non-decreasing and concave, and $\varphi(y, z) = \varphi_1(y) + \varphi_2(z)$,

the functions \mathbf{u} and $\boldsymbol{\alpha}$ obtained in Proposition 2.5 are actually an energetic evolution. The approach is exactly the same of [16]. We first notice that $(\varphi 4)$ implies $(\varphi 1)$, $(\varphi 2)$ and $(\varphi 3)$.

Theorem 2.12. *Let the prescribed displacement \bar{u} belong to $AC([0, T])$ and the initial data $\mathbf{u}^0, \boldsymbol{\alpha}^0$ fulfil (1.13) together with the stability condition (1.15). Assume E_i satisfies (1.3), w_i satisfies (1.7), and φ satisfies $(\varphi 4)$. Then the pair $(\mathbf{u}, \boldsymbol{\alpha})$ obtained in Proposition 2.5 is an energetic evolution.*

If in addition φ_2 is strictly increasing, then the function γ obtained in Proposition 2.5 coincides with the historical variable δ_h .

Proof. Thanks to Theorem 2.11 we only need to show the validity of (GS) and (EB) in Definition 1.5. We first focus on (GS); so we fix $t \in [0, T]$ and two functions $\tilde{\mathbf{u}} \in [H^1_{0, \bar{u}(t)}(0, L)]^2$, $\tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2$ such that $\alpha_i(t) \leq \tilde{\alpha}_i \leq 1$ in $[0, L]$ for $i = 1, 2$. Since the triplet $(\mathbf{u}, \boldsymbol{\alpha}, \gamma)$ satisfies (GS') we know that:

$$\mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] \leq \mathcal{E}[\tilde{\mathbf{u}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}[\tilde{\delta}, \gamma(t) \vee \tilde{\delta}] - \mathcal{K}[\delta(t), \gamma(t)],$$

thus we conclude if we prove

$$\mathcal{K}[\tilde{\delta}, \gamma(t) \vee \tilde{\delta}] - \mathcal{K}[\delta(t), \gamma(t)] \leq \mathcal{K}[\tilde{\delta}, \delta_h(t) \vee \tilde{\delta}] - \mathcal{K}[\delta(t), \delta_h(t)]. \quad (2.13)$$

With this aim, exploiting $(\varphi 4)$, in particular the monotonicity and concavity of φ_2 , and recalling that $\gamma(t) \geq \delta_h(t)$, we get:

$$\begin{aligned} \varphi_2(\gamma(t) \vee \tilde{\delta}) &= \varphi_2(\gamma(t) + [\tilde{\delta} - \gamma(t)]^+) \leq \varphi_2(\gamma(t) + [\tilde{\delta} - \delta_h(t)]^+) \\ &\leq \varphi_2(\gamma(t)) + \varphi_2(\delta_h(t) + [\tilde{\delta} - \delta_h(t)]^+) - \varphi_2(\delta_h(t)) \\ &= \varphi_2(\gamma(t)) + \varphi_2(\delta_h(t) \vee \tilde{\delta}) - \varphi_2(\delta_h(t)). \end{aligned}$$

The above inequality implies:

$$\begin{aligned} \mathcal{K}[\tilde{\delta}, \gamma(t) \vee \tilde{\delta}] - \mathcal{K}[\tilde{\delta}, \delta_h(t) \vee \tilde{\delta}] &= \int_0^L \left(\varphi_2(\gamma(t, x) \vee \tilde{\delta}(x)) - \varphi_2(\delta_h(t, x) \vee \tilde{\delta}(x)) \right) dx \\ &\leq \int_0^L \left(\varphi_2(\gamma(t, x)) - \varphi_2(\delta_h(t, x)) \right) dx \\ &= \mathcal{K}[\delta(t), \gamma(t)] - \mathcal{K}[\delta(t), \delta_h(t)], \end{aligned}$$

which is equivalent to (2.13).

We now prove (EB). Since the triplet $(\mathbf{u}, \boldsymbol{\alpha}, \gamma)$ satisfies (EB'), it is enough to prove

$$\mathcal{K}[\delta(t), \gamma(t)] = \mathcal{K}[\delta(t), \delta_h(t)], \quad \text{for every } t \in [0, T]. \quad (2.14)$$

Since $\gamma(t) \geq \delta_h(t)$ we easily deduce $\mathcal{K}[\delta(t), \gamma(t)] \geq \mathcal{K}[\delta(t), \delta_h(t)]$. To get the other inequality we first observe that arguing exactly as in the proof of Proposition 2.10, but replacing γ with δ_h (indeed we have just proved (GS)) we get:

$$\mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \delta_h(t)] \geq \mathcal{E}[\mathbf{u}^0, \boldsymbol{\alpha}^0] + \mathcal{D}[\boldsymbol{\alpha}^0] + \mathcal{K}[\delta^0, \delta^0] + \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t).$$

Combining the above inequality with (EB') we finally obtain

$$\mathcal{K}[\delta(t), \delta_h(t)] \geq \mathcal{K}[\delta(t), \gamma(t)],$$

hence (2.14) holds true.

If in addition φ_2 is strictly increasing, then (2.14) implies $\gamma(t) = \delta_h(t)$ since both functions are continuous in $[0, L]$. Thus we conclude. \square

3. PDE FORM OF ENERGETIC EVOLUTIONS

In this section we compute the Euler–Lagrange equations coming from the global stability condition (GS'). More precisely we prove that any generalised energetic evolution $(\mathbf{u}, \boldsymbol{\alpha}, \gamma)$ must satisfy, in a suitable weak formulation, the following system of equilibrium equations governing the stresses σ_i (see Proposition 3.3):

$$\begin{cases} -\sigma_1(t)' + \partial_y \varphi(\delta(t), \gamma(t)) \operatorname{sgn}(u_1(t) - u_2(t)) = 0, & \text{in } [0, L], \\ -\sigma_2(t)' - \partial_y \varphi(\delta(t), \gamma(t)) \operatorname{sgn}(u_1(t) - u_2(t)) = 0, & \text{in } [0, L], \end{cases} \quad \text{for every } t \in [0, T], \quad (3.1a)$$

where $\operatorname{sgn}(\cdot)$ denotes the signum function, together with a Karush–Kuhn–Tucker condition describing the evolution of the damage variables (if regular in time, see Propositions 3.4 and 3.5):

$$\begin{cases} \dot{\alpha}_i(t) \geq 0, & \text{in } [0, L], \\ -\alpha_i(t)'' + \frac{1}{2} E_i'(\alpha_i(t)) (u_i(t)')^2 + w_i'(\alpha_i(t)) \geq 0, & \text{in } [0, L], \\ \left[-\alpha_i(t)'' + \frac{1}{2} E_i'(\alpha_i(t)) (u_i(t)')^2 + w_i'(\alpha_i(t)) \right] \dot{\alpha}_i(t) = 0, & \text{in } [0, L], \end{cases} \quad \text{for a.e. } t \in [0, T]. \quad (3.1b)$$

The results of this section will be crucial for the achievement of our goal, namely the equivalence between the fictitious historical variable γ and the concrete one δ_h , under meaningful assumptions on φ . The argument based on temporal regularity of generalised energetic evolutions will be developed in Section 4.

We recall that, given the loading–unloading density $\varphi: \mathcal{T} \rightarrow [0, +\infty)$, we denote by ψ its restriction to the diagonal, namely $\psi(z) = \varphi(z, z)$, for $z \in [0, +\infty)$. Throughout the section the main assumptions on φ (and ψ) are:

$$\text{the function } \psi \text{ belongs to } C^1([0, +\infty)); \quad (3.2a)$$

$$\text{for every } z \in (0, +\infty) \text{ the map } \varphi(\cdot, z) \text{ belongs to } C^1([0, z]); \quad (3.2b)$$

$$\text{for every } z \in (0, +\infty) \text{ there hold } \partial_y \varphi(z, z) = \psi'(z) \text{ and } \partial_y \varphi(0, z) = 0; \quad (3.2c)$$

$$\text{the partial derivative } \partial_y \varphi \text{ belongs to } C^0(\mathcal{T} \setminus (0, 0)) \text{ and it is bounded in } \mathcal{T}. \quad (3.2d)$$

We notice that the above conditions are slightly more general than properties $(\varphi 5)$ – $(\varphi 8)$ listed in Section 1, since we do not require any convexity assumption (which will be instead employed in Section 4).

We start the analysis with a simple but useful lemma.

Lemma 3.1. *Let $f, g \in \mathbb{R}$ such that $f \geq |g|$ and assume the function $\varphi: \mathcal{T} \rightarrow [0, +\infty)$ satisfies:*

$$\text{the function } z \mapsto \varphi(z, z) =: \psi(z) \text{ is differentiable in } [0, +\infty); \quad (3.3a)$$

$$\text{for every } z \in (0, +\infty) \text{ the map } \varphi(\cdot, z) \text{ is differentiable in } [0, z]; \quad (3.3b)$$

$$\text{for every } z \in (0, +\infty) \text{ there hold } \partial_y \varphi(z, z) = \psi'(z). \quad (3.3c)$$

Then for every $v \in \mathbb{R}$ one has:

$$\lim_{h \rightarrow 0^+} \frac{\varphi(|g + hv|, f \vee |g + hv|) - \varphi(|g|, f)}{h} = \begin{cases} \partial_y \varphi(|g|, f) \operatorname{sgn}(g)v, & \text{if } f > |g| > 0, \\ \psi'(|g|) \operatorname{sgn}(g)v, & \text{if } f = |g| > 0, \\ \partial_y \varphi(0, f)|v|, & \text{if } f > |g| = 0, \\ \psi'(0)|v|, & \text{if } f = |g| = 0. \end{cases}$$

Proof. We denote by I the limit we want to compute and we distinguish among all the different cases. We first assume that $f > |g|$, so we get:

- if $g = 0$, then $I = \lim_{h \rightarrow 0^+} \frac{\varphi(h|v|, f) - \varphi(0, f)}{h} = \partial_y \varphi(0, f)|v|$;
- if $g > 0$, then $I = \lim_{h \rightarrow 0^+} \frac{\varphi(g + hv, f) - \varphi(g, f)}{h} = \partial_y \varphi(g, f)v = \partial_y \varphi(|g|, f) \operatorname{sgn}(g)v$;
- if $g < 0$, then $I = \lim_{h \rightarrow 0^+} \frac{\varphi(|g| - hv, f) - \varphi(|g|, f)}{h} = -\partial_y \varphi(|g|, f)v = \partial_y \varphi(|g|, f) \operatorname{sgn}(g)v$.

If instead $f = |g|$ we have:

- if $g = 0$, then $I = \lim_{h \rightarrow 0^+} \frac{\varphi(h|v|, h|v|) - \varphi(0, 0)}{h} = \lim_{h \rightarrow 0^+} \frac{\psi(h|v|) - \psi(0)}{h} = \psi'(0)|v|$;
- if $g > 0$ and $v \geq 0$, then $I = \lim_{h \rightarrow 0^+} \frac{\varphi(g + hv, g + hv) - \varphi(g, g)}{h} = \psi'(g)v = \psi'(|g|) \operatorname{sgn}(g)v$;
- if $g > 0$ and $v < 0$, then $I = \lim_{h \rightarrow 0^+} \frac{\varphi(g + hv, g) - \varphi(g, g)}{h} = \partial_y \varphi(g, g)v = \psi'(g)v = \psi'(|g|) \operatorname{sgn}(g)v$;
- if $g < 0$ and $v \geq 0$, then $I = \lim_{h \rightarrow 0^+} \frac{\varphi(|g| - hv, |g|) - \varphi(|g|, |g|)}{h} = -\partial_y \varphi(|g|, |g|)v = \psi'(|g|) \operatorname{sgn}(g)v$;
- if $g < 0$ and $v < 0$, then $I = \lim_{h \rightarrow 0^+} \frac{\psi(|g| - hv) - \psi(|g|)}{h} = -\psi'(|g|)v = \psi'(|g|) \operatorname{sgn}(g)v$.

So we conclude. \square

As an immediate corollary we deduce:

Corollary 3.2. *Let f, g be two measurable functions such that $f \in L^\infty(0, L)$ and $f \geq |g|$ a.e. in $[0, L]$, and assume φ satisfies (3.2a), (3.2b) and (3.2c). Then for every $v \in L^\infty(0, L)$ it holds:*

$$\lim_{h \rightarrow 0^+} \frac{\mathcal{K}[|g + hv|, f \vee |g + hv|] - \mathcal{K}[|g|, f]}{h} = \int_{\{|g| > 0\}} \partial_y \varphi(|g(x)|, f(x)) \operatorname{sgn}(g(x))v(x) \, dx + \psi'(0) \int_{\{f=0\}} v(x) \, dx.$$

Proof. We notice that, by the explicit expression of \mathcal{K} given by (1.8), the limit we want to compute can be written as

$$\lim_{h \rightarrow 0^+} \int_0^L \frac{\varphi(|g(x) + hv(x)|, f(x) \vee |g(x) + hv(x)|) - \varphi(|g(x)|, f(x))}{h} \, dx.$$

Assumptions (3.2a) and (3.2b) allow us to pass to the limit inside the integral, thus we conclude by means of Lemma 3.1 and exploiting (3.2c). \square

We are now in a position to state and prove the first result of this section, namely a weak form of the Euler–Lagrange equation for the displacement \mathbf{u} , or better for the stress $\boldsymbol{\sigma}$.

Proposition 3.3. *Let $E_i \in C^0([0, 1])$ and assume φ satisfies (3.2a), (3.2b) and (3.2c). Let $(\mathbf{u}, \boldsymbol{\alpha}, \gamma)$ satisfy (CO') and (GS') of Definition 1.8. Then for every $t \in [0, T]$ and for every $v \in [H_0^1(0, L)]^2$ it holds:*

$$\left| \int_0^L \sum_{i=1}^2 \sigma_i(t) v'_i \, dx + \int_{\{\delta(t) > 0\}} [\partial_y \varphi(\delta(t), \gamma(t)) \operatorname{sgn}(u_1(t) - u_2(t))] (v_1 - v_2) \, dx \right| \leq \psi'(0) \int_{\{\gamma(t)=0\}} |v_1 - v_2| \, dx, \quad (3.4)$$

where the stresses σ_i have been introduced in (1.5).

In particular, for every $t \in [0, T]$ the sum of the stresses $\sum_{i=1}^2 \sigma_i(t)$ is constant in $[0, L]$.

Proof. We fix $t \in [0, T]$ and by choosing $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}(t)$ in (GS') we get for every $h > 0$ and $\mathbf{v} \in [H_0^1(0, L)]^2$:

$$\begin{aligned} & \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \\ & \leq \mathcal{E}[\mathbf{u}(t) + h\mathbf{v}, \boldsymbol{\alpha}(t)] + \mathcal{K}[|u_1(t) - u_2(t) + h(v_1 - v_2)|, \gamma(t) \vee |u_1(t) - u_2(t) + h(v_1 - v_2)|]. \end{aligned}$$

Letting $h \rightarrow 0^+$ we thus deduce

$$\begin{aligned} 0 & \leq \lim_{h \rightarrow 0^+} \frac{\mathcal{E}[\mathbf{u}(t) + h\mathbf{v}, \boldsymbol{\alpha}(t)] - \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)]}{h} \\ & + \lim_{h \rightarrow 0^+} \frac{\mathcal{K}[|u_1(t) - u_2(t) + h(v_1 - v_2)|, \gamma(t) \vee |u_1(t) - u_2(t) + h(v_1 - v_2)|] - \mathcal{K}[\delta(t), \gamma(t)]}{h}. \end{aligned}$$

The first limit is trivially equal to $\int_0^L \sum_{i=1}^2 \sigma_i(t) v'_i dx$, while for the second one we employ Corollary 3.2 and we finally obtain:

$$\begin{aligned} 0 & \leq \int_0^L \sum_{i=1}^2 \sigma_i(t) v'_i dx + \int_{\{\delta(t) > 0\}} [\partial_y \varphi(\delta(t), \gamma(t)) \operatorname{sgn}(u_1(t) - u_2(t))] (v_1 - v_2) dx \\ & + \psi'(0) \int_{\{\gamma(t) = 0\}} |v_1 - v_2| dx. \end{aligned}$$

By following the same argument with $-\mathbf{v}$, we prove (3.4).

In particular if $v_1 = v_2 =: v$ we deduce that

$$\int_0^L \left(\sum_{i=1}^2 \sigma_i(t) \right) v' dx = 0, \quad \text{for every } v \in H_0^1(0, L),$$

and so $\sum_{i=1}^2 \sigma_i(t)$ is constant in $[0, L]$. \square

We want to point out that if $\psi'(0)$ were equal to 0 (usually false in a cohesive setting), then inequality (3.4) would actually be equivalent to the system (3.1a). The simplifications brought by the assumption $\psi'(0) = 0$ can be also found in [3], where it has been used for numerical reasons, and in [35], where it has been exploited to perform an approximation argument.

In our work, however, we do not need that additional assumption, indeed inequality (3.4) will be enough for our purposes.

Next proposition deals with the damage variable $\boldsymbol{\alpha}$:

Proposition 3.4. *Assume $E_i, w_i \in C^1([0, 1])$ and let $(\mathbf{u}, \boldsymbol{\alpha}, \gamma)$ satisfy (CO') and (GS') of Definition 1.8. Then, for every $t \in [0, T]$ and for every $\boldsymbol{\beta} \in [H^1(0, L)]^2$ such that $\beta_i \geq 0$ for $i = 1, 2$, it holds:*

$$\sum_{i=1}^2 \left(\frac{1}{2} \int_{\{\alpha_i(t) < 1\}} E'_i(\alpha_i(t)) (u_i(t)')^2 \beta_i dx + \int_{\{\alpha_i(t) < 1\}} w'_i(\alpha_i(t)) \beta_i dx + \int_{\{\alpha_i(t) < 1\}} \alpha_i(t)' \beta'_i dx \right) \geq 0.$$

Proof. We fix $t \in [0, T]$ and by choosing $\tilde{\mathbf{u}} = \mathbf{u}(t)$ in (GS') we get

$$\mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] \leq \mathcal{E}[\mathbf{u}(t), \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}], \quad \text{for every } \tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2 \text{ s.t. } \alpha_i(t) \leq \alpha_i \leq 1. \quad (3.5)$$

We now fix $\boldsymbol{\beta} \in [H^1(0, L)]^2$ such that $\beta_i \geq 0$ and given $h > 0$ we define $\tilde{\boldsymbol{\alpha}}^h(t, x)$ as the vector in \mathbb{R}^2 whose components are $(\alpha_i(t, x) + h\beta_i(x)) \wedge 1$. By plugging $\tilde{\boldsymbol{\alpha}}^h(t)$ in (3.5) as a test function

and letting $h \rightarrow 0^+$ we thus deduce:

$$\begin{aligned}
0 &\leq \liminf_{h \rightarrow 0^+} \frac{\mathcal{E}[\mathbf{u}(t), \tilde{\boldsymbol{\alpha}}^h(t)] - \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}^h(t)] - \mathcal{D}[\boldsymbol{\alpha}(t)]}{h} \\
&= \liminf_{h \rightarrow 0^+} \sum_{i=1}^2 \left(\frac{1}{2} \int_0^L \frac{E_i(\tilde{\alpha}_i^h(t)) - E_i(\alpha_i(t))}{h} (u_i(t)')^2 dx + \int_0^L \frac{w_i(\tilde{\alpha}_i^h(t)) - w_i(\alpha_i(t))}{h} dx \right. \\
&\quad \left. + \frac{1}{2} \int_0^L \frac{(\tilde{\alpha}_i^h(t)')^2 - (\alpha_i(t)')^2}{h} dx \right) = \liminf_{h \rightarrow 0^+} (I_h + II_h + III_h).
\end{aligned} \tag{3.6}$$

We study the limits of I_h , II_h , III_h separately. Since E_i, w_i are in $C^1([0, 1])$ we can pass the limit inside the integral in both I_h and II_h . We also notice that given $f \in C^1([0, 1])$, $a \in [0, 1]$ and $b \geq 0$ one has

$$\lim_{h \rightarrow 0^+} \frac{f((a + hb) \wedge 1) - f(a)}{h} = \begin{cases} f'(a)b, & \text{if } a \in [0, 1), \\ 0, & \text{if } a = 1. \end{cases}$$

Thus we deduce that

$$\lim_{h \rightarrow 0^+} I_h = \sum_{i=1}^2 \frac{1}{2} \int_{\{\alpha_i(t) < 1\}} E_i'(\alpha_i(t)) (u_i(t)')^2 \beta_i dx, \quad \text{and} \quad \lim_{h \rightarrow 0^+} II_h = \sum_{i=1}^2 \int_{\{\alpha_i(t) < 1\}} w_i'(\alpha_i(t)) \beta_i dx. \tag{3.7}$$

To deal with III_h we first observe that $\tilde{\alpha}_i^h(t)' = \begin{cases} \alpha_i(t)' + h\beta_i', & \text{a.e. in } \{\alpha_i(t) + h\beta_i < 1\}, \\ 0, & \text{a.e. in } \{\alpha_i(t) + h\beta_i \geq 1\}, \end{cases}$ and so

$$\begin{aligned}
III_h &= \sum_{i=1}^2 \left(\int_{\{\alpha_i(t) + h\beta_i < 1\}} \alpha_i(t)' \beta_i' dx + \frac{h}{2} \int_{\{\alpha_i(t) + h\beta_i < 1\}} (\beta_i')^2 dx - \frac{1}{2h} \int_{\{\alpha_i(t) + h\beta_i \geq 1\}} (\alpha_i(t)')^2 dx \right) \\
&\leq \sum_{i=1}^2 \left(\int_{\{\alpha_i(t) + h\beta_i < 1\}} \alpha_i(t)' \beta_i' dx + \frac{h}{2} \int_0^L (\beta_i')^2 dx \right).
\end{aligned}$$

By an easy application of dominated convergence theorem we hence obtain

$$\limsup_{h \rightarrow 0^+} III_h \leq \sum_{i=1}^2 \int_{\{\alpha_i(t) < 1\}} \alpha_i(t)' \beta_i' dx, \tag{3.8}$$

and collecting (3.6), (3.7) and (3.8) we conclude. \square

The last result of the section is a byproduct of the energy balance (EB'), assuming a priori that a generalised energetic evolution possesses a certain regularity in time. This kind of regularity will be however proved in Section 4 under suitable convexity assumptions on the data, thus this a priori requirement is not restrictive.

We refer to the Appendix for the definition and the main properties of absolutely continuous functions in Banach spaces, concepts we use in the next proposition.

Proposition 3.5. *Assume $E_i, w_i \in C^1([0, 1])$ and that φ satisfies (3.2) and ($\varphi 3$). Let $(\mathbf{u}, \boldsymbol{\alpha}, \gamma)$ be a generalised energetic evolution such that:*

$$\mathbf{u}, \boldsymbol{\alpha} \in AC([0, T]; [H^1(0, L)]^2), \quad \text{and} \quad \gamma \in C^0([0, T], C^0([0, L])).$$

Then for a.e. $t \in [0, T]$ one has:

$$\begin{aligned} & \bullet \frac{1}{2} \int_0^L E'_i(\alpha_i(t))(u_i(t)')^2 \dot{\alpha}_i(t) \, dx + \int_0^L w'_i(\alpha_i(t)) \dot{\alpha}_i(t) \, dx + \int_0^L \alpha_i(t)' \dot{\alpha}_i(t)' \, dx = 0, \quad \text{for } i = 1, 2; \\ & \bullet \lim_{h \rightarrow 0} \int_0^L \frac{\varphi(\delta(t), \gamma(t+h)) - \varphi(\delta(t), \gamma(t))}{h} \, dx = 0. \end{aligned} \quad (3.9)$$

Proof. First of all we notice that the temporal regularity we are assuming on \mathbf{u} and $\boldsymbol{\alpha}$ ensures that the maps $t \mapsto \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)]$ and $t \mapsto \mathcal{D}[\boldsymbol{\alpha}(t)]$ are absolutely continuous in $[0, T]$. Moreover for almost every time $t \in [0, T]$ the following expressions for their derivatives can be easily obtained:

$$\frac{d}{dt} \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)] = \sum_{i=1}^2 \left(\frac{1}{2} \int_0^L E'_i(\alpha_i(t))(u_i(t)')^2 \dot{\alpha}_i(t) \, dx + \int_0^L E_i(\alpha_i(t)) u_i(t)' \dot{u}_i(t)' \, dx \right); \quad (3.10a)$$

$$\frac{d}{dt} \mathcal{D}[\boldsymbol{\alpha}(t)] = \sum_{i=1}^2 \left(\int_0^L w'_i(\alpha_i(t)) \dot{\alpha}_i(t) \, dx + \int_0^L \alpha_i(t)' \dot{\alpha}_i(t)' \, dx \right). \quad (3.10b)$$

By (EB'), since the work of the prescribed displacement $\mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}]$ is absolutely continuous by definition, we now deduce that also the map $t \mapsto \mathcal{K}[\delta(t), \gamma(t)]$ is absolutely continuous in $[0, T]$. Moreover we know that δ belongs to $AC([0, T]; H_0^1(0, L))$, indeed both u_1 and u_2 are absolutely continuous with values in $H^1(0, L)$ by assumption. Thus for almost every $t \in [0, T]$ there exists the derivative of $\mathcal{K}[\delta(t), \gamma(t)]$ and we can compute:

$$\begin{aligned} \frac{d}{dt} \mathcal{K}[\delta(t), \gamma(t)] &= \lim_{h \rightarrow 0} \int_0^L \frac{\varphi(\delta(t+h), \gamma(t+h)) - \varphi(\delta(t), \gamma(t))}{h} \, dx \\ &= \int_0^L \partial_y \varphi(\delta(t), \gamma(t)) \dot{\delta}(t) \, dx + \lim_{h \rightarrow 0} \int_0^L \frac{\varphi(\delta(t), \gamma(t+h)) - \varphi(\delta(t), \gamma(t))}{h} \, dx, \end{aligned} \quad (3.11)$$

where we exploited the continuity assumption of both $\partial_y \varphi$ and γ .

Differentiating (EB'), using (3.10) and (3.11), and recalling that the sum of the stresses σ_i is constant in $[0, L]$ by Proposition 3.3, we deduce, for almost every $t \in [0, T]$,

$$\begin{aligned} 0 &= \sum_{i=1}^2 \left(\frac{1}{2} \int_0^L E'_i(\alpha_i(t))(u_i(t)')^2 \dot{\alpha}_i(t) \, dx + \int_0^L w'_i(\alpha_i(t)) \dot{\alpha}_i(t) \, dx + \int_0^L \alpha_i(t)' \dot{\alpha}_i(t)' \, dx \right) \\ &+ \int_0^L \sum_{i=1}^2 E_i(\alpha_i(t)) u_i(t)' \dot{u}_i(t)' \, dx + \int_0^L \partial_y \varphi(\delta(t), \gamma(t)) \dot{\delta}(t) \, dx - \dot{u}(t) \sum_{i=1}^2 \sigma_i(t, 0) \\ &+ \lim_{h \rightarrow 0} \int_0^L \frac{\varphi(\delta(t), \gamma(t+h)) - \varphi(\delta(t), \gamma(t))}{h} \, dx. \end{aligned} \quad (3.12)$$

The term in the third line of (3.12) is nonnegative by means of ($\varphi 3$) and the fact that γ is non-decreasing (in time). We thus conclude if we show that also the sum of the terms in the second line and each of the two terms (for $i = 1, 2$) in the sum in the first line are nonnegative.

We first focus on the first line. We notice that for $i = 1, 2$, the function $\dot{\alpha}_i(t) \in H^1(0, L)$ is nonnegative and vanishes on the set $\{\alpha_i(t) = 1\}$; indeed α_i is non-decreasing in time and it is always less or equal than 1. This means that we can use it as a test function in Proposition 3.4, getting for a.e. $t \in [0, T]$:

$$\frac{1}{2} \int_0^L E'_i(\alpha_i(t))(u_i(t)')^2 \dot{\alpha}_i(t) \, dx + \int_0^L w'_i(\alpha_i(t)) \dot{\alpha}_i(t) \, dx + \int_0^L \alpha_i(t)' \dot{\alpha}_i(t)' \, dx \geq 0.$$

As regards the sum of the terms in the second line in (3.12), we actually prove it is equal to zero. To this aim we make use of Proposition 3.3 choosing as test functions $v_i(x) = \dot{u}_i(t, x) - \dot{u}(t)x/L \in H_0^1(0, L)$, so that $|v_1 - v_2| = |\dot{u}_1(t) - \dot{u}_2(t)|$. We indeed notice that $|v_1 - v_2| = 0$ on the set $\{\gamma(t) = 0\}$: if x belongs to that set, then $u_1(\tau, x) = u_2(\tau, x)$ for every $\tau \in [0, t]$, and thus $\dot{u}_1(t, x) = \dot{u}_2(t, x)$. So we deduce for a.e. $t \in [0, T]$:

$$\begin{aligned} 0 &= \int_0^L \sum_{i=1}^2 \sigma_i(t) \left(\dot{u}_i(t)' - \frac{\dot{u}(t)}{L} \right) dx + \int_{\{\delta(t) > 0\}} [\partial_y \varphi(\delta(t), \gamma(t)) \operatorname{sgn}(u_1(t) - u_2(t))] (\dot{u}_1(t) - \dot{u}_2(t)) dx \\ &= \int_0^L \sum_{i=1}^2 E_i(\alpha_i(t)) u_i(t)' \dot{u}_i(t)' dx + \int_0^L \partial_y \varphi(\delta(t), \gamma(t)) \dot{\delta}(t) dx - \dot{u}(t) \sum_{i=1}^2 \sigma_i(t, 0). \end{aligned}$$

In the above equality we first used the fact that by definition

$$\dot{\delta}(t) = (\dot{u}_1(t) - \dot{u}_2(t)) \operatorname{sgn}(u_1(t) - u_2(t)), \quad \text{in } \{\delta(t) > 0\},$$

and then we exploited the assumption $\partial_y \varphi(0, z) = 0$ for $z > 0$.

So the proof is complete. \square

4. TEMPORAL REGULARITY AND EQUIVALENCE BETWEEN γ AND δ_h

In this last section we finally develop the strategy which will allow to show that the fictitious historical variable γ actually coincides with the concrete one δ_h in some meaningful cases, see Theorems 4.7 and 4.8. The argument, which exploits the results of Section 3, is based on the regularity in time of generalised energetic evolutions; this feature, as noticed in [32], is a peculiarity of systems governed by convex energies. For this reason, in this section we need to strengthen the assumptions on the data, requiring for $i = 1, 2$:

$$E_i \in C^2([0, 1]) \text{ is convex and satisfies } \frac{1}{2} E_i''(y) E_i(y) - E_i'(y)^2 > 0 \text{ for every } y \in [0, 1]; \quad (4.1)$$

$w_i \in C^1([0, 1])$ satisfies (1.7) and is uniformly convex with parameter $\mu_i > 0$, namely

$$w_i(\theta y^a + (1 - \theta) y^b) \leq \theta w_i(y^a) + (1 - \theta) w_i(y^b) - \frac{\mu_i}{2} \theta(1 - \theta) |y^a - y^b|^2, \text{ for every } \theta, y^a, y^b \in [0, 1]. \quad (4.2)$$

We notice that (4.1) implies (1.3), while (4.2) is trivially satisfied for instance by the simple example $w_i(y) = \frac{y^2 + y}{2}$. We also define

$$M_i := \max_{y \in [0, 1]} E_i''(y), \quad m_i := \min_{y \in [0, 1]} \left(\frac{1}{2} E_i''(y) E_i(y) - E_i'(y)^2 \right), \quad (4.3)$$

which are strictly positive by (4.1), and we finally denote by μ the minimum between μ_1 and μ_2 , namely

$$\mu := \mu_1 \wedge \mu_2 > 0. \quad (4.4)$$

Remark 4.1 (Hardening Materials). Condition (4.1) is a characteristic of the so called hardening materials, namely those materials for which the compliance $S(y) := E(y)^{-1}$ is strictly concave. Indeed by simple calculations one has:

$$S''(y) = -\frac{2}{E(y)^3} \left(\frac{1}{2} E''(y) E(y) - E'(y)^2 \right),$$

from which $S'' < 0$ if and only if (4.1) is satisfied. Temporal regularity of evolutions is expected only for this kind of materials, indeed in the opposite framework of softening materials (with convex compliance S) discontinuous evolutions are common due to snap-back phenomena (see also the analysis of [36]).

Of course we also need some sort of convexity for the loading–unloading density φ . However, we recall that usually it originates from a concave function ψ , see Remark 1.3; thus, in order to keep that crucial property, we only require a weak form of convexity assumption on ψ , already adopted in [34]:

the function ψ is λ -convex for some $\lambda > 0$, namely for every $\theta \in [0, 1]$ and $z^a, z^b \in [0, +\infty)$

$$\psi(\theta z^a + (1 - \theta)z^b) \leq \theta\psi(z^a) + (1 - \theta)\psi(z^b) + \frac{\lambda}{2}\theta(1 - \theta)|z^a - z^b|^2, \quad (4.5a)$$

while for φ itself, in addition to (3.2), we assume:

$$\text{for every } z \in (0, +\infty) \text{ the map } \varphi(\cdot, z) \text{ is non-decreasing and convex.} \quad (4.5b)$$

Remark 4.2. Coupling (3.2) with (4.5) we have thus recovered the assumptions $(\varphi 5)$ – $(\varphi 8)$ listed in Section 1. We point out again that they are satisfied by the prototypical example of loading–unloading density φ given by (1.11).

A crucial condition on the involved parameters will be given by

$$\frac{m_1}{M_1} \wedge \frac{m_2}{M_2} > \lambda \frac{L^2}{\pi^2}. \quad (4.6)$$

It morally says that the convexity of the internal energy \mathcal{E} , represented by $\frac{m_1}{M_1} \wedge \frac{m_2}{M_2}$, is stronger than the concavity of \mathcal{K} , represented by λ , and thus the overall behaviour is the one of a convex energy.

Remark 4.3. As already observed in [32, 36], a simple example of functions satisfying (4.1) is given by

$$E_i(y) = \frac{a_i}{(1 + y)^{b_i}}, \quad \text{with } a_i > 0 \text{ and } b_i \in (0, 1).$$

In this case indeed it holds

$$\frac{1}{2}E_i''(y)E_i(y) - E_i'(y)^2 = \frac{a_i^2}{2} \frac{b_i(1 - b_i)}{(1 + y)^{2(1+b_i)}} \geq \frac{a_i^2}{2} \frac{b_i(1 - b_i)}{4^{1+b_i}} = m_i.$$

Moreover $M_i = \max_{y \in [0, 1]} E_i''(y) = a_i b_i (1 + b_i)$, so that

$$\frac{m_i}{M_i} = \frac{a_i}{2} \frac{1 - b_i}{1 + b_i} \frac{1}{4^{1+b_i}}.$$

In the particular case in which $a_1 = a_2 =: a$ and $b_1 = b_2 = 1/2$ we get $\frac{m_1}{M_1} = \frac{m_2}{M_2} = \frac{a}{48}$ and so condition (4.6) can be written as:

$$\frac{a}{\lambda L^2} > \frac{48}{\pi^2},$$

and can be achieved by increasing the parameter a or by decreasing λ or the length of the bar L .

For convenience, in this section we also introduce the notation of the “shifted” energy, see also [30], Remark 3.2. For $t \in [0, T]$ and $x \in [0, L]$ we define the function $\bar{\mathbf{u}}_D(t, x) := \left(\frac{\bar{u}(t)}{L}x, \frac{\bar{u}(t)}{L}x \right)$ and we present the shifted variable $\mathbf{v}(t) = \mathbf{u}(t) - \bar{\mathbf{u}}_D(t)$, which has zero boundary conditions and hence it belongs to $[H_0^1(0, L)]^2$. We finally introduce the shifted energy:

$$\mathcal{E}_D[t, \mathbf{v}(t), \boldsymbol{\alpha}(t)] := \mathcal{E}[\mathbf{v}(t) + \bar{\mathbf{u}}_D(t), \boldsymbol{\alpha}(t)] = \mathcal{E}[\mathbf{u}(t), \boldsymbol{\alpha}(t)],$$

and we want to highlight its explicit dependence on time given by the prescribed displacement and encoded by the function $\bar{\mathbf{u}}_D$. Written in this form, the energy allows us to recast the work of the external prescribed displacement (1.14) in the following way:

$$\mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t) = \int_0^t \partial_t \mathcal{E}_D[\tau, \mathbf{v}(\tau), \boldsymbol{\alpha}(\tau)] d\tau. \quad (4.7)$$

Moreover, by simple computations, it is easy to see that for almost every time $\tau \in [0, T]$ and for every $t \in [0, T]$ the following inequality holds true:

$$\begin{aligned} & |\partial_t \mathcal{E}_D[\tau, \mathbf{v}(\tau), \boldsymbol{\alpha}(\tau)] - \partial_t \mathcal{E}_D[\tau, \mathbf{v}(t), \boldsymbol{\alpha}(t)]| \\ & \leq C |\dot{\bar{\mathbf{u}}}(\tau)| \left(\|\boldsymbol{\alpha}(\tau) - \boldsymbol{\alpha}(t)\|_{[H^1(0, L)]^2}^2 + \|\mathbf{v}(\tau)' - \mathbf{v}(t)'\|_{[L^2(0, L)]^2}^2 \right)^{\frac{1}{2}}, \end{aligned} \quad (4.8)$$

where $C > 0$ is a suitable positive constant.

Furthermore we also notice that the global stability condition (GS') of Definition 1.8 can be rewritten as:

for every $t \in [0, T]$ one has $\gamma(t) \geq \delta(t)$ in $[0, L]$ and

$$\mathcal{E}_D[t, \mathbf{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \leq \mathcal{E}_D[t, \tilde{\mathbf{v}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}[\tilde{\delta}, \gamma(t) \vee \tilde{\delta}],$$

for every $\tilde{\mathbf{v}} \in [H_0^1(0, L)]^2$ and for every $\tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2$ such that $\alpha_i(t) \leq \tilde{\alpha}_i \leq 1$ in $[0, L]$ for $i = 1, 2$;

We finally have all the ingredients to start the analysis regarding the temporal regularity of generalised energetic evolutions. We first state a useful lemma, whose simple proof can be found for instance in [25], Lemma 5.6, or in [27], Lemma 4.3.

Lemma 4.4. *Let $(X, \|\cdot\|)$ be a normed space and let $f: [a, b] \rightarrow X$ be a bounded measurable function such that:*

$$\|f(t) - f(s)\|^2 \leq \int_s^t \|f(t) - f(\tau)\| g(\tau) d\tau, \quad \text{for every } a \leq s \leq t \leq b,$$

for some nonnegative $g \in L^1(a, b)$. Then actually it holds:

$$\|f(t) - f(s)\| \leq \int_s^t g(\tau) d\tau, \quad \text{for every } a \leq s \leq t \leq b.$$

We are now in a position to state and prove the first result of this section, which yields temporal regularity of generalised energetic evolutions under the convexity assumptions we stated before. The argument is based on the ideas of [32], adapted to our setting where also a cohesive energy (concave by nature) is taken into account.

Proposition 4.5 (Temporal Regularity). *Assume E_i satisfies (4.1), w_i satisfies (4.2), and assume $\varphi \in C^0(\mathcal{T})$ satisfies $(\varphi 3)$, $(\varphi 5)$ – $(\varphi 8)$. Let $(\mathbf{u}, \boldsymbol{\alpha}, \gamma)$ be a generalised energetic evolution related to the prescribed displacement $\bar{\mathbf{u}} \in AC([0, T])$. If condition (4.6) on the parameters is satisfied, then both the displacements \mathbf{u} and the damages $\boldsymbol{\alpha}$ belong to $AC([0, T]; [H^1(0, L)]^2)$, and so one also has $\delta \in AC([0, T]; H^1(0, L))$ and $\delta_h \in AC([0, T]; C^0([0, L]))$.*

If in addition the family $\{\gamma(t) \wedge \bar{\delta}\}_{t \in [0, T]}$ is equicontinuous and for every $y \in [0, \bar{\delta}]$ the map $\varphi(y, \cdot)$ is strictly increasing in $[y, \bar{\delta}]$, then the function $\gamma \wedge \bar{\delta}$ belongs to $C^0([0, T]; C^0([0, L]))$.

Remark 4.6. We want to point out that the additional requirement of equicontinuity of the family $\{\gamma(t) \wedge \bar{\delta}\}_{t \in [0, T]}$, although can not be derived directly from the Definition 1.8 of generalised energetic evolutions, is automatically satisfied by the limit function γ obtained in Proposition 2.5 thanks to the uniform estimate (2.5b). Thus it is not restrictive.

Proof of Proposition 4.5. We first consider, for $i = 1, 2$, the Hessian matrix of the function $[0, 1] \times \mathbb{R} \ni (\alpha, v) \mapsto \frac{1}{2}E_i(\alpha)v^2$, denoted by $H_i(\alpha, v)$, and its quadratic form, namely the map:

$$(x, y) \mapsto \langle (x, y), H_i(\alpha, v)(x, y) \rangle = \frac{1}{2}E_i''(\alpha)v^2x^2 + 2E_i'(\alpha)vxy + E_i(\alpha)y^2.$$

By (4.1) it must be $E_i''(\alpha) > 0$ for every $\alpha \in [0, 1]$ and so we can write:

$$\begin{aligned} \langle (x, y), H_i(\alpha, v)(x, y) \rangle &= \frac{2}{E_i''(\alpha)} \left[\left(\frac{1}{2}E_i''(\alpha)v x + E_i'(\alpha)y \right)^2 + \left(\frac{1}{2}E_i''(\alpha)E_i(\alpha) - E_i'(\alpha)^2 \right) y^2 \right] \\ &\geq 2\frac{m_i}{M_i}y^2. \end{aligned}$$

Thanks to this estimate on the Hessian matrix it is easy to infer that for every $t \in [0, T]$, for every $\theta \in [0, 1]$ and for every $\mathbf{v}^a, \mathbf{v}^b \in [H_0^1(0, L)]^2$ and $\boldsymbol{\alpha}^a, \boldsymbol{\alpha}^b \in [H_{[0,1]}^1(0, L)]^2$ it holds:

$$\begin{aligned} &\mathcal{E}_D[t, \theta\mathbf{v}^a + (1-\theta)\mathbf{v}^b, \theta\boldsymbol{\alpha}^a + (1-\theta)\boldsymbol{\alpha}^b] \\ &\leq \theta\mathcal{E}_D[t, \mathbf{v}^a, \boldsymbol{\alpha}^a] + (1-\theta)\mathcal{E}_D[t, \mathbf{v}^b, \boldsymbol{\alpha}^b] - \frac{m_1}{M_1} \wedge \frac{m_2}{M_2} \theta(1-\theta) \|(\mathbf{v}^a)' - (\mathbf{v}^b)'\|_{[L^2(0, L)]^2}^2. \end{aligned} \quad (4.9)$$

By means of (4.2) we also deduce that for every $t \in [0, T]$, for every $\theta \in [0, 1]$ and for every $\boldsymbol{\alpha}^a, \boldsymbol{\alpha}^b \in [H_{[0,1]}^1(0, L)]^2$ we have:

$$\mathcal{D}[\theta\boldsymbol{\alpha}^a + (1-\theta)\boldsymbol{\alpha}^b] \leq \theta\mathcal{D}[\boldsymbol{\alpha}^a] + (1-\theta)\mathcal{D}[\boldsymbol{\alpha}^b] - \frac{\mu \wedge 1}{2} \theta(1-\theta) \|\boldsymbol{\alpha}^a - \boldsymbol{\alpha}^b\|_{[H^1(0, L)]^2}^2. \quad (4.10)$$

Finally, by (3.2c), (4.5a) and (4.5b) (which are implied by $(\varphi 5)$ – $(\varphi 8)$) we deduce that for every $z \in [0, +\infty)$ the function $y \mapsto \varphi(y, z \vee y)$ is λ -convex in $[0, +\infty)$; thus for every $t \in [0, T]$, for every $\theta \in [0, 1]$ and for every nonnegative $\delta^a, \delta^b \in H_0^1(0, L)$ it holds:

$$\begin{aligned} &\mathcal{K}[\theta\delta^a + (1-\theta)\delta^b, \gamma(t) \vee (\theta\delta^a + (1-\theta)\delta^b)] \\ &\leq \theta\mathcal{K}[\delta^a, \gamma(t) \vee \delta^a] + (1-\theta)\mathcal{K}[\delta^b, \gamma(t) \vee \delta^b] + \frac{\lambda}{2} \theta(1-\theta) \|\delta^a - \delta^b\|_{L^2(0, L)}^2. \end{aligned} \quad (4.11)$$

We now fix $t \in [0, T]$, $\theta \in (0, 1)$, $\tilde{\mathbf{v}} \in [H_0^1(0, L)]^2$, $\tilde{\boldsymbol{\alpha}} \in [H^1(0, L)]^2$ such that $\alpha_i(t) \leq \tilde{\alpha}_i(t) \leq 1$ for $i = 1, 2$, and we consider as competitors in (GS') the functions $\theta\tilde{\mathbf{v}} + (1-\theta)\mathbf{v}(t)$ and $\theta\tilde{\boldsymbol{\alpha}} + (1-\theta)\boldsymbol{\alpha}(t)$; by means of (4.9), (4.10) and (4.11), together with $(\varphi 3)$ and (4.5b), we thus get:

$$\begin{aligned} &\mathcal{E}_D[t, \mathbf{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \\ &\leq \mathcal{E}_D[t, \theta\tilde{\mathbf{v}} + (1-\theta)\mathbf{v}(t), \theta\tilde{\boldsymbol{\alpha}} + (1-\theta)\boldsymbol{\alpha}(t)] + \mathcal{D}[\theta\tilde{\boldsymbol{\alpha}} + (1-\theta)\boldsymbol{\alpha}(t)] \\ &\quad + \mathcal{K}[|\theta(\tilde{v}_1 - \tilde{v}_2) + (1-\theta)(v_1(t) - v_2(t))|, \gamma(t) \vee |\theta(\tilde{v}_1 - \tilde{v}_2) + (1-\theta)(v_1(t) - v_2(t))|] \\ &\leq \theta\mathcal{E}_D[t, \tilde{\mathbf{v}}, \tilde{\boldsymbol{\alpha}}] + (1-\theta)\mathcal{E}_D[t, \mathbf{v}(t), \boldsymbol{\alpha}(t)] - \frac{m_1}{M_1} \wedge \frac{m_2}{M_2} \theta(1-\theta) \|(\tilde{\mathbf{v}})' - (\mathbf{v}(t))'\|_{[L^2(0, L)]^2}^2 \\ &\quad + \theta\mathcal{D}[\tilde{\boldsymbol{\alpha}}] + (1-\theta)\mathcal{D}[\boldsymbol{\alpha}(t)] - \frac{\mu \wedge 1}{2} \theta(1-\theta) \|\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}(t)\|_{[H^1(0, L)]^2}^2 \\ &\quad + \theta\mathcal{K}[\tilde{\delta}, \gamma(t) \vee \tilde{\delta}] + (1-\theta)\mathcal{K}[\delta(t), \gamma(t)] + \frac{\lambda}{2} \theta(1-\theta) \|\tilde{\delta} - \delta(t)\|_{L^2(0, L)}^2. \end{aligned} \quad (4.12)$$

We now exploit the well known sharp Poincaré inequality:

$$\int_a^b f(x)^2 dx \leq \frac{(b-a)^2}{\pi^2} \int_a^b f'(x)^2 dx, \text{ for every } f \in H_0^1(a, b),$$

to deduce that

$$\|\tilde{\delta} - \delta(t)\|_{L^2(0, L)}^2 \leq 2\frac{L^2}{\pi^2} \|(\tilde{\mathbf{v}})' - (\mathbf{v}(t))'\|_{[L^2(0, L)]^2}^2. \quad (4.13)$$

By plugging (4.13) in (4.12), dividing by θ and then letting $\theta \rightarrow 0^+$ we finally deduce:

$$\begin{aligned} & \left(\frac{m_1}{M_1} \wedge \frac{m_2}{M_2} - \lambda \frac{L^2}{\pi^2} \right) \|(\tilde{\mathbf{v}})' - (\mathbf{v}(t))'\|_{[L^2(0,L)]^2}^2 + \frac{\mu \wedge 1}{2} \|\tilde{\boldsymbol{\alpha}} - \boldsymbol{\alpha}(t)\|_{[H^1(0,L)]^2}^2 \\ & + \mathcal{E}_D[t, \mathbf{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(t)] \\ & \leq \mathcal{E}_D[t, \tilde{\mathbf{v}}, \tilde{\boldsymbol{\alpha}}] + \mathcal{D}[\tilde{\boldsymbol{\alpha}}] + \mathcal{K}[\tilde{\delta}, \gamma(t) \vee \tilde{\delta}]. \end{aligned} \quad (4.14)$$

For the sake of simplicity we denote by c the minimum between $\frac{m_1}{M_1} \wedge \frac{m_2}{M_2} - \lambda \frac{L^2}{\pi^2}$ and $\frac{\mu \wedge 1}{2}$, and we notice that c is strictly positive by (4.6). We now fix two times $0 \leq s \leq t \leq T$. Exploiting (4.14) at time s with $\tilde{\mathbf{v}} = \mathbf{v}(t)$ and $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}(t)$, and recalling (EB') and (4.7) we obtain:

$$\begin{aligned} & c \left(\|\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)\|_{[H^1(0,L)]^2}^2 + \|\mathbf{v}(t)' - \mathbf{v}(s)'\|_{[L^2(0,L)]^2}^2 \right) \\ & \leq \mathcal{E}_D[s, \mathbf{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{D}[\boldsymbol{\alpha}(t)] + \mathcal{K}[\delta(t), \gamma(s) \vee \delta(t)] - \mathcal{E}_D[s, \mathbf{v}(s), \boldsymbol{\alpha}(s)] - \mathcal{D}[\boldsymbol{\alpha}(s)] - \mathcal{K}[\delta(s), \gamma(s)] \\ & \leq \mathcal{E}_D[s, \mathbf{v}(t), \boldsymbol{\alpha}(t)] - \mathcal{E}_D[t, \mathbf{v}(t), \boldsymbol{\alpha}(t)] + \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](t) - \mathcal{W}[\mathbf{u}, \boldsymbol{\alpha}](s) \\ & \leq \int_s^t |\partial_t \mathcal{E}_D[\tau, \mathbf{v}(\tau), \boldsymbol{\alpha}(\tau)] - \partial_t \mathcal{E}_D[\tau, \mathbf{v}(t), \boldsymbol{\alpha}(t)]| d\tau. \end{aligned}$$

By using (4.8) we thus deduce:

$$\begin{aligned} & \|\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)\|_{[H^1(0,L)]^2}^2 + \|\mathbf{v}(t)' - \mathbf{v}(s)'\|_{[L^2(0,L)]^2}^2 \\ & \leq \frac{C}{c} \int_s^t |\dot{u}(\tau)| \left(\|\boldsymbol{\alpha}(\tau) - \boldsymbol{\alpha}(t)\|_{[H^1(0,L)]^2}^2 + \|\mathbf{v}(\tau)' - \mathbf{v}(t)'\|_{[L^2(0,L)]^2}^2 \right)^{\frac{1}{2}} d\tau. \end{aligned}$$

By means of (1.16a) we can apply Lemma 4.4 getting:

$$\left(\|\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)\|_{[H^1(0,L)]^2}^2 + \|\mathbf{v}(t)' - \mathbf{v}(s)'\|_{[L^2(0,L)]^2}^2 \right)^{\frac{1}{2}} \leq \frac{C}{c} \int_s^t |\dot{u}(\tau)| d\tau,$$

and so we infer that $\boldsymbol{\alpha}$ belongs to $AC([0, T]; [H^1(0, L)]^2)$ and \mathbf{v} belongs to $AC([0, T]; [H_0^1(0, L)]^2)$. By construction we also have

$$\begin{aligned} \|\mathbf{u}(t) - \mathbf{u}(s)\|_{[H^1(0,L)]^2} & \leq \|\mathbf{v}(t) - \mathbf{v}(s)\|_{[H^1(0,L)]^2} + \|\mathbf{u}_D(t) - \mathbf{u}_D(s)\|_{[H^1(0,L)]^2} \\ & \leq \|\mathbf{v}(t) - \mathbf{v}(s)\|_{[H^1(0,L)]^2} + C|\bar{u}(t) - \bar{u}(s)|, \end{aligned}$$

so also \mathbf{u} belongs to $AC([0, T]; [H^1(0, L)]^2)$ and as a simple byproduct we obtain that δ is $AC([0, T]; H^1(0, L))$.

Since $H^1(0, L) \subseteq C^0([0, L])$, in particular there exists a nonnegative function $\phi \in L^1(0, T)$ such that

$$\|\delta(t) - \delta(s)\|_{C^0([0,L])} \leq \int_s^t \phi(\tau) d\tau, \quad \text{for every } 0 \leq s \leq t \leq T. \quad (4.15)$$

We now show that the same inequality holds true for δ_h in place of δ . We thus fix $0 \leq s \leq t \leq T$ and $x \in [0, L]$. If $\delta_h(t, x) = \delta_h(s, x)$ there is nothing to prove, so let us assume $\delta_h(t, x) > \delta_h(s, x)$. By definition of δ_h and since now we know that δ is continuous both in time and space we deduce that

$$\delta_h(t, x) = \max_{\tau \in [0, t]} \delta(\tau, x) = \delta(t_x, x), \quad \text{for some } t_x \in [s, t].$$

So we have

$$\delta_h(t, x) - \delta_h(s, x) \leq \delta(t_x, x) - \delta(s, x) \leq \int_s^{t_x} \phi(\tau) d\tau \leq \int_s^t \phi(\tau) d\tau.$$

We have thus proved the validity of (4.15) with δ_h in place of δ , and hence δ_h belongs to $AC([0, T]; C^0([0, L]))$.

We only need to prove that $\gamma \wedge \bar{\delta} \in C^0([0, T]; C^0([0, L])$ under the additional assumptions that $\{\gamma(t) \wedge \bar{\delta}\}_{t \in [0, T]}$ is an equicontinuous family and $\varphi(y, \cdot)$ is strictly increasing in $[y, \bar{\delta})$ for any given $y \in [0, \bar{\delta})$. For the sake of clarity we prove it only in the case $\bar{\delta} = +\infty$; in the other situation the result can be obtained arguing in the same way and recalling equality (1.12). To this aim we observe that, by equicontinuity, for every $t \in [0, T]$ the right and the left limits $\gamma^+(t)$ and $\gamma^-(t)$ are continuous in $[0, L]$. By monotonicity and using classical Dini's theorem we hence obtain

$$\gamma^\pm(t) = \lim_{h \rightarrow 0^\pm} \gamma(t+h), \quad \text{uniformly in } [0, L]. \quad (4.16)$$

So we conclude if we prove that $\gamma^+(t) = \gamma^-(t)$.

Arguing as in the proof of Proposition 3.5, since \mathbf{u} and $\boldsymbol{\alpha}$ are in $AC([0, T]; [H^1(0, L)]^2)$, we deduce by (EB') that the map $t \mapsto \mathcal{K}[\delta(t), \gamma(t)]$ is continuous in $[0, T]$, and thus for every $t \in [0, T]$ we have:

$$\lim_{h \rightarrow 0^+} \mathcal{K}[\delta(t+h), \gamma(t+h)] = \lim_{h \rightarrow 0^-} \mathcal{K}[\delta(t+h), \gamma(t+h)].$$

By using (4.16) we can pass to the limit inside the integral getting

$$\int_0^L \varphi(\delta(t), \gamma^+(t)) \, dx = \int_0^L \varphi(\delta(t), \gamma^-(t)) \, dx.$$

Since $\varphi(y, \cdot)$ is strictly increasing we conclude. \square

Thanks to the temporal regularity obtained in the previous proposition we are able to prove our main results. The first theorem ensures the equality between γ and δ_h (actually between $\gamma \wedge \bar{\delta}$ and $\delta_h \wedge \bar{\delta}$, which however are the meaningful ones, see Remark 1.9) assuming a priori equicontinuity on the family $\{\gamma(t)\}_{t \in [0, T]}$, which is however not restrictive due to Remark 4.6; a similar argument to the one adopted here, but in an easier setting, can be found in [37], Proposition 2.7. The second theorem states that the generalised energetic evolution obtained in Section 2 as limit of discrete minimisers is actually an energetic evolution. We thus reach our goal, avoiding the assumption ($\varphi 4$), and considering the list of reasonable assumptions ($\varphi 5$)–($\varphi 9$) (actually we replace ($\varphi 9$) by the weaker (4.17)) which for instance are satisfied by the example provided in Remark 1.3.

Theorem 4.7 (Equivalence between γ and δ_h). *Let the prescribed displacement \bar{u} belong to the space $AC([0, T])$. Assume E_i satisfies (4.1), w_i satisfies (4.2), and $\varphi \in C^0(\mathcal{T})$ satisfies ($\varphi 5$)–($\varphi 8$), plus the following uniform strict monotonicity with respect to z :*

for every compact set $K \in \{z > y \geq 0\} \cap \overline{\mathcal{T}}_{\bar{\delta}}$ there exists a positive constant $C_K > 0$ such that

$$\varphi(y, z_2) - \varphi(y, z_1) \geq C_K(z_2 - z_1)$$

for every $(z_2, y), (z_1, y) \in K$ satisfying $z_2 \geq z_1$.

(4.17)

Assume also condition (4.6) on the parameters. Then, given a generalised energetic evolution $(\mathbf{u}, \boldsymbol{\alpha}, \gamma)$ such that the family $\{\gamma(t) \wedge \bar{\delta}\}_{t \in [0, T]}$ is equicontinuous, the function $\gamma \wedge \bar{\delta}$ coincides with $\delta_h \wedge \bar{\delta}$.

Proof. For the sake of clarity we prove the result only in the case $\bar{\delta} = +\infty$, being the other situation analogous by (1.12).

We know that $\gamma \geq \delta_h$ and that $\gamma(0) = \delta_h(0) = \delta^0$ and $\gamma(t, 0) = \gamma(t, L) = \delta_h(t, 0) = \delta_h(t, L) = 0$ for every $t \in [0, T]$. Moreover by Proposition 4.5 we know that both γ and δ_h are continuous on $[0, T] \times [0, L]$.

We thus assume by contradiction there exists $(\bar{t}, \bar{x}) \in (0, T] \times (0, L)$ for which $\gamma(\bar{t}, \bar{x}) > \delta_h(\bar{t}, \bar{x})$; by continuity we thus deduce there exists $\eta > 0$ such that

$$\gamma(t, x) > \delta_h(t, x) \geq \delta(t, x), \quad \text{for every } (t, x) \in [\bar{t} - \eta, \bar{t}] \times [\bar{x} - \eta, \bar{x} + \eta].$$

By assumption (4.17) we hence infer the existence of constant $c_\eta > 0$ for which

$$\begin{aligned} \varphi(\delta(s, x), \gamma(t, x)) - \varphi(\delta(s, x), \gamma(s, x)) &\geq c_\eta(\gamma(t, x) - \gamma(s, x)), \\ \text{for every } \bar{t} - \eta \leq s \leq t \leq \bar{t} \text{ and } x \in [\bar{x} - \eta, \bar{x} + \eta]. \end{aligned} \quad (4.18)$$

We now recall that by Proposition 4.5 we know the map $t \mapsto \mathcal{K}[\delta(t), \gamma(t)]$ is absolutely continuous in $[0, T]$. So for every $0 \leq s \leq t \leq T$ we can estimate:

$$\begin{aligned} &\int_0^L (\varphi(\delta(s), \gamma(t)) - \varphi(\delta(s), \gamma(s))) \, dx \\ &= \mathcal{K}[\delta(t), \gamma(t)] - \mathcal{K}[\delta(s), \gamma(s)] + \int_0^L (\varphi(\delta(s), \gamma(t)) - \varphi(\delta(t), \gamma(t))) \, dx \\ &\leq \int_s^t \frac{d}{d\tau} \mathcal{K}[\delta(\tau), \gamma(\tau)] \, d\tau + C \|\delta(t) - \delta(s)\|_{C^0([0, L])} \leq \int_s^t \phi(\tau) \, d\tau, \end{aligned} \quad (4.19)$$

where $\phi \in L^1(0, T)$ is a suitable nonnegative function.

Combining (4.18) and (4.19) we now obtain:

$$c_\eta \int_{\bar{x} - \eta}^{\bar{x} + \eta} (\gamma(t) - \gamma(s)) \, dx \leq \int_s^t \phi(\tau) \, d\tau, \quad \text{for every } \bar{t} - \eta \leq s \leq t \leq \bar{t},$$

hence $\gamma \in AC([\bar{t} - \eta, \bar{t}]; L^1(\bar{x} - \eta, \bar{x} + \eta))$.

By means of (3.9) we now deduce that for a.e. $t \in [\bar{t} - \eta, \bar{t}]$ we have:

$$0 = \lim_{h \rightarrow 0} \int_0^L \frac{\varphi(\delta(t), \gamma(t+h)) - \varphi(\delta(t), \gamma(t))}{h} \, dx \geq c_\eta \limsup_{h \rightarrow 0} \int_{\bar{x} - \eta}^{\bar{x} + \eta} \frac{\gamma(t+h) - \gamma(t)}{h} \, dx \geq 0,$$

namely for almost every $t \in [\bar{t} - \eta, \bar{t}]$ the function γ is strongly differentiable in $L^1(\bar{x} - \eta, \bar{x} + \eta)$ and $\dot{\gamma}(t) = 0$. By Proposition A.3 we now obtain

$$\gamma(t) = \gamma(\bar{t} - \eta) + \int_{\bar{t} - \eta}^t \dot{\gamma}(\tau) \, d\tau = \gamma(\bar{t} - \eta), \quad \text{for every } t \in [\bar{t} - \eta, \bar{t}], \text{ as an equality in } L^1(\bar{x} - \eta, \bar{x} + \eta).$$

In particular, since γ is continuous, we deduce that $\gamma(\bar{t}, \bar{x}) = \gamma(\bar{t} - \eta, \bar{x})$.

Since δ_h is non-decreasing we can iterate all the previous argument, finally getting $\gamma(\bar{t}, \bar{x}) = \gamma(0, \bar{x})$. But this is absurd, indeed it implies:

$$\delta^0(\bar{x}) = \gamma(0, \bar{x}) = \gamma(\bar{t}, \bar{x}) > \delta_h(\bar{t}, \bar{x}) \geq \delta_h(0, \bar{x}) = \delta^0(\bar{x}),$$

and so we conclude. \square

Theorem 4.8 (Existence of Energetic Evolutions). *Let the prescribed displacement \bar{u} belong to the space $AC([0, T])$ and the initial data $\mathbf{u}^0, \boldsymbol{\alpha}^0$ fulfil (1.13) together with the stability condition (1.15). Assume E_i satisfies (4.1), w_i satisfies (4.2), and $\varphi \in C^0(\mathcal{T})$ satisfies $(\varphi 2)$, $(\varphi 5)$ – $(\varphi 8)$ and (4.17). Assume also condition (4.6) on the parameters. Then the pair $(\mathbf{u}, \boldsymbol{\alpha})$ composed by the functions obtained in Proposition 2.5 is an energetic evolution, since it holds $\gamma \wedge \bar{\delta} = \delta_h \wedge \bar{\delta}$.*

Moreover \mathbf{u} and $\boldsymbol{\alpha}$ belong to $AC([0, T]; [H^1(0, L)]^2)$, and so in particular the historical slip δ_h is in $AC([0, T]; C^0([0, L]))$.

Proof. The result is a simple byproduct of Theorem 2.11 together with Proposition 4.5 and Theorem 4.7 (we also recall (1.12)). We indeed notice that the equicontinuity assumption on the family $\{\gamma(t) \wedge \bar{\delta}\}_{t \in [0, T]}$ (actually on the whole $\{\gamma(t)\}_{t \in [0, T]}$) is automatically satisfied by the

limit function γ obtained in Proposition 2.5 thanks to the uniform bounds (2.5b) on the discrete interpolations. \square

Conclusions. The obtained results put the basis for further investigations. First of all we may mention the generalisation of the simple one-dimensional model here presented to higher dimensional settings, as the ones numerically investigated in [2]. A second line of exploration could also be the analysis of the problem in case of complete damage, meant as complete loss of material stiffness.

Moreover, it would be interesting to extend the proposed approach to classical problems of cohesive fracture mechanics. In this case, dissipation combined with irreversible effects introduces difficulties, at least when dealing with global minimisers of the energy, in considering loading-unloading cohesive laws that reflect the real behaviour of materials rather than hypotheses dictated by mere mathematical assumptions. The main difference provided by cohesive fracture models with respect to the considered problem of cohesive interface relies in the reduced dimension of the fracture, which is a $(d - 1)$ -dimensional object in a d -dimensional material. This feature involves the use of weaker topologies, which can not be directly treated following our argument, and thus requires further adaptations to transfer our results.

APPENDIX A. ABSOLUTELY CONTINUOUS AND BV-VECTOR VALUED FUNCTIONS

In this Appendix we briefly present the main definitions and properties of vector valued absolutely continuous functions and functions of bounded variation we used throughout the paper. A deeper and more detailed analysis can be found in the Appendix of [15], to which we refer for all the proofs and examples. Here $(X, \|\cdot\|)$ will denote a Banach space, and by X^* we mean its topological dual. The duality product between $w \in X^*$ and $x \in X$ is finally denoted by $\langle w, x \rangle$.

Definition A.1. A function $f: [0, T] \rightarrow X$ is said to be:

- a function of bounded variation ($BV([0, T]; X)$) if

$$V_X(f; 0, T) := \sup_{\substack{\text{finite partitions} \\ \text{of } [0, T]}} \sum \|f(t_k) - f(t_{k-1})\| < +\infty;$$

- absolutely continuous ($AC([0, T]; X)$) if there exists a nonnegative function $\phi \in L^1(0, T)$ such that

$$\|f(t) - f(s)\| \leq \int_s^t \phi(\tau) \, d\tau, \quad \text{for every } 0 \leq s \leq t \leq T;$$

- in the space $\widetilde{W}^{1,p}(0, T; X)$, $p \in [1, +\infty]$, if there exists a nonnegative function $\phi \in L^p(0, T)$ such that

$$\|f(t) - f(s)\| \leq \int_s^t \phi(\tau) \, d\tau, \quad \text{for every } 0 \leq s \leq t \leq T;$$

- in the Sobolev space $W^{1,p}(0, T; X)$, $p \in [1, +\infty]$, if there exists a function $g \in L^p(0, T; X)$ such that

$$f(t) = f(0) + \int_0^t g(\tau) \, d\tau, \quad \text{for every } t \in [0, T].$$

As in the classical case $X = \mathbb{R}$ any function of bounded variation belongs to $L^\infty(0, T; X)$, it admits right and left (strong) limits at every $t \in [0, T]$ and the set of its discontinuity points is at most countable. To gain the well known property of almost everywhere differentiability also in the vector valued framework it is instead crucial to require X to be reflexive (see the examples in [15]).

Proposition A.2. *If X is reflexive, then any function f belonging to $BV([0, T]; X)$ is weakly differentiable almost everywhere in $[0, T]$. Moreover $\|\dot{f}(t)\| \leq \frac{d}{dt}V_X(f; 0, t)$ for a.e. $t \in [0, T]$ and in particular $\dot{f} \in L^1(0, T; X)$.*

We now focus our attention on absolutely continuous and Sobolev functions. By the very definition it is easy to see that any absolutely continuous function is also of bounded variation; furthermore the spaces $AC([0, T]; X)$ and $\widetilde{W}^{1,1}(0, T; X)$ coincide, while $\widetilde{W}^{1,\infty}(0, T; X)$ is the space of Lipschitz functions from $[0, T]$ to X . Moreover for every $p \in [1, +\infty]$ the inclusion $W^{1,p}(0, T; X) \subseteq \widetilde{W}^{1,p}(0, T; X)$ always holds, but in general is strict.

Next proposition states that the Sobolev space $W^{1,p}(0, T; X)$ is actually characterised by the strong differentiability of its elements.

Proposition A.3. *Let $p \in [1, +\infty]$ and let f be a function from $[0, T]$ to X . Then the following are equivalent:*

- (i) $f \in W^{1,p}(0, T; X)$;
- (ii) $f \in \widetilde{W}^{1,p}(0, T; X)$ and it is strongly differentiable for a.e. $t \in [0, T]$;
- (iii) for every $w \in X^*$ the map $t \mapsto \langle w, f(t) \rangle$ is absolutely continuous in $[0, T]$, f is weakly differentiable for a.e. $t \in [0, T]$ and $\dot{f} \in L^p(0, T; X)$.

If one of the above condition holds, then one has

$$f(t) = f(0) + \int_0^t \dot{f}(\tau) d\tau, \quad \text{for every } t \in [0, T]. \quad (\text{A.1})$$

In the reflexive case, as in Proposition A.2, we gain differentiability of absolutely continuous functions and so we deduce the equivalence between the two spaces $\widetilde{W}^{1,p}(0, T; X)$ and $W^{1,p}(0, T; X)$.

Proposition A.4. *If X is reflexive, then for every $p \in [1, +\infty]$ the Sobolev space $W^{1,p}(0, T; X)$ coincides with $\widetilde{W}^{1,p}(0, T; X)$.*

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