Regularity of solution maps of differential inclusions under state constraints*

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Abstract

Consider a differential inclusion under state constraints

$$x'(t) \in F(t, x(t)),\ x(t) \in K,$$

where F is a closed convex, not necessarily bounded set-valued map, which is measurable in t and $k(t)$-Lipschitz in x (with $k(\cdot) \in L^1$) and $K \subset \mathbb{R}^n$ is a closed set with smooth boundary.

We provide sufficient conditions for the the set-valued map $\xi_0 \rightsquigarrow \mathcal{S}^K_{[t_0,T]}(\xi_0)$ associating to each initial point $\xi_0 \in K$ the set of all the solutions of the above constrained differential inclusion starting at ξ_0 to be pseudo-Lipschitz on K.

Key words: differential inclusions, state constraints, pseudo-Lipschitz dependence on initial conditions.

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Introduction

Let $t_0 < T$, $K \subset \mathbb{R}^n$ and $F : [t_0, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a set-valued map. We associate to these data a Cauchy problem for constrained differential inclusion

$$(1) \qquad \begin{cases} x'(t) \in F(t, x(t)) \\ x(t_0) = \xi_0 \in K \\ x(t) \in K, \end{cases}$$

where the relation $x(t) \in K$ has to be satisfied for all $t \in [t_0, T]$. Denote by $\mathcal{S}^K_{[t_0,T]}(\xi_0)$ the set of all solutions to (1) defined on the time interval $[t_0, T]$.

The aim of the present work is to show that, when the boundary of the subset K is sufficiently smooth and F satisfies an inward pointing condition, then the solution map $\xi_0 \rightsquigarrow \mathcal{S}^K_{[t_0,T]}(\xi_0) \subset W^{1,1}$ is pseudo-Lipschitz even if the images $F(t, x)$ are unbounded (the corresponding definition is recalled in section 1 below). To guarantee such regularity is useful for different control problems. For instance it yields local Lipschitz continuity of value function of constrained Bolza problem, which in turn helps derivation of the optimal synthesis. We refer to [12, 7], where local Lipschitz continuity of the value function of unconstrained Mayer's problem is used to derive necessary and sufficient conditions for optimality in the form of a new differential inclusion. Other areas of application are the robust control and differential games under state constraints.

The problem of regularity of the solution map with respect to initial conditions in the absence of constraints is very well investigated in the literature. In particular the celebrated Filippov's Theorem (see e.g. [3] or [5]) states that if the solutions of the differential inclusion are not subject to state constraints and $F(t, \cdot)$ is $k(t)-$Lipschitz with $k \in L^1(t_0, T)$, then we do have Lipschitz dependence on initial data.

When state constraints are present, H.M. Soner, while investigating in [18] continuity of the value function of an optimal control problem, proposed a sufficient condition for continuous dependence of solutions on initial point for control systems, i.e. when $F(t, x) = f(x, U)$. More precisely, he assumed that the boundary of the bounded set K is C^2 and that the set $f(x, U)$ has a vector pointing strictly inside of K for every boundary point of K to deduce continuous dependence of the solution map on initial conditions. Since then a number of papers treated this problem, studying, under various assumptions regularity of value functions (see, for instance, [8, 16, 1, 15, 6]. In particular in [1], by using an indirect argument, M. Arisawa and P.L. Lions investigate autonomous control systems in a smooth set on unbounded time interval $[t_0, +\infty)$ and give an estimate on controlled trajectories (depending on starting point). Subsequently, an explicit construction of trajectories in the general setting of differential inclusions was proposed by H. Frankowska and F. Rampazzo [15]. In this last work Filippov's and Filippov-Wazewski's Theorems are extended to the case when the state is required to stay in a closed, not necessarily smooth, domain and F has closed and bounded images.

In the above papers the construction of admissible trajectories is anticipative. However some applications require to build explicitely solutions in a nonanticipative way. For this reason, a different construction was introduced by P. Bettiol, P.

Cardaliaguet and M. Quincampoix in [6] in the context of differential games in order to obtain a nonanticipative property (crucial to define suitable strategies). In this last paper F is supposed to be bounded, to have convex compact images and constraints are given by an inequality.

In the present work we extend and simplify the approach developed in [6]. Now the set-valued map F might be also unbounded, but its images still have to be convex and closed. As usually we also assume that F is measurable with respect to the time variable t and $k(t)$-Lipschitz with respect to the state x and that it satisfies an inward pointing condition on sufficiently smooth boundary of K.

This paper is organized as follows. Section 1 is devoted to preliminaries concerning set-valued maps and convex analysis. In section 2 we state our main result, that is proved in section 3.

1 Preliminaries

If X is a normed vector space, we denote the closed unit ball in X by B_X, or, simply, by B when the space X is made precise. For a (closed) ball of radius R centered in $x_0 \in X$ we write $B(x_0; R)$. Here and below $d_K(x) = \mathrm{dist}(x; K) := \inf_{y\in K} ||y - x||$ is the distance function from a point $x \in X$ to a subset $K \subset X$. By X^* we denote the (topological) dual of X and by $\langle\cdot,\cdot\rangle : X^* \times X \to \mathbb{R}$ the usual duality pairing.

Let us recall a definition concerning set-valued maps (cf. [5]).

Definition 1.1 *Assume that X, Y are normed vector spaces and $F : X \rightsquigarrow Y$ is a set-valued map. Given $y \in F(x)$, F is called* pseudo-Lipschitz *around $(x, y) \in \mathrm{Graph}(F)$ if there exists a positive constant L and neighborhoods $\mathcal{U} \subset \mathrm{Dom}(F)$ of x and $\mathcal{V}$ of y such that*

$$\forall x_1, x_2 \in \mathcal{U}, \quad F(x_1) \cap \mathcal{V} \subset F(x_2) + L||x_1 - x_2||B_Y.$$

We recall that this notion is more general that the Lipschitz regularity: F is Lipschitz around $x \in X$ if there exists a positive constant L and a neighborhood $\mathcal{U} \subset \mathrm{Dom}(F)$ of x such that

$$\forall x_1, x_2 \in \mathcal{U}, \quad F(x_1) \subset F(x_2) + L||x_1 - x_2||B_Y.$$

Now, we give a short description of some fundamental concepts we use in our article.

Definition 1.2 *Take $K \subset X$ and $x \in \overline{K}$. Then*

1. *the contingent cone to K at x, $T_K(x)$, is defined by*

$$T_K(x) := \{v \mid \liminf_{h\to 0^+} d_K(x + hv)/h = 0\};$$

2. *the Clarke tangent cone to K at x, $C_K(x)$, is defined by*

$$C_K(x) := \{v \mid \lim_{h\to 0^+,\, K\ni y\to x} d_K(y + hv)/h = 0\};$$

3. *the (Clarke) normal cone to K at x, $N_K(x)$, is defined by*

$$N_K(x) := \{p \in X^* \mid \forall v \in C_K(x)\ \langle p, x\rangle \leq 0\}.$$

We recall that a closed set K is called *sleek* if the set-valued map

$$K \ni x \rightsquigarrow T_K(x)$$

is lower semicontinuous. A very important property of sleek subsets is that the contingent and the Clarke tangent cone to K at x do coincide for every $x \in K$. For many properties of tangent and normal cones we refer the reader for instance to the books [3] and [5].

Let us consider two subsets K_1 and K_2 of a Banach space X and suppose that

$$\text{(2)} \qquad \begin{cases} (i) & K_1 \text{ and } K_2 \text{ are a closed convex subsets,} \\ (ii) & \text{there exists } x_0 \in K_1 \text{ and } \eta > 0 \text{ such that } x_0 + \eta B \subset K_2. \end{cases}$$

Lemma 1.3 *Under assumption (2), if $x \in \partial(K_1 \cap K_2)$ and $n \in N_{K_1 \cap K_2}(x)$, then we have $n = n_1 + n_2$ for some $n_1 \in N_{K_1}(x)$, $n_2 \in N_{K_2}(x)$.*

For a proof of Lemma 1.3 see for instance [4] (Theorem 16 page 174).

Lemma 1.4 *Assume (2) and let $x \in \partial(K_1 \cap K_2)$ and $n \in N_{K_1 \cap K_2}(x)$ be such that $||n|| = 1$ and $n = n_1 + n_2$ for some $n_1 \in N_{K_1}(x)$, $n_2 \in N_{K_2}(x)$. Then the following inequality holds true:*

$$||n_i|| \leq 2 + \frac{||x_0 - x||}{\eta} \qquad \text{for } i = 1, 2.$$

Proof. Set $y_0 := x_0 - x$. Notice that by assumptions (2) $y_0 \in T_{K_1}(x)$ and $y_0 + \eta B \subset T_{K_2}(x)$. For all $b \in B$, we get

$$\text{(3)} \qquad \begin{aligned} \langle n_1, \eta b\rangle &= \langle n_1 + n_2, \eta b\rangle - \langle n_2, \eta b\rangle \leq \\ &\leq \eta ||n|| ||b|| + \langle n_2, y_0 - \eta b\rangle - \langle n_2, y_0\rangle \leq \\ &\leq \eta - \langle n_1 + n_2, y_0\rangle + \langle n_1, y_0\rangle \leq \\ &\leq \eta + ||y_0||, \end{aligned}$$

and, so, for any $b \in B$

$$\langle n_1, b\rangle \leq 1 + \frac{||y_0||}{\eta} = 1 + \frac{||x_0 - x||}{\eta}.$$

Therefore, we obtain

$$||n_1|| \leq 1 + \frac{||x_0 - x||}{\eta}$$

and

$$||n_2|| \leq 1 + ||n_1|| \leq 2 + \frac{||x_0 - x||}{\eta}.$$

□

Lemma 1.5 *Let K be a closed convex cone. Then, for any $x \in K$ and for any $n \in N_K(x)$ we have $\langle n, x\rangle = 0$.*

Proof. Since both x and $-x$ belong to $T_K(x)$ then we get the statement.

□

2 Main Result

Given $K \subset \mathbb{R}^n$ with $\partial K \neq \emptyset$, by b_K we denote the oriented distance function that is defined as follows (cf. [10]):

$$b_K(x) := d_K(x) - d_{\complement K}(x) \tag{4}$$

where $d_K(x)$ is the distance from x to K.

We shall impose the following assumptions on K

$$K \text{ is of class } \mathcal{C}^{1,1}_{\text{loc}}, \tag{5}$$

in the following sense: for any $r > 0$ there exists $\eta_r > 0$ such that $b_K \in \mathcal{C}^{1,1}$ on $\partial K \cap rB + \eta_r B$ (see [11]).

For any $x \in \partial K$ we denote by $\nu(x)$ the exterior unit normal to ∂K. We recall that if b_K is of class $\mathcal{C}^{1,1}$ in some neighborhood of $x \in \partial K$, then we get $\nabla b_K = \nu \circ p_{\partial K}$ where $p_{\partial K}$ is the projection onto ∂K (see [11]). Moreover, for any $x \in \partial K$ we have $N_K(x) = \mathbb{R}_+\nu(x)$ and $T_K(x)$ is the tangent half-space to K at x.

Let us assume that $t_0 < T < +\infty$. As far as the set-valued map F is concerned, we will suppose

$$\left\{ \begin{array}{ll} (i) & F \text{ is a closed convex set-valued map (it may be unbounded)}, \\ (ii) & \forall\, x \in \mathbb{R}^n,\ F(\cdot, x) \text{ is (Lebesgue) measurable}, \\ (iii) & \exists\, k(\cdot) \in L^1\big((t_0,T);\mathbb{R}_+\big) \text{ such that for a. e. } t \in [t_0,T] \\ & F(t,\cdot) \text{ is } k(t)\text{−Lipschitz}, \\ (iv) & \exists\, \eta > 0,\ M > 0 \text{ such that } \forall (t,x) \in [t_0,T] \times \partial K \\ & \exists \hat{f}(t,x) \in F(t,x) \text{ satisfying } \hat{f}(t,x) + \eta B \subset T_K(x) \text{ and} \\ & \sup_{v \in \partial T_K(x) \cap F(t,x)} |\hat{f}(t,x) - v| \leq M. \end{array} \right. \tag{6}$$

Remark 2.1 *The estimate in assumption (iv) of (6) is satisfied, for instance, if there exists a real number $M_0 > 0$ such that*

$$\left\{ \begin{array}{l} \sup_{(s,x) \in [t_0,T] \times \partial K} |\hat{f}(s,x)| \leq M_0 \\ \sup_{v \in \partial T_K(x) \cap F(s,x)} |v| \leq M_0. \end{array} \right.$$

We recall that by $\mathcal{S}^K_{[t_0,T]}(\xi_0)$ we denote the set of solutions on $[t_0, T]$ of the system

$$(7) \qquad \begin{cases} x'(t) \in F(t, x(t)) \\ x(t) \in K \\ x(t_0) = \xi_0 \in K. \end{cases}$$

By $W^{1,1}$ we denote the Sobolev space $W^{1,1}((t_0, T); \mathbb{R}^n)$ endowed with the usual norm $||x||_{W^{1,1}} = ||x||_{L^1((t_0,T);\mathbb{R}^n)} + ||x'||_{L^1((t_0,T);\mathbb{R}^n)}$.

Now, we state our main result.

Theorem 2.2 *Assume (5), (6). Then, the set-valued map*

$$K \ni \xi \rightsquigarrow \mathcal{S}^K_{[t_0,T]}(\xi) \subset W^{1,1}$$

is pseudo-Lipschitz around $(\xi_0, x_0(\cdot))$ for all $x_0(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\xi_0)$.

The proof of Theorem 2.2 is provided in the next section. For this aim we parametrize the set-valued map F in a measurable/Lipschitz way. Once we deal with control systems, we generalize an approach first introduced in the article [6]. It is based on the construction for each $\bar{\xi}_0 \in K$ near ξ_0 of a suitable admissible trajectory that satisfies (7) with ξ_0 replaced by $\bar{\xi}_0$ at time t_0.

3 Construction of constrained trajectories

Let us start with some basic results. The first statement concerns a regularity property for the set-valued map $(t, x) \rightsquigarrow F(t, x) \cap T_K(x)$ that holds true in a more general context than assumption (5): namely, when K is sleek.

Lemma 3.1 *Assume that K is sleek, that $F = F(t, x)$ satisfies (i)-(ii) of (6) and is lower semicontinuous in x. If for any $(t, x) \in [t_0, T] \times K$*

$$F(t, x) \cap \mathrm{Int}(T_K(x)) \neq \emptyset,$$

then we have:

$$\begin{cases} (i)\ \forall x \in K \text{ the set-valued map } t \rightsquigarrow F(t, x) \cap T_K(x) \text{ is measurable,} \\ (ii)\ \forall t \text{ the set-valued map } K \ni x \rightsquigarrow F(t, x) \cap T_K(x) \text{ is lower semicontinuous.} \end{cases}$$

Proof. (i) follows from the Characterization Theorem of measurable set-valued maps (cf. Theorem 8.1.4 in [5]).

Concerning (ii), let us fix $t \in [t_0, T]$. First we notice that, since K is sleek, $K \ni x \rightsquigarrow T_K(x)$ is lower semicontinuous and, so, also $K \ni x \rightsquigarrow \mathrm{Int}(T_K(x))$ is lower semicontinuous.

We claim that the set-valued map $K \ni x \rightsquigarrow F(t, x) \cap \mathrm{Int}(T_K(x))$ is lower semicontinuous. Indeed, if $v \in F(t, x) \cap \mathrm{Int}(T_K(x))$, then there exists $\delta > 0$ such that $B(v; \delta) \subset \mathrm{Int}(T_K(y))$ for all $y \in K$ sufficiently close to x (due to the fact that

$\mathrm{Int}(T_K(\cdot))$ is lower semicontinuous). Since $F(t,\cdot)$ is lower semicontinuous, for all y sufficiently close to x we have $B(v;\delta) \cap F(t,y) \neq \emptyset$ and then

$$B(v;\delta) \cap F(t,y) \cap \mathrm{Int}(T_K(y)) \neq \emptyset.$$

Finally, since $F(t,x) \cap T_K(x) = \overline{F(t,x) \cap \mathrm{Int}(T_K(x))}$, consequently the statement follows.

□

For all $(t,x) \in [t_0,T] \times \partial K$, given $f(t,x) \in F(t,x)$, define a new vector field $\bar{f}(t,x)$ by

$$\bar{f}(t,x) := \pi_{F(t,x)\cap T_K(x)}\big(f(t,x)\big). \tag{8}$$

Notice that if $f(t,x) \in F(t,x) \cap T_K(x)$ then $\bar{f}(t,x) = f(t,x)$. The following result provides an estimate of the difference between $\bar{f}$ and f when $x \in K$ and $f(t,x) \notin F(t,x) \cap T_K(x)$.

Lemma 3.2 *Under assumptions (5) and (6), for all* $(t,x) \in [t_0,T] \times \partial K$ *we have:*

$$|f(t,x) - \bar{f}(t,x)| \leq \left(2 + \frac{M}{\eta}\right) \langle \nu(x), f(t,x) \rangle \tag{9}$$

whenever $f(t,x) \notin F(t,x) \cap T_K(x)$.

Proof. First, recall that $N_K(x) = \mathbb{R}_+ \nu(x)\ \forall x \in \partial K$. Let us define

$$n(t,x) := \frac{f(t,x) - \bar{f}(t,x)}{|f(t,x) - \bar{f}(t,x)|}. \tag{10}$$

Then $n(t,x)$ is the unit outward normal to $F(t,x) \cap T_K(x)$ at $\bar{f}(t,x)$. By Lemmas 1.3 and 1.4, there exists

$$\begin{cases} n_1(t,x) \in N_{F(t,x)}\big(\bar{f}(t,x)\big) \\ n_2(t,x) \in N_{T_K(x)}\big(\bar{f}(t,x)\big) \subset N_K(x) \end{cases} \tag{11}$$

such that

$$n(t,x) = n_1(t,x) + n_2(t,x) \qquad \text{and} \qquad |n_i(t,x)| \leq 2 + \frac{M}{\eta} \ \text{ for } i = 1,2,$$

where M is given by (iv) of assumption (6). Thus, we get

$$|f(t,x) - \bar{f}(t,x)| = \langle n_1(t,x), f(t,x) - \bar{f}(t,x) \rangle + \langle n_2(t,x), f(t,x) - \bar{f}(t,x) \rangle.$$

But, since $\langle n_1(t,x), f(t,x) - \bar{f}(t,x) \rangle \leq 0$ and thanks to Lemma 1.5 $\langle n_2(t,x), \bar{f}(t,x) \rangle = 0$, we obtain

$$|f(t,x) - \bar{f}(t,x)| \leq \langle n_2(t,x), f(t,x) \rangle$$

and, so, the result follows.

□

The next step is to obtain a suitable control system that is provided by a measurable/Lipschitz parameterization of our starting set-valued map F. The passage to control systems is due to technical reasons: this new point of view helps both in construction of a suitable constrained trajectory and in computing of $W^{1,1}$-estimates of the obtained trajectories, as we will show in the next subsection. The Parametrization of Unbounded Maps Theorem (see Theorem 9.7.1 in [5]) implies that there exists a measurable/Lipschitz parametrization f of F. Namely, there exists a single-valued map

$$f : [t_0, T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

such that

$$(12) \qquad \begin{cases} (i) & \forall (t,x) \in [t_0,T] \times \mathbb{R}^n,\ F(t,x) = f(t,x,\mathbb{R}^n) \\ (ii) & \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^n,\ f(\cdot,x,u) \text{ is measurable} \\ (iii) & \forall (t,u) \in [t_0,T] \times \mathbb{R}^n,\ f(t,\cdot,u) \text{ is } ck(t) - \text{Lipschitz} \\ (iv) & \forall (t,x) \in [t_0,T] \times \mathbb{R}^n,\ f(t,x,\cdot) \text{ is } c - \text{Lipschitz} \end{cases}$$

for some constant c independent of F.

Now, if $x_0(\cdot)$ is a solution of system (1) on $[t_0,T]$, then in terms of measurable/Lipschitz parametrization we found, it means that for a.e. $t \in [t_0,T]$

$$x_0'(\cdot) \in f(t, x_0(t), \mathbb{R}^n).$$

By applying, for instance, the Filippov Selection Theorem (see Theorem 8.2.10 in [5]), there exists a measurable selection $u_0(\cdot)$ such that

$$x_0'(\cdot) = f(t, x_0(\cdot), u_0(\cdot)).$$

Therefore, we can pass to study our initial problem from a control point of view. This is the aim of the following subsection.

3.1 Controlled systems

In this section we set $\tilde{F}(t,x) := f(t,x,U)$ where U is a complete separable metric space,

$$(13) \qquad \begin{array}{rccc} f: & \mathbb{R} \times \mathbb{R}^n \times U & \longrightarrow & \mathbb{R}^n \\ & (t,x,u) & \longmapsto & f(t,x,u) \end{array}$$

and we assume that

$$(14) \qquad \begin{cases} (i) & f(t,x,U) \text{ is closed and convex (not necessarily bounded),} \\ (ii) & f \text{ is (Lebesgue) measurable with respect to } t,\ k(t)\text{-Lipschitz in } x \\ & \text{with } k(\cdot) \in L^1\big((t_0,T);\mathbb{R}_+\big) \text{ and continuous in } u, \\ (iii) & \exists\, \eta > 0,\ M > 0 \text{ such that } \forall (t,x) \in [t_0,T] \times \partial K \\ & \text{one can find } u_{t,x} \in U \text{ satisfying } f(t,x,u_{t,x}) + \eta B \subset T_K(x) \text{ and} \\ & \sup_{v \in \partial T_K(x) \cap f(t,x,U)} |f(t,x,u_{t,x}) - v| \leq M. \end{cases}$$

Remark 3.3 *Notice that assumptions (14) are consistent with the setting of the starting differential inclusion problem: namely, if a set-valued map F satisfies (6) then its measurable/Lipschitz parametrization f satisfies (14) with maybe different $k(\cdot)$.*

We denote by $x[t_0, \xi_0, u(\cdot)](\cdot)$ the trajectory controlled by $u(\cdot)$ that starts from ξ_0 at time t_0, where $u : [t_0, T] \longrightarrow U$ is a measurable function. If $x(\cdot) = x[t_0, \xi_0, u(\cdot)](\cdot)$ satisfies system

$$\text{(15)} \qquad \begin{cases} x'(t) = f(t, x(t), u(t)) \\ x(t) \in K \;\forall\, t \in [t_0, T] \\ x(t_0) = \xi_0 \in K \end{cases}$$

then the trajectory $x[t_0, \xi_0, u(\cdot)](\cdot)$ is called *admissible* as well as the corresponding control $u(\cdot)$, and we write $x[t_0, \xi_0, u(\cdot)](\cdot) \in \mathcal{S}^K_{[t_0,T]}(\xi_0)$ and $u(\cdot) \in \mathcal{U}^K_{[t_0,T]}(\xi_0) := \{u : [t_0, T] \longrightarrow U$ measurable s.t. $x[t_0, \xi_0, u(\cdot)](t) \in K \;\forall t \in [t_0, T]\}$. In order to simplify notations we use $x(\cdot)$ instead of $x[t_0, \xi_0, u(\cdot)](\cdot)$ when the initial data t_0, ξ_0 and the control $u(\cdot)$ are made precise.

Take a trajectory-control pair $(x_0(\cdot), u_0(\cdot))$ of (15). Consider a new initial condition $\bar{\xi}_0 \in K$ sufficiently close to ξ_0.

The idea of construction of a new trajectory $\bar{x}(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\bar{\xi}_0)$ is very natural (cf. [6]). Roughly speaking, we control the new trajectory $\bar{x}(\cdot)$ as long as possible by $u_0(\cdot)$, but, whenever we can not use $u_0(\cdot)$ anymore, we project the vector field onto $\tilde{F}(t, x) \cap T_K(x)$. More precisely, if $\bar{x}(t) \in \mathrm{Int}(K)$ or $\bar{x}(t) \in \partial K$ and $f(t, \bar{x}(t), u_0(t)) \in \tilde{F}(t, \bar{x}(t)) \cap T_K(\bar{x}(t))$, then we use $u_0(t)$; otherwise, if $\bar{x}(t) \in \partial K$ but $f(t, \bar{x}(t), u_0(t)) \notin \tilde{F}(t, \bar{x}(t)) \cap T_K(\bar{x}(t))$, then we use a control $\bar{u}(t)$ that realizes the projection of $f(t, \bar{x}(t), u_0(t))$ on $\tilde{F}(t, \bar{x}(t)) \cap T_K(\bar{x}(t))$: i.e., $f(t, \bar{x}(t), \bar{u}(t)) = \pi_{\tilde{F}(t,\bar{x}(t)) \cap T_K(\bar{x}(t))}\big(f(t, \bar{x}(t), u_0(t))\big)$.

Let us define the function $\rho : [t_0, T] \times K \longrightarrow \mathbb{R}$ by

$$\text{(16)} \qquad \rho(t, x) := \mathrm{dist}\big(f(t, x, u_0(t)); \tilde{F}(t, x) \cap T_K(x)\big).$$

Proposition 3.4 *Let us assume (14) and that K is sleek. Then, we obtain the following properties.*

i) *The function $\rho = \rho(t, x)$ is Lebesgue-Borel measurable (in (t, x)) and upper semicontinuous in x.*

ii) *The set-valued map*

$$(t, x) \rightsquigarrow B(f(t, x, u_0(t)); \rho(t, x))$$

is Lebesgue-Borel measurable (in (t, x)) and upper semicontinuous with respect to x.

iii) *The set-valued map*

$$(t, x) \rightsquigarrow G(t, x) := B(f(t, x, u_0(t)); \rho(t, x)) \cap \tilde{F}(t, x)$$

is Lebesgue-Borel measurable (in (t, x)) and upper semicontinuous with respect to x.

Proof. i) By Corollary 8.2.13 in [5] we get that $\rho(t,x)$ is Lebesgue-Borel measurable. Let us fix t. For $x \in K$ take $v_0 \in \tilde{F}(t,x) \cap T_K(x)$ such that

$$|f(t,x,u_0(t)) - v_0| = \rho(t,x).$$

Since the set-valued map $x \rightsquigarrow \tilde{F}(t,x) \cap T_K(x)$ is lower semicontinuous by Lemma 3.1, for any $\varepsilon > 0$, when $y \in K$ is close enough to x, we have

$$\text{dist}(v_0; \tilde{F}(t,y) \cap T_K(y)) \leq \varepsilon$$

and, in particular, there exists $v \in \tilde{F}(t,y) \cap T_K(y)$ such that $|v_0 - v| \leq \varepsilon$. Therefore, by using that f is $k(t)$-Lipschitz continuous with respect to x, we get

$$\begin{aligned} &|f(t,y,u_0(t)) - v| \leq \\ &\quad \leq |f(t,y,u_0(t)) - f(t,x,u_0(t))| + |f(t,x,u_0(t)) - v_0| + |v_0 - v| \leq \\ &\quad \leq k(t)|y-x| + \rho(t,x) + \varepsilon \end{aligned} \tag{17}$$

and, so, for any $\varepsilon > 0$ and all $y \in K$ sufficiently close to x we have

$$\rho(t,y) \leq |f(t,y,u_0(t)) - v| \leq \rho(t,x) + k(t)|y-x| + \varepsilon.$$

ii) Without any loss of generality we may assume that $k(t) \geq 1$ for all $t \in [t_0, T]$. Again, by applying Corollary 8.2.13 of [5] $B(\cdot,\cdot)$ turns out to be Lebesgue-Borel measurable. If we fix t, for any $\varepsilon > 0$ there exists $0 < \delta \leq \frac{\varepsilon}{4k(t)}$ such that if we take y with $|y - x| \leq \delta$ then $\forall v \in B(f(t,y,u_0(t)); \rho(t,y))$, by using i), we get

$$\begin{aligned} &|f(t,x,u_0(t)) - v| \leq \\ &\quad \leq |f(t,x,u_0(t)) - f(t,y,u_0(t))| + |f(t,y,u_0(t)) - v| \leq \\ &\quad \leq k(t)|y-x| + \rho(t,y) \leq \\ &\quad \leq \rho(t,x) + 2k(t)|y-x| + \tfrac{\varepsilon}{2}. \end{aligned} \tag{18}$$

Therefore, since for y close enough to x

$$B(f(t,y,u_0(t)); \rho(t,y)) \subset B(f(t,x,u_0(t)); \rho(t,x)) + \varepsilon B,$$

we obtain that $(t,x) \rightsquigarrow B(f(t,x,u_0(t)); \rho(t,x))$ is upper semicontinuous with respect to x.

iii) Theorem 8.2.4 in [5] guarantees that G is Lebesgue-Borel measurable. On the other hand notice that the set-valued map $\tilde{F}(t,\cdot)$ has closed graph, so the statement immediately follows from ii) (cf. Theorem 1 page 41 in [3]).

□

Remark 3.5 *Notice that for all $(t,x) \in [t_0,T] \times K$*

$$G(t,x) \cap T_K(x) \neq \emptyset.$$

Furthermore, consistently with what is done for vector fields in (8), thanks to Theorem 8.1.3 in [5] we can find a Lebesgue-Borel measurable selection $u(t,x)$ with the following property: $u(t,x) = u_0(t)$ if $f(t,x,u_0(t)) \in \tilde{F}(t,x) \cap T_K(x)$ and $u(t,x)$ is such that $f(t,x,u(t,x)) = \pi_{F(t,x)\cap T_K(x)}\big(f(t,x,u_0(t))\big)$ otherwise.

Lemma 3.6 *Assume (5) and (14). Then, we have*

$$\rho(t,x) \le \mu(t)(1+|x|)$$

for some $\mu \in L^1$.

Proof. First notice that, thanks to Lemma 3.2, we obtain

$$\begin{aligned} \rho(t,x) &\le |f(t,x,u_0(t)) - f(t,x,u(t,x))| \le \\ &\le \max\left\{0; \left(2+\tfrac{M}{\eta}\right)\langle \nu(x), f(t,x,u_0(t))\rangle\right\}. \end{aligned} \tag{19}$$

On the other hand, we have

$$|f(t,x,u_0(t))| \le |f(t,x_0(t),u_0(t))| + k(t)(|x_0(t)| + |x|).$$

So, we conclude choosing $\mu(t) = \left(2+\frac{M}{\eta}\right)(|f(t,x_0(t),u_0(t))| + k(t)|x_0(t)| + k(t))$ (notice that $\mu(\cdot)$ depends on η, M, $k(\cdot)$, $|x_0'(\cdot)|$ and $|x_0(t_0)|$).

□

Remarks 3.7 *Let us assume (5) and (14) and consider the differential inclusion*

$$\begin{cases} x'(t) \in G(t,x(t)) \\ x(t_0) = \bar{\xi}_0 \in K. \end{cases} \tag{20}$$

(a) *If* $v \in G(t,x)$ *where* $(t,x) \in [t_0,T] \times K$ *then*

$$|v| \le |f(t,x,u_0(t))| + |\rho(t,x)| \le |f(t,x_0(t),u_0(t))| + k(t)|x - x_0(t)| + |\rho(t,x)|,$$

therefore, by using Lemma 3.6, we have

$$G(t,x) \subset \bar{\mu}(t)(1+|x|)B$$

for $\bar{\mu}(\cdot) := \left(3+\frac{M}{\eta}\right)\left(|x_0'(\cdot)| + k(\cdot)|x_0(\cdot)| + k(\cdot)\right)$ *(exactly as for* μ, $\bar{\mu} \in L^1$ *and depends only on* η, M, $k(\cdot)$, $|x_0'(\cdot)|$ *and* $|x_0(t_0)|$*). Thus, by the Measurable Viability Theorem in [14], there exists a solution* $\bar{x}(\cdot)$ *to system (20) such that* $\bar{x}([t_0,T]) \subset K$.

(b) *Let us consider a ball* $B(\xi_0, r_0)$ *for some radius* $r_0 > 0$. *By (a) above and by the Gronwall inequality, there exists* $R_0 > |\xi_0|$ *depending only on* r_0 *and* $\bar{\mu}(\cdot)$ *such that for any solution* $\bar{x}(\cdot)$ *to the differential inclusion (20) where* $\bar{\xi}_0 \in B(\xi_0, r_0) \cap K$, *then we obtain*

$$\bar{x}([t_0,T]) \subset R_0 B \cap K.$$

(c) *If we denote by* $\bar{u}(\cdot)$ *the control corresponding to a solution* $\bar{x}(\cdot)$ *of the differential inclusion (20), then (by using notations introduced in Remark 3.5)* $f(t,\bar{x}(t),\bar{u}(t)) = f(t,\bar{x}(t),u(t,\bar{x}(t)))$ *a.e. on* $[t_0,T]$. *Notice that such a way of* *constructing an admissible control* $\bar{u}(\cdot) \in \mathcal{U}^K_{[t_0,T]}(\bar{\xi}_0)$ *is nonanticipative in the following sense. One can define a set-valued map (cf. [6])*

$$\Sigma : \mathcal{U}^K_{[t_0,T]}(\xi_0) \rightsquigarrow \mathcal{U}^K_{[t_0,T]}(\bar{\xi}_0)$$

setting

$$\Sigma(u_0(\cdot)) := \left\{ u(\cdot) \in \mathcal{U}^K_{[t_0,T]}(\bar{\xi}_0) \text{ s.t. } \left\{ \begin{array}{l} x'(t) = f(t, x(t), u(t)) \in G(t, x(t)) \\ x(t) \in K \\ x(t_0) = \bar{\xi}_0 \end{array} \right. \right\}.$$

The set-valued map Σ *turns out to be nonexpansive with nonempty* $(*)$*-closed values in the sense of the article [9]. Therefore, thanks to Plaskacz Lemma in [9] we get a selection* σ *of* Σ*: i.e., a single valued map* $\sigma : \mathcal{U}^K_{[t_0,T]}(\xi_0) \to \mathcal{U}^K_{[t_0,T]}(\bar{\xi}_0)$ *that to any* $u_0(\cdot) \in \mathcal{U}^K_{[t_0,T]}(\xi_0)$ *associates* $\sigma(u_0(\cdot)) \in \Sigma(u_0(\cdot))$*. Furthermore,* σ *is nonanticipative, namely if* $u_1(\cdot), u_2(\cdot) \in \mathcal{U}^K_{[t_0,T]}(\xi_0)$ *and* $u_1(\cdot) = u_2(\cdot)$ *a.e. on* $[t_0, T]$ *then also* $\sigma(u_1(\cdot)) = \sigma(u_2(\cdot))$ *a.e. on* $[t_0, T]$*. This property plays a fundamental role in constructing nonanticipative strategies in Differential Games Theory, for instance.*

(d) *The proof of existence of a trajectory-control pair* $(\bar{x}(\cdot), \bar{u}(\cdot)) \in \mathcal{S}^K_{[t_0,T]}(\bar{\xi}_0) \times \mathcal{U}^K_{[t_0,T]}(\bar{\xi}_0)$ *is direct and constructive.*

In what follows, as in Remarks 3.7, we take $\bar{\xi}_0 \in B(\xi_0, r_0) \cap K$ and $(\bar{x}(\cdot), \bar{u}(\cdot)) \in \mathcal{S}^K_{[t_0,T]}(\bar{\xi}_0) \times \mathcal{U}^K_{[t_0,T]}(\bar{\xi}_0)$ such that

$$\left\{ \begin{array}{l} \bar{x}'(t) = f(t, \bar{x}(t), \bar{u}(t)) \in G(t, \bar{x}(t)) \\ \bar{x}(t) \in K \\ \bar{x}(t_0) = \bar{\xi}_0. \end{array} \right. \tag{21}$$

Notice that if t is such that $\bar{x}(t) \in \mathrm{Int}(K)$, then $f(t, \bar{x}(t), \bar{u}(t)) = f(t, \bar{x}(t), u_0(t))$ and, therefore,

$$|\bar{x}'(t) - x_0'(t)| \leq k(t)|\bar{x}(t) - x_0(t)|.$$

On the other hand, when $\bar{x}(t) \in \partial K$ we have the following two Lemmas, that provide very useful estimates.

Lemma 3.8 *Assume (5) and (14). Then, for almost every* $t \in \{s \in [t_0, T] : \bar{x}(s) \in \partial K\}$ *we have*

$$|\bar{x}'(t) - x_0'(t)| \leq k(t)|\bar{x}(t) - x_0(t)| + c\,\langle \nu(\bar{x}(t)), f(t, \bar{x}(t), u_0(t)) \rangle \tag{22}$$

where $c := 2 + \frac{M}{\eta}$*.*

Proof. Recall that the trajectory $\bar{x}(\cdot)$, controlled by $\bar{u}(\cdot)$, satisfies system (21). Suppose t is such that $\bar{x}(t) \in \partial K$, $\bar{x}'(t)$ exists and $\bar{x}'(t) = f(t, \bar{x}(t), \bar{u}(t))$. Then, we get $\langle \nu(\bar{x}(t)), \bar{x}'(t)\rangle = 0$ and either $f(t,\bar{x}(t),u_0(t)) \in T_K(\bar{x}(t)) \cap \tilde{F}(t,\bar{x}(t))$, in which case $f(t,\bar{x}(t),\bar{u}(t)) = f(t,\bar{x}(t),u_0(t))$ and $\forall c \geq 0$

$$\begin{aligned}
&|f(t,\bar{x}(t),\bar{u}(t)) - f(t,x_0(t),u_0(t))| \leq \\
&\quad \leq |f(t,\bar{x}(t),\bar{u}(t)) - f(t,\bar{x}(t),u_0(t))| + |f(t,\bar{x}(t),u_0(t)) - f(t,x_0(t),u_0(t))| \leq \\
&\quad \leq k(t)|\bar{x}(t) - x_0(t)| + c\,\langle \nu(\bar{x}(t)), \bar{x}'(t)\rangle \leq \\
&\quad \leq k(t)|\bar{x}(t) - x_0(t)| + c\,\langle \nu(\bar{x}(t)), f(t,\bar{x}(t),u_0(t))\rangle\,;
\end{aligned}$$

or $f(t,\bar{x}(t),u_0(t)) \notin T_K(\bar{x}(t)) \cap \tilde{F}(t,\bar{x}(t))$ and, then, by Lemma 3.2, for a.e. $t \in [t_0,T]$ and $c = 2 + \frac{M}{\eta}$

$$|f(t,\bar{x}(t),\bar{u}(t)) - f(t,x_0(t),u_0(t))| \leq k(t)|\bar{x}(t) - x_0(t)| + c\,\langle \nu(\bar{x}(t)), f(t,\bar{x}(t),u_0(t))\rangle\,.$$

□

Let us fix $R = 2R_0$ where R_0 is defined as in (b) of Remarks 3.7. By assumption (5) there exist η_R such that $b_K \in \mathcal{C}^{1,1}$ on $(\partial K \cap RB) + \eta_R B$. Then, let us consider a function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n,\mathbb{R})$ such that

$$\begin{cases} \psi(x) = 1 \text{ if } b_K(x) \geq -\frac{\eta_R}{2} \\ \psi(x) = 0 \text{ if } b_K(x) \leq -\eta_R \end{cases}$$

We set

$$\phi(x) := b_K(x)\psi(x). \tag{23}$$

Notice that ϕ turns out to be $\mathcal{C}^{1,1}$ on $K \cap RB$. Let us denote by L and L_1 the Lipschitz constants of ϕ and $\nabla\phi$ (on $K \cap RB$), respectively.

Lemma 3.9 *Suppose that (5) and (14) hold true. Then, for $\ell(\cdot) := k(\cdot) + ck(\cdot) + cL_1|x_0'(\cdot)| \in L^1$ we obtain*

$$|\bar{x}'(t) - x_0'(t)| \leq \ell(t)|\bar{x}(t) - x_0(t)| + c\,\langle \nabla\phi(x_0(t)), x_0'(t)\rangle$$

for almost every $t \in \{s \in [t_0,T] : \bar{x}(s) \in \partial K\}$, where $c = 2 + \frac{M}{\eta}$.

Proof. As in the proof of Lemma 3.8, we assume t is such that $\bar{x}(t) \in \partial K$, $\bar{x}'(t)$ exists and $\bar{x}'(t) = f(t,\bar{x}(t),\bar{u}(t))$. First notice that we have

$$\begin{aligned}
&\langle \nabla\phi(\bar{x}(t)), f(t,\bar{x}(t),u_0(t))\rangle \leq \\
&\quad \leq \langle \nabla\phi(\bar{x}(t)), x_0'(t)\rangle + k(t)|\nabla\phi(\bar{x}(t))||\bar{x}(t) - x_0(t)| \leq \\
&\quad \leq \langle \nabla\phi(x_0(t)), x_0'(t)\rangle + L_1|x_0'(t)||\bar{x}(t) - x_0(t)| + k(t)|\bar{x}(t) - x_0(t)|.
\end{aligned}$$

Then, by using Lemma 3.8, we obtain

$$\begin{aligned}
&|\bar{x}'(t) - x_0'(t)| \leq \\
&\quad \leq k(t)|\bar{x}(t) - x_0(t)| + \\
&\qquad + c\Big(\langle \nabla\phi(x_0(t)), x_0'(t)\rangle + L_1|x_0'(t)||\bar{x}(t) - x_0(t)| + k(t)|\bar{x}(t) - x_0(t)|\Big).
\end{aligned}$$

Setting $\ell(\cdot) := k(\cdot) + ck(\cdot) + cL_1|x_0'(\cdot)|$, we get the result.

□

Theorem 3.10 *Assume (5) and (14) and let $x_0(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\xi_0)$. Then for every $r_0 > 0$ there exists an increasing absolute continuous function $\ell_0(\cdot)$ (depending on r_0, c, L, L_1, $k(\cdot)$ and $|x_0'(\cdot)|$) such that for any initial point $\bar{\xi}_0 \in K \cap B(\xi_0; r_0)$ we can find an admissible trajectory $\bar{x}(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\bar{\xi}_0)$ with the property*

$$|\bar{x}(t) - x_0(t)| \leq \ell_0(t)|\bar{\xi}_0 - \xi_0|.$$

Proof. First, we recall that, once we fix $r_0 > 0$, then there exists $R_0 > 0$ such that for any solution $\bar{x}(\cdot)$ to the differential inclusion (21) with $\bar{\xi}_0 \in B(\xi_0, r_0) \cap K$ we have $\bar{x}([t_0, T]) \subset R_0 B \cap K$ (cf. (b) of Remarks 3.7). Moreover, we can construct a suitable regularization of the signed distance function, ϕ, as in (23).

Let $\bar{x}(\cdot)$ be the trajectory controlled by $\bar{u}(\cdot)$ that satisfies system (21). Define $t_1 = \inf\{t \in [t_0, T] \ : \ \bar{x}(t) \in \partial K\}$. Then for all $t \in [t_0, t_1]$

$$|\bar{x}(t) - x_0(t)| \leq |\bar{x}(t_0) - x_0(t_0)| + \int_{t_0}^{t} k(s)|\bar{x}(s) - x_0(s)| \ ds.$$

So, by Gronwall inequality, we obtain

$$|\bar{x}(t) - x_0(t)| \leq |\bar{x}(t_0) - x_0(t_0)|e^{\int_{t_0}^{t} k(s) \ ds}$$

for all $t \in [t_0, t_1]$.

If $t_1 = T$, then the proof ends. Otherwise, if $t_1 < T$, then by Lemma 3.9 for all $t \in [t_1, T]$ we get

$$\begin{aligned} &|\bar{x}(t) - x_0(t)| \leq \\ &\quad \leq |\bar{x}(t_0) - x_0(t_0)| + \textstyle\int_{t_0}^{t} \ell(s)|\bar{x}(s) - x_0(s)| \ ds + \\ &\quad\quad + c \textstyle\int_{\{s \in [t_1, t] \ : \ \bar{x}(s) \in \partial K\}} \langle \nabla\phi(x_0(s)), x_0'(s) \rangle \ ds. \end{aligned}$$

Fix $t \in]t_1, T]$ and set $t_2 := \sup\{s \in [t_1, t] \ : \ \bar{x}(s) \in \partial K\}$. Then, we have

$$\begin{aligned} &|\bar{x}(t) - x_0(t)| \leq \\ &\quad \leq |\bar{x}(t_0) - x_0(t_0)| + \textstyle\int_{t_0}^{t} \ell(s)|\bar{x}(s) - x_0(s)| \ ds + \\ &\quad\quad + c \textstyle\int_{\{s \in [t_1, t_2] \ : \ \bar{x}(s) \in \partial K\}} \langle \nabla\phi(x_0(s)), x_0'(s) \rangle \ ds. \end{aligned} \tag{24}$$

Notice that

$$\begin{aligned} \textstyle\int_{t_1}^{t_2} \langle \nabla\phi(x_0(s)), x_0'(s) \rangle \ ds \ &= \phi(x_0(t_2)) - \phi(x_0(t_1)) \leq -\phi(x_0(t_1)) \leq \\ &\leq L|\bar{x}(t_1) - x_0(t_1)| \leq Le^{\int_{t_0}^{t_1} k(s) \ ds}|\bar{x}(t_0) - x_0(t_0)|. \end{aligned}$$

Thus, we have

(25)

$$\begin{aligned} &\textstyle\int_{\{s \in [t_1, t_2] \ : \ \bar{x}(s) \in \partial K\}} \langle \nabla\phi(x_0(s)), x_0'(s) \rangle \ ds = \\ &\quad = \textstyle\int_{t_1}^{t_2} \langle \nabla\phi(x_0(s)), x_0'(s) \rangle \ ds - \textstyle\int_{\{s \in [t_1, t_2] \ : \ \bar{x}(s) \in \mathrm{Int}(K)\}} \langle \nabla\phi(x_0(s)), x_0'(s) \rangle \ ds \leq \\ &\quad \leq Le^{\int_{t_0}^{t_1} k(s) \ ds}|\bar{x}(t_0) - x_0(t_0)| - \textstyle\int_{\{s \in [t_1, t_2] \ : \ \bar{x}(s) \in \mathrm{Int}(K)\}} \langle \nabla\phi(x_0(s)), x_0'(s) \rangle \ ds. \end{aligned}$$

In order to estimate the integral in the right hand side of (25), we have to apply the following lemma.

Lemma 3.11 *Under assumptions of Theorem 3.10 we have*

$$-\int_{\{s\in(t_1,t_2)\,:\,\bar{x}(s)\in\mathrm{Int}(K)\}}\langle\nabla\phi(x_0(s)),x_0'(s)\rangle\;ds\le\int_{t_1}^{t}\ell_1(s)|\bar{x}(s)-x_0(s)|\;ds\tag{26}$$

where we take $\ell_1(s):=Lk(s)+L_1|x_0'(s)|$.

Proof of Lemma 3.11. For a.e. $s\in\{t\;:\;\bar{x}(t)\in\mathrm{Int}(K)\}$, if $\bar{x}(s)\in\mathrm{Int}(K)$, then

$$\begin{aligned}&\langle-\nabla\phi(x_0(s)),x_0'(s)\rangle\le\\&\quad\le-\langle\nabla\phi(\bar{x}(s)),x_0'(s)\rangle+L_1|x_0'(s)||\bar{x}(s)-x_0(s)|\le\\&\quad\le-\langle\nabla\phi(\bar{x}(s)),\bar{x}'(s)\rangle+Lk(s)|\bar{x}(s)-x_0(s)|+L_1|x_0'(s)||\bar{x}(s)-x_0(s)|.\end{aligned}$$

To conclude we have to evaluate the integral $-\int_{\{s\in(t_1,t_2)\,:\,\bar{x}(s)\in\mathrm{Int}(K)\}}\langle\nabla\phi(\bar{x}(s)),\bar{x}'(s)\rangle\;ds$. For this aim, consider the open set

$$\mathcal{O}:=\{s\in(t_1,t_2)\;:\;\bar{x}(s)\in\mathrm{Int}(K)\}\,.$$

Then, there exists an at most countable family of disjoint open intervals (a_i,b_i) such that $\bar{x}(a_i)\in\partial K$, $\bar{x}(b_i)\in\partial K$ and $\mathcal{O}=\cup_{i=1}^{\infty}(a_i,b_i)$. Then, we obtain

$$\begin{aligned}&-\int_{\{s\in(t_1,t_2)\,:\,\bar{x}(s)\in\mathrm{Int}(K)\}}\langle\nabla\phi(\bar{x}(s)),\bar{x}'(s)\rangle\;ds=\\&\quad=-\sum_{i=1}^{\infty}\int_{a_i}^{b_i}\langle\nabla\phi(\bar{x}(s)),\bar{x}'(s)\rangle\;ds=\\&\quad=-\sum_{i=1}^{\infty}\left(\phi(\bar{x}(b_i))-\phi(\bar{x}(a_i))\right)=0.\end{aligned}$$

Therefore, we get the estimate we state.

□

Going back to our proof of Theorem 3.10, by using (24), (25) and (26), we get that for any $t\in[t_1,T]$

$$\begin{aligned}&|\bar{x}(t)-x_0(t)|\le\\&\quad\le(cL+1)e^{\int_{t_0}^{t_1}k(s)\;ds}|\bar{x}(t_0)-x_0(t_0)|+\\&\qquad+\int_{t_0}^{t}(\ell(s)+c\ell_1(s))|\bar{x}(s)-x_0(s)|\;ds.\end{aligned}\tag{27}$$

By Gronwall inequality we deduce that

$$|\bar{x}(t)-x_0(t)|\le(cL+1)e^{\int_{t_0}^{t_1}k(s)\;ds}\left[\int_{t_0}^{t}(\ell(s)+c\ell_1(s))e^{\int_s^t(\ell(\tau)+c\ell_1(\tau))\;d\tau}\;ds+1\right]|\bar{x}(t_0)-x_0(t_0)|$$

and, so, the proof is complete just taking

$$\ell_0(t):=(cL+1)e^{\int_{t_0}^{t_1}k(s)\;ds}\left[\int_{t_0}^{t}(\ell(s)+c\ell_1(s))e^{\int_s^t(\ell(\tau)+c\ell_1(\tau))\;d\tau}\;ds+1\right].$$

□

Corollary 3.12 *Under assumptions of Theorem 3.10, we have*

$$||\bar{x}(\cdot) - x_0(\cdot)||_{W^{1,1}} \le L_0|\bar{\xi}_0 - \xi_0| \tag{28}$$

for some positive constant L_0 *depending only on* $c,\ r_0,\ L,\ L_1,\ k(\cdot),\ |x_0'(\cdot)|$.

Proof. Notice that by Lemma 3.9 for any $t \in [t_0, T]$ we get

$$\int_{t_0}^{t} |\bar{x}'(s)-x_0'(s)|\ ds \le \int_{t_0}^{t} \ell(s)|\bar{x}(s)-x_0(s)|\ ds + c\int_{\{s\in[t_1,t]\ :\ \bar{x}(s)\in\partial K\}} \langle \nabla\phi(x_0(s)), x_0'(s)\rangle\ ds, \tag{29}$$

where $t_1 = \inf\{t \in [t_0, T]\ :\ \bar{x}(t) \in \partial K\}$. By using the same arguments in the proof of Theorem 3.10, we can estimate the second integral on the right hand side of (29), obtaining

$$\int_{t_0}^{t} |\bar{x}'(s)-x_0'(s)|\ ds \le cLe^{\int_{t_0}^{t_1} k(s)\ ds}|\bar{x}(t_0)-x_0(t_0)| + \int_{t_0}^{t} (\ell(s)+c\ell_1(s))|\bar{x}(s)-x_0(s)|\ ds.$$

By applying the estimate provided in the proof of Theorem 3.10 on the term $|\bar{x}(s) - x_0(s)|$ in the previous estimate, we have

$$\int_{t_0}^{T} |\bar{x}'(s) - x_0'(s)|\ ds \le \left[cLe^{\int_{t_0}^{t_1} k(s)\ ds} + \int_{t_0}^{T} (\ell(s) + c\ell_1(s))\ell_0(s)\ ds\right] |\bar{x}(t_0) - x_0(t_0)|.$$

Therefore, we get

$$\begin{aligned} &||\bar{x}(\cdot) - x_0(\cdot)||_{W^{1,1}} \le \\ &\qquad \le \left[\ell_0(T)(T - t_0) + cLe^{\int_{t_0}^{t_1} k(s)\ ds} + \textstyle\int_{t_0}^{T} (\ell(s) + c\ell_1(s))\ell_0(s)\ ds\right] |\bar{x}(t_0) - x_0(t_0)|. \end{aligned}$$

and, so, we get (28) where the constant L_0 depends on $c,\ r_0,\ L,\ L_1,\ k(\cdot),\ |x_0'(\cdot)|$.

□

Finally, the following Corollary states that the set-valued map $\xi \rightsquigarrow \mathcal{S}^K_{[t_0,T]}(\xi) \subset W^{1,1}$ is pseudo-Lipschitz around $(\xi_0, x_0(\cdot))$.

Corollary 3.13 *Under assumptions of Theorem 3.10, there exists a radius* $0 < r \le r_0$ *such that for any* $\xi_1,\ \xi_2 \in K \cap B_{\mathbb{R}^n}(\xi_0; r)$ *and for every trajectory* $y_1(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\xi_1) \cap B_{W^{1,1}}(x_0(\cdot); L_0 r)$, *then we can find* $x_2(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\xi_2)$ *such that*

$$||y_1(\cdot) - x_2(\cdot)||_{W^{1,1}} \le \tilde{L}_0|\xi_1 - \xi_2| \tag{30}$$

where $\tilde{L}_0$ *is a positive constant that depends on* $r,\ c,\ L,\ L_1,\ k(\cdot)$ *and* $|x_0'(\cdot)|$.

Proof. First, notice that $\mathcal{S}^K_{[t_0,T]}(\xi_1) \cap B_{W^{1,1}}(x_0(\cdot); L_0 r) \neq \emptyset$. Indeed, by the previous Corollary 3.12 for any $\xi_1 \in K \cap B_{\mathbb{R}^n}(\xi_0; r)$ there exist a constant $L_0 > 0$ and an admissible trajectory $x_1(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\xi_1)$ such that

$$||x_1(\cdot) - x_0(\cdot)||_{W^{1,1}} \le L_0|\xi_1 - \xi_0| \le L_0 r.$$

Now, let us take $y_1(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\xi_1) \cap B_{W^{1,1}}(x_0(\cdot); L_0 r)$. In particular, we have

$$||y_1'(\cdot)||_{L^1} \le ||x_0'(\cdot)||_{L^1} + L_0 r. \tag{31}$$

Now, by the same arguments we used in the proof of Theorem 3.10 with $(\xi_1, y_1(\cdot))$ instead of $(\xi_0, x_0(\cdot))$ and ξ_2 instead of $\bar{\xi}_0$, we construct suitable controlled trajectories. Therefore, by using the same arguments of Remarks 3.7, we deduce that there exists $x_2(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\xi_2)$ such that $|x_2'(t)| \le \tilde{\mu}(t)(1+|x_2(t)|)$ where $\tilde{\mu}(\cdot) = \left(3 + \frac{M}{\eta}\right)\big(|y_1'(\cdot)| + k(\cdot)|y_1(\cdot)| + k(\cdot)\big)$.

Thanks to (31) and the Gronwall inequality, for a suitable choice of $r > 0$, there exists $R > 0$ such that $y_1([t_0,T]) \subset RB_{\mathbb{R}^n} \cap K$ for all $y_1(\cdot) \in \mathcal{S}^K_{[t_0,T]}(\xi_1) \cap B_{W^{1,1}}(x_0(\cdot); L_0 r)$, and $x_2([t_0,T]) \subset RB_{\mathbb{R}^n} \cap K$ for any $\xi_2 \in B_{\mathbb{R}^n}(\xi_0; r)$.

So, in order to get (30), we just apply Theorem 3.10 and Corollary 3.12 with $(\xi_1, y_1(\cdot))$ instead of $(\xi_0, x_0(\cdot))$ and $(\xi_2, x_2(\cdot))$ instead of $(\bar{\xi}_0, \bar{x}(\cdot))$. The existence of a $\tilde{L}_0$, that depends only on r, c, L, L_1, $k(\cdot)$ and $|x_0'(\cdot)|$, is guaranteed by (31).

□

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