ON THE BLOW-UP OF $GSBV$ FUNCTIONS UNDER SUITABLE GEOMETRIC PROPERTIES OF THE JUMP SET

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ABSTRACT. In this paper we investigate the fine properties of functions under suitable geometric conditions on the jump set. Precisely, given an open set $\Omega \subset \mathbb{R}^n$ and given $p > 1$ we study the blow-up of functions $u \in GSBV(\Omega)$, whose jump sets belong to an appropriate class $\mathcal{J}_p$ and whose approximate gradients are p-th power summable. In analogy with the theory of p-capacity in the context of Sobolev spaces, we prove that the blow-up of u converges up to a set of Hausdorff dimension less than or equal to $n-p$. Moreover, we are able to prove the following result which in the case of $W^{1,p}(\Omega)$ functions can be stated as follows: whenever u_k strongly converges to u, then up to subsequences, u_k pointwise converges to u except on a set whose Hausdorff dimension is at most $n-p$.

Keywords: blow-up, special bounded variation, indecomposable set, jump set, perimeter, rectifiable set, capacity, Cheeger's constant, isoperimetric profile, Poincaré's inequality.

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INTRODUCTION

The following result concerning the Lebesgue points of a Sobolev function is well known (see [9, 7, 20, 23, 11]): given $1 < p < n$, if $u \in L^1_{loc}(\mathbb{R}^n)$ and its first order distributional derivatives are p-th power locally summable, then there exists a set A with $\dim_{\mathcal{H}}(A) \leq n-p$, namely with Hausdorff dimension at most $n-p$, such that every $x \in \mathbb{R}^n \setminus A$ is a Lebesgue point for u. More precisely, for every $x \in \mathbb{R}^n \setminus A$ there exists a real number a such that:

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B_r(x)} |u(y) - a| \, dy = 0. \tag{0.1}$$

By a change of variables, if we call u_x the function constantly equal to a, the convergence in (0.1) can be rephrased by saying that $u_{r,x}(y) := u(x+ry)$, namely the *blow-up* of u at x, converges in $L^1(B_1(0))$ to u_x, i.e.

$$\lim_{r \to 0^+} \int_{B_1(0)} |u(x+ry) - u_x(y)| \, dy = 0. \tag{0.2}$$

Roughly speaking, (0.2) says in a precise way that the values of u near x are close to a single constant. The aim of this paper is to investigate the local behavior of functions when we introduce also a jump discontinuity set.

Given $\Omega \subset \mathbb{R}^n$ an open set, for every function u which belongs to the space $GSBV(\Omega)$ and whose approximate gradient ∇u belongs to $L^1(\Omega;\mathbb{R}^n)$, by using the general theory developed in [1] one can deduce that at every point x it holds

$$u_{r,x} \to u_x \text{ in measure in } B_1(0), \text{ as } r \to 0^+,$$

except on a set A with $\mathcal{H}^{n-1}(A) = 0$. Furthermore, if x is a Lebesgue point then u_x is a constant function, while if $x \in J_u$ then u_x assumes two different values on two disjoint subsets of $B_1(0)$ separated by an $(n-1)$-dimensional hyperplane passing through the origin. In this situation u_x may assume from one or two values.

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In this work we focus our attention on the space $GSBV^p(\Omega)$ when $1 < p \le n$. Precisely, we investigate under which hypothesis on the jump set, the p-th power summability of the approximate gradient guarantees $\dim_{\mathcal{H}}(A) \le n - p$.

To illustrate the result we are going to prove, let us consider the following example. Consider $\Gamma_0 \subset \mathbb{R}^2$ the union of three half lines starting from the origin. Let $\Gamma \subset \mathbb{R}^3$ be defined by $\Gamma_0 \times \mathbb{R}$ and let l be the straight line $\{(0,0,t) \mid t \in \mathbb{R}\}$. The set Γ disconnects $\mathbb{R}^3 \setminus \Gamma$ into three connected components $\Omega_1, \Omega_2, \Omega_3$. Whenever u is a locally integrable function in $GSBV(\Omega)$ with $J_u \subset \Gamma$, thanks to a well known property of locally integrable $GSBV$-functions ∇u coincides with the distributional gradient in each open sets Ω_i; then by using Poincaré-Wirtinger inequality on balls, it is easy to prove that every $u \in GSBV^p(\mathbb{R}^3) \cap L^1_{loc}(\mathbb{R}^3)$ with $J_u \subset \Gamma$ satisfies $u \llcorner \Omega_i \in W^{1,p}_{loc}(\Omega_i)$ for $i = 1, 2, 3$. Using a reflection argument, through an obvious modification of the result in [9], there exists a set A with $\dim_{\mathcal{H}}(A) \le 3 - p$ such that if $x \in \mathbb{R}^3 \setminus A$ then the blow-up of u at x converges. In addition, on the points $x \in l \setminus A$ the limit u_x can assume three different values α_i each on the set $\Omega_i \cap B_1(0)$, $i = 1, 2, 3$. Therefore, the family of all possible limits u_x is richer than the previous cases.

Nevertheless, the p-th power summability of the approximate gradient is in general not enough to guarantee the convergence of the blow-up at every point except on a set of Hausdorff dimension $(n - p)$. Consider for example $u := \mathbb{1}_E$, the characteristic function of a set with finite perimeter. Clearly ∇u is p-summable for every $p \ge 1$, but from the theory of sets of finite perimeter, we know that the blow-up of u in general converges only up to an $\mathcal{H}^{n-1}$-negligible set. Precisely, it is possible to construct a set $E \subset \mathbb{R}^2$ with finite perimeter and such that, by setting $u = \mathbb{1}_E$, the set of points x where $u_{r,x}$ does not converge has Hausdorff dimension exactly equal to 1 (see Section 7). Therefore, it is reasonable to think that the geometry of the jump set affects the local behavior of the functions.

In Definition 3.6, for every $1 < p \le n$ we introduce the class $\mathcal{J}_p$ of all admissible jump sets, for which the following two main results hold true.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $\Gamma \in \mathcal{J}_p$ $(1 < p \le n)$. If $u \in GSBV^p(\Omega;\Gamma)$, then there exists a set A_u with Hausdorff dimension at most $n - p$, such that for every $x \in \Omega \setminus A_u$ there exists a function $u_x(\cdot)\colon B_1(0) \to \mathbb{R}$*

$$u_{r,x} \to u_x, \text{ in measure in } B_1(0), \tag{0.3}$$

as $r \to 0^+$.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be open and let $\Gamma \in \mathcal{J}_p$ with $(1 < p \le n)$. Suppose $(u_k)_{k=1}^\infty \subset GSBV^p(\Omega;\Gamma) \cap L^p(\Omega)$ is such that*

$$\|u_k - u\|_{L^p} + \|\nabla u_k - \nabla u\|_{L^p} \to 0, \text{ as } k \to \infty.$$

Then there exists a subsequence $(k_j)_j$, such that for every $x \in \Omega$ except on a set with Hausdorff dimension at most $n - p$ we have

$$(u_{k_j})_x \to u_x \text{ in measure in } B_1(0), \text{ as } j \to \infty, \tag{0.4}$$

where in (0.4) $(u_k)_x$ is the one given by (0.3) where u is replaced by u_k.

Theorem 1 can be seen as the analogous of the result (0.2) mentioned above. In the context of Sobolev spaces this is obtained through the theory of capacity, by exploiting the well known fact that smooth functions are dense in $W^{1,p}(\Omega)$. However, at the best of our knowlegde, it is not known whether there exist dense subspaces of $GSBV^p(\Omega;\Gamma)$ made of regular functions u with the additional constraint $J_u \subset \Gamma$ (see Remark 5.22). For this reason, we decide to perform a different analysis based on Geometric Measure Theory techniques. In particular we prove a weak version of Poincaré's inequality on balls, which guarantees that the L^0-distance of u from a particular piecewise constant function can be controlled in terms of the L^p-norm of its approximate gradient plus a small volume error (see Theorem 4.4). This tool, together with a fine analysis of the blow-up of u permits us to obtain the conclusion of Theorem 1. The dimension $n - p$ is optimal, since in the $W^{1,p}(\Omega)$ setting, i.e. when $\Gamma = \emptyset$, we already know that it is sharp (see Remark 4.10).

Theorem 2 is reminiscent of the following result in the context of Sobolev space: if a sequence u_k in $W^{1,p}(\Omega)$ strongly converges to u, then, up to subsequences, the precise value of $u_k(x)$ defined by (0.2) converges to the precise value of $u(x)$, except on a set of zero p-capacity (see for example [11, Lemma 4.8]). In order to prove Theorem 2, we use a suitable notion of capacity (see Definition (5.7)), which allows us to deduce the convergence (0.4) for every x except on a set of capacity zero. The relation between this novel notion of capacity and the Hausdorff measure (see Theorem 5.16) enables us to deduce Theorem 2.

The class $\mathcal{J}_p$ is composed of all $(\mathcal{H}^{n-1}, n-1)$-rectifiable sets (see [8, Subsection 3.2.14]) with finite $\mathcal{H}^{n-1}$-measure, which satisfy a suitable geometric condition at every point except on a set with Hausdorff dimension $n-p$ (see Definition 3.6). For example, finite union of $(n-1)$-dimensional manifolds of class C^1 belong to $\mathcal{J}_p$ for every $1 < p \leq n$. More in general, finite unions of graphs of Sobolev functions in $W^{2,p}$ belong to $\mathcal{J}_p$ (see Example 6.3). As pointed out in Remark 6.2, whenever $n > 2p+1$, the graph of a $W^{2,p}$-function might have topological closure with arbitrarily large n-dimensional Lebesgue measure. This shows that a generic set in $\mathcal{J}_p$ *does not need to be essentially closed*. In addition, in Example 6.5 we are able to construct a set in $\mathbb{R}^2$ which cannot be written as a finite union of graphs, but nevertheless it belongs to $\mathcal{J}_p$ for every $1 < p \leq 2$.

In order to define the property which characterizes the sets in $\mathcal{J}_p$, we make use of the theory of indecomposable sets, for which we introduce a geometric quantity called *upper isoperimetric profile* (see (2.12)). This quantity plays a similar role to that of the *Cheeger's constant* in the context of Riemannian manifolds. Roughly speaking, if $\Gamma \in \mathcal{J}_p$ then for every x up to a set of Hausdorff dimension $n-p$, the set $B_1(0) \setminus (\Gamma - x)/r$ can be overrun by N_x indecomposable sets (possibly depending on x), say $(F_{r,i})_{i=1}^{N_x}$, in such a way that the upper isoperimetric profile of the sets $F_{r,i}$ does not vanish as $r \to 0^+$. We call this property *non vanishing upper isoperimetric profile* (see Definition 3.2). This property *is optimal* in view of Theorem 1. More precisely, we construct a counterexample to Theorem 1 which shows that, essentially, *the notion of non vanishing upper isoperimetric profile cannot be weakened* (see Example 6.7).

1. Preliminary results

In this first section we recall some properties about sets of finite perimeter. In particular we focus our attention on the concept of *indecomposable set*, which will play an important role for the rest of the paper. We end this section by recalling some fundamental tools and results about the space $GSBV(\Omega)$ which will be useful in the sequel.

1.1. Sets of finite perimeter.

Given Ω an open set of $\mathbb{R}^n$ we recall that a $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$ has finite perimeter in Ω if

$$P(E;\Omega) := \sup_{\substack{\varphi \in C^1_c(\Omega;\mathbb{R}^n)\\ \|\varphi\|_\infty \leq 1}} \int_E \operatorname{div}\varphi \, dx < \infty,$$

where div denotes the divergence operator defined as usual, i.e. $\operatorname{div}\varphi := \sum_{i=1}^n \frac{\partial \varphi^i}{\partial x_i}$. If $\Omega = \mathbb{R}^n$ we simply write $P(E)$ to denote $P(E;\mathbb{R}^n)$. Whenever E has finite perimeter, by means of Riesz's representation Theorem, we know that the distributional gradient of the characteristic function of E, i.e. $D\mathbb{1}_E$, can be represented as a measure in $\mathcal{M}_b(\Omega;\mathbb{R}^n)$ (the space of all $\mathbb{R}^n$-valued bounded Radon measures on Ω). In particular, by denoting the total variation of $D\mathbb{1}_E$ as $|D\mathbb{1}_E|$, then for every Borel set $B \subset \Omega$ the relative perimeter of E in B is defined as

$$P(E;B) := |D\mathbb{1}_E|(B).$$

We denote by $\partial^* E$ the reduced boundary of E, defined as those $x \in \Omega$ for which there exists $\nu_E(x) \in \mathbb{S}^{n-1}$ such that

$$\lim_{r \to 0^+} \frac{D\mathbb{1}_E(B_r(x))}{|D\mathbb{1}_E|(B_r(x))} = \nu_E(x). \tag{1.1}$$

The unitary vector $\nu_E(x)$ is the *measure-theoretic inner normal* of E at x.

1.2. **Structure properties.** Following the notation in [8, Subsection 2.10.19], given $x \in \Omega$, whenever $0 \leq \alpha \leq n$, we denote the α-dimensional upper and lower densities of μ at x, respectively, as

$$\Theta^{*\alpha}(\mu, x) := \limsup_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha},$$
$$\Theta^{\alpha}_*(\mu, x) := \liminf_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha}.$$

If the upper and lower density coincide, the α-dimensional density of μ at x is defined as

$$\Theta^{\alpha}(\mu, x) := \lim_{r \to 0^+} \frac{\mu(B_r(x))}{\omega_\alpha r^\alpha}.$$

Given $0 \leq \delta \leq 1$ and given a set $A \subset \Omega$ we denote the point of density δ of A as

$$A^{(\delta)} := \{x \in \Omega \mid \Theta^n(\mathcal{L}^n \llcorner A, x) = \delta\},$$

where $\mathcal{L}^n$ is the n-dimensional Lebesgue outer measure.

De Giorgi's structure Theorem holds true (see for example [1, Theorem 3.59]).

Theorem 1.1. *Let Ω be an open set of $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ with $P(E;\Omega) < \infty$. Then $\partial^* E$ is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable and*

$$|D\mathbb{1}_E| = \mathcal{H}^{n-1} \llcorner \partial^* E.$$

In addition, for every $x \in \partial^* E$ the following properties hold

(a) the sets $(E-x)/r$ locally converge in measure in $\mathbb{R}^n$ as $r \to 0^+$ to the halfspace H orthogonal to $\nu_E(x)$ and containing $\nu_E(x)$;
(b) $\Theta^{(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) = 1$.

We shall make use of the following two results. The first is due to Federer and concerns the structure of sets having finite perimeter. The second can be seen as a sort of Leibniz's formula for the intersection of two sets of finite perimeter.

Theorem 1.2. *Let Ω be an open set of $\mathbb{R}^n$ and let $E \subset \mathbb{R}^n$ with $P(E;\Omega) < \infty$. Then*

- $\mathcal{H}^{n-1}(E^{(1/2)} \Delta \partial^* E) = 0$;
- $\mathcal{H}^{n-1}(\Omega \setminus [E^{(1)} \cup E^{(1/2)} \cup E^{(0)}]) = 0$.

Proof. See for example [1, Theorem, 3.61]. □

Proposition 1.3 (Leibniz's formula)**.** *Let Ω be an open set of $\mathbb{R}^n$ and let $E, F \subset \mathbb{R}^n$ with $P(E;\Omega), P(F;\Omega) < \infty$. Then $P(E \cap F;\Omega) < \infty$ and moreover*

$$\begin{aligned}\mathcal{H}^{n-1} \llcorner \partial^*(E \cap F) = \mathcal{H}^{n-1} \llcorner \partial^* E \cap F^{(1)} + \mathcal{H}^{n-1} \llcorner \partial^* F \cap E^{(1)} \\ + \mathcal{H}^{n-1} \llcorner \{\nu_E = \nu_F\}.\end{aligned} \tag{1.2}$$

Proof. See [18, Theorem 16.3] □

1.3. **Caccioppoli's partition and indecomposable sets.** First of all let us recall the definition of Caccioppoli's partition (see [1] for a reference).

Definition 1.4 (Caccioppoli's partition)**.** Let Ω be an open set of $\mathbb{R}^n$. We say that a $\mathcal{L}^n$-measurable partition $(E_i)_{i=1}^\infty$ of Ω is a Caccioppoli's partition if

$$\sum_{i=1}^{\infty} P(E_i;\Omega) < \infty.$$

Moreover we say that a Caccioppoli's partition is ordered if $|E_i| \geq |E_j|$ whenever $i \leq j$.

Definition 1.5 (Indecomposability)**.** Let Ω be an open set of $\mathbb{R}^n$ and let $F \subset \Omega$ with $P(F;\Omega) < \infty$. We say that F is *indecomposable* if for every set E satisfying

$$E \subset F, \quad P(F;\Omega) = P(E;\Omega) + P(F \setminus E;\Omega), \tag{1.3}$$

then $|E| = 0$ or $|E \Delta F| = 0$.

Remark 1.6. The notion of indecomposability can be found for example in [14] and it is in perfect agreement with the following fact (see [6, Proposition 2.12]): the set F is indecomposable if and only if any $u \in BV(\Omega)$ with $|Du|(F) = 0$ is necessarily constant on F.

In particular this tells us that every connected open set $U \subset \Omega$ with finite perimeter is indecomposable.

Remark 1.7. For every set $E \subset F$ it holds $P(F;\Omega) \le P(E;\Omega) + P(F \setminus E;\Omega)$. This means that condition (1.3) is equivalent to

$$E \subset F, \quad P(F;\Omega) \ge P(E;\Omega) + P(F \setminus E;\Omega).$$

Moreover, condition (1.3) can be equivalently stated for a countable family $(E_i)_{i=1}^{\infty}$. This means that F is indecomposable if and only if the following conditions

$$\bigcup_{i=1}^{\infty} E_i = F, \quad |E_i \cap E_j| = 0 \ (i \ne j), \quad \sum_{i=1}^{\infty} P(E_i;\Omega) = P(F;\Omega), \tag{1.4}$$

imply that there exists i_0 such that $|E_{i_0} \Delta F| = 0$ and $|E_i| = 0$ for $i \ne i_0$.

Indeed condition (1.4) clearly implies (1.3). While if F is indecomposable, by setting $E := E_1$, (1.4) tells us

$$P(E;\Omega) + P(F \setminus E;\Omega) \le P(F;\Omega),$$

which implies

$$P(E;\Omega) + P(F \setminus E;\Omega) = P(F;\Omega).$$

By the indecomposability of F we deduce that one between E or $F \Delta E$ has zero Lebesgue measure. If $|F \Delta E| = 0$ we are done. Otherwise $|E| = 0$ and we can proceed as before by defining $E := E_2$. Clearly, if this procedure does not stop, then $|F| = 0$ and we are done. Otherwise if it stops at $i_0 \in \mathbb{N}$ this means that $|F \Delta E_{i_0}| = 0$ and we are done.

We conclude this subsection with two technical propositions.

Proposition 1.8. *Let Ω be an open set of $\mathbb{R}^n$ and let $F \subset \Omega$ be indecomposable. Suppose $E \subset \Omega$ is a set having finite perimeter in Ω and such that*

$$|E \cap F| > 0 \ \text{ and } \ |F \setminus E| > 0. \tag{1.5}$$

Then it holds $\mathcal{H}^{n-1}(\partial^* E \cap F^{(1)}) > 0$.

Proof. We can consider the measurable partition of F given by $F = (E \cap F) \cup (F \setminus E)$. By hypothesis $|E \cap F|, |F \setminus E| > 0$. Using Leibniz's formula (1.2) we can write

$$\partial^*(E \cap F) = [\partial^* E \cap F^{(1)}] \cup [\partial^* F \cap E^{(1)}] \cup [\{\nu_E = \nu_F\}],$$

and

$$\partial^*(F \setminus E) = [\partial^* E \cap F^{(1)}] \cup [\partial^* F \cap E^{(0)}] \cup [\{\nu_E = -\nu_F\}].$$

Since $\partial^* F \cap E^{(1)}$, $\{\nu_E = \nu_F\}$, $\partial^* F \cap E^{(0)}$ and $\{\nu_E = -\nu_F\}$ are pairwise disjoint subsets of $\partial^* F$, if $\mathcal{H}^{n-1}(\partial^* E \cap F^{(1)}) = 0$ then

$$\begin{aligned} P(E \cap F;\Omega) + P(F \setminus E;\Omega) &= \mathcal{H}^{n-1}(\partial^* F \cap E^{(1)}) + \mathcal{H}^{n-1}(\partial^* F \cap E^{(0)}) \\ &\quad + \mathcal{H}^{n-1}(\{\nu_E = \nu_F\}) + \mathcal{H}^{n-1}(\{\nu_E = -\nu_F\}) \\ &\le P(F;\Omega), \end{aligned}$$

which by Remark 1.7 implies (1.3) and this together with (1.5) is in contradiction with the indecomposability of F. □

Proposition 1.9. *Let Ω be an open set of $\mathbb{R}^n$ and let $E, E' \subset \Omega$ with $P(E;\Omega), P(E';\Omega) < \infty$ and such that $\partial^* E' \subseteq \partial^* E$. Let $F \subset E$ be an indecomposable set. Then one and only one of the following holds*

(1) $F \subseteq E'$
(2) $F \subseteq E \setminus E'$.

Proof. It is enough to show that $|F \cap E'| \neq 0$ implies $F \subseteq E'$.

Suppose not. Then $|F \cap E'| > 0$ and also $|F \setminus E'| > 0$. By Leibniz's formula both $F \cap E'$ and $F \setminus E'$ are sets having finite perimeter in Ω. Moreover, by Proposition 1.8 we would have also $\mathcal{H}^{n-1}(\partial^* E' \cap F^{(1)}) > 0$. But since $F \subset E$ then $F^{(1)} \subset E^{(1)}$, and this implies $\mathcal{H}^{n-1}(\partial^* E' \cap E^{(1)}) > 0$ which is in contradiction with the hypothesis $\partial^* E' \subset \partial^* E$. This proves the proposition. □

1.4. **Indecomposable components.** The following result is a well known fact about the decomposability property of sets of finite perimeter. Precisely, every set with finite perimeter E can be decomposed into a countable family of indecomposable sets (F_i) such that

$$\mathbb{1}_E = \sum_{i=1}^{\infty} \mathbb{1}_{F_i}, \quad \text{and} \quad P(E) = \sum_{i=1}^{\infty} P(F_i).$$

This result was first announced (with a sketch of the proof) in [8, Subsection 4.2.25] in the more general setting of integral currents of $\mathbb{R}^n$. A complete proof in the context of sets of finite perimeter in $\mathbb{R}^n$ can be found in [14]. We are interested in the same result when $\mathbb{R}^n$ is replaced by a generic Lipschitz-regular open set Ω. Namely, whenever $E \subset \Omega$ is such that $P(E;\Omega) < \infty$, then there exists a countable family of indecomposable subsets of Ω, say (F_i), such that

$$\mathbb{1}_E = \sum_{i=1}^{\infty} \mathbb{1}_{F_i}, \quad \text{and} \quad P(E;\Omega) = \sum_{i=1}^{\infty} P(F_i;\Omega).$$

This fact can be deduced by using [14, Lemma 3.1, Corollary 3.2, Proposition 3.3], and we decide to give a complete proof in the next proposition.

Proposition 1.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz-regular domain, and let $E \subset \Omega$ be such that $P(E;\Omega) < \infty$. Then there exists a Caccioppoli's indecomposable partition of E, which means a countable family $(F_i)_{i=1}^{\infty}$ of indecomposable sets such that*

(1) $E \cap F_i^{(1)} = F_i$, for every $i \in \mathbb{N}$;
(2) $\mathcal{H}^{n-1}(E \cap E^{(1)} \setminus \bigcup_{i=1}^{\infty} F_i) = 0$;
(3) $F_i \cap F_j \cap E = \emptyset$, for $i \neq j$;
(4) $\sum_{i=1}^{\infty} P(F_i;\Omega) = P(E;\Omega)$;
(5) $\mathcal{H}^{n-1}((\Omega \cap \partial^* E) \setminus \bigcup_{i=1}^{\infty} \partial^* F_i) = 0$;
(6) $\mathcal{H}^{n-1}((\Omega \cap \partial^* F_i) \setminus \partial^* E) = 0$ for every $i \in \mathbb{N}$;
(7) $\mathcal{H}^{n-1}(\Omega \cap \partial^* F_i \cap \partial^* F_j) = 0$ for $i \neq j$.

Moreover the family $(F_i)_{i=1}^{\infty}$ is unique up to permutation of indices in the sense that given any family of indecomposable sets $(F'_i)_{i=1}^{\infty}$ satisfying 1-4 then there exists a bijection $\pi\colon \mathbb{N} \to \mathbb{N}$ such that

$$|F_i \Delta F'_{\pi(i)}| = 0 \text{ for every } i \in \mathbb{N}.$$

Remark 1.11. Conditions 2 and 3 say in a more precise way that $\mathbb{1}_E = \sum_{i=1}^{\infty} \mathbb{1}_{F_i}$.

Proof. We first prove that conditions 1-3 hold true. Since Ω is Lipschitz-regular, we know that E is a set of finite perimeter in $\mathbb{R}^n$, i.e. $P(E) < \infty$ (see [20, Subsection 6.5.1 Lemma 1]). By applying [14, Proposition 3.3] to the set $E \subset \mathbb{R}^n$, we deduce that there exists a countable family $(\tilde{F}_i)_{i\in I}$ satisfying 1 and 2 with the additional property $\tilde{F}_i \neq \emptyset$ for every $i \in I$, and

$$\sum_{i\in I} P(\tilde{F}_i) = P(E). \tag{1.6}$$

Now if the cardinality of I is a natural number N, we define $F_i := \tilde{F}_{\pi(i)}$ where π is any bijection of $\{1,\dots,N\}$ onto I, and $F_i = \emptyset$ for every $i > N$. While if the cardinality of I is equal to the cardinality of $\mathbb{N}$, we define $F_i := \tilde{F}_{\pi(i)}$ where π is any bijection of $\mathbb{N}$ onto I. Clearly the family $(F_i)_{i=1}^{\infty}$ satisfies 1 and 2. We show that it satisfies also 3. Indeed, we can use [14, Lemma 3.1] which says that

$$\mathcal{H}^{n-1}\left([\Omega \cap (\partial^* E \cup E^{(1)})] \setminus \bigcup_{i=1}^{\infty} \partial^* F_i \cup F_i^{(1)} \right) = 0,$$

where now $\partial^* E$ has to be intended as the reduced boundary of E as a subset of $\mathbb{R}^n$. Since $F_i^{(1)} \cap \partial^* E = \emptyset$ for every i, the only possibility is that

$$\mathcal{H}^{n-1}\left(\partial^* E \setminus \bigcup_{i=1}^{\infty} \partial^* F_i\right) = 0, \tag{1.7}$$

which by (1.6) implies

$$\bigcup_{i=1}^{\infty} \partial^* F_i = \partial^* E \quad \text{and} \quad \mathcal{H}^{n-1}(\partial^* F_i \cap \partial^* F_j) = 0, \text{ for } i \neq j,$$

and in particular that

$$\mathcal{H}^{n-1}(\partial^* F_i \setminus \partial^* E) = 0, \text{ for every } i \in \mathbb{N}. \tag{1.8}$$

If $|F_i \cap F_j| > 0$ for some $i \neq j$ then by using Proposition 1.8, we deduce that $\mathcal{H}^{n-1}(\partial^* F_i \cap F_j^{(1)}) > 0$ which is in contradiction with (1.8). Finally, since by (1) every F_i coincides with its measure theoretic interior on the points of E, this shows condition 3.

We claim that

$$\sum_{i=1}^{\infty} P(F_i; \Omega) = P(E; \Omega).$$

To show this, notice that by applying Leibniz's formula (1.2) to the couple of sets Ω and E (both seen as sets with finite perimeter in $\mathbb{R}^n$), since $E \subset \Omega$ we can write

$$\mathcal{H}^{n-1} \llcorner \partial^* E = \mathcal{H}^{n-1} \llcorner (\partial^*(E \cap \Omega)) = \mathcal{H}^{n-1} \llcorner (\partial^* E \cap \Omega^{(1)}) + \mathcal{H}^{n-1} \llcorner (\{\nu_E = \nu_\Omega\}), \tag{1.9}$$

and since by 1 we have $F_i \subset \Omega$ for every $i \in \mathbb{N}$, then we have also

$$\mathcal{H}^{n-1} \llcorner \partial^* F_i = \mathcal{H}^{n-1} \llcorner (\partial^*(F_i \cap \Omega)) = \mathcal{H}^{n-1} \llcorner (\partial^* F_i \cap \Omega^{(1)}) + \mathcal{H}^{n-1} \llcorner (\{\nu_{F_i} = \nu_\Omega\}). \tag{1.10}$$

By using [14, Corollary 3.2] together with (1.9) and (1.10) we deduce that

$$\mathcal{H}^{n-1}\left(\{\nu_E = \nu_\Omega\} \setminus \bigcup_{i=1}^{\infty} \{\nu_{F_i} = \nu_\Omega\}\right) = 0. \tag{1.11}$$

We can write

$$P(E) = \mathcal{H}^{n-1}(\partial^* E) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega^{(1)}) + \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\}) = P(E; \Omega) + \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\})$$

By using the lower semicontinuity of the perimeter and (1.11) we can continue the previous inequality

$$\begin{aligned} P(E) = P(E; \Omega) + \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\}) &\leq \sum_{i=1}^{\infty} P(F_i; \Omega) + \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}) \\ &= \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\partial^* F_i \cap \Omega^{(1)}) + \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}) \\ &= \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\partial^* F_i) = \sum_{i=1}^{\infty} P(F_i) = P(E), \end{aligned}$$

where we have also used that, since Ω is Lipschitz-regular, then $\mathcal{H}^{n-1}(\partial^* F_i \cap \Omega^{(1)}) = P(F_i; \Omega)$. By using again the lower semicontinuity of the perimeter and (1.11), we deduce that the only possibility for which (1.4) is actually an equality is that

$$P(E; \Omega) = \sum_{i=1}^{\infty} P(F_i; \Omega) \quad \text{and} \quad \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\}) = \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}),$$

which in particular implies our claim.

Properties 5-7 simply follow by [14, Corollary 3.2].

It remains to prove that a family of indecomposable sets $(F_i)_{i=1}^\infty$ satisfying 1-4 is unique. As before, since Ω is Lipschitz-regular, then the (F_i) are actually sets having finite perimeter in $\mathbb{R}^n$. Then, in view of [14, Proposition 3.3] it is enough to prove that

$$\sum_{i=1}^{\infty} P(F_i) = P(E). \tag{1.12}$$

In this case, suppose that $(F_i)_i$ and $(F'_i)_i$ are two sequences of sets satisfying 1-4. By removing the sets in $(F_i)_i$ and in $(F'_i)_i$ equal to the emptyset we end up with two families $(F_i)_{i\in I}$ and $(F'_i)_{i\in I'}$, both satisfying 1-4 with the additional condition $F_i, F'_i \neq \emptyset$ for every $i \in I$ and $i \in I'$. Therefore, we are in position to apply the uniqueness result [14, Proposition 3.3] which says that there exists a bijection $\pi\colon I \to I'$ such that

$$|F_i \Delta F'_{\pi(i)}| = 0, \text{ for every } i \in I,$$

and this would be enough to obtain uniqueness.

Now we show (1.12). Since by hypothesis

$$\sum_{i=1}^{\infty} P(F_i;\Omega) = P(E;\Omega),$$

by (1.9) and by (1.10) together with the fact $\mathcal{H}^{n-1}(\partial^* F_i \cap \Omega^{(1)}) = P(F_i;\Omega)$, it is enough to show

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}) = \mathcal{H}^{n-1}(\{\nu_E = \nu_\Omega\}).$$

First of all since $|F_i \cap F_j| = 0$ it is easy to see that

$$\mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\} \cap \{\nu_{F_j} = \nu_\Omega\}) = 0 \text{ for } i \neq j. \tag{1.13}$$

In particular this implies that

$$\sum_{i=1}^{\infty} P(F_i) = \sum_{i=1}^{\infty} P(F_i;\Omega) + \mathcal{H}^{n-1}(\{\nu_{F_i} = \nu_\Omega\}) < \infty. \tag{1.14}$$

We claim that

$$\mathcal{H}^{n-1}\left(\{\nu_E = \nu_\Omega\} \Delta \bigcup_{i=1}^{\infty} \{\nu_{F_i} = \nu_\Omega\}\right) = 0.$$

By using (1.14) we can apply [14, Lemma 3.1] and arguing as before this implies (1.11). To prove

$$\mathcal{H}^{n-1}\left(\bigcup_{i=1}^{\infty} \{\nu_{F_i} = \nu_\Omega\} \setminus \{\nu_E = \nu_\Omega\}\right) = 0,$$

we show that $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_i} = \nu_\Omega\}$ belongs to $\partial^* E$ for every $i \in \mathbb{N}$. For this purpose, define $\Gamma := \bigcup_{i=1}^\infty \partial^* F_i$. Thanks to (1.14) Γ is a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. By properties 1-4, we can apply [14, Corollary 3.2] to deduce that

$$\mathcal{H}^{n-1}(\Omega \cap \partial^* F_i \cap \partial^* F_j) = 0 \text{ for } i \neq j.$$

This together with (1.13) tells us that

$$\mathcal{H}^{n-1}(\partial^* F_i \cap \partial^* F_j) = 0 \text{ for } i \neq j. \tag{1.15}$$

This last condition allows us to define an orientation of Γ, namely a measurable map $\nu\colon \Gamma \to \mathbb{S}^{n-1}$, in the following way

$$\nu(x) := \nu_{F_i}(x), \text{ for } x \in \partial^* F_i. \tag{1.16}$$

If we set $u_i := \mathbb{1}_{\cup_{j=1}^i F_j}$, then we have $u_i \in SBV_p^p(\Omega;\Gamma)$ for every $p \geq 1$. Since for every i $\nabla u_i = \nabla u = 0$ and

$$u_i \to u \text{ strongly in } L^1,$$

then we can apply [21, Remark 4.9] to deduce

$$u_i^\pm \to u^\pm \text{ in } \mathcal{H}^{n-1}\text{-measure on } \Gamma. \tag{1.17}$$

Now fix $i_0 \in \mathbb{N}$. By (1.15) and the definition of $u^{\pm}$ (see Definition 1.16), for $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_{i_0}} = \nu_\Omega\}$ $u_i^+(x) = 1$ and $u_i^-(x) = 0$ for every $i \geq i_0$. Hence, by (1.17) this means also that $u^+(x) = 1$ and $u^-(x) = 0$ for $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_{i_0}} = \nu_\Omega\}$. By definition of $u^{\pm}$ (see Definition 1.16) we deduce that $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_{i_0}} = \nu_\Omega\}$ is a point of density $1/2$ for E, and by Theorem 1.2 also that $\mathcal{H}^{n-1}$-a.e. $x \in \{\nu_{F_{i_0}} = \nu_\Omega\}$ belongs to $\partial^* E$. Thanks to the arbitrariness of i_0 we conclude the proof.

□

Definition 1.12 (Indecomposable components)**.** Let Ω be an open set of $\mathbb{R}^n$ and let $E \subset \Omega$ with $P(E;\Omega) < \infty$. Let $(F_i)_{i=1}^\infty$ be the unique (up to permutation of indices) indecomposable partition of E given by Proposition 1.10. Then, for every $i \in \mathbb{N}$ we say that F_i is an *indecomposable component* of E.

The following proposition says that if E_0 is an indecomposable set, then whenever $|E_r \Delta E_0| \to 0$ and $P(E_r;\Omega) \to P(E_0;\Omega)$ as $r \to 0^+$, then for every r it is possible to select an indecomposable component F_r of E_r, such that $|F_r \Delta E| \to 0$ and $P(F_r;\Omega) \to P(E;\Omega)$ as $r \to 0^+$.

Proposition 1.13. *Let Ω be a bounded Lipschitz regular domain of $\mathbb{R}^n$, and let $(E_r)_{r\in(0,1)}$ be a family of sets contained in Ω with $P(E_r;\Omega) < \infty$. For each $r \in (0,1)$ let $(F_{r,i})_{i=1}^\infty$ be the Caccioppoli's indecomposable partition of E_r given by Proposition 1.10. Let $E_0 \subset \Omega$ be an indecomposable set. Suppose that*

(1) $\lim_{r\to 0^+} |E_r \Delta E_0| = 0$
(2) $\lim_{r\to 0^+} P(E_r;\Omega) = P(E_0;\Omega)$.

Then, for each $r \in (0,1)$ there exists $\sigma_r \in \mathbb{N}$ such that

$$\lim_{r\to 0^+} |F_{r,\sigma_r} \Delta E_0| = 0, \tag{1.18}$$

and

$$\lim_{r\to 0^+} P(F_{r,\sigma_r};\Omega) = P(E_0;\Omega). \tag{1.19}$$

Proof. Suppose that our proposition does not hold. Then there exists a $\delta > 0$ such that

$$\limsup_{r\to 0^+} \left(\inf_{i\in\mathbb{N}} |F_{r,i} \Delta E_0| \right) \geq \delta.$$

This implies the existence of a subsequence $(r_m)_{m=1}^\infty$ such that

$$|F_{r_m,i} \Delta E_0| > \delta, \tag{1.20}$$

for every $m \in \mathbb{N}$ and for every $i \in \mathbb{N}$.

Consider the Caccioppoli's partition of Ω made of $(F_{r_m,i})_{i=1}^\infty \cup \Omega \setminus E_{r_m}$. Since Ω has finite Lebesgue measure, this partition can be ordered. Thus we can apply the compactness theorem for Caccioppoli's ordered partitions (see [1, Theorem, 4.19] and [1, Remark, 4.20]), to find a Caccioppoli's (ordered) partition of Ω, say $(F_{0,i})_{i=1}^\infty$ where one of the $F_{0,i}$ must be equal to $\Omega \setminus E_0$, such that up to subsequences we have

$$\lim_{m\to\infty} |F_{r_m,i} \Delta F_{0,i}| = 0 \text{ for every } i \in \mathbb{N}. \tag{1.21}$$

By removing the set $(\Omega \setminus E_0)$ from the partition, we obtain an ordered measurable partition of E_0, which we still call $(F_{0,i})_{i=1}^\infty$.

By (1.20), there exists a family $I \subset \mathbb{N}$ with cardinality strictly greater than 1, such that $F_{0,i} \neq \emptyset$ for every $i \in I$.

Using the lower semicontinuity of the perimeter and property (4) of Proposition 1.10, we can write

$$\begin{aligned} \sum_{i\in I} P(F_{0,i};\Omega) \leq \sum_{i=0}^\infty \liminf_{m\to\infty} P(F_{k_m,i};\Omega) &\leq \liminf_{m\to\infty} \sum_{i=1}^\infty P(F_{k_m,i};\Omega) \\ &\leq \liminf_{m\to\infty} P(E_{k_m};\Omega) \\ &= P(E_0;\Omega), \end{aligned} \tag{1.22}$$

since $(F_{0,i})_{i\in I}$ is a (measurable) partition of E_0, (1.22) implies

$$\sum_{i\in I} P(F_{0,i};\Omega) = P(E_0;\Omega), \tag{1.23}$$

and by Remark 1.7 this is in contradiction with the indecomposability of E_0. Hence this proves (1.18).

Finally we notice that

$$\begin{aligned} P(E_0;\Omega) \le \liminf_{r\to 0^+} P(F_{r,\sigma_r};\Omega) &\le \limsup_{r\to 0^+} P(F_{r,\sigma_r};\Omega) \\ &\le \limsup_{r\to 0^+} \sum_{i=0}^{\infty} P(F_{r,i};\Omega) \\ &= \limsup_{r\to 0^+} P(E_r;\Omega) \\ &= P(E_0;\Omega), \end{aligned}$$

and this gives (1.19). □

1.5. *GSBV* **functions.** For the general theory concerning the space of generalised functions of special bounded variation $GSBV(\Omega)$, we refer to [1]. In order to give a precise meaning of jump set and of approximate gradient in the context of $GSBV$ functions, we need to recall the notion of approximate limit ([1, Section 4.5]).

Definition 1.14 (Upper and lower approximate limit)**.** Given an $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}$ the upper approximate limit of u at $x \in \Omega$ is defined as

$$u^+(x) := \text{ap-}\limsup_{y\to x} u(y) := \inf\{t \in \mathbb{R} \mid \Theta^n(\mathcal{L}^n \llcorner \{u > t\}, x) = 0\}$$

while the lower approximate limit of u at x is defined as

$$u^-(x) := \text{ap-}\liminf_{y\to x} u(y) := \sup\{t \in \mathbb{R} \mid \Theta^n(\mathcal{L}^n \llcorner \{u < t\}, x) = 0\}.$$

In addition, we say that u admits an approximate limit equal to $a \in \overline{\mathbb{R}}$ at x, and we write

$$\text{ap-}\lim_{y\to x} u(y) = a,$$

if $u^+(x) = u^-(x) = a$ (the case $a = \pm\infty$ are not excluded).

Definition 1.15 (Approximate continuity)**.** Let Ω be an open set of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}$ we define the approximate continuity set as the set of points $x \in \Omega$ for which there exists $a \in \overline{\mathbb{R}}$ such that

$$\text{ap-}\lim_{y\to x} u(y) = a.$$

The approximate discontinuity set S_u is defined as the complement in Ω of the approximate continuity set, i.e.

$$S_u := \{x \in \Omega \mid u^-(x) < u^+(x)\}.$$

When $x \in \Omega \setminus S_u$ we denote the approximate limit of u at x as $\tilde{u}(x)$

We are now in position to remind the definitions of jump set and of approximate gradient for $GSBV$-functions.

Definition 1.16 (Jump set)**.** Let Ω be an open set of $\mathbb{R}^n$. For every $\mathcal{L}^n$-measurable function $u\colon \Omega \to \mathbb{R}$ we define the approximate jump set J_u, as the set of points $x \in \Omega$ for which there exist $a, b \in \overline{\mathbb{R}}$ with $a < b$ and $\nu \in \mathbb{S}^{n-1}$ such that

$$\text{ap-}\lim_{\substack{(y-x)\cdot\nu>0 \\ y\to x}} v(y) = a \quad \text{and} \quad \text{ap-}\lim_{\substack{(y-x)\cdot\nu<0 \\ y\to x}} v(y) = b. \tag{1.24}$$

If $x \in J_u$ then we write $a = u^+(x)$ and $b = u^-(x)$. The vector ν, uniquely determined by this condition, is denoted by $\nu_u(x)$. The jump of u is the function $[u]\colon J_u \to \mathbb{R}$ defined by $[u](x) := u^+(x) - u^-(x)$ for every $x \in J_u$.

Definition 1.17 (Approximate differentiability)**.** Let $u\colon \Omega \to \mathbb{R}$ be a $\mathcal{L}^n$-measurable function and $x \in \Omega \setminus S_u$. Then u is approximately differentiable at x if $\tilde{u}(x) \in \mathbb{R}$ and there exists a linear map $L\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$\operatorname{ap-}\lim_{y\to x} \frac{|u(y) - \tilde{u}(x) - L(y-x)|}{|y-x|} = 0.$$

In this case the approximate gradient of u at x is defined as $\nabla u(x) := L$.

Definition 1.18 ($GSBV$ functions)**.** Let Ω be an open set of $\mathbb{R}^n$. We say that a function $u\colon \Omega \to \mathbb{R}$ belongs to $GSBV(\Omega)$, if for every $M \in \mathbb{N}$ the truncated function $u^M := (u \vee M) \wedge -M$ belongs to $SBV_{loc}(\Omega)$.

Now we recall the main result about the fine properties of $GSBV$ functions (see [1, Theorem 4.34]).

Theorem 1.19 (Fine properties)**.** *Let $u \in GSBV(\Omega)$, let $M \in \mathbb{N}$. Then*

(1) $S_u = \bigcup_{M\in\mathbb{N}} S_{u^M}$ and

$$u^+(x) = \lim_{M\to+\infty} (u^M)^+(x), \quad u^-(x) = \lim_{M\to+\infty} (u^M)^-(x);$$

(2) S_u is countably $\mathcal{H}^{n-1}$-rectifiable, $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ and

$$Tan(S_u, x) = (\nu_u(x))^\perp, \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in S_u;$$

(3) u is weakly approximate differentiable $\mathcal{L}^n$-a.e. in Ω and

$$\nabla u(x) = \nabla u^M(x), \text{ for } \mathcal{L}^n\text{-a.e. } x \in \{|u| \le M\}.$$

The following compactness result holds true.

Theorem 1.20. *Let Ω be an open set of $\mathbb{R}^n$ and let $(u_k)_{k=1}^\infty$ be a sequence of functions in $GSBV(\Omega)$. Suppose that there exists $p > 1$ such that*

$$\sup_{k\in\mathbb{N}} \big(\|u\|_{L^p} + \|\nabla u\|_{L^p} + \mathcal{H}^{n-1}(J_u)\big) < \infty. \tag{1.25}$$

Then there exists $u \in GSBV(\Omega)$ such that, up to passing through a subsequence, we have

$$\lim_{k\to\infty} u_k(x) = u(x), \quad \mathcal{L}^n\text{-a.e.} \quad \text{and} \quad \nabla u_k \rightharpoonup \nabla u, \text{ weakly in } L^1(\Omega), \text{ as } k \to \infty,$$

and

$$\liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}) \ge \mathcal{H}^{n-1}(J_u).$$

Proof. It is a particular case of [1, Theorem 4.36]. ☐

Finally, we introduce suitable subspaces of $GSBV(\Omega)$.

Definition 1.21. Given $\Gamma \subset \Omega$ a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$, we define for every $p \ge 1$

- $GSBV^p(\Omega) := \{u \in GSBV(\Omega) \mid \nabla u \in L^p(\Omega;\mathbb{R}^n)\}$;
- $GSBV^p_p(\Omega) := \{u \in GSBV(\Omega) \mid u \in L^p(\Omega),\ \nabla u \in L^p(\Omega;\mathbb{R}^n)\}$;
- $GSBV(\Omega;\Gamma) := \{u \in GSBV(\Omega) \mid J_u \subseteq \Gamma\}$;
- $GSBV^p(\Omega;\Gamma) := \{u \in GSBV^p(\Omega) \mid J_u \subset \Gamma\}$;
- $GSBV^p_p(\Omega;\Gamma) := \{u \in GSBV^p_p(\Omega) \mid J_u \subset \Gamma\}$.

Remark 1.22. Using [4, Proposition 2.3] and Theorem 1.20, it is possible to prove that $GSBV^p_p(\Omega;\Gamma)$ endowed with the norm

$$\|u\|_p := \|u\|_{L^p} + \|\nabla u\|_{L^p},$$

is a Banach space.

2. Weak Poincaré's inequality for indecomposable sets

This section is devoted to the proof of a weak version of the Poincaré's inequality for indecomposable sets. We recall that given a connected Lipschitz-regular bounded open set Ω, Poincaré's inequality allows to control the L^p-distance of a function u from its average in term of the L^p-norm of its gradient. Namely, for every $u \in W^{1,p}(\Omega)$ it holds

$$\left(\int_\Omega \left|u - \fint u\right|^p dx\right)^{\frac{1}{p}} \leq C(\Omega, p)\left(\int_\Omega |\nabla u|^p\, dx\right)^{\frac{1}{p}}. \tag{2.1}$$

In our case we want to derive a similar inequality in the context of $GSBV^p$-functions, when Ω is replaced by a generic indecomposable set of finite measure. Since in general a function $u \in GSBV^p(\Omega)$ is not even integrable, the L^p-distance on the left hand side of (2.1) will be substituted by the L^0-distance, which is the one that induces the convergence in measure (see Definition 5.4). Precisely, we prove the following inequality

$$\left(\int_F |u - m|^p \wedge 1\, dx\right)^{\frac{1}{p}} \leq C(F, p, \lambda)\left(\int_F |\nabla u|^p\, dx\right)^{\frac{1}{p}} + (2\lambda|F|)^{\frac{1}{p}}, \tag{2.2}$$

for every indecomposable set $F \subset \Omega$ with $|F| < \infty$ and for every $u \in GSBV^p(\Omega)$ such that $\mathcal{H}^{n-1}(J_u \cap F^{(1)}) = 0$. The real number m is the median of u on F (see Definition 2.8), ∇u is the approximate gradient of u (see Section 1.5), and λ is any positive real number in $(0, 1/2]$. The integral on the left hand side of (2.2) is equivalent to the L^0-distance on F (see (5.3)) between u and m. The function $C(F, p, \cdot)$ is decreasing and in general may blow up as $\lambda \to 0^+$. Inequality (2.2) tells us that if $\int_F |\nabla u|^p\, dx$ is sufficiently small, then u is close to a single constant on F. This information will play a crucial role in order to derive the first main result of this paper, namely Theorem 1.

2.1. The upper isoperimetric profile.

Given $F \subset \Omega$ an indecomposable set with finite measure, we want to introduce an isoperimetric quantity h_F, which is a function $h_F \colon (0, \frac{1}{2}] \to (0, \infty)$, and which plays a similar role to the so called *Cheeger's constant*. We recall that when Ω is a bounded open set of $\mathbb{R}^n$, $(n \geq 2)$ the Cheeger's constant is defined as (see [16],[17])

$$h(\Omega) := \inf\left\{\frac{P(E)}{|E|} \mid E \subset \Omega,\, |E| > 0\right\}. \tag{2.3}$$

Let us remind that the Cheeger's constant was introduced in [3] to study lower bounds for the smallest eigenvalue of the Laplace operator on compact Riemannian manifolds without boundary. As a consequence, one obtains the validity of a Poincaré's inequality with optimal constant uniformly bounded from below by a geometric constant. Precisely, for the case of Ω bounded open set of $\mathbb{R}^n$, let $\lambda_p(\Omega)$ be the smallest "eigenvalue" of the p-laplacian with Dirichlet boundary condition $(1 \leq p < \infty)$, i.e.

$$\lambda_p(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)} \frac{\|\nabla u\|_{L^p}^p}{\|u\|_{L^p}^p}.$$

Then arguing as in [3] (see [15] [13]) one can easily show that

$$\lambda_p(\Omega) \geq \frac{h(\Omega)^p}{p^p}.$$

In our case, since we are interested in a weaker version of Poincaré's inequality for indecomposable sets without the assumption of Dirichlet boundary conditions, we need to work with a different notion of Cheeger's constant. Before starting with the definition, we need to prove a lower-semicontinuity property, which can be seen as a generalisation of the well known result of lower semicontinuity of the perimeter: given a sequence of sets (E_k) such that $\lim_{k\to\infty} |E_k \Delta E| = 0$, then for every open set Ω

$$\liminf_{k\to\infty} P(E_k; \Omega) \geq P(E; \Omega).$$

Proposition 2.1 (Lower semicontinuity). *Let Ω be an open set of $\mathbb{R}^n$. Let $(E_k)_{k=1}^\infty$,$(E_k')_{k=1}^\infty$ and E, E' be subsets of Ω with finite perimeter in Ω such that $E_k' \subset E_k$ and*

(1) $\lim_{k\to\infty} |E_k \Delta E| = 0$;
(2) $\lim_{k\to\infty} P(E_k;\Omega) = P(E;\Omega)$;
(3) $\lim_{k\to\infty} |E'_k \Delta E'| = 0$.

Then it holds the following lower semicontinuity property

$$\liminf_{k\to\infty} \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) \geq \mathcal{H}^{n-1}(\partial^* E' \cap E^{(1)}). \tag{2.4}$$

Proof. Using the Leibniz's formula (1.2) we can write

$$\begin{aligned} P(E'_k;\Omega) = P(E'_k \cap E_k;\Omega) &= \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + \mathcal{H}^{n-1}(\partial^* E_k \cap E_k'^{(1)}) \\ &\quad + \mathcal{H}^{n-1}(\{\nu_{E'_k} = \nu_{E_k}\}). \end{aligned} \tag{2.5}$$

Since $E'_k \subset E_k$ then $E_k'^{(1)} \subset E_k^{(1)}$, hence $E_k'^{(1)} \cap E_k^{(1/2)} = \emptyset$. This implies $\mathcal{H}^{n-1}(\partial^* E_k \cap E_k'^{(1)}) = 0$. Moreover, since $E'_k \subset E_k$ then $\mathcal{H}^{n-1}(\{\nu_{E'_k} \neq \nu_{E_k}\}) = 0$. Therefore (2.5) can be rewritten as

$$P(E'_k;\Omega) = \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + \mathcal{H}^{n-1}(\partial^* E'_k \cap \partial^* E_k). \tag{2.6}$$

Analogously we have

$$P(E_k \setminus E'_k;\Omega) = \mathcal{H}^{n-1}(\partial^*(E_k \setminus E'_k) \cap E_k^{(1)}) + \mathcal{H}^{n-1}(\partial^*(E_k \setminus E'_k) \cap \partial^* E_k).$$

Since $\mathcal{H}^{n-1}(\partial^*(E_k \setminus E'_k) \cap E_k^{(1)}) = \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)})$, then we can rewrite the previous equality as

$$P(E_k \setminus E'_k;\Omega) = \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + \mathcal{H}^{n-1}(\partial^*(E_k \setminus E'_k) \cap \partial^* E_k). \tag{2.7}$$

We claim that

$$\mathcal{H}^{n-1}(\partial^* E_k \setminus (\partial^* E'_k \cup \partial^*(E_k \setminus E'_k))) = 0 \tag{2.8}$$

and

$$\mathcal{H}^{n-1}((\partial^* E_k \cap \partial^* E'_k) \cap (\partial^* E_k \cap \partial^*(E_k \setminus E'_k)) = 0. \tag{2.9}$$

To show this, notice that by Theorem 1.2 for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$, if $x \in E_k^{(1/2)}$, then

$$\{x \in (E'_k)^{(0)} \text{ or } x \in (E'_k)^{(1/2)}\} \text{ and } \{x \in (E_k \setminus E'_k)^{(0)} \text{ or } x \in (E_k \setminus E'_k)^{(1/2)}\}.$$

But if $x \in E_k^{(1/2)}$ it cannot happen $x \in (E'_k)^{(0)}$ and $x \in (E_k \setminus E'_k)^{(0)}$, otherwise $x \in E_k^{(0)}$ which is a contradiction. This proves (2.8). Also, if $x \in E_k^{(1/2)}$ then it cannot happen $x \in (E'_k)^{(1/2)}$ and $x \in (E_k \setminus E'_k)^{(1/2)}$, otherwise $x \in E_k^{(1)}$ which is again a contradiction. This proves (2.9).

By (2.8) and (2.9), summing (2.6) with (2.7) we obtain for every $k \in \mathbb{N}$

$$P(E'_k;\Omega) + P(E_k \setminus E'_k;\Omega) = 2\mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + P(E_k;\Omega). \tag{2.10}$$

Since $E' \subset E$, repeating the same argument we have also in this case

$$P(E';\Omega) + P(E \setminus E';\Omega) = 2\mathcal{H}^{n-1}(\partial^* E' \cap E^{(1)}) + P(E;\Omega). \tag{2.11}$$

Finally if we call $l := \liminf_{k\to\infty} \mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)})$ (without loss of generality we can assume $l \in \mathbb{R}$), using (2.10) and the lower semicontinuity of the perimeter on Ω, we can write

$$\begin{aligned} 2\mathcal{H}^{n-1}(\partial^* E' \cap E^{(1)}) + P(E;\Omega) &= P(E';\Omega) + P(E \setminus E';\Omega) \\ &\leq \liminf_{k\to\infty}(P(E'_k;\Omega) + P(E_k \setminus E'_k;\Omega)) \\ &= \liminf_{k\to\infty}(2\mathcal{H}^{n-1}(\partial^* E'_k \cap E_k^{(1)}) + P(E_k;\Omega)) \\ &= 2l + \lim_{k\to\infty} P(E_k;\Omega) \\ &= 2l + P(E;\Omega), \end{aligned}$$

which is our desired result. □

Remark 2.2. If the sets $(E_k)_k$ of the previous proposition are open, say for example $(U_k)_k$, and such that $\mathcal{H}^{n-1}(U_k^{(1)}\Delta U_k)=0$, then $P(E'_k;U_k)=\mathcal{H}^{n-1}(\partial^* E'_k\cap U_k^{(1)})$ for every k and by the previous lower semi-continuity result we have

$$\liminf_{k\to\infty} P(E'_k;U_k)\geq P(E';U),$$

where we have also used $\mathcal{H}^{n-1}(\partial^* E'\cap U^{(1)})\geq P(E';U)$.

With the next definition we introduce the *upper isoperimetric profile.*

Definition 2.3 (Upper isoperimetric profile)**.** Let Ω be an open set of $\mathbb{R}^n$ $(n\geq 2)$ and let F be *an indecomposable set* of Ω with $|F|<\infty$. For every $\lambda\in(0,1/2]$ we define

$$h_F(\lambda):=\inf\left\{\frac{\mathcal{H}^{n-1}(\partial^* E\cap F^{(1)})}{|E|}\;\middle|\; E\subset F,\ \lambda|F|\leq|E|\leq|F|/2,\ P(E;\Omega)<\infty\right\}. \tag{2.12}$$

We call the function $h_F\colon(0,1/2]\to\mathbb{R}^+$ the *upper isoperimetric profile* of F.

Remark 2.4. The upper isoperimetric profile is a non decreasing function. Moreover, if we take an indecomposable open set $U\subset\Omega$ such that $|U|<\infty$ and $\mathcal{H}^{n-1}(U^{(1)}\Delta U)=0$, then (2.12) reduces to

$$h_U(\lambda):=\inf\left\{\frac{P(E;U)}{|E|}\;\middle|\; E\subset U,\ \lambda|U|\leq|E|\leq|U|/2,\ P(E;\Omega)<\infty\right\}.$$

Notice that $\inf_{\lambda>0}h_U(\lambda)$ is not the Cheeger's constant in (2.3), since we look only at the relative perimeter of E inside U, while in (2.3) one is interested in the whole perimeter of E.

Notice also that in literature (in particular in the context of Riemannian manifolds) the isoperimetric profile at λ is defined by considering the infimum among all sets E with fixed volume $|E|=\lambda|F|$. Since we ask for $|E|\geq\lambda|F|$ we decide to call it upper isoperimetric profile.

Finally, the next proposition is the core result of this subsection.

Proposition 2.5. *Let Ω be an open set of $\mathbb{R}^n$ $(n\geq 2)$ and let F be an indecomposable set of Ω with $|F|<\infty$. Then $h_F(\lambda)>0$ for every $\lambda\in(0,1/2]$.*

In particular, it holds the following relative isoperimetric inequality

$$|E|\leq\frac{1}{h_F(\lambda)}\mathcal{H}^{n-1}(\partial^* E\cap F^{(1)}), \tag{2.13}$$

for every $E\subset F$ with $\lambda|F|\leq|E|\leq|F|/2$ and $P(E;\Omega)<\infty$.

Proof. Let $\lambda\in(0,1/2]$ and consider

$$h_F(\lambda)=\inf_{\substack{\lambda|F|\leq|E|\leq|F|/2\\ E\subset F}}\frac{\mathcal{H}^{n-1}(\partial^* E\cap F^{(1)})}{|E|}. \tag{2.14}$$

Clearly $h_F(\lambda)$ is finite. We want to show that it is strictly positive. Consider a minimizing sequence $(E_k)_{k\in\mathbb{N}}$ i.e.

$$h_F(\lambda)=\lim_{k\to\infty}\frac{\mathcal{H}^{n-1}(\partial^* E_k\cap F^{(1)})}{|E_k|};$$

since

$$\begin{aligned}P(E_k;\Omega)&=\mathcal{H}^{n-1}(\partial^* E_k\cap F^{(1)})+\mathcal{H}^{n-1}(\{\nu_F=\nu_{E_k}\})\\&\leq P(F;\Omega)+(h_F(\lambda)+\epsilon)|E_k|\\&\leq P(F;\Omega)+(h_F(\lambda)+\epsilon)(|F|/2),\end{aligned}$$

then by using [1, Theorem 3.39], up to subsequences there exists a set $E_\infty\subset F$ having finite perimeter with $\lambda|F|\leq|E_\infty|\leq|F|/2$ and such that $\lim_{k\to\infty}|E_k\Delta E_\infty|=0$. Moreover thanks to Proposition 2.5 we have

$$h_F(\lambda)=\lim_{k\to\infty}\frac{\mathcal{H}^{n-1}(\partial^* E_k\cap F^{(1)})}{|E_k|}\geq\frac{\mathcal{H}^{n-1}(\partial^* E_\infty\cap F^{(1)})}{|E_\infty|},$$

which means

$$h_F(\lambda) = \frac{\mathcal{H}^{n-1}(\partial^* E_\infty \cap F^{(1)})}{|E_\infty|}.$$

Finally, since $\lambda|F| \le |E_\infty| \le |F|/2$ and $F = E_\infty \cup (F \setminus E_\infty)$, by Proposition 1.8, the indecomposability of F forces $\mathcal{H}^{n-1}(\partial^* E_\infty \cap F^{(1)}) > 0$. This concludes the proof. □

Remark 2.6. Notice that $\inf_{\lambda>0} h_F(\lambda)$ might be equal to zero. Indeed consider two sequences of positive real numbers $(l_n)_{n=1}^\infty$ and $(\delta_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty l_n^2 < \infty$ and $\lim_{n\to\infty} \delta_n/l_n^2 = 0$. Define an open set $U \subset \mathbb{R}^2$ made of an union of disjoint open squares Q_n of side l_n, each connected to an open big rectangle through small bridges of size δ_n as in figure (1).

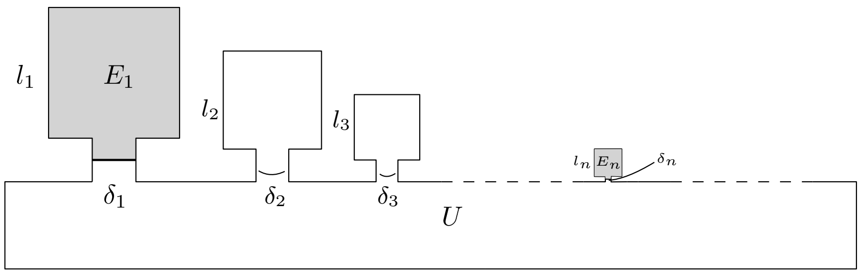

FIGURE 1. *Indecomposable set* U *with* $\inf_{\lambda>0} h_U(\lambda) = 0$.

By our choice of l_n, U is a connected open set with finite perimeter, hence by Remark 1.6 it is indecomposable.

For every $n \in \mathbb{N}$ we define $E_n \subset U$ to be the square of side l_n union half of the n-th bridge as in figure (1). By our choice of l_n and δ_n we have

$$\inf_{\lambda>0} h_U(\lambda) \le \inf_{n\in\mathbb{N}} \frac{\mathcal{H}^1(\partial^* E_n \cap U^{(1)})}{|E_n|} = 0.$$

However, Proposition 2.5 tells us that this can happen only for sequences (E_n) such that $|E_n| \to 0$.

Moreover, by using the Coarea Formula, it can be proved that $\inf_{\lambda>0} h_F(\lambda) > 0$ if and only if for every $u \in BV(\Omega)$ the following Poincaré's inequality holds true

$$\int_F |u - m|\, dx \le c\, |Du|(F^{(1)}),$$

where m is the median of u on F (see Definition 2.8). In this case the best constant c which satisfies the previous inequality is exactly $\inf_{\lambda>0} h_F(\lambda)$.

Remark 2.7. Given F an indecomposable set of $\mathbb{R}^n$, then simply by definition, we have the following scaling property of the relative upper isoperimetric profile:

$$h_F(\cdot) = r\, h_{\frac{F-x}{r}}(\cdot),$$

for every $r > 0$, $x \in \mathbb{R}^n$.

2.2. **Weak Poincaré's inequality.** We are now in position to prove the weak version of Poincaré's inequality (2.2). Before we need the following definition.

Definition 2.8. Let $u\colon \Omega \to \overline{\mathbb{R}}$ be a measurable function. Given a measurable set $F \subset \Omega$ we define the *median* of u on F as

$$m(u, F) := \inf \left\{ t \in \mathbb{R} \mid |\{u > t\} \cap F| \le \frac{|F|}{2} \right\}.$$

Remark 2.9. It holds

$$|\{u > t\} \cap F| \le \frac{|F|}{2} \quad \text{for} \quad t \ge m(u, F) \quad \text{and} \quad |\{u > t\} \cap F| > \frac{|F|}{2} \quad \text{for} \quad t < m(u, F). \tag{2.15}$$

Theorem 2.10. *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Given an indecomposable set $F \subset \Omega$ with $|F| < \infty$ and $\mathcal{H}^{n-1}(\Gamma \cap F^{(1)}) = 0$, then for every $u \in GSBV^p(\Omega;\Gamma)$ $(p \geq 1)$ and for every $\lambda \in (0, 1/2]$, there exists a measurable set $F^\lambda \subset F$ such that*

$$|F \setminus F^\lambda| \leq 2\lambda |F|, \tag{2.16}$$

and the following inequality holds true

$$\left(\int_{F^\lambda} |u-m|^p \, dx\right)^{\frac{1}{p}} \leq \frac{p}{h_F(\lambda)} \left(\int_{F^\lambda} |\nabla u|^p \, dx\right)^{\frac{1}{p}}, \tag{2.17}$$

where $m := m(u, F)$.

Proof. Let $v \in GSBV^p(\Omega;\Gamma)$ be a positive function such that

$$|\{v > t\} \cap F| \leq \frac{|F|}{2} \text{ for every } t > 0. \tag{2.18}$$

Define

$$s := \inf\{t : |\{v > t\} \cap F| \leq \lambda |F|\}, \tag{2.19}$$

and notice that

$$|\{v > t\} \cap F| \leq \lambda |F| \text{ for } t \geq s, \qquad |\{v > t\} \cap F| > \lambda |F| \text{ for } t < s. \tag{2.20}$$

If we set $v^s := v \wedge s$ we can write

$$\int_{F \cap \{v \leq s\}} v \, dx \leq \int_F v^s \, dx = \int_0^s |F \cap \{v^s > t\}| \, dt. \tag{2.21}$$

Since $\{v > t\} = \{v^s > t\}$ for every $t \in (0, s)$ and $v^s \in SBV(\Omega)$, then by (2.18), (2.20), and the definition of $h_F(\cdot)$ we have

$$\mathcal{H}^{n-1}(\partial^*\{v^s > t\} \cap F^{(1)}) = \mathcal{H}^{n-1}(\partial^*\{v > t\} \cap F^{(1)}) \geq h_F(\lambda)|F \cap \{v > t\}| = h_F(\lambda)|F \cap \{v^s > t\}|.$$

Then by (2.21) we can use the Coarea Formula for BV functions (see [1, Theorem 3.40]) to obtain

$$\begin{aligned} \int_{F^{(1)} \cap \{v \leq s\}} v \, dx &\leq \frac{1}{h_F(\lambda)} \int_0^s \mathcal{H}^{n-1}(\partial^*\{v^s > t\} \cap F^{(1)}) \, dt \\ &= \frac{1}{h_F(\lambda)} |Dv^s|(F^{(1)}) \\ &= \frac{1}{h_F(\lambda)} \int_{F^{(1)} \cap \{v \leq s\}} |\nabla v| \, dx, \end{aligned} \tag{2.22}$$

where for the last equality we used $\mathcal{H}^{n-1}(\Gamma \cap F^{(1)}) = 0$ together with the decomposition of the variation measure in BV.

Now define $(u-m)_+^p := [(u-m) \vee 0]^p$. Since by (2.15)

$$|\{(u-m)_+^p > t\} \cap F| \leq \frac{|F|}{2} \text{ for } t > 0,$$

we can apply (2.22) to the function $(u-m)_+^p$ instead of v to deduce that there exists $s^+ \geq 0$ satisfying (2.20) (where v is replaced by $(u-m)_+^p$ and s by s^+) and such that, thanks to the chain rule in BV (see [1, Theorem 3.99]), we can write

$$\int_{F \cap \{0 < (u-m)_+^p \leq s^+\}} (u-m)_+^p \, dx \leq \frac{p}{h_F(\lambda)} \int_{F \cap \{0 < (u-m)_+^p \leq s^+\}} (u-m)_+^{p-1} |\nabla u| \, dx \tag{2.23}$$

where we used that both integrals vanish on the set $\{(u-m)_+^p = 0\}$ and that $|F \Delta F^{(1)}| = 0$. Analogously, if we set $(u-m)_-^p := |(u-m) \wedge 0|^p$ by (2.15)

$$|\{(u-m)_-^p > t\} \cap F| \leq \frac{|F|}{2}, \text{ for } t > 0.$$

Arguing as before there exists $s^- > 0$ such that

$$\int_{F\cap\{0<(u-m)_-^p\le s^-\}} (u-m)_-^p \, dx \le \frac{p}{h_F(\lambda)} \int_{F\cap\{0<(u-m)_-^p\le s^-\}} (u-m)_-^{p-1}|\nabla u| \, dx. \tag{2.24}$$

If we set $F^\lambda := \{m-(s^-)^{1/p} \le u \le m+(s^+)^{1/p}\} \cap F$ by (2.20) we have $|F\setminus F^\lambda| \le 2\lambda|F|$. By summing the previous two inequalities and by using Hölder inequality we deduce

$$\Big(\int_{F^\lambda} |u-m|^p \, dx\Big)^{\frac{1}{p}} \le \frac{p}{h_F(\lambda)}\Big(\int_{F^\lambda} |\nabla u|^p \, dx\Big)^{\frac{1}{p}}, \tag{2.25}$$

which immediately implies (2.17).

□

Corollary 2.11 (Weak Poincaré's inequality)**.** *Under the same hypothesis of Theorem 2.10 we have for every $\lambda \in (0,1/2]$ and for every $u \in GSBV^p(\Omega;\Gamma)$ with $\mathcal{H}^{n-1}(\Gamma\cap F^{(1)}) = 0$*

$$\Big(\int_F |u-m|^p \wedge 1 \, dx\Big)^{\frac{1}{p}} \le \frac{p}{h_F(\lambda)} \Big(\int_F |\nabla u|^p \, dx\Big)^{\frac{1}{p}} + (2\lambda|F|)^{\frac{1}{p}}, \tag{2.26}$$

where $m := m(u,F)$.

Proof. Given $u \in GSBV^p(\Omega;\Gamma)$, we can consider F^λ and m as in Theorem 2.10. Then we can write

$$\begin{aligned}\Big(\int_F |u-m|^p \wedge 1 \, dx\Big)^{\frac{1}{p}} &\le \Big(\int_{F^\lambda} |u-m|^p \, dx\Big)^{\frac{1}{p}} + |F\setminus F^\lambda|^{\frac{1}{p}} \\ &\le \frac{p}{h_F(\lambda)} \Big(\int_F |\nabla u|^p \, dx\Big)^{\frac{1}{p}} + (2\lambda|F|)^{\frac{1}{p}},\end{aligned} \tag{2.27}$$

which is exactly (2.26). □

3. The class $\mathcal{J}_p$

In this section we define the class of admissible jump sets $\mathcal{J}_p$ for which Theorems 1 and 2 hold true. We start with the notion of *non vanishing upper isoperimetric profile*, but before we need the following definitions.

Definition 3.1. We say that a set $A \subset B_1(0)$ is *conical*, if

$$|(A\cap\lambda A)\Delta\lambda A| = 0 \text{ for every } \lambda \in (0,1).$$

Moreover, given an open set $\Omega \subset \mathbb{R}^n$, given a set $A \subset \Omega$, and given a ball $B_r(x) \subset \Omega$ we will use the following notation

$$A_{r,x} := \frac{A-x}{r} \cap B_1(0),$$

and we will always make use of the following identity

$$A_{\lambda r,x} = \frac{A_{r,x}\cap B_\lambda(0)}{\lambda},$$

for every $\lambda \in (0,1]$.

Definition 3.2 (Non vanishing upper isoperimetric profile)**.** Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$. Given $x \in \Omega$ we say that Γ has a non vanishing upper isoperimetric profile at x if there exists $N_x \in \mathbb{N}$ such that

(1) for every $1 \le j \le N_x$ there exists $(F_{r,j})_{0<r\le r_x}$ ($r_x > 0$) a family of indecomposable subsets of $B_1(0)$, with the following properties
   (1.1) $\mathcal{H}^{n-1}(\Gamma_{r,x}\cap F_{r,j}^{(1)}) = 0,\ r \in (0,r_x)$;
   (1.2) $\liminf_{r\to 0^+} h_{F_{r,j}}(\lambda) > 0, \quad \lambda \in (0,1/2]$;

(2) there exists a measurable partition of $B_1(0)$ made of (nonempty) conical sets $(E_{0,j})_{j=1}^{N_x}$ with the following property
   (2.1) $\lim_{r\to 0^+} |F_{r,j}\Delta E_{0,j}| = 0$.

Remark 3.3. In order to prevent misunderstandings, we want to emphasize that since $F_{r,j}$ are subsets of the unitary ball, then in the definition of $h_{F_{r,j}}$ the infimum in (2.12) has to be taken among all sets with finite perimeter in B_1. Moreover, for a given Γ neither the family $(F_{r,j})$ nor $(E_{0,j})$ are unique. Nevertheless, if Γ has a non vanishing upper isoperimetric profile at x, then there exists the minimum number $N_x(\geq 1)$ for which (1) and (2) hold, and this number clearly depends on the geometry of Γ.

Remark 3.4. The property of non vanishing upper isoperimetric profile is stable under inclusion, in the sense that whenever $\Gamma' \subset \Gamma$ and Γ has a non vanishing upper isoperimetric profile at x, then also Γ' satisfies the same property at x.

We give a basic example which clarifies the concept of non vanishing upper isoperimetric profile.

Example 3.5. Let $M \subset \Omega$ be an $(n-1)$-dimensional manifold of class C^1. Then M has a non vanishing upper isoperimetric profile for every $x \in \Omega$. To show this, let us first suppose $x \in M$. Then if we call $\nu(x)$ a unit normal to M at x, we know that there exists a sufficiently small value $r_x > 0$ and a C^1 function $f\colon \nu(x)^\perp \to \mathbb{R}$ such that

$$B_r(x) \cap M = B_r(x) \cap \mathrm{graph}(f), \ \ r \in (0, r_x).$$

By writing the generic point $y \in \mathbb{R}^n$ as $y = (z,t)$ where $y \in \nu(x)^\perp$ and $t \in \mathbb{R}$, we define

$$F_1 := \{y \in B_{r_x}(x) \mid t < f(z)\} \qquad F_2 := \{y \in B_{r_x}(x) \mid t > f(z)\},$$

and

$$N_x = 2, \qquad F_{r,1} := (F_1)_{r,x}, \qquad F_{r,2} := (F_2)_{r,x}, \qquad r \in (0, r_x);$$
$$E_{0,1} := \{y \in B_1(0) \mid \nu(x)\cdot y < 0\}, \qquad E_{0,2} := \{y \in B_1(0) \mid \nu(x)\cdot y > 0\}.$$

To prove condition (1.2), one can use the C^1 regularity of f and an argument similar to the one in Example 6.5, to deduce that the open sets $F_{r,j}$ $(j = 1,2)$ admit a Poincaré's inequality of the form

$$\int_{F_{r,j}} \left| u - \fint_{F_{r,j}} u \right| dx \leq c\,|Du|(F_{r,j}), \ \ u \in BV(B_1) \tag{3.1}$$

where $c > 0$ is a constant independent on $r \in (0, r_x)$. So given $E \subset F_{r,j}$ a set of finite perimeter in B_1, we can use $\mathbb{1}_E$ instead of u in (3.1) to deduce that

$$\min\{|E|, |F_{r,j} \setminus E|\} \leq c\,|D\mathbb{1}_E|(F_{r,j}) = c\,\mathcal{H}^{n-1}(\partial^* E \cap F_{r,j}) = c\,\mathcal{H}^{n-1}(\partial^* E \cap F_{r,j}^{(1)}),$$

where the right-most equality follows from the fact $F_{r,j} = F_{r,j}^{(1)}$. This implies

$$\liminf_{r \to 0^+} h_{F_{r,j}}(\lambda) \geq \frac{1}{c},$$

for every $\lambda \in (0, 1/2]$.

Another possibility to prove that M has a non-vanishing upper isoperimetric profile at $x \in M$, is to notice that since M is an $(n-1)$-manifold of class C^1, we can always find a set of finite perimeter $E \subset \Omega$ such that $M \subset \partial^* E$. In this case we can make use of Proposition 3.7, which says that $\partial^* E$ admits a non-vanishing upper isoperimetric profile at every point $x \in \partial^* E$. Since the property of non-vanishing upper isoperimetric profile is stable under inclusion (Remark 3.4), this means that also M satisfies this property for every $x \in M$.

Finally, the case $x \in \Omega \setminus M$ is much easier. Indeed, by the closeness of M there exists $r_x > 0$ small enough such that $B_r(x) \cap M = \emptyset$ for every $r \in (0, r_x)$. Then it is enough to set

$$N_x = 1, \qquad F_{r_1} := B_1(0), \qquad r \in (0, r_x);$$
$$E_{0,1} := B_1(0).$$

Now we are in position to introduce the space of all the admissible jump sets Γ.

Definition 3.6 (Admissible jump sets)**.** Let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$ and let $1 < p \leq n$. We say that Γ belongs to $\mathcal{J}_p$ if for every $x \in \Omega \setminus S_\Gamma$, where S_Γ is a set of Hausdorff dimension at most $n - p$, Γ has a non vanishing upper isoperimetric profile at x.

We will use the next two proposition to construct examples of sets living in $\mathcal{J}_p$ (see Section 6).

Proposition 3.7. *Let Ω be an open set of $\mathbb{R}^n$ and let $E \subset \Omega$ with $P(E;\Omega) < \infty$. Then the reduced boundary $\partial^* E$ has a non vanishing upper isoperimetric profile at every point x belonging to the following set*

$$\{x \in \Omega \mid \Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) = 0\} \cup \partial^* E.$$

Proof. First we deal with the case $x \in \partial^* E$.

We denote as H the half space given by Theorem 1.1 such that

$$\lim_{r\to 0^+} |(E_{r,x}\Delta H)\cap B_1(0)| = 0, \qquad \lim_{r\to 0^+} P(E_{r,x}; B_1(0)) = P(H; B_1(0)). \tag{3.2}$$

Clearly $H \cap B_1(0)$ and $B_1(0) \setminus H$ are conical and indecomposable sets. Thus, we can apply Proposition 1.13 to find two families $F_{r,1}$ and $F_{r,2}$ made of indecomposable components of $E_{r,x}$ and $B_1(0) \setminus E_{r,x}$, respectively, such that

$$\lim_{r\to 0^+} |(F_{r,1}\Delta H)\cap B_1(0)| = 0, \quad \lim_{r\to 0^+} P(F_{r,1}; B_1(0)) = P(E_{0,1}; B_1(0)), \tag{3.3}$$

$$\lim_{r\to 0^+} |F_{r,2}\Delta(B_1(0)\setminus H)| = 0, \quad \lim_{r\to 0^+} P(F_{r,2}; B_1(0)) = P(E_{0,2}; B_1(0)). \tag{3.4}$$

Given $r_x > 0$ such that $B_{r_x}(x) \subset \Omega$, we set

$$E_1 := E \cap B_{r_x}(x), \quad E_2 := B_{r_x}(x) \setminus E,$$

and

$$E_{0,1} = H \cap B_1(0), \quad E_{0,2} = B_1(0) \setminus H.$$

This choice guarantees also (1.1) and (2.1) of Definition 3.2.

Finally, in order to show (1.2) of Definition 3.2, we claim that

$$\liminf_{r\to 0^+} h_{F_{r,1}}(\lambda) \geq h_H(\lambda), \;\; \lambda \in (0, 1/2], \tag{3.5}$$

and

$$\liminf_{r\to 0^+} h_{F_{r,2}}(\lambda) \geq h_{B_1\setminus H}(\lambda), \;\; \lambda \in (0, 1/2]. \tag{3.6}$$

We prove for example (3.5). To this purpose fix $\lambda \in \big(0, \frac{1}{2}\big]$ and for every $r \in (0, r_x)$ consider $E_r \subset F_{r,1}$ with $P(E_r; B_1) < \infty$, such that

$$\frac{\mathcal{H}^{n-1}(\partial^* E_r \cap F^{(1)}_{r,1})}{|E_r|} \leq h_{F_{r,1}}(\lambda) + r, \qquad \lambda|F_{r,1}| \leq |E_r| \leq |F_{r,1}|/2. \tag{3.7}$$

We show that for every subsequence (r_m) such that $r_m \to 0^+$ as $m \to \infty$ then

$$\liminf_{m\to\infty} h_{F_{r_m,1}}(\lambda) \geq h_H(\lambda).$$

Without loss of generality we assume

$$\liminf_{m\to\infty} h_{F_{r_m,1}}(\lambda) = \lim_{m\to\infty} h_{F_{r_m,1}}(\lambda) = l < \infty.$$

Since $E_{r_m} \subset F_{r_m,1}$, by using Leibniz's formula 1.3 the inequalities (3.7) say to us

$$\sup_m P(E_{r_m}; B_1) \leq \sup_m [\, |E_{r_m}|\, h_{F_{r_m,1}}(\lambda) + P(F_{r_m,1}; B_1)] < \infty.$$

This means that thanks to the compactness result [1, Theorem 3.39], eventually passing through another subsequence, we have $\lim_{m\to\infty} |E_{r_m}\Delta E_0| = 0$ for some set $E_0 \subset H$ with finite perimeter in $B_1(0)$ and with $\lambda|H| \leq |E_0| \leq |H|/2$. Hence, thanks to (3.3) we are in position to apply the lower semicontinuity result of Proposition 2.1 to obtain

$$\liminf_{m\to\infty} h_{F_{r_m,1}}(\lambda) \geq \liminf_{m\to\infty} \frac{\mathcal{H}^{n-1}(E_{r_m} \cap F^{(1)}_{r_m,1})}{|E_{r_m}|} - r_m \geq \frac{\mathcal{H}^{n-1}(E_0 \cap H^{(1)})}{|E_0|} \geq h_H(\lambda).$$

The same argument shows the validity of (3.6). Since $h_H(\lambda) > 0$, this says that $\partial^* E$ admits a non-vanishing upper isoperimetric profile at x with $N_x = 2$.

In the case $x \in \Omega$ is such that $\Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* E, x) = 0$, we claim that we have two different sub-cases:

$$\lim_{r\to 0^+} |B_1(0) \setminus E_{r,x}| = 0, \tag{3.8}$$

or

$$\lim_{r\to 0^+} |E_{r,x}| = 0. \tag{3.9}$$

Indeed by a simple application of the relative isoperimetric inequality in the unitary ball we can write

$$\min\{|E_{x,r}|, |B_1(0) \setminus E_{x,r}|\}^{\frac{n-1}{n}} \leq C(n)\frac{P(E; B_1(0))}{r^{n-1}} = C(n)\frac{\mathcal{H}^{n-1}(\partial^* E \cap B_1(0))}{r^{n-1}},$$

and by the fact that $r \mapsto |E_{x,r}|$ is a continuous map on $(0, r_x)$ we deduce that one between (3.8) and (3.9) must occur. Suppose for example (3.8) holds. Given $r_x > 0$ such that $B_{r_x}(x) \subset \Omega$, we set

$$E_1 := E \cap B_{r_x}(x), \qquad E_{0,1} = B_1(0).$$

Arguing in the very same way as before, we can make use of Proposition 1.13 to find for every $r \in (0, r_x)$ an indecomposable component of $(E_1)_{r,x}$, say $F_{r,1}$, such that

$$\lim_{r\to 0^+} |F_{r,1} \Delta B_1(0)| = 0, \qquad \lim_{r\to 0^+} P(F_{r,1}; B_1) = 0.$$

Finally, by using again Proposition 2.1 we can prove in the very same way as before that

$$\liminf_{r\to 0^+} h_{F_{r,1}}(\lambda) \geq h_{B_1}(\lambda), \;\; \lambda \in (0, 1/2].$$

Case (3.9) can be treated in the same way. □

Remark 3.8. By Proposition 3.7, the reduced boundary $\partial^* E$ of a set $E \subset \Omega$ with finite perimeter such that $\dim_H(\Omega \setminus \partial^* E \cup \{x \mid \Theta(\mathcal{H}^{n-1} \llcorner \partial^* E; x) = 0\}) = n - p$, belongs to $\mathcal{J}_p$.

4. Properties of the blow-up in $GSBV^p(\Omega)$

This section contains the proof of Theorem 1. We proceed following two main steps: first we show that there exist suitable subsequences of radii (r_i) with $r_i \to 0^+$ as $i \to \infty$, such that if $\Gamma \in \mathcal{J}_p$, then for every $x \in \Omega$, up to a set of Hausdorff dimension $n - p$, the limit $\lim_{i\to\infty} m_j(u, r_i, x)$ exists and it is finite; by combining this result with the weak Poincaré's inequality on balls we are able to deduce our first main result.

4.1. **Weak Poincaré's inequality on balls.** We start this section by proving a weak version of Poincare's inequality on balls. First, we need the following definitions.

Definition 4.1. For a given function $u: \Omega \to \mathbb{R}$, we define $u_{r,x}: B_1(0) \to \mathbb{R}$ as

$$u_{r,x}(y) := u(x + ry),$$

for every $y \in B_1(0)$.

Definition 4.2. Let $\Gamma \in \mathcal{J}_p$ $(1 < p \leq n)$ and let $x \in \Omega \setminus S_\Gamma$. Let $r_x > 0$ and $N_x \in \mathbb{N}$ be given by Definition 3.2. We define for every $r \in (0, r_x)$, $\overline{u}_{r,x}: B_{r_x}(x) \to \mathbb{R}$ as

$$\overline{u}_{r,x}(y) := \begin{cases} m_j(u, r, x) & \text{on } x + rF_{r,j} \\ 0 & \text{otherwise.} \end{cases}$$

where $m_j(u, r, x) := m(u, x + rF_{r,j})$ (see Definition 2.8) and $(F_{r,j})_{j=1}^{N_x}$ are the indecomposable sets given by Definition 3.2.

Remark 4.3. The median of u in F is invariant under rescaling and translations in the sense that

$$m(u, F) = m(u_{r,x}, (F - x)/r),$$

for every $x \in \Omega$ and for every $0 < r < r_x$. This means that the number $m_j(u, r, x)$ of the previous definition is also equal to $m(u_{r,x}, F_{r,j})$.

Theorem 4.4 (Weak Poincare's inequality on balls)**.** *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Suppose that $\Gamma \subset \Omega$ has a non vanishing upper isoperimetric profile at x, then for every $\lambda \in (0, 1/2]$ there exists $r_\lambda > 0$ (depending also on x) such that*

$$\left(\int_{B_r(x)} |u - \overline{u}_{r,x}|^p \wedge 1 \, dy\right)^{\frac{1}{p}} \leq C(p,n) \left[\mathscr{H}_x(\lambda) r \left(\int_{B_r(x)} |\nabla u|^p \, dy\right)^{\frac{1}{p}} + (r^n \lambda)^{\frac{1}{p}}\right], \tag{4.1}$$

for every $r \leq r_\lambda$ and for every $u \in GSBV^p(\Omega; \Gamma)$, where

$$\mathscr{H}_x(\lambda) := \limsup_{r \to 0^+} \left[\max_{j=1,\dots,N_x} \left\{\frac{2}{h_{F_{r,j}}(\lambda)}\right\}\right] < \infty. \tag{4.2}$$

Proof. Fix $\lambda > 0$ and let $x \in \Omega \setminus S_\Gamma$. By property (2.1) of Definition 3.2 we know that there exists $0 < r'_\lambda < r_x$ such that for every $r < r'_\lambda$

$$\sup_{r < r'_\lambda} |B_1(0) \setminus \bigcup_{j=1}^{N_x} F_{r,j}| \leq \lambda,$$

which means

$$\sup_{r < r'_\lambda} |B_r(x) \setminus \bigcup_{j=1}^{N_x} (x + rF_{r,j})| \leq r^n \lambda. \tag{4.3}$$

Moreover by the definition of lim sup we can consider r''_λ small enough such that

$$\sup_{r < r''_\lambda} \left[\max_{j=1,\dots,N_x} \left\{\frac{1}{h_{F_{r,j}}(\lambda)}\right\}\right] \leq \mathscr{H}_x(\lambda) < \infty.$$

Since $u_{r,x} \in GSBV^p(B_1(0); \Gamma_{r,x})$ $(r < r_x)$ and thanks to the fact $\mathcal{H}^{n-1}(\Gamma_{r,x} \cap F^{(1)}_{r,j}) = 0$ for every $1 \leq j \leq N_x$, by applying Theorem 2.10 we know that there exists $F^\lambda_{r,j} \subset F_{r,j}$ with

$$|F_{r,j} \setminus F^\lambda_{r,j}| \leq \lambda |F_{r,j}|, \tag{4.4}$$

such that

$$\int_{F^\lambda_{r,j}} |u(x + ry) - m_{r,j}|^p \, dy \leq \left(\frac{pr}{h_{F_{r,j}}(\lambda)}\right)^p \int_{F^\lambda_{r,j}} |\nabla u(x + ry)|^p \, dy,$$

where $m_{r,j} := m_j(u, r, x)$.

If we define $F^\lambda_r := \bigcup_{j=1}^{N_x} F^\lambda_{r,j}$, then by summing on $j = 1, \dots, N_x$ both sides of the previous inequality, if $r \leq \min\{r'_\lambda, r''_\lambda\}$ we obtain

$$\int_{F^\lambda_r} |u(x + ry) - \overline{u}_{r,x}(x + ry)|^p \, dy \leq (p\mathscr{H}_x(\lambda) r)^p \int_{F^\lambda_r} |\nabla u(x + ry)|^p \, dy,$$

or equivalently

$$\int_{x + rF^\lambda_r} |u(y) - \overline{u}_{r,x}|^p \, dy \leq (p\mathscr{H}_x(\lambda) r)^p \int_{x + rF^\lambda_r} |\nabla u(y)|^p \, dy.$$

Finally, by defining $F_r := \bigcup_{j=1}^{N_x} F_{r,j}$ and by using also (4.3) and (4.4), we can write

$$\begin{aligned}
\int_{B_r(x)} |u(y) - \overline{u}_{r,x}|^p \wedge 1 \, dy &\leq \int_{x + rF^\lambda_r} |u(y) - \overline{u}_{r,x}|^p \, dy + |B_r(x) \setminus (x + rF^\lambda_r)| \\
&\leq (p\mathscr{H}_x(\lambda) r)^p \int_{x + rF^\lambda_r} |\nabla u(y)|^p \, dy + r^n |F_r \setminus F^\lambda_r| + |B_r(x) \setminus (x + rF_r)| \\
&\leq (p\mathscr{H}_x(\lambda) r)^p \int_{B_r(x)} |\nabla u(y)|^p \, dy + (\omega_n + 1) r^n \lambda \\
&\leq C(n,p) \left[(\mathscr{H}_x(\lambda) r)^p \int_{B_r(x)} |\nabla u(y)|^p \, dy + r^n \lambda\right].
\end{aligned}$$

which is exactly (4.1). □

Remark 4.5. Under the hypothesis of the previous theorem, in the case the set Γ satisfies the stronger conditions at x

- $\bigcup_{j=1}^{N_x} F_{r,j} = B_1(0), \quad r \le r_x$;
- $\liminf_{r\to 0^+} \inf_{\lambda>0} h_{F_{r,j}}(\lambda) > 0, \quad j = 1,\dots,N_x$.

then it is not difficult to show that inequality (4.1) can be improved to

$$\Big(\int_{B_r(x)} |u - \overline{u}_{r,x}|^p\,dx\Big)^{\frac{1}{p}} \le C(p,n)r\Big(\int_{B_r(x)} |\nabla u|^p\,dx\Big)^{\frac{1}{p}}.$$

4.2. **Convergence of the blow-up.** The following theorem, which is the core result of this section, tells us that when the integrals $\int_{B_r(x)}|\nabla u|^p$ decays properly as $r\to 0^+$, then the medians $m_j(u,r,x)$ are convergent for suitable subsequences of radii $r_i \to 0^+$. In the proof we will use the following inequality which is true for each quadruple of measurable sets $A,B,C,D \subset \Omega$

$$|A\Delta B| \le |C\Delta D| + |A\Delta C| + |B\Delta D|. \tag{4.5}$$

Inequality (4.5) simply follows by noticing that $|A\Delta B| = \|\mathbb{1}_A - \mathbb{1}_B\|_{L^1}$ and by applying the triangular inequality.

Theorem 4.6. *Let $\Omega\subset\mathbb{R}^n$ be an open set, let $\Gamma\in\mathcal{J}_p$ $(1<p\le n)$ and let $x\in\Omega\setminus S_\Gamma$. Suppose that there exists some $\delta\in(0,p]$ with the following property*

$$\limsup_{r\to 0^+} \frac{1}{r^{n-p+\delta}}\int_{B_r(x)} |\nabla u|^p\,dx = 0. \tag{4.6}$$

Then for every sequence of radii $(r_i)_{i=1}^\infty$ such that

(1) $\left(\frac{1}{2}\right)^{\frac{1}{2n}} < \frac{r_{i+1}}{r_i} \le 1, \quad i\in\mathbb{N}$;
(2) $\sum_{i=1}^\infty (r_i)^{\frac{\delta}{p}} < \infty$;

the sequence of medians $\big(m_j(u,r_i,x)\big)_{i=1}^\infty$ is Cauchy for every $j=1,\dots,N_x$.

Proof. Choose $j\in\{1,\dots,N_x\}$. In order to simplify the notation we write

$$t_{r_i} := m_j(u,r_i,x) \qquad F_r := F_{r,j} \qquad E_0 := E_{0,j} \qquad a_i := \frac{r_i}{r_{i-1}}.$$

Fix $\epsilon>0$ such that for every $n\in\mathbb{N}$ it holds $\epsilon \le \frac{2a_i^n-1}{4a_i^n+4}$ (this is possible since by condition (1) it is enough to chose $0<\epsilon\le\frac{\sqrt{2}-1}{2\sqrt{2}+4}$), and consider $\bar{i}\in\mathbb{N}$ so big that for every $i\ge\bar{i}$

$$h_{F_{r_i}}(\epsilon) \ge \frac{1}{c(\epsilon)} := \frac{1}{2}\liminf_{i\to\infty} h_{F_{r_i}}(\epsilon) > 0.$$

This is possible by the definition of lim inf.

By using Theorem 2.10 with the function $u(x+r(\cdot)) \in GSBV^p(B_1(0);\Gamma_{r,x})$ and the indecomposable set F_{r_i}, we deduce that for every $i\ge\bar{i}$ and for every $\epsilon>0$, there exists $F^\epsilon_{r_i}\subset F_{r_i}\subset B_1(0)$ such that

$$|F^\epsilon_{r_i}| \ge (1-2\epsilon)|F_{r_i}|, \tag{4.7}$$

and

$$\Big(\int_{F^\epsilon_{r_i}} |u_{r_i,x} - t_{r_i}|^p\,dy\Big)^{\frac{1}{p}} \le 2c(\epsilon)p\Big(\int_{F^\epsilon_{r_i}} |\nabla u_{r_i,x}|^p\,dy\Big)^{\frac{1}{p}}. \tag{4.8}$$

Now for each $i\ge\bar{i}$ define

$$\mathcal{F}_i := a_iF^\epsilon_{r_i} \cap F^\epsilon_{r_{i-1}} \subset B_{a_i}(0).$$

Since $\mathcal{F}_i\subset a_iF_{r_i}$, we can give the following estimate

$$\begin{aligned}|\mathcal{F}_i| = |a_iF^\epsilon_{r_i}\cap F^\epsilon_{r_{i-1}}| &= |a_iF_{r_i}\setminus(a_iF_{r_i}\setminus a_iF^\epsilon_{r_i} \cup a_iF_{r_i}\setminus F^\epsilon_{r_{i-1}})|\\ &\ge |a_iF_{r_i}| - |a_iF_{r_i}\setminus a_iF^\epsilon_{r_i}| - |a_iF_{r_i}\setminus F^\epsilon_{r_{i-1}}|.\end{aligned}$$

By (4.7) and the fact $|F_{r_i}\Delta E_0|\to 0$ we can write

$$|a_iF_{r_i}\setminus a_iF^\epsilon_{r_i}| = a_i^n|F_{r_i}\setminus F^\epsilon_{r_i}| \le a_i^n 2\epsilon|F_{r_i}| = a_i^n2\epsilon[|E_0|+o(1)],$$

and by using also inequality (4.5) with $A = F_{r_{i-1}} \cap a_i F_{r_i}$, $B = a_i F_{r_i}$, $C = E_0 \cap a_i E_0$ and $D = a_i E_0$, we can write

$$\begin{aligned} |a_i F_{r_i} \setminus F^{\epsilon}_{r_{i-1}}| &\le |(F_{r_{i-1}} \cap a_i F_{r_i}) \setminus F^{\epsilon}_{r_{i-1}}| + |[(F_{r_{i-1}} \cap a_i F_{r_i}) \Delta a_i F_{r_i}] \setminus F^{\epsilon}_{r_{i-1}}| \\ &\le |F_{r_{i-1}} \setminus F^{\epsilon}_{r_{i-1}}| + |(F_{r_{i-1}} \cap a_i F_{r_i}) \Delta a_i F_{r_i}| \\ &= 2\epsilon |E_0| + |(E_0 \cap a_i E_0) \Delta a_i E_0| + o(1), \end{aligned}$$

and since E_0 is conical, then $|(E_0 \cap a_i E_0)\Delta a_i E_0| = 0$ for every $i \in \mathbb{N}$; as a consequence we can write

$$|a_i F_{r_i} \setminus F^{\epsilon}_{r_{i-1}}| \le 2\epsilon |E_0| + o(1).$$

Putting together our previous estimates we obtain

$$\begin{aligned} |\mathcal{F}_i| &\ge a_i^n |E_0| - a_i^n 2\epsilon |E_0| - 2\epsilon |E_0| + o(1) \\ &= |E_0|(a_i^n - a_i^n \epsilon - 2\epsilon) + o(1). \end{aligned}$$

By our choice of ϵ, we have $a_i^n - \epsilon(2a_i^n + 2) \ge \frac{1}{2}$, hence

$$|\mathcal{F}_i| \ge \frac{1}{2}|E_0| + o(1), \; i \in \mathbb{N}. \tag{4.9}$$

Therefore, for every $i \ge \bar{i}$, we can write

$$\begin{aligned} |t_{r_i} - t_{r_{i-1}}|^p &= \fint_{\mathcal{F}_i} |t_{r_i} - t_{r_{i-1}}|^p \, dy \le 2^{p-1} \fint_{\mathcal{F}_i} |u_{r_{i-1},x} - t_{r_i}|^p \, dy + 2^{p-1} \fint_{\mathcal{F}_i} |u_{r_{i-1},x} - t_{r_{i-1}}|^p \, dy \\ &= \frac{2^{p-1} a_i^n}{|\mathcal{F}_i|} \int_{F^{\epsilon}_{r_i}} |u_{r_i,x} - t_{r_i}|^p \, dy + \frac{2^{p-1}}{|\mathcal{F}_i|} \int_{F^{\epsilon}_{r_{i-1}}} |u_{r_{i-1},x} - t_{r_{i-1}}|^p \, dy, \end{aligned}$$

hence by using (4.8) and $a_i \le 1$ there exists $C = C(p, n, \epsilon) > 0$ such that

$$\begin{aligned} |t_{r_i} - t_{r_{i-1}}|^p &\le \frac{C}{|\mathcal{F}_i|} \left[\int_{F^{\epsilon}_{r_i}} |\nabla u_{r_i,x}|^p \, dy + \int_{F^{\epsilon}_{r_{i-1}}} |\nabla u_{r_{i-1},x}|^p \, dy \right] \\ &= \frac{C}{|\mathcal{F}_i|} \left[r_i^p \int_{F^{\epsilon}_{r_i}} |\nabla u(x + r_i y)|^p \, dy + r_{i-1}^p \int_{F^{\epsilon}_{r_{i-1}}} |\nabla u(x + r_{i-1} y)|^p \, dy \right], \end{aligned}$$

and finally by using (4.9) we have

$$\begin{aligned} |t_{r_i} - t_{r_{i-1}}|^p &= \frac{C}{1/2|E_0| + o(1)} \left[\frac{1}{r_i^{n-p}} \int_{x + r_i F^{\epsilon}_{r_i}} |\nabla u|^p \, dx + \frac{1}{r_{i-1}^{n-p}} \int_{x + r_{i-1} F^{\epsilon}_{r_{i-1}}} |\nabla u|^p \, dx \right] \\ &\le \frac{C r_i^{\delta}}{1/2|E_0| + o(1)} \left[\frac{1}{r_i^{n-p+\delta}} \int_{B_{r_i}(x)} |\nabla u|^p \, dx + \frac{(\frac{1}{2})^{-\delta/2n}}{r_{i-1}^{n-p+\delta}} \int_{B_{r_{i-1}}(x)} |\nabla u|^p \, dx \right] \\ &\le C' r_i^{\delta}, \end{aligned}$$

where, thanks also to (4.6), $C' > 0$ is a constant which depends only on x, j, p, n, ϵ.

These last inequality means

$$\sum_{i \ge \bar{i}}^{\infty} |t_{r_i} - t_{r_{i-1}}| \le C'^{\frac{1}{p}} \sum_{i=1}^{\infty} (r_i)^{\frac{\delta}{p}},$$

and this last series is convergent thanks to our choice of r_i. This implies that the sequence $(t_{r_i})_{i=1}^{\infty}$ is Cauchy. Since $1 \le j \le N_x$ was arbitrary, we prove the theorem.

□

Now we are in position to prove our first main result.

Theorem 4.7. *Let Ω be an open set of $\mathbb{R}^n$, let $\Gamma \in \mathcal{J}_p$ $(1 < p \le n)$ and let $u \in GSBV^p(\Omega; \Gamma)$. Then for every $x \in \Omega$ except on a set of Hausdorff dimension at most $n - p$, there exists a piecewise constant function $u_x(\cdot)\colon B_1(0) \to \mathbb{R}$ such that*

$$\lim_{r \to 0^+} \int_{B_1(0)} |u_{r,x} - u_x| \wedge 1 \, dy = 0. \tag{4.10}$$

Moreover using the notation of Definitions 3.2 and 2.8 we have that

$$u_x(y) = m_j(u,x) \text{ if } y \in E_{0,j}, \tag{4.11}$$

where $m_j(u,x) := \lim_{r\to\infty} m_j(u,r,x)$ *for* $1 \le j \le N_x$.

Proof. For every $\delta > 0$, consider $A_\delta \subset \Omega \setminus S_\Gamma$ the set of points x such that

$$\limsup_{r\to 0^+} \frac{1}{r^{n-p+\delta}} \int_{B_r(x)} |\nabla u|^p \, dx > 0. \tag{4.12}$$

By applying for example [7, Theorem 3, Section 2.4.3] we have $\mathcal{H}^{n-p+\delta}(A_\delta) = 0$. Moreover, since $A_{\delta_1} \subset A_{\delta_2}$ for $\delta_1 \le \delta_2$, we have that if we fix $\delta_0 > 0$ then by setting $A := \bigcap_{\delta>0} A_\delta$ we have

$$\mathcal{H}^{n-p+\delta_0}(A) = 0.$$

Since $\delta_0 > 0$ is arbitrary we deduce

$$\dim_{\mathcal{H}}(A) \le n-p, \tag{4.13}$$

and hence also

$$\dim_{\mathcal{H}}(A \cup S_\Gamma) \le n-p. \tag{4.14}$$

We claim that every $x \in \Omega \setminus (S_\Gamma \cup A)$ satisfies (4.10) and (4.11). To show this, let $(r_i)_{i=1}^\infty$ be a sequence of radii satisfying (1) and (2) of Theorem 4.6 and define $u_x : B_1(0) \to \mathbb{R}$ as

$$u_x(y) := \lim_{i\to\infty} m_j(u,r_i,x), \text{ for } y \in E_{0,j} \text{ and } 1 \le j \le N_x.$$

First of all we prove that

$$\lim_{i\to\infty} \int_{B_1(0)} |u_{r_i,x}(y) - u_x(y)| \wedge 1 \, dy = 0. \tag{4.15}$$

Recalling condition (2.1) of Definition 3.2 and the definition of $\overline{u}_{r_i,x} : B_r(x) \to \mathbb{R}$ (see Definition 4.2), we immediately deduce

$$\lim_{i\to\infty} \int_{B_1(0)} |\overline{u}_{r_i,x}(x+r_i y) - u_x(y)| \wedge 1 \, dy = 0. \tag{4.16}$$

which together with the weak Poincaré's inequality on balls (4.4) gives for every $\lambda \in (0,1/2]$

$$\begin{aligned}
\limsup_{i\to\infty} \int_{B_1(0)} |u_{r_i,x}(y) - u_x(y)| \wedge 1 \, dy &\le \limsup_{i\to\infty} \int_{B_1(0)} |u_{r_i,x}(y) - \overline{u}_{r_i,x}(x+r_i y)| \wedge 1 \, dy \\
= \limsup_{i\to\infty} \fint_{B_{r_i}(x)} |u - \overline{u}_{r_i,x}| \wedge 1 \, dy &\le C(p,n) \limsup_{i\to\infty} \left[\mathscr{H}_x(\lambda) \left(\frac{1}{r_i^{n-p}} \int_{B_{r_i}(x)} |\nabla u|^p \, dx \right)^{\frac{1}{p}} + \lambda^{\frac{1}{p}} \right] \\
&\le C(p,n) \lambda^{\frac{1}{p}}.
\end{aligned}$$

By letting $\lambda \to 0^+$ we deduce (4.15). Now in order to prove (4.10), it is equivalent to prove

$$\lim_{r\to 0^+} \fint_{B_r(x)} |u(y) - u_x((y-x)/r)| \wedge 1 \, dy = 0, \tag{4.17}$$

but since the sets $E_{0,j}$ are conical, then $u_x(y) = u_x((y-x)/r)$ for every $y \in B_r(x)$, and therefore we can rewrite (4.17) as

$$\lim_{r\to 0^+} \fint_{B_r(x)} |u(y) - u_x(y)| \wedge 1 \, dy = 0. \tag{4.18}$$

To show (4.18), simply notice that given any $r \in (0,1)$ then there exists $i \in \mathbb{N}$ such that $r_{i+1} \le r \le r_i$, and therefore by using the lower-bound (1) in Theorem 4.6 on the ratio r_{i+1}/r_i we easily deduce

$$\fint_{B_r(x)} |u - u_x| \wedge 1 \, dy \le \frac{|B_{r_i}|}{|B_{r_{i+1}}|} \fint_{B_{r_i}(x)} |u - u_x| \wedge 1 \, dy \le \sqrt{2} \fint_{B_{r_i}(x)} |u - u_x| \wedge 1 \, dy,$$

which together with (4.15) clearly implies (4.18) and hence (4.10).

Finally, to conclude we need to show that if we set for every $j \in 1, \dots, N_x$ $m_j(u,x) := \lim_{i\to\infty} m_j(u,r_i,x)$, then

$$\lim_{r\to 0^+} m_j(u,r,x) = m_j(u,x). \tag{4.19}$$

In order to show this, notice that the convergence (4.10) implies that for every t except on a countable set A, we have

$$|\{u_{r,x} > t\}\Delta\{u_x > t\}| \to 0, \quad (r \to 0^+).$$

Moreover since $|F_{r,j}\Delta E_{0,j}| \to 0$ as $r \to 0^+$, we have also that for $t \in \mathbb{R}\setminus A$

$$|\{u_{r,x} > t\} \cap F_{r,j}| \to |\{u_x > t\} \cap E_{0,j}|, \quad (r \to 0^+). \tag{4.20}$$

By definition of medians (see Definition 2.8), in order to prove (4.19), we need to show that

$$\inf\{t \in \mathbb{R} \mid |\{u_{r,x} > t\} \cap F_{r,j}| \le |F_{r,j}|/2\}$$

converges as $r \to 0^+$ to

$$\inf\{t \in \mathbb{R} \mid |\{u_x > t\} \cap E_{0,j}| \le |E_{0,j}|/2\}.$$

The convergence (4.20) together with $|F_{r,j}\Delta E_{0,j}| \to 0$ as $r \to 0^+$ imply that if $t \in \mathbb{R}\setminus A$ is such that $|\{u_x > t\} \cap E_{0,j}| < \frac{|E_{0,j}|}{2}$, then for every r close enough to 0^+ we have

$$|\{u_{r,x} > t\} \cap F_{r,j}| < \frac{|F_{r,j}|}{2},$$

analogously if $t \in \mathbb{R}\setminus A$ is such that $|\{u_x > t\} \cap E_{0,j}| > \frac{|E_{0,j}|}{2}$, then for every r close enough to 0^+ we have

$$|\{u_{r,x} > t\} \cap F_{r,j}| > \frac{|F_{r,j}|}{2}.$$

Therefore (4.19) is established once we know that

$$|\{u_x > t\}\cap E_{0,j}| < \frac{|E_{0,j}|}{2}, \text{ for } t > m_j(u,x) \quad \text{and} \quad |\{u_x > t\}\cap E_{0,j}| > \frac{|E_{0,j}|}{2}, \text{ for } t < m_j(u,x).$$

But this last condition is obviously verified since $u_{r,x}$ is constantly equal to $m_j(u,x)$ on $E_{0,j}$. □

Remark 4.8. Since the result of Theorem 4.7 is local, then it still holds for the space $GSBV^p_{loc}(\Omega;\Gamma)$.

Remark 4.9. It is not difficult to show that if we substitute condition (1.2) in the definition of non vanishing upper isoperimetric profile with the stronger conditions

$$\bigcup_{j=1}^{N_x} F_{r,j} = B_1(0), \quad (r \le r_x) \quad \text{and} \quad \liminf_{r\to 0^+} \inf_{\lambda>0} h_{F_{r,j}}(\lambda) > 0, \quad (j = 1,\dots,N_x), \tag{4.21}$$

then, by using Remark 4.5 it is possible to show that the convergence (4.10) actually holds with respect to the L^p-convergence (in Example 6.5 we construct a non trivial admissible jump set, such that it admits a non vanishing upper isoperimetric profile with the stronger condition (4.21) at every point x).

Remark 4.10. When we deal with Sobolev spaces, namely $\Gamma = \emptyset$, Theorem 1 implies the well known result that given $u \in W^{1,p}_{loc}(\Omega)$ then every point x, up to a set of Hausdorff dimension at most $n-p$, is a Lebesgue point of u. The function

$$u(x) = \log\log|x|^{-1},$$

which belongs to $W^{1,2}(B_1(0))$, $B_1(0) \subset \mathbb{R}^2$, shows that the dimension $n-p$ is optimal in Theorem 1.

5. A notion of capacity for functions with prescribed jump

This section is devoted to the proof of Theorem 2. For this purpose we need to introduce a suitable notion of capacity for functions in $GSBV^p(\Omega;\Gamma)$. Let us recall that given $A \subset \mathbb{R}^n$, the classical p-capacity in the context of Sobolev functions is defined as (see for example [9] or [7])

$$\mathrm{Cap}_p(A) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^p\,dx \mid u \in K^p,\ u \geq 1 \text{ a.e. in an open neighborhood of } A\right\}, \tag{5.1}$$

where $K^p := \{u\colon \mathbb{R}^n \to \mathbb{R} \mid u \geq 0,\ u \in L^{p*}(\mathbb{R}^n),\ \nabla u \in L^p(\mathbb{R}^n)\}$. Moreover, the following result can be interpreted as a capacitary version of Chebyshev's inequality (see for example [9, Section 7] or [7, Lemma 1, Section 4.8]).

Proposition 5.1. *Assume $u \in K^p$ and $\epsilon > 0$. Let*

$$A := \{x \in \mathbb{R}^n \mid m(u,r,x) > \epsilon \text{ for some } r > 0\},$$

where $m(u,r,x)$ denotes the median of u on $B_r(x)$ (see Definition 2.8). Then

$$\mathrm{Cap}_p(A) \leq \frac{c}{\epsilon^p}\int_{\mathbb{R}^n} |\nabla u|^p\,dx,$$

where $c = c(n,p)$.

The previous proposition suggests us that given $A \subset \mathbb{R}^n$, if we define a variant of the p-capacity in the following way

$$\mathrm{Cap}'_p(A) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^p\,dx \mid u \in K^p,\ \limsup_{r\to 0^+} m(u,r,x) \geq 1 \text{ for every } x \in A\right\}, \tag{5.2}$$

then

$$\mathrm{Cap}'_p(A) \leq \mathrm{Cap}_p(A) \leq c2^p\,\mathrm{Cap}'_p(A),$$

for some constant $c > 0$ and for every $A \subset \mathbb{R}^n$. Indeed, if $u \in K^p$ is such that $u \geq 1$ a.e. in an open neighborhood of A, clearly u satisfies $\limsup_{r\to 0^+} m(u,r,x) \geq 1$ for every $x \in A$ and we obtain

$$\mathrm{Cap}'_p(A) \leq \mathrm{Cap}_p(A).$$

On the other hand, given $\delta > 0$, let $u \in K^p$ be such that $\limsup_{r\to 0^+} m(u,r,x) \geq 1$ for every $x \in A$ and

$$\int_{\mathbb{R}^n} |\nabla u|^p\,dx < \mathrm{Cap}'_p(A) - \delta.$$

By definition of lim sup for every $x \in A$ there exists r_x such that $m(u,r_x,x) > 1/2$. Therefore

$$A \subset \{x \in \mathbb{R}^n \mid m(u,r,x) > 1/2 \text{ for some } r > 0\},$$

and by the capacitary Chebyshev's inequality the previous inclusion together with the monotonicity of the p-capacity immediately imply

$$\mathrm{Cap}_p(A) \leq c2^p\int_{\mathbb{R}^n} |\nabla u|^p\,dx \leq c2^p(\mathrm{Cap}'_p(A) - \delta).$$

Thanks to the arbitrariness of $\delta > 0$, we deduce

$$\mathrm{Cap}_p(A) \leq c\,\mathrm{Cap}'_p(A).$$

Hence, it is possible to define an equivalent notion of capacity by looking at the medians of u for *every* $x \in A$. Since for technical reason we prefer to define a notion of capacity where the infimum (5.1) does not depend on a *a.e.*-condition, the variant introduced in (5.2) seems to fit better our purpose. However, if we want to mimic definition in (5.2), we should take into account different medians, i.e. $(m_j(u,r,x))_{j=1}^{N_x}$, depending on Γ and x (see Definition 4.2). Since we prefer to define a capacity which is a priori independent on Γ, we decide to give a slightly different definition which is based on the notion of approximate limit (see Definition (5.7)).

5.1. **Convergence with respect to an outer measure.** In this subsection we want to fix the notion of convergence with respect to an outer measure and to define a suitable function space which will be useful in view of Theorem 2.

For convenience of the reader we recall the definition of outer measure.

Definition 5.2 (Outer measure)**.** An outer measure on Ω is any set function $\mu\colon \mathcal{P}(\Omega) \to [0,+\infty]$ satisfying the following properties

(a) $\mu(\emptyset) = 0$;
(b) $\mu(A_1) \le \mu(A_2)$, whenever $A_1 \subset A_2$ (monotonicity);
(c) $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} \mu(A_i)$ (countable sub-additivity).

Definition 5.3. Let $\mu\colon \mathcal{P}(\Omega) \to [0,\infty]$ be an outer measure. Given $A \subset \Omega$, we say that a property $\mathcal{P}(x)$, defined for $x \in A$, holds μ-quasi everywhere, and we use the abbreviation μ-*q.e.*, if there exists a set $N \subset A$, with $\mu(N) = 0$, such that $\mathcal{P}(x)$ holds for every $x \in A \setminus N$.

We recall that the convergence in measure can be metrized.

Definition 5.4. We denote by $L^0(B_1)$ (see [12]) the Fréchet space of all (equivalence classes of) Lebesgue measurable real-functions on B_1 equipped with the topology of convergence in measure. This topology can be defined for example by the Lévy-metric

$$\|u - v\|_{L^0(B_1)} := \int_{B_1} |u - v| \wedge 1 \, dx, \;\; u, v \in L^0(B_1). \tag{5.3}$$

By means of Theorem 1, whenever $\Gamma \in \mathcal{J}_p$, we can associate to each function $u \in GSBV^p(\Omega;\Gamma)$ a map $u_{(\cdot)}\colon \Omega \to L^0(B_1)$ defined everywhere except on a set of Hausdorff dimension $n-p$. Given an outer measure μ on Ω, we want to define a space which contains functions defined μ-q.e. from Ω to $L^0(B_1)$, and to endow such a space with a notion of convergence in μ-measure.

Definition 5.5. Let μ be an outer measure on Ω. Let X be the real vector space of all functions $u\colon \Omega \to L^0(B_1)$, and consider the equivalence relation

$$u \sim v \;\text{ iff }\; \mu(\{x \in \Omega \mid u(x) \neq v(x)\}) = 0. \tag{5.4}$$

We define

$$U_\mu(\Omega; L^0(B_1)) := X/\sim,$$

i.e. the space consisting of all equivalence classes obtained as the quotient of X with respect to $\sim$.

Remark 5.6. Notice that, since μ is an outer measure, (5.4) makes sense even without any measurability conditions on the functions u and v.

Definition 5.7. Let μ be an outer measure on Ω, let $(u_k)_{k=1}^{\infty}$ and u be functions in $U_\mu(\Omega; L^0(B_1))$. We say that (u_k) converges to u in μ-*measure* if

$$\lim_{k\to\infty} \mu(\{x \in \Omega \mid \|u_k - u\|_{L^0(B_1)} > \epsilon\}) = 0, \tag{5.5}$$

for every $\epsilon > 0$.

Convergence in μ-measure implies up to subsequences pointwise convergence μ-q.e.. This is the content of the next proposition.

Proposition 5.8. *Let μ be an outer measure on Ω, let $(u_k)_{k=1}^{\infty}$ and u be functions in $U_\mu(\Omega; L^0(B_1))$. Suppose $u_k \to u$ in μ-measure, then there exists a subsequence (k_j) such that for μ-q.e. $x \in \Omega$*

$$\lim_{j\to\infty} \|u_{k_j}(x) - u(x)\|_{L^0(B_1)} = 0,$$

Proof. For every $j \in \mathbb{N}$ choose $k_j \in \mathbb{N}$ such that

$$\mu\left(\left\{x \in \Omega \mid \|u_{k_j} - u\|_{L^0(B_1)} > \frac{1}{j}\right\}\right) \le \frac{1}{2^j}.$$

Set $A_j := \left\{x \in \Omega \mid \|u_{k_j} - u\|_{L^0(B_1)} \le \frac{1}{j}\right\}$, define $B_i := \bigcap_{j \ge i} A_j$ and finally $B := \bigcup_{i=1}^{\infty} B_i$. Suppose $x \in B$, then $x \in B_i$ for some i and hence $x \in A_j$ for every $j \ge i$. Therefore

$$\|u_{k_j}(x) - u(x)\|_{L^0(B_1)} \le \frac{1}{j}, \text{ for } j \ge i,$$

which means

$$\lim_{j\to\infty} \|u_{k_j}(x) - u(x)\|_{L^0(B_1)} = 0.$$

Finally, we can use the monotonicity and the countable sub-additivity of μ to estimate

$$\mu(\Omega \setminus B) \le \mu(\Omega \setminus B_i) \le \sum_{j \ge i} \mu(A_j) \le \frac{1}{2^{i-1}},$$

and by the arbitrariness of i we deduce $\mu(\Omega \setminus B) = 0$. □

The convergence in μ-measure can be metrized in the following way.

Proposition 5.9. *Let μ be an outer measure on Ω such that $\mu(\Omega) < +\infty$, and let $u, v \in U_\mu(\Omega; L^0(B_1))$. The metric $d(u, v)$ defined by*

$$d(u, v) := \inf_{\delta > 0} \mu(\{\|u - v\|_{L^0(B_1)} > \delta\}) + \delta,$$

induces the convergence in measure (5.5), and it gives to $U_\mu(\Omega; L^0(B_1))$ the structure of a complete metric space.

Proof. We start by proving that $d(\cdot, \cdot)$ is a metric.

First of all suppose that $d(u, v) = 0$, then we want to prove that $\mu(\{\|u - v\|_{L^0(B_1)} > 0\}) = 0$. Indeed, if $d(u, v) = 0$, then for every $\delta > 0$ $\mu(\{\|u - v\|_{L^0(B_1)} > \delta\}) = 0$. Since $\{\|u - v\|_{L^0(B_1)} > 0\} = \bigcup_{k=1}^{\infty} \{\|u - v\|_{L^0(B_1)} > 1/k\}$, we can conclude $\mu(\{\|u - v\|_{L^0(B_1)} > 0\}) = 0$ simply by the sub-additivity of μ.

The equality $d(u, v) = d(v, u)$ is obvious.

Finally we need to prove the triangular inequality. For this purpose notice that for every triple of functions $u, v, g \colon \Omega \to L^0(B_1)$ it holds

$$\{\|u - v\|_{L^0(B_1)} > \delta_1 + \delta_2\} \subset \{\|u - g\|_{L^0(B_1)} > \delta_1\} \cup \{\|g - v\|_{L^0(B_1)} > \delta_2\}.$$

Given $\epsilon > 0$, let δ_1 and δ_2 be positive real numbers such that

$$d(u, g) + \epsilon \ge \mu(\{\|u - g\|_{L^0(B_1)} > \delta_1\}) + \delta_1, \quad d(g, v) + \epsilon \ge \mu(\{\|g - v\|_{L^0(B_1)} > \delta_2\}) + \delta_2.$$

Then

$$\begin{aligned} d(u, v) &= \inf_{\delta > 0} \mu(\{\|u - v\|_{L^0(B_1)} > \delta\}) + \delta \\ &\le \mu(\{\|u - v\|_{L^0(B_1)} > \delta_1 + \delta_2\}) + \delta_1 + \delta_2 \\ &\le [\mu(\{\|u - g\|_{L^0(B_1)} > \delta_1\}) + \delta_1] + [\mu(\{\|g - v\|_{L^0(B_1)} > \delta_2\}) + \delta_2] \\ &\le d(u, g) + d(g, v) + 2\epsilon, \end{aligned}$$

and letting $\epsilon \to 0+$ this implies the triangular inequality.

Given $(u_k)_{k=1}^{\infty} \subset U_\mu(\Omega; L^0(B_1))$ and $u \in U_\mu(\Omega; L^0(B_1))$, we claim that $\lim_{k\to\infty} d(u_k, u) = 0$ if and only if u_k converge to u in μ-measure. Let us first suppose $\lim_{k\to\infty} d(u_k, u) = 0$. Then by definition of $d(\cdot, \cdot)$, it turns out that for every k there exist $\delta_k \searrow 0$ such that

$$\lim_{k\to\infty} \mu(\{\|u_k - u\|_{L^0(B_1)} > \delta_k\}) = 0.$$

Hence, given $\epsilon > 0$ we can find $\overline{k}$ big enough such that for every $k \ge \overline{k}$ $\{\|u_k - u\|_{L^0(B_1)} > \epsilon\} \subset \{\|u_k - u\|_{L^0(B_1)} > \delta_k\}$, which implies

$$\lim_{k\to\infty} \mu(\{\|u_k - u\|_{L^0(B_1)} > \epsilon\}) \le \lim_{k\to\infty} \mu(\{\|u_k - u\|_{L^0(B_1)} > \delta_k\}) = 0.$$

This gives the convergence in μ-measure.

Now suppose that u_k converge to u in μ-measure. Then we can write for every $\epsilon > 0$

$$\lim_{k\to\infty} \inf_{\delta > 0} \mu(\{\|u_k - u\|_{L^0(B_1)} > \delta\}) + \delta \le \lim_{k\to\infty} \mu(\{\|u_k - u\|_{L^0(B_1)} > \epsilon\}) + \epsilon = \epsilon,$$

which immediately implies $\lim_{k\to\infty} d(u_k, u) = 0$.

Finally, we have to prove that $U_\mu(\Omega; L^0(B_1))$ endowed with the metric $d(\cdot,\cdot)$ is complete. For this purpose, suppose that the sequence $(u_k)_{k=1}^\infty$ is Cauchy. Given a sequence $(\lambda_j)_j$ of positive real numbers such that $\sum_{j=1}^\infty \lambda_j < \infty$, there exists a subsequence $(k_j)_j$ such that

$$d(u_{k_{j_1}}, u_{k_{j_2}}) \le \lambda_j, \text{ for every } j_1, j_2 \ge j,$$

which means that or every j there exists $0 < \delta_j \le \lambda_j$ such that (without loss of generality we may also suppose $\delta_j \searrow 0$)

$$\mu(\{\|u_{k_j} - u_{k_{j+1}}\|_{L^0(B_1)} > \delta_j\}) + \delta_j \le \lambda_j. \tag{5.6}$$

Define $A_j := \{\|u_{k_j} - u_{k_{j+1}}\|_{L^0(B_1)} > \delta_j\}$ and set $B_j := \bigcup_{m\ge j+1} A_m$. We claim that u_{k_j} converge pointwise for every $x \in \Omega \setminus \bigcap_{j=1}^\infty B_j$. Indeed, if $x \in \Omega \setminus \bigcap_{j=1}^\infty B_j$ then there exists $\bar{j}$ such that $x \notin B_{\bar{j}}$, hence by the definition of $B_{\bar{j}}$ this implies $x \notin A_j$ for every $j \ge \bar{j}+1$. For this reason we have

$$\|u_{k_j}(x) - u_{k_{j+1}}(x)\|_{L^0(B_1)} \le \delta_j, \text{ for every } j \ge \bar{j}+1,$$

and this immediately implies that $(u_{k_j}(x))_j$ is a Cauchy sequence in $L^0(B_1)$. By the completeness of $L^0(B_1)$ we deduce that there exists a function $u\colon \Omega \setminus \bigcap_{j=1}^\infty B_j \to L^0(B_1)$ such that

$$\lim_{j\to\infty} \|u_{k_j}(x) - u(x)\|_{L^0(B_1)} = 0.$$

Since by the monotonicity of μ we have

$$\mu\left(\bigcap_{j=1}^\infty B_j\right) \le \lim_{j\to\infty} \sum_{m\ge j} \mu(A_m) \le \lim_{j\to\infty} \sum_{m\ge j} \lambda_m = 0,$$

we deduce that the function u is a well defined element of $U_\mu(\Omega; L^0(B_1))$.

We claim that the subsequence $(u_{k_j})_j$ converges in μ-measure to u. Indeed given any $\epsilon > 0$ we have

$$\{\|u_{k_j} - u\|_{L^0(B_1)} > \epsilon\} \subset \{\|u_{k_j} - u\|_{L^0(B_1)} > \delta_j\}$$

for every j big enough such that $\delta_j \le \epsilon$. This means that

$$\lim_{j\to\infty} \mu(\{\|u_{k_j} - u\|_{L^0(B_1)} > \epsilon\}) \le \lim_{j\to\infty} \mu(\{\|u_{k_j} - u\|_{L^0(B_1)} > \delta_j\}).$$

By using (5.6) we can deduce

$$\begin{aligned}
\mu(\{\|u_{k_j} - u\|_{L^0(B_1)} > \delta_j\}) &\le \sum_{m=j}^\infty \mu(\{\|u_{k_m} - u_{k_{m+1}}\|_{L^0(B_1)} > \delta_j\}) \\
&\le \sum_{m=j}^\infty \mu(\{\|u_{k_m} - u_{k_{m+1}}\|_{L^0(B_1)} > \delta_m\}) \\
&\le \sum_{m=j}^\infty \lambda_m,
\end{aligned}$$

which by the fact $\sum_{j=1}^\infty \lambda_j < \infty$ implies our claim. Since we already know that the convergence in μ-measure implies the convergence in the metric $d(\cdot,\cdot)$, we can write

$$\lim_{j\to\infty} d(u_{k_j}, u) = 0.$$

This together with the fact that the sequence $(u_k)_k$ is Cauchy in the metric $d(\cdot,\cdot)$, easily implies that the full sequence satisfies

$$\lim_{k\to\infty} d(u_k, u) = 0,$$

and we are done.

□

Remark 5.10. The space $U_\mu(\Omega; L^0(B_1))$ equipped with the distance defined in the previous proposition is actually a Fréchet space.

5.2. **The outer measure C_p.** Let us start with the definition of capacity.

Definition 5.11 (p-Capacity)**.** Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. We define the p-Capacity $(1 < p \leq n)$ of a set $A \subset \Omega$ as

$$C_p(A) := \inf \left\{ \int_\Omega (|\nabla u|^p + |u|^p)\, dx \mid u \in GSBV(\Omega;\Gamma),\, u^+(x) \geq 1 \text{ on } A \right\}, \tag{5.7}$$

where $u^+(x)$ is the upper approximate limit defined in 1.14.

Remark 5.12. In (5.7) we consider also the L^p-norm of the function, while in (5.1) only the L^p-norm of the gradient is present. This is simply because we want to avoid that functions u belonging to the kernel of ∇ could trivialise the infimum in (5.7). We remember that the kernel of the approximate gradient of $GSBV(\Omega;\Gamma)$ functions is made up of piecewise constant functions whose jump sets are contained in a Caccioppoli's partition subordinated to Γ. This result can be found for example in [2] for SBV functions; the case $GSBV$ can be easily recoverd by a truncation argument. For example, with this choice the scaling property $C_p(\lambda A) = \lambda^{n-p} C_p(A)$ (see [7, Section 4.7.1]) is lost. Anyway, we do not need this property to develop our theory.

Proposition 5.13. *For every set $A \subset \Omega$ we have*

$$C_p(A) = \inf \left\{ \int_\Omega (|\nabla u|^p + |u|^p)\, dx \mid u \in GSBV(\Omega;\Gamma),\, u^+(x) \geq 1 \text{ on } A,\, 0 \leq u \leq 1 \right\}.$$

Proof. Let $u_0^1 := (u \wedge 1) \vee 0$. Since $u^+(x) \geq 1$ if and only $u_0^{1+}(x) \geq 1$, it is enough to notice that if $u \in GSBV^p(\Omega;\Gamma)$ then

$$\int_\Omega |\nabla u_0^1|^p + |u_0^1|^p\, dx \leq \int_\Omega |\nabla u|^p + |u|^p\, dx,$$

and this concludes the proof. □

Proposition 5.14. *$C_p(\cdot)$ is an outer measure on Ω.*

Proof. Clearly $C_p(\cdot)$ is monotone and $C_p(\emptyset) = 0$. Hence we need only to prove the countable sub-additivity.

Let $(A_k)_{k=1}^\infty$ be a countable family of subsets of Ω and define $A := \bigcup_{k=1}^\infty A_k$. Without loss of generality we can assume $\sum_k C_p(A_k) < \infty$. For each k we can find $u_k \in GSBV^p(\Omega;\Gamma)$, $0 \leq u_k \leq 1$, and $u_k^+(x) \geq 1$ on A_k such that

$$\int_\Omega |\nabla u_k|^p + |u_k|^p\, dx \leq C_p(A_k) + \frac{\epsilon}{2^k}.$$

We define $u := \sup_{k \in \mathbb{N}} u_k$, and we claim that $u \in GSBV^p(\Omega;\Gamma)$ and $u^+(x) \geq 1$ on A. Indeed, since the u_k are bounded functions in $GSBV_p^p(\Omega;\Gamma)$, we have $u_k \in SBV(\Omega)$. Therefore by using the chain rule in BV [1, Theorem 3.99], if we set $u_m := \sup_{1 \leq k \leq m} u_k$, we have

$$\int_\Omega |\nabla u_m|^p\, dx \leq \sum_{k=1}^m \int_\Omega |\nabla u_k|^p\, dx,$$

hence

$$\sup_m \int_\Omega |\nabla u_m|^p + |u_m|^p\, dx \leq \sum_{k=1}^\infty C_p(A_k) + \frac{\epsilon}{2^k}. \tag{5.8}$$

Thanks to (5.8) we can use the compactness result [1, Theorem 4.36] for $GSBV(\Omega)$ together with [4, Remark 2.9] to deduce that $u \in GSBV_p^p(\Omega;\Gamma)$ and moreover

$$u_m \to u \text{ strongly in } L^1(\Omega) \qquad \nabla u_m \rightharpoonup \nabla u \text{ weakly in } L^1(\Omega). \tag{5.9}$$

Moreover, if $x \in A$ then $x \in A_k$ for some k, therefore $u_k^+(x) \geq 1$, and since $u \geq u_k$ for every k, we deduce $u^+(x) \geq u_k^+(x)$. Therefore

$$A \subset \{x \in \Omega \mid u^+(x) \geq 1\}.$$

By using the lower semicontinuity of the L^p-norm with respect to the convergence (5.9), we have

$$C_p(A) \le \int_\Omega |\nabla u|^p + |u|^p\, dx \le \liminf_{m\to\infty} \int_\Omega |\nabla u_m|^p + |u_m|^p\, dx \le \sum_{k=1}^{\infty} C_p(A_k) + \epsilon,$$

which implies the countable sub-additivity of $C_p(\cdot)$ thanks to the arbitrariness of ϵ. □

5.3. **Relations between C_p and $\mathcal{H}^{n-p}$.** In this subsection we derive the relation between C_p and $\mathcal{H}^{n-p}$. Let us notice that Proposition 5.15 and property 2 of Theorem 5.16 are obtained mainly as in the Sobolev case, and do not depend on the fact $\Gamma \in \mathcal{J}_p$, while property 1 of Theorem 5.16 strongly relies on the validity of Theorem 1, i.e. on $\Gamma \in \mathcal{J}_p$.

Proposition 5.15. *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. For every $1 < p \le n$ there exists a constant $c = c(n,p) > 0$ such that for every $A \subset \Omega$*

$$C_p(A) \le c\,\mathcal{H}^{n-p}(A).$$

Proof. First of all if $B_r(x) \subset \Omega$, then $C_p(B_r(x))$ can be rewritten as

$$\inf\left\{\int_\Omega (|\nabla v|^p + |v|^p)\, dy \mid v(y) = u\left(\frac{x-y}{r}\right),\ u \in GSBV(\Omega';\Gamma'),\ u^+(x) \ge 1 \text{ on } B_1(0)\right\},$$

where $\Omega' = (\Omega - x)/r$ and $\Gamma' = (\Gamma - x)/r$.

Notice that for $r \le 1$ we have

$$\int_\Omega (|\nabla v|^p + |v|^p)\, dy = r^n \int_{\Omega'} (r^{-p}|\nabla u|^p + |u|^p)\, dy \le r^{n-p} \int_{\Omega'} (|\nabla u|^p + |u|^p)\, dy.$$

Hence, by choosing $u(x) := \mathrm{dist}(x, \mathbb{R}^n \setminus B_2(0)) \wedge 1$ whenever $x \in \Omega'$, it follows

$$C_p(B_r(x)) \le 2^{n+1}\omega_n r^{n-p} \quad (r \le 1).$$

Let $(C_i)_{i=1}^\infty$ be a family of sets contained in Ω which is a cover of A and $\mathrm{diam} C_i \le 1$. For each i there exists a ball $B_{r_i}(x_i)$ such that $C_i \subset B_{r_i}(x_i)$ and $r_i = \mathrm{diam}(C_i)$. Therefore

$$C_p(A) \le \sum_{i=1}^\infty C_p(C_i) \le \sum_{i=1}^\infty C_p(B_{r_i}(x_i)) \le 2^{n+1}\omega_n \sum_{i=1}^\infty r_i^{n-p} \le 2^{2n+1-p}\omega_n \sum_{i=1}^\infty \left(\frac{\mathrm{diam}\, C_i}{2}\right)^{n-p}.$$

Hence, if we set $c := 2^{2n+1-p}\omega_n$ then

$$C_p(A) \le c\,\mathcal{H}^{n-p}(A).$$

□

Whenever $u\colon \Omega \to \mathbb{R}$ is such that u_x is a piecewise constant function of the form of Theorem 4.7, then by definition of upper approximate limit (Definition 1.14), it is easy to see that

$$u^+(x) = \max_{1 \le j \le N_x} m_j(u,x). \tag{5.10}$$

We shall use this simple observation to deduce more precise relations between p-capacity and Hausdorff measure.

Theorem 5.16. *Let Ω be an open set of $\mathbb{R}^n$ and let $\Gamma \subset \Omega$ be a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set with $\mathcal{H}^{n-1}(\Gamma) < \infty$. Then for every $A \subset \Omega$ and for every $1 < p \le n$ we have*

(1) $C_p(A) = 0$ and $\Gamma \in \mathcal{J}_p$ imply $\dim_{\mathcal{H}}(A) \le n - p$;
(2) $\mathcal{H}^{n-p}(A) < \infty$ implies $C_p(A) = 0$.

Proof. Suppose $C_p(A) = 0$ and $\Gamma \in \mathcal{J}_p$. By hypothesis we can find a sequence $(u_k)_{k=1}^\infty \subset GSBV^p(\Omega;\Gamma)$, $0 \le u_k \le 1$, such that

(i) $u_k^+(x) \ge 1$, for every $x \in A$;
(ii) $\int_\Omega (|\nabla u_k|^p + |u_k|^p)\, dx \le \frac{1}{k^2}$, for every $k \in \mathbb{N}$.

Define $u := \sum_{k=1}^{\infty} u_k$. Since by Remark 1.22 $GSBV^p_p(\Omega;\Gamma)$ is a Banach space, by (ii) we deduce that $u \in GSBV^p_p(\Omega;\Gamma)$. Thanks to Theorem 4.7, if we call S_k the set of $x \in \Omega$ where the blow-up of u_k does not exist, then $\dim_{\mathcal{H}}(S_k) \le n-p$. By setting $S := \bigcup_{k=1}^{\infty} S_k$ clearly

$$\dim_{\mathcal{H}}(S) \le n-p. \tag{5.11}$$

Property (i) above together with (5.10) imply that for every k and for every $x \in A \setminus S$ the blow up of u_k at x is of the form

$$(u_k)_x = \sum_{j=1}^{N_x} m_j(u_k,x)\mathbb{1}_{E_{0,j}}, \quad \text{and} \quad \max_{1\le j\le N_x} m_j(u_k,x) \ge 1. \tag{5.12}$$

Since $u_k \ge 0$ for every k, we have $u(y) \ge \sum_{k=1}^{M} u_k(y)$ for every $M \in \mathbb{N}$ and every $y \in B_1(0)$. For this reason, by using the linearity of the blow-up and again (5.10), we have

$$u^+(x) \ge \Big(\sum_{k=1}^{M} u_k(x)\Big)^+ = \max_{1\le j\le N_x} \Big[\sum_{k=1}^{M} m_j(u_k,x)\Big].$$

By letting $M \to \infty$, thanks to (5.12), we deduce that $A \setminus S$ is contained in the set of point x where $u^+(x) = +\infty$. By Theorem 4.7 together with observation (5.10) we deduce that

$$\dim_{\mathcal{H}}(A \setminus S) \le n-p,$$

which together with (5.11) is exactly (1).

To prove 2 we follow the proof given in [9, Section 3]. Suppose $\mathcal{H}^{n-p}(A) < 2^{p-n}\gamma < \infty$ for some $\gamma > 0$. By denoting $\mathcal{S}^{n-p}$ the $(n-p)$-dimensional spherical measure (see [8, Paragraph 2.10.2]), we have

$$\mathcal{S}^{n-p}(A) \le 2^{n-p}\mathcal{H}^{n-p}(A) < \gamma.$$

We claim that for every $m \in \mathbb{N}$ we can find an open set V_m and a function $u_m \in W^{1,p}(\Omega)$ such that

(a) $A \subset V_m = \bigcup_{i=1}^{\infty} B_{r_i}(x_i)$, $\sup_i r_i^p \le (m+1)^{-p}\big(\sum_{k=1}^{m+1} k^{-1}\big)^{-p}$;
(b) $B_{2r_i}(x_i) \subset V_{m-1}$ ($V_m \subset V_{m-1}$);
(c) $u_m^+(x) = 1$ on V_m, $\operatorname{spt}(Du_m) \subset V_{m-1} \setminus V_m$, $\int_\Omega |Du_m|^p\,dx \le c\,\gamma$,

where $c := c(n,p) > 0$.

We start by setting $V_0 := \Omega$ and $u_0 := 1$. Set $\delta_{m+1} := (m+1)^{-1}\big(\sum_{k=1}^{m+1} k^{-1}\big)^{-1}$. To define V_m and u_m, by using

$$\sum_{i=1}^{\infty} \mathcal{S}^{n-p}(A \cap \{x \mid 2^i < \operatorname{dist}(x, V_{m-1}) \le 2^{i-1}\}) < \gamma,$$

we can find a sequence of balls $(B_{r_i}(x_i))_{i=1}^{\infty}$ such that $B_{2r_i}(x_i) \subset V_{m-1}$, $\sup_i r_i \le \delta_m$, and

$$A \subset V_m := \bigcup_{i=1}^{\infty} B_{r_i}(x) \quad \text{and} \quad \sum_{i=1}^{\infty} \omega_{n-p} r_i^{n-p} \le \gamma.$$

Define $h_i \in W^{1,p}(\Omega)$ as

$$h_i(x) = 1 \text{ if } |x-x_i| \le r_i, \quad h_i(x) = 0, \quad \text{if} \quad |x-x_i| \ge 2r_i,$$
$$h_i(x) = 2 - |x-x_i|/r_i \quad \text{if} \quad r_i < |x-x_i| < 2r_i.$$

Since $\int_\Omega |Dh_i|^p\,dx = r_i^{-p}\omega_n[(2r_i)^n - r_i^n] = c\,\omega_{n-p} r_i^{n-p}$, if we define $u_m := \sup_{i=1}^{\infty} h_i$, then

$$\int_\Omega |Du_m|^p\,dx \le \int_\Omega \sum_{i=1}^{\infty} |Dh_i|^p\,dx \le c\,\gamma.$$

In this way (a),(b) and (c) are satisfied.

Define $u := \sum_{k=1}^{\infty} k^{-1}u_k$. Since by construction $\operatorname{spt}(Du_m) \subset V_{m-1}\setminus V_m$ we have $|\operatorname{spt}(Du_m) \cap \operatorname{spt}(Du_{m+1})| = 0$ for every $m \in \mathbb{N}$. Therefore we can write

$$\int_\Omega |Du|^p\,dx = \int_\Omega \sum_{k=1}^{\infty} k^{-p}|Du_k|^p\,dx \le c\,\gamma\,\lambda,$$

where $\lambda := \sum_{k=1}^{\infty} k^{-p}$. By using

$$u(x) \leq \sum_{k=1}^{m} k^{-1} \text{ if } x \in V_{m-1} \setminus V_m,$$

we can estimate

$$\int_\Omega |u|^p\,dx = \int_\Omega \left|\sum_{k=1}^{\infty} k^{-1}u_k\right|^p dx \leq \sum_{m=1}^{\infty} \int_{V_{m-1}\setminus V_m} \left(\sum_{k=1}^{m} k^{-1}\right)^p dx \leq \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} k^{-1}\right)^p |V_{m-1}|. \tag{5.13}$$

If we call $(B_{r_i}(x))$ the balls relative to V_{m-1}, i.e. $V_{m-1} = \bigcup_{i=1}^{\infty} B_{r_i}(x)$, then

$$\left(\sum_{k=1}^{m} k^{-1}\right)^p |V_{m-1}| \leq \left(\sum_{k=1}^{m} k^{-1}\right)^p \sum_{i=1}^{\infty} \omega_n r_i^n \leq \frac{\omega_n}{\omega_{n-p}} \sum_{i=1}^{\infty} m^{-p}\omega_{n-p} r_i^{n-p}.$$

Therefore we can continue inequality (5.13) in the following way

$$\int_\Omega |u|^p\,dx \leq \frac{\omega_n}{\omega_{n-p}} \sum_{m,i=1}^{\infty} m^{-p}\,\omega_{n-p} r_i^{n-p} \leq \frac{\omega_n}{\omega_{n-p}} \sum_{m=1}^{\infty} m^{-p}\gamma = \frac{\omega_n}{\omega_{n-p}}\gamma\,\lambda.$$

We claim that

$$u^+(x) \geq \sum_{k=1}^{m} k^{-1}, \quad x \in V_m. \tag{5.14}$$

To prove (5.14) it is sufficient to show that for every $t < \sum_{k=1}^{m} k^{-1}$ the superlevel $\{u > t\}$ has strictly positive density at every $x \in V_m$.

Using the fact that $V_m \subset V_{m-1}$ together with property (c), we have that

$$u_k^+(x) \geq 1, \quad 1 \leq k \leq m, \quad x \in V_m. \tag{5.15}$$

Hence, by choosing any $t < \sum_{k=1}^{m} k^{-1}$, since

$$\bigcap_{k=1}^{m} \{u_k > (1-\delta)\} \subset \{u > t\},$$

for any $0 < \delta < 1$ such that $\sum_{k=1}^{m}(1-\delta)k^{-1} = t$, and since by (5.15) each set $\{u_k > (1-\delta)\}$ has density greater or equal than one at $x \in V_m$, we deduce that $\{u > t\}$ has strictly positive density at every $x \in V_m$.

For this reason, by definition of p-capacity it immediately follows

$$C_p(V_m) \leq \left(\sum_{k=1}^{m} k^{-1}\right)^{-p} \int_\Omega (|\nabla u|^p + |u|^p)\,dx \leq \left(\sum_{k=1}^{m} k^{-1}\right)^{-p} c'\,(\gamma\,\lambda).$$

Sending $m \to \infty$ in the previous inequality we deduce $C_p(A) = 0$.

□

5.4. **The main result.** Let $1 < p \leq n$ and $\Gamma \in \mathcal{J}_p$. Given $u \in GSBV^p(\Omega;\Gamma)$, by Theorem 4.7 we know that $u_x \in L^0(B_1)$ is well defined for every $x \in \Omega$ up to a singular set of Hausdorff dimension at most $n-p$. If we call S such a singular set, this means that for every $1 < q < p$ we have $\mathcal{H}^{n-q}(S) = 0$, and by Proposition 5.15 also $C_q(S) = 0$. Therefore, for every $1 < q < p$, u_x is a well defined element in the Fréchet space $U_{C_q}(\Omega; L^0(B_1))$ (see Definition 5.5). Unfortunately, we can not conclude the same for $q = p$. For this reason we need to introduce a further outer measure.

Definition 5.17 (Lower p-capacity)**.** Let $\Omega \subset \mathbb{R}^n$ be open, and let $\Gamma \in \mathcal{J}_p$ $(1 < p \leq n)$. Given any set $A \subset \Omega$ we define the *lower p-capacity* as

$$C_p^-(A) := \sup_{1<q<p} C_q(A). \tag{5.16}$$

Proposition 5.18. *$C_p^-(\cdot)$ is an outer measure. In addition,*

$$C_p^-(A) = 0 \quad \textit{iff} \quad C_q(A) = 0 \textit{ for every } 1 < q < p. \tag{5.17}$$

Proof. $C_p^-(\cdot)$ is an outer measure simply because it is obtained as supremum of a family of outer measures. Property (5.17) follows by construction. □

Proposition 5.19. *Let $\Gamma \in \mathcal{J}_p$ $(1 < p \le n)$, then for every $u \in GSBV^p(\Omega;\Gamma)$ we have that u_x is a well defined element in $U_{C_p^-}(\Omega; L^0(B_1))$.*

Proof. By Theorem 4.7 we know that u_x exists everywhere except on a singular set S of Hausdorff dimension at most $n-p$. This means that if we call S the set of points where the blow-up $u_{r,x}$ does not converge, then for every $\delta > 0$ $\mathcal{H}^{n-p+\delta}(S) = 0$. As a consequence by Proposition 5.15 this means also $C_{p-\delta}(S) = 0$. Finally, relation (5.17) gives the conclusion of the theorem. □

Proposition 5.20 (Capacitary Chebyshev's inequality)**.** *Let Ω be a bounded open set of $\mathbb{R}^n$ and let $\Gamma \in \mathcal{J}_p$ with $1 < p \le n$. Then for every $\epsilon > 0$ and for every $1 < q < p$ it holds*

$$C_q\big(\{x \in \Omega \mid \|u_x\|_{L^0(B_1)} > \omega_n \epsilon\}\big) \le \frac{1}{\epsilon^q}\int_\Omega (|\nabla u|^q + |u|^q)\,dx, \tag{5.18}$$

for every $u \in GSBV^p(\Omega;\Gamma)$.

Proof. Renormalizing by ϵ, in order to prove (5.18) it is enough to show the following inclusion

$$\{x \in \Omega \mid \|u_x\|_{L^0(B_1)} > \omega_n\} \subset \{x \in \Omega \mid |u|^+(x) \ge 1\}, \tag{5.19}$$

up to a set of zero C_q-capacity.

By Theorem 4.7 together with Theorem 5.15, we know that except on a C_q-negligible set we have

$$|u|_x(y) = \sum_{j=1}^{N_x} m_j(|u|,x)\mathbb{1}_{E_{0,j}}(y), \;\; y \in B_1(0).$$

Using (5.10) we know that $|u|^+(x) \ge 1$ if and only if at least one of the $(m_j(|u|,x))_{j=1}^{N_x}$ is greater or equal than one.

Now suppose by contradiction that $\max_{1\le j\le N_x} m_j(|u|,x) < 1$ and $\|u_x\|_{L^0(B_1)} > \omega_n$. Then

$$\begin{aligned}\|u_x\|_{L^0(B_1)} &= \Big\|\sum_{j=1}^{N_x} m_j(|u|,x)\mathbb{1}_{F_{r,j}}\Big\|_{L^0(B_1)}\\ &= \sum_{j=1}^{N_x}\int_{F_{r,j}} |m_j(|u|,x)| \wedge 1\,dy\\ &\le \omega_n,\end{aligned}$$

which immediately implies (5.19) and hence the proposition. □

Theorem 5.21. *Let Ω be a bounded open set of $\mathbb{R}^n$ and let $\Gamma \in \mathcal{J}_p$ with $1 < p \le n$. Suppose that $(u_k)_{k=1}^\infty \subset GSBV_p^p(\Omega;\Gamma)$ is such that*

$$\|u_k - u\|_{L^p} + \|\nabla u_k - \nabla u\|_{L^p} \to 0, \;\; \textit{as } k \to \infty.$$

Then $(u_k)_x$ converge to u_x in the Fréchet space $U_{C_p^-}(\Omega; L^0(B_1))$.

Proof. We shall prove that given $\epsilon, \delta > 0$, then there exists $\overline{k}$ such that for every $k \ge \overline{k}$

$$C_p^-\big(\{x \in \Omega \mid \|(u_k)_x - u_x\|_{L^0(B_1)} > \omega_n\epsilon\}\big) \le \delta.$$

Thanks to Theorem 4.7 there exists a set S with $\dim_H(S) \le n-p$ such that $(u_k)_x$ and u_x exist for every $x \in \Omega \setminus S$ and for every $k \in \mathbb{N}$. Moreover by Theorem 5.16 we know that $C_q(S) = 0$ for every $1 < q < p$, which by Proposition 5.18 implies $C_p^-(S) = 0$. Therefore, since by linearity we have for every $x \in \Omega \setminus S$ the relation $(u_k)_x - u_x = (u_k - u)_x$, by using the capacitary Chebyshev's inequality for every $1 < q < p$ we get

$$C_q\big(\{x \in \Omega \mid \|(u_k)_x - u_x\|_{L^0(B_1)} > \omega_n\epsilon\}\big) \le \frac{1}{\epsilon^q}\int_\Omega (|\nabla u_k - \nabla u|^q + |u_k - u|^q)\,dx.$$

Finally, by the definition of C_p^- it is enough to choose $\overline{k}$ big enough such that for every $k \geq \overline{k}$

$$\sup_{1<q<p} \frac{1}{\epsilon^q} \int_\Omega (|\nabla u_k - \nabla u|^q + |u_k - u|^q)\, dx \leq \delta,$$

which is possible since Ω is bounded. □

Putting together Theorems 5.16, Theorem 5.21, and relation (5.17) we are able to prove the second main result of this paper.

Proof of Theorem 2. Let us first suppose Ω bounded. Putting together the previous result with Theorem 5.8 we have that there exists a subsequence $(k_j)_j$ such that

$$\lim_{j\to\infty} \|(u_{k_j})_x - u_x\|_{L^0(B_1)} = 0,$$

for every $x \in \Omega$ except on a C_p^--negligible set S. Putting together Theorem 5.16 with relation (5.17) it easily follows $\dim_{\mathcal{H}}(S) \leq n - p$.

For general Ω, we set $\Omega_i := \Omega \cap B_i(0)$ $(i \in \mathbb{N})$. For every i we can apply the previous result on the bounded open set Ω_i to obtain a sequence $(k_j^i)_{j=1}^\infty$ and a set $S_i \subset \Omega_i$ with $\dim_{\mathcal{H}}(S_i) \leq n-p$, such that

$$\lim_{j\to\infty} \|(u_{k_j^i})_x - u_x\|_{L^0(B_1)} = 0, \text{ for every } x \in \Omega_i \setminus S_i.$$

We can also suppose that $(k_j^{i+1})_{j=1}^\infty$ is a subsequence of $(k_j^i)_{j=1}^\infty$ for every i. By a diagonal argument we define for every $j \in \mathbb{N}$ $k_j := k_j^j$, and we obtain that

$$\lim_{j\to\infty} \|(u_{k_j})_x - u_x\|_{L^0(B_1)} = 0, \text{ for every } x \in \Omega \setminus \bigcup_{i=1}^\infty S_i.$$

Finally, since every S_i has Hausdorff dimension which does not exceed $n-p$, then also $\bigcup_{i=1}^\infty S_i$ has Hausdorff dimension which does not exceed $n-p$. This proves the theorem. □

Remark 5.22. In [5] the authors are able to prove a density result for the space $SBV^p(\Omega)$. More precisely, if Ω is an open set with Lipschitz boundary and $u \in SBV^p(\Omega)$, then there exists a sequence of functions $u_j \in SBV^p(\Omega)$ and of compact C^1 manifolds with C^1 boundary $M_j \subset\subset \Omega$ such that $J_{u_j} \subseteq M_j$ but $\mathcal{H}^{n-1}(M_j \setminus J_{u_j}) = 0$ and

$$u_j \in C^\infty(\Omega \setminus \overline{J_{u_j}}), \quad \|u_j - u\|_{L^1} \to 0, \quad \|\nabla u_j - \nabla u\|_{L^p} \to 0, \quad \mathcal{H}^{n-1}(J_{u_j} \Delta J_u) \to 0.$$

It is natural to ask whether the hypothesis $\mathcal{H}^{n-1}(J_{u_j}\Delta J_u) \to 0$ can be improved to

$$J_{u_j} \subset J_u \text{ for every } j \in \mathbb{N}.$$

In other words we can rephrase this question in the following way: given $\Gamma \subset \Omega$ a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set, then is it true that the closure in SBV^p with respect to the norm given by $\|\nabla u\|_{L^p} + \|u\|_{L^1}$ of all functions v such that

$$v \in C^\infty(\Omega \setminus \overline{J_u}), \quad J_u \subset M \cap \Gamma, \quad M \text{ is any } C^1 \text{ manifolds with } C^1 \text{ boundary}, \tag{5.20}$$

is the whole of $SBV^p(\Omega;\Gamma) \cap L^1(\Omega)$?

The answer is in general no. Consider $\Gamma_0 \subset \mathbb{R}^2$ the union of three half lines starting from the origin. Let $\Gamma \subset \mathbb{R}^3$ be defined by $\Gamma_0 \times \mathbb{R}$ and let l be the straight line $\{(0,0,t) \mid t \in \mathbb{R}\}$. The set Γ disconnects $\mathbb{R}^3 \setminus \Gamma$ into three connected components $\Omega_1, \Omega_2, \Omega_3$. Let $v \colon \mathbb{R}^3 \to \mathbb{R}$ be the function which assumes three different constant values on each of the connected components, say $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_1$. Clearly $v \in SBV^p(\Omega;\Gamma)$ for every $p \in [1,3)$. We claim that for $p \in (2,3)$, the function v cannot be approximated in SBV^p by functions satisfying (5.20). Indeed, any function $u \in SBV^p(\Omega)$ satisfying (5.20) has the property that v_x is defined everywhere, except on a $(3-p)$-dimensional Hausdorff set, and it is a function taking at most two values. By using a slightly modified version of Theorem 2 (where we have to substitute the L^p convergence of the functions with the L^1 convergence), we deduce that any limit u in $SBV^p(\Omega;\Gamma)$ of functions satisfying (5.20), inherits the property that its blow-up converges to a function u_x which takes at most two values for every x except on a set of Hausdorff dimension $3-p$. However for every point $x \in l$, v_x assumes three different values, namely $\alpha_1, \alpha_2, \alpha_3$. Since $\dim_{\mathcal{H}}(l) = 1$, this implies that for every $p \in (2,3)$, v cannot be approximated by functions satisfying (5.20).

6. More on the class $\mathcal{J}_p$

We dedicate this section to construct sets living in $\mathcal{J}_p$. In the second part we present a counterexample to Theorem 1.

6.1. **Some examples.** Let $n \geq 3$ and $1 < p \leq n-1$. We write the generic point $x \in \mathbb{R}^n$ as $x = (y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We define $\mathcal{W}^{1,p}(\mathbb{R}^{n-1})$ as the space of all Sobolev functions $f \in W^{1,p}(\mathbb{R}^{n-1})$ such that for every $y \in \mathbb{R}^{n-1}$ except on a set of Hausdorff dimension $n-1-p$, y is a Lebesgue point for the distributional gradient Df [1].

Now let $f \in \mathcal{W}^{1,p}(\mathbb{R}^{n-1})$ and consider its sub-graph

$$S_f^- := \left\{ x \in \mathbb{R}^n \mid t < f(y),\, y \in \mathbb{R}^{n-1} \right\}.$$

It is well known that S_f^- is a set having locally finite perimeter in $\mathbb{R}^n$.

Consider the following sets

$$A := \left\{ y \in \mathbb{R}^{n-1} \mid \exists\, \tilde{f}(y) \in \mathbb{R},\ \lim_{r \to 0} \fint_{B_r^{n-1}(y)} |f(z) - \tilde{f}(y)|\, dz \to 0 \right\},$$

and

$$B := \left\{ y \in \mathbb{R}^{n-1} \mid \exists\, \tilde{D}f(y) \in \mathbb{R}^n,\ \lim_{r \to 0} \fint_{B_r^{n-1}(y)} |Df(z) - \tilde{D}f(y)|\, dz \to 0 \right\},$$

where $B_r^{n-1}(y)$ is the $(n-1)$-dimensional ball of radius r centered at y. To be precise we will call the graph of f the set of points of the form

$$\mathrm{graph}(f) := \{(y,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y \in A \cap B,\ t = \tilde{f}(y)\}.$$

Proposition 6.1. *Let $f \in \mathcal{W}^{1,p}(\mathbb{R}^{n-1})$ with $1 < p \leq n-1$ and $n \geq 3$. Then* $\mathrm{graph}(f)$ *belongs to $\mathcal{J}_p$.*

Proof. By using the theory of capacity developed in [9] (see also [7, Section 4.7]), and the definition of $\mathcal{W}^{1,p}$, we know that

$$\dim_{\mathcal{H}}(\mathbb{R}^{n-1} \setminus A \cap B) \leq n-1-p.$$

Therefore, it follows for example by [19, Corollary 8.11] that

$$\dim_{\mathcal{H}}([\mathbb{R}^{n-1} \setminus (A \cap B)] \times \mathbb{R}) \leq n-p.$$

We claim that for every $x = (y,t) \in \mathbb{R}^n$ such that $y \in A \cup B$ one and only one of the following conditions occurs

- $x \in \partial^* S_f^-$;
- $\Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* S_f^-, x) = 0$.

By Proposition 3.7, this would imply that $\partial^* S_f^-$ has a non vanishing upper isoperimetric profile at x.

We first prove that for every $x = (y,t) \in (A \cup B) \times \mathbb{R}$ such that $t < \tilde{f}(y)$, it holds

$$\Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* S_f^-, x) = 0, \tag{6.1}$$

or equivalently

$$\lim_{r \to 0^+} \mathcal{H}^{n-1}\big((\partial^* S_f^-)_{r,x}\big) = 0. \tag{6.2}$$

Now, since $\lim_{r \to 0} \fint_{B_r^{n-1}(y)} |f(z) - \tilde{f}(y)|\, dz = 0$, then by a change of variable in the integral we have

$$\lim_{r \to 0^+} \|f(y + r(\cdot)) \to \tilde{f}(y)\|_{L^1(B_1^{n-1}(0))} = 0.$$

In particular, this means that for every $\epsilon > 0$

$$\lim_{r \to 0^+} |\{|f(y + r(\cdot)) - \tilde{f}(y)| > \epsilon\} \cap B_1^{n-1}(0)| = 0. \tag{6.3}$$

[1] By using the theory of capacity it is easy to see that $W^{2,p}(\mathbb{R}^{n-1}) \subset \mathcal{W}^{1,p}(\mathbb{R}^{n-1})$

For every $z \in B_1^{n-1}(0)$ such that $|f(y+rz) - \tilde{f}(y)| \leq \epsilon$ we have

$$\frac{|f(y+rz)-t|}{r} \geq \frac{|\tilde{f}(y)-t|}{r} - \frac{|f(y+rz)-\tilde{f}(y)|}{r} \geq \frac{\tilde{f}(y)-t}{r} - \frac{\epsilon}{r},$$

and if $\epsilon < \frac{\tilde{f}(y)-t}{2}$, by the previous inequalities we deduce

$$\frac{|f(y+rz)-t|}{r} > \frac{\tilde{f}(y)-t}{2r}.$$

Hence, for sufficiently small value of r, we have

$$\frac{|f(y+rz)-t|}{r} > 1.$$

Therefore, for sufficiently small value of r, we have

$$\{|f(y+r(\cdot))-t|/r \leq 1\} \cap B_1^{n-1}(0) \subset \{|f(y+r(\cdot)) - \tilde{f}(y)| > \epsilon\} \cap B_1^{n-1}(0).$$

Notice that

$$(\partial^* S_f^-)_{r,x} \subset \left\{(z,s) \in B_1^{n-1}(0) \times (-1,1) \mid s = \frac{f(y+rz)-t}{r}\right\}.$$

As a consequence, for sufficiently small value of r, we have the following inequalities

$$\mathcal{H}^{n-1}\big((\partial^* S_f^-)_{r,x}\big) \leq \int_{\{|f(y+r(\cdot))-t|/r\leq 1\}\cap B_1^{n-1}(0)} \sqrt{1+|Df(z)|^2}\, dz \tag{6.4}$$

$$\leq \int_{\{|f(y+r(\cdot))-\tilde{f}(y)|>\epsilon\}\cap B_1^{n-1}(0)} \sqrt{1+|Df(z)|^2}\, dz. \tag{6.5}$$

Therefore, by using (6.3) and the definition of B, we deduce

$$\lim_{r\to 0^+} \int_{\{|f(y+r(\cdot))-\tilde{f}(y)|>\epsilon\}\cap B_1^{n-1}(0)} \sqrt{1+|Df(z)|^2}\, dz = 0,$$

which proves claim (6.1).

Analogously one can prove that if $x = (y,t) \in (A \cup B) \times \mathbb{R}$ is such that $\tilde{f}(y) < t$ then (6.1) holds.

Finally it remains to prove that if $x \in \mathrm{graph}(u)$ then $x \in \partial^* S_f^-$. First of all, since y is a Lebesgue point for u and a Lebesgue point for Du, by using [1, Theorem 3.83], u is approximately differentiable at y, i.e.

$$\lim_{r\to 0^+} \int_{B_1^{n-1}(0)} \frac{|f(y+rz) - f(y) - \tilde{D}u(y)\cdot rz|}{r}\, dz = 0.$$

Therefore if we set $L_y(z) := \tilde{D}u(y)\cdot z$, then

$$\frac{f(y+r(\cdot)) - \tilde{f}(y)}{r} \to L_y(\cdot), \quad \text{in } L^1(B_1^{n-1}(0)), \text{ as } r \to 0^+. \tag{6.6}$$

This means that if we define $C_1(0)$ to be the cylinder given by $B_1^{n-1}(0) \times (-1,1)$, we can write

$$\lim_{r\to 0^+} P((S_f^- - x)/r; C_1(0)) = \lim_{r\to 0^+} \int_{B_1^{n-1}(0)} \sqrt{1+|Df(y+rz)|^2}\, dz. \tag{6.7}$$

Moreover, if we call H_x^- the lower half space relative to the unit vector $\frac{(-\tilde{D}u(y),1)}{\sqrt{1+|\tilde{D}u(y)|^2}}$, we can continue equality (6.7) in the following way

$$\begin{aligned}\lim_{r\to 0^+} P((S_f^- - x)/r; C_1(0)) &= \lim_{r\to 0^+} \int_{B_1^{n-1}(0)} \sqrt{1+|Df(y+rz)|^2}\, dz \\ &= P(H_x^-; C_1(0)),\end{aligned} \tag{6.8}$$

where we used that y is a Lebesgue point for Du. Putting together (6.6) with (6.8) we deduce

(i) $(S_f^- - x)/r \to H_x^-$ in measure in $C_1(0)$ as $r \to 0^+$;
(ii) $\lim_{r\to 0^+} P((S_f^- - x)/r; C_1(0)) = P(H_x^-; C_1(0))$.

Since $B_1(0) \subset C_1(0)$, condition (i) implies in particular

$$(S_f^- - x)/r \to H_x^-, \text{ in measure in } B_1(0), \text{ as } r \to 0^+. \tag{6.9}$$

Moreover, since $P(H_x^-; \partial B_1(0)) = 0$ we have

$$\begin{aligned} P(H_x^-; C_1(0)) &= \lim_{r\to 0^+} P((S_f^- - x)/r; C_1(0)) \\ &\geq \limsup_{r\to 0^+} [P((S_f^- - x)/r; B_1(0)) + P((S_f^- - x)/r; C_1(0) \setminus \overline{B}_1(0))] \\ &\geq \limsup_{r\to 0^+} P((S_f^- - x)/r; B_1(0)) + \liminf_{r\to 0^+} P((S_f^- - x)/r; C_1(0) \setminus \overline{B}_1(0)) \\ &\geq \liminf_{r\to 0^+} P((S_f^- - x)/r; B_1(0)) + \liminf_{r\to 0^+} P((S_f^- - x)/r; C_1(0) \setminus \overline{B}_1(0)) \\ &\geq P(H_x^-; B_1(0)) + P(H_x^-; C_1(0) \setminus \overline{B}_1(0)) \\ &= P(H_x^-; C_1(0)), \end{aligned}$$

which implies

$$\limsup_{r\to 0^+} P((S_f^- - x)/r; B_1(0)) = \liminf_{r\to 0^+} P((S_f^- - x)/r; B_1(0)) = P(H_x^-; B_1(0)). \tag{6.10}$$

Putting together (6.9) and (6.10) we can apply [1, Proposition 1.62] to the measures $D\mathbb{1}_{(S_f^- - x)/r}$ $(0 < r < 1)$, to deduce that

$$D\mathbb{1}_{(S_f^- - x)/r}(B_1(0)) \to D\mathbb{1}_{H_x^-}(B_1(0)) = \omega_{n-1}\,\nu_{H_x}(x), \quad \text{as } r \to 0^+, \tag{6.11}$$

where ν_{H_x} is the inner unit vector relative to H_x^-. Finally, by (6.10) we deduce

$$\lim_{r\to 0^+} \frac{P(S_f^-; B_r(x))}{r^{n-1}} = \omega_{n-1},$$

which together with (6.11) implies

$$\lim_{r\to 0+} \frac{D\mathbb{1}_{S_f^-}(B_r(x))}{|D\mathbb{1}_{S_f^-}|(B_r(x))} = \lim_{r\to 0+} \frac{r^{n-1}\,\mu_r(B_1(0))}{P(S_f^-; B_r(x))} = \lim_{r\to 0+} \frac{\mu_r(B_1(0))}{\omega_{n-1}} = \nu_{H_x}(x),$$

and this is exactly (1.1), hence we can conclude $x \in \partial^* S_f^-$. □

Remark 6.2. Since for $n - 1 - 2p > 0$ it is possible to construct functions $u \in W^{2,p}(\mathbb{R}^{n-1})$ such that the topological closure of their graphs have arbitrarily large n-dimensional Lebesgue measure, with the previous example we have shown that a generic set in $\mathcal{J}_p$ is not essentially closed.

Proposition 6.3. *Let Ω be an open set of $\mathbb{R}^n$ ($n \geq 3$), and let $(\Gamma_i)_{i=1}^M$ ($M \in \mathbb{N}$), be sets such that for every i there exists $\xi_i \in \mathbb{S}^{n-1}$ and $f_i \in \mathcal{W}^{1,p}(\xi_i^\perp)$ ($1 < p \leq n-1$) with $\Gamma_i :=$ graph$(f_i) \cap \Omega$. Then $\Gamma := \bigcup_{i=1}^M \Gamma_i$ belongs to $\mathcal{J}_p$*

Proof. Proposition 6.1 shows that for every $x \in \Omega$ and for every $1 \leq i \leq M$, except an $(n-p)$-dimensional Hausdorff set, one and only one of the following conditions occurs

- $x \in \partial^* S_{f_i}^-$;
- $\Theta^{*(n-1)}(\mathcal{H}^{n-1} \llcorner \partial^* S_{f_i}, x) = 0$.

Now fix such an $x \in \Omega$. By reordering the indices i we may suppose for example that there exists $k \in \mathbb{N}$ such that for every $1 \leq i \leq k$ $x \in \partial^* S_{f_i}^-$ and for every $k < i \leq M$ $\Theta^{*(n-1)}(S_{f_i}, x) = 0$. Without loss of generality we may also suppose that if $1 \leq i_1 < i_2 \leq k$ and Γ_{i_1} Γ_{i_2} have the same tangent space at x, then the measure-theoretic normals of $S_{f_{i_1}}^-$ and $S_{f_{i_2}}^-$ are the same at x. For the same reason, without loss of generality, we may suppose that for every $k < i \leq M$ x is a point of density 1 for $S_{f_i}^-$.

Given $r > 0$ such that $B_r(x) \subset \Omega$, we set for $1 \leq i \leq k$

$$E_i^- := S_{f_i}^- \cap B_r(x) \text{ and } E_i^+ := S_{f_i}^+ \cap B_r(x).$$

and

$$E_{0,i}^- := \{y \in B_1(0) \mid \nu_{\Gamma_i}(x) \cdot y < 0\} \text{ and } E_{0,i}^+ := \{y \in B_1(0) \mid \nu_{\Gamma_i}(x) \cdot y > 0\},$$

For $k < i \leq M$ we set

$$E_{i,1} := S_{f_i}^- \cap B_r(x),$$

and

$$E_{0,i} := B_1(0).$$

By eventually reordering again the first k indices, we may assume that there exist $k_1, k_2, \dots, k_m$ ($m \leq k$) such that

$$\nu_{\Gamma_{i_1}} = \nu_{\Gamma_{i_2}} \text{ if and only if } k_j \leq i_1, i_2 < k_{j+1}.$$

Now we want to define the sets $F_{r,j}$ and $E_{0,j}$ of Definition 3.2. For this purpose let us denote as Σ_2^M the family of maps from $\{1, \dots, M\}$ into $\{-, +\}$. Given $\sigma \in \Sigma_2^M$ we define

$$E_\sigma = \bigcap_{i=1}^{M} E_i^{\sigma(i)},$$

and

$$E_{0,\sigma} = \bigcap_{i=1}^{M} E_{0,i}^{\sigma(i)},$$

whenever $E_{0,\sigma} \neq \emptyset$.

We have $1 \leq N_x \leq 2^M$ (here, instead of j, we have indexed our sets with the parameter σ running in Σ_2^M). Notice that

$$\lim_{r \to 0^+} |(E_\sigma)_{r,x} \Delta E_{0,\sigma}| = 0.$$

Moreover $E_{0,\sigma}$ are conical and indecomposable sets, since they are intersection of half spaces.

Also, by our choice of $x \in \Omega$, we have that

$$\lim_{r \to 0^+} P((E_i^{\pm})_{r,x}; B_1(0)) = P(E_{0,i}^{\pm}; B_1(0)), \ i = 1, \dots, M.$$

By construction we have also that, since $E_{0,\sigma} \neq \emptyset$, then $\sigma(i_1) = \sigma(i_2)$ for every $k_j \leq i_1, i_2 < k_{j+1}$ and for every $j = 1, \dots, m$. This means that the family $(E_{0,i}^{\sigma(i)})_{i=1}^M$ satisfies also point (3) of Lemma 6.4. Therefore we can deduce that

$$\lim_{r \to 0^+} P((E_\sigma)_{r,x}; B_1(0)) = P(E_{0,\sigma}; B_1(0)).$$

Hence, we are in position to apply Proposition 1.13 and to deduce that for every $\sigma \in \Sigma_2^M$ such that $E_{0,\sigma} \neq \emptyset$, there exist indecomposable components of $(E_\sigma)_{r,x}$, say $F_{r,\sigma}$ such that

$$\lim_{r \to 0^+} |F_{r,\sigma} \Delta E_{0,\sigma}| = 0, \tag{6.12}$$

and

$$\lim_{r \to 0^+} P(F_{r,\sigma}; B_1(0)) = P(E_{0,\sigma}; B_1(0)). \tag{6.13}$$

This gives immediately condition (1.1) and (2.1) of Definition 3.2.

Finally, by (6.13) we can use the same argument as in the proof of Proposition 3.7 to deduce

$$\liminf_{r \to 0^+} h_{F_{r,\sigma}}(\lambda) \geq h_{E_{0,\sigma}}(\lambda), \ \lambda \in (0, 1/2],$$

which implies condition (1.2) of Definition 3.2 since $h_{E_{0,\sigma}}(\lambda) > 0$ for every $\lambda \in (0, 1/2]$. □

Lemma 6.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $(E_{r,i})_{i=1}^M$ ($M \in \mathbb{N}$) be sets having finite perimeter in Ω. Suppose that there exist sets $(E_{0,i})_{i=1}^M$ having finite perimeter in Ω such that*

(1) $\lim_{r \to 0^+} |E_{r,i} \Delta E_{0,i}| = 0, \ 1 \leq i \leq M;$
(2) $\lim_{r \to 0^+} P(E_{r,i}; \Omega) = P(E_{0,i}; \Omega), \ 1 \leq i \leq M;$
(3) $\mathcal{H}^{n-1}(\partial^* E_{0,i_1} \cap \partial^* E_{0,i_2} \cap \{\nu_{E_{0,i_1}} \neq \nu_{E_{0,i_2}}\}) = 0, \ 1 \leq i_1 < i_2 \leq M.$

Then we have

$$\lim_{r \to 0^+} P\Big(\bigcap_{i=1}^{M} E_{r,i}; B_1(0)\Big) = P\Big(\bigcap_{i=1}^{M} E_{0,i}; B_1(0)\Big).$$

Proof. We proceed by induction on M. For $M = 1$ there is nothing to prove. By induction suppose that our statement holds for $M-1$, then we want to show that it still holds for M. For this purpose, suppose to have $(E_{r,i})_{i=1}^{M}$ satisying (1)-(3). If we consider the first $M-1$ sets $(E_{r,i})_{i=1}^{M-1}$, then clearly they still satisfy (1)-(3), hence by inductive hypothesis we have

$$\lim_{r\to 0^+} P\Big(\bigcap_{i=1}^{M-1} E_{r,i}; B_1(0)\Big) = P\Big(\bigcap_{i=1}^{M-1} E_{0,i}; B_1(0)\Big). \tag{6.14}$$

If we define $E'_r := \bigcap_{i=1}^{M-1} E_{r,i}$, $E'_0 := \bigcap_{i=1}^{M-1} E_{0,i}$ and $E_r := E_{r,M}$, $E_0 := E_{0,M}$, then we have that the couple E_r, E'_r still satisfies (1)-(3): the first is clearly satisfied; the second follows from (6.14); for the third just notice that if $x \in \partial^* E'_0 \cap \partial^* E_0$ then there must exists $1 \le i \le M-1$ such that $x \in \partial^* E_{0,i} \cap \partial^* E_{0,M}$, therefore if $\nu_{E'_0}(x) = -\nu_{E_0}(x)$ also $\nu_{E_{0,i}}(x) = -\nu_{E_{0,M}}(x)$. This immediately implies $\mathcal{H}^{n-1}(\partial^* E'_0 \cap \partial^* E_0 \cap \{\nu_{E'_0} \ne \nu_{E_0}(x)\}) = 0$. Hence, we are reduced to prove our statement for $M = 2$.

In order to do that, we notice that by Theorem 1.2 the following identities hold

$$\begin{aligned} P(E'_r; B_1(0)) &= \mathcal{H}^{n-1}(\partial^* E'_r \cap E_r^{(1)}) + \mathcal{H}^{n-1}(\partial^* E'_r \cap E_r^{(0)}) \\ &\quad + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} = \nu_{E_r}\}) + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\}), \end{aligned} \tag{6.15}$$

and analogously

$$\begin{aligned} P(E_r; B_1(0)) &= \mathcal{H}^{n-1}(\partial^* E_r \cap E_r'^{(1)}) + \mathcal{H}^{n-1}(\partial^* E_r \cap E_r'^{(0)}) \\ &\quad + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} = \nu_{E_r}\}) + \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\}). \end{aligned} \tag{6.16}$$

Summing both sides of (6.15) and (6.16), and using Leibniz formulas (1.2) for the reduced boundary of an intersection of sets with finite perimeter we get

$$\begin{aligned} P(E'_r; B_1(0)) + P(E_r; B_1(0)) &= P(E'_r \cap E_r; B_1(0)) + P(E_r'^c \cap E_r^c; B_1(0)) \\ &\quad + 2\mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\}). \end{aligned} \tag{6.17}$$

Taking the lim sup on both sides we get

$$\begin{aligned} P(E'_0; B_1(0)) + P(E_0; B_1(0)) &= \limsup_{r\to 0^+} [P(E'_r \cap E_r; B_1(0)) + P(E_r'^c \cap E_r^c; B_1(0)) \\ &\quad + 2\mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\})] \\ &\ge \liminf_{r\to 0^+} [P(E'_r \cap E_r; B_1(0)) + P(E_r'^c \cap E_r^c; B_1(0)) \\ &\quad + 2\mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\})] \\ &\ge P(E'_0 \cap E_0; B_1(0)) + P(E_0'^c \cap E_0^c; B_1(0)) \\ &\quad + 2\liminf_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\}) \\ &= P(E'_0; B_1(0)) + P(E_0; B_1(0)) \\ &\quad + 2\liminf_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\}), \end{aligned} \tag{6.18}$$

where in the last equality we use again identity (6.17) for E'_0 E_0, and the fact that $\mathcal{H}^{n-1}(\partial^* E'_0 \cap \partial^* E_0 \cap \{\nu_{E'_0} \ne \nu_{E_0}(x)\}) = 0$.

By (6.18) we immediately deduce

$$\liminf_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\}) = 0.$$

Moreover since (6.18) is true for every subsequence $r_j \to 0^+$ we can choose r_j such that

$$\limsup_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\}) = \lim_{j\to\infty} \mathcal{H}^{n-1}(\partial^* E'_{r_j} \cap \partial^* E_{r_j} \cap \{\nu_{E'_{r_j}} \ne \nu_{E_{r_j}}\}),$$

and we immediately deduce that

$$\lim_{r\to 0^+} \mathcal{H}^{n-1}(\partial^* E'_r \cap \partial^* E_r \cap \{\nu_{E'_r} \ne \nu_{E_r}\}) = 0.$$

Using this last information again in (6.18), we obtain

$$\lim_{r\to 0^+} [P(E'_r \cap E_r; B_1(0)) + P(E_r'^c \cap E_r^c; B_1(0))] = P(E'_0 \cap E_0; B_1(0)) + P(E_0'^c \cap E_0^c; B_1(0)),$$

which by the lower semicontinuity of the perimeter implies separately

$$\lim_{r\to 0^+} P(E'_r \cap E_r; B_1(0)) = P(E'_0 \cap E_0; B_1(0)),$$

and

$$\lim_{r\to 0^+} P(E'^c_r \cap E^c_r; B_1(0)) = P(E'^c_0 \cap E^c_0; B_1(0)).$$

This is exactly our desired result. □

The purpose of the previous propositions was to show that the class $\mathcal{J}_p$ is much richer than the class of C^1-manifolds. Nevertheless, we were able to cover condition (1.2) of Definition 3.2, by using the convergence of the perimeter

$$\lim_{r\to 0^+} P(F_{r,j}; B_1(0)) = P(E_{0,j}; B_1(0)),\ 1 \le j \le N_x. \tag{6.19}$$

However, we want to show that (6.19) is not necessary in order to have a non-vanishing upper isoperimetric profile at x. In the next example we exhibit a rectifiable set Γ in $\mathbb{R}^2$ such that there exists a set of Hausdorff dimension α $(0 < \alpha < 1)$ on which Γ admits an asymptotic upper isoperimetric profile but the limit (6.19) diverges to $+\infty$.

Example 6.5 (Cantor's home). We work in $\mathbb{R}^2$. We define a sequence of closed sets, say $(J_n)_{n=1}^\infty$, following the usual way to construct the *Cantor's middle third set* (see [19, Subsection 4.10]): let $J_1 := [0, 1]$ and define $J_n := \frac{J_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{J_{n-1}}{3}\right)$. Set $C_n := \bigcap_{k=1}^n J_n$.

Now fix $2 < s < 3$ and consider by induction the following sets:

$$\mathscr{C}_1 := J_1 \times \left[0, \frac{1}{s-1}\right],$$

and

$$\mathscr{C}_n := \mathscr{C}_{n-1} \setminus (C_{n-1} \setminus C_n \times (-\infty, s_n)), \quad (n \ge 2)$$

where

$$s_n := \frac{s^{1-n}}{(s-1)} = \sum_{i=n}^\infty \frac{1}{s^i} \quad \text{and} \quad \frac{1}{s-1} = \sum_{i=1}^\infty \frac{1}{s^i}.$$

We define the Cantor's home $\mathscr{C} \subset \mathbb{R}^2$ as

$$\mathscr{C} := \bigcap_{n=1}^\infty \mathscr{C}_n.$$

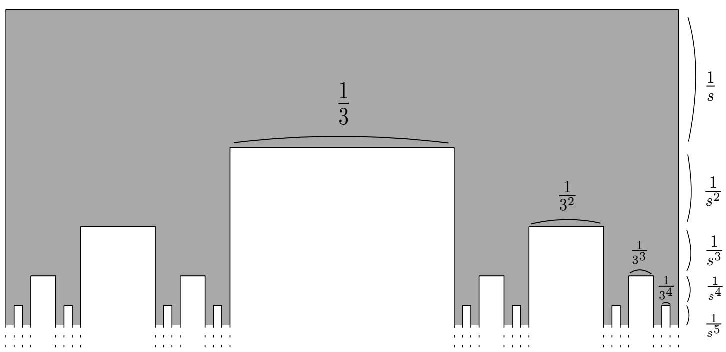

FIGURE 2. *Cantor's home*

By construction $\mathscr{C}$ is a closed set and $P(\mathscr{C}) < \infty$. Indeed it can be easily verified that

$$P(\mathscr{C}_{n+1}) = P(\mathscr{C}_n) + \frac{2^n}{(s-1)s^n},\ n = 1, 2, \ldots,$$

which means

$$P(\mathscr{C}) \le \liminf_{n\to\infty} P(\mathscr{C}_n) = P(\mathscr{C}_1) + \lim_{n\to\infty} \sum_{i=1}^n \frac{2^n}{(s-1)s^n} < \infty,$$

where in the last inequality we have used $s > 2$.

We claim that $\partial^* \mathscr{C}$ admits a non vanishing upper isoperimetric profile at every $x \in \mathbb{R}^2$. As a consequence $\partial^* \mathscr{C} \in \mathscr{J}_p$ for every $p > 1$.

To prove our claim, notice that if we call $C \subset [0,1]$ the Cantor's set, i.e.

$$C = \bigcap_{n=1}^{\infty} J_n, \tag{6.20}$$

then it is easy to see that for every $x \in \mathbb{R}^2 \setminus (C \times \{0\})$ our claim is satisfied. Therefore, we need only to prove that for $x \in C \times \{0\}$ our claim holds.

If $x \in C \times \{0\}$, by using the fact that the number of connected components of $J_n \cap (x_1 - \frac{r}{2}, x_1 + \frac{r}{2})$ can be asymptotically estimated by $2^{n - \log_{1/3} r}$ as $n \to \infty$, together with the fact that $s < 3$, it is possible to check that

$$\Theta^2(\mathcal{L}^2 \llcorner \mathscr{C}, x) = 0, \tag{6.21}$$

while

$$\Theta^1(\mathcal{H}^1 \llcorner \partial^* \mathscr{C}, x) = +\infty. \tag{6.22}$$

Now we denote the generic point $x \in \mathbb{R}^2$ as $x = (x_1, x_2)$ for $x_1, x_2 \in \mathbb{R}$, and we prove that conditions (1) and (2) of Definition 3.2 are satisfied with $N_x = 1$. Instead of the balls $B_r(x)$ we prefer to work with the squares $Q_r(x)$. It is clear that everything will be true also for balls.

Pick $x \in C \times \{0\}$. Set

- $E_{0,1} := Q_1(0)$;
- $F_{r,1} := Q_1(0) \setminus \frac{\mathscr{C} - x}{r}$ for every $r > 0$.

First of all, since for each r the sets $F_{r,1}$ are connected open sets with finite perimeter, then they are indecomposable (see Remark 1.6). Moreover, conditions (1.1) and (2.1) immediately follow from construction and from (6.21), respectively.

In order to show that also condition (1.2) is satisfied, first of all notice that for each $r > 0$ the sets $F_{r,1}$ are open connected and of finite perimeter, hence in particular they are indecomposable. We claim that for every $r > 0$ and every $\lambda \in (0, 1/2]$ we have

$$h_{F_{r,1}}(\lambda) \geq \frac{1}{3}. \tag{6.23}$$

In order to show (6.23) we shall prove that for every $r > 0$

$$\min\{|E|, |F_{r,1} \setminus E|\} \leq 3\mathcal{H}^1(\partial^* E \cap F_{r,1}^{(1)}), \ E \subset F_{r,1}. \tag{6.24}$$

This can be achieved by proving that for every $r \in (0,1)$ it holds the following Poincaré's inequality

$$\int_{F_{r,1}} |u - \overline{u}| \, dx \leq 3|Du|(F_{r,1}), \ u \in BV(Q_1(0)), \tag{6.25}$$

where $\overline{u} := \fint_{F_{r,1}} u$. Then (6.24) follows by choosing $u = \mathbb{1}_E$ in (6.25), since in this case

$$\int_{F_{r,1}} |u - \overline{u}| \, dx \geq \min\{|E|, |F_{r,1} \setminus E|\},$$

and

$$\mathcal{H}^{n-1}(\partial^* E \cap F_{r,1}^{(1)}) \geq \mathcal{H}^{n-1}(\partial^* E \cap F_{r,1}) = |Du|(F_{r,1}).$$

Given $t \in \mathbb{R}$ we write

$$F_t := \{x_2 \in \mathbb{R} \mid (t, x_2) \in F\}, \text{ and } F^t := \{x_1 \in \mathbb{R} \mid (x_1, t) \in F\}.$$

Notice that for each $r > 0$ the sets $F_{r,1}$ have the following two properties

(1) $(x_1, x_2) \in F_{r,1}$ and $(x_1, y_2) \in F_{r,1}$ implies $(x_1, \lambda x_2 + (1-\lambda) y_2) \in F_{r,1}$ for every $\lambda \in [0,1]$;
(2) $(x_1, x_2) \in F_{r,1}$, $(y_1, x_2) \in F_{r,1}$ and $x_2 \in (-1/2, 0)$ implies $(\lambda x_1 + (1-\lambda) y_1, x_2) \in F_{r,1}$ for every $\lambda \in [0,1]$.

We show that any set $F \subset Q_1(0)$ satisfying (1) and (2) admits a Poincaré's inequality like (6.25).

Indeed we have

$$\int_F |u-\overline{u}|\,dx = 2\int_{-1/2}^0 \left[\fint_F \left| u(x_1,x_2) - \left(\fint_F u(y_1,y_2)\,dy_1 d\,y_2 \right) \right| dx_1\,dx_2 \right] dt$$
$$\le 2\int_{-1/2}^0 \left[\int_F \left(\fint_F |u(x_1,x_2)-u(y_1,y_2)|\,dy_1 d\,y_2 \right) dx_1\,dx_2 \right] dt.$$

If $t \in (-1/2, 0)$, by using the triangle inequality, we can write

$$|u(x_1,x_2)-u(y_1,y_2)| \le |u(x_1,x_2)-u(x_1,t)| + |u(x_1,t)-u(y_1,t)| + |u(y_1,t)-u(y_1,y_2)|,$$

hence by using the Fundamental Theorem of Calculus we have

$$\int_F |u-\overline{u}|\,dx \le 2\int_{-1/2}^0 \left[\int_F \left(\fint_F |D_2 u|(F_{x_1})\,dy_1 d\,y_2 \right) dx_1\,dx_2 \right] dt$$
$$+ 2\int_{-1/2}^0 \left[\int_F \left(\fint_F |D_1 u|(F^t)|\,dy_1 d\,y_2 \right) dx_1\,dx_2 \right] dt$$
$$+ 2\int_{-1/2}^0 \left[\int_F \left(\fint_F |D_2 u|(F_{y_1})\,dy_1 d\,y_2 \right) dx_1\,dx_2 \right] dt,$$

and finally by using Fubini's Theorem

$$\int_F |u-\overline{u}|\,dx \le \int_{-1/2}^{1/2} |D_2 u|(F_{x_1})|\mathcal{L}^1(F_{x_1})\,dx_1 + 2|F|\int_{-1/2}^0 |D_1 u|(F^t)\,dt$$
$$+ \int_{-1/2}^{1/2} |D_2 u|(F_{y_1})|\mathcal{L}^1(F_{y_1})\,dy_1$$
$$\le |D_2 u|(F) + 2|F||D_1 u|(F) + |D_2 u|(F)$$
$$\le (1+2|F|)|Du|(F).$$

Since $|F| \le 1$, this shows exactly (6.25).

Remark 6.6. In the previous example, the set $\mathscr{C}$ has a non vanishing upper isoperimetric profile for every $x \in \mathbb{R}^2$ but with the stronger condition

$$\liminf_{r\to 0^+} \inf_{\lambda>0} h_{F_{r,1}}(\lambda) > 0.$$

Therefore by Remark 4.9 if we set $\Gamma = \mathscr{C}$ in Theorem 1, then (0.3) actually holds with respect to the strong L^p-convergence.

6.2. **A counterexample.** Now we want to exploit the idea of the previous example to show that, the indecomposability condition together with condition (1.2) of Definition 3.2 are crucial in order to get the validity of Theorem 1.

Example 6.7 (Optimality for the class $\mathscr{J}_p$). We start by showing that the indecomposability assumption on the sets $(F_{r,j})$ in Definition 3.2, cannot be removed in order to get the validity of Theorem 1. For this purpose we shall construct a countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable set Γ in $\mathbb{R}^2$ with $\mathcal{H}^{n-1}(\Gamma) < \infty$, such that there exists $\Gamma_0 \subset \mathbb{R}^2$ with $\dim_{\mathcal{H}}(\Gamma_0) = \log_3(2)$, with the following properties

(a) for every $x \in \Gamma_0$ there exists a family of sets $F_r \subset B_1(0)$ $(r>0)$ satisfying $\lim_{r\to 0^+} |F_r \Delta B_1(0)| = 0$;
(b) there exists a function $u \in SBV^2(\mathbb{R}^2;\Gamma)$ such that for every $x \in \Gamma_0$ the blow-up $u_{r,x}$ does not converge.

To construct such a Γ, we start by considering $\tilde{\mathscr{C}} \subset \mathbb{R}^2$ the reflection of the Cantor's home given in Example 6.5 with respect to the x_1 axis, i.e.

$$\tilde{\mathscr{C}} := \{(x_1,x_2) \in \mathbb{R}^2 \mid (x_1,-x_2) \in \mathscr{C}\}.$$

We define $E := \mathscr{C} \cup \tilde{\mathscr{C}}$. The set E can be seen also as the limit in measure of $\mathscr{C}_n \cup \overline{\mathscr{C}}_n$ when $n \to \infty$, where $\mathscr{C}_n$ is the approximated Cantor's home at the n-th step (see Example 6.5) and $\tilde{\mathscr{C}}_n$ is its reflection with respect to the x_1-axis (see Figure 3). Clearly E has a non vanishing upper isoperimetric profile for every $x \in \mathbb{R}^2 \setminus (C \times \{0\})$, where C denotes the Cantor's set (see (6.20)). By arguing as in Example 6.5 we know that E is a set of finite perimeter and

$$\Theta^n(\mathcal{L}^n \llcorner E, x) = 0, \quad x \in C \times \{0\}.$$

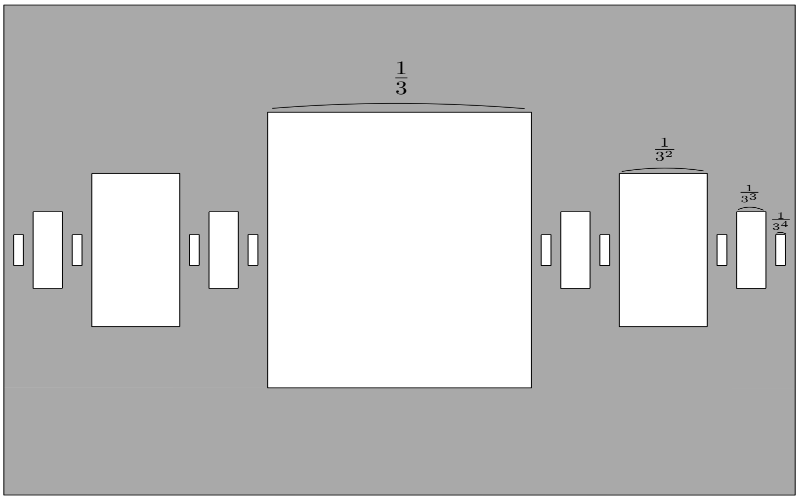

Figure 3. *Approximation of E at the fifth step.*

Now let (C_n) and (s_n) be the sequence of sets and the sequence of numbers defined in Example 6.5, respectively. Define the following function

$$u(x) := \begin{cases} 1 & \text{if } x \in C_{2n-1} \setminus C_{2n} \times (-s_{2n}, s_{2n}) \\ -1 & \text{if } x \in C_{2n} \setminus C_{2n+1} \times (-s_{2n+1}, s_{2n+1}) \\ 0 & \text{otherwise.} \end{cases} \tag{6.26}$$

By Remark 1.22 we know that $u \in SBV^2(\mathbb{R}^2; \partial^* E)$. Call $f \colon \mathbb{R} \to \mathbb{R}$ the restriction of u to the x_1 axis, i.e.

$$f(\cdot) := u(\cdot, 0).$$

Notice that given $x \in C \times \{0\}$, then if the blow-up of u at $x = (x_1, x_2)$ converges as $r \to 0^+$ then also the blow-up of f at x_1 must converge. To see this, by using the fact that the parameter s of Example 6.5 has been chosen strictly less than 3, it is not difficult to show that

$$\lim_{r \to 0^+} \mathcal{L}^1(\{y_1 \in (-1, +1) \mid u_{r,x}(y_1, y_2) = f_{r,x_1}(y_1) \text{ for every } y_2 \}) = 2.$$

This means that any limit u_x in $L^1(B_1)$ of $(u_{r,x})$ must be constant along the segments orthogonal to the x_1-axis and contained in B_1, and moreover it must satisfy

$$\lim_{r \to 0^+} \|f_{r,x_1}(\cdot) - u_x(\cdot, 0)\|_{L^1} = 0.$$

Therefore, given $x = (x_1, x_2)$, if we want to prove that $(u_{r,x})$ does not converge, we can reduce ourselves to prove that f_{r,x_1} does not converge.

In view of the previous observation, we claim that for every $x_1 \in C \times \{0\}$, except on a countable set A, f_{r,x_1} does not converge as $r \to 0^+$. For this purpose, we show that given $x_1 \in C \setminus A$, then for every $\epsilon > 0$ there exists a couple of radii $r, r' \leq \epsilon$ such that

$$\|f_{r,x_1} - f_{r',x_1}\|_{L^0(B_1)} \geq \frac{1}{2}. \tag{6.27}$$

To see this it is convenient to write every point in the Cantor's set in base 3. This means that given $x_1 \in C$, then there exists a map σ defined on every positive integer number with values in $\{0,2\}$, such that

$$x_1 = \sum_{i=1}^{\infty} \frac{\sigma(i)}{3^i}.$$

Define the set A to be the set of points x_1 in the Cantor's set such that there exists $i_0 \in \mathbb{N}$ (depending on x_1) such that for every $i \geq i_0$ the function σ alternates consecutively the values 0 and 2. Clearly the set A is countable. We want to prove our claim on every point $x_1 \in C \setminus A$.

Let $x_1 \in C \setminus A$, then by definition of A there exists a sufficiently large value of n such that $1/3^n \leq \epsilon/2$ and $\sigma(n) = \sigma(n+1) = 0$ or $\sigma(n) = \sigma(n+1) = 2$. Let us suppose to be in the case $\sigma(n) = 2$ (the case $\sigma(n) = 0$ can be treated in the same way). Since $x_1 \in C$, then x_1 belongs to a connected component (an interval) of J_{n-1}, say I (J_n are those defined in Example 6.5). We can consider a partition of I made of three closed intervals I_1, I_2, I_3 (with overlapping end-points) each of length $\frac{|I|}{3}$ where I_1 is the most-left one, I_3 is the most right one, and I_2 is in between. By construction of the sets (J_n), we know that $J_n \cap I = I_1 \cup I_3$, and since $\sigma(n) = 2$ then $x_1 \in I_3$. By (6.26), we know that f takes value 1 or -1 on I_2. Let us suppose for example 1. As before we can consider a partition of I_3 made of three closed intervals $I_{3,1}, I_{3,2}, I_{3,3}$ (with overlapping end-points) each of length $|I_3|/3$ where $I_{3,1}$ is the most-left one, $I_{3,3}$ is the most right one, and $I_{3,2}$ is in between. In addition, by (6.26), we know that f assumes the value -1 on $I_{3,2}$. As before, since $\sigma(n+1) = 2$, then $x_1 \in I_{3,3}$.

Now call a the left end-point of I_2 and b the left end-point of $I_{3,2}$. Clearly we have the following estimates

$$(x_1 - a) \leq \frac{2}{3^n} \quad \text{and} \quad (x_1 - b) \leq \frac{2}{3^{n+1}}. \tag{6.28}$$

Moreover, if we set $r = x_1 - a$ and $r' = x_1 - b$, then we can write

$$\begin{aligned} \|f_{r,x_1} - f_{r',x_1}\|_{L^0(B_1)} &= \fint_{B_1} |f(x_1 + (x_1 - a)y_1) - f(x_1 + (x_1 - b)y_1)| \wedge 1 \, dy_1 \\ &= \fint_{B_{(x_1-a)}} |f(x_1 + y_1) - f(x_1 + (x_1 - b)/(x_1 - a)y_1)| \wedge 1 \, dy_1. \end{aligned}$$

Using the fact that, by construction, the dilated interval $\frac{(x_1-a)}{(x_1-b)}(I_{3,2} - x_1)$ has left end-point coincident with the left end-point of the interval $I_2 - x_1$, and that $|f(x_1 + y_1) - f(x_1 + (x_1 - b)/(x_1 - a)y_1)| = 2$ on $(I_2 - x_1) \cap \frac{(x_1-a)}{(x_1-b)}(I_{3,2} - x_1)$, we can continue the previous inequality by writing

$$\begin{aligned} \|f_{x_1,(x_1-a)} - f_{x_1,(x_1-b)}\|_{L^0(B_1)} &\geq \frac{1}{(x_1 - a)} \left| (I_2 - x_1) \cap \frac{(x_1 - a)}{(x_1 - b)}(I_{3,2} - x_1) \right| \\ &\geq \frac{1}{(x_1 - a)} \min \left\{ |I_2|, \frac{(x_1 - a)}{(x_1 - b)} |I_{3,2}| \right\} \\ &= \min \left\{ \frac{3^{-n}}{(x_1 - a)}, \frac{3^{-(n+1)}}{(x_1 - b)} \right\} \\ &\geq \frac{1}{2}, \end{aligned}$$

where for the last inequality we use (6.28). This proves our claim and shows that at every $x \in (C \setminus A) \times \{0\}$ the blow-up $u_{r,x}$ does not converge as $r \to 0^+$.

Finally, by setting $\Gamma := \partial^* E$, $\Gamma_0 := (C \setminus A) \times \{0\}$, and $F_r := B_1(0) \setminus E_{r,x}$ we obtain (a) and (b). As a consequence we deduce that $\Gamma \notin \mathcal{J}_p$ for every $p \in (2 - \log_3(2), 2]$. In this case it is clear that what fails in the definition of non vanishing upper isoperimetric profile is the indecomposability of the sets $(F_{r,x})_{r>0}$ for every $x \in C \times \{0\}$.

Exploiting the previous idea, it is possible to construct sets Γ Γ_0, and a function u satisfying (a) and (b) with the additional property that the sets F_r are indecomposable. This together with Theorem 1, immediately implies that on every point of Γ_0 the set Γ satisfies all the properties of Definition 3.2 except (1.2). This shows that condition (1.2) is crucial in view of Theorem 1.

The idea is to connect each white rectangle of E (see Figure 3) by small bridges without altering the local behavior of the set E on points of the set $C \times \{0\}$. To do this, define for each $n \geq 1$, $\delta_n := 1/7^n$. We start by connecting the two white rectangles whose horizontal sides have length $1/3^2$ with the white rectangle whose horizontal sides have length $1/3$ (see figure 3) by subtracting from the set E a thin horizontal bridge in the following way

$$E_1 := E \setminus (1/3^2, 1 - 1/3^2) \times (s_3 - \delta_1, s_3).$$

By induction, if we call the n-th thin bridge $R_n := (1/3^{n+1}, 1 - 1/3^{n+1}) \times (s_{n+2} - \delta_n, s_{n+2})$ (s_n are defined in Example 6.5), then we define for general n (see Figure 4 for $n = 3$)

$$E_n := E_{n-1} \setminus R_n.$$

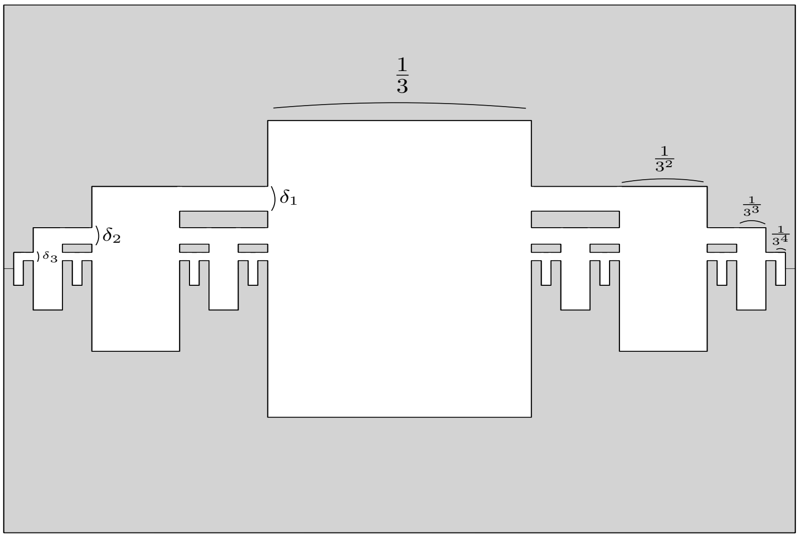

FIGURE 4. *The set* E_3

Since by the choice of (δ_n) the rectangles (R_n) are pairwise disjoint, by subtracting from E_{n-1} the rectangle R_n, one adds an amount of perimeter which is exactly $2(2^{n+1} - 2)/3^{n+1}$, i.e.

$$P(E_n) = P(E_{n-1}) + 2(2^{n+1} - 2)/3^{n+1}.$$

This means that by defining

$$E' := \bigcap_{n=1}^{\infty} E_n,$$

then E' is a closed set of finite perimeter in $\mathbb{R}^2$.

Since $E' \subset E$, this means that we still have that for every $x \in C \times \{0\}$ it holds

$$\Theta^n(\mathcal{L}^n \llcorner E', x) = 0,$$

but with the additional property that for every $r > 0$ the open sets $B_1(0) \setminus E'_{r,x}$ are connected and with finite perimeter and hence indecomposable (see Remark 1.6). The connectedness comes from the fact that if Q_1 is a connected components (white rectangle) of the set $[C_{n-1} \setminus C_n] \times (-s_n, s_n)$ for some n (where (C_n) and (s_n) are defined in Example 6.5), and Q_2 is a connected components (white rectangle) of the set $[C_{m-1} \setminus C_m] \times (-s_m, s_m)$ for some m, both Q_1 Q_2 with non empty intersection with $B_r(x)$ $(x \in C \times \{0\})$, then there must exist a sufficiently large $M \geq \max\{n, m\}$ for which the bridge R_M connects $Q_1 \cap B_r(x)$ with $Q_2 \cap B_r(x)$.

Now we define the function $v \in SBV^2(\mathbb{R}^2; \partial^* E')$ in the following way. If $x \notin \bigcup_{n=1}^{\infty} R_n \cap E$ we define $v(x) := u(x)$ where u is the function defined in (6.26). If $x \in R_n \cap E$ for some n, then by construction there exists a connected components of $J_{n+1} \subset [0, 1]$ (see Example 6.5), say I, such that $x \in I \times (s_{n+2} - \delta_n, s_{n+2})$. We have two cases: suppose that $I \times (s_{n+2} - \delta_n, s_{n+2})$ connects two rectangles where v has the same value, i.e. -1 or $+1$, then we simply define $v(x)$ to be exactly -1 or $+1$, respectively; otherwise suppose that v changes value (for example from -1 to $+1$), then if we call $p\colon I \to \mathbb{R}$ the linear interpolation between -1 and $+1$ we define

$$v(x) := p(\pi_1(x)),$$

where $\pi_1 \colon \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the first component (we proceed analogously if v changes value from $+1$ to -1).

Clearly, since v differs from u on a set which is contained in E, and $\Theta^n(\mathcal{L}^n \llcorner E, x) = 0$ for every $x \in C \times \{0\}$, then the blow-up of v has the same behavior of the blow-up of u at each point in $C \times \{0\}$. It remains to prove that $\nabla v \in L^2(\mathbb{R}^2)$. But by our choice of δ_n, an easy computation shows that we can estimate from above

$$\int_{\mathbb{R}^2} |\nabla v|^2 \, dx \leq 2 \sum_{n=1}^{\infty} \frac{6^{n+1}}{7^n} < \infty.$$

7. Non convergence of the blow-up

In this last part we construct a set in $E \subset \mathbb{R}^2$ with the property that its blow-up $(E - x)/r$ does not converge in measure on every point of a set having Hausdorff dimension equal to 1. To show this, we need the following theorem which can be deduced from the results obtained in [22] (see also [10] for a simpler proof). Anyway we decide to present this result with an alternative proof which is more convenient for our purpose.

Theorem 7.1. *Let $N \subset (0,1)$. Then N has zero Lebesgue measure if and only if there exists a Lipschitz function $u\colon (0,1) \to \mathbb{R}$ such that u is not differentiable from the right at every point of N.*

Proof. If f is lipschitz then the set of point where it is not right differentiable has Lebesgue measure zero from Rademacher's Theorem.

Now let $N \subset \mathbb{R}$ be such that $|N| = 0$. We claim that there exists a Borel set $F \subset \mathbb{R}$ such that for every $x \in N$ we have

$$0 = \liminf_{r \to 0^+} \frac{|F \cap (x, x+r)|}{r} < \limsup_{r \to 0^+} \frac{|F \cap (x, x+r)|}{r} = 1. \tag{7.1}$$

To prove this, notice that for every $0 < \epsilon \leq 1/6$, since $|N| = 0$, we can find a cover of N made of open and disjoint subintervals of $(0,1)$, say $(I_i)_{i=1}^{\infty}$, such that

(1) $\sum_{i=1}^{\infty} |I_i| \leq \epsilon$;
(2) $N \cap I_i \subset \bigcup_{j=1}^{\infty} \{x \in I_i \mid 1/2^{s(j+1)} < \mathrm{dist}(x, \mathbb{R} \setminus I_i) < 1/2^{sj}\}$, for some $s \in [1/2, 1]$ depending on I_i.

Indeed (1) simply follows by the fact that $|N| = 0$. Moreover, let $a < b$ be the end points of the intervals I_i; then since $|N| = 0$ we have that for every j

$$|\{s \in [1/2, 1] \mid a \in N - 1/2^{sj}\}| = 0,$$

therefore also

$$\left| \bigcup_{j=1}^{\infty} \{s \in [1/2, 1] \mid a \in N - 1/2^{sj}\} \right| = 0.$$

For this reason by choosing $s \in \bigcup_{j=1}^{\infty} \{s \in [1/2, 1] \mid a \in N - 1/2^{sj}\}$ we obtain that

$$a + 1/2^{sj} \notin N, \quad j = 1, 2, 3, \ldots .$$

By repeating the same argument for the right end point we can find $s \in [1/2, 1]$ satisfying (2).

Define $\mathcal{I}^1 \subset (0,1)$ to be a cover of N made of open intervals satisfying (1) and (2) with $\epsilon = 1/6$. By induction we define $\mathcal{I}^i$ in the following way. For every $I \in \mathcal{I}^{i-1}$ we consider the set

$$N_j := N \cap \{x \in I \mid 1/2^{s(j+1)} < \mathrm{dist}(x, \mathbb{R} \setminus I) < 1/2^{sj}\},$$

where $s \in [1/2,1]$ is relative to I. Since $|N_j| = 0$ we can use the claim to find a cover of N_j made of open and disjoint subintervals of $\{x \in I \mid 1/2^{s(j+1)} < \text{dist}(x, \mathbb{R} \setminus I) < 1/2^{sj}\}$, say $(I_i)_{i=1}^{\infty}$, satisfying (1) and (2) with $\epsilon = 1/6^j$. Finally, we call $\mathcal{I}^i$ the family made of all open intervals obtained as in the previous procedure, by letting I varies in $\mathcal{I}^{i-1}$ and j varies in $\mathbb{N}$.

We set

$$F := \bigcup_{i=1}^{\infty} \Big(\bigcup_{I \in \mathcal{I}^{2i-1}} I \setminus \bigcup_{I \in \mathcal{I}^{2i}} I \Big), \tag{7.2}$$

and we claim that F does the job. Clearly F is Borel since every $\mathcal{I}^i$ is a family made of open intervals. Moreover, whenever $x \in N$, since for every i the family $\mathcal{I}_i$ covers N, then for every i there exists $\overline{I} \in \mathcal{I}^i$ such that $x \in \overline{I}$. Moreover, by property (2) there exists j such that $x \in \{x \in \overline{I} \mid 1/2^{s(j+1)} < \text{dist}(x, \mathbb{R} \setminus \overline{I}) < 1/2^{sj}\}$ for some $s \in [1/2, 1]$. Since

$$\Big(x, x + \frac{1}{2^{s(j+2)}} \Big) \subset \bigcup_{k=j-1}^{j+1} \{x \in \overline{I} \mid 1/2^{s(k+1)} < \text{dist}(x, \mathbb{R} \setminus \overline{I}) < 1/2^{sk}\},$$

and since by construction the intervals $I \in \mathcal{I}_{i+1}$ which are contained in the set $\{x \in \overline{I} \mid 1/2^{s(k+1)} < \text{dist}(x, \mathbb{R} \setminus \overline{I}) < 1/2^{sk}\}$ are such that $\sum |I| \le \epsilon$ with $\epsilon = 1/6^k$, we can write

$$\Big| \bigcup_{I \in \mathcal{I}^{i+1}} I \cap \Big(x, x + \frac{1}{2^{s(j+2)}} \Big) \Big| \le \sum_{k=j-1}^{j+1} \frac{1}{6^k}$$

If i is odd, by (7.2) we have that $\overline{I} \setminus \bigcup_{I \in \mathcal{I}_{i+1}} I \subset F$, and then

$$\begin{aligned} \frac{|F \cap (x, x + 1/2^{s(j+2)})|}{1/2^{s(j+2)}} &\ge 2^{s(j+2)} \Big(\frac{1}{2^{s(j+2)}} - \sum_{k=j-1}^{j+1} \frac{1}{6^k} \Big) = 1 - \frac{2^{s(j+2)}}{6^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k} \\ &= 1 - \frac{2^{(s-1)j+3s}}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k} \\ &\ge 1 - \frac{2^3}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k}, \end{aligned} \tag{7.3}$$

where for the last inequalty we use $s \in [1/2, 1]$. Therefore for each i odds there exists a corresponding j_i, with $j_i \to \infty$ as $i \to \infty$, satisfying (7.3), hence by letting $i \to \infty$ among all odds numbers, this proves

$$\limsup_{r \to 0^+} \frac{|F \cap (x, x+r)|}{r} = 1.$$

If i is even, by (7.2) we have that $F \cap \overline{I} \subset \bigcup_{I \in \mathcal{I}_{i+1}} I$, and then

$$\begin{aligned} \frac{|F \cap (x, x + 1/2^{s(j+2)})|}{1/2^{s(j+2)}} &\le 2^{s(j+2)} \sum_{k=j-1}^{j+1} \frac{1}{6^k} = \frac{2^{s(j+2)}}{6^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k} \\ &= \frac{2^{(s-1)j+3s}}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k} \\ &\le \frac{2^3}{3^{j-1}} \sum_{k=1}^{3} \frac{1}{6^k}, \end{aligned} \tag{7.4}$$

where for the last inequalty we use $s \in [1/2, 1]$. Arguing as before, this proves

$$\liminf_{r \to 0^+} \frac{|F \cap (x, x+r)|}{r} = 0.$$

Finally, define

$$u(t) := |F \cap (0, t)|, \ t \in (0, 1).$$

Clearly u is 1-Lipschitz and moreover $(u(x+r)-u(x))/r = |(F \cap (x, x+r))|/r$ for every $0 < r < 1-x$. By (7.1) we immediately deduce that u is not right differentiable at every $x \in N$. ☐

Theorem 7.2. *There exists a set $E \subset \mathbb{R}^2$ of finite perimeter, such that the set of points where its blow-up $(E-x)/r$ does not converge locally in measure has Hausdorff dimension* 1.

Proof. Let $N \subset (0,1)$ be a set of Hausdorff dimension equal to 1. It can be easily constructed as a countable union of sets N_k with positive $\mathcal{H}^{1-1/k}$-measure. Clearly N has zero Lebesgue measure.

Let $u\colon (0,1) \to \mathbb{R}$ be the 1-lipschitz function given by the previous proposition, which is not right differentiable at every point of N. Define $E := \{x \in \mathbb{R}^2 \mid 0 < x_1 < 1,\ 0 < x_2 < u(x_1)\}$. We claim that at every point x of the form $x_1 \in N$ and $x_2 = u(x_1)$ the blow-up of E at x does not converge in measure.

Indeed, since u is 1-Lipschitz, then in the square $(-1,1)^2 \subset \mathbb{R}^2$ the set $(E-x)/r$ can be described as the subgraph of the function $y_1 \to (u_{r,x_1}(y_1) - u(x_1))/r$ for $y_1 \in (-1,1)$ and for every $r < \min\{x_1, 1-x_1\}$. Hence the convergence of $(E-x)/r$ locally in measure implies in particular the convergence in $L^1_{loc}((0,1))$ of the sequence $((u_{r,x_1}(y_1) - u(x_1))/r)$ to some function $v(y_1)$. Moreover, since for every $\lambda > 0$ we have

$$v(\lambda y_1) = \lim_{r\to 0^+} (u_{r,x_1}(\lambda y_1) - u(x_1))/r = \lambda \lim_{r\to 0^+} (u_{\lambda r,x_1}(y_1) - u(x_1))/\lambda r = \lambda v(y_1),$$

then v is positively one-homogeneous. But since u is 1-Lipschitz, then the L^1_{loc} convergence can be improved to a uniform convergence on the closed interval $[0,1]$, i.e.

$$\lim_{r\to 0^+} \sup_{y_1 \in [0,1]} \left| \frac{u(x_1 + r y_1) - u(x_1)}{r} - v(y_1) \right| = 0,$$

Thanks to the positively one homogeneity of v, this immediately implies the right differentiability of u at x_1 with $u'(x_1) = v(1)$ which is a contradiction. This proves the theorem. ☐

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