

# On topological M-theory

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## ABSTRACT

We construct a gauge fixed action for topological membranes on  $G_2$ -manifolds such that its bosonic part is the standard membrane theory in a particular gauge. We prove that quantum mechanically the path-integral in this gauge localizes on associative submanifolds. Moreover on  $M \times S^1$  the theory naturally reduces to the standard A-model on Calabi-Yau manifold and to a membrane theory localized on special Lagrangian submanifolds. We discuss some properties of topological membrane theory on  $G_2$ -manifolds. We also generalize our construction to topological  $p$ -branes on special manifolds by exploring a relation between vector cross product structures and TFTs.

# 1 Introduction

The notion of topological M-theory has been introduced in [16] (also for earlier proposal see [18]) as unifying description of the topological A- and B-models. This is very much in analogy with the connection between the physical superstring and the physical M-theory. In [16] the analysis has been done at the classical level of the “effective” actions. Different arguments in favor of topological M-theory have been proposed in [19, 9, 28].

In this note we propose a microscopic description of topological M-theory on seven dimensional  $G_2$ -manifold as a topological membrane theory. Namely we construct the gauge fixed action  $S_{GF}$  for the topological membrane

$$S_{top} = \int X^*(\Phi), \quad (1.1)$$

where  $\Phi$  is a closed three form associated with a  $G_2$ -structure. The bosonic part of  $S_{GF}$  turns out to be the standard membrane theory in a particular gauge. Moreover on  $CY_6 \times S^1$  the action  $S_{GF}$  naturally reduces to A-model. The proposed membrane theory is localized on associative cycles. It is well-known that membrane instantons on  $G_2$ -manifold are given by associative three submanifolds [11]. The contribution of membrane instantons on  $G_2$ -manifold to the superpotential of  $\mathcal{N} = 1$  compactifications of M-theory have been studied in [21] and [10].

Actually, in the present work we do not couple the topological membrane model to  $3D$  gravity on the world-volume and therefore a full comparison with the topological string can not be performed yet. We plan to discuss this problem in a separate work.

Recently using the Mathai-Quillen formalism the authors [5] proposed a gauge fixed action for the topological membrane. However the bosonic part of this model is unusually highly polynomial and this obscures the relation to the usual membrane theory, since in [5] agreement is found only up to quadratic order. Moreover, the relation with A-model is shown at the level of zero section and not for the gauge fixed action. Further we will comment more on the relation between our construction and the one proposed in [5]. In [7] the authors discuss topological membranes using the Green-Schwarz formalism.

The structure of the paper is as follows. In Section 2 we recall the description of membrane instantons on manifolds with  $G_2$ -structure. On a  $G_2$ -manifold the instantons correspond to three dimensional associative submanifolds. In Section 3 we construct the gauge fixed action for the topological membrane (1.1). In our treatment we follow closely the lagrangean approach advocated by Baulieu and Singer in [8] for the topological sigma model. Its extension to our membrane theory proves that the path-integral evaluation in our gauge is localized on membrane instantons, that is on associative submanifolds. We discuss also further properties

of the gauge fixed action. In the next Section we go through the Hamiltonian treatment and make a few comments regarding the previous work [14]. Section 5 is devoted to the discussion of observables and moduli spaces. In Section 6 we collect some observations about the general relations between vector product structures and TFTs generalizing our construction. The Appendices collect some relevant properties of  $G_2$ -manifolds and vector cross product structures.

## 2 Membrane instantons on $G_2$ -manifolds

In this Section we present a natural and elementary approach to the classical aspects of membrane instantons.

Let us consider the Euclidean membrane theory defined by the following Nambu-Goto action

$$S = \int d^3\sigma \sqrt{\det(\partial_\alpha X^\mu g_{\mu\nu} \partial_\beta X^\nu)}, \quad (2.2)$$

where  $\alpha = 0, 1, 2$ . In (2.2) we have chosen units such that the membrane tension is one. Introducing the auxiliary world-volume metric  $h_{\alpha\beta}$  the action (2.2) can be obtained as the stationary value of

$$S = \frac{1}{2} \int d^3\sigma \sqrt{h} (h^{\alpha\beta} \partial_\alpha X^\mu g_{\mu\nu} \partial_\beta X^\nu - 1). \quad (2.3)$$

under arbitrary variations of  $h_{\alpha\beta}$ . Let us now fix the gauge symmetry. Unlike the string case, there is not enough gauge symmetry to fix the whole auxiliary metric  $h_{\alpha\beta}$  which has six independent components. However using reparameterization symmetry we can fix the components  $h_{0\beta}$  to be

$$h_{0a} = 0, \quad h_{00} = \det(h_{ab}) \quad (2.4)$$

where  $h_{ab}$ , with  $a, b = 1, 2$ , are the remaining spatial components of the auxiliary metric. Once we have chosen this gauge, no further components of  $h_{\alpha\beta}$  can be fixed. Globally this gauge can be only chosen when the membrane world-volume is of the form  $\Sigma_2 \times S^1$  (also  $S^1$  can be replaced by either an interval or a real line) with  $\Sigma_2$  being a Riemann surface. Therefore this lagrangean approach to gauge fixing is equivalent to the usual hamiltonian one. After fixing the stationary condition for the remaining  $h_{ab}$  in this gauge the membrane action becomes

$$S = \frac{1}{2} \int d^3\sigma \left( \dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) \right), \quad (2.5)$$

where  $\dot{X}^\mu \equiv \partial_0 X^\mu$ , which is the well known gauge fixed Euclidean membrane action on the Riemannian manifold  $(M, g)$ .

Now consider the following bound

$$\int d^3\sigma \left( \dot{X}^\mu \pm \Phi^\mu{}_{\nu\rho} \partial_1 X^\nu \partial_2 X^\rho \right) g_{\mu\lambda} \left( \dot{X}^\lambda \pm \Phi^\lambda{}_{\sigma\tau} \partial_1 X^\sigma \partial_2 X^\tau \right) \geq 0, \quad (2.6)$$

where  $\Phi_{\mu\nu\rho}$  is a 3-form and  $g_{\mu\nu}$  a Riemannian metric on  $M$ . If  $M$  is a seven dimensional manifold with  $G_2$  structure given by  $\Phi$  and  $g$  via the vector cross product relation (see Appendix B), then the bound (2.6) can be rewritten as follows

$$\frac{1}{2} \int d^3\sigma \left( \dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) \right) \geq \mp \frac{1}{6} \int X^*(\Phi). \quad (2.7)$$

If  $\Phi$  is a closed form then the term on the right hand side of (2.7) is topological. Thus for the manifold with  $G_2$ -structure  $(M, g, \Phi)$  such that  $d\Phi = 0$  we obtain the following membrane instantons

$$\dot{X}^\mu \pm \Phi^\mu{}_{\nu\rho} \partial_1 X^\nu \partial_2 X^\rho = 0, \quad (2.8)$$

which minimize the Euclidean action (2.5). We call a map  $X$  from  $\Sigma_3$  to  $M$  which satisfies (2.8) an associative map.

Actually, we can show that the condition (2.8) is equivalent to the calibration condition

$$d \text{vol}(\Sigma_3) = \mp \frac{1}{6} X^*(\Phi), \quad (2.9)$$

where  $d \text{vol}$  is the volume element induced by  $g$  and the pull-back is along the associative map. In fact by multiplying (2.8) by  $g_{\mu\nu} \partial_\alpha X^\nu$  we get  $h_{0\alpha} = g_{\mu\nu} \partial_\alpha X^\mu \dot{X}^\nu = 0$  and  $(\dot{X})^2 \pm \frac{1}{6} X^*(\Phi) = 0$ . On the other hand, by squaring (2.8) and eliminating  $X^*(\Phi)$  by the previous equations we get for the induced metric  $h_{00} = \det(h_{ab})$ . Hence

$$d \text{vol}(\Sigma_3) = \sqrt{h} = h_{00} = \mp \frac{1}{6} X^*(\Phi) \quad (2.10)$$

If we require the manifold  $M$  to be of  $G_2$ -holonomy<sup>1</sup> (i.e. either  $\nabla_\mu \Phi_{\nu\rho\sigma} = 0$  or  $d\Phi = 0$  and  $d*\Phi = 0$ ) then the instantons (2.8) are interpreted as associative submanifolds of  $M$ , namely submanifolds calibrated by  $\Phi$  [22].

Next consider  $M_6$  to be a Calabi-Yau threefold with Kähler form  $\omega$  and holomorphic three form  $\Omega$ . Then there is a natural  $G_2$ -structure  $\Phi$  on the product  $M_7 = M_6 \times S^1$  given by

$$\Phi = \text{Re } \Omega + dX^7 \wedge \omega, \quad (2.11)$$

where we choose the coordinates  $\mu = (N, 7)$  with the uppercase Latin letters denoting the coordinates along  $M_6$  and  $X^7$  a coordinate along  $S^1$ . This induces the product metric on

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<sup>1</sup>From now on we refer to the manifolds of  $G_2$ -holonomy as  $G_2$ -manifolds.

$M_6 \times S^1$ , with the flat metric on  $S^1$

$$g_{\mu\nu} = \begin{pmatrix} g_{MN} & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.12)$$

With the appropriate orientation on  $M_6 \times S^1$  we have

$$*\Phi = -dX^7 \wedge \text{Im} \Omega + \frac{1}{2}\omega^2. \quad (2.13)$$

Thus  $M_6 \times S^1$  is a  $G_2$ -manifold. On  $M_6 \times S^1$  the associative 3-cycles wrapping  $S^1$  are of the form  $\Sigma_3 = \Sigma_2 \times S^1$ , where  $\Sigma_2$  is an (anti)holomorphic curve in  $M_6$ , while the associative 3-cycles localized along  $S^1$  correspond to special Lagrangian submanifolds with phase zero. Correspondingly the Euclidean action (2.5) reduces on  $M_6$  either to string theory in conformal gauge

$$S = \frac{k}{2} \int d^2\sigma \left( \dot{X}^N g_{NM} \dot{X}^M + \partial_1 X^N g_{NM} \partial_1 X^M \right) \quad (2.14)$$

where  $k$  is the  $S^1$  winding, or to membrane theory in the same type of gauge as before

$$S = \frac{1}{2} \int d^3\sigma \left( \dot{X}^N g_{NM} \dot{X}^M + \det(\partial_a X^N g_{NM} \partial_b X^M) \right). \quad (2.15)$$

These reductions can also be done at the level of membrane instantons (2.8). Thus the holomorphic curves (calibrated by  $\omega$ ) are the instantons for (2.14) and the special Lagrangian submanifolds with phase zero (calibrated by  $\text{Re} \Omega$ ) are instantons for (2.15). However the special Lagrangian submanifolds with phase zero are not the most general instantons for the action (2.15) – see Section 6 for further discussion.

Indeed there is a family of  $G_2$ -structures on  $M_6 \times S^1$  [25]

$$\Phi_\theta = \text{Re}(e^{i\theta}\Omega) + dX^7 \wedge \omega, \quad (2.16)$$

where now the Calabi-Yau structure on  $M_6$  is given by  $e^{i\theta}\Omega$  and  $\omega$ . It is well-known that one can change the holomorphic form  $\Omega$  by a multiplicative phase while preserving the Ricci-flat metric. With this new  $G_2$ -structure one can repeat the same considerations as above. However now the associative manifolds localized in  $S^1$  correspond to special Lagrangian submanifolds of  $M_6$  with the phase  $\theta$ , i.e. calibrated by  $\text{Re}(e^{i\theta}\Omega)$ .

### 3 The gauged fixed action

In this Section we consider the gauge fixing for the following topological membrane theory

$$S = -\frac{1}{6} \int_{\Sigma_3} X^*(\Phi), \quad (3.17)$$

where  $\Phi$  is the three form associated to a  $G_2$ -structure on a seven dimensional manifold  $M_7$ . We assume that  $\Phi$  is closed. In our treatment we closely follow Baulieu and Singer [8] generalizing their method to membranes.

The gauge symmetry of the action is

$$\delta X^\mu = \epsilon^\mu. \quad (3.18)$$

The corresponding BRST operator  $s$  is defined as follows

$$sX^\mu = \psi^\mu, \quad s\psi^\mu = 0, \quad s\bar{\psi}^\mu = b^\mu, \quad sb^\mu = 0, \quad (3.19)$$

where  $\psi^\mu$  is the ghost associated to  $\epsilon^\mu$ . The ghost numbers are respectively 0, 1, -1, 0 for  $X^\mu, \psi^\mu, \bar{\psi}^\mu, b^\mu$ . We will now fix the symmetry (3.18) and obtain a gauged fixed action which is quadratic in the velocities. We choose the following gauge function

$$\mathcal{F}^\mu = \dot{X}^\mu + \Phi^\mu_{\nu\rho} \partial_1 X^\nu \partial_2 X^\rho + \frac{1}{2} \Gamma^\mu_{\sigma\rho} \bar{\psi}^\sigma \psi^\rho, \quad (3.20)$$

where  $\Gamma_{\mu\nu\rho} = 1/2(g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu})$  with  $g_{\mu\nu,\rho} \equiv \partial_\rho g_{\mu\nu}$ . The quadratic term in the ghosts in (3.20) is necessary for manifest general covariance. The BRST invariant gauge fixed action is obtained by adding to the classical action (3.17) an  $s$ -exact gauge fixing term and reads

$$S_{GF} = -\frac{1}{6} \int X^*(\Phi) + \int d^3\sigma \ s \left( \bar{\psi}^\mu (g_{\mu\nu} \dot{X}^\nu + \Phi_{\mu\nu\rho} \partial_1 X^\nu \partial_2 X^\rho + \frac{1}{2} \Gamma_{\mu\sigma\rho} \bar{\psi}^\sigma \psi^\rho - \frac{1}{2} g_{\mu\nu} b^\nu) \right). \quad (3.21)$$

Using the definition (3.19) and eliminating  $b^\mu$  by its algebraic equation of motion

$$b^\mu = \dot{X}^\mu + \Phi^\mu_{\nu\rho} \partial_1 X^\nu \partial_2 X^\rho + \Gamma^\mu_{\sigma\rho} \bar{\psi}^\sigma \psi^\rho \quad (3.22)$$

we arrive to the following gauged fixed action

$$S_{GF} = \int d^3\sigma \left( \frac{1}{2} \dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \frac{1}{2} \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) - \bar{\psi}^\mu g_{\mu\nu} \nabla_0 \psi^\nu \right. \\ \left. - \Phi_{\mu\nu\rho} \bar{\psi}^\mu \nabla_a \psi^\nu \partial_b X^\rho \epsilon^{ab} - \frac{1}{2} \epsilon^{ab} \partial_a X^\nu \partial_b X^\rho \bar{\psi}^\mu \psi^\lambda \nabla_\lambda \Phi_{\mu\nu\rho} + \frac{1}{4} \mathcal{R}_{\mu\sigma\lambda\rho} \bar{\psi}^\mu \psi^\rho \bar{\psi}^\sigma \psi^\lambda \right), \quad (3.23)$$

where

$$\nabla_\alpha \psi^\mu = \partial_\alpha \psi^\mu + \Gamma^\mu_{\rho\lambda} \partial_\alpha X^\rho \psi^\lambda \quad (3.24)$$

and

$$\mathcal{R}^\mu_{\sigma\rho\lambda} = \Gamma^\mu_{\lambda\sigma,\rho} - \Gamma^\mu_{\rho\sigma,\lambda} + \Gamma^\mu_{\rho\tau} \Gamma^\tau_{\lambda\sigma} - \Gamma^\mu_{\lambda\tau} \Gamma^\tau_{\rho\sigma}. \quad (3.25)$$

For sake of simplicity, from now on we assume that  $\Phi$  is also coclosed, i.e.  $\nabla_\lambda \Phi_{\mu\nu\rho} = 0$  and thus the manifold is of  $G_2$ -holonomy. Assuming  $G_2$ -holonomy the gauge fixed action (3.23) becomes

$$S_{GF} = \int d^3\sigma \left( \frac{1}{2} \dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \frac{1}{2} \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) - \bar{\psi}^\mu g_{\mu\nu} \nabla_0 \psi^\nu \right)$$

$$-\Phi_{\mu\nu\rho}\bar{\psi}^\mu\nabla_a\psi^\nu\partial_bX^\rho\epsilon^{ab} + \frac{1}{4}\mathcal{R}_{\mu\sigma\lambda\rho}\bar{\psi}^\mu\psi^\rho\bar{\psi}^\sigma\psi^\lambda). \quad (3.26)$$

The action (3.26) is invariant under the following BRST symmetry

$$sX^\mu = \psi^\mu, \quad s\psi^\mu = 0, \quad s\bar{\psi}^\mu = \dot{X}^\mu + \Phi^\mu{}_{\nu\rho}\partial_1X^\nu\partial_2X^\rho + \Gamma^\mu{}_{\sigma\rho}\bar{\psi}^\sigma\psi^\rho, \quad (3.27)$$

which is nilpotent on-shell only unlike (3.19). The action (3.26) can be rewritten as follows

$$S_{GF} = -\frac{1}{6}\int X^*(\Phi) + \frac{1}{2}\int d^3\sigma s\left(\bar{\psi}^\mu(g_{\mu\nu}\dot{X}^\nu + \Phi_{\mu\nu\rho}\partial_1X^\nu\partial_2X^\rho)\right). \quad (3.28)$$

Thus due to standard arguments (e.g., see [30]) the model is localized on the solutions of (2.8) which correspond to associative three manifolds. In (3.28) the topological term depends only on the cohomology class of  $\Phi$  and the homotopy class of the map  $X$ .

If we assume that  $H^3(M_7, \mathbb{Z}) = \mathbb{Z}$ , we can normalize  $\Phi$  such that the periods of  $\frac{1}{6}\Phi$  are integer multiples of  $2\pi$ , that is

$$\frac{1}{6}\int_{\Sigma_3} X^*(\Phi) = 2\pi n, \quad n \in \mathbb{Z} \quad (3.29)$$

where  $n$  is the instanton number for the associative map (2.8). Therefore the path integral is reduced to a sum of the integrals over the moduli space  $\mathcal{M}_n$  of associative maps of degree  $n$ . Actually (3.29) is not well defined at the quantum mechanical level due to a parity anomaly [6] (see Sect.5 for further discussions). The amplitudes of our topological theory do not depend on the way we describe the associative maps. Namely in (2.8) we choose a specific coordinate with distinguished direction  $\alpha = 0$ . However any variation with respect to this choice appears in the path integral as a BRST-exact term. Thus indeed our specific parametrization of associative maps is irrelevant for the theory.

The action (3.26) has another set of BRST transformations

$$\bar{s}X^\mu = \bar{\psi}^\mu, \quad \bar{s}\bar{\psi}^\mu = 0, \quad \bar{s}\psi^\mu = \dot{X}^\mu - \Phi^\mu{}_{\nu\rho}\partial_1X^\nu\partial_2X^\rho - \Gamma^\mu{}_{\sigma\rho}\bar{\psi}^\sigma\psi^\rho, \quad (3.30)$$

which are nilpotent only on-shell. The two BRST transformations above form the following on-shell algebra

$$s\bar{s} + \bar{s}s = 2\partial_0. \quad (3.31)$$

We will comment more on the transformations (3.27) and (3.30) in Section 4.

Furthermore we can consider the theory (3.26) with its BRST symmetry (3.27) on  $M_7 = M_6 \times S^1$  on which we assume the structure (2.11)-(2.13). There are two interesting sectors: the configurations which wrap  $S^1$  and the configurations which are localized on  $S^1$ . First let

us consider the configurations which wrap  $S^1$ . Assuming that  $X^7 = k\sigma_2$  with  $\psi^7 = \bar{\psi}^7 = 0$  and the other fields independent on  $\sigma_2$ , the action (3.26) reduces to

$$S_{GF} = k \int d^2\sigma \left( \frac{1}{2} \dot{X}^N g_{NM} \dot{X}^M + \frac{1}{2} \partial_1 X^N g_{NM} \partial_1 X^M - \bar{\psi}^N g_{NM} \nabla_0 \psi^M - \omega_{MN} \bar{\psi}^M \nabla_1 \psi^N + \frac{1}{4} \mathcal{R}_{MNP L} \bar{\psi}^M \psi^L \bar{\psi}^N \psi^P \right), \quad (3.32)$$

where we have used (2.11) and (2.12). In its turn the BRST transformations (3.27) reduce to

$$sX^M = \psi^M, \quad s\psi^M = 0, \quad s\bar{\psi}^M = \dot{X}^M - J^M_N \partial_1 X^N + \Gamma^M_{NL} \bar{\psi}^N \psi^L, \quad (3.33)$$

where  $J^M_N$  is the complex structure on  $M_6$  such that  $\omega_{NM} = -g_{NL} J^L_M$ . Let us introduce the complex coordinates  $I = (i, \bar{i})$  with respect to  $J$  and redefine our fields as follows

$$\psi^i = i\alpha\chi^i, \quad \psi^{\bar{i}} = i\tilde{\alpha}\chi^{\bar{i}}, \quad \bar{\psi}^i = -\frac{1}{\tilde{\alpha}}\psi^i_z, \quad \bar{\psi}^{\bar{i}} = -\frac{1}{\alpha}\psi^{\bar{i}}_z, \quad (3.34)$$

where  $\alpha$  and  $\tilde{\alpha}$  are some non-zero constants. In the complex coordinates and new fields the action (3.32) becomes

$$S_{GF} = k \int d^2\sigma \left( \frac{1}{2} \partial_z X^N g_{NM} \partial_{\bar{z}} X^M + i\psi^i_z g_{i\bar{j}} \nabla_z \chi^{\bar{j}} + i\psi^{\bar{i}}_z g_{\bar{i}j} \nabla_{\bar{z}} \chi^j - \mathcal{R}_{i\bar{k}l\bar{s}} \psi^i_z \psi^{\bar{k}}_z \chi^l \chi^{\bar{s}} \right), \quad (3.35)$$

where we introduced  $\partial_z = \partial_0 + i\partial_1$ ,  $\nabla_z = \nabla_0 + i\nabla_1$  and their complex conjugates. In the new notation the BRST transformations become

$$sX^i = i\alpha\chi^i, \quad sX^{\bar{i}} = i\tilde{\alpha}\chi^{\bar{i}}, \quad s\chi^i = s\chi^{\bar{i}} = 0, \\ s\psi^i_z = -\tilde{\alpha}\partial_{\bar{z}} X^i - i\alpha\Gamma^i_{nl} \chi^n \psi^l_z, \quad s\psi^{\bar{i}}_z = -\alpha\partial_z X^{\bar{i}} - i\tilde{\alpha}\Gamma^{\bar{i}}_{\bar{n}l} \chi^{\bar{n}} \psi^l_z. \quad (3.36)$$

Indeed the action (3.35) and the transformations (3.36) are exactly the same as the topological A-model in [30].

Next consider the membranes which are localized on  $S^1$ , i.e.  $X^7 = const.$ ,  $\psi^7 = 0$  and  $\bar{\psi}^7 = 0$ . In this case the action (3.26) is reduced to

$$S_{GF} = \int d^3\sigma \left( \frac{1}{2} \dot{X}^N g_{NM} \dot{X}^M + \frac{1}{2} \det(\partial_a X^N g_{NM} \partial_b X^M) - \bar{\psi}^N g_{NM} \nabla_0 \psi^M - (Re \Omega)_{MNL} \bar{\psi}^M \nabla_a \psi^N \partial_b X^L \epsilon^{ab} + \frac{1}{4} \mathcal{R}_{MNLS} \bar{\psi}^M \psi^S \bar{\psi}^N \psi^L \right) \quad (3.37)$$

where we have used (2.11) and (2.12). The BRST transformation becomes

$$sX^M = \psi^M, \quad s\psi^M = 0, \quad s\bar{\psi}^M = \dot{X}^M + (Re \Omega)^M_{NL} \partial_1 X^N \partial_2 X^L + \Gamma^M_{NL} \bar{\psi}^N \psi^L. \quad (3.38)$$



The action (3.37) is  $s$ -exact modulo a topological term, i.e.

$$S_{GF} = \frac{1}{6} \int X^*(Re\Omega) + \frac{1}{2} \int d^3\sigma s \left( \bar{\psi}^M (g_{MN} \dot{X}^N + (Re\Omega)_{MNL} \partial_1 X^N \partial_2 X^L) \right). \quad (3.39)$$

This membrane theory is localized on the configurations

$$\dot{X}^M + (Re\Omega)_{NL}^M \partial_1 X^N \partial_2 X^L = 0, \quad (3.40)$$

which are special Lagrangian submanifolds with phase zero (i.e., calibrated by  $Re\Omega$ ).

If on  $M_6 \times S^1$  we choose the different  $G_2$ -structure (2.16) then the membrane theory on  $M_6$  would be a bit different: in all equations (3.37)-(3.40)  $Re\Omega$  should be replaced by  $Re(e^{i\theta}\Omega)$ . Now the membrane theory is localized on special Lagrangian manifolds with phase  $\theta$ .

To conclude this Section we would like to make a comment on Mathai-Quillen formalism. Indeed the action (3.26) could be constructed within this formalism if we would choose the condition (2.8) as the appropriate zero section.

## 4 Hamiltonian treatment

In this Section we sketch the Hamiltonian treatment of the model (3.26). Indeed the Hamiltonian formalism is useful for the geometrical interpretation of BRST transformations (3.27) and (3.30).

Starting from the gauge fixed action (3.26) we define the momenta

$$p_\mu = g_{\mu\nu} \dot{X}^\nu - \Gamma_{\lambda\mu\nu} \bar{\psi}^\lambda \psi^\nu, \quad p_{\psi^\mu} = g_{\mu\nu} \bar{\psi}^\nu, \quad (4.41)$$

where we defined the odd momenta  $p_\psi$  using the right derivative of  $S_{GF}$  with respect to  $\partial_0 \psi^\mu$ . The canonical commutation relations are

$$\{X^\mu(\sigma), p_\nu(\sigma')\} = \delta^\mu_\nu \delta^2(\sigma - \sigma'), \quad \{\psi^\mu(\sigma), \bar{\psi}^\nu(\sigma')\}_+ = \delta^\mu_\nu \delta^2(\sigma - \sigma') \quad (4.42)$$

with other being trivial and  $\{ , \}_+$  denotes the odd Poisson bracket. The Hamiltonian  $H_{GF}$  corresponding to  $S_{GF}$  is obtained by Legendre transform and can be written as follows

$$H_{GF} = \int d^2\sigma \left( \frac{1}{2} (p_\mu + \Gamma_{\lambda\mu\rho} \bar{\psi}^\lambda \psi^\rho) g^{\mu\nu} (p_\nu + \Gamma_{\sigma\nu\tau} \bar{\psi}^\sigma \psi^\tau) - \frac{1}{2} \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) \right. \\ \left. + \Phi_{\mu\nu\rho} \bar{\psi}^\mu \nabla_a \psi^\nu \partial_b X^\rho \epsilon^{ab} - \frac{1}{4} \mathcal{R}_{\mu\sigma\lambda\rho} \bar{\psi}^\mu \bar{\psi}^\sigma \psi^\lambda \psi^\rho \right). \quad (4.43)$$

In the phase space the generator of BRST transformations (3.27) is

$$Q = \int d^2\sigma \psi^\mu (p_\mu + \Phi_{\mu\nu\rho} \partial_1 X^\nu \partial_2 X^\rho), \quad (4.44)$$

where one should be careful in working with the contravariant and covariant versions of  $\bar{\psi}$ . The anti-BRST transformations (3.30) are generated by

$$\bar{Q} = \int d^2\sigma \bar{\psi}_\tau g^{\tau\mu} (p_\mu - \Phi_{\mu\nu\rho} \partial_1 X^\nu \partial_2 X^\rho + \Gamma_{\sigma\mu\rho} \bar{\psi}^\sigma \psi^\rho). \quad (4.45)$$

Indeed  $\bar{Q}$  is minus adjoint operator of  $Q$ . This can be shown directly if  $Q$  is understood as an operator on the states  $K_{\mu_1 \dots \mu_r} \psi^{\mu_1} \dots \psi^{\mu_r} |0\rangle$  where  $K$  is an  $r$ -form on  $M$  and  $\bar{\psi}^\mu |0\rangle = 0$ . The inner product of two forms is defined as usual,  $\int K' \wedge *K$ , with  $*K = (*K)_{\mu_1 \dots \mu_{d-r}} \bar{\psi}^{\nu_1} \dots \bar{\psi}^{\nu_r} \epsilon^{\mu_1 \dots \mu_{d-r} \nu_1 \dots \nu_r}$ , where  $d = 7$  is the space-time dimension. The operator  $Q$  acts on the differential forms as a de Rham differential and  $\bar{Q}$  as minus its adjoint. This explains the choice of the bilinear fermionic term in the gauge function (3.20). In fact, analogously to the discussion in [8] one can check that this the only choice for which  $\bar{Q} = -Q^\dagger$ .

The Hamiltonian (4.43)  $H_{GF}$ , the BRST generator  $Q$  and the anti-BRST generator  $\bar{Q}$  satisfy the following on-shell relations

$$H_{GF} = \frac{1}{2} \{Q, \bar{Q}\}_+, \quad Q^2 = 0, \quad \bar{Q}^2 = 0. \quad (4.46)$$

As result of this  $H_{GF}$  is BRST and anti-BRST invariant, i.e.  $\{H_{GF}, Q\} = \{H_{GF}, \bar{Q}\} = 0$ .

In principle we could proceed with the construction of  $S_{GF}$  via Hamiltonian formalism. The model is described by the following first class constraints

$$J_\mu = p_\mu + \Phi_{\mu\nu\rho} \partial_1 X^\nu \partial_2 X^\rho, \quad (4.47)$$

which has been discussed in [14]. Since these constraints satisfy the following Poisson brackets

$$\{J_\mu(\sigma), J_\nu(\sigma')\} = 0, \quad (4.48)$$

introducing the ghosts  $\psi$  one can construct the BRST charge in the minimal sector [23] (4.44). There should exist a non-minimal extension and a suitable gauge-fixing such that after integrating out the non-minimal sector one recovers the gauge-fixed BRST structure described above. Some of the aspects of the Hamiltonian analysis of this and related systems has been discussed in [14]. Indeed the correct reduction of this model on  $M_6 \times S^1$  works at the level constraints (4.47) as well.

Moreover the Hamiltonian point of view suggests that if one wishes to include the flux 4-form  $H$  into consideration then the topological model is defined as

$$S_{top} = \int_{\Sigma_3} X^*(\Phi) - \int_{\Sigma_4} X^*(H), \quad (4.49)$$

such that  $\Phi$  is a three-form associated with a  $G_2$ -structure,  $d\Phi = H$  and  $\partial\Sigma_4 = \Sigma_3$ <sup>2</sup>. The dimensional reduction on  $M_7 = M_6 \times S^1$  of this model should give a topological sigma model for generalised complex geometries. We hope to discuss this extension of our model elsewhere.

## 5 Observables and moduli spaces

In this Section we collect some generalities on the topological membrane theory constructed in Section 3. We start by discussing the observables in the theory. For a nontrivial element  $[K] \in H^q(M)$  we can formally define the following cocycles

$$\begin{aligned} \mathcal{C}_3^{q-3} &= \frac{1}{6} K_{\mu_1 \dots \mu_q} dX^{\mu_1} \wedge dX^{\mu_2} \wedge dX^{\mu_3} \psi^{\mu_4} \dots \psi^{\mu_q}, \\ \mathcal{C}_2^{q-2} &= \frac{1}{2} K_{\mu_1 \dots \mu_q} dX^{\mu_1} \wedge dX^{\mu_2} \psi^{\mu_3} \psi^{\mu_4} \dots \psi^{\mu_q}, \\ \mathcal{C}_1^{q-1} &= K_{\mu_1 \dots \mu_q} dX^{\mu_1} \psi^{\mu_2} \psi^{\mu_3} \psi^{\mu_4} \dots \psi^{\mu_q}, \\ \mathcal{C}_0^q &= K_{\mu_1 \dots \mu_q} \psi^{\mu_1} \psi^{\mu_2} \psi^{\mu_3} \psi^{\mu_4} \dots \psi^{\mu_q}, \end{aligned} \quad (5.50)$$

where in  $\mathcal{C}_i^{q-i}$  the upper index stands for the ghost number and the lower index for the degree of the differential form on  $\Sigma_3$ . Using the transformations (3.19) we can derive the decent equations for  $\mathcal{C}_i^{q-i}$

$$s\mathcal{C}_3^{q-3} = \frac{1}{q-2} d\mathcal{C}_2^{q-2}, \quad s\mathcal{C}_2^{q-2} = \frac{1}{q-1} d\mathcal{C}_1^{q-1}, \quad s\mathcal{C}_1^{q-1} = \frac{1}{q} d\mathcal{C}_0^q, \quad s\mathcal{C}_0^q = 0. \quad (5.51)$$

Thus  $\mathcal{C}_0^q$  are BRST-invariant local observables labeled by the elements of the de Rham complex  $H^\bullet(M)$ . From  $\mathcal{C}_i^{q-i}$  with  $i > 0$  we can construct BRST-invariant non-local observables as integrals

$$\int_{c_i} \mathcal{C}_i^{q-i}, \quad (5.52)$$

---

<sup>2</sup>The action (4.49) is invariant due to the identity

$$\delta \int_{\Sigma_n} X^*(\Phi) = \int_{\partial\Sigma_n} X^*(i_{\delta_X} \Phi) + \int_{\Sigma_n} X^*(i_{\delta_X} d\Phi)$$

where  $c_i$  is  $i$ -cycle on  $\Sigma_3$ . However not all observables have non-vanishing correlators in the theory. To study this we need to address the ghost number anomaly. The action (3.26) has at the classical level a ghost number conservation law, with  $\psi$  having ghost number 1,  $\bar{\psi}$  having ghost number  $-1$  and  $X$  having ghost number 0. The BRST transformation  $s$  (3.27) changes the ghost number by 1. Notice that all the observables, but  $C_i^0$  with  $i = 1, 2, 3$ , defined in (5.50) have a non-vanishing ghost number. Thus, in order to have non-vanishing correlators there should be a compensating ghost number anomaly. The linearized equations for the fermionic fluctuations around the instanton background are

$$D\psi^\mu = \nabla_0\psi^\mu + \Phi^\mu{}_{\nu\rho}\epsilon^{ab}\nabla_a\psi^\nu\partial_bX^\rho = 0, \quad (5.53)$$

$$D^\dagger\bar{\psi}^\mu = \nabla_0\bar{\psi}^\mu - \Phi^\mu{}_{\nu\rho}\epsilon^{ab}\nabla_a\bar{\psi}^\nu\partial_bX^\rho = 0. \quad (5.54)$$

The equation (5.53) is the first order variation of the associative map (2.8). As such,  $\psi$  can be interpreted as a section of the tangent bundle to the moduli space  $\mathcal{M}$  of associative maps. Indeed the operator  $D^\dagger$  is the adjoint of  $D$ , and thus the ghost number anomaly is given by the index  $ind(D)$ . Since our theory lives in three dimensions  $ind(D)$  vanishes by index theorem. Thus the correlators of the operators in (5.50) are vanishing, except  $C_i^0$  with  $i = 1, 2, 3$ , where  $C_3^0$  corresponds to our classical action (3.17) and its variations in  $H^3(M)$ . The non-trivial topological observable is the partition function, which computes the Euler characteristic of the moduli space of associative maps. In fact, the zero modes of (5.53) and (5.54) can be soaked up by the  $\mathcal{R}\psi^2\bar{\psi}^2$  term of the action (3.26). The situation is very similar to supersymmetric quantum mechanics (for a review, see [13]).

The above discussion was rather formal and we did not analyse the moduli space of associative maps. Not so much is known about these moduli spaces, see for example the work by McLean [27] and the recent works [1, 2]. Actually our Dirac operator  $D$  can be mapped to McLean operator since both deal with the deformations of associative manifolds. Indeed, instead of (2.8) we could have chosen a different parametrization for the associative maps corresponding to the static gauge

$$X^\alpha = \sigma^\alpha, \quad (5.55)$$

$$\dot{X}^i \pm \Phi^i{}_{aj}\epsilon^{ab}\partial_bX^j = 0, \quad (5.56)$$

where  $\alpha = 0, 1, 2$  and  $i = 3, \dots, 6$ . The BRST variation of (5.56) gives the McLean twisted Dirac operator. In the gauge (5.56) our model reproduces the results of the supermembrane theory in [10] and also [5].

Although the membrane theory does not display a ghost number anomaly, it suffers a global anomaly under parity transformations [6]. This has a counterpart in the A-model upon  $S^1$  compactification. For membranes wrapping the  $S^1$  with a given winding number  $k$

the spectral flow of the operator (5.53) should match the index of the  $\nabla_z$  operator in (3.35) by arguments similar to those in [3]. This should allow to recover the ghost number anomaly of the A model and the corresponding non-trivial correlators coming from (5.50).

## 6 Vector cross products and TFTs

In Sections 2 and 3 we discussed the membrane instantons on  $G_2$ -manifolds and constructed topological membrane theory which localizes on these instantons. The whole construction is very similar to A-model (topological sigma model) [29, 8]. Indeed there is whole set of topological  $p$ -brane models which follows the same pattern. These models are based on the geometrical notion of vector cross product structure. In what follows we sketch the main steps of the construction of topological theories based both on real and complex vector cross products.

We start with the case of a real cross vector product (see Appendix for the definition and properties). Consider the Nambu-Goto  $p$ -brane theory on the manifold  $M$  with Riemannian metric  $g$

$$S = \int d^{p+1}\sigma \sqrt{\det(\partial_\alpha X^\mu g_{\mu\nu} \partial_\beta X^\nu)}, \quad (6.57)$$

where  $\alpha = 0, 1, \dots, p$ . In analogy with the membrane case there is a gauge in which the  $p$ -brane action has the following form

$$S = \frac{1}{2} \int d^{p+1}\sigma \left( \dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) \right), \quad (6.58)$$

with  $\dot{X}^\mu = \partial_0 X^\mu$  and  $a, b = 1, \dots, p$ . Assuming that there is a  $(p+1)$ -form on  $M$  we can write down the bound

$$\int d^{p+1}\sigma \left( \dot{X}^\mu \pm \phi^\mu_{\nu_1 \dots \nu_p} \partial_1 X^{\nu_1} \dots \partial_p X^{\nu_p} \right) g_{\mu\lambda} \left( \dot{X}^\lambda \pm \phi^\lambda_{\sigma_1 \dots \sigma_p} \partial_1 X^{\sigma_1} \dots \partial_p X^{\sigma_p} \right) \geq 0. \quad (6.59)$$

If  $\phi$  and  $g$  correspond to a vector cross product structure on  $M$  then the bound (6.59) can be rewritten as follows

$$\frac{1}{2} \int d^{p+1}\sigma \left( \dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) \right) \geq \mp \frac{1}{(p+1)!} \int X^*(\phi). \quad (6.60)$$

Moreover if  $d\phi = 0$  the right-hand side is a topological term. The bound (6.60) is saturated if

$$\dot{X}^\mu \pm \phi^\mu_{\nu_1 \dots \nu_p} \partial_1 X^{\nu_1} \dots \partial_p X^{\nu_p} = 0, \quad (6.61)$$

which we call  $p$ -brane instanton. Geometrically it corresponds to a submanifold of  $M$  calibrated by  $\phi$  [26].

Following the considerations from Section 2 we consider the topological  $p$ -brane theory

$$S_{top} = -\frac{1}{(p+1)!} \int X^*(\phi), \quad (6.62)$$

where  $\phi$  is a closed  $(p+1)$ -form corresponding to a cross vector product on  $M$ . The action (6.62) is invariant under the gauge symmetry  $\delta X^\mu = \epsilon^\mu$ . The corresponding BRST transformations are defined as in (3.19). Choosing the gauge function as

$$\mathcal{F}^\mu = \dot{X}^\mu + \phi^\mu_{\nu_1 \dots \nu_p} \partial_1 X^{\nu_1} \dots \partial_p X^{\nu_p} + \frac{1}{2} \Gamma^\mu_{\sigma\rho} \bar{\psi}^\sigma \psi^\rho \quad (6.63)$$

we define the gauge fixed action as follows

$$S_{GF} = -\frac{1}{(p+1)!} \int X^*(\phi) + \int d^{p+1}\sigma \, s \left( \bar{\psi}^\mu (g_{\mu\nu} \dot{X}^\nu + \phi_{\mu\nu_1 \dots \nu_p} \partial_1 X^{\nu_1} \dots \partial_p X^{\nu_p} + \frac{1}{2} \Gamma_{\mu\sigma\rho} \bar{\psi}^\sigma \psi^\rho - \frac{1}{2} g_{\mu\nu} b^\nu) \right). \quad (6.64)$$

Eliminating  $b$  by its algebraic equation we arrive to the following gauge fixed action

$$S_{GF} = \int d^{p+1}\sigma \left( \frac{1}{2} \dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \frac{1}{2} \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) - \bar{\psi}^\mu g_{\mu\nu} \nabla_0 \psi^\nu + \frac{1}{4} \mathcal{R}_{\mu\sigma\lambda\rho} \bar{\psi}^\mu \psi^\rho \bar{\psi}^\sigma \psi^\lambda - \frac{1}{(p-1)!} \phi_{\mu\nu_1 \nu_2 \dots \nu_p} \bar{\psi}^\mu \nabla_{a_1} \psi^{\nu_1} \partial_{a_2} X^{\nu_2} \dots \partial_{a_p} X^{\nu_p} \epsilon^{a_1 a_2 \dots a_p} - \frac{1}{(p-2)!} \nabla_\lambda \phi_{\mu\nu_1 \nu_2 \dots \nu_p} \bar{\psi}^\mu \psi^\lambda \partial_{a_1} X^{\nu_1} \dots \partial_{a_p} X^{\nu_p} \epsilon^{a_1 a_2 \dots a_p} \right), \quad (6.65)$$

where  $\nabla_\alpha \psi$  is defined in (3.24). The action (6.65) is invariant under the following BRST transformations

$$sX^\mu = \psi^\mu, \quad s\psi^\mu = 0, \quad s\bar{\psi}^\mu = \dot{X}^\mu + \phi^\mu_{\nu_1 \dots \nu_p} \partial_1 X^{\nu_1} \dots \partial_p X^{\nu_p} + \Gamma^\mu_{\sigma\rho} \bar{\psi}^\sigma \psi^\rho, \quad (6.66)$$

which are nilpotent on-shell. The action (6.65) can be rewritten as

$$S_{GF} = -\frac{1}{(p+1)!} \int X^*(\phi) + \frac{1}{2} \int d^{p+1}\sigma \, s \left( \bar{\psi}^\mu (g_{\mu\nu} \dot{X}^\nu + \phi_{\mu\nu_1 \dots \nu_p} \partial_1 X^{\nu_1} \dots \partial_p X^{\nu_p}) \right) \quad (6.67)$$

and therefore the model is localized on the  $p$ -brane instantons (6.61). Due to standard arguments the theory does not depend on the way we describe  $p$ -brane instantons. Namely any change in the distinguished direction  $\alpha = 0$  in the path integral will contribute a BRST-exact term and thus is irrelevant.

An interesting question is: how generic is our construction? Actually, all real vector cross products have been classified by Brown and Gray [15] (see the list in the Appendix B). There are four different cases for which the cross product exists. The first case corresponds to  $\phi$

being the volume form on  $M$ . In this case the TFT we constructed corresponds to  $p$ -branes embedded into a  $p + 1$  dimensional space  $M$ . Some of the aspects of this TFT has been discussed in [12]. The second case corresponds to a symplectic manifold with  $\phi$  being a closed non-degenerate 2-form. The corresponding TFT is just topological sigma model (A-model) [29]. The remaining two vector cross product structures are the exceptional cases. The first corresponds to seven dimensional manifolds with  $G_2$ -structure and  $\phi$  is the three form  $\Phi$ . This is the theory we constructed in Section 3. The second exceptional case corresponds to eight dimensional manifolds with  $Spin(7)$ -structure where  $\phi$  the associated 4-form  $\Psi$  (the Cayley form). In this case our model describes 3-branes in a  $Spin(7)$ -manifold. This is presumably the microscopic description of the recently proposed topological F-theory [4]. Therefore we refer to this theory as topological F-theory on  $Spin(7)$ -manifolds. This TFT is localized on Cayley 4-manifolds (i.e, those calibrated by  $\Psi$ ). It is not hard to repeat for the topological F-theory the analysis which we have done in Sections 3-5 for topological M-theory. In particular one can consider the reduction of F-theory on

$$CY_3 \times T^2 \implies CY_3 \times S^1 \implies CY_3,$$

where  $CY_3 \times T^2$  is a  $Spin(7)$ -manifold. This reduction will produce the whole Zoo of TFTs on  $CY_3$  which were discussed briefly at the Hamiltonian level in [14]. One can perform the reduction also at the level of the gauge fixed action in a similar way as in Section 3.

So far we have considered the real cross vector product structures. On Hermitian manifolds  $(g, J, M)$  one can introduce the complex version [26] of cross vector products (see the Appendix B for the definition). In this case the complex vector product<sup>3</sup> is given by a holomorphic  $p$ -form which is either a holomorphic volume form or a holomorphic symplectic form on  $M$ . Indeed it is straightforward to generalize our construction to the topological action

$$S_{top} = -\frac{1}{(p+1)!} \int X^*(Re(e^{i\theta}\Omega)), \quad (6.68)$$

where  $\Omega$  is a closed form corresponding to a complex vector cross product on  $(g, J, M)$ . The construction of the gauge fixed action goes along the lines we have presented above and thus we give only the final result of the construction. The gauge fixed action is

$$\begin{aligned} S_{GF} = & \int d^{p+1}\sigma \left( \frac{1}{2} \dot{X}^\mu g_{\mu\nu} \dot{X}^\nu + \frac{1}{2} \det(\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu) - \bar{\psi}^\mu g_{\mu\nu} \nabla_0 \psi^\nu \right. \\ & + \frac{1}{4} \mathcal{R}_{\mu\sigma\lambda\rho} \bar{\psi}^\mu \psi^\rho \bar{\psi}^\sigma \psi^\lambda - \frac{1}{(p-1)!} (Re(e^{i\theta}\Omega))_{\mu\nu_1\nu_2\dots\nu_p} \bar{\psi}^\mu \nabla_{a_1} \psi^{\nu_1} \partial_{a_2} X^{\nu_2} \dots \partial_{a_p} X^{\nu_p} \epsilon^{a_1 a_2 \dots a_p} \\ & \left. - \frac{1}{(p-2)!} \nabla_\lambda (Re(e^{i\theta}\Omega))_{\mu\nu_1\nu_2\dots\nu_p} \bar{\psi}^\mu \psi^\lambda \partial_{a_1} X^{\nu_1} \partial_{a_2} X^{\nu_2} \dots \partial_{a_p} X^{\nu_p} \epsilon^{a_1 a_2 \dots a_p} \right), \quad (6.69) \end{aligned}$$

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<sup>3</sup>If  $M$  is a Kähler manifold with complex vector cross product then  $M$  is either Calabi-Yau with a holomorphic volume form or hyperkähler with holomorphic symplectic form.

which is invariant under the following BRST transformation

$$sX^\mu = \psi^\mu, \quad s\psi^\mu = 0, \quad s\bar{\psi}^\mu = \dot{X}^\mu + (Re(e^{i\theta}\Omega))^\mu_{\nu_1\dots\nu_p} \partial_1 X^{\nu_1} \dots \partial_p X^{\nu_p} + \Gamma^\mu_{\sigma\rho} \bar{\psi}^\sigma \psi^\rho. \quad (6.70)$$

In this construction it is essential that the metric  $g$  is Hermitian and  $\Omega$  is either a holomorphic symplectic form or a holomorphic volume form. The present model is localized on submanifolds calibrated by  $Re(e^{i\theta}\Omega)$ .

## 7 Conclusions

In this work we have constructed the gauge fixed action for the topological membrane on  $G_2$ -manifolds. The bosonic part of the action is the standard membrane action in a particular gauge. This TFT is localized on associative maps and its partition function computes the Euler characteristic of the corresponding moduli space<sup>4</sup>.

Indeed our model plays the analogous rôle for topological M theory as the topological sigma model for the topological string. Therefore in order to complete the program of giving a microscopic description of topological M theory in terms of membranes, a crucial issue is the coupling with three-dimensional topological gravity. In analogy with the topological A string, the contribution of the constant maps to the partition function of this complete membrane model should give the volume of the target  $G_2$  manifold, and hence the Hitchin functional considered in [16].

The coupling of our model to 3D gravity requires a covariant world-volume gauge-fixed description. The relevant three-dimensional gauge theory should be a  $BF$  theory with  $SU(2)$  gauge group<sup>5</sup>. This coupling will contribute to the three-dimensional parity anomaly in such a way that the complete model on  $S^1$  at fixed winding will match the ghost anomaly of the topological string and hopefully reproduce the non-trivial correlators of the latter theory.

We also generalized our approach to topological  $p$ -brane theories corresponding to real and complex vector cross product structures on  $M$ . In particular, there is a well-defined topological 3-brane theory on  $Spin(7)$ -manifolds, which is possibly relevant for topological F theory and whose quantum mechanical properties deserve further study. Notice that we expect this theory to display a non-vanishing ghost anomaly, thus completing the analogous of the dimensional ladder of [3, 17].

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<sup>4</sup>For example, the contribution of the constant maps to the free energy of both our model and the one presented in [5] does not give the volume of the  $G_2$ -manifold due to presence of fermionic zero modes.

<sup>5</sup>The possible relevance of  $BF$  theory in the context of topological M theory was put forward in [16, 9].



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## A $G_2$ -manifolds

In this Appendix we collect the relevant information about seven dimensional manifolds with  $G_2$ -structure. For further details the reader may consult [24].

Let  $e_1, e_2, \dots, e_7$  denote the standard basis of  $\mathbb{R}^7$  and let  $e^1, e^2, \dots, e^7$  denote the corresponding dual basis. Define an element in  $\Lambda^3((\mathbb{R}^7)^*)$

$$\Phi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where  $e^{ijk} \equiv \frac{1}{3!} e^i \wedge e^j \wedge e^k$ . The group  $G_2$  is defined as follows

$$G_2 = \{g \in GL(7, \mathbb{R}), g^*(\Phi_0) = \Phi_0\},$$

i.e.  $G_2$  is the stabilizer subgroup of  $\Phi_0$  in  $GL(7, \mathbb{R})$ .

A smooth seven dimensional manifold  $M$  has  $G_2$ -structure if its tangent frame bundle reduces to a  $G_2$  bundle. Equivalently,  $M$  has a  $G_2$ -structure if there is a three form  $\Phi \in \Omega^3(M)$  such that at each point  $x \in M$  the pair  $(T_x M, \Phi_x)$  is isomorphic to  $(T_0 \mathbb{R}^7, \Phi_0)$ .

A manifold with  $G_2$ -structure  $(M, \Phi)$  is called  $G_2$ -manifold if the holonomy group of the Levi-Civita connection of the metric  $g$  lies inside of  $G_2$ . Equivalently  $(M, \Phi)$  is a  $G_2$ -manifold if  $d\Phi = d * \Phi = 0$ .

The crucial property of  $\Phi$  and  $g$  on manifolds with  $G_2$ -structure we use in the calculation is the following one

$$\Phi_{\mu\nu\rho} u^\nu v^\rho \Phi^\mu_{\lambda\sigma} u^\lambda v^\sigma = \det \begin{pmatrix} u^\mu g_{\mu\nu} u^\nu & u^\mu g_{\mu\nu} v^\nu \\ v^\mu g_{\mu\nu} u^\nu & v^\mu g_{\mu\nu} v^\nu \end{pmatrix}.$$

This corresponds to the property that there is a vector cross product structure on  $M$ , see next Appendix.

## B Vector cross product structure

In this Appendix we review the real and complex vector cross product structures on smooth manifolds.

We start from the real version of vector cross product. We all are familiar with the usual vector cross product  $\times$  of two vectors in  $\mathbb{R}^3$ , which satisfies

- $u \times v$  is bilinear and skew symmetric
- $u \times v \perp u, v$ ; so  $(u \times v) \cdot v = 0$  and  $(u \times v) \cdot u = 0$
- $(u \times v) \cdot (u \times v) = \det \begin{pmatrix} u \cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{pmatrix}$

The generalization of vector cross product to a Riemannian manifold leads to the following definition by Brown and Gray [15]

**Definition 1** *On  $d$ -dimensional Riemannian manifold  $M$  with a metric  $g$  an  $p$ -fold vector cross product is a smooth bundle map*

$$\chi : \wedge^p TM \rightarrow TM$$

*satisfying*

$$\begin{aligned} g(\chi(v_1, \dots, v_p), v_i) &= 0, \quad 1 \leq i \leq p \\ g(\chi(v_1, \dots, v_p), \chi(v_1, \dots, v_p)) &= \|v_1 \wedge \dots \wedge v_p\|^2 \end{aligned}$$

*where  $\|\dots\|$  is the induced metric on  $\wedge^p TM$ .*

Equivalently the last property can be rewritten in the following form

$$g(\chi(v_1, \dots, v_p), \chi(v_1, \dots, v_p)) = \det(g(v_i, v_j)) = \|v_1 \wedge \dots \wedge v_p\|^2.$$

The first condition in the above definition is equivalent to the following tensor  $\phi$

$$\phi(v_1, \dots, v_p, v_{p+1}) = g(\chi(v_1, \dots, v_p), v_{p+1})$$

being a skew symmetric tensor of degree  $p + 1$ , i.e.  $\phi \in \Omega^{p+1}(M)$ . Thus in what follows we consider a  $(p + 1)$ -form  $\phi$  which defines the  $p$ -fold vector cross product. Alternatively a vector cross product form can be defined via a form  $\phi \in \Omega^{p+1}(M)$  satisfying the following property

$$\|i_{e_1 \wedge e_2 \wedge \dots \wedge e_p} \phi\| = 1$$

for any orthonormal set  $e_1, e_2, \dots, e_p \in T_x M$  and any  $x \in M$ .

Cross products on real spaces were classified by Brown and Gray [15]. The global vector cross products on manifolds were first studied by Gray [20]. They fall into four categories:

(1)  $p = d - 1$  and  $\phi$  is the volume form of the manifold.

(2)  $d$  is even and  $p = 1$ . In this case we have a one-fold cross product  $J : TM \rightarrow TM$ . Such a map satisfies  $J^2 = -1$  and is an almost complex structure. The associated 2-form is the Kähler form.

(3) The first of two exceptional cases is a 2-fold cross product ( $p = 2$ ) on a 7-manifold. Such a structure is called a  $G_2$ -structure and the associated 3-form is called a  $G_2$ -form (that is  $\Phi$  in the notation of Appendix A and in the main text).

(4) The second exceptional case is 3-fold cross product ( $p = 3$ ) on an 8-manifold. This is called a  $Spin(7)$ -structure and the associated 4-form is called  $Spin(7)$ -form.

The complex version of vector cross product has been introduced in [26]. Consider a Hermitian manifold  $(g, J, M)$  and define the complex vector cross product as a holomorphic  $(p + 1)$ -form satisfying

$$\|i_{e_1 \wedge e_2 \wedge \dots \wedge e_p} \phi\| = 2^{(p+1)/2}$$

for any orthonormal tangent vectors  $e_1, e_2, \dots, e_p \in T_x^{1,0} M$ , for any  $x \in M$ . One can show from this definition that  $\phi$  can be either a holomorphic symplectic form or a holomorphic volume form [26]. Thus the examples of manifolds equipped with the complex vector cross product structure are hyperkähler and Calabi-Yau manifolds.

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